

## Global Universality at the Onset of Chaos: Results of a Forced Rayleigh-Bénard Experiment

Mogens H. Jensen, Leo P. Kadanoff, and Albert Libchaber

*The James Franck Institute, The University of Chicago, Chicago, Illinois 60637*

Itamar Procaccia

*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*

and

Joel Stavans

*The James Franck Institute, The University of Chicago, Chicago, Illinois 60637*

(Received 15 October 1985)

We study an experimental orbit on a two-torus with a golden-mean winding number obtained from a forced Rayleigh-Bénard system at the onset of chaos. This experimental orbit is compared with the orbit generated by a simple theoretical model, the circle map, at its golden-mean winding number at the onset of chaos. The "spectrum of singularities" of the two orbits are compared. Within error, these are identical. Since the spectrum characterizes the metric properties of the entire orbit, this result confirms theoretical speculations that these orbits, taken as a whole, enjoy a kind of universality.

PACS numbers: 47.20.+m, 05.45.+b, 47.25.-c

In the study of the transition to chaos most theoretical attention has been paid to the behavior near special points in phase space. Thus Feigenbaum<sup>1</sup> concentrated upon the region of the maximum of the period-doubling map, Shenker<sup>2</sup> looked near the inflection point of the circle map, etc. In experimental situations, such distinguished points in phase space are not readily discernible. If one is to look experimentally for universality, one would do well to seek more global,<sup>3</sup> but still universal, features of the phase-space orbits.

In this Letter we report experimental results which, together with theoretical analysis, show that critical orbits in phase space at the onset of chaos exhibit global universal properties. The example discussed here is the cycle with golden-mean winding number at the point of breakdown of a 2-torus. The experiment is a periodically forced Rayleigh-Bénard system with mercury as a fluid. Recent measurements on this system revealed two scaling indices<sup>4</sup>: the index for the ratio between two successive Fibonacci resonances<sup>2</sup> and the dimension of the structure of mode locking.<sup>5</sup> Both were in agreement with the indices found for circle maps. We therefore compare the experimentally observed critical orbit with the corresponding orbit in the circle map.

In order to examine global scaling properties it is not sufficient to measure the dimension of the attracting set; the set certainly contains more topological information than can be characterized by a single number.<sup>1b</sup> To achieve a characterization that more fully describes those properties of such sets which remain unchanged under smooth changes of coordinates, it has been proposed to use a continuous spectrum of scaling indices.<sup>6</sup>

These spectra display the range of scaling indices and their density in the set. To clarify what we mean, consider the experimental cycle displayed in Fig. 1. One sees with bare eyes that the time series is concentrated with various intensities in different regions. The spectrum that we use quantifies this variation in density on the attractor, and allows us to show the similarity of

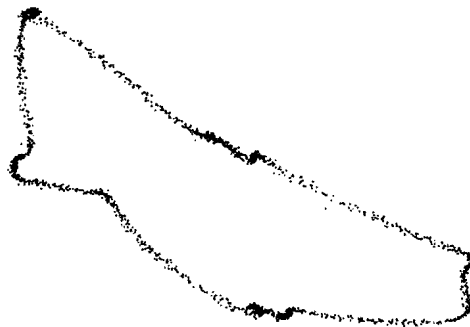


FIG. 1. The experimental attractor in two dimensions. 2500 points are plotted. Note the variation in the density of points on the attractor. Part of this variation is, however, due to the projection of the attractor onto the plane. The attractor is nonintersecting in three dimensions, in which it was embedded for the numerical analysis. In the absence of experimental noise the points should fall on a single curve. The smearing of the observed data set is mostly due to the slow drift in the experimental system during the run over about 2 h. Our method of analyzing the data to secure  $f$  vs  $\alpha$  (see Fig. 2) is intended to minimize the effect of the slow drift. This is realized by estimating the recurrence times which experimentally are matters of minutes rather than hours.

this cycle to sets produced by model equations that describe the onset of chaos via quasiperiodicity.<sup>2,7,8</sup> In fact, the approach proposed here constitutes a rare opportunity for an extensive quantitative comparison of experiments with universal results obtained from theoretical models.

The experiment which yielded the critical golden-mean trajectory has been described previously.<sup>4</sup> The experiment studies a small-aspect-ratio Rayleigh-Bénard system of size  $0.7 \times 0.7 \times 1.4 \text{ cm}^3$  with two convective rolls present. For a low-Prandl-number fluid like mercury, as the heat flux increases beyond the convection threshold  $R_c$ , the system undergoes a Hopf bifurcation, called the oscillatory instability, into a time-dependent periodic mode. This mode is characterized by an ac vertical vorticity otherwise absent in the static roll pattern. This oscillation is one of our two oscillators (frequency  $\approx 230 \text{ mHz}$ ). The second oscillator is introduced electromagnetically, mercury being an electrical conductor. An ac current sheet is passed through the mercury and the system is immersed in an horizontal magnetic field ( $H \approx 200 \text{ G}$ ) parallel to the rolls' axes. The geometry of electrode and field is such that the Lorentz force on the fluid produce ac vertical vorticity. In this way the oscillators are dynamically coupled. During the experiment the Rayleigh number is kept fixed at  $R = 4.09R_c$  giving a large amplitude to the first oscillator.

The nonlinear interaction between the oscillators is controlled by the amplitude of the injected ac current. A signal is obtained from the experiment by means of a thermal probe located in the bottom plate of the cell. The winding number, which is the ratio between the two frequencies, is kept close to the golden mean, i.e., within  $10^{-4}$ . Time series are obtained by observation of the temperature signal at discrete times separated by the period of the forcing.

The theoretical work is aimed at estimating in a quantitative fashion how "bunched" the density on the orbit might be. In technical terms this bunching is a description of singularities in the probabilities of the orbit points.<sup>9</sup> Less technically, one can view a particular orbit point  $x_i$ , in a phase space like that of Fig. 1, and ask what is the probability for other points falling within the small distance,  $l$ , of this one. Call this probability  $p_i(l)$ . One can describe this probability by defining an index  $\alpha_i(l)$  via

$$p_i(l) = l^{\alpha_i(l)}. \quad (1)$$

In typical sets the scaling index  $\alpha_i$  takes, for small  $l$ , a range of values between  $\alpha_{\min}$  and  $\alpha_{\max}$ . We refer to this situation as a spectrum of singularities.

To analyze the experimental time series, we make use of a key theoretical idea that fractal sets in general, and critical orbits in particular, can be described as interwoven sets of singularities.<sup>9</sup> The density of singu-

larities of type  $\alpha$ ,  $\alpha_{\min} < \alpha < \alpha_{\max}$ , is determined by an index  $f$  that can be interpreted as the dimension of the set of singularities of this type. In other words, if the system is divided into pieces of size  $l$ , then the number of times,  $n(\alpha, l)$ , that  $\alpha$  takes on a value between  $\alpha$  and  $\alpha + d\alpha$  is of the form

$$n(\alpha, l) = d\alpha \rho(\alpha) l^{-f(\alpha)}, \quad (2)$$

where  $\rho(\alpha)$  is nonsingular with respect to  $l$ . The intuitive meaning of  $\alpha_{\max}$  is that it is associated with the most rarefied regions of the measure, whereas  $\alpha_{\min}$  with the most concentrated. Typically,  $f(\alpha_{\max}) = f(\alpha_{\min}) = 0$ . Other types of singularities between  $\alpha_{\max}$  and  $\alpha_{\min}$  live on subsets of dimension  $f$ ,  $0 < f < D_0$  ( $D_0$  being the dimension of the set). The functions  $f(\alpha)$  are universal functions for critical cycles like the trajectory with golden-mean winding number at the onset of chaos via quasiperiodicity.<sup>2</sup> Another key point is that these functions are smooth, in contrast to the universal scaling functions of the type suggested by Feigenbaum,<sup>1</sup> which are nowhere differentiable (see for example Fig. 10 of Ref. 1b). The reason for this important difference is that the latter functions are constructed by following the *local* changes in scaling everywhere,<sup>1</sup> whereas the former are based on finding the *global density* of scaling indices of each type.

The  $\alpha_i(l)$  of Eq. (1) are estimated in a very simple fashion: Start from the point  $x_i$  on the trajectory. Count the number of steps along the time series required before a point returns to within  $l$  of the starting point. We call the number the recurrence time and denote it  $m_i$ . We shall now make use of the fact that the orbit is conjugate to a pure rotation<sup>2,7,8</sup> with an irrational winding number, and is therefore ergodic. Thus, we simply estimate  $p_i(l)$  as the inverse recurrence time,  $(m_i)^{-1}$ , and find

$$\alpha_i(l) = -\ln m_i / \ln l. \quad (3)$$

In principle, the remainder of the analysis is very simple. One estimates how many  $\alpha_i(l)$  values live in a given range, substitutes that estimate into Eq. (2), and then chooses some very small value of  $l$  to find  $f(\alpha)$ . In practice, given only a moderate amount of data, one cannot obtain a good estimate of  $f(\alpha)$  by this direct method. Instead, we employ<sup>6</sup> an indirect method which smooths the data and gives an efficient calculation of  $f(\alpha)$ . To obtain this smoothness, we use the data to calculate the auxiliary quantity

$$\Gamma(q, l) = \langle p_i(l)^q \rangle = \langle m_i^{1-q} \rangle, \quad (4)$$

where the brackets represent an average over all the trajectory elements  $i$ . The whole point of using the "partition function" Eq. (4) is that it is a smooth function of  $l$  and  $q$  and, for  $l \ll 1$ , is given by a power

of  $l$ ,

$$\Gamma(q, l) \sim l^{\tau(q)}. \quad (5)$$

This  $\tau(q)$  is related to the generalized dimensions,  $D_q$ , of Hentschel and Procaccia<sup>10</sup> by  $D_q = (q-1)^{-1}\tau(q)$ .

From the point of view of this paper  $\tau(q)$  is not important in itself. Instead it is a kind of generating function which can be used to determine the function  $f(\alpha)$  via the pair of formulas (derived in Ref. 6),

$$\alpha(q) = d\tau(q)/dq, \quad (6)$$

$$f(q) = \tau(q) - q d\tau/dq.$$

This is essentially a Legendre transformation, as used in statistical thermodynamics. Once  $q$  is eliminated from the pair of Eqs. (6), we discover that we have  $f(\alpha)$  defined in a range of  $\alpha$  values  $\alpha_{\min} < \alpha < \alpha_{\max}$ .

An example of a theoretical  $f(\alpha)$  curve calculated in this manner is shown as the curve in Fig. 2. The attractor is the critical cycle of the circle map<sup>2,7,8</sup>  $\theta_{n+1} = \theta_n + \Omega - (K/2\pi) \sin 2\pi\theta_n$  at the critical value  $K=1$  and  $\Omega = \Omega_{\text{gm}}$ , where the orbit has a golden-mean winding number. Here,  $\tau(q)$  was evaluated by calculation of the average Eq. (4) for several  $l$  values and then finding  $\tau$  as the slope of a straight line fit of a plot of  $\ln\Gamma$  vs  $\ln l$ . The value of  $\alpha_{\max}$ , which is also  $D_{-\infty}$  agrees with the theoretical expectation<sup>6</sup>

$$\alpha_{\max} = \ln\omega^*/\ln\alpha_{\text{gm}}^{-1} = 1.8980 \dots, \quad (7)$$

where  $\omega^*$  is the golden mean  $\omega^* = (\sqrt{5}-1)/2$  and  $\alpha_{\text{gm}}$  is the universal local scale factor<sup>2</sup> in the vicinity of the critical point  $\theta=0$ ,  $\alpha_{\text{gm}} = 1.2885 \dots$ . This is the most rarefied region in the trajectory. This region is mapped onto the most concentrated region in the set<sup>6</sup> which is characterized by the  $\alpha$  value  $\alpha_{\min} = \ln\omega^*/\ln\alpha_{\text{gm}}^{-3} = 0.6326 \dots$ . The curve turns around at the value of  $f$  which is  $f = D_0 = 1$ . This is also to be expected since the support of the measure is the circle, which is one dimensional.

The experimental data were similarly analyzed. On the basis of a time series of 2500 points embedded in a three-dimensional space we first calculated  $\Gamma$  as required by Eq. (4). Plotting again  $\ln\Gamma$  vs  $\ln l$ , we typically fitted the  $\tau$ 's with over fifty different values of  $l$  ranging over two decades. The  $f(\alpha)$  values were computed via Eqs. (6) with the result shown as the dots in Fig. 2. For small  $q$  ( $|q| < 1$ ) the scaling was in general best for the largest values of  $l$ . As  $|q|$  was increased, the best scaling regime gradually moved towards lower values of  $l$ . This is expected since high  $|q|$  values correspond to isolated regimes on the attractor. The accuracy of the fits was always very good for positive  $q$ 's (corresponding to the leftmost branch of the curve), and we estimate the error bar on the point  $(D_{\infty}, 0)$  to be a few percent. The accuracy was less for negative values of  $q$  (corresponding to the rightmost branch) and the error bar on the point

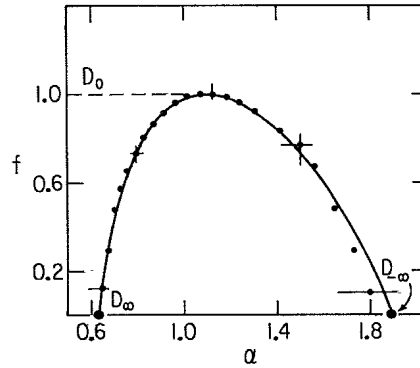


FIG. 2. The  $f(\alpha)$  spectrum calculated for a critical circle map with golden-mean winding number is shown by the curve (Ref. 6). The curve ends in the points  $(D_{\infty}, 0)$  and  $(D_0, 0)$ , which are shown by the two large dots. The  $f(\alpha)$  estimates for the experimental time series are marked by the smaller dots. The error bars are estimated by varying the range of  $l$  used to fit the data.

$(D_{-\infty}, 0)$  is around (10–12)%. The accuracy of the maximum point of the curve (i.e.,  $D_0$ ) is indicated by a vertical error bar. Theory and experiment agree. This agreement supports our conjecture that this Rayleigh-Bénard system at the onset of chaos and the critical circle map belong to the same universality class. We note in passing that from the value of  $\alpha_{\max}$  (and also of  $\alpha_{\min}$ ) one can read immediately  $\alpha_{\text{gm}}$ , cf. Eq. (7). To the best of our knowledge this is the first direct measurement of this universal scaling number.

To conclude we note that the raw experimental orbit in its reconstructed phase space looks nothing like the orbit of the circle map. To the eye, it does not appear to lie on a circle. It is twisted and contorted in a complicated way. Our results demonstrate, however, that from the metric point of view these two sets are the same within experimental accuracy. To date we are not aware of any other approach that can lead to such a strong conclusion.

We thank Thomas C. Halsey for many stimulating discussions. This work has been partially supported by the Materials Research Laboratory of the University of Chicago, the National Science Foundation through Grants No. DMR 83-16626 and No. DMR 83-16204, the Office of Naval Research, and the Minerva Foundation, Munich, Germany.

<sup>1a</sup>M. J. Feigenbaum, *J. Stat. Phys.* **19**, 25 (1978), and *J. Stat. Phys.* **21**, 669 (1979).

<sup>1b</sup>M. J. Feigenbaum, *Los Alamos Sci.* **1**, 4 (1980).

<sup>2</sup>Scott J. Shenker, *Physica (Amsterdam)* **5D**, 405 (1982).

<sup>3</sup>In parameter space however, global universal features have been proposed [M. H. Jensen, P. Bak, and T. Bohr,

Phys. Rev. Lett. **50**, 1637 (1983), and Phys. Rev. A **30**, 1960 (1984); P. Cvitanović, M. H. Jensen, L. P. Kadanoff, and I. Procaccia, Phys. Rev. Lett. **55**, 343 (1985)] and measured experimentally [J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985); see also A. Libchaber, C. Laroche, and S. Fauve, Physica (Amsterdam) **7D**, 73 (1983), and J. Phys. (Paris), Lett. **43**, L211 (1982)] for the structure of modelocking.

<sup>4</sup>Stavans, Heslot, and Libchaber, and Libchaber, Laroche, and Fauve, Ref. 3.

<sup>5</sup>Jensen, Bak, and Bohr, and Cvitanović *et al.*, Ref. 3.

<sup>6</sup>T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A (to be published).

<sup>7</sup>M. J. Feigenbaum, L. P. Kadanoff, and Scott J. Shenker, Physica (Amsterdam) **5D**, 370 (1982).

<sup>8</sup>S. Ostlund, D. Rand, J. P. Sethna, and E. D. Siggia, Phys. Rev. Lett. **49**, 132 (1982), and Physica (Amsterdam) **8D**, 303 (1983).

<sup>9</sup>T. C. Halsey, P. Meakin, and I. Procaccia, to be published.

<sup>10</sup>H. G. E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983); I. Procaccia, Phys. Scr. **T9**, 40 (1985).