

Three is much more than two in coarsening dynamics of cyclic competitions

Namiko Mitarai,^{*} Ivar Gunnarson, Buster Niels Pedersen, Christian Anker Rosiek, and Kim Sneppen[†]

Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen, Denmark

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The classical game of rock-paper-scissors has inspired experiments and spatial model systems that address the robustness of biological diversity. In particular, the game nicely illustrates that cyclic interactions allow multiple strategies to coexist for long-time intervals. When formulated in terms of a one-dimensional cellular automata, the spatial distribution of strategies exhibits coarsening with algebraically growing domain size over time, while the two-dimensional version allows domains to break and thereby opens the possibility for long-time coexistence. We consider a quasi-one-dimensional implementation of the cyclic competition, and study the long-term dynamics as a function of rare invasions between parallel linear ecosystems. We find that increasing the complexity from two to three parallel subsystems allows a transition from complete coarsening to an active steady state where the domain size stays finite. We further find that this transition happens irrespective of whether the update is done in parallel for all sites simultaneously or done randomly in sequential order. In both cases, the active state is characterized by localized bursts of dislocations, followed by longer periods of coarsening. In the case of the parallel dynamics, we find that there is another phase transition between the active steady state and the coarsening state within the three-line system when the invasion rate between the subsystems is varied. We identify the critical parameter for this transition and show that the density of active boundaries has critical exponents that are consistent with the directed percolation universality class. On the other hand, numerical simulations with the random sequential dynamics suggest that the system may exhibit an active steady state as long as the invasion rate is finite.

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I. INTRODUCTION

Coarsening is important in a number of dynamical systems [1,2] and may be used to differentiate observed phenomenology into appropriated universality classes [3]. It appears in decay towards equilibrium in diverse phenomena such as spinodal decomposition, segregation of grains [4], opinions [5], languages [6], and populations [7–9], as well as in the ongoing tendency of biological competition to decrease species abundance in ecological models [10,11].

The coarsening has been extensively studied for voter models [3,12] and extended voter models with cyclic competition, especially for the three-species cyclic competition or the rock-paper-scissors game [13,14]. For the three-species competition in one dimension, the number of separated populations coarsens as $t^{-3/4}$ for the random sequential dynamics, where t counts the number of update attempts per site. In contrast, the parallel dynamics provides a slower coarsening characterized by $t^{-1/2}$ [13,14]. One can counteract the coarsening in one dimension by introducing an explicit mutation rate between species [15] or by introducing mobility [16], both of which can lead to an active steady state with the coexistence of all three species. Another more widely studied way is to extend it to the two-dimensional space, where the species domains are occasionally broken up into smaller patches, which in turn allow the long-time coexistence of all three species [17]. This has motivated the extensive study of nonhierarchical ecosystem models as a mechanism to support the coexistence of species in ecology research [10,18–23], and cycles have

been proposed to act as engines of increased diversity in two-dimensional ecologies [24,25].

In this paper, we consider the cyclic predatory relations between species in a quasi-one-dimensional ecology. We demonstrate that when one extends a simple one-dimensional ecology to three parallel ecologies with weak coupling between them, one obtains a hugely increased lifetime of all species. We find that this increase in lifetimes is closely connected with ongoing “fragmentationlike” events where invasion from one linear ecology to another initiates a positive feedback driven by a growing divergence to the third linear ecology. As the ecologies diverge, more successful invasions take place between them. This opens the possibility for the creation of new patches of species within each ecology, and thereby opens the possibility for a system where the overall invasion activity remains high. We quantitatively characterize the transition from the coarsening state to the active state in the parallel update case by changing the invasion rate between linear subsystems, and we show that the critical behavior is consistent with the directed percolation (DP) universality class [26,27]. We further demonstrate that the random sequential update tends to make the system reach an active steady state as long as the invasion rate is finite.

II. COUPLED LINEAR SYSTEMS OF CYCLIC COMPETITION

We consider a system composed of several one-dimensional lattices that each have a length L in the x direction. We focus, in particular, on a stack of $n = 3$ of these systems positioned on top of each other in the y direction, as shown in Fig. 1(a). Periodic boundary conditions are imposed in both the x and y directions. The simulation is initialized by assigning each lattice site to be occupied by one of the three species A , B , or

^{*}mitarai@nbi.dk

[†]sneppen@nbi.dk

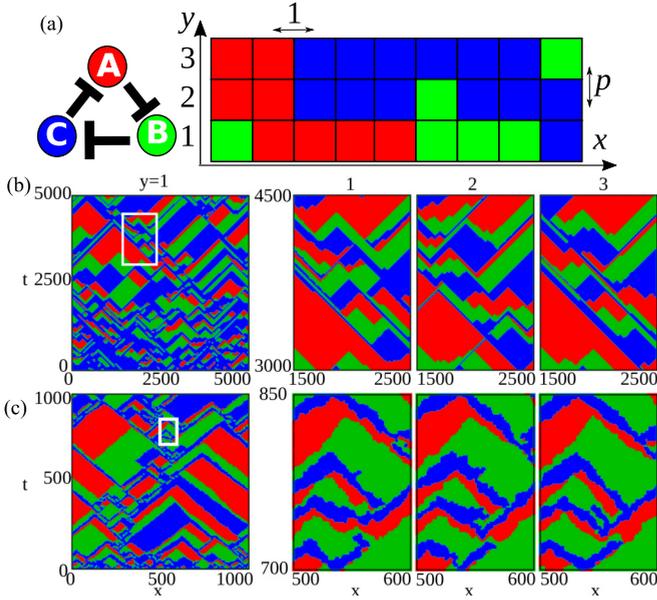
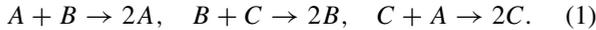


FIG. 1. Model description and spatiotemporal plot with $n = 3$. (a) Schematic description of the model. (b) Spatiotemporal plot from the parallel dynamics with $p = 0.00025$ and $L = 5000$. The left panel shows the whole system until $t = 5000$ for $y = 1$. The three panels to the right show a magnification of part of the system (marked as white box in the left panel) at the corresponding locations for $y = 1, 2, 3$. (c) Same as (b), but for random sequential update with $p = 0.05$ and $L = 1000$.

C with equal probability. The species interaction is cyclic as given by the following rule:



The interactions are limited to the nearest-neighbor sites, and further limited in the vertical direction by a parameter p that controls the vertical invasion rate relative to the interaction rate along the x direction.

The update can be either parallel dynamics or random sequential dynamics. In the case of parallel dynamics, all of the bonds in the x direction are updated simultaneously according to Eq. (1). For example, if the configuration is ABC , then after one update B will be replaced with A , and C will be replaced with B , simultaneously; therefore, boundaries between different species that move in the same direction will never collide. Then bonds in the vertical direction are selected with probability p per bond (i.e., pnL bonds are selected on average) and updated sequentially according to Eq. (1). This defines one time step in the model.

The random sequential dynamics is defined as follows: (i) Choose a random bond in the x direction and update its two neighbors according to the reactions in Eq. (1). (ii) With a probability p , choose a random bond in the vertical direction and update its two neighbors according to the reaction in Eq. (1). One time step is defined here as nL repetitions of (i) and (ii).

Irrespective of the updating rule, the system consists of domains that each consist of populations of one of the three species. These domains are separated by domain boundaries

that move either left or right, as one of the populations systematically displaces the other.

In the pure one-dimensional case, i.e., $n = 1$, coarsening happens through the collision between two moving boundaries. Such collisions eliminate the population located in the domain between the boundaries. For parallel dynamics, the boundaries move at the same speed and coarsening only occurs through the collision between a right moving boundary and a left moving one, resulting in the annihilation of both. For the random sequential update, the collision of two boundaries moving in the same direction is also possible due to the fluctuating speed. Such a collision creates one new boundary that moves in the opposite direction of its two parents. This makes the coarsening in the random sequential dynamics faster than that in the parallel dynamics [13].

When parallel linear systems are added, occasional interaction between the subsystems can create new boundaries. When $n = 2$, the introduction of p can increase the fragmentation of the domains temporarily in early time compared to the $n = 1$ case, but in the long term the synchronization of the two subsystems are enhanced, as shown in the example spatiotemporal plots in Figs. 2(a) and 2(b). The time evolution

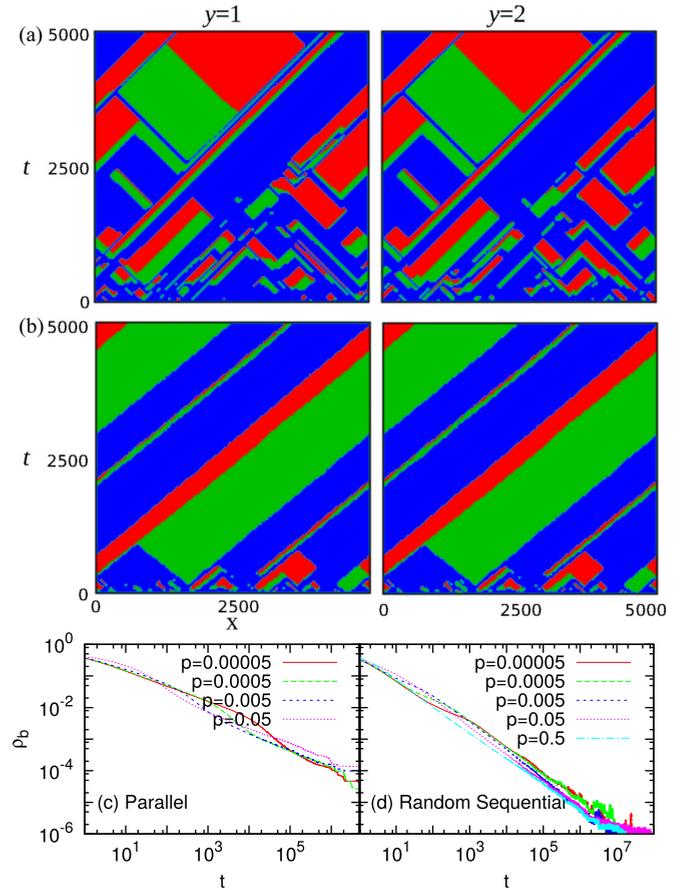


FIG. 2. Simulation results for $n = 2$. (a) Spatiotemporal plot for each subsystem with the parallel dynamics with $p = 0.00025$ and $L = 5000$. (b) Same as (a), but for random sequential update. (c), (d) The boundary density ρ_b as a function of time for (c) the parallel dynamics and (d) the random sequential dynamics with $L = 2^{24}$ for various values of p .

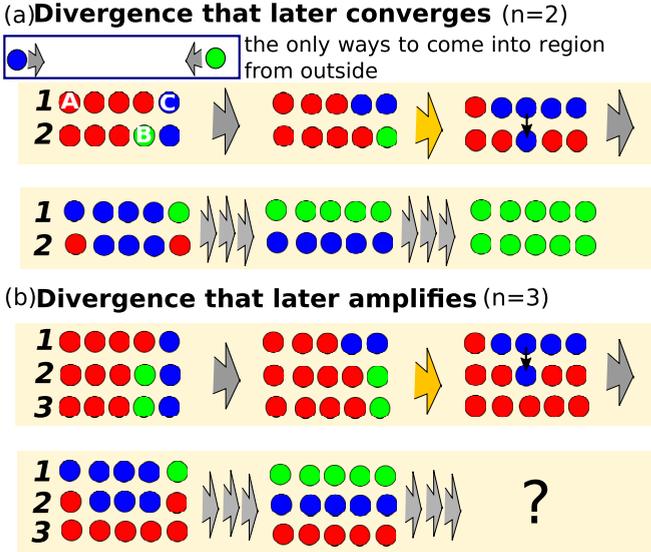


FIG. 3. Qualitative difference between the (a) $n = 2$ case and (b) $n = 3$ case. The vertical transfers are shown as black arrows. (a) In the initial situation, the difference in the considered region is only one site. Outside of the considered region, if there are green B species to the right, they come in to the considered region in later time by eliminating the blue C species. The red A species cannot come in from the right, since the boundary between blue C and red A moves to the right. Similarly, from the left of the considered region, only blue C can come in to the considered region by eliminating the red A species. As the time goes, the red A species will disappear from the considered region, allowing the green B to occupy all of the considered sites in the long run. (b) When $n = 3$, the considered region is able to keep all three species even starting from a very similar situation as (a), opening the possibility for the creation of new boundaries.

of the density of the domain boundaries ρ_b , defined as the number of domain boundaries in one of the subsystems divided by L , is shown for $n = 2$ for the parallel dynamics [Fig. 2(c)] and the random sequential dynamics [Fig. 2(d)]. We see that the coarsening continues until only the boundaries moving in parallel are left in the parallel dynamics or until only a small number (order 10) of boundaries are left in the random sequential dynamics where the noise masks the coarsening. We could not find a value of p that can stop the coarsening, either in the parallel or the random sequential update.

Interestingly, we find qualitatively different results for $n = 3$; see Figs. 1(b) and 1(c). Irrespective of whether one considers parallel or random sequential dynamics, some low p values make the three-line system develop into an active steady state. In this state, the coarsening described above is balanced by ongoing fragmentation events that create new domains. These events are initiated by occasional small differences between the three linear ecosystems, which subsequently cause larger divergences between the systems. We also see this active steady state behavior for $n > 3$ systems (data not shown), demonstrating that it is the transition from $n \leq 2$ to $n \geq 3$ that fundamentally changes the overall system behavior.

Let us consider the difference between the $n = 2$ case and the $n = 3$ case. In order to stop coarsening, there should be possibilities for amplification of the difference between the subsystems over time. The only nontrivial single site difference

is the situation shown in Fig. 3(a). This initial state allows a temporal increase of the difference due to the diverging boundary motions. However, at some point, a vertical invasion of a species C (blue) from subsystem 1 to subsystem 2 happens, making both subsystems dominated by species C (blue). The only possibility to change this convergence to uniformity is the species B (green) that could come in from the right side of subsystem 1 [because it may not have species A (red) that protects against invasion of B (green)]. But then later a vertical invasion would trigger a spread of the species B (green) for both subsystems, allowing the species B (green) to occupy all of the considered sites. Therefore, one cannot keep the difference between the subsystems with $n = 2$, and the system will always tend to coarsen over the long time.

On the contrary, a similar situation for the $n = 3$ case can keep the difference among the subsystems. As shown in Fig. 3(b), the additional third subsystem can keep species A (red) in the considered region, and the configuration in which all three species are present in the considered region enables various ways to create new boundaries. The ability to keep all of the species in the same region is needed to keep the active steady state.

The transition to the active steady state in the $n = 3$ case is quantitatively different between the parallel and random sequential update. Note that Fig. 1(b) for parallel dynamics shows the result with $p = 0.00025$ with $L = 5000$, while Fig. 1(c) for random sequential dynamics shows the result with $p = 0.05$ with $L = 1000$. We also find some qualitative difference in the transition between the coarsening state and the active state when varying p . In the subsequent sections, we further quantify the coarsening dynamics for the two updating rules.

III. PARALLEL DYNAMICS

In the case of the parallel dynamics, there is no noise in the horizontal movement of the boundaries. Thus if we initially synchronize all three subsystems (i.e., $S_i^1 = S_i^2 = S_i^3$ for all i , with S_i^k being the species name at the site with $x = i$ and $y = k$), then the subsystems will stay synchronized and the dynamics will be identical to the one-dimensional system, irrespective of the value of p . Therefore, to maintain the active steady state, there must remain differences among the three subsystems, where a case study was already shown in Fig. 3(b).

Figure 4(a) shows the motion of the boundaries ($S_i^k \neq S_{i+1}^k$) from the simulation shown in Fig. 1(b). The boundaries between the domains in subsystem 1 are shown. The green symbols mark boundary sites where one of the subsystems is different, whereas thick blue symbols mark sites where all subsystems differ. The red crosses show the successful invasions of the species to line 1 from subsystem 2 or 3. We observe that such vertical transfers create new interfaces.

The competition between the gradual alignment and the creation of new boundaries by vertical invasion causes a transition in behavior between a low p case, where there is a sustained activity, to a high p case, where the system persistently coarsens. Figure 4(b) shows the development in the boundary density ρ_b for different values of p in a large system. When p is increased from $p = 0.000005$ to $p = 0.00025$, we see that the system settles in a steady state

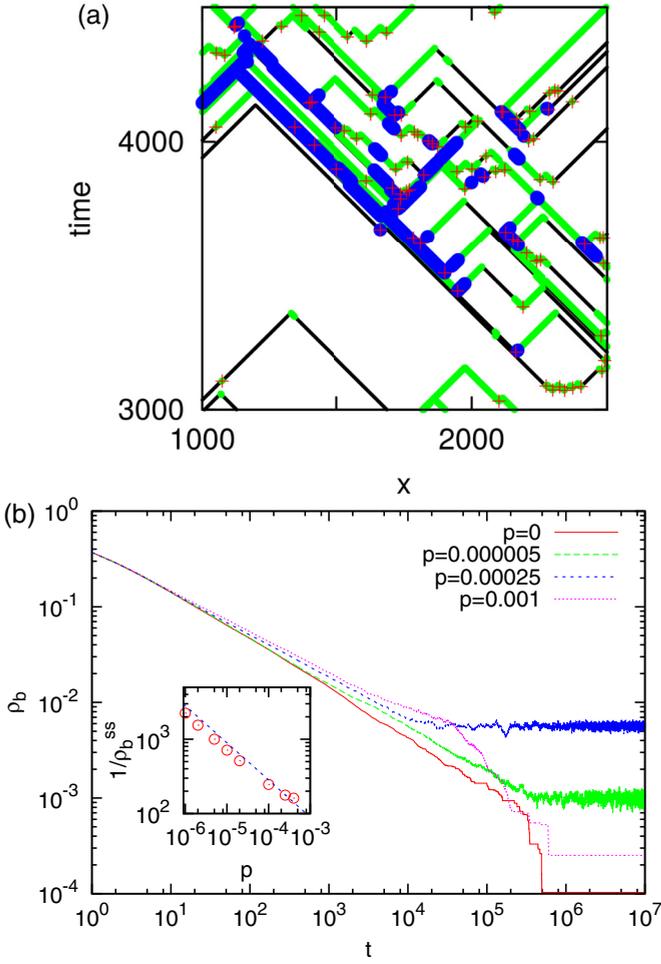


FIG. 4. (a) Development of used boundaries, with invasion probability $p = 0.00025$. Thin black symbols mark the points where the boundary sites have the same species in all three subsystems. Thick green circles mark boundary sites where one subsystem deviates, whereas thick blue circles mark boundary sites where all subsystems carry different species. The red crosses show events where species from subsystem 2 or 3 invade subsystem 1. (b) Density of boundaries per subsystem ρ_b simulated with system size $L = 2^{20}$ and $n = 3$. The $p = 0$ case is equivalent to the $n = 1$ case with long-time coarsening as $t^{-1/2}$. Inset: The average domain size in the active steady state $1/\rho_b^{ss}$ vs p . The dashed line shows $2\sqrt{2}/\sqrt{p}$ (see text).

with constant number of domains (boundaries). Also we see that an increased p increases the number of such boundaries. However, further increase to $p = 0.001$ shows the collapse of the active steady state to the coarsening mode that is also found for the case of isolated subsystems ($p = 0$). This is because high p makes gradual alignment happen too often compared to the creation of new boundaries. Then, in the high p case, all of the subsystems act as the same in the long term, which is equivalent to the one-dimensional case.

Considering the system in an active steady state, we can estimate the average time Δt it takes between the first creation of divergent boundaries at time t^* [an event such as the vertical transfer in Fig. 3(b)] to the first transfer of divergent states among the three subsystems. This time is given by the transfer rate p per site in a linearly growing divergent region between

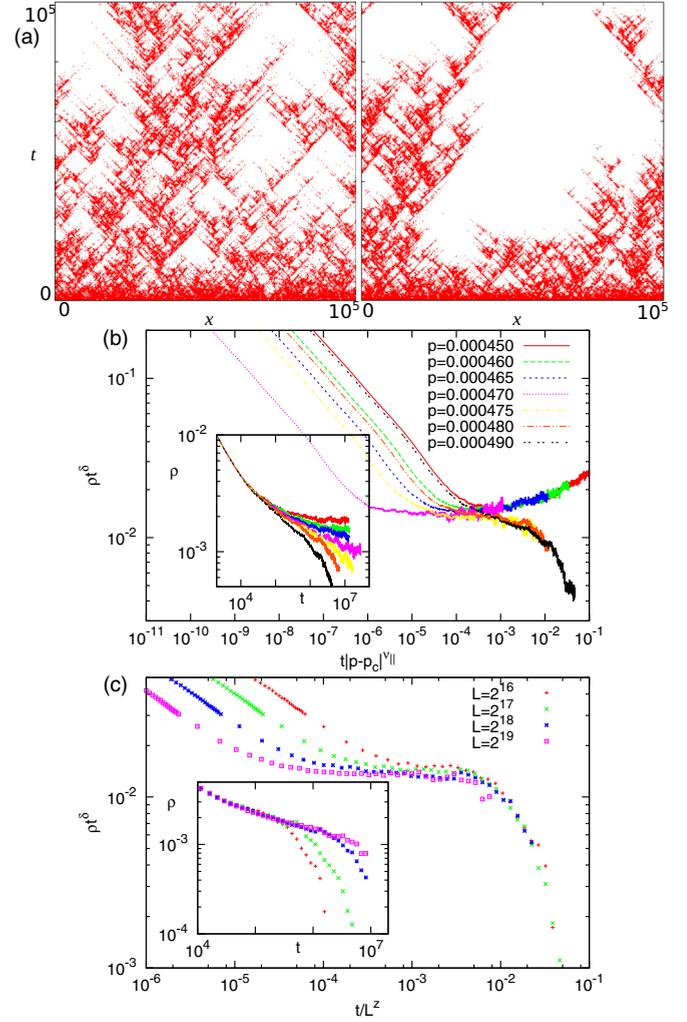


FIG. 5. (a) Large-scale behavior of the vertical transfer events for parallel update. The red dots show the invasions of species to subsystem 1 from subsystem 2 or 3. The figure shows a part of a $L = 1\,000\,000$ system. The left panel is for $p = 0.00025$ and the right panel is for $p = 0.0005$. (b) Scaling plot for the simulations of a system size $L = 2^{26}$ using p values close to the critical invasion rate p_c . We fitted this to be $p_c = 0.000471$ and used 1+1 dimensional DP-scaling exponents $\delta = 0.0159$ and $\nu_{||} = 1.733$ for rescaling. The inset shows the time evolution of ρ without such rescaling. (c) Finite size scaling plot at $p = 0.000475$ with $z = 1.581$. The inset shows the time evolution of ρ without rescaling.

two subsystems. Assuming that the event occurs when the cumulative probability is one, we expect

$$\int_{t^*}^{t^*+\Delta t} p t dt \approx 1,$$

which gives $\Delta t \approx \sqrt{2/p}$. As the boundary motion is ballistic, a new transfer occurs between a divergent region of size $\Delta \ell \approx 2\Delta t \approx 2\sqrt{2/p}$. Or said in another way, then the average steady state domain size for small p should scale as $\Delta \ell$, and one indeed sees it in the inset of Fig. 4(b).

The spatiotemporal plots of successful vertical invasions are shown in Fig. 5(a) for $p = 0.00025$ (left) and $p = 0.0005$ (right). One observes ballistic lines that follow the

propagating boundaries until they occasionally disappear due to local synchronization among the three subsystems. Also, one observes the occasional creation of new boundaries from old ones. These creation events seem to occur as dense bursts of activity. The large-scale structure of this spreading birth and death process reminds us of the directed percolation (DP) class of models in 1+1 dimension [27].

For small p [Fig. 5(a), left], the alignment is so slow that vertical transfer dominates and activity is sustained. At large p values, the faster alignment between the three subsystems makes it more difficult to maintain sufficient divergence to sustain on-going vertical transfer events [Fig. 5(a), right] and ultimately the whole system aligns to form a few parallel moving domains. It should be noted that the smallest “unit” of this DP-like structure is not one site, but is instead given by the length and time scale of $1/p \sim 10^3$.

We conjecture that the transition from the active state to the coarsening state belongs to the DP universality class. Note that the absorbing state at large p is the state where all three subsystems are synchronized, still leaving the possibility for some diversity with some moving but synchronized boundaries. We, therefore, chose to study the density of the *active sites* ρ , defined as the density of the boundaries that contains sites which are not completely aligned with other subsystems. The development of ρ in the inset of Fig. 5(b) illustrates transient coarsening up to $t \sim 10^4$, after which it changes to either a steady state density or collapses to zero density. We rescaled these data by using 1+1 dimensional DP exponents [27,28] $\delta = \beta/v_{\parallel} = 0.0159$ and $v_{\parallel} = 1.733$. By fitting the critical p at the transition, p_c , to 0.000471, we obtain a data collapse that is consistent with the DP-universality class, except for the initial transient regime [Fig. 5(b)]. Figure 5(c) shows the finite size scaling using another DP-scaling exponent $z = v_{\parallel}/v_{\perp} = 1.581$. Thus both the time coarsening and the finite size dependence are consistent with the DP-universality class.

IV. RANDOM SEQUENTIAL DYNAMICS

The behavior of the random sequential dynamics is more complicated because three subsystems can spontaneously desynchronize due to the randomness of the movement of the boundaries in the respective subsystems. For example, if all three subsystems are locally identical with two left moving interfaces ABC each, it is possible that the two boundaries in subsystem 1 merge spontaneously to make AAC at the next time step. This suddenly creates one right moving interface, very similar to the situation in Fig. 3(b), allowing further diversification. Collapse of the interfaces that are moving in the same direction is the reason why the random sequential dynamics coarsens faster than the parallel dynamics in a one-dimensional system. With vertical coupling, however, it also provides an additional way to create more boundaries.

First of all, this allows the three-line system to maintain an active steady state for much higher p values than in the parallel update case, a robustness that reflects the more frequent diversification events. Furthermore, the fully synchronized state is no longer an absorbing state. Numerically this seems to result in the loss of a clear transition with changing value

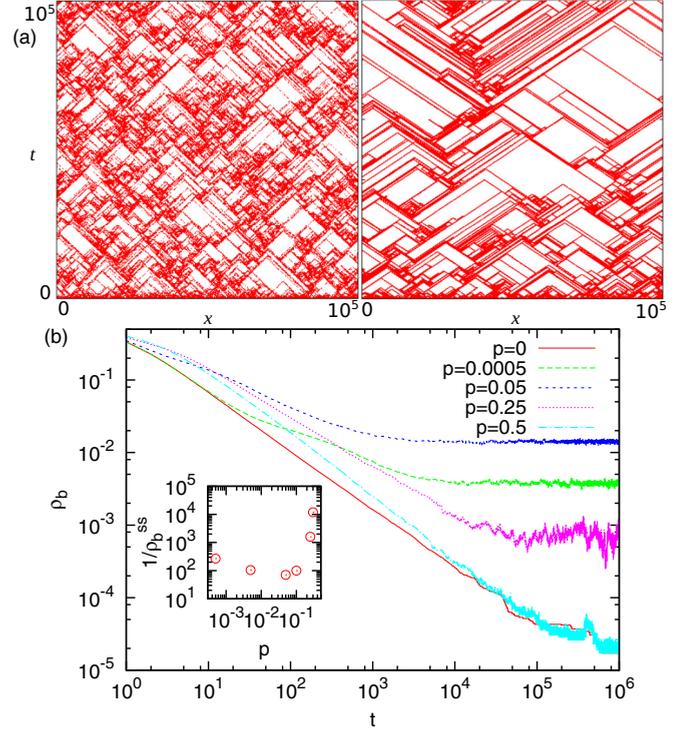


FIG. 6. (a) Large-scale structure of the vertical transfer events for $L = 100\,000$ with random sequential update. The red dots show the invasion of species to subsystem 1 from subsystems 2 and 3. The left panel shows $p = 0.0005$ and the right panel shows $p = 0.25$. (b) Density of interface per subsystem with $L = 2^{20}$ and $n = 3$. The random sequential update for $p = 0$ (equivalent to $n = 1$, with long-time behavior $t^{-3/4}$), $p = 0.0005, 0.05, 0.25$, and 0.5 . Inset shows the average domain size in the steady state $1/\rho_b^{ss}$ as a function of p . The data for $p = 0.3$ was obtained from the simulation with $L = 2^{24}$.

of p . This manifests itself in the spatiotemporal plot of the successful invasion to subsystem 1 from other subsystems shown in Fig. 6(a). In contrast to the parallel dynamics case in Fig. 5(b), we see continuous ballistic trajectories that closely follow the moving domain boundaries in one of the subsystems. This is because the random fluctuations of the boundary motions keep desynchronizing the subsystems to allow vertical transfers, until the boundary disappears. In other words, in the random update case, an active site can only be annihilated by meeting another active site. This in itself is qualitatively different from the DP-universality class.

The time evolution of the boundary density ρ_b is shown in Fig. 6(b). The $p = 0$ (pure one-dimensional) case shows coarsening that declines as $t^{-3/4}$ in the long-time limit. Introducing a small finite p increases ρ_b compared to the $p = 0$ case. Increasing p to 0.0005 and further to 0.05 allowed the system to reach a steady state with a constant ρ_b . Further increase of p decreased ρ_b at the steady state, but no sudden collapse was observed. The corresponding nonmonotonous behavior of the average steady state domain size $1/\rho_b^{ss}$ as a function of p is shown in the inset. Simulations with larger p always allowed us to find correspondingly larger systems with an active steady state within the range that we could test numerically (we tried up to size $L = 2^{26}$).

V. DISCUSSION

We have shown that the transition from coarsening of domains of the rock-paper-scissors game to an active steady state with a finite level of interface density requires at least three coupled linear subsystems. With such systems, it becomes difficult to synchronize all three, which in turn gives rise to the creation of new domains through the mutual invasions.

With the parallel dynamics, the complete synchronization of the three linear subsystems acts as an absorbing state, and the system exhibits a transition from the active steady state where subsystems never synchronize to the absorbing state. We have shown that the transition is consistent with the DP-universality class in the 1+1 dimension.

It has been conjectured that the short-range process is a requirement for the DP-universality class [27]. The observation of the DP class in the present model was unexpected because of the apparent long-range correlation between ballistic boundaries. When subsystems are coupled by rare invasions, however, the invasion from other subsystems breaks up this correlation, and the interaction between domains becomes “short range” when viewed on length scales larger than $1/p$. Since the critical p happens to be about 0.0005, the DP behavior appears only after a long transient in large systems.

When the update is random and sequential, the synchronized state is no longer an absorbing state. It is then possible that the active steady state may exist as long as p is finite. Since

p is the rate per site for the vertical invasion, we can, in principle, consider $p \rightarrow \infty$ limit, where all three subsystems stay synchronized. Note that this limit is not exactly the same as the pure one-dimensional system since if a boundary of one of the three subsystems proceeds more than average by chance, that will be copied to other subsystems immediately, namely, the fluctuation tends to make the interface motion slightly faster, which may result in faster coarsening than a one-dimensional system. We did not identify any transition in the system’s behavior at finite p in the random sequential dynamics. We speculate that this type of system deviates fundamentally from the DP class because the active (desynchronized) boundary cannot “die” by itself, but rather needs another active site to be eliminated. The feature that there is no spontaneous death process differentiates it from the DP process.

The overall lesson from this work is that three is much more than two and provides an engine for sustained yet dynamic heterogeneity in the spatial rock-paper-scissors game. Thus parallel systems open the possibility for a qualitatively different way of sustaining patchiness from increasing the number of species in a cycle of invasions [13,14].

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