UNIVERSITY OF
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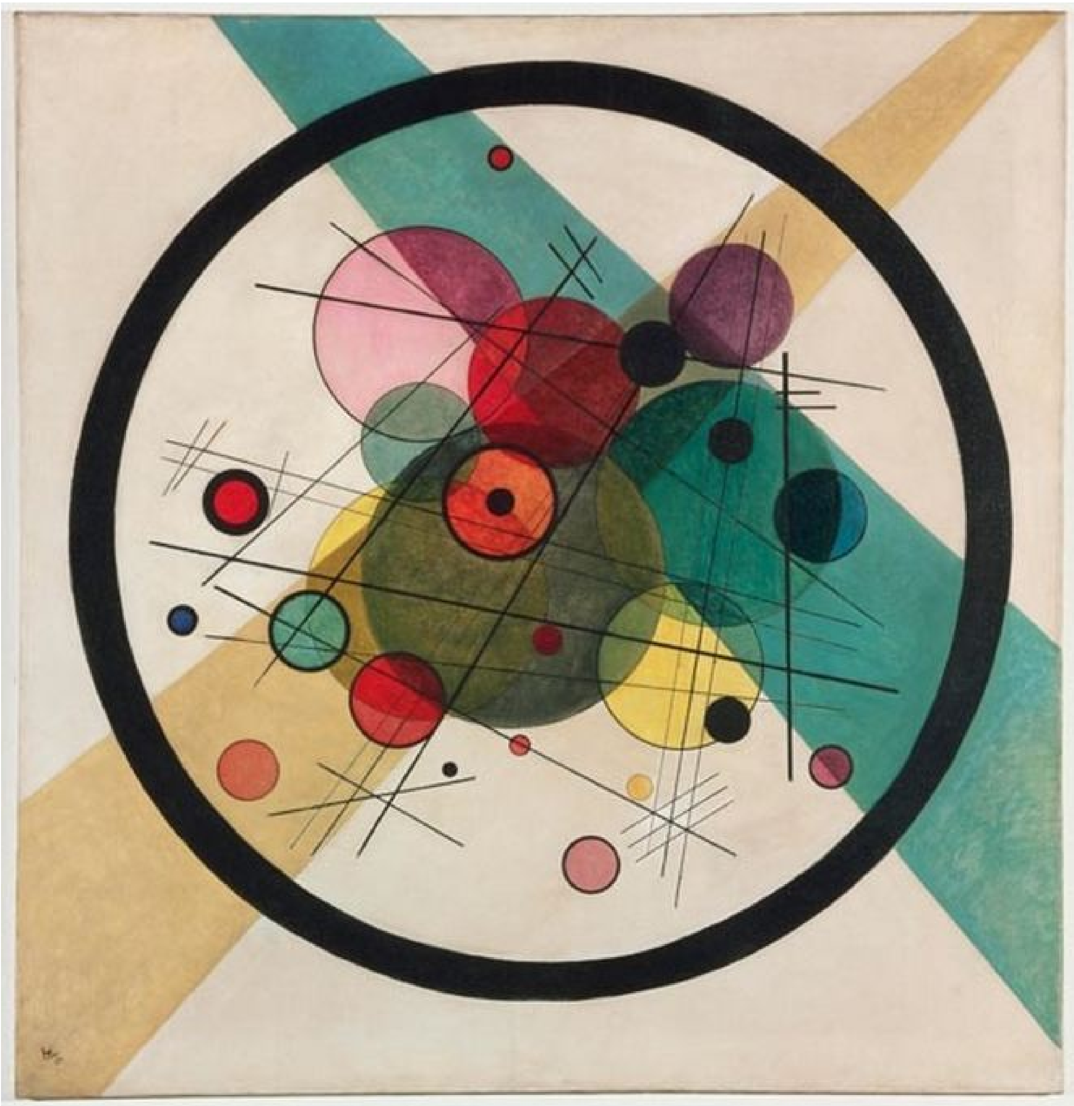
On Charged Lifshitz Holography

Emil Have



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I just think too many nice things have happened in string theory for it to be all wrong. Humans do not understand it very well, but I just don't believe there is a big cosmic conspiracy that created this incredible thing that has nothing to do with the real world.

— Edward Witten

ABSTRACT

We begin by studying holographic renormalization of a free Einstein-Maxwell-Dilaton theory, which we call the electromagnetic uplift. Imposing asymptotically locally AdS boundary conditions, we employ a generalized version of the Hamilton-Jacobi approach to holographic renormalization and find a novel counterterm for $d = 4$. We construct the associated Fefferman-Graham expansions recursively and identify the Ward identities. We also comment on a subtlety regarding holographic renormalization of p-form fields and provide a conjecture based on our results for one-forms. We present a simple method to determine the counterterm action for massless p-form fields in AdS_{p+2} and illustrate the approach for $d = 2$.

We then develop charged Lifshitz holography for $z = 2$ by performing a Scherk-Schwarz reduction of the electromagnetic uplift. The Lifshitz space-time in the reduced theory is shown to correspond to a $z = 0$ Schrödinger geometry in the electromagnetic uplift. The sources are identified as the leading components of the bulk fields in a vielbein formalism and are shown to transform under a $U(1)$-extended Schrödinger group. We show that the new sources can be identified with the fields of Galiliean Electrodynamics (GED). Since the Scherk-Schwarz reduction becomes null on the boundary, we are directly lead to the conclusion that the boundary geometry becomes torsional Newton-Cartan (TNC), which we also explore from the perspective of gauging the Schrödinger algebra. We then determine the VEVs along with all Ward identities, and we show that the integrated Weyl anomaly becomes an action describing Hořava-Lifshitz gravity coupled to GED on a Newton-Cartan geometry. Finally, based on a dimensional analysis of GED on anisotropic backgrounds, we provide a conjecture for the extension of charged Lifshitz holography to general values of z.

We provide the required background in holography, Newton-Cartan geometry and holographic renormalization. We also provide a survey of pure Lifshitz holography for arbitrary values of z.

We are all agreed that your theory is crazy.
The question which divides us is whether
it is crazy enough to have a chance of being correct.
My own feeling is that it is not crazy enough.

— Niels Bohr

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Finally, I want to thank my family for their love and support through the years.

Special thanks goes to Nanna. You mean the world to me.

1 Although, I'd prefer it to be even.

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NOTATION & CONVENTIONS

We use the mostly positive Minkowski metric—for $\mathbb{R}^{d,1}$,

$$\eta_{\mu\nu} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{d \text{ entries}}). \tag{i}$$

Further, we make use of the summation convention, i.e. all repeated indices (of any kind!) are summed over unless explicitly stated. In the chiral—or Weyl—representation the four-dimensional Dirac matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{ii}$$

with Pauli matrices

$$(\sigma^\mu)_{\alpha\dot{\beta}} = (1, \sigma^i)_{\alpha\dot{\beta}}, \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (1, -\sigma^i)^{\dot{\alpha}\beta}. \tag{iii}$$

They satisfy the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}$. Objects contracted with γ-matrices may be written in Feynman slash notation, e.g. $\not\partial := \gamma^\mu \partial_\mu$. The van der Waerden indices are raised and lowered using the two dimensional Levi-Civita symbol

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta} = -\varepsilon_{\dot\alpha\dot\beta}. \tag{iv}$$

Further, we can define representations of generators of Lorentz transformations in terms of the Pauli matrices,

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\right), \tag{v}$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{4}\left(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\right). \tag{vi}$$

We denote antisymmetrization of indices with "[]" and symmetrization with "()", e.g.

$$T_{[\mu\ldots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}} \tag{vii}$$

where S_n is the symmetric group on n symbols. The Riemann tensor is defined via

$$[\nabla_\mu, \nabla_\nu] X_\rho = R_{\mu\nu\sigma}{}^\rho X_\rho - 2\Gamma^\rho_{[\mu\nu]}\nabla_\rho X_\sigma, \tag{viii}$$

for an arbitrary one-form X_ρ, where

$$R_{\mu\nu\sigma}{}^\rho X_\rho = -\partial_\mu \Gamma^\rho_{\nu\sigma} + \partial_\nu \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{ix}$$

from which the Ricci tensor obtained via the contraction $R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma$. Given a metric $g_{\mu\nu}$, the Christoffel symbols are given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{x}$$

We will, unless stated otherwise, adhere to the following index convention

$\mu, \nu, \ldots$: General space-time indices. In the context of Lifshitz holography in chapters 5 and 6, the three-dimensional indices on radial hypersurfaces.
$i, j, \ldots$: Spatial part of the general space-time indices. The time compoenent will be denoted t.
$\underline{a}, \underline{b}, \ldots$: Tangent space indices including time.
$a, b, \ldots$: Tangent space indices excluding time. The time component is denoted "0".
$\mathcal{M}, \mathcal{N}, \ldots$: Five-dimensional space-time indices. Used in bulk models for the uplift in chapter 6.
u: The compact direction in five-dimensional models.
$A, B, \ldots$: Four-dimensional space-time indices. The same as $\mathcal{M}, \mathcal{N}, \ldots$, but excludes the radial direction.
$M, N, \ldots$: Four-dimensional space-time indices. The same as $\mathcal{M}, \mathcal{N}, \ldots$, but excludes the compact direction u.

We point out that in chapter 3, the index structure is different: there, we use $\mu, \nu, \ldots$ to denote $(d+1)$-dimensional space-time indices, while index $i, j, \ldots$ are used for d-dimensional indices that exclude the radial direction. We also point out that tangent space indices $a, b, \ldots$ in this chapter incluces time.

Note also that in chapter 2, we use z to denote the radial coordinate in Poincaré coordinates, while in later chapters—notably chapters 3, 5, and 6, we use r. In particular, since in the context of Lifshitz holography, z is the symbol for the dynamic exponent, we hope that this does not cause any confusion.

Throughout the thesis, we will use $\simeq$ to indicate the leading term. For example, if a quantity X has an expansion in a parameter r near $r = 0$ of the form $X = X_{(0)} r^{-2} + X_{(1)} r + X_{(2)} r^{75} + \cdots$, we write $X \simeq X_{(0)} r^{-2}$.

Unless stated otherwise, we work in natural units, where $\hbar = c = k = 1$.

1 INTRODUCTION

The seminal paper by Maldacena in 1997 on the AdS/CFT correspondence [1] sparked a veritable revolution in theoretical physics. The correspondence provides a concrete realization of the holographic principle put forth by 't Hooft and Susskind in [2, 3], and was the culmination of the second superstring revolution initiated by Witten in [4].

While the most general version of the AdS/CFT correspondence posits an exact equivalence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory, a more moderate and well-established version of the correspondence relates weakly coupled gravitational theories in asymptotically locally AdS—the *bulk*—to strongly coupled field theories on the boundary (usually of either flat or spherical topology). This feature of the duality is extremely useful: it offers tantalizing opportunities of describing strongly coupled field theories using *weakly coupled*—that is to say classical—gravitational theories, and many of the most interesting not-yet well understood physical phenomena belong in this class of theories, e.g. high T_c superconductivity—explored holographically in e.g. [5–8]—and the $\mathcal{N} = 4$ quark-gluon plasma, for which the viscosity to entropy ratio was calculated holographically in the seminal work [9] (see [10] for a review). The attempt to describe strongly coupled condensed matter systems using the AdS/CFT correspondence has garnered much attention in recent years [11–14] following the groundbreaking work of Sachdev and Herzog in [15].

However, most strongly coupled condensed matter systems are non-relativistic in nature. This is an inherent problem and would seem to render the application of holography in the context of such systems void, as succinctly pointed out by Nobel laureate Phil Anderson in [16]:

> *"As a very general problem with the AdS/CFT approach in condensed-matter theory, we can point to those telltale initials "CFT"—conformal field theory. Condensed-matter problems are, in general, neither relativistic nor conformal. Near a quantum critical point, both time and space may be scaling, but even there we still have a preferred coordinate system and, usually, a lattice. There is some evidence of other linear-T phases to the left of the strange metal about which they are welcome to speculate, but again in this case the condensed-matter problem is overdetermined by experimental facts. "* —Philip W. Anderson.

This is of course a legitimate point of critique. While holographic systems in an AdS/CFT context are conformal, relativistic and, to make matters worse, supersymmetric, at short distances, many of these symmetries are broken at large distances in the presence of a finite chemical potential or temperature. This long distance behaviour can, in fact, be captured by gravity duals, showcasing the power of the fluid/gravity correspondence [10, 11, 17, 18].

Another—and more recent—way forward is to directly consider bulk space-times, whose asymptotic behaviours are different from AdS as emphasized in [19–22]. Such space-times include Schrödinger, Lifshitz and hyperscaling violating geometries, which are all characterized by a so-called dynamical exponent z, which expresses the anisotropy between space and time on the boundary. Developing such notions of non-relativistic holography turns out to be extremely challenging, and despite a flurry of research (see e.g. [23–29]), this is still very much a work in progress, and many aspects remain poorly understood. It is important to emphasize in this context that there are no conventional tools available to the condensed matter theorist that allows him/her to attack such problems, and as such these holographic approaches are unique in their scope.

In addition to the applications of non-relativistic holography to condensed matter problems, it is also of intense theoretical interest to ascertain the validity of the holographic principle in concrete settings that go beyond the original AdS setting, and circumstantial evidence for the principle in the form of exotic gravity/field-theory dualities constitute important milestones on the road towards quantum gravity.

For Lifshitz bulk geometries, which will be the focus of this thesis, such a non-relativistic holographic correspondence for $z = 2$ was developed in [23, 24], where it was shown that this model could be embedded in a higher-dimensional model with conventional AdS geometry. By considering perturbations around a $z = 0$ Schrödinger geometry (which has AdS boundary conditions) in the higher-dimensional theory, it was shown that a Scherk-Schwarz reduction produced a corresponding $z = 2$ Lifshitz geometry with associated perturbations, and this mapping allowed for a complete identification of the holographic dictionary for $z = 2$ Lifshitz holography. A crucial observation was the realization that the boundary is described by a novel extension of Newton-Cartan (NC) geometry (originally developed by Cartan in [30, 31]) called torsional Newton-Cartan (TNC) geometry. This analysis was subsequently extended to generic values of $z > 1$ in [29] (see also [32, 33]). The sources and VEVs of Lifshitz holography possess a Schrödinger symmetry, and it was shown in [34] that TNC geometry as it appears in Lifshitz holography can be obtained by gauging the Schrödinger algebra for suitable values of z.

Concretely, it was recently shown in [35, 36] (see also [37, 38]) that TNC provides a framework for field theory analyses of the (fractional) quantum Hall effect. It has also been used in connection with other problems involving strongly correlated electrons [39, 40]. In these approaches, the symmetry of the problem is used as a guiding principle, and various quantities such as the Hall viscosity and the Hall conductance can be computed as responses to the geometric data.

1.1 OUTLINE & SUMMARY

Below, we provide an outline of the contents of each chapter. Note that chapters 3, and 6 in particular, contain mainly new material—these results will be the subject of [41], to appear. We also remark that each chapter has its own outlook section, where we comment on interesting extensions of the analyses provided in the chapter.

In chapter 2—along with the companion appendix D, where a string theoretic derivation is presented—we provide a review of the AdS/CFT correspondence. This includes an investigation of the holographic principle in section 2.1 and a motivation of the correspondence based on the interpretation of the radial AdS direction as a renormalization group scale in section 2.2. We then formulate a precise version of the correspondence in section 2.3 and explore the holographic dictionary in sections 2.4 and 2.5. These sections review material found mainly in [11, 12, 42–46], and provides a foothold in holography.

We conclude in section 2.6 with a brief discussion of Witten diagrams and provide a concrete computation of a three-point function, following the approach of [47].

In chapter 3, we provide a detailed survey of holographic renormalization for AdS space-times. In the supplementary appendix F, we—following mainly the review [48]—review the original Fefferman-Graham (FG) approach, which, although conceptually straightforward, is computationally forbidding. We also provide a survey of the de Boer-Verlinde-Verlinde (dBVV) method following [49], which relies on an ansatz and a bilinear operation that we have named the deWitt bracket[1]. We then show in section 3.1 how holographic renormalization can be understood in terms of Hamilton-Jacobi (HJ) theory, and discuss how the solution of the HJ equation conveniently involves the introduction of a suitably chosen operator $\mathfrak{F}$ in terms of which the (bare) on-shell action and other relevant quantities can be expanded in eigen-modes. The discussion is based on [50, 51]. In section 3.2, we discuss the renormalization of the free Einstein-Maxwell-Dilaton model (EMD)—which we will call the electromagnetic uplift in chapter 6—and derive a set of counterterms, which has not previously appeared in the literature. We also construct novel FG expansions and determine the new Ward identities satisfied by the VEVs. Finally, in section 3.3, we discuss renormalization of p-form fields based on our results for the Maxwell field and provide a conjecture. These results do not exist in the literature, but are under investigation by other people, and will be the subject of an upcoming paper[2] [52]. We conclude with the observation that for $d = 2$, we can renormalize an EMD model by hodge dualization, and we generalize this observation to an easy way of obtaining the counterterm action for free p-form fields in AdS_{p+2}. To our knowledge, this has not appeared in the literature previously.

In chapter 4, we tell the tale of Newton-Cartan geometry from the perspective of *gauging algebras*. We start in section 4.1 with an introductory calculation that shows how Riemannian geometry—the arena of general relativity—is obtained by gauging the Poincaré group in a process analogous how one obtains gauge theories from a quantum field theory perspective. This section is based on appendix A in [53] and [54, 55], but is significantly more detailed. In section 4.2, we then discuss generalities of non-relativistic space-times and their relation Newton-Cartan geometry, before showing in section 4.2.2 that the gauging procedure applied to the Galilei and Bargmann algebras precisely gives TNC geometry. These sections are based on [34, 53, 56] and summarize the results therein. In section 4.3, we showcase how TNC geometry can be obtained from Lorentzian geometry via null reduction, and in the following section 4.4, we explore how non-relativistic field theories couple to TNC geometry from the perspective of null reduction. Both the preceding sections are based mainly on [57].

1 This object, seemingly, did not have a name. Since it involves the deWitt metric with parameter $d - 1$, we have named this operation the deWitt bracket.
2 I thank Kostas Skenderis for telling me about this.

In chapter 5, we investigate Lifshitz holography as developed by Hartong, Kiritsis and Obers in [29, 32] for generic values of the critical exponent z. We begin by a lighting review of Lifshitz field theory in section 5.1, before we turning to the actual holographic analysis in section 5.2. The bulk consists of an Einstein-Proca-Dilaton (EPD) model, which is shown to support Lifshitz solutions for generic values of z. We then identify the sources as the leading parts of the bulk fields in a vielbein formalism and see that they transform under the Schrödinger algebra, and we discuss how the boundary geometry becomes TNC in section 5.2.4. We identify the general properties of the VEVs by assuming holographic renormalizabiltiy in section 5.3, which also allows us to determine the TNC covariant Ward identities. The analysis of this chapter is a much more detailed version of [29].

We end the thesis in chapter 6, where we develop charged Lifshitz holography for $z = 2$ by generalizing the results of [23, 24, 57]. In section 6.1, we begin with a discussion of Galilean Electrodynamics (GED) coupled to TNC geometries, which was recently developed in [58]. We then provide a review of pure $z = 2$ Lifshitz holography and relate it to the uplift (see also appendix H), after which we Scherk-Schwarz reduce the electromagnetic uplift that we renormalized in chapter 3, which gives rise to a new reduced EPD-Maxwell-scalar model acting as the bulk theory for charged Lifshitz holography. In section 6.3, we show that this allows for a complete identification of the sources of charged Lifshitz holography, and we show that the new sources transform as the fields of GED under local transformations. This novel result is one the main contributions of this thesis. We then demonstrate explicitly that the reduction employed becomes null on the boundary, which is also discussed in [23, 24, 57]. This means that the results of chapter 4 directly leads us to conclude that the boundary geometry becomes TNC. In section 6.4.2, we, following [34], explore how the boundary geometry emerges from from gauging the $z = 2$ Schrödinger algebra. In section 6.5, we show that the integrated dimensionally reduced Weyl anomaly takes the form of an action for Hořava-Lifshitz (HL) gravity coupled to GED on TNC geometry, which is also a new result. We then consider in section 6.6 how the higher-dimensional FG expansions that we generated in chapter 3 lead to expansions for the Lifshitz bulk fields, which will allow us to see the source structure of section 6.3 appear explicitly, which is a new analysis. We then work out all the new VEVs corresponding to the sources and determine novel Ward identities. Finally, we provide a conjecture for general-z charged Lifshitz holography based on a dimensional analysis of GED coupled to anisotropic backgrounds.

2 ASPECTS OF THE ADS/CFT CORRESPONDENCE

In this chapter, we provide background in the AdS/CFT correspondence which will be useful for the understanding of Lifshitz holography. This material contained here is based on a motley collection of sources, chiefly [11, 42–45].

We begin by reviewing the holographic principle in section 2.1, which, in its most general sense, roughly states that a volume of spacetime is equivalently described by the codimension one boundary of the volume, as argued by 't Hooft and Susskind, and we briefly mention a generalization known as the covariant entropy bound. This section is a review of the discussions in [2, 3, 46].

Based on [12], we then motivate the AdS/CFT correspondence in section 2.2 by interpreting the radial AdS direction as a renormalization scale, which naturally leads to AdS geometry.

Next, in section 2.3, we provide an overview of the AdS/CFT correspondence and discuss the relation to string theory and various supersymmetric field theories. This analysis follows mainly [42–44]. In the companion appendix D, we present a fairly detailed "stringy" derivation of the correspondence from D3 branes, which are considered from the point of view of both open and closed strings, which, when unified, give the correspondence.

We then proceed to consider the holographic dictionary in its various manifestations, starting with the field-operator correspondence in section 2.4 and culminating with the GKPW rule in section 2.5 as well as a somewhat detailed summary of the dictionary in table 2.3. These analyses are based primarily on [11, 43]

We end the chapter in section 2.6 with a small discussion of Witten diagrams and showcase the computation of a concrete three-point diagram. The calculation follows [47].

2.1 THE HOLOGRAPHIC PRINCIPLE

The idea of holography originates with 't Hooft's exposition in [2]: given some quantum theory, we can—invoking an analogue of the third law of thermodynamics—relate the entropy S to the total number of degrees of freedom; in particular, if[1] $\mathcal{N}$ is the dimension of the Hilbert space (i.e. the number of states), the following relation holds

$$e^S = \mathcal{N}. \tag{2.1.1}$$

By the the covariant entropy bound, or simply the Bekenstein bound[2], the entropy of some system cannot exceed that of a black hole which is given by the Bekenstein-Hawking relation [60]

$$S \leq S_{\text{BH}} = \frac{A}{4G}, \tag{2.1.2}$$

where A is the area of the black hole and G is Newton's constant. Combining the spherical entropy bound with the relation between entropy and degrees of freedom (2.1.1) leads us to conclude that the number of states is bounded by $\mathcal{N} \leq e^{\frac{A}{4G}}$.The ideas put forward by 't Hooft inspired Susskind to write the iconic article *The World as a Hologram* [3], where he proffers the following bit of reasoning: supposing that the world—which we can take to be d-dimensional—is a lattice of binary quantum degrees of freedom (i.e. "spin-like") and assume that the lattice spacing is the Planck length, ℓ_p, since smaller distances cannot be resolved in quantum gravity. Thus, the number of quantum states in a volume V is $\mathcal{N}(V) = 2^n$, where $n = \frac{V}{\ell_p^d}$ denotes the number of lattice sites in V, which—in contradistinction to what we found above—implies the following entropy bound:

$$S \leq \log \mathcal{N}(V) = \frac{V}{\ell_p^d} \log 2. \tag{2.1.3}$$

Rather than being bounded by the area A, the largest possible entropy scales as the *volume* V! As long as the system is larger than the Planck scale, it holds that $V \geq A$—and so this "field-theoretic" derivation predicts a larger entropy bound.

1 Although standard in this context, it is unfortunate that the dimensionality of the Hilbert space is denoted by $\mathcal{N}$—a symbol usually reserved to denote to the number of supercharges in a given theory. We hope this does not cause confusion.

2 This bound relies on certain assumptions [59]: the system must be of constant, finite size and have limited self-gravity—that is, gravity must be weak compared to the other forces acting in the system.

To see this, we invoke a basic principle of quantum mechanics: unitarity. Suppose that the field-theoretic entropy estimate holds. Then by the bound (2.1.1), the dimensionality of the Hilbert space describing the region is $\mathcal{N} \sim e^V$ for V the volume. But, supposing that the region was converted into a black hole, the Bekenstein-Hawking entropy relation implies that the region is now described by a Hilbert space of dimension $e^{A/4G}$: the number of states has decreased and the Hilbert spaces are no longer isomorphic, violating unitarity. Insisting on unitary quantum mechanical evolution, we are let to conclude that the Hilbert space must have had dimensionality $e^{A/4G}$ to start with. This observation leads to a preliminary version of the holographic principle:

Susskind–'t Hooft Holographic Principle: *A region of spacetime with boundary of area A is fully described by at most $A/4G$ degrees of freedom.*

The spherical entropy bound, however, turns out to be violated in some instances—the simplest of which is the following: consider a system in the midst of a gravitational collapse. Before the system is destroyed on the black hole singularity, its surface area becomes arbitrarily small, and since entropy cannot decrease, the bound is violated. This spurred the discovery of a more universal[3] entropy bound: *the covariant entropy bound*, put forward by Bousso in [59]. It is worth noting, however, that no known fundamental derivation of the covariant entropy bound exists; if it holds true it must eventually be explained by a theory unifying gravity and quantum mechanics.

Since its proposal, several circumstances under which the bound holds has been uncovered by Flanagan *et al.* in [62], and further proof of its validity for free matter fields in the limit of weak gravitational back-reaction was given in [63]. Also, the covariant entropy bound has been shown [64] to reduce to other entropy bounds, which were observed to hold in some settings—in particular the spherical entropy bound. For more details, we refer the reader to [46, 63, 64].

Taking the holographic principle at face value, then, we are lead to conclude that if we have a theory of quantum gravity on some manifold $\mathcal{M}$, the theory will be entirely determined by some other theory living on the boundary $\partial\mathcal{M}$, and the theories are said to be *dual*.

2.2 MOTIVATION OF THE CORRESPONDENCE: GEOMETRIZATION OF RENORMALIZATION GROUP FLOW

Although the AdS/CFT has its origins in string theory—as we demonstrate in appendix D—it is possible to motivate the correspondence without explicit reference to string theory. The following line of reasoning was put forth by Horowitz and Polchinski in [65] (see also [11, 12, 66]): since any quantum theory of gravity contains a massless spin-two graviton[4], one could theorize that the graviton somehow arises as a composite of two spin-one gauge bosons[5]. A priori, this seems to be excluded by the Weinberg-Witten theorem [68], which states that

Theorem (Weinberg-Witten) *A quantum field theory with a Poincaré covariant and conserved energy-momentum tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum*[6].

General relativity circumvents this theorem by having either a vanishing energy-momentum tensor[7] or by having a reparametrization non-invariant *matter stress tensor* when additional fields are present.

However, the crucial assumption made in the Weinberg-Witten theorem is that the graviton moves in the same spacetime as the gauge boson, so by making the graviton live in a higher dimensional space, for example, we can again circumvent the theorem. As we saw above, the holographic principle constrains the entropy of any system to be at most that of a black hole occupying the space of the system, implying that the theory of quantum gravity lives in one dimension higher than the gauge theory. The extra dimension in the quantum gravity, as we now demonstrate, has a nice interpretation

3 Although a recent paper [61] suggests speculative scenarios in which this *new* bound might be broken.
4 The usual way of realizing this is by linearizing Einsteins equations, gauge fixing and then noting that the resulting polarization tensor for the perturbation (which we think of as the graviton) is a spin-two representation.
5 Interestingly, in perturbative quantum gravity, this composite behavior is actually observed at the amplitude-level: so-called KLT (Kawai, Lewellen, Tye) relations relate gravity amplitudes to products of Yang-Mills amplitudes; this *double-copy* structure has led to the paradigm (Gravity) = (Yang-Mills)^2; see e.g. [67] for details.
6 This momentum would then be given by $P^\mu = \int d^d x \, T^{0\mu}$.
7 By the metric equation of motion, $0 = \frac{\delta S}{\delta g_{\mu\nu}} \sim T^{\mu\nu}$.

in terms of the Wilsonian renormalization energy scale u of the gauge theory. In particular, the RG equations for a generic coupling constant λ are *local*,

$$u\partial_u \lambda(u) = \beta(\lambda(u)), \tag{2.2.1}$$

where $\partial_u = \frac{\partial}{\partial u}$ and β is the beta-function which encodes the energy-scale dependency of the coupling (see e.g. [69, 70]). Furthermore, if we want gravity from pure Yang-Mills, we must require that it be strongly coupled so that quantum effects are dominant which are not well understood—classical Yang-Mills and general relativity are certainly not the same thing. To simplify our considerations, we take the simplest possible RG flow of vanishing β-function, leading to a conformal field theory. This implies that—in a Lorentz-invariant theory, which we tacitly assume to be dealing with—the scale transformation $x^\mu \to \lambda x^\mu$, with $\mu = 1, \dots, d$ and $\lambda \in \mathbb{R}$, is a symmetry, and, if u is an energy scale, it must behave under scale transformations as $u \to u/\lambda$ by dimensional analysis, giving rise to a $SO(1,1)$ symmetry. Writing down a $(d+1)$-dimensional metric with u as the extra direction, which is Poincaré invariant and respects $SO(1,1)$ symmetry, we are invariably led to the result $ds^2 = \frac{u^2}{L^2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{L^2}{u^2}\mathrm{d}u^2$, which, on making the change of coordinates $z := L^2/u$ takes the form

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2\right), \tag{2.2.2}$$

for L the AdS radius, which we recognize as the metric (of the Poincaré patch) of AdS$_{d+1}$ (see Appendix A). This beautiful relation between the renormalization group (RG) equations and geometry (i.e. general relativity (GR)) is occasionally stylized as

$$\mathrm{RG} = \mathrm{GR}. \tag{2.2.3}$$

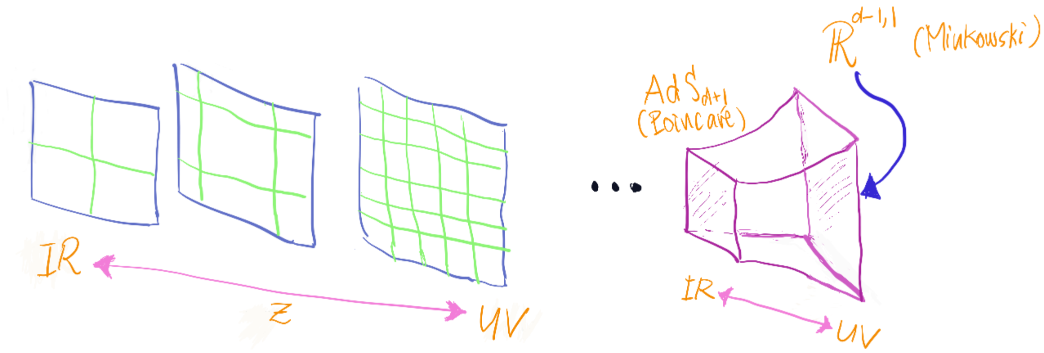


Figure 2.1: The left figure is an illustration of how the extra "radial" direction of the bulk acts as the resolution scale of the field theory. This course-graining naturally leads to AdS space in the Poincaré patch, as shown on the right, which has boundary $\mathbb{R}^{d-1,1}$.

The metric (2.2.2) is a solution to a large class of dynamical theories fulfilling the "Landau criterion" of being invariant under diffeomorphisms while at the same time having a minimal number of derivatives; these have the schematic Einstein-Hilbert form

$$S = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{-g}\,[R - 2\Lambda + \dots], \tag{2.2.4}$$

where $2\Lambda = -d(d-1)/L^2$ in order for the AdS metric (2.2.2) to be a solution. The AdS length scale L represents the radius of curvature, i.e. $R \sim L^{-2}$, implying that the gravity theory is *classical* (and in particular "weakly coupled") in the regime $L \gg \ell_p$. As we shall see, the dual CFT is precisely *strongly coupled* in this limit. We now turn to a more formal discussion of the AdS/CFT correspondence.

2.3 THE ADS5/CFT4 CORRESPONDENCE

While there are in fact various incarnations of the AdS/CFT correspondence[8], we focus our exposition on the *vanilla* version: AdS$_5$/CFT$_4$, which relates the dynamics of $(3+1)$-dimensional $\mathcal{N} = 4$

8 For example, AdS$_7$/CFT$_6$, which involves a horizon limit of M5 branes, where the CFT is a 6d $(2,0)$-superconformal field theory; see [42] for a review. This field theory has extremely interesting ties to the geometric Langlands duality, see [71].

super Yang-Mills theory (SYM)—see appendix C for a short review—to type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. The *derivation* for this correspondence—which is, admittedly, somewhat heuristic[9]—is considered in appendix D and involves two perspectives of D3 branes. We state the correspondence below:

The AdS$_5$/CFT$_4$ Conjecture: *$\mathcal{N} = 4$ SYM with gauge group $SU(N)$ and coupling g_{YM} is dynamically equivalent to type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and string coupling g_s on $AdS_5 \times S^5$ with AdS radius L and N units of $F_{(5)}$ flux on S^5. The parameters of each side of the correspondence are mapped to each other via*

$$g_{YM}^2 = 4\pi g_s, \qquad \textit{and} \qquad 2g_{YM}^2 N = \frac{L^4}{(\alpha')^2}. \tag{2.3.1}$$

In particular, the results of appendix D allow us identify three versions of the conjecture (see also e.g. [43]):

Table 2.1: Incarnations of the AdS$_5$/CFT$_4$ Correspondence. Here, $\lambda = g_{\mathrm{YM}}^2 N$ is the t'Hooft coupling (cf. appendices C and D).

	$\mathcal{N} = 4$ SYM	IIB on $\mathrm{AdS}_5 \times S^5$
Strongest form	any N and λ	Quantum string theory, $g_s \neq 0, \alpha'/L^2 \neq 0$
Strong form	$N \to \infty$, λ fixed but arbitrary	Classical string theory, $g_s \to 0, \alpha'/L^2 \neq 0$
Weak form	$N \to \infty$, λ large	Classical supergravity, $g_s \to 0, \alpha'/L^2 \to 0$

The derivation found in appendix D presents a heuristic derivation of the weakest form of the conjecture as described above.

2.3.1 *Counting Degrees of Freedom and the UV/IR Connection*

To furnish a realization of the holographic principle, the Bekenstein-Hawking relation (2.1.2) should be satisfied, that is; the area of the boundary A should equal the number of degrees of freedom N_d, or, in other words, the maximum entropy, which we schematically write as

$$\frac{A}{4G} = N_d. \tag{2.3.2}$$

Naïvely, both sides of the above equality are infinite: on the boundary $\mathcal{N} = 4$ SYM, the entropy is infinite since the theory is conformal, which means that it has degrees of freedom at arbitrarily small scales, while the boundary area of AdS has infinite area. To verify (2.3.2), we therefore require regularization, which, following [12, 73], is achieved for the boundary by replacing it with a sphere just inside the boundary at $z = \delta$, the resulting area of which is given by $A \sim L^3/\delta^3$. The total number of *cells* making up the sphere is $\sim \delta^{-3}$, while the number of field degrees of freedom for $SU(N)$ $\mathcal{N} = 4$ SYM is $\sim N^2$, implying that (up to numerical factors)

$$N_d \sim \frac{N^2}{\delta^3} \sim \frac{AN^2}{L^3} \sim \frac{AL^5}{(\alpha')^2 g_s^2} \sim A/G_5, \tag{2.3.3}$$

where we have used (D.1.26) and identified the five-dimensional Newton constant $G_5 = (\alpha')^2 g_s^2 L^{-5}$. This is precisely the desired result.

Now, from the discussion in section 2.2, we immediately infer what has become known as the UV/IR connection: when the energy scale u is small—corresponding to the IR of the field theory—the corresponding AdS space has a large value of z, i.e. we are in the deep interior; the UV—and *vice versa*. This is precisely the UV/IR connection of [73] (see also [72, 74]), where they argue for it by using the behaviours of certain correlators in supergravity and Yang-Mills theory.

2.4 THE HOLOGRAPHIC DICTIONARY FOR ADS/CFT I: THE FIELD/OPERATOR MAP

The AdS/CFT correspondence provides a map between operators of $\mathcal{N} = 4$ SYM in certain representations of $PSU(2,2|4)$ and supergravity fields showing up in the Kaluza-Klein tower of type IIB

9 But the correspondence has so far passed all tests; many of which have been very non-trivial [43, 72].

SUGRA on AdS_5 reduced on S^5. The principal[10] field theory operators involved in the mapping are gauge invariant[11] 1/2 BPS primary operators $\mathcal{O}_{\Delta_i}$ of conformal weight Δ_i, i.e. the operators satisfy $[D, \mathcal{O}_{\Delta_i}(0)] = -i\Delta_i \mathcal{O}_{\Delta_i}(0)$, $[K_\mu, \mathcal{O}_{\Delta_i}(0)] = 0$, as well as $[S_{a\alpha}, \mathcal{O}_\Delta\} = 0 = [\bar{S}^a_{\dot{\alpha}}, \mathcal{O}_\Delta\}$ for all $a = 1, \ldots, \mathcal{N}$ and all $\alpha, \dot{\alpha} = 1, 2$, as well as $[\mathcal{Q}^a_\alpha, \mathcal{O}\} = 0$ for at least one of the Poincaré supercharges (see also appendix B). The 1/2 BPS operators of conformal weight $\Delta = k$ all have the form[12] [43]

$$\mathcal{O}_\Delta(x) = \mathrm{Str}\left[\phi^{\{i_1} \cdots \phi^{i_k\}}\right], \tag{2.4.1}$$

where the ϕ^i are the scalar fields of $\mathcal{N} = 4$. Operators of this form are dual to the single particle (elementary) fields of type IIB SUGRA on $\mathrm{AdS}_5 \times S^5$, while higher trace operators are dual to bound states of one-particle states. A concrete mapping can be found in [45]. In particular, to actually derive the mapping, type IIB SUGRA is Kalaza-Klein compactified on S^5, leading to an expansion of the SUGRA fields in spherical harmonics Y_Δ on the sphere, which are labelled by the rank Δ of the totally symmetric traceless representations of $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$. Taking y to be the coordinates on S^5, we get

$$\varphi(x, y) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(z) Y_\Delta(y). \tag{2.4.2}$$

The reduced fields may acquire mass from the compactification process, and an involved calculation shows that the conformal dimension of the field theory operator $\mathcal{O}_\Delta$ is mapped to the mass as depicted in the table below (generalized to the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence) [43].

Table 2.2: Relation between mass and conformal dimension for various fields.

Type of field	Relation between m and Δ
Scalars, massive spin-2 fields	$m^2L^2 = \Delta(\Delta - d)$
Massless spin-2 fields	$\Delta = d,$
p-form fields	$m^2L^2 = (\Delta - p)(\Delta + p - d)$
Spin-1/2, spin-3/2	$\lvert m\rvert\, L = \Delta - d/2$
Rank s symmetric traceless tensor	$m^2L^2 = (\Delta + s - 2)(\Delta - s + 2 - d)$

Unprotected non-BPS operators, such as the Konishi operator [75], are believed to be dual to massive type IIB string modes not present in the low energy SUGRA description.

2.4.1 *Boundary Asymptotics*

Our considerations above were motivated by symmetry arguments alone. We can, however, be more explicit by examining the boundary behaviour of the SUGRA fields [12, 76]. For simplicity, we consider a toy model consisting of a massive scalar φ in the bulk

$$S = -\frac{1}{2}\int_{\mathrm{AdS}} \mathrm{d}z\mathrm{d}^d x\, \sqrt{-g}\left[g^{mn}\partial_m\varphi\partial_m\varphi + m^2\varphi^2\right]. \tag{2.4.3}$$

The scalar equation of motion becomes the usual Klein-Gordon equation,

$$\left(\Box_g - m^2\right)\varphi = 0, \tag{2.4.4}$$

where the Laplacian reads

$$\Box_g\varphi = \frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}g^{mn}\partial_n\varphi\right), \tag{2.4.5}$$

10 The single trace 1/2 BPS operators are the principal fields in the sense that higher trace operators may be constructed in terms of them using the OPE [45].

11 There is no gauge group on the SUGRA side of the correspondence, so the operators on the field theory side must be gauge invariant.

12 Gauge invariance and the requirement of superconformal primarity constrains the single trace 1/2 BPS operators to involve only scalars, see [45] for details.

with g_{mn} the AdS$_{d+1}$ metric in Poincaré coordinates, $ds^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)$, so that

$$\Box_g = \frac{z^{d+1}}{L^{d+1}}\partial_m\left(\frac{L^{d+1}}{z^{d+1}}g^{mn}\partial_n\right) = \frac{z^{d+1}}{L^{d+1}}\partial_z\left(\frac{L^{d-1}}{z^{d-1}}\partial_z + \frac{L^{d-1}}{z^{d-1}}\partial^2\right) \tag{2.4.6}$$

$$= \frac{z^2}{L^2}\left(\partial_z^2 - (d-1)z^{-1}\partial_z + \partial^2\right), \tag{2.4.7}$$

where $\partial^2 = \eta_{\mu\nu}\partial^\mu\partial^\nu$. Employing a plane wave ansatz $\varphi(z,x) = e^{ip_\mu x^\mu}\varphi_p(z)$, the Klein-Gordon equation becomes a Bessel-like ODE,

$$z^2\partial_z^2\varphi_p(z) - (d-1)z\partial_z\varphi_p(z) - (m^2L^2 + p^2z^2)\varphi_p(z) = 0, \tag{2.4.8}$$

which can be solved with e.g. *Mathematica*—the solution is a superposition of two Bessel functions (each of which corresponds to a solution itself), which gives rise to the $z \to 0$ boundary expansion (to lowest order in z)

$$\varphi(z,x) \sim \varphi_{(0)}(x)z^{\Delta_-} + \varphi_{(+)}(x)z^{\Delta_+}, \tag{2.4.9}$$

where $\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2L^2}$, and where the fields $\varphi_{(0)}$ and $\varphi_{(+)}$ live on the boundary—i.e. they are a function only of the boundary coordinates. The solution that goes as $\sim z^{\Delta_+}$ is known as the normalizable solution, while the solution that behaves as $\sim z^{\Delta_-}$ is called the non-normalizable solution[13]. Requiring the quantity in the square root to be positive in order to avoid imaginary fields, we get the Breitenholmer-Freedman bound [77],

$$m^2L^2 \geq -\frac{d^2}{4}, \tag{2.4.10}$$

which implies that AdS is stable in the presence of negative mass-squared scalars as long as the bound above is not violated[14]. Note further that $\Delta_+ \geq \Delta_-$ as well as $\Delta_- = d - \Delta_+$, which implies that under boundary conformal rescalings $x \to x' = \lambda x$, the boundary field $\varphi_{(0)}(x)$ transforms as (defining $z' = \lambda z$)

$$\varphi'_{(0)}(\lambda x) = \lim_{z'\to 0}\left(z'\right)^{-\Delta_-}\varphi'(z',x') = \lambda^{-\Delta_-}\lim_{z\to 0}z^{-\Delta_-}\varphi(z,x) = \lambda^{d-\Delta_+}\phi_{(0)}(x), \tag{2.4.11}$$

where we have used invariance of the bulk field $\varphi(z,x)$ under the AdS isometry $(z,x) \to (z',x') = (\lambda z, \lambda x)$, i.e. $\varphi'(z',x') = \varphi(z,x)$. From this we infer that $\varphi_{(0)}$ transforms as a source for a conformal primary operator with dimension Δ_+, leading us to identify the boundary field $\varphi_{(0)}$ as a source for a dual field theory operator $\mathcal{O}_{\Delta_+}$; and a similar analysis reveals that $\varphi_{(+)}$ is the vacuum expectation value (VEV) of $\mathcal{O}_{\Delta_+}$.

A careful analysis reveals, however, that this identification of source and VEV is only true for the mass range $-d^2/4 \leq m^2L^2 < 0$ with $\Delta < d$. For $(d-2)/2 \leq \Delta < d/2$, it turns out that we have to identify the conformal dimension of the field with Δ_-, so on the overlap $-d^2/4 < m^2L^2 \leq -d^2/4 + 1$, the identification of VEV and source of the field theory operator can be interchanged (see [43] for details.)

For completeness, we bring a version of the AdS/CFT dictionary:

13 This nomenclature comes from the fact that the action evaluated on the normalizable solution is finite, while the action evaluated on the other is not.

14 In fact, violating this bound will give rise to tachyons, just as $m^2 < 0$ in ordinary field theory. Such violations allow us to encode spontaneous symmetry breaking in the boundary field theory from a bulk perspective [11].

Table 2.3: The basic dictionary for the AdS/CFT correspondence. Adapted from [11].

Boundary: Field theory	**Bulk: Supergravity**
Scalar operator/order parameter $\mathcal{O}$	Scalar field
Source of operator	Boundary value of field (leading coefficient of non-normalizable solution)
VEV of operator	Boundary value of radial momentum of the field (leading coefficient of the normalizable solution; sub-leading to the non-normalizable solution)
Conformal dimension of operator	Mass of field (see table 2.2)
spin/charge of operator	Spin/charge of field
Energy momentum tensor $T^{\mu\nu}$	Metric field g_{mn}
Global internal symmetry current J^μ	Maxwell field A_m
Fermionic operator $\mathcal{O}_\psi$	Dirac field ψ
RG flow	Evolution in the radial AdS direction
No. of degrees of freedom	Radius L of AdS space
Global spacetime symmetry	Local isometry
Global internal symmetry	local gauge symmetry
Finite temperature	Black hole Hawking temperature (or radius of compact Euclidean time circle)
Chemical potential/charge density	Boundary values of electrostatic potential A_t (time component)
Free energy	On-shell value of action
Phase transition	Instability of black holes
Wilson line along $\mathcal{C}$	String worldsheet with endpoints on $\mathcal{C}$
Entanglement entropy of area $\mathcal{A}$	Minimal surface Σ with boundary $\partial\Sigma = \mathcal{A}$
Quantum anomalies	Chern-Simons terms

2.5 THE HOLOGRAPHIC DICTIONARY FOR ADS/CFT II: THE GKPW RULE

The GKPW rule—named after Gubser, Klebanov, Polyakov and Witten, who discovered it in 1998 [78, 79]—links the partition functions of the two sides of the AdS/CFT correspondence. On the field theory side, the $\varphi_{(0)}$ acts as a source for the operator $\mathcal{O}_\Delta$—as we argued above—so the partition function (in Euclidean signature) for the CFT takes the form

$$Z_{\text{CFT}}[\varphi_{(0)}] = e^{-W[\varphi_{(0)}]} = \left\langle \exp\left(\int \mathrm{d}^d x\ \varphi_{(0)}(x)\mathcal{O}_\Delta\right)\right\rangle_{\text{CFT}} . \tag{2.5.1}$$

On the AdS side—for the weak form of the conjecture[15]—the partition function is given in terms of $S^{\text{on-shell}}_{\text{SUGRA}}[\varphi]$, where φ are fields (with possible indices suppressed) on AdS$_5$ reduced on S^5. The AdS/CFT correspondence, then, translates into the following statement relating the generating functional $W[\varphi_{(0)}]$ of connected diagrams to the SUGRA action

$$W[\varphi_{(0)}] = S^{\text{on-shell}}_{\text{SUGRA}}[\varphi]\Big|_{\lim\limits_{z\to 0} z^{\Delta-d}\varphi(z,x)=\varphi_{(0)}(x)} . \tag{2.5.3}$$

However, the on-shell value of the SUGRA action is fraught with IR divergences due to the infinite volume of AdS, while the field theory action will contain UV divergences. The method of holographic

15 For the strongest form of the conjecture, the correspondence relates the partition function of the CFT to the partition function of the full (type IIB) string theory. Letting the generalized quantities (i.e. not necessarily scalars) $\mathcal{O}_\Delta$ and φ be dual to each other in the usual sense, the strongest form can be expressed as

$$\left\langle \exp\left(\int \mathrm{d}^d x\ \varphi_{(0)}(x)\mathcal{O}_\Delta\right)\right\rangle_{\text{CFT}} = Z_{\text{IIB String}}\Big|_{\lim\limits_{z\to 0} z^{\Delta-d}\varphi(z,x)=\varphi_{(0)}(x)} . \tag{2.5.2}$$

The string partition function is not known. The weak form of the correspondence, which we expound on presently, amounts to a saddle point approximation of the string partition function.

renormalization (see chapter 3) can then be applied to subtract divergences from the bulk action, which, by the AdS/CFT correspondence, simultaneously renormalizes the field theory. When this process has been carried out, we can compute n-point correlation functions as follows,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\ldots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\text{Connected}} = (-1)^n \left.\frac{\delta S^{\text{on-shell,ren}}_{\text{SUGRA}}[\varphi_{\Delta_i}]|_{\lim\limits_{z\to 0} z^{\Delta-d}\varphi_{\Delta_i}(z,x)=\varphi_{(0)\Delta_i}(x)}}{\delta\varphi_{(0)\Delta_1}(x_1)\cdots\delta\varphi_{(0)\Delta_n}(x_n)}\right|_{\varphi_{(0)\Delta_i}=0}. \tag{2.5.4}$$

Remarkably, holographic renormalization is not required for the two-point function for a massless scalar[16] [44]; so without having to deal with the hassle that is holographic renormalization just yet, we may apply the formalism above to compute the two-point function for a massless scalar in the CFT. Using Euclidean AdS$_{d+1}$ (Lobachevski space) with metric $ds^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)$, it is now convenient to introduce the bulk-to-boundary propagator, K_Δ, which relates the bulk field $\varphi_\Delta(z,x)$—subject to the boundary condition $\varphi(z,x)\sim\varphi_{(0),\Delta}(x)z^{d-\Delta}$ for $z\to 0$—to the boundary field $\varphi_{(0),\Delta}(x)$,

$$\varphi_\Delta(z,x) = \oint_{\partial\text{AdS}} \mathrm{d}^d y\; K_\Delta(z,x;y)\varphi_{(0),\Delta}(y). \tag{2.5.5}$$

In order to actually determine the bulk-to-boundary propagator, we first identify the bulk-to-bulk propagator $G_\Delta(z,x;w,y)$. Now, the Klein-Gordon equation for a massless scalar with a source reads $\Box_g\varphi_\Delta(z,x) = J(z,x)$, the solution to which can be written as

$$\varphi_\Delta(z,x) = \int_{\text{AdS}} \mathrm{d}w\mathrm{d}^d y\;\sqrt{g}G_\Delta(z,x;w,y)J(w,y), \tag{2.5.6}$$

implying that the bulk-to-bulk propagator satisfies,

$$\Box_g G_\Delta(z,x;w,y) = \frac{\delta(z-w)\delta^d(x-y)}{\sqrt{g}}. \tag{2.5.7}$$

The solution to this differential equation is given by a hypergeometric function in the chordal distance ξ of the geodesic connecting the points (z,x) and (w,y) [45]. From $G_\Delta(z,x;w,y)$, we can get the bulk-to-boundary propagator by taking the limit $w\to 0$ in an appropriate way[17]; the result is[18]

$$K_\Delta(z,x;y) = \underbrace{\frac{\Gamma(d)}{\pi^{d/2}\Gamma(d/2)}}_{=:C_d}\left(\frac{z}{z^2+(x-y)^2}\right)^d. \tag{2.5.9}$$

On the boundary, $z\to 0$, the Lorentzian form of the propagator implies that

$$\lim_{z\to 0} K_\Delta(z,x;y) = \overbrace{z^{\Delta-d}\delta^d(x-y)}^{\text{General case for massive scalar}} = \delta^d(x-y) \tag{2.5.10}$$

Equipped with the bulk-to-boundary propagator, and its boundary behavior, we're ready to calculate the massless two-point function. The on-shell (Euclidean) action reads

$$S_{\text{on-shell}} = -\frac{1}{2}\int_{\text{AdS}} \mathrm{d}z\mathrm{d}^d x\;\sqrt{g}g^{mn}\partial_m\varphi\partial_m\varphi \tag{2.5.11}$$

$$= \underbrace{\frac{1}{2}\int_{\text{AdS}} \mathrm{d}z\mathrm{d}^d x\;\sqrt{g}\varphi\Box_g\varphi}_{=0} + \frac{1}{2}\oint_{\partial\text{AdS}} \mathrm{d}^d x\;\sqrt{h}\varphi\hat{\partial}\varphi \tag{2.5.12}$$

$$= \frac{1}{2}\oint_{\partial\text{AdS}} \mathrm{d}^d x\;\sqrt{h}\varphi\hat{\partial}\varphi, \tag{2.5.13}$$

16 By table 2.2, masslessness implies that $\Delta = d$.

17 An illuminating way of deriving the precise relation between K_Δ and G_Δ involves Green's second identity,

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\left(\phi(\Box - m^2)\psi - (\phi\leftrightarrow\psi)\right) = \int_{\partial\mathcal{M}} \mathrm{d}^d x\;\sqrt{-\gamma}\left(\phi n\cdot\partial\psi - (\phi\leftrightarrow\psi)\right),$$

where n is the normal to $\partial\mathcal{M}$ and then setting $\phi = G_\Delta$ and $\psi = K_\Delta$; see e.g. [43] for details.

18 For a massive scalar with conformal dimension Δ, the result reads

$$K_\Delta(z,x;y) = \underbrace{\frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-d/2)}}_{=:C_\Delta}\left(\frac{z}{z^2+(x-y)^2}\right)^\Delta. \tag{2.5.8}$$

where $h_{\mu\nu}$ is the induced metric on the boundary, $z \to 0$, which gives $h_{\mu\nu} = \frac{L^2}{z^2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, and where $\hat{\partial} = n^m \partial_m$ is the normal derivative outward directed from the boundary, which thus clearly points in the z direction, $n^m = (n, 0, \dots, 0)$ while normality dictates that $1 = g_{mn}n^m n^n = n^2 \frac{L^2}{z^2}$, so $n = \frac{z}{L}$. Hence, since the leading behavior as $z \to 0$ of the field is $\varphi(z,x) \sim z^{d-\Delta}\varphi_{(0)}(x) = \varphi_{(0)}(x)$,

$$\sqrt{h}\varphi(z,x)\hat{\partial}\varphi(z,x) = \left(\frac{L}{z}\right)^{d-1} \varphi(z,x)\partial_z\varphi(z,x) = \left(\frac{L}{z}\right)^{d-1} \varphi(z,x)\partial_z \oint_{\partial\mathrm{AdS}} \mathrm{d}^d y\, K_\Delta(z,x;y)\varphi_{(0)}(y) \tag{2.5.14}$$

$$= \oint_{\partial\mathrm{AdS}} \mathrm{d}^d y\, C_d \left(\frac{L}{z}\right)^{d-1} \varphi(z,x) \frac{d\left((x-y)^2 - z^2\right) z^{d-1}}{\left((x-y)^2 + z^2\right)^{d+1}} \varphi_{(0)}(y) \tag{2.5.15}$$

$$\xrightarrow{z\to 0} C_d d L^{d-1} \oint_{\partial\mathrm{AdS}} \mathrm{d}^d y\, \frac{\varphi_{(0)}(x)\varphi_{(0)}(y)}{(x-y)^{2d}}, \tag{2.5.16}$$

which means that the on-shell action takes the form,

$$S_{\text{on-shell}} = \frac{C_d d L^{d-1}}{2} \oint_{\partial\mathrm{AdS}} \oint_{\partial\mathrm{AdS}} \mathrm{d}^d x \mathrm{d}^d y\, \frac{\varphi_{(0)}(x)\varphi_{(0)}(y)}{(x-y)^{2d}}. \tag{2.5.17}$$

With this, the two-point function for scalar conformal operators $\mathcal{O}_d$ of dimension $\Delta = d$ can be determined using the GKPW rule (2.5.4):

$$\langle \mathcal{O}_d(x_1)\mathcal{O}_d(x_2)\rangle = \left.\frac{\delta S_{\text{on-shell}}}{\delta\varphi_{(0)}(x_1)\delta\varphi_{(0)}(x_2)}\right|_{\varphi_{(0)}=0} = \frac{C_d d L^{d-1}}{(x_1 - x_2)^{2d}}, \tag{2.5.18}$$

in agreement with the conformal result (B.1.16). For a massive scalar dual to operators of weight Δ (the operators need to have the same conformal dimension, otherwise the two-point function vanishes identically, see appendix B), the result is [43]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle = L^{d-1} C_\Delta \frac{2\Delta - d}{(x_1 - x_2)^{2\Delta}}, \tag{2.5.19}$$

which reduces to our result for $\Delta = d$. Naïvely, the procedure for the massless scalar should work for the massive scalar case as well, but it does not give the correct prefactor: in that case, we have to regulate the bulk-to-boundary propagator using holographic renormalization, which will be the topic of chapter 3.

2.6 WITTEN DIAGRAMS

Introduced by Witten in [79], Witten diagrams are pictorial representations of the GKPW relation (2.5.1) and (2.5.4)—i.e. a way to obtain CFT correlation functions from diagrams in AdS. In the supergravity approximation, (2.5.3)—corresponding to the large-N limit of the CFT—only tree diagrams contribute, with loop diagrams inducing $1/N$ corrections to the boundary correlators. Pictorially, we may represent AdS as a sphere and draw the supergravity fields in a manner similar to the one employed when drawing Feynman diagrams (see e.g. [69, 70] for standard references), but with external lines connecting to the boundary. For concreteness, consider a scalar operator of dimension Δ, which is dual to the bulk field φ with interaction

$$S_{\text{int}} = \int \mathrm{d}^{d+1}x\, \sqrt{-g}\left(-\frac{\lambda}{3!}\varphi^3\right). \tag{2.6.1}$$

The Feynman rules for Witten diagrams are summarized in table 2.4 below.

Table 2.4: Feynman rules for Witten Diagrams.

Internal line	Bulk-to-bulk propagator $G_\Delta(z,x;w,y)$ (see (2.5.7))
Vertex	$\int \mathrm{d}z\mathrm{d}^d x \sqrt{-g}\lambda$, i.e. we integrate over *all* of AdS
External line (i.e. connected to b'dary)	A bulk-to-boundary propagator $K_\Delta(z,x;y)$ (see (2.5.8))

Since their introduction, many exciting developments involving Witten diagrams have taken place. For example, in [80] (see also [81]), it is proposed that the holographic dual of conformal blocks are so-called *geodesic Witten diagrams*, that is, Witten diagrams where vertices are integrated—not over all of AdS—but rather over geodesics connecting each pair of boundary points. Other interesting developments (see e.g. [82, 83]) include the use of the Mellin transform to deduce a more useful set of Feynman rules in "Mellin space"—leading to the concept of *Mellin amplitudes*—as well as an implementation of BCFW-like (see fig. C.1) recursive techniques applicable for Witten diagrams [84].

2.6.0.1 *Holographic Computation of the CFT Three-Point Correlator*

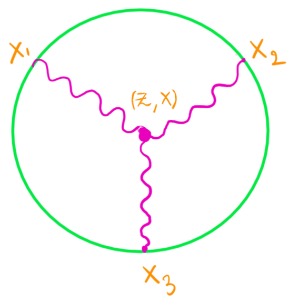


Figure 2.2: Witten diagram for $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle$ due to the interaction (2.6.1).

To showcase the power of the formalism, we now proceed to calculate the three-point function using Witten diagrams (see also [47, 85]). We already know the result: it is given by (B.1.17). Taking the interaction for simplicity to be (2.6.1), where φ is dual to an operator $\mathcal{O}_\Delta$ of dimension Δ, the leading order contribution to the correlation function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle$ is given by the single diagram of figure 2.2. By use of the Feynman rules of table 2.4, the value of this diagram is readily obtained[19]:

$$\langle \mathcal{O}(x_1)_\Delta \mathcal{O}_\Delta(x_2) \mathcal{O}_\Delta(x_3) \rangle = \int_{\text{AdS}} \mathrm{d}z\mathrm{d}^d x \ \sqrt{-g}\lambda \prod_{i=1}^{3} K_\Delta(z, x; x_i). \tag{2.6.2}$$

To tackle this integral, we—following [47]—use Feynman parametrization, allowing us to perform the integral over AdS to obtain[20]

$$\langle \cdots \rangle = C_\Delta^3 \lambda \frac{\pi^{d/2} \Gamma\left(3\Delta/2 - d/2\right) \Gamma\left(3\Delta/2\right)}{2\Gamma\left(\Delta\right)^3} \int_0^\infty \left(\prod_i \mathrm{d}\alpha_i \right) \delta \left(\sum_j \alpha_j - 1 \right) \frac{\prod_k \alpha_k^{\Delta - 1}}{\left[\sum_{\ell < q} \alpha_\ell \alpha_q x_{\ell q}^2 \right]}, \tag{2.6.3}$$

where we have defined $x_{ij} = x_i - x_j$. Changing integration variables to β_i defined via $\alpha_1 = \beta_1$ and $\alpha_i = \beta_1 \beta_i$ for $i \in \{2, 3\}$, the integral over the β_i takes the form

$$\langle \cdots \rangle = C_\Delta^3 \lambda \frac{\pi^{d/2} \Gamma\left(3\Delta/2 - d/2\right) \Gamma\left(3\Delta/2\right)}{2\Gamma\left(\Delta\right)^3} \int_0^\infty \left(\prod_{i>1} \mathrm{d}\beta_i \right) \frac{\prod_{k>1} \beta_k^{\Delta-1}}{\left[\sum_{\ell>1} \beta_\ell \left(x_{1\ell}^2 + \sum_{q>\ell} \beta_q x_{\ell q}^2 \right) \right]} \tag{2.6.4}$$

$$= -\frac{\lambda \Gamma\left((3\Delta - d)/2\right)}{2\pi^d} \left(\frac{\Gamma(\Delta/2)}{\Gamma\left(\Delta - d/2\right)} \right)^3 \frac{1}{(x_{12} x_{13} x_{23})^\Delta}, \tag{2.6.5}$$

in agreement with the result obtained from conformal field theory, cf. (B.1.17).

2.7 OUTLOOK

The AdS/CFT correspondence remains a veritable cornucopia for new, interesting physics and is an extremely active area of research. Although still not fully understood, significant progress has been achieved in the years following Maldacena's foundational work, and many formal aspects are now quite well understood.

19 Note in particular that there is no contribution from the kinetic term, see e.g. [72] for details.

20 The standard method to compute the three-point function exploits inversion symmetry to constrain the spatial dependence, see e.g. [43, 45].

It has been successfully applied to explain the low viscosity of the quark gluon plasma (see [9]), and a whole program exploring the application of holography in strongly coupled QCD—known as AdS/QCD—has since flourished.

It has also been applied to strongly coupled field theories that arise in condensed matter theories (this endeavour is known as AdS/CMT, see [11] for a recent review), notably holographic superconductors [5, 6] (these are the subject of appendix E).

The AdS/CFT correspondence has been likened to the "hydrogen atom of holography": as a specific realization of the holographic principle, it may well serve as a stepping stone to a much grander array of holographic correspondences, and, in particular, as the most promising route towards an understanding of quantum gravity. Already many other holographic dualities have been proposed, and many of these are independent of string theory. The main topic of this thesis is one such realization of the holographic principle: Lifshitz holography.

3 HOLOGRAPHIC RENORMALIZATION

This chapter—along with the supplementary appendix F—explores many facets of holographic renormalization. This procedure is required to remove the divergences due to the infinite volume of AdS space—corresponding to UV divergences in the dual field theory (recall our interpretation of the radial direction as an energy scale in section 2.3.1).

There are many approaches to this endeavour, and in section F.1 of the the supplementary appendix F, we review the original approach: the *Fefferman-Graham* (FG) approach, developed by de Haro et al. in [86], which relies on the FG theorem of differential geometry [87]. This theorem roughly states that the metric for certain classes of geometries has a certain universal expansion in an appropriate radial coordinate. We introduce the method and provide a detailed exposition of the renormalization of pure gravity in five-dimensional asymptotically locally AdS in section F.1.1. Since the FG approach is computationally involved, we will occasionally make use of the symbolic differential geometry package *xAct* for Mathematica [88].

In section F.2 in the same appendix, we review the de Boer-Verline-Verlinde (dBVV) ansatz method, where one writes down an ansatz for the counterterm. This approach involves a bilinear and symmetric operation that we have named the deWitt bracket, which allows for a recursive determination of the unknown coefficients in the counterterm ansatz. This analysis is based on [49, 89].

We start this chapter with a brief review (following [90–92]) of Hamilton-Jacobi (HJ) theory in section 3.1.1, serving as an appetizer to the HJ approach to holographic renormalization presented in section 3.1.2, which is based on [50, 51, 93]. This method requires us to determine the Hamiltonian of the model we consider, and to this end we employ a radial ADM formalism and illustrate the principles using a generic model consisting of a scalar ϕ with some potential $V(\phi)$ coupled to gravity. The HJ equation implies a set of constraints, which can be solved recursively which allows the identification of the divergent parts of the action. In order to solve the constraints order by order, we need a good way to define what we mean by "order": to that end, we introduce an operator $\mathfrak{S}$—which is conveniently (for our purposes) chosen to be the dilatation operator δ_D—that we use to sort various quantities such as the action and the Hamiltonian in eigenmodes.

In section 3.2, we present the main new result of this chapter: using the HJ approach, we calculate a novel counterterm for a five-dimensional free Einstein-Maxwell-Dilaton (EMD) model—which we call the electromagnetic uplift due to its rôle as an upliftable model for charged Lifshitz holography in chapter 6. We also work out the novel Ward identities satisfied by the VEVs and construct the new FG expansions, which is achieved via the so-called flow equations; a natural ingredient in the HJ approach.

We also comment on a new subtlety regarding the renormalization of p-form fields. This has gone unnoticed in the literature, and will be the topic of an upcoming article by Skenderis and Papadimitriou [52]. Based on insights from our renormalization of the electromagnetic uplift, which contains a one-form, we provide a conjecture for the general p-form case: in particular we present four new regions of parameter space, where holographic renormalization works differently depending on the values of d and p.

Finally, in section 3.3, we illustrate a manifestation of this subtlety by considering hodge dualization of a biscalar theory to an EMD model in $d = 2$ and generalize the approach to a simple way of determining the counterterm action for a massless p-form in AdS$_{p+2}$. This has, to our knowledge, not appeared in the literature previously.

3.1 HOLOGRAPHIC RENORMALIZATION AS HAMILTON-JACOBI THEORY

In this section, we describe a method originally devised by Papadimitriou and Skenderis in [94, 95] and further refined in [50, 51]. This method was used by Ross in [27] (see also [96]) to renormalize a particular Einstein-Proca model which supports Lifshitz solutions; this we review in appendix G. For expositional reasons, we consider free scalars coupled to gravity, while our final application of the procedure concerns the electromagnetic uplift, the renormalization of which is new.

3.1.1 *Hamilton-Jacobi Theory in a Nutshell*

We begin with a review of Hamilton-Jacobi theory as it appears in classical mechanics. Our presentation is based on [90–92].

Consider a mechanical system described by the action functional,

$$S = \int_\gamma \mathrm{d}t' \; L(q, \dot{q}; t'), \tag{3.1.1}$$

where $\dot{q}^i = \partial_t q^i$ and γ is a curve connecting the initial and final positions, (q_0, t_0) and (q_1, t_1), in the configuration space V, which is an n-dimensional manifold. The coordinates on V are the generalized coordinates q^i, and to each generalized coordinate q_i corresponds a canonical momentum, given by $p_i = \frac{\mathrm{d}L}{\mathrm{d}\dot{q}^i}$. The p_i and the q^i are independent variables and are the natural coordinates of the Hamiltonian formalism, where we trade the Lagrangian for the Hamiltonian via the Legendre transformation,

$$H(q, p; t) = p_i \dot{q}^i(q, p) - L(p, q; t), \tag{3.1.2}$$

which dictates the time evolution through Hamilton's equations,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{3.1.3}$$

Note that these imply an additional "Hamilton equation" for explicitly time dependent systems: since $\dot{H} = \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$, we infer that $-H$ is the canonical momentum conjugate to t.

When $H(p, q)$ does not explicitly depend on time, the set $\{p_i, q^i\}$ combine to form the $2n$-dimensional *phase space* of our system, which is isomorphic to the cotangent bundle of the configuration space, T^*V.

If the Hamiltonian *does* depend explicitly on time, we must include time as a generalized coordinate, and our extended configuration space $\overline{V} = V \times \mathbb{R}$ becomes $(n+1)$-dimensional, with a corresponding extended phase space $T^*\overline{V}$. Note that the Hamiltonian is a mapping giving some real number for each point (p_i, q^i) in phase space, i.e. $H : T^*\overline{V} \to \mathbb{R}$. The (extended) cotangent bundle is naturally endowed with a symplectic structure (see [97] for a review on symplectic geometry), i.e. a closed two-form ω^2 given by

$$\omega^2 = \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H \wedge \mathrm{d}t, \tag{3.1.4}$$

which, due to closure, is locally expressible as

$$\omega^2 = \mathrm{d}\lambda, \tag{3.1.5}$$

where $\lambda = p_i \mathrm{d}q^i - H\mathrm{d}t$ is the presymplectic one-form[1]. We remark that the presymplectic form is equal to the exterior derivative of the action (3.1.1) considered as a function of final position (i.e, with the initial position fixed) [92]. Consider now a diffeomorphism on $T^*\overline{V}$ defining a set of new coordinates $\{P_i(q^i, p_i; t), Q_I(q^i, p_i; t)\}$ (leaving t untouched). By demanding that the Lagrangian changes only up to a total derivative[2],

$$p_i \dot{q}^i - H(p, q; t) = P_i \dot{Q}^i - \widetilde{H}(P, Q; t) + \frac{\mathrm{d}}{\mathrm{d}t} F(q, Q; t), \tag{3.1.6}$$

we can ensure that the principle of stationary action again produces Hamilton's equations (3.1.3) in the new coordinates,

$$\dot{Q}^i = \frac{\partial \widetilde{H}}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \widetilde{H}}{\partial Q^i}. \tag{3.1.7}$$

A diffeomorphism satisfying the above is called *canonical*. Now, observe that $\frac{\mathrm{d}}{\mathrm{d}t}F(q, Q; t) = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial Q^i}\dot{Q}^i + \frac{\partial F}{\partial t}$, which we can plug into the requirement (3.1.6) to obtain

$$\left(p_i - \frac{\partial F}{\partial q^i}\right)\dot{q}^i - H(p, q; t) = \left(P_i + \frac{\partial F}{\partial Q^i}\right)\dot{Q}^i - \widetilde{H}(P, Q; t) + \frac{\partial F}{\partial t}. \tag{3.1.8}$$

1 The presymplectic form is also known as the Liouville form, and, occasionally, as the tautological form [97].

2 Note that we have a lot of freedom in choosing the variables occurring in F. In [91], the functions are numbered according to their dependence; for example our F here is called F_1.

Now, the old and the new coordinates are independent, implying that the identity above is true identically only if the coefficients of $\dot{q}^i$ and $\dot{Q}^i$ each vanish, leaving us with

$$p_i = \frac{\partial}{\partial q^i} F(q, Q; t), \qquad P_i = \frac{\partial}{\partial Q^i} F(q, Q; t), \qquad \widetilde{H}(P, Q; t) = H(p, q; t) + \frac{\partial}{\partial t} F(q, Q; t). \tag{3.1.9}$$

In particular, we see that any such F generates some canonical function—consequently it is referred to as the generating function. Now, write

$$F(q, Q; t) = -P_i Q^i + \mathcal{S}. \tag{3.1.10}$$

By taking the exterior derivative of F, which is a section of $T^*\overline{V}$, it is easy to show that $S = S(q, P; t)$—that is, the rewriting (3.1.10) is nothing but a Legendre transformation from the set of coordinates $(q, Q; t)$ to $(q, P; t)$— and that

$$p_i = \frac{\partial \mathcal{S}}{\partial q^i}, \qquad Q^i = \frac{\partial \mathcal{S}}{\partial P_i}, \qquad \frac{\partial \mathcal{S}}{\partial t} = \frac{\partial F}{\partial t}. \tag{3.1.11}$$

As before, *any* $\mathcal{S}$ generates some canonical transformation, and we choose S such that $\widetilde{H}$ is identically zero, which means that $\dot{Q}^i = 0 = \dot{P}_i$, i.e. they are constants of motion, and we write $P_i = \alpha_i$ and $Q^i = \beta^i$. The condition $\widetilde{H} = 0$ is can now be recast in the form

$$H\left(\frac{\partial \mathcal{S}}{\partial q}, q; t\right) + \frac{\partial \mathcal{S}}{\partial t} = 0, \tag{3.1.12}$$

where we have used the third relation of (3.1.9) as well as the expression for p_i in (3.1.11). This is the Hamilton-Jacobi equation. It is a first-order partial differential equation for $\mathcal{S}$, known as *Hamilton's principal function*. Now, note that we can write $\mathcal{S} = \mathcal{S}(q, \alpha)$ as well inverting F to obtain $q^i = q^i(\alpha, \beta; t)$, which is solution to the mechanical problem in the original coordinates and thus leads to a trajectory γ along which the action is stationary. Along γ, where $P_i = \alpha_i$, we can compute the time derivative of $\mathcal{S}$:

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = \frac{\partial \mathcal{S}}{\partial q^i}\dot{q}^i + \frac{\partial \mathcal{S}}{\partial t} = p_i \dot{q}^i - H = L, \tag{3.1.13}$$

where we have used the relation for p_i in (3.1.11) and the Hamilton-Jacobi equation (3.1.12), implying that

$$\mathcal{S}(t) = \mathcal{S}(q(t), \alpha, t) = \int_{\gamma(t_0, t)} \mathrm{d}t'\, L(t') + \mathcal{S}(t_0), \tag{3.1.14}$$

where $\gamma(t_0, t)$ is the part of the trajectory γ starting at t_0 and ending at t. Thus, we conclude that *Hamilton's principal function is the on-shell action*. This insight is the crucial ingredient in the Hamilton-Jacobi approach to holographic renormalization.

3.1.2 *Hamiltonian Gravity and the Hamilton-Jacobi Equation*

In this section, we set up a HJ description of holographic renormalization, where r plays the rôle of time. Consider gravity coupled to (possibly massive) free scalars in a $(d+1)$-dimensional non-compact manifold $\mathcal{M}$, which is described by the action[3]

$$S = -\int_{\mathcal{M}} \mathrm{d}^{d+1}x\, \sqrt{g}\left(R^{(g)} - \frac{1}{2}\partial_\mu \partial^\mu \phi + V(\phi)\right) + \int_{\partial\mathcal{M}} \mathrm{d}^d x\, \sqrt{\gamma}\, 2K. \tag{3.1.15}$$

The next ingredient is the identification of a radial coordinate r such that $r \to \infty$ corresponds to the boundary $\partial\mathcal{M}$ of $\mathcal{M}$, which we require only to cover an open chart $\mathcal{M}_\varepsilon$ in the vicinity of $\partial\mathcal{M}$. There are additional subtleties involved if the boundary $\partial\mathcal{M}$ consists of several disconnected pieces; in that case, a different radial coordinate will have to be used for each component [50]. When such complications have been dealt with, we apply a radial ADM formalism (for a review of the ADM

3 Note that we'll work in Euclidean signature, but it is straightforward to generalize this to Lorentzian signature [48]; in particular, we can interchange the two with no complications, so we will not be particularly careful about this.

formalism, see [98]), where the metric is parametrized in terms of the lapse N, the shift N_i and the induced metric γ_{ij} on radial leaves Σ_r (i.e. hypersurfaces of constant r),

$$ds^2 = (N^2 + N_i N^i)\mathrm{d}r^2 + 2N_i \mathrm{d}r\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.1.16}$$

implying that the metric $g_{\mu\nu}$ is equivalently described by the triplet $\{N, N_i, \gamma_{ij}\}$. The curvatures pertaining to $g_{\mu\nu}$ are then expressible in terms of the curvatures on Σ_r and the extrinsic curvature K_{ij}, which is given by

$$K_{ij} = \frac{1}{2}\pounds_n g_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right), \tag{3.1.17}$$

where $\dot{\gamma}_{ij} = \partial_r \gamma_{ij}$, $\pounds_n$ is the Lie derivative in the direction of the normal vector n, and D_i is the covariant derivative with respect to the induced metric γ_{ij}. The unit normal vector n^μ to Σ_r is given by

$$n^\mu = \left(1/N, -N^i/N\right). \tag{3.1.18}$$

Following [50], we now consider some useful identities in the radial ADM formalism that we are employing. Note first that

$$g = \begin{pmatrix} N^2 + N_i N^i & N_i \\ N_i & \gamma_{ij} \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 1/N^2 & -N^i/N^2 \\ -N^i/N^2 & \gamma^{ij} + N^i N^j/N^2 \end{pmatrix}, \tag{3.1.19}$$

which means that

$$\sqrt{g} = N\sqrt{\gamma}, \tag{3.1.20}$$

which follows from the useful identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det\left(A - BD^{-1}C\right). \tag{3.1.21}$$

From the metric (3.1.19), we get the following Christoffel symbols,

$$\Gamma^r_{rr} = N^{-1}\left(\dot{N} + N^i\partial_i N - N^i N^j K_{ij}\right), \tag{3.1.22}$$
$$\Gamma^r_{ri} = N^{-1}\left(\partial_i N - N^j K_{ij}\right), \tag{3.1.23}$$
$$\Gamma^r_{ij} = N^{-1}K_{ij}, \tag{3.1.24}$$
$$\Gamma^i_{rr} = -N^{-1}N^i\dot{N} - ND^iN - N^{-1}N^iN^j\partial_j N + \dot{N}^i + N^j D_j N^i + 2NN^jK^i_j + N^{-1}N^iN^kN^lK_{kl}, \tag{3.1.25}$$
$$\Gamma^i_{rj} = -N^{-1}N^i\partial_j N + D_j N^i + N^{-1}N^iN^kK_{kj} + NK^i_j, \tag{3.1.26}$$
$$\Gamma^k_{ij} = \Gamma^{(\gamma)k}_{ij} + N^{-1}N^kK_{ij}. \tag{3.1.27}$$

In terms of the ADM variables, the Ricci scalar of g is decomposed in the following manner,

$$R^{(g)} = R^{(\gamma)} + K^2 - K_{ij}K^{ij} + \nabla_\mu \zeta^\mu, \tag{3.1.28}$$

where $\zeta^\mu = -2Kn^\mu + 2n^\rho\nabla_\rho n^\mu$, which implies that

$$\zeta^r = -2Kn^r + 2n^\rho\nabla_\rho n^r = -2K/N + 2n^\rho\left(\partial_\rho n^r + \Gamma^r_{\rho\lambda}n^\lambda\right) \tag{3.1.29}$$
$$= -2K/N + \overbrace{2n^r\left(\partial_\rho n^r + \Gamma^r_{rr}n^r\right) + 2n^i\left(\partial_i n^r + \Gamma^r_{ir}n^r + \Gamma^r_{ij}n^j\right)}^{=0} \tag{3.1.30}$$
$$= -2K/N, \tag{3.1.31}$$

which means that the Gibbons-Hawking term precisely cancels the total derivative of the Ricci scalar (3.1.28); see also [99]. The total action is then expressible as

$$S = \int \mathrm{d}r\, L, \tag{3.1.32}$$

where

$$L = \int_{\Sigma_r} \mathrm{d}^d x \sqrt{\gamma} N \left(R^{(\gamma)} + K^2 - K_{ij}K^{ij} - \frac{1}{2N^2}\left(\dot{\phi} - N^i \partial_i \phi\right)^2 - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - V(\phi) \right), \quad (3.1.33)$$

where we have used the metric (3.1.19) to write

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2. \quad (3.1.34)$$

From the Lagrangian (3.1.33), we can read off—using the extrinsic curvature (3.1.17)—the conjugate momenta to the induced metric and the scalar,

$$\pi^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta L}{\delta \dot{\gamma}_{ij}} = K^{ij} - K\gamma^{ij}, \quad (3.1.35)$$

$$\pi_\phi = \frac{1}{\sqrt{\gamma}}\frac{\delta L}{\dot{\phi}} = N^{-1}\left(\dot{\phi} - N^i\partial_i\phi\right), \quad (3.1.36)$$

so we can apply the Legendre transform to our Lagrangian (3.1.33) to obtain the Hamiltonian on the radial hypersurface Σ_r,

$$H = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{\gamma}\left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\phi\dot{\phi}\right) - L = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{\gamma}\left(N\mathcal{H} + N_i\mathcal{H}^i\right), \quad (3.1.37)$$

where

$$\mathcal{H} = \pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{2}\pi_\phi^2 + R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi - V(\phi), \quad (3.1.38)$$

$$\mathcal{H}^i = -2D_j\pi^{ij} + \pi_\phi\partial^i\phi, \quad (3.1.39)$$

where we have used the extrinsic curvature (3.1.17) and the momenta (3.1.35)–(3.1.36) to get $\mathcal{H}$, whereas we have integrated (covariantly) by parts to get $\mathcal{H}^i$. From the Hamiltonian (3.1.37), the equations of motion for the Lagrange multiplier fields N and N^i immediately give the constraints

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0, \quad (3.1.40)$$

known as the Hamiltonian and momentum constraint, respectively. The constraints (3.1.40) imply that the Hamiltonian (3.1.37) vanishes on the constraint surface; this is a consequence of diffeomorphism invariance and is, as we shall see, closely related to the Hamilton-Jacobi equation. The constraints of (3.1.40) are so-called *first-class constraints*[4] [100], which generate diffeomorphisms along the radial direction and along Σ_r, respectively, via the Poisson bracket. Locally, we can use diffeomorphism invariance to gauge fix the shift and lapse in the following manner [89],

$$N = 1, \quad N^i = 0. \quad (3.1.41)$$

Henceforth, we will adopt the gauge choice of (3.1.41). Now, from the Hamiltonian densities (3.1.38) and (3.1.39), it is clear the Hamiltonian (3.1.37) has no explicit radial dependence, so the radial Hamilton-Jacobi equation assumes the form

$$H = 0, \quad (3.1.42)$$

which is the same as the constraints (3.1.40); crucially, however, the canonical momenta are here determined in terms of Hamilton's principal function $\mathcal{S}$ (compare with the expressions in (3.1.11)),

$$\pi^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \qquad \pi_\phi = \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}. \quad (3.1.43)$$

The prescription (3.1.43) then implies that the constraints (3.1.40) become functional differential equations for $\mathcal{S}$, and the Hamiltonian constraint thus reads

$$\left[\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right]\frac{1}{\gamma}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}} + \frac{1}{2\gamma}\left(\frac{\delta\mathcal{S}}{\delta\phi}\right)^2 + R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi - V(\phi) = 0, \quad (3.1.44)$$

4 In the sense that their Poisson brackets with the constraints (3.1.40) vanishes *weakly*, for details see [100].

which is the starting point of the dBVV-DeWitt[5] approach to holographic renormalization, which we review in appendix F. As we saw in section (3.1.1), a solution $\mathcal{S}$ to the HJ equation provides a solution to Hamilton's equations—in fact, it is be to identified with the on-shell action—and thus it also solves the second order equations of motion (i.e. the Euler-Lagrange equations); in particular, Hamilton's first equation $\dot{\mathbf{q}} = \frac{\delta H}{\delta \mathbf{p}}$ leads to the first order *flow equations*

$$\dot{\gamma}_{ij} = 2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}, \tag{3.1.45}$$

$$\dot{\phi} = \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}. \tag{3.1.46}$$

Equivalently, the equations above may be obtained by inserting the momenta (3.1.43) into their representations (3.1.35) and (3.1.36) with the gauge choice (3.1.41); this is straightforward for $\dot{\phi}$, but $\dot{\gamma}_{ij}$ is related to the extrinsic curvature as detailed in (3.1.17), which implies that $\frac{1}{2}\dot{\gamma}_{ij} = K_{ij}$, and thus the relation for π^{ij} (3.1.35) leads to $\dot{\gamma}_{ij} = 2\pi_{ij} - \frac{2}{1-d}\pi$, which is the same as (3.1.45).

In asymptotically locally AdS (AlAdS) geometries (see appendix A) in particular[6], these flow equations allow one to reconstruct the Fefferman–Graham expansions of the fields under scrutiny, which we pursue in section 3.2.3.

The application of Hamilton-Jacobi theory in holographic renormalization hinges on the fact that Hamilton's principal function on Σ_r is equivalent to the on-shell action S on Σ_r, as we showed in section 3.1.1. More explicitly, the regularized action, i.e. the on-shell action evaluated on Σ_r, which is given by

$$S_{\text{reg}}[\gamma(r,x),\phi(r,x)] = \int^r \mathrm{d}r'\, L_{\text{os}}, \tag{3.1.47}$$

satisfies the HJ equation and may thus be identified with $\mathcal{S}$. So, if we can determine the principal function, we can identify the divergences of the on-shell action and thus construct the appropriate counterterm; denoting the divergent part of the principal function $\mathcal{S}_{\text{loc}}$, which is a local functional of the data on Σ_r, the counterterm takes the form

$$S_{\text{ct}} = -\mathcal{S}_{\text{loc}}. \tag{3.1.48}$$

It is worth noting that it is possible to add finite local terms to the bulk action[7], corresponding to choosing a specific renormalization scheme, i.e.

$$S_{\text{ct}} = -\left(\mathcal{S}_{\text{loc}} + \mathcal{S}_{\text{scheme}}\right). \tag{3.1.49}$$

In the present work, we exclusively use minimal subtraction, i.e. we subtract only the divergences of the bare action. Once the counterterm has been obtained, the renormalized action on-shell on Σ_r is given by

$$S_{\text{ren}} = S_{\text{reg}} + S_{\text{ct}} = \int \mathrm{d}^d x\, \sqrt{\gamma}\left(\gamma_{ij}\Pi^{ij} + \phi\Pi_\phi\right), \tag{3.1.50}$$

where the renormalized canonical momenta Π^{ij} and Π_ϕ are closely related to the renormalized one-point functions of the operators dual to the fields.

3.1.3 *Recursive Solution of the HJ Equation*

In order to provide a general algorithm for determining the principal function $\mathcal{S}$, we write $\mathcal{S}$ as an expansion of some functional operator δ, i.e.

$$\mathcal{S} = \mathcal{S}_{(\alpha_0)} + \mathcal{S}_{(\alpha_1)} + \mathcal{S}_{(\alpha_2)} + \cdots, \tag{3.1.51}$$

5 The quantity in square brackets in (3.1.44) bears a striking resemblance to the standard Wheeler-DeWitt metric [101]; in fact, it is the deWitt metric with parameter $d-1$ (see [102]).

6 It is also possible for more general backgrounds to generate analogues of the FG expansions using this method, but then one has to solve the differential equations. For AlAdS boundary conditions, the FG theorem provides a highly useful ansatz.

7 This corresponds to a special choice of integration constants in the complete integral of the HJ equation.

with

$$\delta\mathcal{S}_{(\alpha_k)} = \lambda_k \mathcal{S}_{(\alpha_k)}, \tag{3.1.52}$$

that is, every term in the expansion (3.1.51) is an eigenfunction of δ with eigenvalue λ_k, where all the λ_k's are distinct. It is now convenient to introduce a density $\mathscr{L}$ such that the principal function reads

$$\mathcal{S} = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{\gamma}\mathscr{L}, \tag{3.1.53}$$

where we *require that* $\sqrt{\gamma}$ be an eigenfunction of our operator δ with eigenvalue λ_γ; this is a constraint on the possible form of δ. This implies that we have an expansion of the form

$$\mathscr{L} = \mathscr{L}_{(\alpha_0)} + \mathscr{L}_{(\alpha_1)} + \mathscr{L}_{(\alpha_2)} + \cdots, \tag{3.1.54}$$

where

$$\mathcal{S}_{(\alpha_k)} = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{\gamma}\mathscr{L}_{(\alpha_k)}, \tag{3.1.55}$$

implying that the $\sqrt{\gamma}\mathscr{L}_{(\alpha_k)}$ are only defined up to a total divergence, which is not generically an eigenfunction of δ. To remedy this, we note first that for arbitrary variations of the on-shell action, the momentum relations (3.1.43) imply that

$$\pi^{ij}\delta\gamma_{ij} + \pi_\phi \delta\phi = \frac{1}{\sqrt{\gamma}}\delta\left(\sqrt{\gamma}\mathscr{L}\right) + \frac{1}{\sqrt{\gamma}}\partial_i v^i(\delta\gamma, \delta\phi), \tag{3.1.56}$$

for some vector field $v^i(\delta\gamma, \delta\phi)$. Applying this for δ, we find that

$$\begin{aligned}\pi^{ij}_{(\alpha_k)}\delta\gamma_{ij} + \pi_{\phi(\alpha_k)}\delta\phi &= \frac{1}{\sqrt{\gamma}}\delta\left(\sqrt{\gamma}\mathscr{L}_{(\alpha_k)}\right) + \frac{1}{\sqrt{\gamma}}\partial_i \tilde{v}^i(\delta\gamma, \delta\phi) &(3.1.57)\\ &= \left(\lambda_\gamma + \lambda_k\right)\mathscr{L}_{(\alpha_k)} + \frac{1}{\sqrt{\gamma}}\partial_i \tilde{v}^i(\delta\gamma, \delta\phi), &(3.1.58)\end{aligned}$$

where we have used the Leibniz property of δ, and where $\tilde{v}^i_{(\alpha_k)}$ is in general different from $v^i_{(\alpha_k)}$, since δ acting on $\sqrt{\gamma}\mathscr{L}$ may involve a total derivative. However, since we are ultimately interested in the *action*, these total derivatives are mere "artefacts", and we may choose—without loss of generality—the total derivative such that the following extremely useful relation holds:

$$\pi^{ij}_{(\alpha_k)}\delta\gamma_{ij} + \pi_{\phi(\alpha_k)}\delta\phi = (\lambda_\gamma - \lambda_k)\mathscr{L}_{(\alpha_k)}, \tag{3.1.59}$$

which is crucial since it allows us to relate the Hamiltonian constraint (3.1.44) directly to the on-shell action via the $\mathscr{L}_{(\alpha_k)}$. We now proceed to discuss two concrete realizations of the operator δ.

3.1.4 *The Induced Metric Expansion*

Used in [51] to renormalize general dilaton-axion gravity, this approach relies on choosing δ to be

$$\delta_\gamma = \int_{\Sigma_r} \mathrm{d}^d x \, 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}, \tag{3.1.60}$$

which satisfies both the Leibniz property and has $\sqrt{\gamma}$ as an eigenfunction, since

$$\delta_\gamma\sqrt{\gamma} = \gamma_{ij}\gamma^{ij}\sqrt{\gamma} = d\sqrt{\gamma}, \tag{3.1.61}$$

i.e. $\lambda_\gamma = d$. The expansion in eigenmodes of δ_γ treats the scalar field non-perturbatively, so we do not need to specify the potential explicitly, which makes this method quite general—in particular, this method applies also to asymptotically non-AdS backgrounds, e.g. non–conformal branes [103]. For the gravity-scalar system we consider, the resulting expansion is a derivative expansion[8], and each factor of the metric is associated with two derivatives, i.e. in terms of the form $\partial_i\phi\partial^i\phi$, and therefore

8 This is no longer the case if we include a Maxwell field A_i, since then we have terms of the schematic form $\gamma^{ij}A_iA_j$.—we will explore this scenario in section 3.2.1.

we choose the label $\alpha_k = 2k$ to count derivatives. The statement corresponding to (3.1.58) with $\mathfrak{d}$ given by (3.1.60) thus takes the form

$$2\pi^i_{i(2k)} = (d-2k)\mathcal{L}_{(2k)}. \tag{3.1.62}$$

Now, the zeroth order contribution contains no derivatives and is therefore solely expressed in terms of ϕ. We define the *superpotential* $U(\phi) := \mathcal{L}_{(0)}$, so that

$$\mathcal{S}_{(0)} = \int_{\Sigma_r} \mathrm{d}^d x\, \sqrt{\gamma} U(\phi). \tag{3.1.63}$$

Plugging the above into the functional Hamiltonian constraint of (3.1.44) and collecting terms with no derivatives, we get

$$2(U'(\phi))^2 - \frac{d}{(d-1)}U(\phi)^2 - 4V(\phi) = 0, \tag{3.1.64}$$

where we have used that $\frac{\delta\mathcal{S}_{(0)}}{\delta\gamma_{ij}} = \frac{1}{2}\sqrt{\gamma}\gamma^{ij}U(\phi)$. It is important to note that we need only solve this equation near the boundary, which simplifies the analysis substantially. Given $U(\phi)$, we can insert the δ_γ–eigenmode expansion of the principal function into (3.1.44) to obtain the following first order linear inhomogeneous recursive functional differential equations

$$2U'(\phi)\frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\phi}\int_{\Sigma_r} \mathrm{d}^d x\, \sqrt{\gamma}\mathcal{L}_{(2n)} - \frac{d-2n}{d-1}U(\phi)\mathcal{L}_{(2n)} = \mathcal{R}_{(2n)}, \qquad n > 0, \tag{3.1.65}$$

where

$$\mathcal{R}_{(2)} = \frac{1}{2}\partial_i\phi\partial^i\phi - R^{(\gamma)}, \tag{3.1.66}$$

$$\mathcal{R}_{(2n)} = -\sum_{m=1}^{n-1}\left(\pi^{ij}_{(2m)}\pi_{(2(n-m))ij} - \frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))} + \frac{1}{2}\pi_{\phi(2m)}\pi_{\phi(2(n-m))}\right). \tag{3.1.67}$$

Solving these equations is, however, tricky, and it is often advantageous to choose $\mathfrak{d}$ to be the dilatation operator (which, as we shall see, occasionally coincides with δ_γ), which, in this context, is written as δ_D, since in many cases it reduces the functional differential equations to algebraic equations which are much easier to solve. We explore the consequences of this choice in the section below.

3.2 THE δ_D EXPANSION & HOLOGRAPHIC RENORMALIZATION OF THE ELECTROMAGNETIC UPLIFT

In this section, we perform a novel renormalization of an EMD model where the dilaton does not couple to the electromagnetic field nor the cosmological constant: we refer to this model as the electromagnetic uplift. We need the renormalized action of this model when considering charged Lifshitz holography via Sherk-Schwarz reduction, which will be the topic of chapter 6. As remarked above, choosing $\mathfrak{d}$ to be the dilatation operator δ_D is often a good choice, and one we will make in this section. The addition of a free Maxwell field leads to an extra constraint encoding gauge invariance of the action in addition to those encountered in (3.1.40). This constraint is not trivially satisfied even after fixing the gauge to an axial gauge $A_r = 0$ (which supplements the ADM gauge (3.1.41)), since this choice leaves residual gauge transformations, under which we must also require invariance.

Some aspects of holographic renormalization for EMD models has been explored in previous literature: it was considered in a restricted setting in [104], while the general principles were laid down in [93].

Holography including renormalization for a different class of EMD models from the perspective of dimensional reduction was considered [105]; this approach somewhat reminiscent of the structure we will discuss in chapter 6.

3.2.1 *Holographic Renormalization of the Electromagnetic Uplift*

The model we consider is

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\left(R^{(g)} + d(d-1) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + S_{\mathrm{GH}}, \tag{3.2.1}$$

where S_{GH} is the Gibbons-Hawking boundary term, and where the field strength is defined in the usual manner $F_{ij} = 2\partial_{(i}A_{j)}$. Since we will eventually set $d = 4$, we will assume that d is even. The equations of motion for this action read

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) + \frac{d(d-1)}{2} + \frac{1}{2}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}\right), \tag{3.2.2}$$

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right), \qquad \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \tag{3.2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor. Performing an ADM-decomposition of the metric,

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = N^2\mathrm{d}r^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}r\right)\left(\mathrm{d}x^j + N^j\mathrm{d}r\right), \tag{3.2.4}$$

and using the results of section 3.1.2, the total action may be written as

$$S = \int \mathrm{d}r\ L, \tag{3.2.5}$$

where [28]

$$L = \frac{1}{2\kappa^2}\int_{\Sigma_r}\mathrm{d}^dx\sqrt{-\gamma}N\Bigg(K^2 - K^{ij}K_{ij} - \frac{1}{2N^2}\left(\dot\phi - N^i\partial_i\phi\right)^2 - \frac{1}{2N^2}\left(F_{ri} - N^kF_{ki}\right)\left(F_r{}^i - N^lF_l{}^i\right) \tag{3.2.6}$$

$$+ R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{4}F_{ij}F^{ij} + d(d-1)\Bigg), \tag{3.2.7}$$

where we note that A_r is not dynamical, i.e. it acts as a Lagrange multiplier. From the Lagrangian (3.2.7), we may immediately determine the canonical momenta

$$\pi^{ij} = \frac{1}{\sqrt{-\gamma}}\frac{\delta L}{\delta\dot\gamma_{ij}} = \frac{1}{2\kappa^2}\left(K\gamma^{ij} - K^{ij}\right), \tag{3.2.8}$$

$$\pi^i = \frac{1}{\sqrt{-\gamma}}\frac{\delta L}{\delta\dot A_i} = -\frac{1}{2\kappa^2}N^{-1}\left(F_r{}^i - N^kF_k{}^i\right), \tag{3.2.9}$$

$$\pi_\phi = \frac{1}{2\kappa^2}N^{-1}\left(\dot\phi - N^i\partial_i\phi\right). \tag{3.2.10}$$

These can be inverted to obtain expressions for the generalized velocities in terms of the momenta; this produces the flow equations that we encountered in (3.1.45)–(3.1.46),

$$\dot\gamma_{ij} = -4\kappa^2N\left(\pi_{ij} - \frac{1}{d-1}\pi\gamma_{ij}\right) + D_iN_j + D_jN_i, \tag{3.2.11}$$

$$\dot A_i = -2\kappa^2N\pi_i + \partial_iA_r + N^kF_{ki}, \tag{3.2.12}$$

$$\dot\phi = -2\kappa^2N\pi_\phi + N^i\partial_i\phi, \tag{3.2.13}$$

which allow us to determine the Hamiltonian via the Legendre transform,

$$H = \int_{\Sigma_r}\mathrm{d}^dx\ \sqrt{-\gamma}\left(\dot\gamma_{ij}\pi^{ij} + \dot A_i\pi^i + \dot\phi\pi_\phi\right) - L = \int_{\Sigma_r}\mathrm{d}^dx\ \sqrt{-\gamma}\left(N\mathcal{H} + N_i\mathcal{H}^i + A_r\mathcal{F}\right), \tag{3.2.14}$$

with the densities in front of the Lagrange multipliers $\{N, N_i, A_r\}$ are given by

$$\mathcal{H} = -2\kappa^2\left(\pi^{ij}\pi_{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi^i\pi_i\right) + \frac{1}{2\kappa^2}\left(-R^{(\gamma)} + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}F_{ij}F^{ij} - d(d-1)\right), \tag{3.2.15}$$

$$\mathcal{H}^i = -2D_j\pi^{ji} + F^{ij}\pi_j + \pi_\phi\partial^i\phi, \tag{3.2.16}$$

$$\mathcal{F} = -D_i\pi^i. \tag{3.2.17}$$

The equations of motion for the Lagrange multipliers $\{N, N_i, A_r\}$ impose the following first-class constraints,

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0, \quad \mathcal{F} = 0. \tag{3.2.18}$$

These are equivalent to the Hamilton-Jacobi equation $H = 0$, where the canonical momenta in H are expressed in the usual way as functional derivatives of Hamilton's principal function $\mathcal{S}[\gamma, A, \phi]$, leading to a functional differential equation. However, the momentum constraint $\mathcal{H}^i = 0$ simply requires $\mathcal{S}$ to be diffeomorphism invariant on constant-r leaves Σ_r, whereas $\mathcal{F} = 0$ amounts to the statement that $\mathcal{S}$ is gauge invariant—something we will explicitly see later. In order to proceed, we adopt the gauge choice

$$N = 1, \quad N^i = 0, \quad A_r = 0, \tag{3.2.19}$$

which simplifies the analysis substantially. On AlAdS backgrounds, the equations of motion (3.2.2)–(3.2.3) imply that the fields involved in our model have the following asymptotic behaviour[9]

$$\gamma_{ij} \simeq e^{2r} g_{(0)ij}(x), \quad \phi \simeq \phi_{(0)}(x), \quad A_i \simeq A_{(0)i}(x), \tag{3.2.20}$$

where we use domain-wall coordinates (cf. (A.1.9)). The radial derivative, given by

$$\delta_r = \int_{\Sigma_r} \mathrm{d}^d x \, \left(\dot{\gamma}_{ij} \frac{\delta}{\delta \gamma_{ij}} + \dot{\phi} \frac{\delta}{\delta \phi} + \dot{A}_i \frac{\delta}{\delta A_i} \right), \tag{3.2.21}$$

asymptotes—since by (3.2.20) $\dot{\gamma}_{ij} \simeq 2\gamma_{ij}$, $\dot{\phi} \simeq 0$ and $\dot{A}_i \simeq 0$—the dilatation operator,

$$\delta_r \simeq \int_{\Sigma_r} \mathrm{d}^d x \, 2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} = \delta_D. \tag{3.2.22}$$

Note that for this model, the dilatation operator in fact coincides with the induced metric operator δ_γ of (3.1.60) to lowest order. The next step is to express the principal function as

$$\mathcal{S} = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{-\gamma} \mathscr{L}, \tag{3.2.23}$$

where $\mathscr{L}$ is expanded in eigenmodes of δ_D in the following manner, where, due to the fact that d is even, we have anticipated logarithmic terms in accordance with the general FG scheme applicable for AlAdS boundary conditions,

$$\mathscr{L} = \cdots + \mathscr{L}_{(-1)} + \mathscr{L}_{(0)} + \mathscr{L}_{(1)} + \mathscr{L}_{(2)} + \cdots + \widetilde{\mathscr{L}}_{(d)} \log e^{-2r} + \mathscr{L}_{(d)} + \cdots, \tag{3.2.24}$$

where

$$\delta_D \mathscr{L}_{(w)} = -w \mathscr{L}_{(w)}, \quad w < d, \tag{3.2.25}$$

$$\delta_D \widetilde{\mathscr{L}}_{(d)} = -d \widetilde{\mathscr{L}}_{(d)}, \tag{3.2.26}$$

$$\delta_D \mathscr{L}_{(d)} = -d \mathscr{L}_{(d)} - 2\widetilde{\mathscr{L}}_{(d)}. \tag{3.2.27}$$

The transformation (3.2.27) is not entirely obvious, so let us derive it. Due to diffeomorphism invariance, the on-shell action does not—as we have seen—depend explicitly on the cut-off r, so the only dependence enters via the fields. This implies that the radial derivative of the bare on-shell action is given by δ_r, as given in (3.2.21), which to leading order is equivalent to the dilatation operator as demonstrated in (3.2.22). Thus, we find that

$$\delta_r \left(\widetilde{\mathscr{L}}_{(d)} \log e^{-2r} + \mathscr{L}_{(d)} \right) = \delta_r \left(\widetilde{\mathscr{L}}_{(d)} \right) \log e^{-2r} - 2\widetilde{\mathscr{L}}_{(d)} + \delta_r \mathscr{L}_{(d)}, \tag{3.2.28}$$

which, when set equal to $\log e^{-2r} \delta_D \widetilde{\mathscr{L}}_{(d)} + \delta_D \mathscr{L}_{(d)}$, produces the transformation property (3.2.27). The expansion of $\mathscr{L}$ in (3.2.24) leads to the following expansions of the momenta,

$$\pi^{ij} = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma_{ij}} \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{-\gamma} \mathscr{L} = \cdots + \pi^{ij}_{(1)} + \pi^{ij}_{(2)} + \pi^{ij}_{(3)} + \pi^{ij}_{(4)} + \cdots + \widetilde{\pi}^{ij}_{d+2} \log e^{-2r} + \pi^{ij}_{(d+2)} + \cdots, \tag{3.2.29}$$

$$\pi_\phi = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta \phi} \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{-\gamma} \mathscr{L} = \cdots + \pi_{\phi(-1)} + \pi_{\phi(0)} + \pi_{\phi(1)} + \cdots \widetilde{\pi}_{\phi(d)} \log e^{-2r} + \pi_{\phi(d)} + \cdots, \tag{3.2.30}$$

$$\pi^i = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta A_i} \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{-\gamma} \mathscr{L} = \cdots + \pi^i_{(-1)} + \pi^i_{(0)} + \cdots \widetilde{\pi}^i_{(d)} \log e^{-2r} + \pi^i_{(d)} + \cdots. \tag{3.2.31}$$

9 This statement is only valid for $d \geq 2$, which has interesting consequences that we will explore in sections 3.2.2 and 3.2.3 and in particular in section 3.3. We will also briefly comment on the generalization for p-forms.

Applying the dilatation operator (3.2.22) to the principal function (3.2.23), we get the following useful relation for $w < d$:

$$\left[2\gamma_{ij}\pi^{ij}\right]_{(w)} = (d-w)\mathscr{L}_{(w)}, \qquad w < d \tag{3.2.32}$$

$$\Rightarrow 2\gamma_{ij}\pi^{ij}_{(w+2)} = 2\pi_{(w)} = (d-w)\mathscr{L}_{(w)}, \quad w < d, \tag{3.2.33}$$

where $[XY]_{(w)}$ means the piece in the δ_D expansion of XY with dilatation weight w, i.e. $\delta_D[XY]_{(w)} = -w[XY]_{(w)}$, and where $\pi_{(w)} = \gamma_{ij}\pi^{ij}_{(w+2)}$ is the trace of the momentum, which, due to the asymptotic behaviour of the fields (3.2.20), has scaling weight w. For $w = d$, we find, by use of the scaling relations for the eigenmodes (3.2.26) and (3.2.27), the following relation,

$$2\pi_{(d)} = \frac{1}{\sqrt{-\gamma}}\delta_D \int_{\Sigma_r} \mathrm{d}^d x \sqrt{-\gamma}\left(\widetilde{\mathscr{L}}_{(d)} \log e^{-2r} + \mathscr{L}_{(d)}\right) \tag{3.2.34}$$

$$= -2\widetilde{\mathscr{L}}_{(d)}, \tag{3.2.35}$$

which is closely related to the conformal anomaly. Note in particular, we don't need to care about $w > d$, since these terms will vanish as $r \to \infty$. Now let's rewrite the Hamiltonian

$$\mathcal{H} = -2\kappa^2\left(\pi^{ij}\pi_{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi^i\pi_i\right) + \frac{1}{2\kappa^2}\left(-R^{(\gamma)} + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}F_{ij}F^{ij} - d(d-1)\right) \tag{3.2.36}$$

$$= K_{ij}\pi^{ij} - 2\kappa^2\left(\frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi^i\pi_i\right) + \frac{1}{2\kappa^2}\left(-R^{(\gamma)} + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}F_{ij}F^{ij} - d(d-1)\right), \tag{3.2.37}$$

where we have used our expression for π^{ij} (3.2.8) to rewrite the gravitational kinetic term. We can also expand this in dilatation weights,

$$\mathcal{H} = \sum_w \mathcal{H}_{(w)}, \quad \delta_D\mathcal{H}_{(w)} = -w\mathcal{H}_{(w)}, \tag{3.2.38}$$

and we then impose the Hamiltonian constraint $\mathcal{H} = 0$, which, due to unitarity of δ_D implies that each $\mathcal{H}_{(w)}$ vanishes individually,

$$\mathcal{H}_{(w)} = 0, \quad \forall w. \tag{3.2.39}$$

The next step is to introduce vielbeine[10], $\gamma_{ij} = e_i^a e_j^b \eta_{ab}$. This is advantageous since quantities with flat indices retain their scaling weight when indices are raised and lowered—in contrast to quantities carrying curved indices. This does, however, change the scaling weights of quantities with indices (but does so once and for all). Clearly, e_i^a has scaling weight -1, while e_a^i has scaling weight 1, so $A_a = A_i e_a^i$ also has scaling weight 1. The expansions of the momenta (3.2.29) and (3.2.31) in flat indices become (the expansion for π_ϕ carries no indices and is unchanged)

$$\pi^{ab} = e_i^a e_j^b \pi^{ij} = \cdots + \pi^{ab}_{(-1)} + \pi^{ab}_{(0)} + \pi^{ab}_{(1)} + \pi^{ab}_{(2)} + \cdots + \widetilde{\pi}^{ab}_{(d)} \log e^{-2r} + \pi^{ab}_{(d)} + \cdots, \tag{3.2.40}$$

$$\pi^a = e_i^a \pi^i = \cdots + \pi^a_{(-2)} + \pi^a_{(-1)} + \cdots \widetilde{\pi}^a_{(d-1)} \log e^{-2r} + \pi^a_{(d-1)} + \cdots \tag{3.2.41}$$

The asymptotic behaviours of the fields (3.2.20) determine—via the momentum relations (3.2.8)–(3.2.10)—the scaling weight at which the asymptotic expansions (3.2.29)–(3.2.31) begin, which in turn, through the useful relation (3.2.33), determines the scaling weight at which the expansion (3.2.24) of $\mathscr{L}$ begins. Similarly, they determine where the expansion (3.2.38) of $\mathcal{H}$ begins. We find that the asymptotic identification $\partial_r \simeq \delta_D$ implies

$$\pi^{ij} = \frac{1}{2\kappa^2}\left(K\gamma^{ij} - K^{ij}\right) = \frac{1}{2\kappa^2}\left(\frac{1}{2}\dot{\gamma}_{lk}\gamma^{lk}\gamma^{ij} - \frac{1}{2}\dot{\gamma}_{lk}\gamma^{li}\gamma^{kj}\right) \tag{3.2.42}$$

$$\overset{\partial_r \simeq \delta_D}{\simeq} \frac{d-1}{2\kappa^2}\gamma^{ij}, \tag{3.2.43}$$

so the expansion of π^{ij} starts at scaling weight two, which in turn means that π^{ab} starts at scaling weight zero. Similarly, we find for $\pi_{(\phi)}$ that the lowest term has the same scaling weight as ϕ itself

10 This is not necessary and is mainly an "aesthetic choice" in the sense that we find it to make the analysis conceptually cleaner; in the approaches taken in [28, 50, 51, 106] this is not done.

and is proportional to that scaling weight, which means that $\pi_{\phi(0)} = 0$, and the eigenmode expansion consequently begins at scaling weight one. Similarly, for the gauge field momentum, we find that

$$\pi^i = -\frac{1}{2\kappa^2} F_{rj}\gamma^{ji} = -\frac{1}{2\kappa^2}\dot{A}_j\gamma^{ji} \overset{\partial_r \simeq \delta_D}{\simeq} 0, \tag{3.2.44}$$

i.e. the expansion for π^i starts at scaling weight one, implying that the expansion for π^a starts at scaling weight zero. Combining our results, we find that the expansions (3.2.30), (3.2.40) and (3.2.41) take the form

$$\pi_\phi = \pi_{\phi(1)} + \cdots \tilde{\pi}_{\phi(d)} \log e^{-2r} + \pi_{\phi(d)} + \cdots, \tag{3.2.45}$$

$$\pi^{ab} = \pi^{ab}_{(0)} + \pi^{ab}_{(1)} + \pi^{ab}_{(2)} + \cdots + \tilde{\pi}^{ab}_{(d)} \log e^{-2r} + \pi^{ab}_{(d)} + \cdots, \tag{3.2.46}$$

$$\pi^a = \pi^a_{(0)} + \cdots \tilde{\pi}^a_{(d-1)} \log e^{-2r} + \pi^a_{(d-1)} + \cdots, \tag{3.2.47}$$

where, explicitly (compare (3.2.43))

$$\pi^{ab}_{(0)} = \frac{d-1}{2\kappa^2}\eta^{ab}, \tag{3.2.48}$$

whereas π^a_{-1} and $\pi_{\phi(0)}$ were determined to be zero by the same analysis. Since the trace of $\pi^{ab}_{(w)}$ is related to the principal function density $\mathscr{L}_{(w)}$ via (3.2.33), this in turn means that the expansion for $\mathscr{L}$ (3.2.24) starts at scaling weight zero: using (3.2.48) the expression for $\mathscr{L}_{(0)}$ can be read off immediately,

$$d\mathscr{L}_{(0)} = 2\eta_{ab}\pi^{ab}_{(0)} = \frac{d(d-1)}{\kappa^2} \Rightarrow \mathscr{L}_{(0)} = \frac{d-1}{\kappa^2}. \tag{3.2.49}$$

In general, we find the $\mathscr{L}_{(w)}$'s by relating the expressions for these in terms of the momenta traces $\pi_{(w)}$ (3.2.33) to the constraints $\mathcal{H}_{(w)} = 0$. This is achieved by noting that the Hamiltonian constraint (3.2.39) can be recast in the form

$$\mathcal{H}_{(w)} = \left[K_{ab}\pi^{ab} - 2\kappa^2\left(\frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi^a\pi_a\right) + \frac{1}{2\kappa^2}\left(-R^{(\gamma)} + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{4}F_{ab}F^{ab} - d(d-1)\right)\right]_{(w)} = 0, \tag{3.2.50}$$

for all w, which, using the flat index momenta (3.2.45)–(3.2.47), means that[11]

$$\mathcal{H}_{(0)} = K_{(0)ab}\pi^{ab}_{(0)} - \kappa^2\pi^a_{(0)}\pi_{(0)a} - \frac{d(d-1)}{2\kappa^2}, \tag{3.2.51}$$

$$\mathcal{H}_{(1)} = 2K_{(0)ab}\pi^{ab}_{(1)} - 2\kappa^2\pi^a_{(1)}\pi_{(0)a} \tag{3.2.52}$$

$$\mathcal{H}_{(2)} = 2K_{(0)ab}\pi^{ab}_{(2)} + K_{(1)ab}\pi^{ab}_{(1)} - \kappa^2\pi^2_{\phi(1)} - \kappa^2\pi^a_{(1)}\pi_{(1)a} - 2\kappa^2\pi^a_{(2)}\pi_{(0)a} + \frac{1}{2\kappa^2}\left(\frac{1}{2}\partial_i\phi\partial^i\phi - R^{(\gamma)}\right), \tag{3.2.53}$$

$$\mathcal{H}_{(3)} = \sum_{n=0}^{3} K_{(n)ab}\pi^{ab}_{(3-n)} - 2\kappa^2\pi_{\phi(1)}\pi_{\phi(2)} - 2\kappa^2\pi^a_{(2)}\pi_{(1)a}, \tag{3.2.54}$$

$$\mathcal{H}_{(4)} = \sum_{n=0}^{3} K_{(n)ab}\pi^{ab}_{(4-n)} - 2\kappa^2\left(\pi_{\phi(1)}\pi_{\phi(3)} + \frac{1}{2}\pi^2_{\phi(2)} + \frac{1}{2}\pi^a_{(2)}\pi_{(2)a} + \pi^a_{(1)}\pi_{(3)a} + \pi^a_{(0)}\pi_{(4)a}\right) + \frac{1}{8\kappa^2}F_{ab}F^{ab}, \tag{3.2.55}$$

$$\mathcal{H}_{(w)} = \sum_{n=0}^{w} K_{(n)ab}\pi^{ab}_{(w-n)} - 2\kappa^2\left[\frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi^a\pi_a\right]_{(w)}, \quad 4 < w \leq d. \tag{3.2.56}$$

Note that $\pi^a_{(w)}$ is the flat version of $\pi^i_{(w+1)}$, implying that $\pi^a_{(0)}$, for example, comes from $\mathcal{S}_{(1)}$. We also remark that due to the fact that both ϕ and A_i have scaling weight zero (cf. the asymptotic behaviour (3.2.20)), the scaling weight of various quantities such as $\mathcal{H}_{(w)}$ will vanish for odd values of w, i.e. $\mathcal{H}_{(2n+1)} = 0$ for $n \in \mathbb{Z}$, something we will also find below.

11 Note that $R^{(\gamma)}$ has scaling weight two, since the variation with respect to the metric γ_{ij} reads $\delta R^{(\gamma)} = -R^{(\gamma)ij}\delta\gamma_{ij}$, implying that $\delta_D R^{(\gamma)} = -2R^{(\gamma)}$, i.e. $R^{(\gamma)}$ has scaling weight two.

In order to simplify the constraints (3.2.51)–(3.2.56), we have made use of the fact that $K_{(n)ab}\pi^{ab}_{(m)} = K_{(m)ab}\pi^{ab}_{(n)}$, which follows from the expression for π^{ij} in (3.2.8), since

$$K_{(n)ab}\pi^{ab}_{(m)} = \frac{1}{2\kappa^2}k_{(n)ab}\left(K_{(m)}\eta^{ab} - K^{ab}_{(m)}\right) \tag{3.2.57}$$

$$= \frac{1}{2\kappa^2}\left(K_{(n)}K_{(m)} - K_{(n)ab}K^{ab}_{(m)}\right), \tag{3.2.58}$$

which is clearly symmetric under the interchange $n \leftrightarrow m$. We then observe that the leading behaviour K_{ab} is determined by the asymptotic identification $\partial_r \simeq \delta_D$, which gives us

$$K_{ij} \simeq \gamma_{ij} \Rightarrow K_{(0)ab} = \eta_{ab}. \tag{3.2.59}$$

This allows us to express the relation (3.2.33) in the following manner

$$2K_{(0)ab}\pi^{ab}_{(w)} = (d-w)\mathscr{L}_{(w)}, \tag{3.2.60}$$

$$2K_{(0)ab}\pi^{ab}_{(d)} = -2\widetilde{\mathscr{L}}_{(d)}. \tag{3.2.61}$$

We note that the combination $2K_{(0)ab}\pi^{ab}_{(w)}$ is ubiquitous in the constraints (3.2.51)–(3.2.56). The way we obtain $\mathscr{L}_{(w)}$ is then by imposing $\mathcal{H}_{(w)} = 0$ and recognizing the expression for $\mathscr{L}_{(w)}$ as given in (3.2.60); i.e. the constraint $\mathcal{H}_{(0)}$ translates into the following expression by use of (3.2.60):

$$0 = \mathcal{H}_{(0)} = \frac{d}{2}\mathscr{L}_{(0)} - \kappa^2\pi^a_{(0)}\pi_{(0)a} - \frac{d(d-1)}{2\kappa^2} \Rightarrow \mathscr{L}_{(0)} = \frac{d-1}{\kappa^2} + \frac{2}{d}\kappa^2\pi^a_{(0)}\pi_{(0)a}. \tag{3.2.62}$$

However, since $\mathscr{L}_{(0)}$ is given by (3.2.49), it follows that

$$\pi^a_{(0)} = 0, \tag{3.2.63}$$

i.e. $\mathscr{L}_{(1)}$ does not depend on A_i. The level one constraint $\mathcal{H}_{(1)} = 0$ simplifies by use of (3.2.63) to $2\pi_{(1)} = 0$, that is, by the relation (3.2.33), $(d-1)\mathscr{L}_{(1)} = 0$, so that

$$\mathscr{L}_{(1)} = 0 \therefore \pi_{\phi(1)} = 0 = \pi^{ab}_{(1)}. \tag{3.2.64}$$

Combining the results (3.2.63) and (3.2.64) implies that the level two constraint $\mathcal{H}_{(2)} = 0$ now reads

$$0 = \mathcal{H}_{(2)} = 2\pi_{(2)} - \kappa^2\pi^a_{(1)}\pi_{(1)a} + \frac{1}{2\kappa^2}\left(\frac{1}{2}\partial_i\phi\partial^i\phi - R^{(\gamma)}\right) \tag{3.2.65}$$

$$= (d-2)\mathscr{L}_{(2)} - \kappa^2\pi^a_{(1)}\pi_{(1)a} + \frac{1}{2\kappa^2}\left(\frac{1}{2}\partial_i\phi\partial^i\phi - R^{(\gamma)}\right). \tag{3.2.66}$$

Now, since $\pi^a_{(1)} = e^a_i\pi^i_{(2)} = \frac{1}{\sqrt{-\gamma}}e^a_i\frac{\delta\mathcal{S}_{(2)}}{\delta A_i}$, the above turns, in principle, into a functional differential equation for $\mathscr{L}_{(2)}$. However, we can use the $U(1)$ constraint $\mathcal{F} = 0$ (cf. (3.2.18)) to determine $\pi^i_{(2)}$. This constraint can be recast in the form

$$D_i\pi^i_{(w)} = D_i\left[\frac{\delta\mathcal{S}_{(w)}}{\delta A_i}\right] = 0, \qquad \forall w. \tag{3.2.67}$$

Now, the most general expression involving A that can appear in the action at scaling weight two has the form

$$\mathcal{S}_{(2)} \supset \sum_{n_1}\phi^{n_1}c^{(1)}_{n_1}\partial^i A_i + \sum_{n_2}\phi^{n_1}c^{(2)}_{n_2}A^iA_i + \sum_{n_3}\phi^{n_3}c^{(3)}_{n_3}A^i\partial_i\phi + \sum_{n_4}\phi^{n_4}c^{(4)}_{n_4}D_iA^i, \tag{3.2.68}$$

but, crucially, none of these terms—by themselves or in any combination—are in compliance with the requirement (3.2.67), i.e. all the $c^{(m)}$'s above vanish. Equivalently, we note that the gauge choice[12] $A_r = 0$ does not completely fix the gauge, as we now show: under a general gauge transformation, the gauge field transforms as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda, \tag{3.2.69}$$

12 This is an example of an axial gauge, i.e. a gauge of the form $n^\mu A_\mu = 0$ for some normal vector n^μ; for details see e.g. [69].

for some function λ. The condition $A_r = 0$ is easily realised by choosing λ to be of the form

$$\lambda = -\int \mathrm{d}r\, A_r + f(x,t,y), \tag{3.2.70}$$

where f represents *the residual gauge symmetry*. Requiring our on-shell action to be invariant under residual gauge transformations, we recover the same result as from the $\mathcal{F} = 0$ constraint that $\mathcal{S}_{(2)}$ cannot involve A_i, that is

$$\pi^i_{(2)} = 0, \qquad \text{or in flat indices} \qquad \pi^a_{(1)} = 0, \tag{3.2.71}$$

leading us to conclude that

$$\mathscr{L}_{(2)} = \frac{1}{2(d-2)\kappa^2}\left(R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi\right). \tag{3.2.72}$$

The level three constraint vanishes—using $\pi^{ab}_{(1)} = 0 = \pi^a_{(1)}$, cf. (3.2.64), (3.2.71), as well as the useful relation for the $\mathscr{L}_{(w)}$'s (3.2.33)—identically

$$\mathscr{L}_{(3)} = 0. \tag{3.2.73}$$

We now specialize to $d = 4$. We have already computed the first two $\mathscr{L}$'s; in $d = 4$, they take the form

$$\mathscr{L}_{(0)} = \frac{3}{\kappa^2}, \tag{3.2.74}$$
$$\mathscr{L}_{(2)} = \frac{1}{4\kappa^2}\left(R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi\right). \tag{3.2.75}$$

At level four, the constraint reads—after taking into account all our findings:

$$0 = \mathcal{H}_{(4)} = 2K_{(0)ab}\pi^{ab}_{(4)} + K_{(2)ab}\pi^{ab}_{(2)} - \kappa^2\pi^2_{\phi(2)} + \frac{1}{8\kappa^2}F_{ab}F^{ab} \tag{3.2.76}$$
$$= -2\widetilde{\mathscr{L}}_{(4)} + K_{(2)ab}\pi^{ab}_{(2)} - \kappa^2\pi^2_{\phi(2)} + \frac{1}{8\kappa^2}F_{ab}F^{ab}, \tag{3.2.77}$$

implying that

$$\widetilde{\mathscr{L}}_{(4)} = \frac{1}{2}K_{(2)ab}\pi^{ab}_{(2)} - \frac{\kappa^2}{2}\pi^2_{\phi(2)} + \frac{1}{8\kappa^2}F_{ab}F^{ab}. \tag{3.2.78}$$

This involves only quantities that can be determined from $\mathscr{L}_{(2)}$ (cf. (3.2.72)). First note that the level two constraint $\mathcal{H}_{(2)} = 0$ implies that for $d = 4$, we get $\pi^a_{(2)a} = \mathscr{L}_{(2)}$, and so

$$\pi^a_{(2)a} = \pi_{(2)} = \frac{1}{4\kappa^2}\left(R^{(\gamma)} - \frac{1}{2}\partial_i\phi\partial^i\phi\right). \tag{3.2.79}$$

Then, observe that

$$\pi^{ab} = \frac{1}{2\kappa^2}\left(K\eta^{ab} - K^{ab}\right) \therefore K_{(2)} = \frac{2\kappa^2}{3}\pi_{(2)}, \tag{3.2.80}$$

implying that $K_{(2)}$ becomes

$$K_{(2)} = \frac{R^{(\gamma)}}{6} - \frac{1}{12}\partial_i\phi\partial^i\phi. \tag{3.2.81}$$

Now we can determine $\pi_{(2)ab}$ from our knowledge of $\mathscr{L}_{(2)}$; since $\mathcal{S}_{(2)} = \int_{\Sigma_r}\mathrm{d}^4x\,\sqrt{-\gamma}\mathscr{L}_{(2)}$, we see that

$$\pi_{(2)ab} = e^i_a e^j_b \frac{1}{\sqrt{-\gamma}}\frac{\delta\mathcal{S}_{(2)}}{\delta\gamma^{ij}} \tag{3.2.82}$$
$$= e^i_a e^j_b \frac{1}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma^{ij}}\frac{1}{2\kappa^2}\int_{\Sigma_r}\mathrm{d}^4x\,\sqrt{-\gamma}\left(\frac{R^{(\gamma)}}{2} - \frac{1}{4}\partial_i\phi\partial^i\phi\right) \tag{3.2.83}$$
$$= \frac{1}{2\kappa^2}\left(\frac{1}{8}\eta_{ab}\partial_i\phi\partial^i\phi - \frac{1}{4}R^{(\gamma)}\eta_{ab} - \frac{1}{4}\partial_a\phi\partial_b\phi + \frac{1}{2}R^{(\gamma)}_{ab}\right). \tag{3.2.84}$$

Our expression for the momentum above allows us to find the scaling weight two extrinsic curvature,

$$\pi_{(2)ab} = \frac{1}{2\kappa^2}\left(K_{(2)}\eta_{ab} - K_{(2)ab}\right), \tag{3.2.85}$$

and so, using the trace of $K_{(2)}$ (3.2.81) and our expression for $\pi_{(2)ab}$ (3.2.84), we get

$$K_{(2)ab} = \frac{1}{24}\eta_{ab}\partial_i\phi\partial^i\phi - \frac{1}{12}\eta_{ab}R^{(\gamma)} - \frac{1}{4}\partial_a\phi\partial_b\phi + \frac{1}{2}R^{(\gamma)}_{ab}. \tag{3.2.86}$$

We are now in a position to work out the combination $K_{(2)ab}\pi^{ab}_{(2)}$ appearing in the anomaly (3.2.78):

$$K_{(2)ab}\pi^{ab}_{(2)} = -\frac{1}{2\kappa^2}\left(\frac{1}{24}\left(\partial_i\phi\partial^i\phi\right)^2 + \frac{1}{12}R^{(\gamma)}\partial_i\phi\partial^i\phi - \frac{1}{12}R^{(\gamma)2} - \frac{1}{4}R^{(\gamma)}_{ij}\partial^i\phi\partial^j\phi + \frac{1}{4}R^{(\gamma)}_{ij}R^{(\gamma)ij}\right), \tag{3.2.87}$$

The final piece is a Laplacian term, coming from

$$\pi_{\psi(2)} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{(2)}}{\delta\phi} = -\frac{1}{4\kappa^2}\Box^{(\gamma)}\phi, \tag{3.2.88}$$

where we have used that $\Box^{(\gamma)}\phi = \frac{1}{\sqrt{-\gamma}}\partial_i\left(\sqrt{-\gamma}\partial^i\phi\right)$. Combining everything, the anomaly (3.2.78) takes the form

$$\widetilde{\mathscr{L}}_{(4)} = \frac{1}{8\kappa^2}F_{ij}F^{ij} \tag{3.2.89}$$

$$-\frac{1}{4\kappa^2}\left(\frac{1}{24}\left(\partial_i\phi\partial^i\phi\right)^2 + \frac{1}{12}R^{(\gamma)}\partial_i\phi\partial^i\phi - \frac{1}{12}R^{(\gamma)2} - \frac{1}{4}R^{(\gamma)}_{ij}\partial^i\phi\partial^j\phi + \frac{1}{4}R^{(\gamma)}_{ij}R^{(\gamma)ij}\right) \tag{3.2.90}$$

$$-\frac{1}{32\kappa^2}\left(\Box^{(\gamma)}\phi\right)^2. \tag{3.2.91}$$

Note that we have $\log e^{-2r}$, so when rewriting to Poincaré coordinates (for details, see appendix A), there will be an additional factor of two. Combining all our findings, the counterterm becomes (including a minus, since we need to subtract the divergences)

$$S_{\text{ct}} = \frac{1}{\kappa^2}\int_{\Sigma_r} \mathrm{d}^4x\,\sqrt{-\gamma}\left(-\frac{1}{4}\left(R^{(\gamma)} + 12 - \frac{1}{2}\partial_i\phi\partial^i\phi\right) - \frac{1}{4}\mathcal{A}\log e^{-2r}\right), \tag{3.2.92}$$

where the anomaly can be rewritten as

$$\mathcal{A} = -\frac{1}{4}\left(Q_{ij}Q^{ij} - \frac{1}{3}Q^2 - 2F_{ij}F^{ij} + \frac{1}{2}\left(\Box^{(\gamma)}\phi\right)^2\right), \tag{3.2.93}$$

where

$$Q_{ij} = R^{(\gamma)}_{ij} - \frac{1}{2}\partial_i\phi\partial_j\phi. \tag{3.2.94}$$

In order to obtain the actually renormalized action on the boundary, we take the limit $r \to \infty$, i.e.

$$\hat{S}_{\text{ren}} := \lim_{r\to\infty} S_{\text{ren}} = \lim_{r\to\infty}\int_{\Sigma_r} \mathrm{d}^dx\,\sqrt{-\gamma}\mathscr{L}_{(d)} \tag{3.2.95}$$

$$= \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^5x\,\sqrt{-g}\left(R^{(g)} + 12 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \hat{S}_{\text{GH}} + \hat{S}_{\text{ct}}, \tag{3.2.96}$$

where we have used that $\mathscr{L}_{(d)}$ is the first term that does not produce a divergence, while all higher order terms will simply vanish, and where quantities with hats have had the limit $r \to \infty$ applied to them.

Removing the field strength from the counterterm (3.2.92), our result agrees with previous results for Einstein-Dilaton theory, which was considered in [51, 57, 107]. While at a first glance it may seem surprising that for $d = 4$, the only modification to the counterterm action due to the addition of a Maxwell field occurs in the anomaly, this is in hindsight obvious: there is simply no gauge invariant combination with scaling weight 2. If we take $d = 6$, the situation changes, and, roughly speaking, what was $\widetilde{\mathscr{L}}_{(4)}$ in $d = 4$ becomes $\mathscr{L}_{(4)}$ in $d = 6$ (with some coefficients changed).

3.2.2 *Renormalized Ward Identities & One-Point Functions*

In this section, we derive the Ward identities of the EMD model of (3.2.1), which are essentially the boundary versions of the first-class constraints (3.2.18). Varying the renormalized on-shell action on $\mathcal{M}$ with $\partial\mathcal{M} = \Sigma_r$, we get

$$\delta S_{\text{ren}} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^5x\,\sqrt{-g}\left(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\phi\delta\phi + \mathcal{E}_\mu\delta A^\mu\right) \tag{3.2.97}$$

$$+\frac{1}{2\kappa^2}\int_{\Sigma_r}\mathrm{d}^4x\,\sqrt{-\gamma}\left(\frac{1}{2}\langle T^{ij}\rangle_{\text{ren}}\,\delta\gamma^{ij} + \langle\mathcal{O}_\phi\rangle_{\text{ren}}\,\delta\phi + \langle J^i\rangle_{\text{ren}}\,\delta A_i + \mathcal{A}\delta r\right), \tag{3.2.98}$$

where $\mathcal{E}_{\mu\nu}$, $\mathcal{E}_\phi$ and $\mathcal{E}_\mu$ are the equations of motion for the respective fields (3.2.2)–(3.2.3), and where the renormalized responses are given in terms of dilatation operator expansion eigenmodes via the form of the on-shell renormalized action (3.2.95)

$$\langle T^{ij}\rangle_{\text{ren}} = \frac{4\kappa^2}{\sqrt{-\gamma}}\frac{\delta S_{\text{ren}}}{\delta\gamma_{ij}} = 4\kappa^2\pi^{ij}_{(6)}, \tag{3.2.99}$$

$$\langle\mathcal{O}_\phi\rangle_{\text{ren}} = \frac{2\kappa^2}{\sqrt{-\gamma}}\frac{\delta S_{\text{ren}}}{\delta\phi} = 2\kappa^2\pi_{\phi(4)}, \tag{3.2.100}$$

$$\langle J^i\rangle_{\text{ren}} = \frac{2\kappa^2}{\sqrt{-\gamma}}\frac{\delta S_{\text{ren}}}{\delta A_i} = 2\kappa^2\pi^i_{(4)}. \tag{3.2.101}$$

These expressions are all evaluated at the regularized hypersurface Σ_r. Using the asymptotic behaviours

$$\gamma_{ij} \simeq e^{2r}\gamma_{(0)ij}, \quad \phi \simeq \phi_{(0)}, \quad A_i \simeq A_{(0)i}, \quad \sqrt{-\gamma} \simeq e^{4r}\sqrt{-\gamma_{(0)}}, \tag{3.2.102}$$

it follows that, since S_{ren} approaches the finite quantity $\hat{S}_{\text{ren}}$ as $r\to\infty$, we must multiply the responses with a suitably chosen factor in order to obtain finite values—i.e. the actual VEVs—in the limit $r\to\infty$, i.e.

$$\langle\hat{T}^{ij}\rangle_{\text{ren}} := \lim_{r\to\infty} e^{6r}\langle T^{ij}\rangle_{\text{ren}} = \frac{4\kappa^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta\gamma_{(0)ij}} = 4\kappa^2\hat{\pi}^{ij}_{(6)} \tag{3.2.103}$$

$$\langle\hat{\mathcal{O}}_\phi\rangle_{\text{ren}} := \lim_{r\to\infty} e^{4r}\langle\mathcal{O}_\phi\rangle_{\text{ren}} = \frac{2\kappa^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta\phi_{(0)}} = 2\kappa^2\hat{\pi}_{\phi(4)}, \tag{3.2.104}$$

$$\langle\hat{J}^i\rangle_{\text{ren}} := \lim_{r\to\infty} e^{4r}\langle J^i\rangle_{\text{ren}} = \frac{2\kappa^2}{\sqrt{-\gamma_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta A_{(0)i}} = 2\kappa^2\hat{\pi}^i_{(4)}. \tag{3.2.105}$$

Now, consider the momentum constraint $\mathcal{H}^i = 0$,

$$-2D_j\pi^j_i + F_{ij}\pi^j + \pi_\phi\partial_i\phi = 0, \tag{3.2.106}$$

the weight four part of which reads[13]

$$-2D_j\pi^j_{(4)i} + F_{ij}\pi^j_{(4)} + \pi_{\phi(4)}\partial_i\phi = 0, \tag{3.2.107}$$

which, using the explicit VEVs (3.2.99)–(3.2.101), becomes

$$-D_j\langle T^j{}_i\rangle_{\text{ren}} + F_{ij}\langle J^j\rangle_{\text{ren}} + \langle\mathcal{O}_\phi\rangle_{\text{ren}}\,\partial_i\phi = 0, \tag{3.2.108}$$

which we can multiply with e^{4r} and take the limit $r\to\infty$ to obtain the diffeomorphism Ward identity,

$$-D_{(0)j}\langle\hat{T}^j_i\rangle_{\text{ren}} + F_{(0)ij}\langle\hat{J}^j\rangle_{\text{ren}} + \langle\hat{\mathcal{O}}_\phi\rangle_{\text{ren}}\,\partial_i\phi_{(0)} = 0. \tag{3.2.109}$$

We now proceed to derive the trace Ward identity. Using the relation (3.2.61), which states that

$$2\pi^i_{(4)i} = -2\widetilde{\mathscr{L}}_{(4)}, \tag{3.2.110}$$

13 Note that when lowering the i-index on $\pi^{ij}_{(4)}$, we get $\pi^j_{(4)i}$ due to the fact that the metric has weight -2; also note that F_{ij} has weight zero.

we immediately obtain the trace Ward identity—again after multiplying by e^{4r} and taking the limit $r \to \infty$:

$$\langle \hat{T}^i{}_i \rangle_{\text{ren}} = \hat{\mathcal{A}}, \tag{3.2.111}$$

where

$$\hat{\mathcal{A}} = -2 \lim_{r\to\infty} e^{4r} \widetilde{\mathscr{L}}_{(4)}, \tag{3.2.112}$$

is the conformal anomaly on the boundary. Finally, the constraint $\mathcal{F}$ with

$$\mathcal{F} = -D_i \pi^i, \tag{3.2.113}$$

turns into the $U(1)$ gauge transformation Ward identity, since at weight 4, it reads

$$0 = D_i \pi^i_{(4)} = D_i \langle \hat{J}^i \rangle_{\text{ren}}, \tag{3.2.114}$$

which, when multiplied with e^{dr} and the limit $r \to \infty$ is taken produces the result

$$D_{(0)i} \langle \hat{J}^i \rangle_{\text{ren}} = 0, \tag{3.2.115}$$

which is the $U(1)$ Ward identity.

3.2.3 *Generating the $d = 4$ Fefferman-Graham Expansions*

In order to construct the FG expansions of the fields, we are going to apply the gauge-fixed first-order flow equations (3.2.11)–(3.2.13), which for our purposes read

$$\dot{\gamma}_{ij} = -4\kappa^2 \left(\gamma_{ik}\gamma_{jl} - \frac{1}{3}\gamma_{ij}\gamma_{kl} \right) \left(\pi^{kl}_{(2)} + \pi^{kl}_{(4)} + \widetilde{\pi}^{kl}_{(6)} \log e^{-2r} + \pi^{kl}_{(6)} + \cdots \right) \tag{3.2.116}$$

$$= -4\kappa^2 \left(\pi_{(-2)ij} - \frac{1}{3}\pi_{(0)}\gamma_{ij} + \pi_{(0)ij} - \frac{1}{3}\pi_{(2)}\gamma_{ij} + \log e^{-2r} \left(\tilde{\pi}_{(2)ij} - \frac{1}{3}\tilde{\pi}_{(4)}\gamma_{ij} \right) + \pi_{(2)ij} - \frac{1}{3}\pi_{(4)}\gamma_{ij} + \cdots \right), \tag{3.2.117}$$

$$\dot{\phi} = -2\kappa^2 \left(\pi_{\phi(2)} + \widetilde{\pi}_{\phi(4)} \log e^{-2r} + \pi_{\phi(4)} + \cdots \right), \tag{3.2.118}$$

$$\dot{A}_i = -2\kappa^2 \left(\widetilde{\pi}_{(2)i} \log e^{-2r} + \pi_{(2)i} + \cdots \right). \tag{3.2.119}$$

In section 3.2.1, we found the following expressions

$$\pi^{ij}_{(2)} = \frac{3}{2\kappa^2}\gamma^{ij}, \tag{3.2.120}$$

$$\pi^{ij}_{(4)} = \frac{1}{2\kappa^2} \left(\frac{1}{8}\partial_k\phi\partial^k\phi\gamma^{ij} - \frac{1}{4}R^{(\gamma)}\gamma^{ij} - \frac{1}{4}\partial^i\phi\partial^j\phi + \frac{1}{2}R^{(\gamma)ij} \right), \tag{3.2.121}$$

$$\tilde{\pi}^{ij}_{(6)} = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta\gamma_{ij}} \int \sqrt{-\gamma}\widetilde{\mathscr{L}}_{(4)}, \tag{3.2.122}$$

$$\pi_{\phi(2)} = -\frac{1}{4\kappa^2}\Box^{(\gamma)}\phi, \tag{3.2.123}$$

$$\tilde{\pi}_{\phi(4)} = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta\phi} \int \sqrt{-\gamma}\widetilde{\mathscr{L}}_{(4)}, \tag{3.2.124}$$

$$\tilde{\pi}_{(2)i} = \frac{1}{2\kappa^2} D_j F^j{}_i. \tag{3.2.125}$$

We now introduce very general formal expansions for the fields involved,

$$\gamma_{ij} = \gamma^{(0)}_{ij} + \gamma^{(1)}_{ij} + \cdots, \qquad \phi = \phi^{(0)} + \phi^{(1)} + \cdots, \qquad A_i = A^{(0)}_i + A^{(1)}_i + \cdots, \tag{3.2.126}$$

where each order is assumed to be subleading in r to the previous, i.e. *these raised labels are not scaling weights*. When inserting these into the flow equations above, we obtain a series of differential equations which can be solved order-by-order. However, we can do better: making explicit FG ansätze for (3.2.126) (*in these expressions, the labels in the subscripts are not scaling weights, rather they label the respective terms in the FG expansions just as in* (F.1.12)),

$$\gamma_{ij} = e^{2r}\left(h_{(0)ij} + e^{-2r}h_{(2)ij} + e^{-4r}\left(\log e^{-2r}h_{(4,1)ij} + h_{(4)ij}\right) + \cdots\right), \tag{3.2.127}$$

$$\phi = \phi_{(0)} + e^{-2r}\phi_{(2)} + e^{-4r}\left(\log e^{-2r}\phi_{(4,1)} + \phi_{(4)}\right) + \cdots, \tag{3.2.128}$$

$$A_i = A_{(0)i} + e^{-2r}\left(\log e^{-2r}A_{(2,1)i} + A_{(2)i}\right) + \cdots, \tag{3.2.129}$$

the differential equations become algebraic. Using the momentum (3.2.120), we find the differential equation for the lowest order of the metric $\dot{\gamma}^{(0)}_{ij} = 2\gamma^{(0)}_{ij}$, which is solved by $\gamma^{(0)} = e^{2r}h_{(0)ij}$, where $h_{(0)ij}$ plays the rôle of an integration constant, which is in agreement with the ansatz (3.2.127). The same happens for the scalar and the Maxwell field, i.e. we get $\phi^{(0)} = \phi_{(0)}$ and $A^{(0)}_i = A_{(0)i}$. At second order, we get the differential equation for the metric

$$\dot{\gamma}^{(1)}_{ij} = 2\gamma^{(1)}_{ij} + R^{(h_{(0)})}_{ij} - \frac{1}{2}\partial_i\phi_{(0)}\partial_j\phi_{(0)} - \frac{1}{6}R^{(h_{(0)})}h_{(0)ij} + \frac{1}{12}h_{(0)ij}\left(\partial\phi_{(0)}\right)^2, \tag{3.2.130}$$

where we have kept only the lowest order terms in the expansion. Using the FG ansatz (3.2.127), the left-hand side above vanishes and provides us with the relation

$$h_{(2)ij} = -\frac{1}{2}\left(R^{(h_{(0)})}_{ij} - \frac{1}{2}\partial_i\phi_{(0)}\partial_j\phi_{(0)} - \frac{1}{6}R^{(h_{(0)})}h_{(0)ij} + \frac{1}{12}h_{(0)ij}\left(\partial\phi_{(0)}\right)^2\right). \tag{3.2.131}$$

The second order equation for the scalar reads

$$\dot{\phi}^{(1)} = \frac{1}{2}\Box_{(\gamma)}\phi = \frac{1}{2}\frac{1}{\sqrt{-\gamma}}\partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_j\phi\right), \tag{3.2.132}$$

which, when using the FG ansatz becomes

$$2e^{2r}\phi_{(2)} = \frac{e^{2r}}{2}\frac{1}{\sqrt{-h_{(0)}}}\partial_i\left(\sqrt{-h_{(0)}}h^{ij}_{(0)}\partial_j\phi_{(0)}\right) + \mathcal{O}(e^{4r}), \tag{3.2.133}$$

so that

$$\phi_{(2)} = \frac{1}{4}\Box_{(0)}\phi_{(0)}. \tag{3.2.134}$$

Moving to the next order, we find it useful to consider the the flow equation (3.2.117) in the form

$$\dot{\gamma}_{ij} = -4\kappa^2\Bigg(\overbrace{\pi_{(-2)ij} - \frac{1}{3}\pi_{(0)}\gamma_{ij}}^{(*)} + \overbrace{\pi_{(0)ij} - \frac{1}{3}\pi_{(2)}\gamma_{ij}}^{(\dagger)} + \log e^{-2r}\overbrace{\left(\tilde{\pi}_{(2)ij} - \frac{1}{3}\tilde{\pi}_{(4)}\gamma_{ij}\right)}^{\ddagger} + \overbrace{\pi_{(2)ij} - \frac{1}{3}\pi_{(4)}\gamma_{ij}}^{(**)} + \cdots\Bigg). \tag{3.2.135}$$

The differential equation at the next order consequently takes the form

$$\dot{\gamma}^{(2)}_{ij} = \overbrace{2\gamma^{(2)}_{ij}}^{\text{from }(*)} + e^{-2r}\log e^{-2r}\left[\text{lowest order terms from }(\ddagger)\right] + e^{-2r}\left[\text{terms from expansion of }(\dagger)\right] \tag{3.2.136}$$

$$+ e^{-2r}\left[\text{lowest order terms from }(**)\right]. \tag{3.2.137}$$

Now, the Fefferman-Graham ansatz implies that $\gamma^{(2)}_{ij} = e^{-2r}\left(\log e^{-2r}h_{(4,1)ij} + h_{(4)ij}\right)$, which means that after subtracting $2\gamma^{(2)}_{ij}$ (in red) on both sides, we obtain

$$-4e^{-2r}\log e^{-2r}h_{(4,1)ij} - e^{-2r}\left(4h_{(4)ij} + 2h_{(4,1)ij}\right) = \text{appropriate terms from }(*), (\dagger), (\ddagger), (**), \tag{3.2.138}$$

where "appropriate terms" means terms that have the correct radial dependence when all fields are FG expanded. In particular, we can read off the value of $h_{(4,1)ij}$ by computing the term involving the logarithm on the right-hand side; this gives[14]

$$h_{(4,1)ij} = \frac{1}{8}D^k_{(0)}\left(D_{(0)i}h_{(2)jk} + D_{(0)j}h_{(2)ik} - D_{(0)k}h_{(2)ij}\right) - \frac{1}{8}D_{(0)i}D_{(0)j}h^k_{(2)k} \tag{3.2.139}$$

$$+ \frac{1}{2}h_{(2)ik}h^k_{(2)j} - \frac{1}{4}\partial_{(i}\phi_{(0)}D_{(0)j)}\phi_{(2)} - h_{(0)ij}\left(\frac{1}{8}h^{kl}_{(2)}h_{(2)kl} + \frac{1}{4}\phi^2_{(2)}\right) \tag{3.2.140}$$

$$+ \frac{1}{6}F_{(0)ik}F_{(0)j}{}^k - \frac{1}{24}h_{(0)ij}F_{(0)kl}F^{kl}_{(0)}, \tag{3.2.141}$$

where $F_{(0)ij} = 2\partial_{(i}A_{(0)j)}$ and where we have used that $\gamma^{ij} = e^{-2r}h^{ij}_{(0)} + \mathcal{O}(e^{-4r}\log e^{-2r})$ (see appendix F). Now, note that the terms contributing from $(**)$ are

$$(**) = e^{-2r}\left[\langle\hat{T}_{ij}\rangle_{\text{ren}} - \frac{1}{3}h_{(0)ij}\,\langle\hat{T}^i{}_i\rangle_{\text{ren}}\right] + \text{higher orders}, \tag{3.2.142}$$

where we have used the relation for the VEV (3.2.103). Combining this with with our result for $h_{(4,1)ij}$, we now find [86]

$$h_{(4)ij} = X_{ij} - \frac{1}{4}\,\langle\hat{T}_{ij}\rangle_{\text{ren}}, \tag{3.2.143}$$

where

$$X_{ij} = \frac{1}{2}h_{(2)ik}h^k_{(2)j} - \frac{1}{4}h^k_{(2)k}h_{(2)ij} + \frac{1}{8}h_{(0)ij}\hat{\mathcal{A}} - \frac{3}{2}h_{(4,1)ij}, \tag{3.2.144}$$

with the boundary anomaly $\hat{\mathcal{A}}$ defined in (3.2.112). We could now repeat the analysis above for the scalar, but nothing is changed from the case with no Maxwell field, so we refer to the standard treatments [51, 86], where the full FG expansion for the scalar can be found. The result is:

$$\phi_{(4,1)} = -\frac{1}{8}\left[\Box_{(0)}\phi_{(2)} + 2\phi_{(2)}h^i_{(2)i} + \frac{1}{2}\partial^i\phi_{(0)}D_{(0)i}h^j_{(2)j} - h^{ij}_{(2)}D_{(0)i}\partial_j\phi_{(0)} - \partial^i\phi_{(0)}D^j_{(0)}h_{(2)ij}\right], \tag{3.2.145}$$

$$\langle\hat{\mathcal{O}}_\phi\rangle_{\text{ren}} = 4\phi_{(4)} + \phi_{(2)}h^i_{(2)i} + 6\phi_{(4,1)}. \tag{3.2.146}$$

Now, we turn our attention to the Maxwell field. This expansion has not appeared in the literature before to our knowledge. Looking at the FG expansion for the Maxwell field, we see that $A^{(2)}_i = e^{-2r}\left(\log e^{-2r}A_{(2,1)i} + A_{(2)i}\right)$, so the flow equation (3.2.119) takes the form

$$\dot{A}^{(2)}_i = -2e^{-2r}\left(A_{(2,1)i} + A_{(2)i}\right) - 2e^{-2r}\log e^{-2r}A_{(2,1)i} \tag{3.2.147}$$

$$= -D_{(0)j}F_{(0)ki}h^{kj}_{(0)}e^{-2r}\log e^{-2r} - e^{-2r}\,\langle\hat{J}_i\rangle_{\text{ren}} + \text{higher order terms}, \tag{3.2.148}$$

from which we may immediately read off the component $A_{(2,1)i}$:

$$A_{(2,1)i} = \frac{1}{2}D_{(0)j}F^j_{(0)i}, \tag{3.2.149}$$

and

$$A_{(2)i} = \frac{1}{2}\,\langle\hat{J}_i\rangle_{\text{ren}} - A_{(2,1)i}. \tag{3.2.150}$$

3.3 HOLOGRAPHIC RENORMALIZATION OF p-FORMS & THE SCALAR-VECTOR DUALITY

3.3.1 *From the Maxwell One-Form to p-Forms: A Conjecture*

Note that the FG expansion for the Maxwell field (3.2.129) for $d = 4$ has its logarithmic term appear at order $\mathcal{O}(e^{-2r})$, which follows from the flow equation (3.2.119). This is a consequence of the fact

14 We have employed *xAct* [88] and *xTras* [108] to obtain these results as well as used known results for the case without the Maxwell field, see e.g. [51, 57, 86]. Note also the extra factor of two that comes from our convention of writing $\log e^{-2r}$, which in Poincaré coordinates reads $2\log r_{\text{Poin.}}$; when we'll use these results in chapter 6, we will multiply all logarithmic terms by two, since we will change conventions. We hope this does not cause any confusion.

that the Maxwell field is a one-form, since in a general (even) dimension, the associated momentum π^i admits a δ_D-expansion of the form[15]

$$\pi^i = \cdots + \log e^{-2r} \widetilde{\pi}^i_{(d)} + \pi^i_{(d)} + \cdots, \tag{3.3.1}$$

which means that $\pi_i = \gamma_{ij}\pi^j = \cdots + \log e^{-2r} \widetilde{\pi}_{(d-2)i} + \pi_{(d-2)i}$. Now, A_i is roughly related to π_i via the flow equations, and so we see that something peculiar happens when $d = 2$: the VEV part of the expansion occurs at scaling weight zero. More explicitly, the equation of motion for A_μ (3.2.3) on a AdS$_3$ background in domain wall coordinates with the ansatz $A_\mu \simeq e^{cr} A_{(0)\mu}$ for, say, the time component reads

$$0 = e^{-2r}\partial_r \left(e^{2r} e^{-2r} F_{rt} \right) = e^{(c-2)r} c^2 A_{(0)}, \tag{3.3.2}$$

implying a double root at $c = 0$, in agreement with our somewhat heuristic argument above. So, both the source and the VEV carry the same scaling weight, which is at first sight a little puzzling. It turns out, however, that the case $d = 2$ can be dealt with using Hodge dualization, which we will consider in section 3.3.2. More generally, for a Maxwell field in an AdS$_{d+1}$ background, we see that the scaling weight of the VEV—that is to say the power of e^{-r} multiplying it—is given by

$$w_{\text{VEV}} = d - 2. \tag{3.3.3}$$

Repeating the calculation for $d = 1$, it is now no longer true that A_i has scaling weight zero, i.e. that $A_i \simeq A_{(0)i}$ as in (3.2.20); instead we find that it has scaling weight 1 which changes the entire analysis of section 3.2.1.

We now conjecture a generalization this to p-forms: assuming that the equations of motion for a p-form $C^{(p)}$ exhibit the same structure, it is clear that the scaling weight of the VEV gets modified to

$$w_{\text{VEV}} = d - 2p, \tag{3.3.4}$$

which leads us to conjecture[16] that, for general p, holographic renormalization works differently depending on whether

$$(i): d - 2p > 0 \qquad \text{or} \qquad (ii): d - 2p < 0 \qquad \text{or} \qquad (iii): d - 2p = 0. \tag{3.3.5}$$

3.3.2 *The $d = 2$ Scalar-Vector Duality & Its Generalization*

When $d = 2$—i.e. when we are in case (iii) of (3.3.5) for $p = 1$—we saw above that both the source and the VEV carry the same scaling weight. We now show how this situation can be taken care of by Hodge dualization: in three dimensions, the electromagnetic part of the Wick-rotated[17] action is

$$S_A = \frac{1}{4}\int_{\mathcal{M}} \star F \wedge F = \frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^3x\, \sqrt{g} F_{\mu\nu}F^{\mu\nu}. \tag{3.3.6}$$

The field strength two-form F is related to a dual scalar φ via[18]

$$F^{\mu\nu} = \epsilon^{\mu\nu\rho}\partial_\rho\varphi / \sqrt{g}, \tag{3.3.7}$$

so that

$$S_A = \frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^3x\, \sqrt{g}\epsilon_{\mu\nu\sigma}\epsilon^{\mu\nu\rho}\partial_\rho\varphi\partial^\sigma\varphi \tag{3.3.8}$$

$$= \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^3x\, \sqrt{g}\partial_\rho\varphi\partial^\rho\varphi, \tag{3.3.9}$$

where we have used the identity $\epsilon_{\mu\nu\sigma}\epsilon^{\mu\nu\rho} = 2\delta^\rho_\sigma$. In two spatial dimensions renormalizing our EMD model is thus equivalent to renormalizing gravity coupled to two massless scalars. The action describing such a theory is given by

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^3x\, \sqrt{-g}\left(R^{(g)} + 2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \right). \tag{3.3.10}$$

15 The first term is $\pi^i_{(4)}$ in dimensions above two.

16 I thank Kostas Skenderis for telling me about this condition. Whether the mechanism in the case of p-forms is really as simple as the one I have described remains to be seen. The general analysis of holographic renormalization of p-forms will be the topic of an upcoming paper by Skenderis and Papadimitriou [52].

17 We Wick rotate so that the Hodge dual is simpler.

18 With the convention $\epsilon^{01r} = 1$.

The ADM parametrized momenta are given by

$$\pi^{ij} = \frac{1}{2\kappa^2}\left(K\gamma^{ij} - K^{ij}\right), \quad \pi_\phi = \frac{1}{2\kappa^2}N^{-1}\left(\phi - N^i\partial_i\phi\right), \quad \pi_\varphi = \frac{1}{2\kappa^2}N^{-1}\left(\varphi - N^i\partial_i\varphi\right). \tag{3.3.11}$$

The Hamiltonian is given by

$$H = \int_{\Sigma_r} \mathrm{d}^2x\, \sqrt{-\gamma}\left(N\mathcal{H} + N_i\mathcal{H}^i\right), \tag{3.3.12}$$

where

$$\mathcal{H} = -2\kappa^2\left(\pi^{ij}\pi_{ij} - \pi^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}\pi_\varphi^2\right) + \frac{1}{2\kappa^2}\left(-R^{(\gamma)} + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{2}\partial_i\varphi\partial^i\varphi - 2\right), \tag{3.3.13}$$

$$\mathcal{H}^i = -2D_j + \pi_\phi\partial^i\phi + \pi_\varphi\partial^i\varphi. \tag{3.3.14}$$

As usual, the equations of motion for the Lagrange multiplier fields impose the first class constraints,

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0. \tag{3.3.15}$$

Gauge-fixing to $N = 1$ and $N_i = 0$, we impose AlAdS boundary conditions, which read

$$\gamma_{ij} \simeq e^{2r}g_{(0)ij}(x), \quad \phi \simeq \phi_{(0)}(x) \quad \varphi \simeq \varphi_{(0)}(x), \tag{3.3.16}$$

where x represents the coordinates on Σ_r. Carrying out the same procedure as in section 3.2, we find that

$$\mathscr{L}_{(0)} = \frac{1}{2\kappa^2}, \tag{3.3.17}$$

$$\widetilde{\mathscr{L}}_{(2)} = \frac{1}{\kappa^2}R^{(\gamma)} - \frac{1}{2\kappa^2}\left(\partial_i\phi\partial^i\phi + \partial_i\varphi\partial^i\varphi\right). \tag{3.3.18}$$

Hodge dualizing back to the vector theory, we obtain

$$\partial_i\varphi = \frac{1}{2}\epsilon_{\mu\nu i}F^{\mu\nu}\sqrt{g}, \tag{3.3.19}$$

$$\therefore \frac{1}{2\kappa^2}\partial_i\varphi\partial^i\varphi = \frac{1}{8\kappa^2}\epsilon_{\mu\nu i}\epsilon^{\rho\lambda i}F^{\mu\nu}F_{\rho\lambda} = \frac{1}{8\kappa^2}\left[2\epsilon_{rji}\epsilon^{rki}F^{rj}F_{rk}\right] = \frac{1}{8\kappa^2}\left[4F^{ri}F_{ri}\right] = \frac{1}{2\kappa^2}F^{ri}F_{ri}, \tag{3.3.20}$$

implying that the anomaly takes the form

$$\widetilde{\mathscr{L}}_{(2)} = \frac{1}{\kappa^2}R^{(\gamma)} - \frac{1}{2\kappa^2}\partial_i\phi\partial^i\phi - \frac{1}{2\kappa^2}F^{ri}F_{ri}. \tag{3.3.21}$$

To generalize this to p-forms, we note that the field strength $(p+1)$-form $F^{(p+1)} = \mathrm{d}C^{(p)}$ of the p-form $C^{(p)}$ in $d = 2p$, corresponding to case (iii) of (3.3.5), is dual to a p-form, which is not immediately useful. A more fruitful observation is that in $d = p+1$, the field strength $F^{(p+1)}$ is again dual to a scalar in the sense of (3.3.7), which means that the methods of this section generalize to a way of easily performing holographic renormalization of free massless p-form fields in AdS_{p+2}, since in this case it is equivalent to renormalizing a free massless scalar.

3.4 OUTLOOK

The results of this chapter will be important when we consider charged Lifshitz holography in chapter 6, since the renormalized electromagnetic uplift can be Scherk-Schwarz reduced to an Einstein-Proca-dilaton-Maxwell-scalar model, which admits Lifshitz solutions that correspond to $z = 0$ Schrödinger spactimes[19] in the higher dimensional theory.

More generally, there are many unsolved problems in holographic renormalization. As we demonstrated, even in the case of AlAdS spacetimes, there are subtleties regarding the renormalization of p-form fields that are not widely known. Holographic renormalization has also been undertaken for other geometries of holographic interest: Schrödinger space-times were considered in e.g. [109–111], while Lifshitz spacetimes were dealt with in e.g. [25, 27, 28, 106], but there are still several outstanding

19 And these are AlAdS.

problems. In Appendix G we provide a review of the renormalization of Lifshitz geometries using the HJ approach developed in this chapter.

An interesting generalization of the procedure described in this chapter was introduced in [28, 106] in the context of renormalization of EPD models with Lifshitz boundary conditions, and involves the identification of yet another operator $\mathfrak{d}'$ that *commutes* with, say, δ_D (generally, whatever we choose for $\mathfrak{d}$). One then performs a dual expansion in (common) eigenmodes of the two operators, which simplifies the analysis. It would be interesting to explore if such methods are also advantageous in the context of the models discussed in this chapter.

4 NEWTON-CARTAN GEOMETRY

Newton-Cartan (NC) geometry was originally developed by Cartan to provide a geometric framework for Newtonian gravity [30, 31], and in the context of non-relativistic holography, NC geometry plays a prominent rôle as the boundary geometry.

Specifically, it was shown in [23, 24] that the boundary geometry of asymptotically locally $z = 2$ Lifshitz space-times is described by an extension of NC geometry known as twistless torsional Newton-Cartan geometry (TTNC), while for for generic values of the dynamical exponent, $1 < z \leq 2$, the boundary geometry is described by torsional Newton-Cartan geometry (TNC) [29, 32]—this we explore in chapters 5 and 6.

Since TNC geometry plays a vital rôle in Lifshitz holography, we devote this chapter to its description. However, TNC geometry is both interesting and useful to study in its own right—independent of holography, it has, for example, recently been shown that TNC geometry provides an effective description of the quantum Hall effect [35, 36]. Interestingly, it was shown in [53] that making TNC dynamical corresponds to Hořava-Lifshitz (HL) gravity.

We begin the chapter in section (4.1) with a survey of how Riemannian geometry emerges from gauging the Poincaré algebra, which will be useful when replicating the procedure in order to obtain TNC geometry. This section is a significantly expanded analysis of appendix A of [53] and [54].

We then turn to a general description of *non-relativistic space-times* in section 4.2, which are the non-relativistic analogues of the familiar Lorentzian spacetimes from general relativity. Our exposition is based on [24, 33, 53] and involves elements of [112], although we do not use their terminology. Geometrically, non-relativistic space-times are described with TNC geometry, and we provide a classification of these non-relativistic structures depending on the properties of the so-called clock form, τ, which, as we will see in section 4.2.4, is intimately connected with the torsion.

In section 4.2.2, we begin with a discussion of how to obtain the Bargmann and Galilei algebras via Inönü-Wigner contraction of a centrally extended Poincaré algebra. We then proceed to gauge both the Galilei and the Bargmann algebras in sections 4.2.3 and 4.2.4, which will give us TNC geometry in the same way as Riemannian geometry appears from the gauging of the Poincaré algebra. This analysis is based mainly on [34, 53, 54, 56].

We then turn our attention to how TNC geometry can be obtained via *null reduction* of Lorentzian spacetimes in section 4.3, which will be of great use in chapter 6.

Finally, in section 4.4, we explore how field theories couple to Newton-Cartan geometries from the perspective of null reduction, which again will be a great aid to our considerations in chapter 6. These last two sections both follow [57].

4.1 WARM-UP: GAUGING THE POINCARÉ ALGEBRA TO OBTAIN GENERAL RELATIVITY

Following [54, 55] and Appendix A of [53], we proceed to derive the framework of general relativity from properties of the Poincaré algebra (see e.g. [113] for a nice review). The d-dimensional Poincaré algebra $\mathfrak{iso}(d-1,1)$ is generated by P_a and J_{ab} with the following non-zero commutators,

$$[J_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a, \tag{4.1.1}$$

$$[J_{ab}, J_{cd}] = \eta_{ac}J_{bd} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac}. \tag{4.1.2}$$

Note that the tangent space indices $a, b, c, \ldots$ in this section *include time*, in contradistinction to the rest of this chapter. The first step of the gauging procedure is to introduce the Lie algebra valued connection by

$$\mathcal{A}_\mu = P_a e^a_\mu + \frac{1}{2}M_{ab}\omega_\mu{}^{ab}, \tag{4.1.3}$$

where antisymmetry of M means that we take $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$. Mimicking the approach taken for Yang-Mills theory[1], we fist observe that the connection behaves under *local* $ISO(d-1,1)$ transformations as

$$\mathcal{A}_\mu \to L(x)\mathcal{A}_\mu L^{-1}(x) - L(x)\partial_\mu L^{-1}(x), \qquad L(x) \in ISO(d-1,1). \tag{4.1.4}$$

1 For a nice review, see e.g. the book [114].

Infinitesimally, we may write $L(x) = \mathbb{1} + \Lambda(x)$ for $\Lambda(x) \in \mathfrak{iso}(d-1,d)$, and so, under infinitesimal transformations[2], the connection transforms as

$$\delta\mathcal{A}_\mu = \partial_\mu\Lambda + [\mathcal{A}_\mu, \Lambda], \tag{4.1.5}$$

where, since $\Lambda \in \mathfrak{iso}(d-1,1)$, we may write

$$\Lambda = P_a\zeta^a(x) + \frac{1}{2}J_{ab}\sigma^{ab}(x). \tag{4.1.6}$$

Since

$$\delta\mathcal{A}_\mu = P_a\delta e^a_\mu + \frac{1}{2}J_{ab}\delta\omega_\mu{}^{ab}, \tag{4.1.7}$$

the transformation (4.1.5) with Λ given by (4.1.6) implies that

$$\delta e^a_\mu = \partial_\mu\zeta^a - \omega_\mu{}^{ab}\zeta^b + \overbrace{\sigma^a{}_b e^b_\mu}^{(*)}, \tag{4.1.8}$$

$$\delta\omega_\mu{}^{ab} = \partial_\mu\sigma^{ab} + \sigma_c{}^a\omega_\mu{}^{bc} - \sigma_c{}^b\omega_\mu{}^{ac}. \tag{4.1.9}$$

To get a feel for how these are obtained, we show how to find the term $(*)$. This term comes from the commutator $[\mathcal{A}_\mu, \Lambda] = \frac{e^a_\mu\sigma^{bc}}{2}[P_a, J_{bc}] + \ldots$, where we can use the commutator relation (4.1.1) to rewrite

$$\frac{e^a_\mu\sigma^{bc}}{2}[P_a, J_{bc}] = \frac{e^a_\mu\sigma^{bc}}{2}\eta_{a[c}P_{b]} = e^a_\mu\sigma^b{}_a P_b, \tag{4.1.10}$$

where we have used antisymmetry of σ^{ab}. In order to make connection to local space-time diffeomorphisms, we introduce a new set of local (infinitesimal) transformations, denoted $\bar{\delta}$, where we replace the local translation parameter ζ^a in Λ with a space-time vector ξ^μ via $\zeta^a = \xi^\mu e^a_\mu$, which makes the emergence of diffeomorphisms manifest[3], as we demonstrate. Writing

$$\Lambda = \xi^\mu\mathcal{A}_\mu + \Sigma, \text{ where } \Sigma = \frac{1}{2}\overbrace{\left(\sigma^{ab} - \xi^\mu\omega_\mu{}^{ab}\right)}^{=:\lambda^{ab}}J_{ab}, \tag{4.1.11}$$

we are now ready to define $\bar{\delta}\mathcal{A}_\mu$:

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu\mathcal{F}_{\mu\nu}, \tag{4.1.12}$$

where $\mathcal{F}_{\mu\nu}$ is the curvature of $\mathcal{A}_\mu$, defined in the usual manner,

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{4.1.13}$$

Writing out $\bar{\delta}$, we find

$$\bar{\delta}\mathcal{A}_\mu = \partial_\mu\Lambda + [\mathcal{A}_\mu, \Lambda] - \xi^\nu\mathcal{F}_{\mu\nu} \tag{4.1.14}$$

$$= \mathcal{A}_\nu\partial_\mu\xi^\nu + \cancel{\xi^\nu\partial_\mu\mathcal{A}_\nu} + \partial_\mu\Sigma + \cancel{\xi^\nu[\mathcal{A}_\mu, \mathcal{A}_\nu]} + [\mathcal{A}_\mu, \Sigma] - \cancel{\xi^\nu\partial_\mu\mathcal{A}_\nu} + \xi^\nu\partial_\nu\mathcal{A}_\mu - \cancel{\xi^\nu[\mathcal{A}_\mu, \mathcal{A}_\nu]} \tag{4.1.15}$$

$$= \pounds_\xi\mathcal{A}_\mu + \partial_\mu\Sigma + [\mathcal{A}_\mu, \Sigma], \tag{4.1.16}$$

where, as claimed, the transformation of $\mathcal{A}_\mu$ under diffeomorphisms generated by ξ—that is to say, the term $\pounds_\xi\mathcal{A}_\mu$—appears explicitly. We also note that the parameters λ^{ab} now correspond to infinitesimal local Lorentz transformations generated by J_{ab}.

Now, going back to the curvature $\mathcal{F}_{\mu\nu}$, we may conveniently express it in terms of curvatures pertaining to the generators P and J,

$$\mathcal{F}_{\mu\nu} = R_{\mu\nu}{}^a(P)P_a + \frac{1}{2}R_{\mu\nu}{}^{ab}(J)J_{ab}. \tag{4.1.17}$$

2 That is, ignoring terms of order $\mathcal{O}(\Lambda^2, \Lambda\partial\Lambda)$.

3 This approach first appeared in [53] and was not used in e.g. [34, 54]. We shall refer to this trick as *the $\bar{\delta}$-method*.

The explicit expressions of the curvatures are determined by plugging the connection (4.1.3) into the curvature (4.1.13) and using the Poincaré algebra:

$$\mathcal{F}_{\mu\nu} = \partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{4.1.18}$$

$$= 2P_a\partial_{[\mu}e^a_{\nu]} + J_{ab}\partial_{[\mu}\omega_{\nu]}{}^{ab} + \frac{1}{2}e^a_\mu\omega_\nu{}^{bc}\overbrace{[P_a, J_{bc}]}^{=-2\eta_{a[b}P_{c]}} + \frac{1}{4}\omega_\mu{}^{ab}\omega_\nu{}^{cd}\overbrace{[J_{ab}, J_{cd}]}^{=-4\eta_{[a[c}J_{d]b]}} + \frac{1}{2}e^a_\nu\omega_\mu{}^{bc}\overbrace{[J_{bc}, P_a]}^{2\eta_{a[b}P_{c]}} \tag{4.1.19}$$

$$= 2P_a\left(\partial_{[\mu}e^a_{\nu]} - \omega_{[\mu}{}^{ab}e_{\nu]b}\right) + J_{ab}\left(\partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ca}\omega_{\nu]}{}^{b}{}_{c}\right), \tag{4.1.20}$$

leading us to conclude that

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu}e^a_{\nu]} - 2\omega_{[\mu}{}^{ab}e_{\nu]b}, \qquad R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ca}\omega_{\nu]}{}^{b}{}_{c}, \tag{4.1.21}$$

where the factor of two in the expression for $R_{\mu\nu}{}^{ab}(J)$ comes from its definition in (4.1.17). Under $\bar{\delta}$-transformations, the quantities e^a_μ and $\omega_\mu{}^{ab}$ transform respectively as a vielbein and a spin connection: writing out (4.1.16), we find

$$\bar{\delta}\mathcal{A}_\mu = P_a\pounds_\xi e^a_\mu + \frac{1}{2}J_{ab}\pounds_\xi\omega_\mu{}^{ab} + \frac{1}{2}J_{ab}\partial_\mu\lambda^{ab} + \frac{1}{2}\lambda^{cd}\left(e^a_\mu[P_a, J_{cd}] + \frac{1}{2}\omega_\mu{}^{ab}[J_{ab}, J_{cd}]\right) \tag{4.1.22}$$

$$= P_a\pounds_\xi e^a_\mu + \frac{1}{2}J_{ab}\pounds_\xi\omega_\mu{}^{ab} + \frac{1}{2}J_{ab}\partial_\mu\lambda^{ab} + P_a\lambda^a{}_b e^b_\mu + \lambda_c{}^b\omega_\mu{}^{ca}J_{ab}, \tag{4.1.23}$$

and so, using $\bar{\delta}\mathcal{A}_\mu = P_a\bar{\delta}e^a_\mu + \frac{1}{2}J_{ab}\bar{\delta}\omega_\mu{}^{ab}$, we immediately get:

$$\bar{\delta}e^a_\mu = \pounds_\xi e^a_\mu + \lambda^a{}_b e^b_\mu, \tag{4.1.24}$$

$$\bar{\delta}\omega_\mu{}^{ab} = \pounds_\xi\omega_\mu{}^{ab} + \partial_\mu\lambda^{ab} + 2\lambda^{[a}{}_c\omega_\mu{}^{|c|b]}. \tag{4.1.25}$$

Thus, e^a_μ does indeed transforms as a vielbein under $\bar{\delta}$–transformations, with λ^{ab} an infinitesimal local Lorentz transformation. Next, we introduce a space-time covariant derivative via

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu, \tag{4.1.26}$$

which transforms (by requirement) covariantly under $\bar{\delta}$–transformations. By this, we mean that

$$\bar{\delta}\left(\mathcal{D}_\mu e^a_\nu\right) = \pounds_\xi\left((\mathcal{D}_\mu e^a_\nu)\right) + \lambda^a{}_b(\mathcal{D}_\mu e^b_\nu). \tag{4.1.27}$$

This allows us determine the $\bar{\delta}$–transformation of the affine connection $\Gamma^\rho_{\mu\nu}$, since

$$\bar{\delta}\left(\mathcal{D}_\mu e^a_\nu\right) = \overbrace{\partial_\mu(\bar{\delta}e^a_\nu)}^{=(*)} - \overbrace{\bar{\delta}\left(\Gamma^\rho_{\mu\nu}e^a_\rho\right)}^{=(\dagger)} - \overbrace{\bar{\delta}\left(\omega_\mu{}^a{}_b e^b_\nu\right)}^{=(\ddagger)}. \tag{4.1.28}$$

Considering each term in isolation and applying the transformation properties of the vielbein and spin connection, (4.1.24) and (4.1.25), we find

$$(*) = (\partial_\mu\xi^\rho)(\partial_\rho e^a_\nu) + \xi^\rho(\partial_\mu\partial_\rho e^a_\nu) + e^a_\rho\partial_\nu\partial_\mu\xi^\rho + e^b_\nu\partial_\mu\lambda^a{}_b + \lambda^a{}_b\partial_\mu e^a_\nu, \tag{4.1.29}$$

whereas

$$(\dagger) = \Gamma^\rho_{\mu\nu}\left(\xi^\sigma\partial_\sigma e^a_\rho + e^a_\sigma\partial_\nu\xi^\sigma + \lambda^a{}_b e^b_\rho\right) + \bar{\delta}\Gamma^\rho_{\mu\nu}e^a_\rho, \tag{4.1.30}$$

and, finally,

$$(\ddagger) = e_{\nu b}\left(\xi^\rho\partial_\rho\omega_\mu{}^{ab} + \omega_\rho{}^{ab}\partial_\mu\xi^\rho + \partial_\mu\lambda^{ab} + \lambda^a{}_c\omega_\mu{}^{cb} - \cancel{\lambda^b{}_c\omega_\mu{}^{ca}}\right) + \omega_\mu{}^a{}_b\left(\xi^\rho\partial_\rho e^b_\nu + e^b_\rho\partial_\nu\xi^\rho + \cancel{\lambda^b{}_c e^c_\nu}\right). \tag{4.1.31}$$

Colour-coded terms are equal, and will cancel out when taking the combination $(*) - (\dagger) - (\ddagger)$, leaving us with

$$\bar{\delta}\left(\mathcal{D}_\mu e^a_\nu\right) = (\partial_\mu\xi^\rho)(\partial_\rho e^a_\nu) + \xi^\rho(\partial_\mu\partial_\rho e^a_\nu) + e^a_\rho(\partial_\nu\partial_\mu\xi^\rho) + \lambda^a{}_b\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}\left(\xi^\sigma\partial_\sigma e^a_\rho + e^a_\sigma\partial_\rho\xi^\sigma + \lambda^a{}_b e^b_\rho\right) - \bar{\delta}\Gamma^\rho_{\mu\nu}e^a_\rho \tag{4.1.32}$$

$$- e_{\nu b}\left(\xi^\rho\partial_\rho\omega_\mu{}^{ab} + \omega_\rho{}^{ab}\partial_\mu\xi^\rho + \lambda^a{}_c\omega_\mu{}^{cb}\right) - \omega_\mu{}^a{}_b\left(\xi^\rho\partial_\rho e^b_\nu + e^b_\rho\partial_\nu\xi^\rho\right). \tag{4.1.33}$$

Now, compare this with the expression for $\bar{\delta}\left(\mathcal{D}_\mu e^a_\nu\right)$ of (4.1.27) written out in all its glory,

$$
\begin{aligned}
\bar{\delta}\left(\mathcal{D}_\mu e^a_\nu\right) &= \pounds_\xi\left(\left(\mathcal{D}_\mu e^a_\nu\right)\right) + \lambda^a{}_b\left(\mathcal{D}_\mu e^b_\nu\right) && (4.1.34)\\
&= \xi^\sigma\left(\partial_\sigma\partial_\mu e^a_\nu\right) - \xi^\sigma e^a_\rho\partial_\sigma\Gamma^\rho_{\mu\nu} - \xi^\sigma\Gamma^\rho_{\mu\nu}\partial_\sigma e^a_\rho - \xi^\sigma\omega_\mu{}^a{}_b\partial_\sigma e^b_\nu - \xi^\sigma e^b_\nu\partial_\sigma\omega_\mu{}^a{}_b && (4.1.35)\\
&\quad + \partial_\mu\xi^\sigma\left[\partial_\sigma e^a_\nu - \Gamma^\rho_{\sigma\nu}e^a_\rho - \omega_\sigma{}^a{}_b e^b_\nu\right] + \partial_\nu\xi^\sigma\left[\partial_\mu e^a_\sigma - \Gamma^\rho_{\mu\sigma}e^a_\rho - \omega_\mu{}^a{}_b e^b_\sigma\right] && (4.1.36)\\
&\quad + \lambda^a{}_b\left(\partial_\mu e^b_\nu - \Gamma^\rho_{\mu\nu}e^b_\rho - \omega_\mu{}^b{}_c e^c_\nu\right). && (4.1.37)
\end{aligned}
$$

Demanding that these be equal, we see that, first off, everything to do with the spin connection already matches, whereas a single term from the Lie derivative is amiss. The constraint that the two $\bar{\delta}$-variations be equal then reduces to

$$
\begin{aligned}
& e^a_\rho(\partial_\nu\partial_\mu\xi^\rho) - \Gamma^\rho_{\mu\nu}\left(\cancel{\xi^\sigma\partial_\sigma e^a_\rho} + e^a_\sigma\partial_\rho\xi^\sigma + \cancel{\lambda^a{}_b e^b_\rho}\right) - \bar{\delta}\Gamma^\rho_{\mu\nu}e^a_\rho \overset{(!)}{=} && (4.1.38)\\
&\quad - \xi^\sigma e^a_\rho\partial_\sigma\Gamma^\rho_{\mu\nu} - \cancel{\xi^\sigma\Gamma^\rho_{\mu\nu}\partial_\sigma e^a_\rho} - \Gamma^\rho_{\sigma\nu}e^a_\rho\partial_\mu\xi^\sigma - \Gamma^\rho_{\mu\sigma}e^a_\rho\partial_\nu\xi^\sigma - \cancel{\lambda^a{}_b\Gamma^\rho_{\mu\nu}e^b_\rho}, && (4.1.39)
\end{aligned}
$$

leading us to conclude that

$$
\bar{\delta}\Gamma^\rho_{\mu\nu} = \partial_\mu\partial_\nu\xi^\rho + \xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\nu}\partial_\mu\xi^\sigma + \Gamma^\rho_{\mu\sigma}\partial_\nu\xi^\sigma - \Gamma^\sigma_{\mu\nu}\partial_\sigma\xi^\rho. \tag{4.1.40}
$$

The next step is to relate the curvatures to the affine connection, which we do by imposing the following vielbein postulate

$$
\mathcal{D}_\mu e^a_\nu = 0, \tag{4.1.41}
$$

which allows for an identification of the affine connection $\Gamma^\rho_{\mu\nu}$ in terms of the spin connection $\omega_\mu{}^{ab}$. Taking the antisymmetric part of the vielbein postulate, we find

$$
\begin{aligned}
0 &= \partial_{[\mu}e^a_{\nu]} - \Gamma^\rho_{[\mu\nu]}e^a_\rho - \omega_{[\mu}{}^a{}_b e^b_{\nu]} && (4.1.42)\\
\therefore R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu}e^a_{\nu]} - 2\omega_{[\mu}{}^a{}_b e^b_{\nu]} = 2\Gamma^\rho_{[\mu\nu]}e^a_\rho, && (4.1.43)
\end{aligned}
$$

where we used the curvatures (4.1.21). Thus, we see that $R_{\mu\nu}{}^a(P)$ is really the torsion. The unique Lorentz invariant tensor we can build out of the vielbeine is the metric, $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, implying that the connection may be written as

$$
\Gamma^\rho_{\mu\nu} = e^\rho_a\partial_\mu e^a_\nu - e^\rho_a\omega_\mu{}^a{}_b e^b_\nu. \tag{4.1.44}
$$

To see what the other curvature represents, we define the "usual" covariant derivative $\nabla_\mu X_\mu := \partial_\mu X_\nu - \Gamma^\rho_{\mu\nu}X_\rho$, the commutator of which is related to the Riemann tensor (ix)

$$
[\nabla_\mu, \nabla_\nu]X_\rho = R_{\mu\nu\rho}{}^\sigma X_\sigma - \Gamma^\sigma_{[\mu\nu]}\nabla_\sigma X_\rho, \tag{4.1.45}
$$

where, explicitly,

$$
R_{\mu\nu\sigma}{}^\rho = 2\partial_{[\nu}\Gamma^\rho_{\mu]\sigma} + 2\Gamma^\rho_{[\rho|\lambda|}\Gamma^\lambda_{\mu]\sigma}, \tag{4.1.46}
$$

which, using the curvatures (4.1.21) and imposing the vielbein postulate (4.1.41), can be rewritten as

$$
R_{\mu\nu\rho}{}^\sigma = -e_{\rho a}e^\sigma_b R_{\mu\nu}{}^{ab}(J), \tag{4.1.47}
$$

implying that $R_{\mu\nu}{}^{ab}(J)$ is the Riemann curvature two-form. Further, the vielbein postulate implies—due to antisymmetry of the spin connection in the flat indices—that the metric is covariantly conserved with respect to ∇_μ, i.e.

$$
\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu}g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu}g_{\lambda\mu} = 2\omega_{\rho ab}e^a_{(\mu}e^b_{\nu)} = 0, \tag{4.1.48}
$$

where we have used the affine connection (4.1.44). This fixes the symmetric part of the connection and makes it equal to the Levi-Civita connection, while leaving the torsion unfixed. However, imposing the curvature constraint $R_{\mu\nu}{}^a(P) = 0$ then corresponds to setting the torsion equal to zero, and, in turn, makes the spin connection entirely determined by the vielbeine in the usual manner. By imposing Einstein's equations, the theory can be put on-shell.

4.2 NON-RELATIVISTIC SPACE-TIMES & NEWTON-CARTAN GEOMETRY

In this section, we introduce the appropriate geometric framework for the description of non-relativistic physics. Our exposition begins with a discussion of non-relativistic space-times and their geometric descriptions—see also [112, 115, 116] for similar work, where a slightly different terminology is used.

We then turn to a more detailed discussion of Newton-Cartan geometry as obtained from gauging appropriate algebras in the spirit of section 4.1. For a complementary analysis of Newton-Cartan geometry using frame bundles, we refer the reader to [117].

4.2.1 *Non-Relativistic Space-Times*

A $(d+1)$-dimensional non-relativistic space-time consists of a triad $(\mathcal{M}, \tau, h^{-1})$, where $\mathcal{M}$ is a $(d+1)$-dimensional manifold, τ is a nowhere-vanishing one-form on $\mathcal{M}$ known as the *clock form*, and h^{-1} is a rank-two contravariant symmetric tensor whose kernel is spanned by τ, i.e. $h^{-1}(\tau, \cdot) = 0$, or, in components[4],

$$h^{\mu\nu}\tau_\nu = 0. \tag{4.2.1}$$

The doublet (τ, h^{-1}) may be thought of as a degenerate metric structure, and thus is not invertible. We may, however, define the *projective inverse dyad* (v, h) satisfying

$$v^\mu \tau_\mu = -1, \quad h_{\mu\rho}h^{\rho\nu} = \delta^\nu_\mu + \tau_\mu v^\nu. \tag{4.2.2}$$

These are not unique [117] and can be boosted to take a different form. The fields (τ, v, h^{-1}, h) (with a proper choice of connection) realize a torsional Newton-Cartan (TNC) geometry, and it is often advantageous to trade the h's for vielbeine, defined through $h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu$ and $h^{\mu\nu} = \delta^{ab}e^\mu_a e^\nu_b$—something we will make extensive use of.

Now, consider two points $A, B \in \mathcal{M}$ connected by a path γ. Parametrizing the path γ by some parameter λ, we interpret the integral

$$T_\gamma = \int_\gamma \tau = \int_\gamma \tau_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\mathrm{d}\lambda, \tag{4.2.3}$$

as the proper time that passes when going from A to B along γ.

Since there is no restriction on the clock form τ, we will refer to this type of space-time as a TNC space-time. It is so far purely "metric" in nature (i.e. we have not introduced any connections, so no notion of parallelism is involved). Depending on the properties of the clock form τ, we now proceed to introduce two special cases of the TNC space-time, which, when introducing a connection, will be in a one-to-one correspondence with the various forms of TNC geometry classified by the torsion of the connection (see section 4.2.3), starting with

Definition: *A **TTNC space-time** is a non-relativistic space-time, whose clock form satisfies the Frobenius integrability condition, $\tau \wedge d\tau = 0$, which in component form reads $\tau_{[\rho}\partial_\mu\tau_{\nu]} = 0$. This is equivalent to the statement that τ be hypersurface orthogonal (HSO).*

By Frobenius' theorem [118], the one-form thus defines a foliation of $\mathcal{M}$ by a family of codimension one hypersurfaces, or *leaves*. Each leaf corresponds to an absolute space, in the following sense: by Frobenius' theorem, we may *locally* write the clock form as $\tau = f\mathrm{d}t$ for some $f, t \in C^\infty(\mathcal{M})$, where f is the time unit and t is the *absolute time*. This absolute time is fixed on each leaf, and it is in this sense that they are absolute spaces. Furthermore, the metric h (or rather, the pullback of h to the appropriate leaf) is non-degenerate on these hypersurfaces since $\tau = \mathrm{d}t = 0$, so the completeness relation (4.2.2) implies that $h_{\mu\nu}$ defines a (torsion free) Riemannian geometry on each absolute space. Finally, the most restrictive case is characterized by a *closed* clock form:

Definition: *An **NC space-time** is a non-relativistic space-time, whose clock form is exact, i.e. $\tau = dt$, where t is the absolute time.*

This means—by Stokes theorem—that the proper time (4.2.3) between two points $A, B \in \mathcal{M}$ is independent of the path one follows, which leads to a concept of absolute time not present in the more

4 Note that $h^{-1} = h^{\mu\nu}\partial_\mu \otimes \partial_\nu$, so $h^{\mu\nu}$ are the components of h^{-1}.

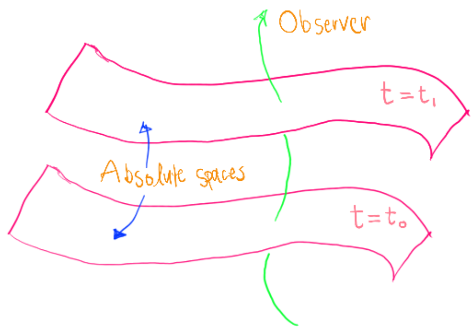


Figure 4.1: Foliation of a TTNC space-time by absolute spaces.

general TNC and TTNC space-times. In order to fully realize a NC geometry, we require a connection. A convenient and illuminating way of obtaining NC geometry in its various incarnations is to gauge an appropriate algebra, which we describe in the next section.

4.2.2 *Newton-Cartan Geometry from Local Algebras*

The most general version of Newton-Cartan geometry, known as *torsional Newton-Cartan geometry* (TNC), can be obtained in several ways. The route we take involves adding torsion to the analysis of [54], which concerns the gauging of the Bargmann algebra. TNC can also be obtained by gauging the Schrödinger algebra which is locally scale invariant; an endeavour undertaken in [34] (see also chapter 6, where we gauge the Schrödinger algebra for $z = 2$). The latter approach is more computationally involved, but reveals a richer structure which is the same as the one encounters in general-z Lifshitz holography.

4.2.2.1 *The Bargmann Algebra from Ínönü-Wigner Contraction*

It is possible [54] to obtain the Bargmann algebra by extending the $(d+1)$-dimensional Poincaré algebra $\mathfrak{iso}(d,1)$ of (4.1.1) via a direct sum with a commutative subalgebra $\mathfrak{g}_N$ generated by N (occasionally referred as the mass for reasons we will explain shortly),

$$\mathfrak{iso}(d,1) \to \mathfrak{iso}(d,1) \oplus \mathfrak{g}_N. \tag{4.2.4}$$

To get to the Bargmann algebra, we perform the following Ínönü-Wigner contraction (see [119])

$$\overbrace{P_0 \to \frac{1}{\varepsilon^2} N + H, \quad P_a \to \frac{1}{\varepsilon} P_a, \quad J_{0a} \to \frac{1}{\varepsilon} G_a,}^{\varepsilon \to 0} \tag{4.2.5}$$

where the contraction parameter ε corresponds to the reciprocal speed of light. This choice is inspired by the non-relativistic approximation of P_0 for a free particle of mass[5] m:

$$P_0 = \sqrt{c^2 P_a P^a + m^2 c^4} \approx mc^2 + \frac{P_a P^a}{2m}. \tag{4.2.6}$$

The algebra thus obtained is the Bargmann algebra $\mathfrak{barg}(d,1)$, and is generated by H (time translations), P_a (spatial rotations), G_a (Galilean boosts), J_{ab} (spatial rotations) and, finally, N (mass/particle number). The contraction produces the following non-vanishing commutators

$$[H, G_a] = P_a, \quad [J_{ab}, G_c] = 2\delta_{c[a} G_{b]}, \quad [J_{ab}, P_c] = 2\delta_{c[a} P_{b]}, \quad [J_{ab}, J_{cd}] = 4\delta_{[a[d} J_{c]b]}, \quad [P_a, G_b] = N\delta_{ab}. \tag{4.2.7}$$

These are obtained from the commutation relations of the Poincaré algebra (4.1.1); for example, we have that

$$[P_a, G_b] = [\varepsilon P_a, \varepsilon J_{0b}] = \varepsilon^2 \Big(\overbrace{\eta_{0a}}^{=0} P_b - \delta_{ab} \left(\frac{1}{\varepsilon^2} N + H \right) \Big) \to \delta_{ab} N. \tag{4.2.8}$$

5 Corresponding to N.

Setting $N = 0$, we obtain the $(d+1)$-dimensional Galilean algebra, $\mathfrak{gal}(d,1)$. It is interesting to note that N is required to obtain massive representations of the Galilean algebra, for which there is a nice argument: following [54], we can understand this by considering the action for a non-relativistic free particle of mass m,

$$S = \int \mathrm{d}t\ L = \frac{1}{2}\int \mathrm{d}t\ m\dot{x}^2, \tag{4.2.9}$$

which is invariant under $\mathrm{Gal}(d,1)$. However, the Lagrangian L is *not* invariant; rather, it transforms as a total derivative under infinitesimal Galilean boosts, $\delta x^a = v^a t$, since

$$\delta L = m\dot{x}_a\delta\dot{x}^a = \frac{\mathrm{d}}{\mathrm{d}t}\left(m x_a v^a\right), \tag{4.2.10}$$

which implies that the naïve Nöther charge $\mathcal{Q}_{\text{naïve}} = p_a\delta x^a = m\dot{x}_a v^a t$ gets an additional boundary term, so that the correct boost Nöther charge takes the form

$$\mathcal{Q}_G = p_a v^a t - m x_a v^a. \tag{4.2.11}$$

The translation Noether charge is straightforward, i.e. $\mathcal{Q}_P = p_a\zeta^a$ for infinitesimal translations $\delta x^a = \zeta^a$, which means that their Poisson bracket reads

$$\{\mathcal{Q}_P, \mathcal{Q}_G\}_{\text{PB}} = \frac{\partial\mathcal{Q}_P}{\partial x^a}\frac{\partial\mathcal{Q}_G}{\partial p_a} - \frac{\partial\mathcal{Q}_P}{\partial p_a}\frac{\partial\mathcal{Q}_G}{\partial x^a} = m v^a\zeta_a, \tag{4.2.12}$$

which is in accord with the commutator (4.2.8).

4.2.3 *Gauging Galilei*

Mimicking the approach in section 4.1, we start by defining the following Lie algebra valued connection,

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a\omega_\mu{}^a + \frac{1}{2}J_{ab}\omega_\mu{}^{ab}. \tag{4.2.13}$$

For $L(x) \in \mathfrak{gal}(d,1)$, the connection transforms as $\mathcal{A}_\mu \to L(x)\mathcal{A}_\mu L^{-1}(x) - L(x)\partial_\mu L^{-1}(x)$, and so, taking $L(x)$ to be infinitesimal, $L(x) = \mathbb{1} + \Lambda(x)$ for $\Lambda(x) \in \mathfrak{gal}_2(d,1)$, we get

$$\delta\mathcal{A}_\mu = \partial_\mu\Lambda + [\mathcal{A}_\mu, \Lambda], \tag{4.2.14}$$

i.e. $\mathcal{A}_\mu$ transforms in the adjoint. Now, since $\Lambda(x) \in \mathfrak{gal}(d,1)$, we may write

$$\Lambda(x) = H\zeta(x) + P_a\zeta^a(x) + G_a\tilde{\lambda}^a(x) + \frac{1}{2}J_{ab}\tilde{\lambda}^{ab}(x). \tag{4.2.15}$$

Next, we make contact with local space-time diffeomorphisms via the introduction of the $\bar{\delta}$-transformation, where we replace the local translation parameters ζ^a in Λ with a space-time vector ξ^μ defined via $\zeta^a = \xi^\mu e^a_\mu$, allowing us to write

$$\Lambda = \xi^\mu\mathcal{A}_\mu + \Sigma, \tag{4.2.16}$$

which implies that

$$\Sigma = G_a\lambda^a + \frac{1}{2}J_{ab}\lambda^{ab}, \tag{4.2.17}$$

where the un-tilded parameters of (4.2.17) are related to their tilded cousins of (4.2.15) in a manner similar to the way local Lorentz transformations emerged in the case of the Poincaré algebra in (4.1.11). The $\bar{\delta}$-transformation—compare eqs. (4.1.12) and (4.1.16)—then becomes

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu\mathcal{F}_{\mu\nu} = \pounds_\xi\mathcal{A}_\mu + \partial_\mu\Sigma + [\mathcal{A}_\mu, \Sigma], \tag{4.2.18}$$

where $\mathcal{F}_{\mu\nu}$ is the Yang-Mills curvature,

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= 2\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] &(4.2.19)\\ &= HR_{\mu\nu}(H) + P_aR_{\mu\nu}{}^a(P) + G_aR_{\mu\nu}{}^a(G) + \frac{1}{2}J_{ab}R_{\mu\nu}{}^{ab}(J). &(4.2.20)\end{aligned}$$

We summarise this construction in the table below:

Table 4.1: Generators of $\mathfrak{gal}(d,1)$ with their associated gauge fields, local parameters and covariant curvatures.

Symmetry	Generators	Gauge Field	Parameters	Curvatures
Time translations	H	τ_μ	$\zeta(x)$	$R_{\mu\nu}(H)$
Spatial translations	P_a	e^a_μ	$\zeta^a(x)$	$R_{\mu\nu}{}^a(P)$
Boosts	G_a	$\omega_\mu{}^a$	$\lambda^a(x)$	$R_{\mu\nu}{}^a(G)$
Spatial rotations	J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x)$	$R_{\mu\nu}{}^{ab}(J)$

Writing out the expression for $\bar{\delta}\mathcal{A}_\mu$ in (4.2.19) with the help of our expressions for $\mathcal{A}_\mu$ and Σ—as we did for Poincaré in (4.1.18)–(4.1.20)—we find the following curvatures:

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}, \tag{4.2.21}$$

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu}e^a_{\nu]} - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]}, \tag{4.2.22}$$

$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{c[a}\omega_{\nu]}{}^{b]}{}_c, \tag{4.2.23}$$

$$R_{\mu\nu}{}^a(G) = 2\partial_{[\mu}\omega_{\nu]}{}^a + 2\omega_{[\mu}{}^b\omega_{\nu]}{}^a{}_b. \tag{4.2.24}$$

Similarly, by writing out $\mathcal{A}_\mu$ and Σ in (4.2.18) and identifying coefficients in front of the generators, we find the following transformations of the gauge fields:

$$\bar{\delta}\tau_\mu = \pounds_\xi\tau_\mu, \tag{4.2.25}$$

$$\bar{\delta}e^a_\mu = \pounds_\xi e^a_\mu + \lambda^a\tau_\mu + \lambda^a{}_b e^b_\mu, \tag{4.2.26}$$

$$\bar{\delta}\omega_\mu{}^{ab} = \pounds_\xi\omega_\mu{}^{ab} + \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]}{}_c, \tag{4.2.27}$$

$$\bar{\delta}\omega_\mu{}^a = \pounds_\xi\omega_\mu{}^a + \partial_\mu\lambda^a - \lambda^b\omega_\mu{}^a{}_b + \lambda^a{}_b\omega_\mu{}^b. \tag{4.2.28}$$

The gauge fields τ_μ and e^a_μ transform under spatial rotations and Galilean boosts as the Newton-Cartan clock form and vielbeine, respectively [54], and we identify them as such. Since they are of rank 1 and rank d, respectively, they are not invertible in a $(d+1)$–dimensional spacetime, but we can define projective inverses v^μ and e^μ_a, satisfying the relations

$$v^\mu\tau_\mu = -1, \qquad v^\mu e^a_\mu = 0, \qquad \tau_\mu e^\mu_a = 0, \qquad e^a_\mu e^\mu_b = \delta^a_b, \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu + v^\mu\tau_\nu. \tag{4.2.29}$$

The projective inverses transform under $\bar{\delta}$-transformations in the following manner,

$$\bar{\delta}v^\mu = \pounds_\xi v^\mu + \lambda^a e^\mu_a, \tag{4.2.30}$$

$$\bar{\delta}e^\mu_a = \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b, \tag{4.2.31}$$

which are derived by considering[6] the relations $0 = \bar{\delta}\left(v^\mu\tau_\mu\right)$ and $\bar{\delta}\left(v^\mu e^a_\mu\right)$ and using the identities (4.2.29). The curvatures (4.2.21)–(4.2.24) are subject to the usual Bianchi identity $\mathrm{d}_D\mathcal{F} = 0$ for d_D the exterior covariant derivative, which in local coordinates reads

$$0 = D_{[\mu}\mathcal{F}_{\nu\rho]} = \partial_{[\mu}\mathcal{F}_{\nu\rho]} + \left[\mathcal{A}_{[\mu}, \mathcal{F}_{\nu\rho]}\right], \tag{4.2.32}$$

where D is the gauge covariant derivative in the adjoint representation. In order to identify the structure of the covariant derivative acting on the gauge fields, we use the standard relation $D_\mu\mathcal{A}_\nu = \partial_\mu\mathcal{A}_\nu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$, which allows us to identify additional terms in the covariant derivatives that ensure that they transform covariantly under $\bar{\delta}$-transformations, (4.2.25)–(4.2.28). By taking into account all the commutators of $\mathfrak{gal}(d,1)$ that are proportional to H and P_a, we find the following expressions for the covariant derivatives,

$$\mathcal{D}_\mu\tau_\nu = \partial_\mu\tau_\nu - \Gamma^\rho_{\mu\nu}\tau_\rho, \tag{4.2.33}$$

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu - \omega_\mu{}^a\tau_\nu, \tag{4.2.34}$$

$$\mathcal{D}_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\lambda}v^\lambda - \omega_\mu{}^a e^\nu_a, \tag{4.2.35}$$

$$\mathcal{D}_\mu e^\nu_a = \partial_\mu e^\nu_a + \Gamma^\nu_{\mu\lambda}e^\lambda_a + \omega_\mu{}^b{}_a e^\nu_b. \tag{4.2.36}$$

6 And, for simplicity, ignoring diffeomorphisms.

We can explicitly check that they transform covariantly; for example, as we had for the Poincaré algebra, $\mathcal{D}_\mu \tau_\nu$ should transform only under diffeomorphisms, i.e. we require that

$$\begin{aligned}
\bar{\delta}\left(\mathcal{D}_\mu \tau_\nu\right) &= \pounds_\xi \left(\mathcal{D}_\mu \tau_\nu\right) && (4.2.37)\\
&= \xi^\rho \partial_\rho \left(\mathcal{D}_\mu \tau_\nu\right) + \partial_\mu \xi^\rho \left(\mathcal{D}_\rho \tau_\nu\right) + \partial_\nu \xi^\rho \left(\mathcal{D}_\mu \tau_\rho\right) && (4.2.38)\\
&= \xi^\rho \left(\partial_\rho \partial_\mu \tau_\nu\right) - \xi^\rho \left(\partial_\rho \Gamma^\lambda_{\mu\nu}\right) \tau_\lambda - \xi^\rho \Gamma^\lambda_{\mu\nu} \partial_\rho \tau_\lambda && (4.2.39)\\
&\quad + \partial_\mu \xi^\rho \left(\partial_\rho \tau_\nu - \Gamma^\lambda_{\rho\nu} \tau_\lambda\right) + \partial_\nu \xi^\rho \left(\partial_\mu \tau_\rho - \Gamma^\lambda_{\mu\rho} \tau_\lambda\right). && (4.2.40)
\end{aligned}$$

By using (4.2.25)–(4.2.28), the $\bar{\delta}$-variation of the covariant derivative may equivalently be expressed in the following manner,

$$\bar{\delta}\left(\mathcal{D}_\mu \tau_\nu\right) = \partial_\mu \bar{\delta} \tau_\nu - \bar{\delta} \Gamma^\rho_{\mu\nu} \tau_\rho - \Gamma^\rho_{\mu\nu} \bar{\delta} \tau_\rho, \tag{4.2.41}$$

which, when equated with (4.2.40), produces the same transformation property (4.1.40) for $\Gamma^\nu_{\mu\lambda}$ that we found in section 4.1. We now impose the vielbein postulates,

$$\begin{aligned}
\mathcal{D}_\mu \tau_\nu &= 0, && (4.2.42)\\
\mathcal{D}_\mu e^a_\nu &= 0. && (4.2.43)
\end{aligned}$$

These postulates allow us to express the affine connection in terms of $\omega_\mu{}^a$ and $\omega_\mu{}^{ab}$ by using the identities (4.2.29). In particular, multiplying the first postulate (4.2.42) with v^λ produces the relation $v^\lambda \partial_\mu \tau_\nu = -\Gamma^\lambda_{\mu\nu} + e^\lambda_a \overbrace{\Gamma^\rho_{\mu\nu} e^a_\rho}^{(*)}$, while the second postulate (4.2.43) can be rearranged to give an expression for $(*)$. This produces the result

$$\Gamma^\lambda_{\mu\nu} = -v^\lambda \partial_\mu \tau_\nu + e^\lambda_a \left(\partial_\mu e^a_\nu - \omega_\mu{}^a{}_b e^b_\nu - \omega_\mu{}^a \tau_\nu\right). \tag{4.2.44}$$

The vielbein postulates also allow for a identification of the curvatures (4.2.21) and (4.2.22) in terms of the affine connection: the postulates (4.2.42) and (4.2.43) imply that

$$\begin{aligned}
R_{\mu\nu}(H) &= 2\Gamma^\rho_{[\mu\nu]} \tau_\rho, && (4.2.45)\\
R_{\mu\nu}{}^a(P) &= 2\Gamma^\rho_{[\mu\nu]} e^a_\rho, && (4.2.46)
\end{aligned}$$

so they are related to the torsion tensor in the following manner; contracting the curvature (6.4.63) with e^λ_a and using the geometric identities (4.2.29), we get

$$\begin{aligned}
R_{\mu\nu}{}^a e^\lambda_a &= 2\Gamma^\rho_{[\mu\nu]} e^a_\rho e^\lambda_a && (4.2.47)\\
&= 2\Gamma^\rho_{[\mu\nu]} \left(\delta^\lambda_\rho + v^\lambda \tau_\rho\right) && (4.2.48)\\
&= 2\Gamma^\lambda_{[\mu\nu]} + v^\lambda \underbrace{2\Gamma^\rho_{[\mu\nu]} \tau_\rho}_{=R_{\mu\nu}(H)} && (4.2.49)\\
\therefore 2\Gamma^\lambda_{[\mu\nu]} &= -v^\lambda R_{\mu\nu}(H) + e^\lambda_a R_{\mu\nu}{}^a(P). && (4.2.50)
\end{aligned}$$

We now introduce a new covariant derivative ∇_μ involving only the affine connection $\Gamma^\rho_{\mu\nu}$, which means that writing out the first vielbein postulate (4.2.42) using the definition (4.2.33) gives us the relation

$$\nabla_\mu \tau_\nu = 0, \tag{4.2.51}$$

while antisymmetry of $\omega_\mu{}^{ab}$ means that, by the second vielbein postulate (4.2.43), the object $h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b$ satisfies

$$\nabla_\mu h^{\nu\rho} = 0,, \tag{4.2.52}$$

where eqs. (4.2.51) and (4.2.52) are the TNC version of metric compatibility. With this new derivative at hand, we now turn to the identification of the other curvatures. To that end, we begin by computing the Riemann tensor,

$$[\nabla_\mu, \nabla_\nu] X_\rho = R_{\mu\nu\sigma}{}^\rho X_\rho - 2\Gamma^\rho_{[\mu\nu]} \nabla_\rho X_\sigma, \tag{4.2.53}$$

for an arbitrary one-form X_ρ, where

$$R_{\mu\nu\sigma}{}^{\rho}X_\rho = -\partial_\mu\Gamma^\rho_{\nu\sigma} + \partial_\nu\Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{4.2.54}$$

which, upon using our expression for $\Gamma^\rho_{\mu\nu}$ (4.2.44) as well as the curvatures (4.2.23)–(4.2.24), produces the result

$$R_{\mu\nu\sigma}{}^{\rho} = e^\rho_a\tau_\sigma R_{\mu\nu}{}^{a}(G) - e_{\sigma a}e^\rho_b R_{\mu\nu}{}^{ab}(J). \tag{4.2.55}$$

4.2.4 *Extension to Bargmann and the Affine Connection*

We now extend the analysis above to the Bargmann algebra (4.2.7) and include the central element N. We denote the associated gauge connection by m_μ, so that

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a\omega_\mu{}^a + \frac{1}{2}J_{ab}\omega_\mu{}^{ab} + Nm_\mu, \tag{4.2.56}$$

$$\Sigma = G_a\lambda^a + \frac{1}{2}J_{ab}\lambda^{ab} + N\sigma, \tag{4.2.57}$$

$$\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{4.2.58}$$

$$= HR_{\mu\nu}(H) + P_aR_{\mu\nu}{}^a(P) + G_aR_{\mu\nu}{}^a(G) + \frac{1}{2}J_{ab}R_{\mu\nu}{}^{ab}(J) + NR_{\mu\nu}(N). \tag{4.2.59}$$

where σ is the parameter associated with the N transformation. As N is central, all the results of the previous section are unchanged; in particular the transformations (4.2.25)–(4.2.28) are the same with the augmentation

$$\bar{\delta}m_\mu = \pounds_\xi m_\mu + \partial_\mu\sigma + e^a_\mu\lambda_a. \tag{4.2.60}$$

To make contact with TNC geometry arising in Lifshitz holography as it appears in [23, 24, 29, 33], it is convenient to introduce a background Stückelberg field χ transforming as

$$\bar{\delta}\chi = \pounds_\xi\chi + \sigma, \tag{4.2.61}$$

and to define

$$M_\mu = m_\mu - \partial_\mu\chi, \tag{4.2.62}$$

which is invariant under local N transformations. It was shown in [34] that this has the effect of replacing m_μ with M_μ everywhere. It is convenient to define a series of Galilean boost invariant objects,

$$\hat{v}^\mu = v^\mu - h^{\mu\nu}M_\nu, \quad \hat{e}^a_\mu = e^a_\mu - M_\nu e^{\nu a}\tau_\mu, \quad \tilde{\Phi} = -v^\mu M_\mu + \frac{1}{2}h^{\mu\nu}M_\mu M_\nu, \tag{4.2.63}$$

$$h^{\mu\nu} = \delta^{ab}e^\mu_a e^\nu_b, \quad \bar{h}_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu - \tau_\mu M_\nu - \tau_\nu M_\mu. \tag{4.2.64}$$

These objects are invariant under local Galilean boosts (we will discuss their properties in a holographic context in section 5.2.4). The quantity $\tilde{\Phi}$ is closely related to the Newtonian potential when the space-time is flat [33, 34]; for this reason we shall refer to it as the Newtonian potential. We note that it represents the component of M_μ that cannot be boosted away. Now, the next crucial observation is that the objects $(\hat{e}^a_\mu, \hat{v}^\mu, \tau_\mu, e^\mu_a)$ form an orthonormal set, i.e. they satisfy the same orthogonality relations as in (4.2.29); for example we have that

$$\hat{v}^\mu\tau_\mu = v^\mu\tau_\mu - \delta^{ab}\overbrace{e^\mu_a\tau_\mu}^{=0}e^\nu_b = v^\mu\tau_\mu = -1. \tag{4.2.65}$$

The invariant objects further satisfy the following useful relations:

$$h^{\nu\rho}\bar{h}_{\rho\mu} = \delta^\nu_\mu + \hat{v}^\nu\tau_\mu, \quad \hat{v}^\mu\bar{h}_{\mu\nu} = 2\tau_\nu\tilde{\Phi}, \quad \hat{e}^a_\mu\hat{e}_{\nu a} = \bar{h}_{\mu\nu} + 2\tilde{\Phi}\tau_\mu\tau_\nu, \quad -\hat{v}^\nu\tau_\mu + \hat{e}^a_\mu e^\nu_a = \delta^\nu_\mu. \tag{4.2.66}$$

The first of these, for example, is derived in the following manner

$$h^{\nu\rho}\bar{h}_{\rho\mu} = e^\nu_a e^a_\mu - e^{a\nu}\overbrace{e^\rho_a\tau_\rho}^{=0}M_\mu - \tau_\mu M_\rho e^{\rho a}e^\nu_a \tag{4.2.67}$$

$$= \delta^\nu_\mu + (v^\nu - M_\rho h^{\rho\nu})\tau_\mu \tag{4.2.68}$$

$$= \delta^\nu_\mu + \hat{v}^\nu\tau_\mu, \tag{4.2.69}$$

where we have used (4.2.29). From the invariant objects we have considered so far, the most general affine and metric compatible (in the sense of (4.2.51) and (4.2.52)) connection was constructed in [33, 53] and has the form

$$\Gamma^{\rho}_{\mu\nu} = -v^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\lambda}\left(\partial_{\mu}h_{\nu\lambda} + \partial_{\nu}h_{\mu\lambda} - \partial_{\lambda}h_{\mu\nu}\right) + W^{\rho}_{\mu\nu}, \tag{4.2.70}$$

$$W^{\rho}_{\mu\nu} = \frac{1}{2}h^{\rho\lambda}\left(\tau_{\mu}K_{\lambda\nu} + \tau_{\nu}K_{\lambda\mu} + L_{\lambda\mu\nu}\right), \tag{4.2.71}$$

$$K_{\mu\nu} = -K_{\nu\mu}, \quad L_{\lambda\mu\nu} = -L_{\nu\mu\lambda}, \tag{4.2.72}$$

where $K_{\mu\nu}$ and $L_{\lambda\mu\nu}$ transform as tensors under diffeomorphisms. The object $W^{\rho}_{\mu\nu}$ is known as the *pseudo-contortion* tensor in analogy with the object that arises in Riemannian geometry [56]. We now see that TNC connections (4.2.70) in general have non-vanishing torsion, since for any choice of pseudo-contortion, we have

$$2\Gamma^{\rho}_{[\mu\nu]}\tau_{\lambda} = \partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}, \tag{4.2.73}$$

which makes the connection between the clock form and torsion manifest. In summary: for Newton-Cartan (NC) geometry, the torsion vanishes since τ is closed, $\mathrm{d}\tau = 0$, while in twistless torsional Newton-Cartan (TTNC) geometry, τ obeys the Frobenius condition $\tau \wedge \mathrm{d}\tau = 0$ implying that the *twist* vanishes (see also chapter 5), $h^{\mu\rho}h^{\nu\sigma}(\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}) = 0$, and, finally, torsional Newton-Cartan (TNC) geometry, where no restrictions are imposed on τ. This result is worthy of tabulation:

Table 4.2: The three Newton-Cartan geometries.

Geometry	Constr. on τ	Abs. time	Abs. space	Torsion
TNC	None	NO	NO	YES
TTNC	$\tau \wedge \mathrm{d}\tau = 0$	NO	YES	YES
NC	$\mathrm{d}\tau = 0$	YES	YES	NO

There exists a unique TNC connection linear in M_{μ} given by [34, 120]

$$\bar{\Gamma}^{\rho}_{\mu\nu} = -\hat{v}^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right), \tag{4.2.74}$$

which, from the perspective of the Noether procedure, is the minimal TNC connection [56]. This connection is obtained by setting

$$K_{\sigma\rho} = 2\partial_{[\sigma}M_{\rho]}, \quad L_{\sigma\mu\nu} = 2M_{\sigma}\partial_{[\mu}\tau_{\nu]} - 2M_{\mu}\partial_{[\nu}\tau_{\sigma]} + 2M_{\nu}\partial_{[\sigma}\tau_{\mu]}. \tag{4.2.75}$$

4.3 TNC GEOMETRY FROM NULL REDUCTION

An alternative route to TNC geometry is through null reduction of a Lorentzian $(d+1)$-dimensional manifold [23, 24, 57, 121, 122]. Consider to this end the following null reduction ansatz for the metric

$$ds^2 = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B = 2\tau_{\mu}\mathrm{d}x^{\mu}\left(\mathrm{d}u - m_{\nu}\mathrm{d}x^{\nu}\right) + h_{\mu\nu}\mathrm{d}x\mu\mathrm{d}x^{\nu} \tag{4.3.1}$$

$$= 2\tau_{\mu}\mathrm{d}x^{\mu}\mathrm{d}u + \bar{h}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \tag{4.3.2}$$

where $A = (u, \mu)$ and

$$h_{\mu\nu} = \delta_{ab}e^a_{\mu}e^b_{\nu}, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)}, \tag{4.3.3}$$

where $a = 1, \cdots, d$. The reduction ansatz (4.3.1) is the most general metric with $\gamma_{uu} = 0$, and we take ∂_u to be a Killing vector for this metric, which thus becomes a null Killing vector of γ_{AB}. The fields are τ_{μ} and e^a_{μ} are the vielbeine of the d-dimensional TNC geometry. The inverse metric is taken to be

$$\gamma^{uu} = 2\widetilde{\Phi}, \quad \gamma^{u\mu} = -\hat{v}^{\mu}, \quad \gamma^{\mu\nu} = h^{\mu\nu}, \tag{4.3.4}$$

where

$$\widetilde{\Phi} = -v^{\rho}m_{\rho} + \frac{1}{2}h^{\rho\sigma}m_{\rho}m_{\sigma}, \tag{4.3.5}$$

$$\hat{v}^{\mu} = v^{\mu} - h^{\mu\nu}m_{\nu}, \tag{4.3.6}$$

$$h^{\mu\nu} = \delta^{ab}e^{\mu}_a e^{\nu}_b, \tag{4.3.7}$$

where the inverse vielbeine v^μ and e^μ_a are defined via the usual relations (4.2.29). The following diffeomorphisms preserve the form of the null Killing vector,

$$u' = u - \sigma(x), \tag{4.3.8}$$
$$x'^\mu = x'^\mu(x). \tag{4.3.9}$$

Under these, the metric transforms as

$$\gamma'_{AB} = \frac{\partial x'^C}{\partial x^A}\frac{\partial x'^D}{\partial x^B}\gamma_{CD}, \tag{4.3.10}$$

so requiring

$$\gamma'_{\mu\nu} = h'_{\mu\nu} - 2\tau'_{(\mu}m'_{\nu)}, \tag{4.3.11}$$

the transformation (4.3.10) implies that under the diffeomorphisms (4.3.8)–(4.3.9)

$$\begin{aligned}\gamma'_{\mu\nu} &= \partial_\mu x'^\rho \partial_\nu x'^\sigma \gamma_{\rho\sigma} + 2\partial_{(\mu}u'\partial_{\nu)}x'^\sigma \gamma_{u\sigma} &(4.3.12)\\ &= h'_{\mu\nu} - 2\partial_\mu x'^\rho \partial_\nu x'^\sigma \tau_{(\rho}m_{\sigma)} + 2\partial_{(\mu}u'\partial_{\nu)}x'^\sigma \tau_\sigma, &(4.3.13)\end{aligned}$$

which, by (4.3.11), implies the transformation properties

$$\tau'_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\tau_\nu, \qquad m'_\mu = \frac{\partial x'^\nu}{\partial x^\mu}m_\nu - \partial_\mu\sigma. \tag{4.3.14}$$

In this sense, the m_μ vector transforms as a $U(1)$ connection under shifts of u. As we have seen, the ansatz (4.3.1) preserves the following (infinitesimal) local tangent space transformations for the vielbeine

$$\delta\tau_\mu = 0, \tag{4.3.15}$$
$$\delta e^a_\mu = \tau_\mu \lambda^a + \lambda^a{}_b e^b_\mu, \tag{4.3.16}$$
$$\delta m_\mu = \partial_\mu \sigma + \lambda_a e^a_\mu. \tag{4.3.17}$$

Similarly, the inverse vielbeine transform according to

$$\delta v^\mu = \lambda^a e^\mu_a, \tag{4.3.18}$$
$$\delta e^\mu_a = \lambda_a{}^b e^\mu_b \tag{4.3.19}$$

It is occasionally useful to work with the boost invariant vielbeine (see also table 5.2) τ_μ, $\hat{e}^a_\mu$ and their inverses $\hat{v}^\mu$, e^μ_a, where

$$\hat{e}^a_\mu = e^a_\mu - m_\nu e^{\nu a}\tau_\mu, \tag{4.3.20}$$

which does not change the orthogonality relations, i.e.

$$\hat{v}^\mu \hat{e}^a_\mu = 0, \qquad \hat{v}^\mu \tau_\mu = -1, \qquad e^\mu_a \tau_\mu = 0, \qquad e^\mu_a \hat{e}^b_\mu = \delta^b_a. \tag{4.3.21}$$

Further, the following spatial metric will prove useful

$$\hat{h}_{\mu\nu} = \delta_{ab}\hat{e}^a_\mu \hat{e}^b_\nu = \bar{h}_{\mu\nu} + 2\tilde{\Phi}\tau_\mu\tau_\nu. \tag{4.3.22}$$

Note in particular that this analysis does not impose any constraints on τ_μ. In order to complete this realization of TNC geometry, we need only introduce the (minimal) connection $\Gamma^\rho_{\mu\nu}$ of (4.2.74). We will encounter null reduction again in chapter 6 in the context of charged Lifshitz holography.

4.4 THE ENERGY-MOMENTUM TENSOR AND FIELD THEORIES ON TNC BACKGROUNDS

In this section, we consider how non-relativistic field theories couple to TNC geometry. There are multiple viable approaches, see e.g. [33, 56]; in what follows we will summarize these results and derive them from null reduction [24, 57]. In this section, we keep the spatial dimension d of the TNC background general. Given some action S describing a field theory coupled to TNC geometry, the energy-momentum tensor is defined as the response to variations with respect to the TNC fields of (4.2.63)–(4.2.64) and (4.3.22)

$$\delta_{\text{bg}} S = \int \mathrm{d}^{d+1}x\, e \left[-\tau_\nu T^\nu{}_\mu \delta\hat{v}^\mu + \left(\hat{h}_{\sigma\nu}\hat{v}^\mu T^\nu{}_\mu\right)\tau_\rho \delta h^{\rho\sigma} + \frac{1}{2}\left(\hat{h}_{\rho\nu}\hat{h}_{\sigma\lambda}h^{\lambda\mu}T^\nu{}_\mu\right)\delta h^{\rho\sigma} + \tau_\mu T^\mu \delta\tilde{\Phi}\right]. \tag{4.4.1}$$

Alternatively, the variation with respect to the unhatted TNC complex can be expressed as

$$\delta_{\text{bg}} S = \int \mathrm{d}^{d+1}x\, e \left[-\mathcal{T}_\mu \delta v^\mu + \frac{1}{2}\mathcal{T}_{\mu\nu}\delta h^{\mu\nu} + T^\mu \delta m_\mu \right]. \tag{4.4.2}$$

Using the definitions of the TNC fields (4.2.63)–(4.2.64), the relation between the two sets of responses reads

$$h^{\nu\rho}\mathcal{T}_{\rho\mu} - v^\nu \mathcal{T}_\mu = T^\nu{}_\mu + T^\nu m_\mu. \tag{4.4.3}$$

It can be shown [24, 57] (see also chapter 6) that the null reduction relates the energy-momentum tensor $T^\nu{}_\mu$ and the mass current T_μ to the higher dimensional energy-momentum tensor $t^A{}_B$ in the following manner

$$t^{\mu u} = 2\tilde{\Phi}T^\mu - \hat{v}^\sigma T^\mu{}_\sigma, \qquad t^{\mu\nu} = -\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho. \tag{4.4.4}$$

Note that t^{uu}—i.e. the response to varying γ_{uu}—is arbitrary since $\gamma_{uu} = 0$, so we need not worry about it. We now proceed to exploit the properties of the higher-dimensional energy-momentum tensor in order to extract information about the reduced objects. First off, symmetry of $t^{\mu\nu}$ implies that $t^{[\mu\nu]} = 0$, or

$$-\hat{v}^\mu T^\nu + h^{\mu\rho}T^\nu{}_\rho + \hat{v}^\nu T^\mu - h^{\nu\rho}T^\mu{}_\rho = 0. \tag{4.4.5}$$

From this, we obtain what turns out to be the boost and rotation Ward identities,

$$0 = -\hat{h}_{\mu\nu}T^\mu + \tau_\mu h^{\rho\sigma}\hat{h}_{\nu\sigma}T^\mu{}_\rho, \qquad 0 = \hat{h}_{\mu\rho}\hat{h}_{\nu\lambda}h^{\lambda\sigma}T^\rho{}_\sigma - (\mu \leftrightarrow \nu), \tag{4.4.6}$$

where the first identity is obtained by contracting (4.4.5) with $\tau_\nu \hat{h}_{\sigma\mu}$ and renaming $\sigma \leftrightarrow \nu$, while the second is obtained by contracting (4.4.5) with $\hat{h}_{\sigma\nu}\hat{h}_{\lambda\mu}$ and renaming $(\sigma,\lambda) \leftrightarrow (\mu,\nu)$ as well as dummy indices in the resulting expression. Continuing in this manner, we now consider the implications of diffeomorphism invariance and tracelessness, i.e. the relations $\nabla_A T^A{}_B = 0 = T^A{}_A$. Lowering the indices of the energy momentum tensor of (4.4.4), one finds the expressions [57]

$$t^u{}_u = 2\tilde{\Phi}\tau_\mu T^\mu - \hat{v}^\nu \tau_\mu T^\mu{}_\nu, \quad t^u{}_\nu = 2\tilde{\Phi}\tau_\mu T^\mu{}_\nu - \hat{v}^\sigma \hat{h}_{\nu\rho}T^\rho{}_\sigma + \tau_\nu \hat{v}^\rho \hat{v}^\sigma t_{\rho\sigma}, \quad t^\mu{}_u = T^\mu, \quad t^\mu{}_\nu = T^\mu{}_\nu. \tag{4.4.7}$$

as well as,

$$\nabla_A t^A{}_u = \partial_\mu \left(eT^\mu\right), \tag{4.4.8}$$

$$\nabla_A t^A{}_\mu = e^{-1}\partial_\nu \left(eT^\nu{}_\mu\right) + T^\rho{}_\nu \left(\hat{v}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu \partial_\mu \tilde{\Phi}, \tag{4.4.9}$$

$$t^A{}_A = -2\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2\tilde{\Phi}\tau_\mu T^\mu. \tag{4.4.10}$$

These become the $U(1)$, diffeomorphism and $z = 2$ dilatation Ward identities respectively, something we will be explicit about in chapter 6, where we will put this analysis to good use in the context of charged Lifshitz holography.

We briefly remark that it is possible to write the diffeomorphism Ward identity (4.4.9) in a more TNC covariant form by introducing the *Riemann-Cartan* connection instead of the minimal connection (4.2.74) [57, 121]. This connection is given by

$$\check{\Gamma}^\lambda_{\mu\rho} = -\hat{v}^\lambda \partial_\mu \tau_\rho + \frac{1}{2}h^{\nu\lambda}\left(\partial_\mu \hat{h}_{\rho\nu} + \partial_\rho \hat{h}_{\mu\nu} - \partial_\nu \hat{h}_{\mu\rho}\right) + h^{\nu\lambda}\tau_\rho K_{\mu\nu}, \tag{4.4.11}$$

where $K_{\mu\nu} = \frac{1}{2}\pounds_{\hat{v}}\hat{h}_{\mu\nu}$ is the extrinsic curvature. The Riemann-Cartan connection obeys a variant of metric compatibility, namely

$$\check{\nabla}_\mu \tau_\nu = 0, \qquad \check{\nabla}_\mu \hat{h}_{\nu\rho} = 0, \qquad \check{\nabla}_\mu \hat{v}^\nu = 0, \qquad \check{\nabla}_\mu h^{\nu\rho} = 0, \tag{4.4.12}$$

and thus the diffeomorphism Ward identity of (4.4.9) takes the form

$$\nabla_A t^A{}_\mu = \check{\nabla}_\mu T^\mu{}_\nu + 2\check{\Gamma}^\rho_{[\mu\rho]}T^\mu{}_\nu - 2\check{\Gamma}^\mu_{[\nu\rho]}T^\rho{}_\mu + \tau_\mu T^\mu \partial_\nu \tilde{\Phi}. \tag{4.4.13}$$

4.5 OUTLOOK

In this chapter, we have described TNC geometry and observed how it arises by gauging local algebras. There are many possible generalizations of this, and many of these have been pursued.

One possible extension involves adding supersymmetry: indeed, in [123] the $\mathcal{N} = 2$ super-Poincaré algebra is contracted to obtain the three-dimensional $\mathcal{N} = 2$ super-Bargmann algebra which is then subsequently gauged to construct supersymmetric Newton-Cartan geometry, while in [124], this analysis was extended to the super-Schrödinger algebra.

Another interesting development would be to gauge the Schrödinger-Virasoro algebra (see [125]) and see what kind of structure would emerge. There has also been some work on non-relativistic twistor theory using TNC geometry [126–128], and it would be interesting to make a connection to that.

Recently, TNC gravity was discussed in [129] from the perspective of Schrödinger field theories, and it would be interesting to work out the relation to HL gravity, since, as mentioned in the introduction, HL gravity and dynamical TNC geometry are equivalent.

In another direction, TNC geometry provides a natural geometric arena for the study of non-relativistic phenomena in general: as mentioned in the introduction, (T)TNC geometry has been used to model the (fractional) quantum Hall effect in [35–37].

5 (PURE) LIFSHITZ HOLOGRAPHY

In this chapter, we provide a review of Lifshitz holography for $d = 3$ as developed in [29, 32].

To whet the reader's appetite, we begin in section 5.1 with a brief exposition of Lifshitz field theory, which is of relevance to condensed matter systems. The analysis follows [130].

The rest of the chapter mainly provides a detailed exposition of the results of [29], incorporating elements from [32, 33, 57]: In section 5.2, we then proceed to develop Lifshitz holography for values of the dynamical exponent $1 < z \leq 2$. The bulk theory consists of a class of Einstein-Proca-Dilaton (EPD) models, described in section 5.2.1, which admit pure Lifshitz vacuum solutions, around which we then consider perturbations and develop the notion of asymptotically locally Lifshitz (AlLif) geometries in section 5.2.2.

This allows us to identify the sources as the leading components of the bulk fields in a near-boundary expansion. There is a caveat to this identification; in particular, it turns out that the the boundary gauge field M_μ, which, roughly speaking, is the source from the Proca field B, appears as a subleading term in the near-boundary expansion of the time-like bulk vielbein E^0. We also demonstrate how the Lorentz group contracts to the Galilei group under imposition of the AlLif boundary conditions and show how the sources transform under the contracted Lorentz group.

Having identified the sources of Lifshitz holography, we further investigate their transformation properties in section 5.2.3 under Stückelberg gauge transformations and diffeomorphisms preserving the radial gauge choice of section 5.2.2 (the so called Penrose-Brown-Henneaux (PBH) transformations), which will give rise to boundary diffeomorphisms and dilatations.

In section 5.2.4, we show how TNC geometry as described in chapter 4 emerges in the context of Lifshitz holography, and discuss the relation to the Schrödinger algebra.

We then consider the VEVs and Ward identities of Lifshitz holography in section 5.3, and see how explicit renormalization of the EPD model can be circumvented by assuming the existence of a counterterm. This allows us to determine general properties of the VEVs and is the topic of section 5.3.1.

We remark that for $z = 2$, it is possible to obtain Lifshitz geometries by means of a Scherk-Schwarz reduction of a five-dimensional $z = 0$ Schrödinger spacetime (which is an example of an AlAdS spacetime), which makes it possible to study Lifshitz holography by using properties of the more familiar AdS/CFT correspondence. This analysis is performed in [23, 24] (building on previous work [107]), and we will have more to say about this approach in chapter 6. In particular, via the reduction procedure, it is possible to obtain an explicit counterterm action from knowledge of the appropriate AdS counterterm, which is hugely advantageous.

More generally, renormalization of the EPD model AlLif boundary conditions was considered in [28]. Note that it is possible to consider Lifshitz holography starting from an Einstein-Proca model in the bulk [27, 131], and we consider the explicit renormalization of such a model in appendix G following [27, 96].

Note also that not much is known about the boundary theory; even in the case $z = 2$, where the boundary theory is a null reduction of $\mathcal{N} = 4$ SYM with a θ-term [24], explicit results are lacking.

In section 5.3.2, we define the Hollands-Ishibashi-Marolf (HIM) boundary stress tensor for our holographic theory. In particular, since energy and mass are no longer equivalent, the analogue of the familiar energy-momentum tensor in general relativity now consists of two distinct objects: the HIM stress tensor and the mass current.

Finally, in section 5.3.3, we derive the Ward identities for Lifshitz holography and covariantize them.

5.1 THE QUANTUM LIFSHITZ MODEL

Introduced in [130], the simplest Lifshitz invariant scalar field theory in $2 + 1$ dimensions is known as the *quantum Lifshitz model* and is governed by the action

$$S = \frac{1}{2} \int \mathrm{d}t\mathrm{d}^2x \left((\partial_t \varphi)^2 - \kappa^2 (\partial_i \partial^i \varphi)^2 \right), \tag{5.1.1}$$

which is invariant under global Lifshitz rescalings with $z = 2$. The corresponding Hamiltonian is given by

$$H = \frac{1}{2}\int d^2x \, \overbrace{\left[\Pi^2 + \kappa^2\right]}^{=:\mathcal{H}}, \tag{5.1.2}$$

where $\Pi = \dot{\varphi}$. The Hamiltonian (5.1.2) furnishes a field theoretic description of *Lifshitz points*, arising in e.g. the description smectic liquid crystals [132]. This model has an intriguing relation to regular two-dimensional conformal field theory due to *detailed balance* [130, 133]. This is essentially due to the fact that the Hamiltonian (5.1.2) can be written in the form

$$\mathcal{H} = Q^\dagger Q, \qquad Q = \frac{1}{\sqrt{2}}\left(\kappa\partial^i\partial_i\varphi - i\Pi\right), \tag{5.1.3}$$

where Π is the momentum operator conjugate to φ, which in the Schrödinger picture is the functional derivative $\Pi = -i\frac{\delta}{\delta\varphi}$. Hamiltonians[1] satisfying detailed balance (i.e. a condition of the form (5.1.3)) have eigenvalues satisfying $E \geq 0$, implying that if we can find a state $|0\rangle$ annihilated by $Q(x)$ for all x, then it is necessarily a ground state; that is to say, the corresponding ground state wave functional $\Psi_0[\varphi] = \langle[\varphi]|0\rangle$ satisfies $Q\Psi_0 = 0$ for all x, which is nothing but a first-order functional differential equation, $\left(\frac{\delta}{\delta\varphi} + \kappa\partial_i\partial^i\varphi\right)\Psi_0[\varphi] = 0$, which is solved by

$$\Psi_0[\varphi] = \frac{1}{\sqrt{\mathcal{Z}}}\exp\left[-\frac{\kappa}{2}\int d^2x \, \partial_i\varphi\partial^i\varphi\right], \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\exp\left[-\kappa\int d^2x \, \partial_i\varphi\partial^i\varphi\right]. \tag{5.1.4}$$

The probability of finding the ground state in the field configuration $|[\varphi]\rangle$ is given by the absolute square of the overlap,

$$|\Psi_0[\varphi]|^2 = \frac{1}{\mathcal{Z}}\exp\left[-\kappa\int d^2x \, \partial_i\varphi\partial^i\varphi\right], \tag{5.1.5}$$

from which it follows that VEVs of local operators $\mathcal{O}[\varphi(x)]$ are given by

$$\langle 0|\mathcal{O}[\varphi(x_1)]\ldots\mathcal{O}[\varphi(x_n)]|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[\varphi]\mathcal{O}[\varphi(x_1)]\ldots\mathcal{O}[\varphi(x_n)]\exp\left[-\kappa\int d^2x \, \partial_i\varphi\partial^i\varphi\right], \tag{5.1.6}$$

and so we infer that the quantum Lifshitz model is in fact equivalent to a two-dimensional theory of a free scalar, which is a CFT. This intriguing connection has been used in [135] to compute the ground state entanglement entropy of the quantum Lifshitz model (see also [136] for more recent results in that direction). Interestingly, supersymmetric Lifshitz field theories have also been studied, see [137].

5.2 LIFSHITZ HOLOGRAPHY

In this section, we describe general-z Lifshitz holography as developed in [29, 32, 33, 57].

5.2.1 *The EPD Model*

The bulk theory of our holographic setup is comprised by the Einstein-Proca-dilaton (EPD) model[2], which is a fully relativistic theory. The class of four-dimensional EPD theories are described by the following family of actions,

$$S = \int d^4x \sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}W(\Phi)B^2 - \frac{k}{2}(\partial\Phi)^2 - V(\Phi)\right), \tag{5.2.1}$$

where k is a convenient parameter (it has value three for the model we consider in chapter 6, for example), and $F = dB$ for the Proca field B_M, where capital Roman indices $M = (r, \mu)$ are used for four-dimensional bulk, and Greek indices μ for the boundary. The functions $Z(\Phi)$ and $W(\Phi)$ are

1 An example of another Hamiltonian of this form is the Rokshar-Kivelson Hamiltonian for the quantum dimer model, see [134].
2 For the Einstein-Proca models that we consider in appendix G, black brane solutions are not known analytically—these were instead studied numerically in [138, 139].

arbitrary positive functions, while the dilatonic potential is negative close to a Lifshitz solution. The equations of motion take the form:

$$G_{MN} = \frac{k}{2}\left(\partial_M\Phi\partial_N\Phi - \frac{1}{2}(\partial\Phi)^2 g_{MN}\right) - \frac{1}{2}V(\Phi)g_{MN} \tag{5.2.2}$$

$$+\frac{1}{2}Z(\Phi)\left(F_{MP}F_N{}^P - \frac{1}{4}F^2 g_{MN}\right) + \frac{1}{2}W(\Phi)\left(B_M B_N - \frac{1}{2}B^2 g_{MN}\right), \tag{5.2.3}$$

$$\frac{k}{\sqrt{-g}}\partial_M\left(\sqrt{-g}\partial^M\Phi\right) = \frac{1}{4}\frac{\mathrm{d}Z}{\mathrm{d}\Phi}F^2 + \frac{1}{2}\frac{\mathrm{d}W}{\mathrm{d}\Phi}B^2 + \frac{\mathrm{d}V}{\mathrm{d}\Phi}, \tag{5.2.4}$$

$$\frac{k}{\sqrt{-g}}\partial_M\left(\sqrt{-g}Z(\Phi)F^{MN}\right) = W(\Phi)B^N. \tag{5.2.5}$$

To determine the first equation of motion, it is most convenient to vary the action using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{MN}\delta g^{MN}$ and $\delta\left(\sqrt{-g}R\right) = G_{MN}\delta g^{MN}$, while the other two equations of motion are conveniently obtained by using the Euler-Lagrange equations , e.g. for the scalar,

$$\frac{\partial\mathcal{L}}{\partial\Phi} = \sqrt{-g}\left(\frac{1}{4}\frac{\mathrm{d}Z}{\mathrm{d}\Phi}F^2 + \frac{1}{2}\frac{\mathrm{d}W}{\mathrm{d}\Phi}B^2 + \frac{\mathrm{d}V}{\mathrm{d}\Phi}\right) = \partial_M\frac{\partial\mathcal{L}}{\partial\left(\partial_M\Phi\right)} = k\partial_M\left(\sqrt{-g}\partial^M\Phi\right), \tag{5.2.6}$$

which, upon division by $\sqrt{-g}$ gives the postulated equation of motion. The model (5.2.1) has a broken $U(1)$ symmetry due to the mass term—this, however, can be remedied by making the following Stückelberg decomposition (see [140] for a review of the formalism) of the massive vector,

$$B_M = A_M - \partial_M\Xi. \tag{5.2.7}$$

Note that Ξ has dimensions of length, while all other fields are dimensionless. The EPD model admits Lifshitz solutions for $z > 1$; these take the form

$$ds^2 = -\frac{1}{r^{2z}}\mathrm{d}t^2 + \frac{1}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{5.2.8}$$

$$B = \frac{A_0}{r^z}\mathrm{d}t, \tag{5.2.9}$$

$$\Phi = \Phi_\star \qquad (\Phi_\star \text{ const.}), \tag{5.2.10}$$

as long as

$$A_0^2 = \frac{2(z-1)}{zZ_0}, \quad \frac{W_0}{Z_0} = 2z, \quad V_0 = -(z^2+z+4), \quad V_1 = (za+2b)(z-1), \tag{5.2.11}$$

where

$$a = \frac{Z_1}{Z_0}, \quad b = \frac{W_1}{W_0}, \tag{5.2.12}$$

where we have defined

$$V_0 := V(\Phi_\star), \quad V_1 := V'(\Phi_\star), \quad V_2 := V''(\Phi_\star). \tag{5.2.13}$$

Note that the last equation in (5.2.11) is a condition on the potential making sure that Lifshitz is a solution to the family of actions (5.2.1). For the details of this analysis, we refer to [32].

5.2.2 *AlLif Boundary Conditions & the Sources*

The spatial anisotropy makes the use of vielbeine convenient [27], so we define asymptotically locally Lifshitz (AlLif) space-times by specifying boundary conditions for the vielbeine. Defining a holographic radial coordinate r by requiring the metric to be asymptotically (conformally) radial, i.e.

$$ds^2 = \frac{\mathrm{d}r^2}{Rr^2} - E^0E^0 + \delta_{ab}E^aE^b, \tag{5.2.14}$$

where $E^0_r = 0 = E^a_r$, and where the function R is analogous to a radial lapse, which is sometimes [27] set to one—this is known as radial gauge. The benefit of leaving R unfixed was realized in [23]. The boundary is located at $r = 0$. The vielbeine are related to the induced metric $\gamma_{\mu\nu}$ on slices of constant r via

$$\gamma_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -E^0 E^0 + \delta_{ab} E^a E^b, \tag{5.2.15}$$

and the boundary conditions for AlLif read

$$R = \mathcal{O}(1), \quad E^0_\mu = \mathcal{O}(r^{-z}), \quad E^a_\mu = \mathcal{O}(r^{-1}), \qquad (z > 1). \tag{5.2.16}$$

These boundary conditions imply that

$$\sqrt{-g} = \mathcal{O}(r^{-z-3}). \tag{5.2.17}$$

For the Proca field, $B = B_r \mathrm{d}r + B_\mu \mathrm{d}x^\mu$, we have that (as was the case for the pure Lifshitz solution)

$$B_\mu = \mathcal{O}(r^{-z}), \tag{5.2.18}$$

which can be obtained from an asymptotic analysis of the equations of motion. Now, due to antisymmetry of the field strength tensor, the equation of motion for the Proca field (5.2.5) implies that

$$0 = \partial_N \left(\sqrt{-g} W(\Phi) B^N \right) = \partial_r \left(\sqrt{-g} W(\Phi) B^r \right) + \partial_\mu \left(\sqrt{-g} W(\Phi) B^\mu \right), \tag{5.2.19}$$

which is a differential equation for B_r in terms of B_μ—using $B_r = g^{rr} B_r = R r^2 B_r$, it takes the form

$$0 = \partial_r \left(\sqrt{-g} W(\Phi) R r^2 B_r \right) + \partial_\mu \left(\sqrt{-g} W(\Phi) B^\mu \right), \tag{5.2.20}$$

which we can readily integrate to obtain (remembering our freedom to add an r-independent function $-f(x)$),

$$0 = \sqrt{-g} W(\Phi) R r^2 B_r - f(x) + \int_0^r \mathrm{d}r' \partial_\mu \left(\sqrt{-g} W(\Phi) B^\mu \right) \tag{5.2.21}$$

$$\therefore B_r = \frac{f(x)}{\sqrt{-g} W(\Phi) R r^2} - \frac{1}{\sqrt{-g} W(\Phi) R r^2} \int_0^r \mathrm{d}r' \partial_\mu \left(\sqrt{-g} W(\Phi) B^\mu \right). \tag{5.2.22}$$

In particular, the extra term becomes important in a radial expansion at order r^{z+1}, and thus it does not affect the leading behaviour. We may express the boundary conditions for B_μ and E^0_μ—eqs. (5.2.18) and (5.2.16), respectively—by inferring the existence of a function $\alpha(x, r)$ such that,

$$B_\mu - \alpha E^0_\mu = o(r^{-z}), \tag{5.2.23}$$

with $\alpha = \mathcal{O}(1)$ near $r = 0$, and where the Landau symbol $o(r^{-z})$ means[3] that whatever is on the LHS grows strictly slower than r^{-z}. The rôle of the function α will be made clear later. The boundary condition on the dilaton reads

$$\Phi \simeq r^\Delta \phi, \qquad \Delta \geq 0, \tag{5.2.24}$$

where $\simeq$ refers to the leading order term in the near-boundary expansion, and $\phi(x)$ is the boundary value of the dilaton. In particular, in order to find Δ, we need to solve the equations of motion (specifically, by considering radial perturbations around the Lifshitz solution (5.2.8)), which requires us to expand the dilation around $\Phi = \Phi_\star$, and this depends on whether $\Delta > 0$ or $\Delta = 0$. Without loss of generality, we can shift Φ and set $\Phi_\star = 0$, which we will do in what follows. In terms of the quantities introduced above, we impose the following boundary conditions for the vielbeine,

$$E^0_\mu \simeq r^{-z} \alpha_{(0)}^{1/3} \tau_\mu, \quad E^a_\mu \simeq r^{-1} \alpha_{(0)}^{-1/3} e^a_\mu, \quad R \simeq R_{(0)}, \tag{5.2.25}$$

where $\alpha_{(0)}$ and $R_{(0)}$ are the leading terms of α and R, respectively, which are fixed by the equations of motion *as long as we are in the range* $1 < z \leq 2$ (see [32] for details), so from now on we specialize to this case. Note also that we do not demand (by hand) that τ_μ be hypersurface orthogonal (HSO)[4],

3 More precisely, $f(x) = o(g(x))$ means that for all $\epsilon > 0$ there exists an $x_0 \in \mathbb{R}$ such that $f(x) < \epsilon g(x)$ for all $x > x_0$. Similarly, if $f(x) = \mathcal{O}(g(x))$ means that there exists a $\epsilon > 0$ for which you can find an $x_0 \in \mathbb{R}$ such that $f(x) \leq g(x)$ for all $x > x_0$. Since both $o(g(x))$ and $\mathcal{O}(g(x))$ are more appropriately viewed as sets, we should strictly speaking write e.g. $f(x) \in o(g(x))$, but we'll stick with the notation $f(x) = o(g(x))$.

4 That is, orthogonal to leaves of constant time.

$\tau \wedge d\tau = 0$; rather we let the equations of motion determine it. It turns out that for $z > 2$, τ_μ is always HSO [29]. In particular, this the HSO condition be rewritten as the vanishing of the *twist* of τ, $\varpi^2 = 0$, where

$$\varpi^2 = \frac{1}{2}\left(\varepsilon^{\mu\nu\rho}\tau_\mu\partial_\nu\tau_\rho\right)^2. \tag{5.2.26}$$

In general, one finds that the solution splits up into four branches (details will be given in [32]), namely

$$(i): 1 < z < 2 \text{ and } \Delta > 0, \quad (ii): 1 < z < 2 \text{ and } \Delta = 0, \tag{5.2.27}$$
$$(iii): z = 2 \text{ and } \Delta > 0, \quad (iv): z = 2 \text{ and } \Delta = 0. \tag{5.2.28}$$

It turns out that case (iv) splits up into two additional branches: either τ is HSO, in which case $W = 4Z^{2/3}$. If this is not satisfied, the twist (5.2.26) must instead be given by $\varpi^2 = -2(Z(\phi))^{2/3} + \frac{1}{2}W(\phi)$, which becomes a constraint on the source ϕ.

We will take α and R to depend on Φ, so in particular, the leading terms will depend on Δ. Inverting (5.2.25), the inverse vielbeine are subject to the boundary conditions,

$$E_0^\mu \simeq -r^z\alpha_{(0)}^{-1/3}v^\mu, \quad E_a^\mu \simeq r\alpha_{(0)}^{1/3}e_a^\mu, \tag{5.2.29}$$

but since

$$1 = \delta_0^0 = E_0^\mu E_\mu^0 = -v^\mu\tau_\mu, \quad 0 = \delta_0^a = E_0^\mu E_\mu^a = -r^{z-1}\alpha_{(0)}^{-2/3}v^\mu e_\mu^a, \tag{5.2.30}$$
$$0 = \delta_a^0 = E_a^\mu E_\mu^0 = r^{1-z}\alpha_{(0)}^{2/3}e_a^\mu\tau_\mu, \quad \delta_b^a = E_b^\mu E_\mu^a = e_b^\mu e_\mu^a, \tag{5.2.31}$$

we obtain the orthogonality relations,

$$v^\mu\tau_\mu = -1, \quad v^\mu e_\mu^a = 0, \quad e_a^\mu\tau_\mu = 0, \quad e_b^\mu e_\mu^a = \delta_b^a. \tag{5.2.32}$$

Further, the relation

$$\delta_\nu^\mu = E_\nu^0 E_0^\mu + E_\nu^a E_a^\mu = -v^\mu\tau_\nu + e_a^\mu e_\nu^a, \tag{5.2.33}$$

implies the important completeness relation

$$e_a^\mu e_\nu^a = \delta_\nu^\mu + v^\mu\tau_\nu. \tag{5.2.34}$$

Following the arguments given in [24], we now scrutinize the behaviour of the boundary vielbeine under local Lorentz transformations (LLTs). Since $z > 1$, the boundary conditions (5.2.16) imply that the time-like vielbein diverges faster than the space-like vielbeine, which means that the local light cones flatten out, and so the Lorentz group contracts to the Galilei group; in other words: $r \to \infty$ corresponds to sending the speed of light to infinity, which, as we have seen, contracts the Lorentz group to the Galilean group. To this end, we start by considering the transformation properties of the bulk vielbeine under LLTs; specifically we consider $SO(2,1)$ transformations leaving the radial direction invariant. For such transformations, we may write

$$E_\mu^0 = \Lambda^0{}_{0'}E_\mu^{0'} + \Lambda^0{}_{a'}E_\mu^{a'}, \tag{5.2.35}$$
$$E_\mu^a = \Lambda^a{}_{0'}E_\mu^{0'} + \Lambda^a{}_{a'}E_\mu^{a'}. \tag{5.2.36}$$

Now, for generic frame indices $(\underline{a}, \underline{b}, \underline{c}, \underline{d})$—which includes time—the defining property for Lorentz transformations, $\eta_{\underline{ab}}\Lambda^{\underline{a}}{}_{\underline{c}}\Lambda^{\underline{b}}{}_{\underline{d}} = \eta_{\underline{cd}}$, leads to

$$-\Lambda^0{}_{0'}\Lambda^0{}_{0'} + \delta_{ab}\Lambda^a{}_{0'}\Lambda^b{}_{0'} = -1, \tag{5.2.37}$$
$$-\Lambda^0{}_{0'}\Lambda^0{}_{a'} + \delta_{ab}\Lambda^a{}_{0'}\Lambda^b{}_{a'} = 0, \tag{5.2.38}$$
$$-\Lambda^0{}_{a'}\Lambda^0{}_{b'} + \delta_{ab}\Lambda^a{}_{a'}\Lambda^b{}_{b'} = \delta_{a'b'}. \tag{5.2.39}$$

The Lorentz transformed boundary conditions take the form,

$$E_\mu^{0'} \simeq r^{-z}\alpha_{(0)}^{1/3}\tau'_\mu, \qquad E_\mu^{a'} \simeq r^{-1}\alpha_{(0)}^{-1/3}e_\mu^{a'}, \tag{5.2.40}$$

since neither r nor $\alpha_{(0)}$ transforms. By comparing powers of r, the above together with the transformations (5.2.35)–(5.2.36) imply that

$$\Lambda^{0}{}_{0'} \simeq \Lambda^{0}_{(0)0'}, \quad \Lambda^{0}{}_{a'} \simeq r^{1-z}\alpha_{(0)}^{2/3}\Lambda^{0}_{(0)a'}, \quad \Lambda^{a}{}_{0'} \simeq r^{z-1}\alpha_{(0)}^{-2/3}\Lambda^{a}_{(0)0'}, \quad \Lambda^{a}{}_{a'} \simeq \Lambda^{a}_{(0)a'}, \tag{5.2.41}$$

but the constraints (5.2.37)–(5.2.39) imply that, for example

$$-\Lambda^{0}_{(0)0'}\Lambda^{0}_{(0)0'} + \delta_{ab}r^{2(z-1)}\Lambda^{a}_{(0)0'}\Lambda^{b}_{(0)0'} = -1. \tag{5.2.42}$$

Since $z > 1$, we can discard the second term altogether, which makes $\Lambda^{a}_{(0)0'}$ a set of two free parameters—implying that

$$\Lambda^{0}_{(0)0'}\Lambda^{0}_{(0)0'} = 1, \tag{5.2.43}$$

prompting us to choose

$$\Lambda^{0}_{(0)0'} = 1, \tag{5.2.44}$$

so that we may recover the identity. Similar considerations lead us to conclude that

$$\Lambda^{0}_{(0)a'} = 0, \tag{5.2.45}$$

$$\delta_{ab}\Lambda^{a}_{(0)a'}\Lambda^{b}_{(0)b'} = \delta_{a'b'}. \tag{5.2.46}$$

These findings agree with those obtained in [24] when setting $z = 2$. This, in conjunction with the transformation of the time-like bulk vielbein (5.2.35), means that

$$\tau_\mu = \tau'_\mu, \tag{5.2.47}$$

while the transformation of the spatial bulk vielbein (5.2.36) implies that

$$e^{a}_{\mu} = \Lambda^{a}_{(0)0'}e^{0'}_{\mu} + \Lambda^{a}_{(0)a'}e^{a'}_{\mu}. \tag{5.2.48}$$

The transformations $\Lambda^{a}_{(0)0'}$ and $\Lambda^{a}_{(0)b}$ correspond to local Galilean boosts and local rotations, respectively (and will collectively be denoted as local Galilean transformations: LGTs). In order to obtain the infinitesimal versions of these local tangent space transformations, we define

$$\Lambda^{a}_{(0)0'} = \varepsilon\lambda^{a} + \mathcal{O}(\varepsilon^2), \tag{5.2.49}$$

$$\Lambda^{a}_{(0)b} = \delta^{a}_{b} + \varepsilon\lambda^{a}{}_{b} + \mathcal{O}(\varepsilon^2). \tag{5.2.50}$$

With these, we can summarize our findings above as

$$\delta\tau_\mu = 0, \tag{5.2.51}$$

$$\delta e^{a}_{\mu} = \lambda^{a}\tau_\mu + \lambda^{a}{}_{b}e^{b}_{\mu}. \tag{5.2.52}$$

where we have used $\tau_\mu = \tau'_\mu$ in the second line. To determine the transformation properties of the inverse vielbeine, we use the orthogonality relations (5.2.32), i.e.

$$0 = \delta(v^\mu e^{a}_{\mu}) = e^{a}_{\mu}\delta v^\mu + v^\mu\delta e^{a}_{\mu} = e^{a}_{\mu}\delta v^\mu + \lambda^{a}\overbrace{v^\mu\tau_\mu}^{=-1} + \lambda^{a}{}_{b}\overbrace{v^\mu e^{b}_{\mu}}^{=0}, \tag{5.2.53}$$

i.e.[5] $\delta v^\mu = \lambda^{a}e^{\mu}_{a}$, since $e^{\mu}_{b}e^{a}_{\mu} = \delta^{a}_{b}$—which we will now vary to get

$$0 = \delta(e^{\mu}_{b}e^{a}_{\mu}) = e^{a}_{\mu}\delta e^{\mu}_{b} + e^{\mu}_{b}\delta e^{a}_{\mu} = e^{a}_{\mu}\delta e^{\mu}_{b} + \lambda^{a}\overbrace{e^{\mu}_{b}\tau_\mu}^{=0} + \overbrace{e^{\mu}_{b}e^{a}_{\mu}\lambda^{a}{}_{b}}^{=\lambda^{a}{}_{b}}, \tag{5.2.54}$$

which, by the same reasoning, means that $\delta e^{\mu}_{a} = -\lambda^{b}{}_{a}e^{\mu}_{b} = \lambda_{b}{}^{a}e^{\mu}_{b}$, where we used antisymmetry of the infinitesimal $SO(2)$ rotations. To sum up our findings:

$$\delta v^\mu = \lambda^{a}e^{\mu}_{a}, \tag{5.2.55}$$

$$\delta e^{\mu}_{a} = \lambda_{a}{}^{b}e^{\mu}_{b}. \tag{5.2.56}$$

5 Note that this is compatible with $0 = \delta(v^\mu\tau_\mu) = \tau_\mu\delta v^\mu$ since $\tau_\mu e^{\mu}_{a} = 0$.

Now, all terms in the near-boundary expansion of the metric (5.2.14) should be invariant under Galilean transformations when expressed in terms of the boundary vielbeine; in particular, at order r^{-2}, we get

$$\mathcal{O}(r^{-2}) : \alpha_{(0)}^{-2/3}\delta_{ab}e_\nu^a e_\mu^b + \cdots , \tag{5.2.57}$$

with the dots denoting the contributions from the expansion of $E_\mu^0 E_\nu^0$. Galilean invariance of the whole expression implies Galilean invariance of each order in r, but the first term of (5.2.57) is not invariant, since by the transformation (5.2.52)

$$\delta\left(e_\mu^a e_\nu^b\right) = \lambda_a \tau_\mu e_\nu^a + \lambda_a \tau_\nu e_\mu^a. \tag{5.2.58}$$

From this we infer the existence of a term coming from $E_\mu^0 E_\nu^0$, the rôle of which is to make the whole expression invariant—we can engineer such a term by demanding

$$E_\mu^0 = r^{-z}\alpha_{(0)}^{1/3} + \cdots + r^{z-2}\alpha_{(0)}^{-1}X_\mu, \tag{5.2.59}$$

and so the complete term at order r^{-2} is

$$\mathcal{O}(r^{-2}) : \alpha_{(0)}^{-2/3}\left(\delta_{ab}e_\mu^a e_\mu^b - \tau_\mu X_\nu - \tau_\nu X_\mu\right), \tag{5.2.60}$$

where invariance implies that under local Galilean transformations, X_μ transforms as

$$\delta X_\mu = e_\mu^a \lambda_a. \tag{5.2.61}$$

Now, since such an object[6] cannot be constructed from the boundary vielbein sources τ_μ and e_μ^a, there must exist a boundary vector field M_μ such that

$$X_\mu = M_\mu + I_\mu, \tag{5.2.62}$$

where I_μ is invariant under LGTs, and we assume without loss of generality[7] that $I_\mu = I\tau_\mu$ for I a scalar. For the time-like bulk vielbein we may consequently write

$$E_\mu^0 = r^{-z}\alpha_{(0)}^{1/3}\tau_\mu + \cdots + r^{z-2}\alpha_{(0)}^{-1}\left(M_\mu + I\tau_\mu\right). \tag{5.2.63}$$

For the massive vector, which is Galilean boost invariant at each order of r, we may write (the choice of coefficients harks back to the relation for the Proca field (5.2.23) and will be justified shortly)

$$B_\mu = r^{-z}\alpha_{(0)}^{4/3}\tau_\mu + \cdots + r^{z-2}\tilde{I}\tau_\mu + \ldots, \tag{5.2.64}$$

where $\tilde{I}$ is another scalar. We are now ready to make more explicit the relation (5.2.23): for a function α of the form

$$\alpha = \alpha_{(0)} + r^{2z-2}\alpha_{(0)}\left(\tilde{I} - I\right) + \ldots, \tag{5.2.65}$$

we get an explicit version of (5.2.23)

$$B_\mu - \alpha(\Phi)E_\mu^0 \simeq -r^{z-2}M_\mu. \tag{5.2.66}$$

Contracting with the bulk vielbeine, we find that the following leading behaviour of the frame components of the Proca field,

$$B_0 = E_0^\mu B_\mu = \mathcal{O}(1), \qquad B_a = E^\mu B_\mu = \mathcal{O}(r^{z-1}), \tag{5.2.67}$$

where we have used (5.2.66) as well as the boundary conditions (5.2.16). Combining these results, we find for the Proca field

$$B^\mu = E_0^\mu B^0 + E_a^\mu B^a = \mathcal{O}(r^z), \tag{5.2.68}$$

6 That is, an object transforming as in (5.2.61).
7 That is, without affecting the properties of the new source M_μ.

and thus, by the relation for B_r (5.2.22), since $\frac{1}{r^2\sqrt{-g}} = \mathcal{O}(r^{z+1})$ and $\int_0^r \mathrm{d}r' \partial_\mu \left(\sqrt{-g} W(\Phi) B^\mu\right) = \mathcal{O}(r^{-2})$, we get

$$B_r = \mathcal{O}(r^{z-1}). \tag{5.2.69}$$

Now, returning the Stückelberg decomposition of B_μ, (5.2.7), the expression (5.2.66) allows us to do the same for M_μ,

$$M_\mu = \tilde{m}_\mu - \partial_\mu \chi, \tag{5.2.70}$$

provided that

$$\partial_\mu \Xi \simeq -r^{z-2} \partial_\mu \chi, \text{ or } \Xi \simeq -r^{z-2} \chi. \tag{5.2.71}$$

This can always be done since the Stückelberg scalar is unphysical and thus we can impose any boundary condition we like. For the radial component of the Proca field, (5.2.69) implies

$$B_r = A_r - \partial_r \Xi = \mathcal{O}(r^{z-1}), \tag{5.2.72}$$

but the scaling of the Stückelberg scalar Ξ (5.2.71) implies that $\partial_r \Xi \simeq -(z-2) r^{z-3} \chi$, so the only way the leading behaviour of B_r can be $\mathcal{O}(r^{z-1})$ is if

$$A_r \simeq -(z-2) r^{z-3} \chi, \tag{5.2.73}$$

such that it cancels in (5.2.71). For the other components of A, the definition of the source M_μ (5.2.66) gives us

$$A_\mu - \cancel{\partial_\mu \Xi} - \alpha(\Phi) E^0_\mu \simeq -r^{z-2} \left(\tilde{m}_\mu - \cancel{\partial_\mu \chi}\right), \tag{5.2.74}$$

$$\therefore A_\mu - \alpha(\Phi) E^0_\mu \simeq -r^{z-2} \tilde{m}_\mu, \tag{5.2.75}$$

where we have used the scaling of the Stückelberg field (5.2.71).

5.2.3 *More Local Transformations of the Sources*

So far, we have seen how the sources behave under local Galilean boosts and rotations. Another local transformation arises as a consequence of working with the Stückelberg field Ξ: this results in a local $U(1)$ symmetry, which acts on A_M and Ξ as

$$\delta A_M = \partial_M \Lambda, \quad \delta \Xi = \Lambda, \tag{5.2.76}$$

but in order to preserve the boundary conditions (5.2.73) and (5.2.71), we take

$$\Lambda \simeq -r^{z-2} \sigma. \tag{5.2.77}$$

In turn, by the definitions of χ and $\tilde{m}_\mu$ in (5.2.71) and (5.2.75), respectively, this implies that under local $U(1)$ transformations, the sources $\tilde{m}_\mu$ and χ transform as

$$\delta \tilde{m}_\mu = \partial_\mu \sigma, \quad \delta \chi = \sigma. \tag{5.2.78}$$

There is another class of local symmetries, namely bulk diffeomorphisms preserving the conformally radial gauge choice of (5.2.14). These are the PBH transformations, which we now describe. Requiring that Rg_{MN} remains in radial gauge after acting on with a diffeomorphism amounts to the statement

$$\delta \left(R g_{rr}\right) = 0, \tag{5.2.79}$$

where

$$\delta \left(R g_{MN}\right) = \pounds_\zeta \left(R g_{MN}\right), \tag{5.2.80}$$

where ζ is the bulk vector generating PBH transformations. The condition (5.2.79) implies that

$$\pounds_\zeta \left(R g_{MN}\right) = \zeta^M \partial_M g_{rr} + 2 \partial_r \zeta^M g_{Mr} \tag{5.2.81}$$

$$= -2\zeta^r \frac{1}{r^3} + \frac{2}{r^2} \partial_r \zeta^r = 0, \tag{5.2.82}$$

so solving the resulting differential equation $\zeta^r = r\partial_r\zeta^r$ gives us $\zeta^r = -r\Lambda_D$ for some arbitrary function[8] $\Lambda_D(x)$ of the boundary coordinates[9]. It can be shown [33] that the general behaviour of the generator of PBH transformations ζ^μ is $\zeta^\mu = \xi^\mu + \mathcal{O}(r^2)$ by considering non-constant Λ_D. Summarizing our results, we have found that

$$\zeta^r = -r\Lambda_D, \quad \zeta^\mu = \xi^\mu + \mathcal{O}(r^2). \tag{5.2.83}$$

The PBH diffeomorphisms generated by ζ^M act on all bulk fields, and by applying the result (5.2.83), we are able to obtain the transformation properties of the sources. Acting on E^0_μ, for example, with PBH diffeomorphisms gives us

$$\pounds_\zeta E^0_\mu = \zeta^M\partial_M E^0_\mu + E^0_\nu\partial_\mu\zeta^\nu = \zeta^r\partial_r E^0_\mu + \pounds_\xi E^0_\mu \tag{5.2.84}$$
$$= -r\Lambda_D\partial_r E^0_\mu + \pounds_\xi E^0_\mu \tag{5.2.85}$$
$$= z\Lambda_D E^0_\mu + \pounds_\xi E^0_\mu, \tag{5.2.86}$$

where we note how the sign and the factor of r works out nicely in the last equality and makes explicit how Λ_D corresponds to dilatations. The complete behaviour of τ_μ under local transformations consequently works out to be:

$$\delta\tau_\mu = \pounds_\xi\tau_\mu + z\Lambda_D\tau_\mu. \tag{5.2.87}$$

Repeating this exercise for all the other sources, we obtain

$$\delta e^a_\mu = \pounds_\xi e^a_\mu + \lambda^a\tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu. \tag{5.2.88}$$
$$\delta M_\mu = \pounds_\xi M_\mu + e^a_\mu\lambda_a + (2-z)\Lambda_D M_\mu, \tag{5.2.89}$$
$$\delta\chi = \pounds_\xi\chi + \sigma + (2-z)\Lambda_D\chi, \tag{5.2.90}$$
$$\delta v^\mu = \pounds_\xi v^\mu + \lambda^a e^\mu_a - z\Lambda_D v^\mu, \tag{5.2.91}$$
$$\delta e^\mu_a = \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b - \Lambda_D, \tag{5.2.92}$$
$$\delta M_a = \pounds_\xi M_a + \lambda_a{}^b M_b + \lambda_a + (1-z)\Lambda_D M_a, \tag{5.2.93}$$
$$\delta\phi = \pounds_\xi\phi - \Delta\Lambda_D\phi, \tag{5.2.94}$$

where $M_a = e^\mu_a M_\mu$. In the above, λ^a corresponds to Galilean boosts (occasionally known as Milne boosts, see e.g. [141]) (G), $\lambda_a{}^b$ to spatial rotations (J), Λ_D to dilatations (D) and σ, at last, corresponds to Stückelberg gauge transformations (N). The transformations (6.3.29)–(6.3.34) are similar to (4.2.25)–(4.2.26) and (4.2.60), (4.2.61), except they have some additional structure. This is due to the fact that G, J, D, N realize a Schrödinger algebra (see [34]), which is a conformal extension the Bargmann algebra that we discussed in chapter 4. The transformation of $\tilde{m}_\mu$ will also be required and is found by using the definition $\tilde{m}_\mu = M_\mu + \partial_\mu\chi$:

$$\delta\tilde{m}_\mu = \delta M_\mu + \partial_\mu\delta\chi \tag{5.2.95}$$
$$= \pounds_\xi(\tilde{m}_\mu - \cancel{\partial_\mu\chi}) + e^a_\mu\lambda_a + (2-z)\Lambda_D(\tilde{m}_\mu - \cancel{\partial_\mu\chi}) + \cancel{\partial_\mu(\pounds_\xi\chi)} + \partial_\mu\sigma + (2-z)\left[\cancel{\Lambda_D\partial_\mu\chi} + \chi\partial_\mu\Lambda_D\right] \tag{5.2.96}$$
$$= \pounds_\xi\tilde{m}_\mu + e^a_\mu\lambda_a + (z-2)\Lambda_D\tilde{m}_\mu + \partial_\mu\sigma + (2-z)\chi\partial_\mu\Lambda_D. \tag{5.2.97}$$

We can collect our findings regarding sources and their scaling dimensions (dilatation weights w, which—due to the sign in (5.2.83)—is defined via $\Lambda_D X = -wX$) in the table below:

Table 5.1: Sources and their scaling dimensions. Note that the sets (τ_μ, e^a_μ) and (v^μ, e^μ_a) are not independent, one must choose to work with one of them.

source	ϕ	τ_μ	e^a_μ	v^μ	e^μ_a	$\tilde{m}_0$	$\tilde{m}_a$	χ
scaling dim.	Δ	$-z$	-1	z	1	$2z-2$	$z-1$	$z-2$

When counting the number of components of the sources, we must choose to work with either (τ_μ, e^a_μ) or (v^μ, e^μ_a). Choosing one of these, we count a total of 14 components[10]. The local transformations of the sources provide additional constraints reducing the number of free sources. The total

8 Which we will later identify to correspond to dilatations.

9 Note that the sign in the identification $\zeta^r = -r\Lambda_D$ is arbitrary. Refs. [29, 33] use—as we do—the convention of (5.2.83), while ref. [32] uses a plus. It does of course not change anything.

10 The specific counting is $\phi : 1, \tau_\mu : 3, e^a_\mu : 6, \tilde{m}_0 : 1, \tilde{m}_a : 2, \chi : 1$, which adds up to 14.

number of symmetry parameters is[11] 8, resulting in a total of $14 - 8 = 6$ free sources; at least as long as we're in the range $1 < z < 2$. For $z = 2$, there is an additional constraint (as described just below (5.2.28)), which consequently reduces the number of free sources to 5. Before turning to the identification of the VEVs corresponding to the sources of table 5.1, we discuss the boundary geometry of our holographic setup.

5.2.4 *Emergence of TNC Geometry & Schrödinger Symmetry*

In this section, we demonstrate the emergence of TNC geometry of chapter 4 as the boundary geometry, following [29, 33, 34].

From the sources we derived in section 5.2.2, we can construct a series of boost invariant objects[12],

$$\hat{v}^\mu = v^\mu - h^{\mu\nu} M_\nu, \quad \hat{e}^a_\mu = e^a_\mu - M_\nu e^{\nu a}\tau_\mu, \quad \tilde{\Phi} = -v^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu, \tag{5.2.98}$$

$$h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b, \quad \bar{h}_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu - \tau_\mu M_\nu - \tau_\nu M_\mu. \tag{5.2.99}$$

These are entirely equivalent to (4.2.63)–(4.2.64). These objects are all invariant under (G, N) and nearly all of them are also invariant under J; for example, the (G, J, N)-transformation[13] of $\hat{v}^\mu$ reads,

$$\begin{aligned}
\delta\hat{v}^\mu &= \delta v^\mu - \delta^{ab}\left(M_b \delta e^\mu_a + e^\mu_a \delta M_b\right) && (5.2.101)\\
&= \lambda^a e^\mu_a - \delta^{ab}\left(M_b \lambda_a{}^c e^\mu_c + e^\mu_a \left(\lambda_b{}^c M_c + \lambda_b\right)\right) && (5.2.102)\\
&= M^a \lambda_a{}^c e^\mu_c + e^\mu_a \lambda^{ac} M_c && (5.2.103)\\
&= M_a \lambda^{ac} e^\mu_c + e^\mu_a \lambda^{ac} M_c && (5.2.104)\\
&= 0, && (5.2.105)
\end{aligned}$$

where we have used that infinitesimal rotations are anti-symmetric[14],

$$\lambda^{ab} = -\lambda^{ba}. \tag{5.2.106}$$

By these invariant objects, we mean quantities of specific dilatation weight invariant under certain symmetries.

Table 5.2: Newton-Cartan complexes and associated invariance. If a quantity is invariant under e.g. G transformations, this will be marked with a "✓" in the corresponding column, while a quantity not invariant under a given transformation will have a "X" in the corresponding column.

Inv. Obj.	G	J	N	Scaling Dim.
$\hat{v}^\mu$	✓	✓	✓	z
$\hat{e}^a_\mu$	✓	X	✓	-1
$\tilde{\Phi}$	✓	✓	✓	$2z-2$
τ_μ	✓	✓	✓	$-z$
$h^{\mu\nu}$	✓	✓	✓	2
$h^{\mu\nu}$	✓	✓	✓	2
$\bar{h}_{\mu\nu}$	✓	✓	✓	-2

11 This time, the explicit counting is $\xi^\mu : 3, \lambda^a : 2, \lambda^{ab} : 1, \Lambda_D : 1, \sigma : 1$, which adds up to 8. Note that antisymmetry of rotations implies that it only has a single free parameter.

12 Note that in contradistinction to chapter 6, the N-invariant objects in (5.2.98)–(5.2.99) will not have a subscript (χ) indicating invariance in this chapter. We hope that this inter-chapter change of notation will not confuse the reader.

13 In our calculations, we use that the (G, J, N)-transformations of the fields involved are as follows:

$$\delta v^\mu = \lambda^a e^\mu_a, \quad \delta M_a = \lambda_a{}^b M_b + \lambda_a, \quad \delta e^\mu_a = \lambda_a{}^b e^\mu_b. \tag{5.2.100}$$

14 This follows from the defining property, $R^T R = \mathbb{1}$. An infinitesimal rotation $R = \mathbb{1} + \omega$ then satisfies $\mathbb{1} + \omega^T + \omega = \mathbb{1}$, implying, as claimed, that $\omega^T = -\omega$.

We now explicitly verify the statements made in table 5.2.

$$\delta_{GJN}\hat{e}^a_\mu = \delta_{GJN}e^a_\mu - \overbrace{\delta_{GJN}\tau_\mu}^{=0} M^a - \tau_\mu \delta_{GJN} M^a \tag{5.2.107}$$

$$= \cancel{\lambda^a \tau_\mu} + \lambda^a{}_b e^b_\mu - \tau_\mu \left(\lambda^{ab} M_b + \cancel{\lambda^a}\right). \tag{5.2.108}$$

Similarly,

$$\delta_{GJN}\tilde{\Phi} = -M_\mu \delta_{GJN} v^\mu - v^\mu \delta_{GJN} M_\mu - M^a \delta_{GJN} M_a \tag{5.2.109}$$

$$= \cancel{-\lambda^a M_a} - \overbrace{v^\mu e^a_\mu}^{=0} \lambda_a + M^a \left(\lambda_a{}^b M_b + \cancel{\lambda_a}\right) \tag{5.2.110}$$

$$= \lambda^{ab} M_a M_b \overset{\text{eq.(5.2.106)}}{=} 0. \tag{5.2.111}$$

And

$$\delta_{GJN} h^{\mu\nu} = e^\nu_b \lambda^{bc} e^\mu_c + e^\mu_a \lambda^{ac} e^\nu_c = 0, \tag{5.2.112}$$

where we have used the property (5.2.106). Finally, we have

$$\delta_{GJN}\bar{h}_{\mu\nu} = \delta_{ab}\left(e^a_\mu \delta_{GJN} e^b_\nu + e^b_\nu \delta_{GJN} e^a_\mu\right) - \tau_\mu \delta_{GJN} M_\nu - \tau_\nu \delta_{GJN} M_\mu \tag{5.2.113}$$

$$= \cancel{\tau_\nu e^a_\mu \lambda_a} + \cancel{e^a_\mu \lambda_{ac} e^c_\nu} + \cancel{\tau_\mu e^b_\nu \lambda_b} + \cancel{e^b_\nu \lambda_{bc} e^c_\mu} - \cancel{\tau_\nu e^b_\mu \lambda_a} - \cancel{\tau_\mu e^a_\nu \lambda_a} = 0. \tag{5.2.114}$$

Further, we have already shown in the geometric relations (5.2.32) and (5.2.34) that these objects share the properties of the TNC fields (cf. (4.2.29)). From the sources, we can again build the (minimal) affine connection of (4.2.74), providing a full realization of TNC geometry on the boundary.

Note that for $z > 2$, and for many cases with $z = 2$ (case (iv) of (5.2.28) with τ_μ HSO), the boundary is described by TTNC geometry. In this case, we can apply a local dilatation to make τ closed, turning TTNC into a local NC geometry. The TTNC torsion can thus be ascribed to the dilatation invariance.

Now, the sources of table 5.1 transform under $\mathfrak{sch}_z(2,1)$, i.e. the $(2+1)$-dimensional Schrödinger algebra with critical exponent z, as we already remarked in section 5.2.3. This is entirely analogous to the Galilean and Bargmann structures considered in section 4.2.2; in particular, mimicking the approach taken when gauging the Galilean group in section 4.2.3, we can write the transformations of the sources (6.3.29)–(6.3.34) under H, P, G, J, N, D as

$$\bar{\delta}\mathcal{A}_\mu = \pounds_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma], \tag{5.2.115}$$

where $\mathcal{A}_\mu, \Sigma \in \mathfrak{sch}_z(2,1)$ with the expressions

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a \omega_\mu{}^a + \frac{1}{2} J_{ab} \omega_\mu{}^{ab} + N m_\mu + D b_\mu, \tag{5.2.116}$$

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N\sigma + D\Lambda_D, \tag{5.2.117}$$

which differs from the Bargmann algebra only in the addition of dilatations D with corresponding connection b_μ. The commutation relations of $\mathfrak{sch}_z(2,1)$ read

$$[D, H] = -zH, \quad [D, P_a] = -P_a, \quad [D, G_a] = (z-1)G_a, \quad [D, N] = (z-2)N, \quad [H, G_a] = P_a, \tag{5.2.118}$$

$$[P_a, G_b] = \delta_{ab} N, \quad [J_{ab}, G_c] = 2\delta_{c[a} G_{b]}, \quad [J_{ab}, P_c] = 2\delta_{c[a} P_{b]}, \quad [J_{ab}, J_{cd}] = 4\delta_{[a[d} J_{c]b]}. \tag{5.2.119}$$

Note that $\tilde{m}_\mu = m_\mu - (z-2)\chi b_\mu$, where χ is the Stückelberg field. Thus, the boundary geometry for arbitrary z can be obtained by gauging $\mathfrak{sch}_z(2,1)$ (with the inclusion of the Stückelberg field χ, just like in section 4.2.4). This is done by following the gauging procedure that we described in chapter 4, and was considered in[15] [34]. Finally, we remark that when $z = 2$, and the boundary geometry is TTNC, there is an additional special conformal transformation K contained in $\mathfrak{sch}_2(2,1)$ that we need to take into account when writing down the gauge connection (5.2.116) [34]—see also chapter 6, where we carry out this procedure explicitly.

15 The approach of [34] differs slightly from the approach we follow in this work. In [34], the behaviour under H, P transformations is inferred via the imposition of curvature constraints rather than by introducing the $\bar{\delta}$-transformation.

5.2.5 *Bulk Metric AlLif Boundary Conditions Revisited*

The analyses of the preceding sections allow us to recast the boundary conditions of section 5.2.2 in terms of the metric and the Proca field. Using (5.2.14) with (5.2.25) and (5.2.63), we find that

$$ds^2 = \frac{\mathrm{d}r^2}{Rr^2} \overbrace{- \alpha_{(0)}^{2/3} r^{-2z} \tau_\mu \tau_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots - r^{-2} \alpha_{(0)}^{-2/3} \left[2\tau_{(\mu} M_{\nu)} + I \tau_\mu \tau_\nu \right] \mathrm{d}x^\mu \mathrm{d}x^\nu + \ldots}^{\text{from } E^0 E^0} + \overbrace{r^{-2} \alpha_{(0)}^{-2/3} e_\mu^a e_{a\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \ldots}^{\text{from } E^a E_a} \tag{5.2.120}$$

$$= \frac{\mathrm{d}r^2}{Rr^2} - \alpha_{(0)}^{2/3} r^{-2z} \tau_\mu \tau_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots + \alpha_{(0)}^{-2/3} r^{-2} \left(\bar{h}_\mu \nu + I \tau_\mu \tau_\nu \right) \mathrm{d}x^\mu \mathrm{d}x^\nu + \ldots, \tag{5.2.121}$$

while for the Proca field, a similar computation reveals that

$$B = B_r \mathrm{d}r + \alpha_{(0)}^{4/3} r^{-z} \tau_\mu \mathrm{d}x^\mu + \cdots + r^{z-2} \tilde{I} \tau_\mu \mathrm{d}x^\mu + \cdots. \tag{5.2.122}$$

5.3 VEVS AND COVARIANT WARD IDENTITIES IN LIFSHITZ HOLOGRAPHY

In this section, we continue our investigation of Lifshitz holography focussing on the VEVs and their Ward identities. The analysis is based on [29], but contains more details.

5.3.1 *VEVs & Circumvention of Renormalization*

We now want to consider the vevs associated with our sources. In principle, we really need the renormalized on-shell action, which would require us to holographically renormalize the EPD model with the Lifshitz solution. However, we can apply general arguments to find the leading behaviour of the VEVs. In particular, as was our modus operandi when doing holographic renormalization in chapter 2, we know that the on-shell action on a radial hypersurface can be written as

$$S = \int_{\Sigma_r} \mathrm{d}^3 x \sqrt{-\gamma} \mathscr{L}, \tag{5.3.1}$$

where

$$\sqrt{-\gamma} \simeq \alpha_{(0)}^{-1/3} r^{-z-2} e. \tag{5.3.2}$$

and $\mathscr{L}$ is a function of the bulk data, which will become the sources on the boundary. This bulk data (cf. section 5.2.3) we take to be

$$\chi = \{E_0^\mu, E_a^\mu, \Phi, A_a, \Xi, \varphi\}, \tag{5.3.3}$$

where we have defined $\varphi = E_0^\mu \left(A_\mu - \alpha(\Phi) E_\mu^0 \right)$, which is intimately connected with the new source $\tilde{m}_\mu$. We note that

$$\varphi = \mathcal{O}(r^{2z-2}). \tag{5.3.4}$$

Now, write

$$\sqrt{-\gamma} \mathscr{L} = \sum_w e \mathscr{L}_{(w)} r^{w-z-2}, \tag{5.3.5}$$

where $e = \det \left(\tau_\mu, e_\mu^a \right) = \sqrt{-\gamma}$. The holographic renormalization procedure, as we have seen, removes all divergent terms[16], that is, terms $\sim r^\#$ for $\# < 0$. On the other hand, terms with r^ϱ where $\varrho > 0$ can be ignored, since they will play no rôle on the boundary. Thus, the term with scaling weight $w = z + 2$ is precisely what determines the renormalized on-shell action. By the same token, varying the on-shell action, we get

$$\delta S_{\text{ren}} = \int_{\Sigma_r} \mathrm{d}^3 x \, e \left[\boldsymbol{\mathcal{V}} \delta \chi - \mathcal{A} \frac{\delta r}{r} \right], \tag{5.3.6}$$

16 As $r \to 0$.

where we have included the $\alpha_{(0)}^{-1/3} r^{-z-2}$-dependence (coming from $\sqrt{-\gamma}$, cf. (5.3.2)) in the "VEV complex" $\mathcal{V}$; we will have to take this into account later when determining the near-boundary behaviour of the VEVs. The VEV complex consists of the responses

$$\mathcal{V} = \{\mathcal{S}^0_\mu, \mathcal{S}^a_\mu, \mathcal{T}_\Phi, \mathcal{T}^a, \mathcal{T}_\Xi, \mathcal{T}_\varphi\}, \tag{5.3.7}$$

which arise when varying with respect to the bulk data (5.3.3), e.g. $\mathcal{S}^0_\mu = \frac{1}{e}\frac{\delta S_{\text{Ren}}}{\delta E^\mu_0}$. Since the variation of the action (5.3.6) is finite by construction, we must demand that, for example, the combination

$$\mathcal{S}^0_\mu \delta E^\mu_0 \tag{5.3.8}$$

does not depend on r, so including the $\alpha_{(0)}^{-1/3} r^{-z-2}$–dependence from the metric determinant in the response and using the relation for the near-boundary behaviour of the bulk fields (5.2.29), we find that the combination

$$\alpha_{(0)}^{-2/3} r^{-z-2+z} \mathcal{S}^0_\mu \tag{5.3.9}$$

should not depend on r and $\alpha_{(0)}$; that is,

$$\mathcal{S}^0_\mu \simeq r^2 \alpha_{(0)}^{2/3} S^0_\mu, \tag{5.3.10}$$

where S^0_μ is the VEV corresponding to e^μ_0. Repeating this analysis for E^μ_a and again using (5.2.29), we find that

$$r^{-z-2+1} \mathcal{S}^0_\mu \tag{5.3.11}$$

should be independent of r, and so

$$\mathcal{S}^a_\mu \simeq r^{z+1} S^a_\mu, \tag{5.3.12}$$

where $r^{z+1} S^a_\mu$ is the vev corresponding to e^μ_a. Continuing this process, we reproduce the results of [29]:

$$\mathcal{T}_\varphi \simeq r^{4-z}\alpha_{(0)}^{2/3} T^0, \quad \mathcal{T}^a \simeq r^3 T^a, \quad \mathcal{T}_\Xi \simeq r^4 \alpha_{(0)}^{1/3}\langle O_\chi\rangle, \quad \mathcal{T}_\Phi \simeq r^{z+2-\Delta}\alpha_{(0)}^{1/3}\langle O_\phi\rangle, \quad \mathcal{A} \simeq r^{z+2}\alpha_{(0)}^{1/3}\mathcal{A}_{(0)}. \tag{5.3.13}$$

This means that we can write the variation of the on-shell action as (on the boundary, i.e. on $\Sigma_0 = \partial\mathcal{M}$)

$$\delta S_{\text{ren}} = \int_{\partial\mathcal{M}} \mathrm{d}^3x\, e\left(-S^0_\mu \delta v^\mu + S^a_\mu \delta e^\mu_a + T^0\delta\tilde{m}_0 + T^a\delta\tilde{m}_a + \langle O_\chi\rangle\delta\chi + \langle\tilde{O}_\phi\rangle\delta\phi - \mathcal{A}_{(0)}\frac{\delta r}{r}\right). \tag{5.3.14}$$

Since the responses were explicitly constructed such that all factors of r and $\alpha_{(0)}$ drop out, the expression for δS_{ren} in (5.3.14) follows immediately—except for the terms involving $\tilde{m}$. We now show how they come about: by (5.2.75), we have

$$\mathcal{T}_\varphi \delta\varphi \simeq T^0 e^\mu_0 \delta\tilde{m}_\mu = T^0\delta\tilde{m}_0. \tag{5.3.15}$$

Also by (5.2.75), $E^0_\mu E^\mu_a = 0$ implies $A_a \simeq -r^{z-2}\tilde{m}_a$, and so

$$\mathcal{T}^a \delta A_a \simeq T^a \delta\tilde{m}_a. \tag{5.3.16}$$

Finally, the vev corresponding to ϕ requires additional terms when $\Delta = 0$ (we indicate this by a tilde on the VEV)

$$\langle\tilde{O}_\phi\rangle = \langle O_\phi\rangle + \delta_{\Delta,0}\left[\frac{1}{3}v^\mu\left(S^0_\mu + T^0\tilde{m}_\mu\right) + \frac{1}{3}e^\mu_a\left(S^a_\mu + T^a\tilde{m}_\mu\right)\right]\frac{\mathrm{d}\log\alpha_{(0)}}{\mathrm{d}\phi}. \tag{5.3.17}$$

where the extra contributions for $\Delta = 0$ come from factors of $\alpha_{(0)}(\phi)$ in the leading behaviour of the sources. Also note that for τ_μ HSO (i.e. for TTNC), it can be shown that $\langle\tilde{O}_\phi\rangle = 0$; this is due to the presence of an extra constraint, and in the upliftable case (see [23, 24] and chapter 6) can be traced back to the FG expansion in the higher-dimensional theory.

We now turn our attention to the behaviour of the VEVs under local transformations: being tensors defined on the boundary, we infer that they transform under the PBH transformations that we considered in section 5.2.3. The radial PBH transformation—corresponding to dilatations—implies a dilatation weight of the vevs corresponding to the power of r in the near-horizon expansions of the corresponding responses of (5.3.13). All in all, one finds[17] [29, 32]

$$\delta S^0_\mu = -T^0\partial_\mu\sigma - 2\Lambda_D S^0_\mu + \dots, \quad \delta S^a_\mu = \lambda^a S^0_\mu + \lambda^a{}_b S^b_\mu - T^a\partial_\mu\sigma - (z+1)\Lambda_D S^a_\mu + \dots, \tag{5.3.18}$$

$$\delta T^0 = (z-4)\Lambda_D T^0 + \dots, \quad \delta T^a = \lambda^a T^0 + \lambda^a{}_b T^b - 3\Lambda_D T^a + \dots, \tag{5.3.19}$$

$$\delta\langle O_\chi\rangle = -4\Lambda_D\langle O_\chi\rangle + \dots, \quad \delta\langle O_\phi\rangle = \delta_{\Delta,0}\frac{\mathrm{d}\log\alpha_{(0)}}{\mathrm{d}\phi}\lambda_a T^a - (z+2-\Delta)\Lambda_D\langle O_\phi\rangle + \dots, \tag{5.3.20}$$

where dots denote Lie derivatives along ξ^μ and possibly derivatives of Λ_D. We note that the VEVs also transform under the Schrödinger group, just like the sources.

5.3.2 *Boundary Energy-Momentum Tensor and Mass Current*

The HIM stress-tensor, which was introduced in [142], is relevant when additional non-scalar fields are present in the bulk model. Following [142], we proceed to define

$$e\tilde{T}^\mu{}_\nu = e^\mu_a\frac{\delta S_{\mathrm{Ren}}}{\delta e^\nu_a} + v^\mu\frac{\delta S_{\mathrm{Ren}}}{\delta v^\nu_0} \tag{5.3.21}$$

$$= e\left(S^a_\nu e^\mu_a - S^0_\nu v^\mu\right). \tag{5.3.22}$$

However, under local Stückelberg gauge transformations (N), this is not invariant, since

$$\delta_N\tilde{T}^\mu{}_\nu = e^\mu_a\delta_N S^a_\nu + S^a_\nu\overbrace{\delta_N e^\mu_a}^{=0} - v^\mu\delta_N S^0_\nu - S^0_\nu\overbrace{\delta_N v^\mu}^{=0} \tag{5.3.23}$$

$$= T^a\partial_\nu\sigma e^\mu_a - v^\mu T^0\partial_\mu\sigma. \tag{5.3.24}$$

However, we can define an "improved" HIM tensor which is invariant under the full Schrödinger group by noting that $\delta_N\chi = \sigma$, implying that the gauge invariant extension of (5.3.24) is

$$T^\mu{}_\nu = \left(S^a_\nu + T^a\partial_\nu\chi\right)e^\mu_a - \left(S^0_\nu + T^0\partial_\nu\chi\right)v^\mu. \tag{5.3.25}$$

Being built of e.g. S^a_ν and v^μ, we infer that $T^\mu{}_\nu$ has scaling dimension $z+2$, making it marginal for two spatial dimensions. The vielbein projections of $T^\mu{}_\nu$ become the energy density $(T^\mu{}_\nu\tau_\mu v^\nu)$, momentum flux $(T^\mu{}_\nu\tau_\mu e^\nu_a)$, energy flux $T^\mu{}_\nu e^a_\mu v^\nu$ as well as stress $(T^\mu{}_\nu e^a_\mu e^\nu_b)$; the mass density is given by $T^0 = T^\mu\tau_\mu$, while the mass flux is $T^a = T^\mu e^a_\mu$. These quantities appear also in e.g. [26, 27]. We summarize our findings in table 5.3 below.

Table 5.3: Tangent space projections of $T^\mu{}_\nu$ and T^μ with associated scaling weights. Quantities in parentheses denote the corresponding symbol in [26].

VEV	$T^\mu{}_\nu\tau_\mu v^\nu$ $(\mathcal{E})$	$T^\mu{}_\nu\tau_\mu e^\nu_a$ $(\mathcal{P}_a)$	$T^\mu{}_\nu e^a_\mu v^\nu$ $(\mathcal{E}^a)$	$T^\mu{}_\nu e^a_\mu e^\nu_b$ $(\Pi^a{}_b)$	$T^\mu\tau_\mu$ (ρ)	$T^\mu e^a_\mu$ (ρ^a)
scaling dim.	$z+2$	3	$2z+1$	$z+2$	$4-z$	3

5.3.3 *Ward Identities*

The Ward identities are obtained by demanding invariance of (5.3.14) under the transformations (6.3.29)-(6.3.34). The derivation of the Ward identities will, it turns out, depend on which of the four cases of (5.2.28) we consider, but the final result will not change [29, 32].

The boost Ward identity, for example, is obtained by replacing the variations in (5.3.14) by their corresponding transformations under Galilean boosts. The only sources transforming under boosts are v^μ, $\tilde{m}_a$ and $\tilde{m}_a$, where the boost transformation properties of the two last follow from (5.2.97), i.e.

$$\delta_G\tilde{m}_a = \delta_G(\tilde{m}_\mu e^\mu_a) = \tilde{m}_\mu\overbrace{\delta_G e^\mu_a}^{=0} + e^\mu_a\overbrace{\delta_G\tilde{m}_\mu}^{=\lambda^b e_{\mu b}} = \lambda_a, \tag{5.3.26}$$

17 Note the typo in [29] where all δ_N transformations, i.e. those involving σ, of the VEVs have the wrong sign.

as well as

$$\delta_G \tilde{m}_0 = \delta_G(-v^\mu \tilde{m}_\mu) = -\tilde{m}_\mu \overbrace{\delta_G v^\mu}^{=\lambda^a e^\mu_a} - v^\mu \overbrace{\lambda^b e_{\mu b}}^{=0,\text{ since } v^\mu e_{\mu b}=0} = -\lambda^a \tilde{m}_a, \tag{5.3.27}$$

which means that

$$0 = \delta_G S_{\text{ren}} = \int \mathrm{d}^3 x\, e \left[-S^0_\mu e^\mu_a - T^0 \tilde{m}_a + \delta_{ab} T^b \right] \lambda^a, \tag{5.3.28}$$

that is to say, the boost Ward identity is

$$-S^0_\mu e^\mu_a - T^0 \tilde{m}_a + \delta_{ab} T^b = 0. \tag{5.3.29}$$

Now, this result can be *covariantized*, by which we mean a rewriting of the result above involving the boost invariant quantities of (5.2.98)–(5.2.99). First, observe that

$$\hat{e}^a_\mu T^\mu = \left(e^a_\mu - (\tilde{m}_\nu - \partial_\nu \chi) e^{\nu a} \tau_\mu \right) \left(-T^0 v^\mu + T^b e^\mu_b \right) \tag{5.3.30}$$

$$= -T^0 (\tilde{m}_\nu - \partial_\nu \chi) e^{\nu a} + T^a, \tag{5.3.31}$$

where we have used the orthogonality property $\tau_\mu e^\mu_a$. Next, note that

$$-\tau_\nu e^{\mu a} T^\nu{}_\mu = -\tau_\nu e^{\mu a} \left[-(S^0_\mu + T^0 \partial_\mu \chi) v^\nu + (S^a_\mu + T^a \partial_\mu \chi) e^\nu_a \right] \tag{5.3.32}$$

$$= -S^0_\mu - T^0 \partial_\mu \chi, \tag{5.3.33}$$

and thus the boost Ward identity (5.3.29) can be expressed covariantly in the form

$$\hat{e}^a_\mu T^\mu = \tau_\mu e^{\nu a} T^\mu{}_\nu. \tag{5.3.34}$$

Repeating the procedure, we see that that the rotation Ward identity—appropriately antisymmetrized (due to antisymmetry of λ_{ab})—reads

$$S^a_\mu e^{\mu b} + T^a \tilde{m}^b - (a \leftrightarrow b) = 0. \tag{5.3.35}$$

Multiplying with $\tilde{m}^b$ and antisymmetrizing, we get the relation

$$S^0_\mu e^{\mu[a} \tilde{m}^{b]} = T^{[a} \tilde{m}^{b]}, \tag{5.3.36}$$

which means that the naïve rotation Ward identity (5.3.35) can be recast in the form

$$S^{[a}_\mu e^{\mu b]} + S^0_\mu e^{\mu[a} \tilde{m}^{b]} = 0. \tag{5.3.37}$$

Now, consider the combination

$$\hat{e}^a_\nu e^{b\mu} T^\nu{}_\mu = \left(e^a_\nu - M_\rho e^{\rho a} \tau_\nu \right) e^{b\mu} \left(-(S^0_\mu + T^0 \partial_\mu \chi) v^\nu + (S^c_\mu + T^c \partial_\mu \chi) e^\nu_c \right) \tag{5.3.38}$$

$$= e^{\mu b} S^a_\mu + e^{\mu b} T^a \partial_\mu \chi - \tilde{m}_\rho e^{\rho a} e^{\mu b} S^0_\mu - \tilde{m}_\rho e^{\rho a} e^{\mu b} T^0 \partial_\mu \chi - \partial_\rho \chi e^{\rho a} e^{\mu b} T^0 \partial_\mu \chi + \partial_\rho \chi e^{\rho a} e^{\mu b} S^0_\mu, \tag{5.3.39}$$

where we immediately note that the next-to-last term (in red) is symmetric and so vanishes upon antisymmetrization, and hence

$$\hat{e}^{[a}_\nu e^{b]\mu} T^\nu{}_\mu = e^{\mu[b} S^{a]}_\mu - \tilde{m}^{[a} e^{\mu b]} S^0_\mu + \partial_\mu \chi \left(e^{\mu b} T^a - \tilde{m}^a e^{\mu b} T^0 + e^{\mu a} e^{\rho b} S^0_\rho \right) \tag{5.3.40}$$

$$- \partial_\mu \chi \left(e^{\mu a} T^b - \tilde{m}^b e^{\mu a} T^0 + e^{\mu b} e^{\rho a} S^0_\rho \right) \tag{5.3.41}$$

$$= e^{\mu[b} S^{a]}_\mu - \tilde{m}^{[a} e^{\mu b]} S^0_\mu + \partial_\mu \chi \left(e^{\mu b} T^a - \tilde{m}^a e^{\mu b} T^0 - e^{\mu b} e^{\rho a} S^0_\rho \right) \tag{5.3.42}$$

$$- \partial_\mu \chi \left(e^{\mu a} T^b - \tilde{m}^b e^{\mu a} T^0 - e^{\mu a} e^{\rho b} S^0_\rho \right) \tag{5.3.43}$$

$$= e^{\mu[b} S^{a]}_\mu - \tilde{m}^{[a} e^{\mu b]} S^0_\mu + \partial_\mu \chi e^{\mu b} \left(T^a - \tilde{m}^a T^0 - e^{\rho a} S^0_\rho \right) \tag{5.3.44}$$

$$- \partial_\mu \chi e^{\mu a} \left(T^b - \tilde{m}^b T^0 - e^{\rho b} S^0_\rho \right) \tag{5.3.45}$$

$$= e^{\mu[b} S^{a]}_\mu - \tilde{m}^{[a} e^{\mu b]} S^0_\mu \tag{5.3.46}$$

$$= e^{\mu[b} S^{a]}_\mu + \tilde{m}^{[b} e^{\mu a]} S^0_\mu, \tag{5.3.47}$$

where we have used the boost Ward identity in the form (5.3.29). Combining our findings, we conclude that the rotation Ward identity is covariantly expressed as

$$0 = \hat{e}_\nu^{[a} e^{b]\mu} T^\nu{}_\mu . \tag{5.3.48}$$

Repeating this procedure, we find that the rest of the Ward identities become [29]

$$\langle O_\chi \rangle = e^{-1}\partial_\mu(eT^\mu) \quad \text{(gauge transformations)}, \tag{5.3.49}$$

$$\mathcal{A}_{(0)} = -z\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\nu e^\nu_a T^\mu{}_\nu + 2(z-1)\tilde{\Phi}\tau_\mu T^\mu \quad \text{(dilatations)}, \tag{5.3.50}$$

$$0 = \nabla_\mu T^\mu{}_\nu + 2\Gamma^\rho_{[\mu\rho]} T^\mu{}_\nu - 2\Gamma^\rho_{[\nu\rho]} T^\rho{}_\nu - T^\mu \hat{e}^a_\mu D_\nu M_a + \tau_\mu T^\mu \partial_\nu \tilde{\Phi} \quad \text{(diffeomorphisms)}, \tag{5.3.51}$$

where $D_\mu M^a = \partial_\mu M^a - \omega_\mu{}^a - \omega_\mu{}^a{}_b M^b$. Note the similarity to the Ward identities derived from null reduction in (4.4.8)–(4.4.10) when setting $z = 2$.

5.4 OUTLOOK

Many interesting avenues for future research into Lifshitz holography exist. Studying holographic entanglement entropy in the context of Lifshitz spacetimes is an interesting unsolved problem, considered in [143, 144] (also considered recently from a field theory perspective in [145, 146]). There are also potentially very interesting ties between Lifshitz holography and Hořava-Lifshitz (HL) gravity: it was shown in [53] that dynamical NC geometry is equivalent to HL gravity, and since NC geometry plays a prominent role in Lifshitz holography, it would be exciting to explore the rôle of HL gravity in Lifshitz holography.

In a similar spirit, having a dynamical NC bulk (i.e. a HL bulk) would an extremely interesting generalization; something that was suggested from the perspective of HL gravity in [147]. It would be extremely interesting to develop upon these results given new knowledge that HL gravity is the same as dynamical TNC geometry and explore the general rôle TNC geometry plays in such holographic setups. In many ways, starting from HL gravity is more natural, and we hope to report on such an analysis in the future.

Furthermore, studying holographic Lifshitz hydrodynamics is also worthwhile; this was done in [148, 149] as well as in [57]. More generally, studying non-relativistic fluid dynamics independently from holography using TNC geometry is the topic of an upcoming paper [150].

6 CHARGED LIFSHITZ HOLOGRAPHY & GALILEAN ELECTRODYNAMICS

In this chapter, we develop charged Lifshitz holography for $z = 2$, building on the insights from the last chapter. The bulk action consists of a generalization of the EPD model considered in chapter 5: an EPD-Maxwell-scalar model, which is related to the EMD model (3.2.1) we renormalized in chapter 3 via Scherk-Schwarz reduction; consequently we will refer to this EMD model as the *electromagnetic uplift*. As we will show, the new sources transform, perhaps not entirely unexpectedly, as the fields of Galilean Electrodynamics (GED), and while there is some freedom in defining the sources, we find that there is a natural choice respecting the symmetries. We also show that with this choice, the (integrated) Weyl anomaly becomes Hořava-Lifshitz (HL) gravity coupled to GED. This generalizes the observation made in [24, 107] that the Weyl anomaly in pure $z = 2$ Lifshitz holography is related to HL gravity.

The analysis is a novel extension of the approach taken in [23, 24], and the findings of this chapter, along with those of chapter 3, will be the subject of [41], to appear.

We begin with an introduction to GED in section 6.1, where we discuss the relation to previous work and show in section 6.1.2 it may be obtained as a null reduction of "ordinary" electromagnetism. In this section, we also provide dictionary between the notation used in this work and that used in previous literature. In section 6.1.3, we provide a new dimensional analysis of GED on anisotropic backgrounds.

In section 6.2, we discuss the Scherk-Schwarz reduction of the five-dimensional electromagnetic uplift. We begin our exposition with a recount of the uplift as used in previous work [23, 24, 57] and show how this leads to a specific EPD model. A detailed derivation of this is presented in appendix H. In section 6.2.2 we discuss the additional subtleties involved with the extra Maxwell field and present the full result of the Scherk-Schwarz reduction of the electromagnetic uplift. We then vary the resulting new action in section 6.2.3 and determine all equations of motion as well as the responses. Finally, in section 6.2.4, we demonstrate how the $z = 2$ Lifshitz solution in the dimensionally reduced model is related to a $z = 0$ Schrödinger geometry, which is asymptotically AdS.

We then turn our attention to the sources in section 6.3. We define all sources in section 6.3.1 and discuss the properties of the new sources; in particular we find that the new sources appear as subleading terms in the near-boundary expansion of the bulk $U(1)$ gauge field. This is entirely analogous to the source m_μ, which also appears as a subleading term in the expansion of the time-like bulk vielbein. In section 6.3.2, we first recall how the known sources transform before we proceed to derive the novel transformation properties of the new sources under all local transformations and demonstrate that they transform as GED fields, which were discussed in section 6.1.2.

In section 6.4, we discuss the boundary geometry of our model. In particular, we show that the Scherk-Schwarz reduction employed in section 6.2 becomes null on the boundary, which, as demonstrated in section 4.3, directly leads to TNC geometry. Building on our considerations in chapter 4, we discuss the relation to the gauging of the Schrödinger algebra for $z = 2$, $\mathfrak{sch}_2(2,1)$, which we show leads to TTNC geometry, since the imposition of certain curvature constraints forces τ_μ to be HSO. The discussion is based [34, 53]. In order to obtain TNC, we must add torsion by hand. We also discuss the Stückelberg symmetry of the central charge.

As advertised, the reduced anomaly takes the form of a Lagrangian describing Hořava-Lifshitz gravity coupled to GED, as we demonstrate in section 6.5. In order to obtain this result, we—following [24]—make a few simplifying assumptions, namely we consider the less general case of NC boundary geometry, where the torsion vanishes, and set the Stückelberg field to zero.

In order to make the connection between the electromagnetic uplift and the reduced model manifest, we determine the near-boundary expansions of the four-dimensional bulk fields in section 6.6. Based on the novel FG expansions of the five-dimensional fields that we determined in chapter 3, we explicitly see the emergence of structure that leads to the definition of the sources in section 6.3. In this section, we also derive a constraint relating the boundary value of the dilaton to the twist, which has the important consequence that ϕ is not an independent field.

In section 6.7, we determine the VEVs corresponding to the sources introduced in section 6.3. In particular, we relate the VEVs to the responses worked out in section 6.2.3 and work out their near-boundary behaviour in section 6.7.1. We then derive the relation between the four-dimensional VEVs and the five-dimensional VEVs of the electromagnetic uplift in section 6.7.2, which allows us to determine all local transformations of the VEVs in section 6.7.3.

Having determined all the VEVs, we derive novel Ward identities for charged Lifshitz holography in section 6.8 by demanding invariance of the variation of the section under the local symmetries. We begin in section 6.8.1 with a definition of the HIM stress-energy tensor and define several GED-augmented versions of this object that will appear in the Ward identities. We also provide the map between these objects and the five-dimensional stress-energy tensor based on our results in section 6.7.2. In section 6.8.2, we then derive all Ward identities and consider their TNC covariant forms. Finally, in section 6.8.3, we consider the Ward identities from a reduction perspective and show how the Ward identities obtained in section 6.8.2 are obtained from the five-dimensional Ward identities derived in chapter 3.

Finally, in section 6.9, we present a conjecture for the extension of these results to general values of z based on our results in section 6.1.3.

6.1 GALILEAN ELECTRODYNAMICS

Galilean electrodynamics (GED) was recently explored in the context of Newton-Cartan geometry in [58] following earlier work in [151] (see also [152] for a historical review of Galilean electromagnetism). In this section, we summarize the construction of [58] with a particular emphasis on the coupling of GED to TNC geometries as obtained from null reduction; this will serve to introduce the notation we will employ in a holographic context later in the chapter.

6.1.1 *Non-Relativistic Electrodynamics & the Unification of Two Limits*

Following [151], we identify two limits of Maxwellian electromagnetism that produce Galilean invariant theories, known as the *electric* and *magnetic* limits, respectively. We will initially work at the level of equations of motion and eventually identify a larger theory admitting an off-shell description into which both these limits can be embedded [153]; this model will be referred to as Galilean Electrodynamics (GED) [58]. In order to discuss relativistic effects, we will keep factors of c explicit in this section.

Denote by $\hat{a}_\mu$ a $U(1)$ gauge field on $\mathbb{R}^{d,1}$ with coordinates (t, x^i). The associated field strength is defined as $\hat{f}_{\mu\nu} = 2\partial_{[\mu}\hat{a}_{\nu]}$, implying that the equations of motion in the absence of sources, $\partial_\mu \hat{f}^{\mu\nu} = 0$, can be written as

$$\partial_i\left(\partial_i\hat{a}_t - \frac{1}{c}\partial_t\hat{a}_i\right) = 0, \qquad \frac{1}{c}\partial_t\left(\partial_i\hat{a}_t - \frac{1}{c}\partial_t\hat{a}_i\right) + \partial_j\hat{f}_{ji} = 0. \tag{6.1.1}$$

The gauge field transforms under gauge transformations as

$$\hat{a}'_t = \hat{a}_t + \frac{1}{c}\partial_t\Gamma, \qquad \hat{a}'_i = \hat{a}_i + \partial_i\Gamma. \tag{6.1.2}$$

Depending on whether $\hat{a}_\mu$ is time-like or space-like, we can take either the electric or magnetic limit: these are summarized in the table below.

Name	Limit structure	EoM
Electric	$\hat{a}_t = -\vartheta, \hat{a}_i = \frac{1}{c}a_i, \Gamma = \frac{1}{c}\Gamma_{(0)}$ $c \to \infty$ with $\vartheta, a_i, \Gamma_{(0)}$ fixed	$\partial_i\partial_i\vartheta = 0 = \partial_t\partial_i\vartheta + \partial_j f_{ji}$ where $f_{ij} = 2\partial_{[i}a_{j]}$
Magnetic	$\hat{a}_t = -\tilde{\varphi}, \hat{a}_i = c a_i, \Gamma = c\Gamma_{(0)}$ $c \to \infty$ with $\tilde{\varphi}, a_i, \Gamma_{(0)}$ fixed	$\partial_i\tilde{E}_i = 0 = \partial_j f_{ji}$ where $\tilde{E}_i = -\partial_i\tilde{\varphi} - \partial_t a_i$

Table 6.1: The electric and magnetic limits and the resulting equations of motion (EoM).

In order to obtain GED, we start with a Maxwell action with an additional free scalar $\hat{\phi}$, the Lagrangian density of which reads

$$\mathcal{L} = \frac{1}{2c^2}\left(\partial_t\hat{a}_i - c\partial_i\hat{a}_t\right)^2 - \hat{f}_{ij}\hat{f}_{ij} + \frac{1}{2c^2}\left(\partial_t\hat{\phi}\right)^2 - \frac{1}{2}\left(\partial_i\hat{\phi}\right)^2, \tag{6.1.3}$$

and take the *GED limit*, defined by

$$\hat{\phi} = c\vartheta, \qquad \hat{a}_t = -c\vartheta - \frac{1}{c}\tilde{\varphi}, \qquad \hat{a}_i = a_i, \qquad c \to \infty \text{ with } \vartheta, \tilde{\varphi}, a_i \text{ fixed}, \tag{6.1.4}$$

giving rise to the GED action,

$$S = \int \mathrm{d}^{d+1}x \left(-\frac{1}{2} f_{ij} f_{ij} - \tilde{E}_i \partial_i \vartheta + \frac{1}{2} \left(\partial_t \vartheta\right)^2 \right). \tag{6.1.5}$$

The equations of motion for this action are identical to those of the electric limit (cf. table 6.1) along with the additional equation of motion for ϑ, which takes the form $\partial_t^2 \vartheta - \partial_i \tilde{E}_i = 0$. In this sense, the electric limit sits inside the GED model. To obtain the magnetic limit, we may simply (and consistently) set $\vartheta = 0$ in the equations of motion for the GED model (6.1.5). For more details, including transformation properties of the fields in the various limits, we refer the reader to [58].

6.1.2 *Null Reduction of Maxwell Electrodynamics on a Curved Background*

A convenient way to obtain GED coupled to TNC geometry is the null reduction (cf. section 4.3) of the Maxwell action on a $(d+2)$-dimensional Lorentzian background described by a metric γ_{AB},

$$S = -\frac{1}{4} \int \mathrm{d}^{d+2}x \sqrt{-\gamma} \mathcal{F}_{AB} \mathcal{F}^{AB}, \tag{6.1.6}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$. The null reduction ansatz has the form

$$ds^2 = \gamma_{AB} \mathrm{d}x^A \mathrm{d}x^B = 2\tau_\mu \mathrm{d}x^\mu \left(\mathrm{d}u - m_\nu \mathrm{d}x^\nu\right) + h_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{6.1.7}$$

$$\sqrt{-\gamma} = e, \qquad \gamma^{uu} = 2\tilde{\Phi}, \qquad \gamma^{u\mu} = -\hat{v}^\mu, \qquad \gamma^{\mu\nu} = h^{\mu\nu}, \tag{6.1.8}$$

$$\mathcal{A}_\mu = a_\mu - \vartheta m_\mu - \tilde{\varphi} \tau_\mu =: \hat{a}_\mu, \qquad \mathcal{A}_u = \vartheta, \tag{6.1.9}$$

where none of the fields depends on u. In the above, $A, B, C, \ldots$ represent $(d+2)$-dimensional space-time indices, while $\mu, \nu, \ldots$ represent $(d+1)$-dimensional space-time indices and exclude the compact direction u,

For easy reference, we summarize some properties of the Newton-Cartan fields that we will use repeatedly in this chapter (see also chapter 4):

$$v^\mu \tau_\mu = -1, \qquad v^\mu \tau_\nu + \delta^\mu_\nu = e^\mu_a e^a_\nu, \qquad e^a_\mu v^\mu = 0, \qquad e^\mu_a \tau_\mu = 0, \qquad e^\mu_a e^b_\mu = \delta^b_a, \tag{6.1.10}$$

and

$$\hat{v}^\mu = v^\mu - e^\mu_a e^{\nu a} m_\nu, \qquad \hat{e}^a_\mu = e^a_\mu - \tau_\mu m_\nu e^{\nu a}, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - 2m_{(\mu}\tau_{\nu)}, \qquad \tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu. \tag{6.1.11}$$

For the curvatures, we therefore obtain

$$\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\hat{a}_{\nu]} =: \hat{f}_{\mu\nu}, \qquad \mathcal{F}_{\mu u} = \partial_\mu \vartheta, \tag{6.1.12}$$

Thus the term $\mathcal{F}_{AB}\mathcal{F}^{AB}$ in the action (6.1.6) becomes

$$\mathcal{F}_{AB}\mathcal{F}^{AB} = \gamma^{AC}\gamma^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD} = \gamma^{\mu\nu}\gamma^{\rho\sigma}\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu\sigma} + 2\gamma^{uu}\gamma^{\mu\nu}\mathcal{F}_{u\mu}\mathcal{F}_{u\nu} + 4\gamma^{u\mu}\gamma^{\nu\rho}\mathcal{F}_{u\nu}\mathcal{F}_{\mu\rho} + 2\gamma^{u\mu}\gamma^{\nu u}\mathcal{F}_{u\nu}\mathcal{F}_{\mu u} \tag{6.1.13}$$

$$= h^{\mu\nu}h^{\rho\sigma}\hat{f}_{\mu\rho}\hat{f}_{\nu\sigma} + 4\tilde{\Phi}h^{\mu\nu}\partial_\mu\vartheta\partial_\nu\vartheta + 4\hat{v}^\mu h^{\nu\rho}\partial_\nu\vartheta\hat{f}_{\mu\rho} - 2\hat{v}^\mu\hat{v}^\nu\partial_\mu\vartheta\partial_\nu\vartheta, \tag{6.1.14}$$

which produces the GED action

$$S_{\text{GED}} = -\int \mathrm{d}^{d+1}x\, e \left[\frac{1}{4}h^{\mu\nu}h^{\rho\sigma}\hat{f}_{\mu\rho}\hat{f}_{\nu\sigma} + \tilde{\Phi}h^{\mu\nu}\partial_\mu\vartheta\partial_\nu\vartheta + \hat{v}^\mu h^{\nu\rho}\partial_\nu\vartheta\hat{f}_{\mu\rho} - \frac{1}{2}\hat{v}^\mu\hat{v}^\nu\partial_\mu\vartheta\partial_\nu\vartheta\right]. \tag{6.1.15}$$

The GED fields transform under $Barg(d,1) \times U_\Gamma(1)$ as[1] [58]

$$\delta a_\mu = \vartheta e^a_\mu \lambda_a + \tau_\mu a_\nu e^\nu_a \lambda^a + \tau_\mu v^\nu \partial_\nu \Gamma + \partial_\mu \Gamma, \qquad \delta\tilde{\varphi} = a_\nu e^\nu_a \lambda^a + v^\nu \partial_\nu \Gamma, \qquad \delta\vartheta = 0, \tag{6.1.16}$$

where λ^a parametrizes Galilean boosts, while Γ represents a $U_\Gamma(1)$ gauge transformation.

In order to facilitate a comparison with the work on GED in [58], we construct a table below providing a dictionary between different notations.

1 Note the typo in the transformation property of a_μ under gauge transformations in eq. (6.6) in [58].

Festuccia et al. [58]	Notation in present work	Description
A_μ	$\mathcal{A}_{(0)\mu}, \hat{a}_\mu$	b'dary value of 3-dimensional part of 5-dim. gauge field
a_μ	a_μ	Spatial part of "GED gauge field"; satisfies $v^\mu a_\mu = 0$
$\tilde{\varphi}$	$\tilde{\varphi}$	Temporal part of GED gauge field
$\overline{A}_\mu$ (Note the bar!)	$a_\mu - \vartheta\tau_\mu$	The GED gauge field
φ	ϑ	The GED scalar; comes from $\mathcal{A}_u$
M_μ	m_μ	Newton-Cartan gauge field
τ_μ	τ_μ	Newton-Cartan clock form
v^μ	v^μ	Inverse NC clock form, $v^\mu\tau_\mu = -1$
e^a_μ	e^a_μ	NC vielbein
e^μ_a	e^μ_a	Inverse NC vielbein, $e^a_\mu e^\mu_b = \delta^a_b$

Table 6.2: Dictionary between [58] and this work.

6.1.3 *Dimensional Analysis & Scaling Weights for Anisotropic Backgrounds*

The GED action (6.1.5) enables us to determine the dimensionality of the GED fields and their associated sources when the background is anisotropic, characterized by time scaling with $z > 1$ and spatial directions scaling with weight one. Restricting to $d = 2$, the measure acquires scaling weight $-(z+2)$. Introducing the scaling weight operator $\Delta[\cdot]$, we observe that for a_i, the relevant piece of the action reads $f_{ij}f_{ij}$. Now, since $\Delta[f_{ij}] = 1 + \Delta[a_i]$ due to the *spatial* derivative, we obtain the relation

$$2 + z = 2 + 2\Delta[a_i] \Rightarrow \Delta[a_i] = \frac{z}{2}. \tag{6.1.17}$$

For ϑ, the piece $(\partial_t\vartheta)^2$ in the action yields

$$2 + z = 2(2 + \Delta[\vartheta]) \Rightarrow \Delta[\vartheta] = \frac{z-2}{2}, \tag{6.1.18}$$

while the term $\partial_i\tilde{\varphi}\partial_i\vartheta$ gives us

$$2 + z = 2 + \Delta[\tilde{\varphi}] + \Delta[\vartheta] = \frac{2+z}{2} + \Delta[\tilde{\varphi}] \Rightarrow \Delta[\tilde{\varphi}] = \frac{2+z}{2}. \tag{6.1.19}$$

We note that this is consistent since the last piece $\partial_t a_i \partial_i \vartheta$ has scaling weight $2 + z/2 + 1 + (z-2)/2 = 2 + z$. To summarize: on anisotropic backgrounds, the fields of GED scale as in table 6.3.

Field	a_i	$\tilde{\varphi}$	ϑ
Weight	$z/2$	$z/2+1$	$z/2-1$

Table 6.3: Scaling Weights of the GED fields.

The GED fields generally couple to "GED charges", which we will simply refer to as VEVs. When considering such couplings, we must add a Lagrangian of the form

$$\mathcal{L}_{\text{VEV}} = j^i a_i + j^0\tilde{\varphi} + \langle O_\vartheta\rangle\,\vartheta. \tag{6.1.20}$$

Performing the same analysis as above, we find that the VEVs have scaling weights as in table 6.4.

VEV	j^i	j^0	$\langle O_\vartheta\rangle$
Weight	$z/2+2$	$z/2+1$	$z/2+3$

Table 6.4: Scaling Weights of the GED VEVs.

6.2 THE ELECTROMAGNETIC UPLIFT

6.2.1 *The Uplift & $z = 2$ Pure Lifshitz Holography*

In this section, we review the results of [23, 24]. This is also the approach of [57], which deals with hydrodynamics in the context of Lifshitz holography. As we will discuss below, when making the choices

$$Z = e^{3\Phi}, \qquad W = 4, \qquad V(\Phi) = 2e^{-3\Phi} - 12e^{-\Phi}, \qquad k = 3, \tag{6.2.1}$$

in the EPD model (5.2.1), i.e. when

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(R - \frac{1}{4}e^{3\Phi}F^2 - 2B^2 - \frac{3}{2}\left(\partial\Phi\right)^2 - 2e^{-3\Phi} - 12e^{-\Phi} \right), \tag{6.2.2}$$

the action (6.2.2) can be *uplifted* to the following five-dimensional action,

$$S_{5\mathrm{d}} = \frac{1}{2\kappa_5^2} \int \mathrm{d}^5x \sqrt{-\mathcal{G}} \left(R^{(\mathcal{G})} + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi \right), \tag{6.2.3}$$

with $\kappa_5^2 = 8\pi G_5$ and $\mathcal{M} = (u, M)$. The reduced model admits $z = 2$ Lifshitz solutions.

The action (6.2.2) is the S^1 Scherk-Schwarz reduction of the uplift (H.2.9) (see appendix H): here, the global symmetry is the shift symmetry of ψ; i.e. the action (H.2.9) is invariant under $\psi \to \psi + \Lambda$ for Λ a constant. When going around the circle once, the reduction scheme tells us that ψ should come back to itself plus a local shift.

Explicitly, the reduction ansatz has the form

$$ds_5^2 = e^{-\Phi}g_{MN}\mathrm{d}x^M\mathrm{d}x^N + e^{2\Phi}\left(\mathrm{d}u + A_M\mathrm{d}x^M\right)^2, \tag{6.2.4}$$

$$\psi = 2u + 2\Xi, \tag{6.2.5}$$

where $\mathcal{M}, \mathcal{N}, \ldots$ are five-dimensional space-time indices, and $M, N, \ldots$ are four-dimensional space-time indices that exclude the compact direction u. The four-dimensional fields g_{MN}, A_M, Ξ and Φ do not depend on the compactified u–direction, which—since we're reducing on a circle of radius L—is periodically identified, $u \sim u + 2\pi L$. Note also that since our normalization is such that $\frac{1}{16\pi G_4} = 1$, the five-dimensional Newton constant satisfies $\frac{2\pi L}{16\pi G_5} = 1$.

The renormalized on-shell four-dimensional EPD $z = 2$ action has the schematic form

$$S_{\mathrm{ren}} = S + S_{\mathrm{gh}} + S_{\mathrm{ct}}, \tag{6.2.6}$$

where $S_{\mathrm{gh}} = 2\int \mathrm{d}^3x\, \sqrt{-h}K$ is the Gibbons-Hawking boundary term.

Performing this reduction gives the action [57] (we present a detailed derivation of this result in appendix H)

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(R - \frac{1}{4}e^{3\Phi}F_{MN}F^{MN} - 2B_MB^M - \frac{3}{2}\partial_M\Phi\partial^M\Phi - V \right) + 2\int \mathrm{d}^3x\, \sqrt{-h}K + S_{\mathrm{ct}}, \tag{6.2.7}$$

where $B_\mu = A_\mu - \partial_\mu\Xi$.

6.2.2 *Scherk-Schwarz Reduction of the Electromagnetic Uplift*

We now add a free abelian gauge field to the five-dimensional model considered above, resulting in what we shall call the *electromagnetic uplift*,

$$S = \frac{1}{2\kappa_5^2} \int \mathrm{d}^5x\, \sqrt{-\mathcal{G}} \left(R + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi - \frac{1}{4}\mathcal{F}_{\mathcal{M}\mathcal{N}}\mathcal{F}^{\mathcal{M}\mathcal{N}} \right), \tag{6.2.8}$$

where $\mathcal{F}_{\mathcal{M}\mathcal{N}}$ is the field strength of the $U(1)$ gauge field $\mathcal{A}_{\mathcal{M}}$, i.e.

$$\mathcal{F}_{\mathcal{M}\mathcal{N}} = 2\partial_{[\mathcal{M}}\mathcal{A}_{\mathcal{N}]}. \tag{6.2.9}$$

The equations of motion are

$$G_{\mathcal{MN}} = \frac{1}{2}\left(\partial_{\mathcal{M}}\psi\partial_{\mathcal{N}}\psi - \frac{1}{2}(\partial\psi)^2\mathcal{G}_{\mathcal{MN}}\right) + 6\mathcal{G}_{\mathcal{MN}} + \frac{1}{2}\left(\mathcal{F}_{\mathcal{MP}}\mathcal{F}_{\mathcal{N}}{}^{\mathcal{P}} - \frac{1}{4}\mathcal{F}^2\mathcal{G}_{\mathcal{MN}}\right), \tag{6.2.10}$$

$$\frac{1}{\sqrt{-\mathcal{G}}}\partial_{\mathcal{M}}\left(\sqrt{-\mathcal{G}}\partial^{\mathcal{M}}\psi\right) = 0, \quad \frac{1}{\sqrt{-\mathcal{G}}}\partial_{\mathcal{M}}\left(\sqrt{-\mathcal{G}}\mathcal{F}^{\mathcal{MN}}\right) = 0, \tag{6.2.11}$$

where $G_{\mathcal{MN}}$ is the five-dimensional Einstein tensor of $\mathcal{G}_{\mathcal{MN}}$. Upon reduction, the action (6.2.8) will generically produce an extra gauge field and a scalar[2] compared to the pure uplift considered in section 6.2.1. The massless scalar $\Upsilon = \mathcal{A}_u$, with its origin in the five-dimensional gauge field, is associated with spontaneous breaking of the five-dimensional $U_{\hat{\Gamma}}(1)$ gauge transformations, $\delta_{\hat{\Gamma}}\mathcal{A}_{\mathcal{M}} = \partial_{\mathcal{M}}\hat{\Gamma}$, that depend on the compactification coordinate u. In the reduced theory, we thus have diffeomorphisms and gauge transformations of the form

$$\hat{\zeta}^N = \zeta^N(x), \qquad \hat{\zeta}^u = \Lambda(x), \qquad \hat{\Gamma} = \Gamma(x). \tag{6.2.12}$$

Now, we want to identify the Maxwell field of the reduced theory. The naïve choice $\mathcal{A}_M$ does not transfrom correctly under KK $U_\Lambda(1)$ gauge transformations—that is, gauge transformations of A, which is a remnant of diffeomorphism invariance in the compact direction. This is an ubiquitous feature of KK reductions when p-form fields are involved, and also happens e.g. when reducing eleven-dimensional supergravity to ten-dimensional type IIB supergravity. The higer-dimensional gauge field $\mathcal{A}_{\mathcal{M}}$ transforms under diffeomorphisms and $U_{\hat{\Gamma}}(1)$ gauge transformations, we may identity the correct lower dimensional gauge field,

$$\delta\mathcal{A}_{\mathcal{M}} = \hat{\zeta}^{\mathcal{N}}\partial_{\mathcal{N}}\mathcal{A}_{\mathcal{M}} + \mathcal{A}_{\mathcal{N}}\partial_{\mathcal{M}}\hat{\zeta}^{\mathcal{N}} + \partial_{\mathcal{M}}\hat{\Gamma}. \tag{6.2.13}$$

We are thus lead to conclude that the naïve reduction $\mathcal{A}_M$ transforms according to

$$\delta\mathcal{A}_M = \zeta^N\partial_N\mathcal{A}_M + \mathcal{A}_N\partial_M\zeta^N + \overbrace{\mathcal{A}_u}^{=\Upsilon}\partial_M\Lambda + \partial_M\Gamma, \tag{6.2.14}$$

which transforms as a lower-dimensional vector field but also under $U_\Lambda(1)$ gauge transformations. The combination

$$C_M = \mathcal{A}_M - \Upsilon A_M, \tag{6.2.15}$$

transforms as desired, so we identify it as the lower-dimensional gauge field. Under u-independent gauge transformations $\hat{\Gamma}(x,u) = \Gamma(x)$—which we will denote $U_\Gamma(1)$—only C_M transforms: $\delta_\Gamma C_M = \partial_M\Gamma$.

We now identify the four-dimensional field strengths, which is closely related to $\mathcal{F}_{MN}$, which is manifestly invariant under $U_{\hat{\Gamma}}(1)$ and $U_\Lambda(1)$ transformations. The frame version of the field strength reads

$$\mathcal{F}_{\underline{ab}} = e^M_{\underline{a}}e^N_{\underline{b}}\left(2\partial_{[M}C_{N]} + 2\Upsilon\partial_{[M}A_{N]}\right), \tag{6.2.16}$$

where H_{MN} is $U_\Lambda(1)$ and $U_\Gamma(1)$ invariant. For the scalar, this produces the term

$$\mathcal{F}_{\underline{a}u} = e^\Phi\partial_{\underline{a}}\Upsilon, \tag{6.2.17}$$

implying that

$$\mathcal{F}^2 = \mathcal{F}_{\underline{ab}}\mathcal{F}^{\underline{ab}} + 2\mathcal{F}_{\underline{a}u}\mathcal{F}^{\underline{a}}{}_u = H^2 + \Upsilon^2F^2 + 2\Upsilon H_{MN}F^{MN} + 2e^{2\Phi}\partial_M\Upsilon\partial^M\Upsilon, \tag{6.2.18}$$

where H is the field strength for C and F is the field strength for A. Using $\frac{\pi L}{\kappa_5^2} = 1$, we can summarize our findings as

$$\frac{1}{2\kappa_5^2}\sqrt{-\mathcal{G}}\frac{1}{4}\mathcal{F}_{\mathcal{MN}}\mathcal{F}^{\mathcal{MN}} \overset{\text{reduces to}}{\longrightarrow} \frac{\sqrt{-g}e^{-\Phi}}{4}\left(H^2 + \Upsilon^2F^2 + 2\Upsilon H_{MN}F^{MN} + 2e^{2\Phi}\partial_M\Upsilon\partial^M\Upsilon\right). \tag{6.2.19}$$

2 More specifically, we get the usual KK tower of massive (i.e. Stückelberged) gauge fields. We keep only the massless vector and the massless scalar.

Since nothing else is changed from uplift discussed in section 6.2.1, we infer that the reduced theory has the form of an EPD-Maxwell-scalar model,

$$S = \int d^4x \sqrt{-g}\Big(R - \frac{1}{4}\left(e^{3\Phi} + e^{-\Phi}Y^2\right)F^2 - 2B^2 - \frac{3}{2}(\partial\Phi)^2 - \frac{1}{4}e^{-\Phi}H^2 \tag{6.2.20}$$

$$- \frac{1}{2}Ye^{-\Phi}H_{MN}F^{MN} - \frac{1}{2}e^{\Phi}(\partial Y)^2 - 2e^{-3\Phi} - 12e^{-\Phi}\Big), \tag{6.2.21}$$

Note that the counterterm is unchanged under addition of the additional Maxwell field, as we demonstrated in section 3.2.1. This is a very non-trivial result, since it implies that the extra massless field in the lower-dimensional theory do not give rise to divergences in the case of Lifshitz boundary conditions. Explicitly, it reads

$$S_{\text{ct}} = 2\int_{\partial\mathcal{M}} d^3x \sqrt{-\tilde{h}}\left(-\frac{1}{4}e^{\Phi/2}\left[R^{(\tilde{h})} - \frac{3}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}e^{3\Phi}F_{\mu\nu}F^{\mu\nu} - 2B_\mu B^\mu + 10e^{-\Phi}\right]\right) \tag{6.2.22}$$

$$- \log r\int_{\partial\mathcal{M}} d^3x \sqrt{-\tilde{h}}e^{-\Phi/2}\mathcal{A}_{\text{red}}, \tag{6.2.23}$$

where $\tilde{h}$ is the boundary metric. Here, r is the radial coordinate (which is inherited from the $z = 0$ Schrödinger space-time), and $\mathcal{A}_{\text{red}}$ is the Scherk-Schwarz reduced Weyl anomaly—we will discuss this later in section 6.5.

6.2.3 *Variation of the Action*

After the reduction, the renormalized action reads

$$S_{\text{ren}} = S + 2\int_{\partial\mathcal{M}} d^3x\sqrt{-\tilde{h}}K + S_{\text{ct}}, \tag{6.2.24}$$

where S is given in (6.2.21), and S_{ct} in (6.2.23). The total variation of the renormalized action consequently takes the form

$$\delta S_{\text{ren}} = \int_{\mathcal{M}} d^4x \sqrt{-g}\left(\mathcal{E}_{MN}\delta g^{MN} + \mathcal{E}^N_{(B)}\delta B_N + \mathcal{E}_\Phi\delta\Phi + \mathcal{E}_Y\delta Y + \mathcal{E}^N_{(C)}\delta C_N\right) \tag{6.2.25}$$

$$+ \int_{\partial\mathcal{M}} d^3x \sqrt{-\tilde{h}}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^\nu\delta B_\nu + T_\Phi\delta\Phi + \mathfrak{T}^\nu\delta C_\nu + \mathcal{T}_Y\delta Y - \frac{\delta r}{r}e^{-\Phi/2}\mathcal{A}_{\text{red}}\right), \tag{6.2.26}$$

where

$$\mathcal{E}_{MN} = G_{MN} + \frac{1}{8}\left(e^{3\Phi} + e^{-\Phi}Y^2\right)g_{MN}F_{PQ}F^{PQ} - \frac{1}{2}\left(e^{3\Phi} + e^{-\Phi}Y^2\right)F_{MP}F_N{}^P + g_{MN}B_PB^P - 2B_MB_N \tag{6.2.27}$$

$$+ \frac{3}{4}g_{MN}\partial_P\Phi\partial^P\Phi - \frac{3}{2}\partial_M\Phi\partial_N\Phi + g_{MN}e^{-\Phi}\left(-6 + e^{-2\Phi}\right) + \frac{1}{8}e^{-\Phi}H_{PQ}H^{PQ}g_{MN} - \frac{1}{2}e^{-\Phi}H_{MP}H_N{}^P \tag{6.2.28}$$

$$+ \frac{1}{4}e^{-\Phi}YH_{PQ}F^{PQ}g_{MN} - e^{-\Phi}YH_{MP}F_N{}^P - \frac{1}{2}e^{\Phi}\partial_MY\partial_NY + \frac{1}{4}e^{\Phi}g_{MN}\partial_PY\partial^PY, \tag{6.2.29}$$

$$\mathcal{E}_\Phi = 3\Box_{(g)}\Phi - \frac{1}{4}\left(3e^{3\Phi} - e^{-\Phi}Y^2\right)F_{MN}F^{MN} + 6e^{-3\Phi} - 12e^{-\Phi} \tag{6.2.30}$$

$$+ \frac{1}{4}e^{-\Phi}H_{MN}H^{MN} + \frac{1}{2}e^{-\Phi}YH_{MN}F^{MN} + \frac{1}{2}e^{\Phi}\partial_MY\partial^MY, \tag{6.2.31}$$

$$\mathcal{E}^N_{(B)} = \nabla_M\left(\left[e^{3\Phi} + e^{-\Phi}Y^2\right]F^{MN} + e^{-\Phi}H^{MN}\right) - 4B^N, \tag{6.2.32}$$

$$\mathcal{E}_Y = \frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}e^{\Phi}\partial^MY\right) - \frac{1}{2}e^{-\Phi}YF^2 - \frac{1}{2}e^{-\Phi}H^{MN}F_{MN} \tag{6.2.33}$$

$$\mathcal{E}^N_{(C)} = \nabla_M\left(e^{-\Phi}H^{MN} + Ye^{-\Phi}F^{MN}\right), \tag{6.2.34}$$

as well as

$$T_{\mu\nu} = -2Kh_{\mu\nu} + 2K_{\mu\nu} - e^{\Phi/2}G^{(\tilde{h})}_{\mu\nu} + 5e^{-\phi/2}\tilde{h}_{\mu\nu} \tag{6.2.35}$$

$$+\frac{1}{2}e^{7\Phi/2}F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{8}e^{7\Phi/2}h_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - e^{\Phi/2}h_{\mu\nu}B_{\rho}B^{\rho} + 2e^{\Phi/2}B_{\mu}B_{\nu} \tag{6.2.36}$$

$$+\frac{1}{2}e^{\Phi/2}\left(\nabla_{(\tilde{h})\mu}\partial_{\nu}\Phi - \tilde{h}_{\mu\nu}\Box_{(\tilde{h})}\Phi\right) + \frac{7}{4}e^{\Phi/2}\partial_{\mu}\Phi\partial_{\nu}\Phi - e^{\Phi/2}\tilde{h}_{\mu\nu}\partial_{\rho}\Phi\partial^{\rho}\Phi, \tag{6.2.37}$$

$$T_{\Phi} = -3n^{M}\partial_{M}\Phi - \frac{1}{4}e^{\Phi/2}R^{(\tilde{h})} - \frac{3}{8}e^{\Phi/2}\partial_{\mu}\Phi\partial^{\mu}\Phi - \frac{3}{2}e^{\Phi/2}\Box_{(\tilde{h})}\Phi \tag{6.2.38}$$

$$+\frac{7}{16}e^{7\Phi/2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^{\Phi/2}B_{\mu}B^{\mu} + \frac{5}{2}e^{-\Phi/2}, \tag{6.2.39}$$

$$\mathcal{T}^{\nu} = -\left(\left[e^{3\Phi} + e^{-\Phi}\Upsilon^{2}\right]F^{M\nu} + e^{-\Phi}H^{M\nu}\right)n_{M} - \frac{1}{2}\nabla_{(\tilde{h})\mu}\left(e^{7\Phi/2}F^{\mu\nu}\right) + 2e^{\Phi/2}B^{\nu}, \tag{6.2.40}$$

$$\mathscr{T}^{\nu} = -\left(e^{-\Phi}H^{M\nu} + 2\Upsilon e^{-\Phi}F^{M\nu}\right)n_{M}, \tag{6.2.41}$$

$$\mathcal{T}_{\Upsilon} = -e^{\Phi}n^{M}\partial_{M}\Upsilon. \tag{6.2.42}$$

Note that we have not varied counterterm in the expressions above, so the expression are correct up to $\log r$-terms. The reduced bulk is described by the metric

$$ds^{2} = e^{\Phi}\frac{\mathrm{d}r^{2}}{r^{2}} + \tilde{h}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \tag{6.2.43}$$

so the vector normal to hyperplanes of constant r is given by

$$n^{M} = -\mathcal{N}\delta^{M}_{r}, \tag{6.2.44}$$

where $\mathcal{N}$ is a normalization constant and the minus is due to the fact that the boundary is located at $r = 0$. The normalization constant is determined by the condition $g_{MN}n^{M}n^{N} = 1$, which translates into the requirement $1 = g_{rr}\mathcal{N}^{2} = \mathcal{N}^{2}e^{\Phi}/r$, which in turn implies that $\mathcal{N} = re^{-\Phi/2}$. This means that we have the following relation for the extrinsic curvature

$$K = \tilde{h}^{\mu\nu}K_{\mu\nu}, \qquad K_{\mu\nu} = -\frac{1}{2}\pounds_{n}\tilde{h}_{\mu\nu}, \qquad n^{M} = -re^{-\Phi/2}\delta^{M}_{r} \tag{6.2.45}$$

When appropriately truncated, these results agree with those of [57].

6.2.4 *Oxidation to a $z = 0$ Schrödinger Geometry*

Starting from the $z = 0$ Schrödinger geometry (see also previous work in [24, 57, 154, 155])

$$ds^{2}_{5} = \frac{\mathrm{d}r^{2}}{r^{2}} + \frac{1}{r^{2}}\left(2\mathrm{d}t\mathrm{d}u + \mathrm{d}x^{2} + \mathrm{d}y^{2}\right) + \mathrm{d}u^{2}, \tag{6.2.46}$$

$$\psi = 2u + 2\Xi_{(0)}, \tag{6.2.47}$$

$$\mathcal{A} = \mathcal{A}_{(0)\mu}\mathrm{d}x^{\mu}, \tag{6.2.48}$$

where $\Xi_{(0)}$ and $\mathcal{A}_{(0)}$ are constants, the Scherk-Schwarz reduction ansatz (H.2.10) gives rise to the Lifshitz solution in the reduced theory:

$$ds^{2} = e^{\Phi_{(0)}}\left(\frac{\mathrm{d}r^{2}}{r^{2}} - e^{-2\Phi_{(0)}}\frac{\mathrm{d}t^{2}}{r^{4}} + \frac{1}{r^{2}}\left(\mathrm{d}x^{2} + \mathrm{d}y^{2}\right)\right), \tag{6.2.49}$$

$$B = e^{-2\Phi_{(0)}}r^{-2}\mathrm{d}t \qquad \text{(Constant Stückelberg field)}, \tag{6.2.50}$$

$$\Phi = \Phi_{(0)}, \tag{6.2.51}$$

$$\Upsilon = \Upsilon_{(0)}, \tag{6.2.52}$$

$$C = \mathcal{A}_{(0)\mu}\mathrm{d}x^{\mu} - \Upsilon_{(0)}e^{-2\Phi_{(0)}}r^{-2}\mathrm{d}t. \tag{6.2.53}$$

This solution can be verified by plugging it into the equations of motion we found above.

The $z = 0$ Schrödinger metric (6.2.46) satisfies the AlAdS requirement (A.2.5), since e.g.

$$R_{trru} = \frac{1}{r^{4}} = \mathcal{G}_{tr}\mathcal{G}_{ru} - \mathcal{G}_{tu}\mathcal{G}_{rr} + \mathcal{O}(r^{-3}), \tag{6.2.54}$$

while a similar analysis for the other seven non-zero components of the Riemann tensor shows that, indeed, the $z = 0$ Schrödinger metric belongs to the class of AlAdS metrics, which is what permits the use of the results obtained in chapter 3.

6.3 SOURCES & THEIR TRANSFORMATIONS

In this section, we find the sources of charged Lifshitz Holography. We will be guided by the fact that we expect to see GED (see [58]) on the boundary, so we can define our sources in such a way that they transform appropriately. Guided by insights from [23, 24] involving the reduced conformal anomaly—which eventually led to the discovery that dynamical Newton-Cartan geometry corresponds to (non-relativistic) Hořava-Lifshitz (HL) gravity [53]—we expect that the anomaly becomes a Lagrangian describing GED coupled to HL gravity.

6.3.1 *Defining the Sources*

We begin our discussion with a recap of the known sources, which were found in [23, 24, 57]. Writing the metric (6.2.43) in terms of vielbeine,

$$ds^2 = e^{\Phi}\frac{\mathrm{d}r^2}{r^2} - E^0E^0 + \delta_{ab}E^aE^b, \tag{6.3.1}$$

and performing a Stückelberg decomposition of the massive vector, $B_\nu = A_\nu - \partial_\nu\Xi$, we find the sources $v^\mu, e^\mu_a, m_\mu, \phi, \chi$, defined as

$$\begin{aligned} E^\mu_0 &\simeq -r^2\alpha_{(0)}^{-1/3}v^\mu, &(6.3.2)\\ E^\mu_a &\simeq r\alpha_{(0)}^{1/3}e^\mu_a, &(6.3.3)\\ A_\mu - \alpha(\Phi)E^0_\mu &\simeq -m_\mu, &(6.3.4)\\ \Phi &\simeq \phi, &(6.3.5)\\ \Xi &\simeq -\chi, &(6.3.6)\\ \varphi &\simeq r^2\alpha_{(0)}^{-1/3}v^\mu m_\mu, &(6.3.7)\\ A_a &\simeq -r\alpha_{(0)}^{1/3}e^\mu_a m_\mu, &(6.3.8)\end{aligned}$$

where (in the notation of chapter 5)

$$\alpha(\Phi) = e^{-3\Phi/2}, \qquad \alpha = \alpha_{(0)} + \cdots + r^2\alpha_{(0)}^{-1/3}\left(\tilde{I} - I\right) + \ldots, \tag{6.3.9}$$

where the dots hide a logarithmic term, and

$$m_\mu = M_\mu + \partial_\mu\chi. \tag{6.3.10}$$

The source M_μ appears in the expansion of the vielbein E^0_μ at order $\mathcal{O}(r^0)$ (see e.g. [33] and our discussion in chapter 5)

$$E^0_\mu = r^{-2}\alpha_{(0)}^{1/3}\tau_\mu + \cdots + \alpha_{(0)}^{-1}M_\mu + \cdots, \tag{6.3.11}$$

where we have omitted logarithmic terms. Note that the dots hide additional N, G, J-invariant terms at order $\mathcal{O}(r^0)$, see section 6.6. Note further that combining the expansion (6.3.11) with the definition of m_μ (6.3.4) and the expansion for α in (6.3.9) gives us the relation

$$A_\mu = \alpha_{(0)}^{4/3}r^{-2}\tau_\mu + \cdots - \partial_\mu\chi + \cdots, \tag{6.3.12}$$

where, again, the dots hide invariant terms of order $\mathcal{O}(r^0)$. Now, from a reduction perspective, the Maxwell field consists of two pieces,

$$C_\mu = \mathcal{A}_\mu - \Upsilon A_\mu, \tag{6.3.13}$$

which we see reflected at the solution level. Guided by the expected behaviour of the anomaly discussed in section 6.5—i.e. that the anomaly should become HL gravity coupled to GED—we define a new source $(a_\mu, \tilde{\varphi})$

$$C_\mu + \Upsilon\alpha(\Phi)E^0_\mu \simeq a_\mu - \tilde{\varphi}\tau_\mu, \tag{6.3.14}$$

where we have performed a decomposition into spatial and temporal directions; that is, in the above we have that $a_\mu v^\mu = 0$. Note that this is completely analogous to the way the source m_μ appears in (6.3.4).

We can isolate the time component of this source by contracting it with E_0^μ, giving

$$\Omega := E_0^\mu C_\mu + \Upsilon\alpha \simeq -r^2 \alpha_{(0)}^{-1/3} \tilde{\varphi}, \tag{6.3.15}$$

which is the analogue of (6.3.7). The relation $E^0_\mu E^\mu_a = \delta^0_a = 0$ further gives us (when applying E^μ_a to (6.3.14))

$$C_a \simeq r\alpha_{(0)}^{1/3} a_a. \tag{6.3.16}$$

This allows us to identify the fall-off of C_μ using the behaviour of A_μ in (6.3.12)

$$\begin{aligned} C_\mu &= -\vartheta \alpha_{(0)}^{4/3} \tau_\mu r^{-2} + \cdots + a_\mu - \vartheta\left(m_\mu - \partial_\mu \chi\right) - \tilde{\varphi}\tau_\mu + \cdots && (6.3.17)\\ &= -\vartheta \alpha_{(0)}^{4/3} \tau_\mu r^{-2} + \cdots + a_\mu - \vartheta M_\mu - \tilde{\varphi}\tau_\mu + \cdots && (6.3.18)\end{aligned}$$

In summary, our holographic setup involves the following sources

$$\begin{aligned} E_0^\mu &\simeq -r^2\alpha_{(0)}^{-1/3} v^\mu, && (6.3.19)\\ E_a^\mu &\simeq r\alpha_{(0)}^{1/3} e_a^\mu, && (6.3.20)\\ A_\mu - \alpha(\Phi) E^0_\mu &\simeq -m_\mu, && (6.3.21)\\ \Phi &\simeq \phi, && (6.3.22)\\ \Xi &\simeq -\chi, && (6.3.23)\\ \varphi &\simeq r^2 \alpha_{(0)}^{-1/3} v^\mu m_\mu, && (6.3.24)\\ \Upsilon &\simeq \vartheta && (6.3.25)\\ C_\mu + \Upsilon\alpha(\Phi) E^0_\mu &\simeq a_\mu - \tilde{\varphi}\tau_\mu, && (6.3.26)\\ E_0^\nu C_\nu + \Upsilon\alpha(\Phi) &\simeq -r^2\alpha_{(0)}^{-1/3}\tilde{\varphi}, && (6.3.27)\\ C_a &\simeq r\alpha_{(0)}^{1/3} a_a, && (6.3.28)\end{aligned}$$

where $a_\mu v^\mu = 0$.

6.3.2 *Transformation Properties*

The known sources (6.7.12)–(6.7.20) transform as follows [24, 57] (see also chapter 5),

$$\begin{aligned} \delta e^a_\mu &= \pounds_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu. && (6.3.29)\\ \delta\chi &= \pounds_\xi \chi + \sigma, && (6.3.30)\\ \delta v^\mu &= \pounds_\xi v^\mu + \lambda^a e^\mu_a - 2\Lambda_D v^\mu, && (6.3.31)\\ \delta e^\mu_a &= \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b - \Lambda_D, && (6.3.32)\\ \delta\phi &= 0, && (6.3.33)\\ \delta m_\mu &= \pounds_\xi m_\mu + e^a_\mu \lambda_a + \partial_\mu \sigma, && (6.3.34)\end{aligned}$$

The rest of this section is devoted to a detailed analysis of the transformation properties of the new sources $a_\mu, \tilde{\varphi}, \vartheta$.

6.3.2.1 *Galilean Boosts and Rotations*

C_μ is invariant under bulk local Lorentz transformations—in particular boosts. These contract to Galilean boosts near the boundary, which means that in a near-boundary expansion, invariance under local Lorentz boosts is translated into invariance under Galilean boosts at each order of r. Using the near-boundary expansion of C_μ (6.3.18), this gives us at order[3] $\mathcal{O}(r^0)$

$$0 = \delta_G\left(a_\mu - \vartheta M_\mu - \tilde{\varphi}\tau_\mu\right) = \delta_G a_\mu - \vartheta \overbrace{\delta_G M_\mu}^{=\lambda_a e^a_\mu} - \tau_\mu \delta_G \tilde{\varphi}. \tag{6.3.35}$$

3 Since ϑ is a scalar, it transforms trivially, $\delta_G \vartheta = 0$.

Thus, see immediately that we must have

$$\delta_G a_\mu = \vartheta \lambda_a e^a_\mu + \tau_\mu \left(\cdot\right), \tag{6.3.36}$$

where $(\cdot)$ is so far undetermined. However, the condition $v^\mu a_\mu = 0$ provides us with the additional constraint

$$0 = \delta_G(v^\mu a_\mu) = a_\nu \delta_G v^\nu + v^\mu \delta a_\mu = a_\nu \lambda^a e^\nu_a - (\cdot), \tag{6.3.37}$$

where we have used the (6.3.36) as well as the orthogonality property $v^\mu e^a_\mu = 0$. Thus the transformation of a_μ becomes

$$\delta_G a_\mu = \vartheta \lambda_a e^a_\mu + \tau_\mu a_\nu e^\nu_a \lambda^a, \tag{6.3.38}$$

and hence (6.3.35) gives us that

$$\delta_G \tilde{\varphi} = a_\nu e^\nu_a \lambda^a. \tag{6.3.39}$$

For rotations, we do the same as above. We use invariance of the bulk field C_μ under rotations, i.e. $\delta_J C_\mu = 0$, which implies that at order $\mathcal{O}(r^0)$ we have

$$0 = \delta_J \left(a_\mu - \vartheta M_\mu - \tilde{\varphi}\tau_\mu\right) = \delta_J a_\mu - \tau_\mu \delta_G \tilde{\varphi}. \tag{6.3.40}$$

As with boosts, rotation invariance at order $\mathcal{O}(r^{-2})$ is manifest. Hence, as before, $\delta_J a_\mu$ will be proportional to τ. Using again the orthogonality property $a_\mu v^\mu = 0$, we find that $\delta_J a_\mu = 0$ since $\delta_J v^\mu = 0$, leading us to conclude that

$$\delta_J a_\mu = 0 = \delta_J \tilde{\varphi}. \tag{6.3.41}$$

6.3.2.2 *Gauge Transformations*

There are two types of gauge transformations: Stückelberg gauge transformations, $U_\Lambda(1)$, and gauge transformations of $C_\mu \to C_\mu + \partial_\mu \Gamma$, which we will call $U_\Gamma(1)$. Starting with $U_\Gamma(1)$, we recall that C_μ transforms—by construction—according to

$$\delta_\Gamma C_\mu = \partial_\mu \Gamma, \tag{6.3.42}$$

where Γ has an expansion of the form (reflecting the structure of C_μ in (6.3.18))

$$\Gamma = \Gamma_{(0)} + \Gamma_{(1,0)} \log r + \Gamma_{(2)} r^2 + \cdots . \tag{6.3.43}$$

At order $\mathcal{O}(r^0)$, the transformation (6.3.42) takes the form

$$\partial_\mu \Gamma = \delta_\Gamma a_\mu - \tau_\mu \delta_\Gamma \tilde{\varphi}, \tag{6.3.44}$$

since neither ϑ nor M_μ transforms under $U_\Gamma(1)$. Combining this with the orthogonality relation,

$$0 = \delta_\Gamma(v^\mu a_\mu) = v^\mu \delta_\Gamma a_\mu, \tag{6.3.45}$$

we see that (6.3.44) implies that $\delta_\Gamma a_\mu$ also involves a term proportional to τ, which we can determine using (6.3.45), giving

$$\delta_\Gamma a_\mu = \partial_\mu \Gamma_{(0)} + \tau_\mu v^\nu \partial_\nu \Gamma_{(0)}, \qquad \delta_\Gamma \tilde{\varphi} = v^\mu \partial_\mu \Gamma_{(0)}, \tag{6.3.46}$$

On the boundary $r \to 0$, the gauge group is truncated, i.e. $U_\Gamma(1) \to U_{\Gamma_{(0)}}(1)$, and, similarly, for Stückelberg gauge transformations, we obtain the group $U_\Lambda(1) \to U_\sigma(1)$, where σ is (minus) the boundary value of Λ.

Repeating the calculation for δ_Λ transformations, we obtain immediately that on the boundary (where $\delta_\Lambda \to \delta_\sigma$),

$$\delta_\sigma a_\mu = 0, \qquad \delta_\sigma \tilde{\varphi} = 0, \qquad \delta_\sigma \vartheta = 0. \tag{6.3.47}$$

6.3.2.3 *PBH Transformations: Diffeomorphisms, Dilatations & Stückelberg Gauge Transformations*

The PBH transformations in the electromagnetic uplift translate into boundary diffeomorphisms, Weyl transformations, and Stückelberg gauge transformations. As we saw in section 5.2.3 (see also appendix A of [24]), the PBH transformations are the diffeomorphisms preserving the radial gauge choice of the FG expansion, i.e. diffeomorphisms acting infinitesimally on the five-dimensional fields in the manner

$$\delta_{\hat{\zeta}}\mathcal{G}_{\mathcal{MN}} = \pounds_{\hat{\zeta}}\mathcal{G}_{\mathcal{MN}}, \tag{6.3.48}$$

$$\delta_{\hat{\zeta}}\mathcal{A}_{\mathcal{M}} = \pounds_{\hat{\zeta}}\mathcal{A}_{\mathcal{M}}, \tag{6.3.49}$$

$$\delta_{\hat{\zeta}}\psi = \pounds_{\hat{\zeta}}\psi, \tag{6.3.50}$$

and satisfying $\pounds_{\hat{\zeta}}\mathcal{G}_{rr} = 0 = \pounds_{\hat{\zeta}}\mathcal{G}_{\mathcal{M}r}$. Solving these equations gives

$$\hat{\zeta}^r \simeq -r\Lambda_D, \qquad \hat{\zeta}^A \simeq \hat{\xi}^A, \tag{6.3.51}$$

where $\hat{\xi}$ and Λ_D are independent of r. This leads to the transformation properties

$$\delta_{\hat{\xi}}\gamma_{(0)AB} = \hat{\xi}^C\partial_C\gamma_{(0)AB} + \gamma_{(0)CB}\partial_A\hat{\xi}^C + \gamma_{(0)AC}\partial_B\hat{\xi}^C + 2\Lambda_D\gamma_{(0)AB}, \tag{6.3.52}$$

$$\delta_{\hat{\xi}}\psi_{(0)} = \hat{\xi}^A\partial_A\psi_{(0)}, \tag{6.3.53}$$

$$\delta_{\hat{\xi}}\mathcal{A}_{(0)A} = \hat{\xi}^B\partial_B\mathcal{A}_{(0)A} + \mathcal{A}_{(0)B}\partial_A\hat{\xi}^B. \tag{6.3.54}$$

In the reduced theory, we take the diffeomorphisms to be independent of the compact direction. Writing $\hat{\xi}^\mu = \xi^\mu$ and $-\sigma$ for the u-component of ξ, we get diffeomorphisms, Weyl transformations and Stückelberg gauge transformations in exactly the same fashion as in section 5.2.3.

6.3.2.4 *Residual Transformations*

In addition to the diffeomorphisms and gauge transformations (6.2.12) in the reduced theory, the reduction is also preserved under the u-dependent transformations $\hat{\zeta}^u = cu$ and $\hat{\Gamma} = \ell u$, where c, ℓ are constants. These are remnants of full symmetry in the electromagnetic uplift and are only preserved to leading order. We will refer to them as δ_c and δ_ℓ, respectively. In [24], δ_c was called *local dilatations*, and the relevant field transformation reads

$$\delta_c\phi = c. \tag{6.3.55}$$

The GED scalar transforms undre δ_ℓ transformations according to

$$\delta_\ell\vartheta = \ell. \tag{6.3.56}$$

Note that when solving the equations of motion in the reduced theory, as we discussed in section 6.2.4, the δ_ℓ symmetry manifests itself by leaving $\Upsilon_{(0)}$—which is assumed to be constant—undetermined by the equations of motion.

6.3.2.5 *Summary of Local Transformations*

Below, we present our results for the transformations of all our sources

$$\delta e^a_\mu = \pounds_\xi e^a_\mu + \lambda^a\tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu. \tag{6.3.57}$$

$$\delta\chi = \pounds_\xi\chi + \sigma, \tag{6.3.58}$$

$$\delta v^\mu = \pounds_\xi v^\mu + \lambda^a e^\mu_a - 2\Lambda_D v^\mu, \tag{6.3.59}$$

$$\delta e^\mu_a = \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b - \Lambda_D e^\mu_a, \tag{6.3.60}$$

$$\delta\phi = \pounds_\xi\phi, \tag{6.3.61}$$

$$\delta m_\mu = \pounds_\xi m_\mu + e^a_\mu\lambda_a + \partial_\mu\sigma, \tag{6.3.62}$$

$$\delta\vartheta = \pounds_\xi\vartheta, \tag{6.3.63}$$

$$\delta\tilde{\varphi} = \pounds_\xi\tilde{\varphi} + a_a\lambda^a + v^\mu\partial_\mu\Gamma_{(0)} - 2\Lambda_D\tilde{\varphi}, \tag{6.3.64}$$

$$\delta a_\mu = \pounds_\xi a_\mu + \vartheta\lambda_a e^a_\mu + \tau_\mu a_\nu e^\nu_a\lambda^a + \partial_\mu\Gamma_{(0)} + \tau_\mu v^\nu\partial_\nu\Gamma_{(0)}, \tag{6.3.65}$$

$$\delta a_a = \pounds_\xi a_a + \vartheta\lambda_a + \lambda_a{}^b a_b - \Lambda_D a_a + e^\mu_a\partial_\mu\Gamma_{(0)}. \tag{6.3.66}$$

Note that the sources $a_\mu, \vartheta, \tilde{\varphi}$ precisely transform as GED fields (6.1.16). We also see that the symmetry group is the direct product group

$$Diff(\mathcal{M}) \times Sch_2(2,1) \times U(1). \tag{6.3.67}$$

Under dilatations, we have

Source	ϕ	τ_μ	e^a_μ	v^μ	e^μ_a	m_0	m_a	χ	a_a	$\tilde{\varphi}$	ϑ
scaling dim.	0	-2	-1	2	1	2	1	0	1	2	0

Table 6.5: The sources and their scaling dimensions

We note that for the new sources, this behaviour precisely agrees with the scaling weights determined in the field theory in table 6.3 when putting $z = 2$.

6.4 BOUNDARY GEOMETRY & $\mathfrak{sch}_2(2,1)$

6.4.1 *Null Reduction on the Boundary*

In this section, we show how the Scherk-Schwarz reduction employed in section (6.2) becomes a null reduction—as considered in section 4.3—on the boundary. Recall that

$$A_\mu - \alpha(\Phi)E^0_\mu \simeq -m_\mu, \qquad B_\mu - \alpha(\Phi)E^0_\mu \simeq -M_\mu, \tag{6.4.1}$$

where $B_\mu = A_\mu - \partial_\mu \Xi$, which implies that

$$\left(B_\mu + \partial_\mu \Xi\right) - \alpha(\Phi)E^0_\mu \simeq -M_\mu - \partial_\mu \chi \iff A_\mu - \alpha(\Phi)E^0_\mu \simeq -\left(M_\mu + \partial_\mu \chi\right), \tag{6.4.2}$$

where, explicitly,

$$m_\mu = M_\mu + \partial_\mu \chi, \qquad \text{or} \qquad M_\mu = m_\mu - \partial_\mu \chi. \tag{6.4.3}$$

The reduction ansatz can be written in the form

$$ds^2_5 = \frac{\mathrm{d}r^2}{r^2} + \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B = e^{-\Phi}\left[e^{\Phi}\frac{\mathrm{d}r^2}{r^2} + \tilde{h}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right] + e^{2\Phi}(\mathrm{d}u + A_\mu \mathrm{d}x^\mu)^2. \tag{6.4.4}$$

where the metric γ_{AB} on leaves of constant r can be written in Fefferman-Graham form (see appendices A and **??**); in particular, the boundary is described by the leading term $\gamma_{(0)AB}$ which has dilatation weight 2, i.e.

$$\gamma_{AB} \simeq r^{-2}\gamma_{(0)AB}. \tag{6.4.5}$$

The four-dimensional part in (C.1.20)—i.e. the quantity in square brackets—can be written in terms of vielbeine in the manner depicted in (6.3.1). We now dissect the ansatz (C.1.20) and write it piece by piece in terms of sources via the near-boundary expansions of the bulk fields worked out in section 6.3. Using the Stückelberg decomposition of the mass gauge field (6.4.3), the first piece of (C.1.20) becomes

$$-e^{-\Phi}E^0_\mu E^0_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu = -\alpha^{2/3}_{(0)}\left(r^{-2}\alpha^{1/3}_{(0)}\tau_\mu + \alpha^{-1}_{(0)}M_\mu + \cdots\right)\left(r^{-2}\alpha^{1/3}_{(0)}\tau_\nu + \alpha^{-1}_{(0)}M_\nu + \cdots\right)\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{6.4.6}$$

$$= \left(-\alpha^{4/3}_{(0)}r^{-4}\tau_\mu\tau_\nu \mathrm{d}x - 2r^{-2}\tau_{(\mu}M_{\nu)} - \alpha^{-1/3}_{(0)}M_\mu M_\nu\right)\mathrm{d}x^\mu\mathrm{d}x^\nu + \cdots, \tag{6.4.7}$$

as well

$$e^{-\Phi}\delta_{ab}E^a_\mu E^b_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu = \alpha^{2/3}_{(0)}\delta_{ab}r^{-2}\alpha^{-2/3}_{(0)}e^a_\mu e^b_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu = r^{-2}e_{a\mu}a^a_\nu =: r^{-2}h_{\mu\nu}, \tag{6.4.8}$$

whereas

$$e^{2\Phi}A_\mu A_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu = e^{2\Phi}\left(B_\mu + \partial_\mu\Xi\right)\left(B_\nu + \partial_\mu\Xi\right)\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{6.4.9}$$

$$= \alpha^{-4/3}_{(0)}\left(\alpha^{4/3}_{(0)}r^{-2}\tau_\mu - \partial_\mu\chi\right)\left(\alpha^{4/3}_{(0)}r^{-2}\tau_\nu - \partial_\nu\chi\right)\mathrm{d}x^\mu\mathrm{d}x^\nu + \cdots \tag{6.4.10}$$

$$= \alpha^{4/3}_{(0)}r^{-4}\tau_\mu\tau_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu - 2r^{-2}\partial_\mu\chi\tau_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu + \alpha^{-4/3}_{(0)}\partial_\mu\chi\partial_\nu\chi\mathrm{d}x^\mu\mathrm{d}x^\nu + \cdots. \tag{6.4.11}$$

The cross term reduces to

$$2e^{2\Phi}A_\mu \mathrm{d}u\mathrm{d}x^\mu = 2e^{2\Phi}\left(B_\mu - \partial_\mu \Xi\right)\mathrm{d}u\mathrm{d}x^\mu \tag{6.4.12}$$

$$= 2\alpha_{(0)}^{-4/3}\left(\alpha_{(0)}^{4/3} r^{-2}\tau_\mu + \partial_\mu \chi\right)\mathrm{d}u\mathrm{d}x^\mu + \cdots \tag{6.4.13}$$

$$= 2r^{-2}\tau_\mu \mathrm{d}u\mathrm{d}x^\mu + 2\alpha_{(0)}^{-4/3}\partial_\mu\chi \mathrm{d}u\mathrm{d}x^\mu + \cdots , \tag{6.4.14}$$

and, finally,

$$e^{2\Phi}\mathrm{d}u^2 = \alpha_{(0)}^{-4/3}\mathrm{d}u^2 + \cdots , \tag{6.4.15}$$

which does not contribute at order r^{-2}. Now, collecting all terms of order r^{-4}, we see that they cancel (cf. equations (6.4.7) and (6.4.11)). The boundary metric $\gamma_{(0)AB}$ is consequently given by

$$ds^2 = \gamma_{(0)AB}\mathrm{d}x^A\mathrm{d}x^B = -2\tau_\mu M_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu - 2\partial_\mu\chi\tau_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu + 2\tau_\mu\mathrm{d}u\mathrm{d}x^\mu + h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{6.4.16}$$

$$= 2\tau_\mu\mathrm{d}x^\mu\left(\mathrm{d}u - (M_\nu + \partial_\nu\chi)\mathrm{d}x^\nu\right) + h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{6.4.17}$$

$$= 2\tau_\mu\mathrm{d}x^\mu\left(\mathrm{d}u - m_\nu\mathrm{d}x^\nu\right) + h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu , \tag{6.4.18}$$

which is identical to the null reduction ansatz employed in section 4.3, which was shown to give rise to TNC geometry: thus we see directly that the boundary geometry is TNC.

6.4.1.1 *Null Reduction of the Connection*

We will now consider explicitly the consequences of the null reduction for the affine connection $\mathring{\Gamma}^C_{AB}$, where we use a '$\circ$' to denote that this is the affine connection of the metric $\gamma_{(0)AB}$. Taking the three legs in non-compact directions, we find that

$$\mathring{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}h^{\lambda\rho}\left(\partial_\mu\bar{h}_{\nu\lambda} + \partial_\nu\bar{h}_{\mu\lambda} - \partial_\lambda\bar{h}_{\mu\nu}\right) - \hat{v}^\lambda\partial_{(\mu}\tau_{\nu)} \tag{6.4.19}$$

$$= \bar{\Gamma}^\lambda_{\mu\nu} + \hat{v}^\lambda\partial_{[\mu}\tau_{\nu]} \tag{6.4.20}$$

$$= \bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{[\mu\nu]} , \tag{6.4.21}$$

where we define

$$\bar{\Gamma}^\lambda_{\mu\nu} = -\hat{v}^\lambda\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\rho}\left(\partial_\mu\bar{h}_{\nu\lambda} + \partial_\nu\bar{h}_{\mu\lambda} - \partial_\lambda\bar{h}_{\mu\nu}\right) , \tag{6.4.22}$$

which is the unique TNC metric compatible connection linear in m_μ [33] that we also discussed in chapter 4. It is (G, J) invariant and satisfies metric compatibility in the usual sense,

$$\nabla_\mu\tau_\nu = 0, \qquad \nabla_\mu h^{\nu\lambda} = 0. \tag{6.4.23}$$

It can be made N invariant by replacing m_μ with M_μ. The other components of $\mathring{\Gamma}^A_{BC}$ read

$$\mathring{\Gamma}^u_{\mu\nu} = -\frac{1}{2}\hat{v}^\lambda\left(\partial_\mu\bar{h}_{\lambda\nu} + \partial_\nu\bar{h}_{\lambda\mu} - \partial_\lambda\bar{h}_{\mu\nu}\right) + 2\tilde{\Phi}\partial_{(\mu}\tau_{\nu)} , \tag{6.4.24}$$

$$\mathring{\Gamma}^\lambda_{u\mu} = h^{\lambda\rho}\partial_{(\mu}\tau_{\rho)} , \tag{6.4.25}$$

$$\mathring{\Gamma}^u_{u\mu} = -\hat{v}^\lambda\partial_{[\mu}\tau_{\lambda]} , \tag{6.4.26}$$

$$\mathring{\Gamma}^u_{uu} = 0 = \mathring{\Gamma}^\mu_{uu} . \tag{6.4.27}$$

6.4.2 *Gauging* $\mathfrak{sch}_2(2,1)$

Complementary to our discussion of TNC geometry in chapter 4 and the emergence of TNC geometry in section 5.2.4 for generic values of $z \in (1, 2]$, we discuss in this section how TTNC explicitly occurs when gauging $\mathfrak{sch}_2(d, 1)$, which was first shown in [34]. In order to realize the more general TNC geometry, torsion has to be added by hand in this approach. For an overview of Schrödinger algebras and their rôle as Newton-Cartan space-time symmetry groups, see e.g. [115, 125].

For $z = 2$, it is possible to add an additional special conformal transformation K, which cannot be done in the general-z case. To keep the analysis as general as possible, we do that in what follows. As such, $\mathfrak{sch}_2(2, 1)$ is generated by: D (dilatations) and K (special conformal transformations), in addition to those of the Bargmann algebra, $\mathfrak{bar}(2,1)$, which is generated by H (time translations), P (spatial translations), G (Galilean boosts), J_{ab} (spatial rotations) and N (central charge). The non-zero commutation relations of $\mathfrak{sch}_2(2,1)$ read

$$[D,H] = -2H, \quad [H,K] = D, \quad [D,K] = 2K, \quad [H,G_a] = P_a \tag{6.4.28}$$
$$[D,P_a] = -P_a, \quad [D,G_a] = G_a, \quad [K,P_a] = -G_a, \quad [P_a,G_b] = \delta_{ab}N, \tag{6.4.29}$$
$$[J_{ab},P_c] = 2\delta_{c[a}P_{b]}, \quad [J_{ab},G_c] = 2\delta_{c[a}G_{b]}, \quad [J_{ab},J_{cd}] = 4\delta_{[a[d}J_{c]b]}. \tag{6.4.30}$$

For $z=2$, the Bargmann central charge N is still a central element. Further, we note in passing that it can be shown [115] that $\mathfrak{sch}_2(2,1)$ admits the following Levi decomposition,

$$\mathfrak{sch}_2(2,1) \cong (\mathfrak{so}(3) \times \mathfrak{sl}(2,\mathbb{R})) \ltimes \left(\mathbb{R}^3 \times \mathbb{R}^3\right). \tag{6.4.31}$$

We will use the $\bar{\delta}$–transformation approach discussed in chapter 4, where contact with diffeomorphisms is manifest; this was not done in [34]. As usual, we start by defining the Lie algebra valued connection

$$\mathfrak{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a\omega_\mu{}^a + \frac{1}{2}J_{ab}\omega_\mu{}^{ab} + Nm_\mu + Db_\mu + Kf_\mu. \tag{6.4.32}$$

For $L(x) \in Sch_2(d,1)$, the connection transforms in the adjoint, $\mathfrak{A}_\mu \to L(x)\mathfrak{A}_\mu L^{-1}(x) - L(x)\partial_\mu L^{-1}(x)$, so taking $L(x)$ to be infinitesimal, $L(x) = \mathbb{1} + \Lambda(x)$ with $\Lambda(x) \in \mathfrak{sch}_2(d,1)$, we get

$$\delta\mathfrak{A}_\mu = \partial_\mu\Lambda + [\mathfrak{A}_\mu, \Lambda], \tag{6.4.33}$$

but, since $\Lambda(x) \in \mathfrak{sch}_2(d,1)$, we may write

$$\Lambda(x) = H\zeta(x) + P_a\zeta^a(x) + G_a\tilde{\lambda}^a(x) + \frac{1}{2}J_{ab}\tilde{\lambda}^{ab}(x) + N\tilde{\sigma}(x) + D\tilde{\Lambda}_D(x) + K\tilde{\Lambda}_K(x). \tag{6.4.34}$$

For the $\bar{\delta}$-transformation, we replace the local translation parameters ζ^a in Λ with a space-time vector ξ^μ defined via $\zeta^a = \xi^\mu e^a_\mu$, which allows us to write

$$\Lambda = \xi^\mu\mathfrak{A}_\mu + \Sigma, \tag{6.4.35}$$

which implies that, as we have seen

$$\Sigma = G_a\lambda^a + \frac{1}{2}J_{ab}\lambda^{ab} + N\sigma + D\Lambda_D + K\Lambda_K, \tag{6.4.36}$$

where the tilde-less parameters of (6.4.36) are related to their tilded counterparts in (6.4.34) in the same as in (4.1.11). The $\bar{\delta}$-transformation then becomes

$$\bar{\delta}\mathfrak{A}_\mu = \delta\mathfrak{A}_\mu - \xi^\nu\mathcal{F}_{\mu\nu} = \pounds_\xi\mathfrak{A}_\mu + \partial_\mu\Sigma + [\mathfrak{A}_\mu, \Sigma], \tag{6.4.37}$$

where $\mathcal{F}_{\mu\nu}$ is the Yang-Mills curvature,

$$\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]} + [\mathfrak{A}_\mu, \mathfrak{A}_\nu] \tag{6.4.38}$$
$$= HR_{\mu\nu}(H) + P_aR_{\mu\nu}{}^a(P) + G_aR_{\mu\nu}{}^a(G) + \frac{1}{2}J_{ab}R_{\mu\nu}{}^{ab}(J) + NR_{\mu\nu}(N) + DR_{\mu\nu}(D) + KR_{\mu\nu}(K). \tag{6.4.39}$$

We summarise this construction in the table below:

Table 6.6: Generators of $\mathfrak{sch}_2(2,1)$ with their associated gauge fields, local parameters and covariant curvatures.

Symmetry	Generators	Gauge Field	Parameters	Curvatures
Time translations	H	τ_μ	$\zeta(x)$	$R_{\mu\nu}(H)$
Spatial translations	P_a	e^a_μ	$\zeta^a(x)$	$R_{\mu\nu}{}^a(P)$
Boosts	G_a	$\omega_\mu{}^a$	$\lambda^a(x)$	$R_{\mu\nu}{}^a(G)$
Spatial rotations	J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x)$	$R_{\mu\nu}{}^{ab}(J)$
Central ch. traf'os	N	m_μ	$\sigma(x)$	$R_{\mu\nu}(N)$
Dilatations	D	b_μ	$\Lambda_D(x)$	$R_{\mu\nu}(D)$
Spec. conf. traf'os	K	f_μ	$\Lambda_K(x)$	$R_{\mu\nu}(K)$

Writing out the expression of (6.4.38) for $\bar{\delta}\mathfrak{A}_\mu$ using our expressions for $\mathfrak{A}_\mu$ and Σ, we find the following expressions for the curvatures:

$$
\begin{aligned}
R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]} - 4b_{[\mu}\tau_{\nu]}, && (6.4.40)\\
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu}e^a_{\nu]} - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]} - 2b_{[\mu}e^a_{\nu]}, && (6.4.41)\\
R_{\mu\nu}{}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{c[a}\omega_{\nu]}{}^{b]}{}_c, && (6.4.42)\\
R_{\mu\nu}{}^a(G) &= 2\partial_{[\mu}\omega_{\nu]}{}^a + 2\omega_{[\mu}{}^b\omega_{\nu]}{}^a{}_b - 2\omega_{[\mu}{}^a b_{\nu]} - 2f_{[\mu}e^a_{\nu]}, && (6.4.43)\\
R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} - 2f_{[\mu}\tau_{\nu]}, && (6.4.44)\\
R_{\mu\nu}(K) &= 2\partial_{[\mu}f_{\nu]} + 4b_{[\mu}f_{\nu]}, && (6.4.45)\\
R_{\mu\nu}(N) &= 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a}. && (6.4.46)
\end{aligned}
$$

Similarly, by writing out $\mathfrak{A}_\mu$ and Σ as they appear in (6.4.37) and identifying coefficients in front of the generators, we find the following transformations of the gauge fields:

$$
\begin{aligned}
\bar{\delta}\tau_\mu &= \pounds_\xi\tau_\mu + 2\Lambda_D\tau_\mu, && (6.4.47)\\
\bar{\delta}e^a_\mu &= \pounds_\xi e^a_\mu + \lambda^a\tau_\mu + \Lambda_D e^a_\mu, && (6.4.48)\\
\bar{\delta}\omega_\mu{}^{ab} &= \pounds_\xi\omega_\mu{}^{ab} + \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]}{}_c, && (6.4.49)\\
\bar{\delta}\omega_\mu{}^a &= \pounds_\xi\omega_\mu{}^a + \partial_\mu\lambda^a - \lambda^b\omega_\mu{}^a{}_b + \lambda^a{}_b\omega_\mu{}^b + \lambda^a b_\mu - \Lambda_D\omega_\mu{}^a + \Lambda_K e^a_\mu, && (6.4.50)\\
\bar{\delta}b_\mu &= \pounds_\xi b_\mu + \partial_\mu\Lambda_D + \Lambda_K\tau_\mu, && (6.4.51)\\
\bar{\delta}f_\mu &= \pounds_\xi f_\mu + \partial_\mu\Lambda_K + 2\Lambda_K b_\mu - 2\Lambda_D f_\mu, && (6.4.52)\\
\bar{\delta}m_\mu &= \pounds_\xi m_\mu + \partial_\mu\sigma + \lambda^a e_{\mu a}. && (6.4.53)
\end{aligned}
$$

The gauge fields τ_μ and e^a_μ transform under spatial rotations and Galilean boosts as Newton-Cartan temporal and spatial vielbeine, respectively [54], and we identify them as such. Since they are of rank 1 and rank d, respectively, in a $(d+1)$–dimensional spacetime, they are not invertible, but we can define projective inverses v^μ and e^μ_a, satisfying the usual set of relations

$$
v^\mu\tau_\mu = -1, \qquad v^\mu e^a_\mu = 0, \qquad \tau_\mu e^\mu_a = 0, \qquad e^a_\mu e^\mu_b = \delta^a_b, \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu + v^\mu\tau_\nu. \tag{6.4.54}
$$

The projective inverses transform under $\bar{\delta}$–transformations in the following manner,

$$
\begin{aligned}
\bar{\delta}v^\mu &= \pounds_\xi v^\mu + \lambda^a e^\mu_a - 2\Lambda_D v^\mu, && (6.4.55)\\
\bar{\delta}e^\mu_a &= \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b - \Lambda_D e^\mu_a, && (6.4.56)
\end{aligned}
$$

which are derived by considering the relations $0 = \bar{\delta}\left(v^\mu\tau_\mu\right)$ and $\bar{\delta}\left(v^\mu e^a_\mu\right)$ and using the identities (6.4.54). So far, all the gauge fields are independent. This can be remedied via the imposition of certain curvature constraints—as detailed in [34]—which allows us to make contact with TTNC. These constraints read

$$
0 = R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(N) = R_{\mu\nu}(D) = R_{0a}{}^a(G) + 2m^b R_{0a}{}^a{}_b(J) + m^b m^c R_{ba}{}^a{}_c(J). \tag{6.4.57}
$$

These constraints—along with some additional constraints coming from imposing the curvature constraints in the Bianchi identities (see [34] for details)—leave us with the independent fields τ_μ, e^a_μ and $v^\mu b_\mu$, which we recognize as the usual fields of TNC geometry except for the odd one out: $v^\mu b_\mu$. However, note that under local transformations, this object transforms in the following manner

$$
\bar{\delta}(v^\mu b_\mu) = \pounds_\xi(v^\mu b_\mu) + v^\mu\partial_\mu\Lambda_D - 2\Lambda_D v^\mu b_\mu + \lambda^a e^\mu_a b_\mu - \Lambda_K, \tag{6.4.58}
$$

leading us to identify the combination $v^\mu b_\mu$ as a Stückelberg field for K transformations. This prompts us to gauge fix it to $v^\mu b_\mu = 0$, thereby introducing the following *compensating special conformal transformation*

$$
\Lambda_K = v^\mu\partial_\mu\Lambda_D + \lambda^a e^\mu_a b_\mu. \tag{6.4.59}
$$

The next step in the gauging procedure is to write down covariant derivatives, which are covariant with respect to the transformations (6.4.47)–(6.4.53). By taking into account all the commutators of

$\mathfrak{sch}_2(d,1)$ that are proportional to H and P_a, we find the following expressions for the covariant derivatives:

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\tau_\rho - 2b_\mu \tau_\nu, \tag{6.4.60}$$

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \tilde{\Gamma}^\rho_{\mu\nu} e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu - \omega_\mu{}^a \tau_\nu - b_\mu e^a_\nu. \tag{6.4.61}$$

Imposing the vielbein postulates, $\mathcal{D}_\mu \tau_\nu = 0 = \mathcal{D}_\mu e^a_\nu$, allows for a identification of the curvatures (6.4.40) and (6.4.41) in terms of the affine connection. Since the postulates imply that

$$\tilde{\Gamma}^\rho_{\mu\nu}\tau_\rho = \partial_\mu \tau_\nu - 2b_\mu \tau_\nu, \qquad \tilde{\Gamma}^\rho_{\mu\nu} e^a_\rho = \partial_\mu e^a_\nu - \omega_\mu{}^a{}_b e^a_\nu - \omega_\mu{}^a \tau_\nu - b_\mu e^a_\nu, \tag{6.4.62}$$

we immediately obtain

$$R_{\mu\nu}(H) = 2\Gamma^\rho_{[\mu\nu]}\tau_\rho, \qquad R_{\mu\nu}{}^a(P) = 2\Gamma^\rho_{[\mu\nu]} e^a_\rho, \tag{6.4.63}$$

implying, via the curvature constraints (6.4.57), that $\tilde{\Gamma}^\rho_{\mu\nu}$ is symmetric. It can be shown that it is uniquely determined by [34]

$$\tilde{\Gamma}^\rho_{\mu\nu} = -\hat{v}^\rho D^{(b)}_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma}\left(D^{(b)}_\mu \bar{h}_{\nu\sigma} + D^{(b)}_\nu \bar{h}_{\mu\sigma} - D^{(b)}_\sigma \bar{h}_{\mu\nu}\right), \tag{6.4.64}$$

where $D^{(b)}_\mu = \partial_\mu + 2b_\mu$ is the dilatation covariant derivative. Dropping all terms involving b_μ, we recover the minimal TNC connection,

$$\bar{\Gamma}^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu \bar{h}_{\nu\rho} + \partial_\nu \bar{h}_{\mu\rho} - \partial_\rho \bar{h}_{\mu\nu}\right), \tag{6.4.65}$$

with torsion

$$\bar{\Gamma}^\rho_{[\mu\nu]} = -\hat{v}^\rho \partial_{[\mu}\tau_{\nu]}. \tag{6.4.66}$$

This intimate relationship between torsion and dilatation covariance leads us—following [34]—to attribute torsion to the gauge field b_μ in the sense above.

The curvature constraint $R_{\mu\nu}(H)$ implies that

$$\partial_{[\mu}\tau_{\nu]} = 2b_{[\mu}\tau_{\nu]}, \tag{6.4.67}$$

which means that

$$\tau \wedge \mathrm{d}\tau = \tau_{[\rho}\partial_\rho \tau_{\nu]} = \frac{1}{3}\left(\tau_\mu \partial_{[\rho}\tau_{\nu]} + \tau_\rho \partial_{[\nu}\tau_{\mu]} + \tau_\nu \partial_{[\mu}\tau_{\rho]}\right) = \frac{2}{3}\left(b_\rho \tau_{[\mu}\tau_{\nu]} + b_\mu \tau_{[\rho}\tau_{\nu]} + b_\nu \tau_{[\mu}\tau_{\rho]}\right) = 0, \tag{6.4.68}$$

implying, by Frobenius' theorem, that τ_μ is hypersurface orthogonal, as we discussed in chapter 4. This leads to a preferred foliation in terms of (Riemannian) slices of absolute simultaneity. Equivalently, the condition (6.4.67) may be expressed as the vanishing of the twist tensor,

$$\varpi^{\mu\nu} = 2h^{\mu\rho}h^{\nu\sigma}\partial_{[\rho}\tau_{\sigma]} = 0. \tag{6.4.69}$$

In this sense, as also emphasized in [32], the more general TNC geometry can be achieved by first gauging the Schrödinger algebra and then adding torsion by hand.

6.4.2.1 *Introducing Stückelberg Symmetry of the Central Charge*

As we have seen, in Lifshitz holography the field m_μ is accompanied by a Stückelberg scalar χ. The purpose of this subsection is to make manifest the connection between TTNC geometry as obtained from gauging $\mathfrak{sch}_2(2,1)$ in section 6.4.2 and the sources found in section 6.3. Since we already discussed this mechanism in section 4.2.4, we will be brief.

By imposing the curvature constraints (6.4.57), we saw how we were left with three independent fields τ_μ, e^a_μ and m_μ. By introducing an additional scalar field χ transforming as

$$\bar{\delta}\chi = \pounds_\xi \chi + \sigma, \tag{6.4.70}$$

we promote N to a Stückelberg symmetry, the effect of which will be the replacement of m_μ by M_μ given by the usual expression

$$M_\mu = m_\mu - \partial_\mu \chi \tag{6.4.71}$$

in all geometric quantities. We stress that it is not necessary to do so, just as when considering a Stückelberged Proca field, one can either choose to work directly with the Proca field or a Maxwell field and the Stückelberg field. We will use a mixture between the two approaches.

6.5 THE ANOMALY & HOŘAVA-LIFSHITZ GRAVITY

6.5.1 *Reduction of the Anomaly*

Holographic renormalization of the electromagnetic uplift produces the boundary anomaly

$$\mathcal{A} = -\frac{1}{4}\left(Q_{(0)AB}Q^{AB}_{(0)} - \frac{1}{3}Q^2_{(0)} - 2\mathcal{F}_{(0)AB}\mathcal{F}^{AB}_{(0)} + \frac{1}{2}\left(\Box_{(\gamma_{(0)})}\psi_{(0)}\right)^2\right), \tag{6.5.1}$$

where

$$Q_{(0)AB} = R^{(\gamma_{(0)})}_{AB} - \frac{1}{2}\partial_A\psi_{(0)}\partial_B\psi_{(0)}. \tag{6.5.2}$$

The anomaly is related to HL gravity, as we now briefly discuss—this was observed in [107] and formulated in a TNC covariant manner in [24]. In the case of ALif, where the torsion vanishes and where χ (i.e. the boundary value of bulk Stückelberg field Ξ) is a constant (for simplicity we will set $\chi = 0$) , the reduced anomaly becomes [24]

$$\mathcal{A} = -\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \frac{1}{2}\left(h^{\mu\nu}\mathcal{K}_{\mu\nu}\right)^2\right) - \frac{1}{24}\left(\mathcal{R} + e^\mu_a e^a_\nu \nabla_\mu\left(h^{\nu\rho}\pounds_v\tau_\rho\right)\right)^2 + e^{-1}\partial_\mu\left(eJ^\mu\right) - 2\mathcal{L}_{\text{GED}} \tag{6.5.3}$$

where $\mathcal{K}_{\mu\nu} = \frac{1}{2}\pounds_v h_{\mu\nu}$ is the TNC extrinsic curvature, and $\mathcal{R}$ is the Ricci scalar of $\bar{\Gamma}$. J^μ is some complicated current (cf. [24]), while $\mathcal{L}_{\text{GED}}$ is the Lagrangian for GED which can be read off from (6.1.15). Combining our findings, we may thus write

$$-\frac{1}{2}\int \mathrm{d}^3x\; e\; \mathcal{A} = S_{\text{PHL}} + S_{\text{GED}}, \tag{6.5.4}$$

where PHL stands for *projectable Hořava-Lifshitz*, see [53] for details.

6.6 HOLOGRAPHIC RECONSTRUCTION OR MOCK FG EXPANSIONS

From the FG expansions of the higher-dimensional fields that we determined in chapter 3, we can work out the corresponding expansions of the four dimensional bulk fields. In particular, having shown in section 6.4.1 that the reduction becomes null on the boundary, we can use our explicit expression for $\gamma_{(0)AB}$ as well as our knowledge of the sources from section 6.3 to determine the expansions. We start by translating our results from chapter 3 into the notation of this chapter.

From a five-dimensional perspective, we have the fields $ds^2_5 = \frac{\mathrm{d}r^2}{r^2} + \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B, \psi, \mathcal{A}_A$, admitting FG expansions of the form

$$\gamma_{AB} = \frac{1}{r^2}\left(\gamma_{(0)AB} + r^2\gamma_{(2)AB} + r^4\left(\log r\,\gamma_{(4,1)AB} - \gamma_{(4)AB}\right)\right) + \mathcal{O}(r^6\log r), \tag{6.6.1}$$

$$\psi = \psi_{(0)} + r^2\psi_{(2)} + r^4\log r\,\psi_{(4,1)} + r^4\psi_{(4)} + \mathcal{O}(r^6\log r), \tag{6.6.2}$$

$$\mathcal{A}_A = \mathcal{A}_{(0)A} + r^2\log r\,\mathcal{A}_{(2,1)A} + r^2\mathcal{A}_{(2)A} + \mathcal{O}(r^4\log r), \tag{6.6.3}$$

where the coefficients are given by (see section 3.2.3 and also footnote 14)

$$\gamma_{(2)AB} = -\frac{1}{2}\left(R^{(\gamma_{(0)})}_{AB} - \frac{1}{2}\partial_A\psi_{(0)}\partial_B\psi_{(0)}\right) + \frac{1}{12}\gamma_{(0)AB}\left(R^{(\gamma_{(0)})} - \frac{1}{2}(\partial\psi_{(0)})^2\right), \tag{6.6.4}$$

$$\psi_{(2)} = \frac{1}{4}\Box_{(\gamma_{(0)})}\psi_{(0)}, \tag{6.6.5}$$

$$\mathcal{A}_{(2,1)A} = \nabla^{(\gamma_{(0)})}_B \mathcal{F}^B_{(0)A}, \tag{6.6.6}$$

$$\mathcal{A}_{(2)A} = \frac{1}{2}\mathcal{J}_{(0)A} - \frac{1}{2}\mathcal{A}_{(2,1)A}, \tag{6.6.7}$$

$$\gamma_{(4,1)AB} = \frac{1}{4}\nabla^{(\gamma_{(0)})C}\left(\nabla^{(\gamma_{(0)})}_A\gamma_{(2)BC} + \nabla^{(\gamma_{(0)})}_B\gamma_{(2)AC} - \nabla^{(\gamma_{(0)})}_C\gamma_{(2)AB}\right) - \frac{1}{4}\nabla^{(\gamma_{(0)})}_A\nabla^{(\gamma_{(0)})}_B\gamma^C_{(2)C} \tag{6.6.8}$$

$$+\gamma_{(2)AC}\gamma^C_{(2)B} - \frac{1}{2}\partial_{(A}\psi_{(0)}\nabla^{(\gamma_{(0)})}_{B)}\psi_{(2)} - \gamma_{(0)AB}\left(\frac{1}{4}\gamma^{CD}_{(2)}\gamma_{(2)CD} + \frac{1}{2}\psi^2_{(2)}\right) \tag{6.6.9}$$

$$+\frac{1}{3}\mathcal{F}_{(0)AC}\mathcal{F}_{(0)B}{}^C - \frac{1}{12}\gamma_{(0)AB}\mathcal{F}_{(0)CD}\mathcal{F}^{CD}_{(0)}, \tag{6.6.10}$$

$$\psi_{(4,1)} = -\frac{1}{4}\left(\Box_{(\gamma_{(0)})}\psi_{(2)} + 2\psi_{(2)}\gamma^A_{(2)A} + \frac{1}{2}\partial^A\psi_{(0)}\nabla^{(\gamma_{(0)})}_A\gamma^B_{(2)B} - \gamma^{AB}_{(2)}\nabla^{(\gamma_{(0)})}_A\partial_B\psi_{(0)} - \partial^A\psi_{(0)}\nabla^{(\gamma_{(0)})B}\gamma_{(0)AB}\right) \tag{6.6.11}$$

$$\psi_{(4)} = \frac{1}{4}\langle\mathcal{O}_\psi\rangle - \frac{1}{4}\psi_{(2)}\gamma^A_{(2)A} - \frac{3}{4}\psi_{(4,1)}. \tag{6.6.12}$$

The fields that we are interested live on three-dimensional surfaces of constant r,

$$h_{\mu\nu} = e^{\Phi}\left(\gamma_{\mu\nu} - e^{-2\Phi}\gamma_{\mu u}\gamma_{\nu u}\right), \qquad A_\mu = e^{-2\Phi}\gamma_{\mu u}, \qquad \Phi = \frac{1}{2}\log\gamma_{uu}, \qquad C_\mu = \mathcal{A}_\mu - \mathcal{A}_u e^{-2\Phi}\gamma_{\mu u} \tag{6.6.13}$$

First off, since $\gamma_{(0)uu} = 0$, the expansion for the dilaton becomes

$$\Phi = \frac{1}{2}\log\left(\gamma_{(2)uu} + r^2\log r\gamma_{(4,1)uu} + r^2\gamma_{(4)uu} + \cdots\right) \tag{6.6.14}$$

$$= \frac{1}{2}\log\left(\gamma_{(2)uu}\right) + \frac{1}{2}r^2\log r\gamma_{(4,1)uu}\gamma^{-1}_{(2)uu} + \frac{1}{2}r^2\gamma_{(4)uu}\gamma^{-1}_{(2)uu} + \cdots \tag{6.6.15}$$

$$= \phi + r^2\log r\Phi_{(2,1)} + r^2\Phi_{(2)} + \cdots, \tag{6.6.16}$$

where

$$e^{2\phi} = \gamma_{(2)uu} = -\frac{1}{2}R^{(\gamma_{(0)})}_{uu} + 1, \qquad \Phi_{(2,1)} = \frac{1}{2}e^{-2\phi}\gamma_{(4,1)uu}, \qquad \Phi_{(2)} = \frac{1}{2}e^{-2\phi}\left(X_{uu} - \frac{1}{4}t_{uu}\right), \tag{6.6.17}$$

and we have used the FG coefficients (6.6.4)–(6.6.12). In particular, in the first relation we have used that $\gamma_{(0)uu} = 0$ and $\psi_{(0)} = 2u - 2\chi$. Note that the Ricci tensor above can be rewritten in the form $R^{(\gamma_{(0)})}_{uu} = 2\mathring{\Gamma}^A_{u[u,A]} + 2\mathring{\Gamma}^A_{B[A}\mathring{\Gamma}^B_{u]u} = \frac{1}{2}\left(\epsilon^{\mu\nu\rho}\tau_\mu\partial_\nu\tau_\rho\right)^2$, where $\epsilon^{\mu\nu\rho} = e^{-1}\varepsilon^{\mu\nu}$ with $\varepsilon^{\mu\nu\rho}$ the Levi-Civita symbol. In particular, this implies that ϕ is not an independent source, which has the important consequence that the corresponding VEV will vanish—we will explore the consequences of this in section 6.7, where we also provide a different reason why the VEV should vanish. This also allows us to identify the coefficients of the expansion of α (cf. eq. (6.3.9)),

$$\alpha(\Phi) = e^{-3\Phi/2} = \overbrace{e^{-3\phi/2}}^{=\alpha_{(0)}(\phi)} - \frac{3}{2}r^2\log r\alpha_{(0)}\Phi_{(2,1)} - \frac{3}{2}r^2\alpha_{(0)}\Phi_{(2)} + \cdots, \tag{6.6.18}$$

so that the combination $\tilde{I} - I$ appearing in (6.3.9) is equal to $\tilde{I} - I = -\frac{3}{2}\alpha^{4/3}_{(0)}\Phi_{(2)}$.

Similarly, for the KK vector, we find the expansion

$$A_\mu = r^{-2}A_{(0)\mu} + \log r A_{(2,1)\mu} + A_{(2)\mu} + r^2\log^2 r A_{(4,2)\mu} + r^2\log r A_{(4,1)\mu} + r^2 A_{(4)\mu} + \cdots, \tag{6.6.19}$$

where

$$A_{(0)\mu} = \alpha_{(0)}^{4/3}\tau_\mu, \qquad A_{(2,1)\mu} = -2\alpha_{(0)}^{4/3}\Phi_{(2,1)}\tau_\mu, \qquad A_{(2)\mu} = \alpha_{(0)}^{4/3}\Upsilon_{(2)\mu u} - 2\Phi_{(2)}A_{(0)\mu}, \tag{6.6.20}$$

$$A_{(4,2)\mu} = \Phi_{(2,1)}^2 A_{(0)\mu}, \qquad A_{(4,1)\mu} = \alpha_{(0)}^{4/3}\Upsilon_{(4,1)\mu u} - 2\Phi_{(2,1)}A_{(2)\mu} - 2\Phi_{(4,1)}A_{(0)\mu}. \tag{6.6.21}$$

Note that $A_{(2)\mu}$ transforms as $\delta_\sigma A_{(2)\mu} = -\partial_\mu\sigma$ under Stückelberg gauge transformations (i.e. diffeomorphisms in the compact direction), which follows from the transformation property $\delta_\sigma R_{\mu u}^{(\gamma_{(0)})} = -R_{uu}^{(\gamma_{(0)})}\partial_\mu\sigma$ and then using the first relation of (6.6.17). For the metric, we get

$$h_{\mu\nu} = r^{-4}h_{(0)\mu\nu} + r^{-2}\log r h_{(2,1)\mu\nu} + r^{-2}h_{(2)\mu\nu} + \mathcal{O}(\log^2 r), \tag{6.6.22}$$

where

$$h_{(0)\mu\nu} = -\alpha_{(0)}^{2/3}\tau_\mu\tau_\nu, \qquad h_{(2,1)\mu\nu} = -h_{(0)\mu\nu}\Phi_{(2,1)}, \qquad h_{(2)\mu\nu} = \alpha_{(0)}^{-2/3}\bar{h}_{\mu\nu} - 2\alpha_{(0)}^{2/3}\tau_\mu\Upsilon_{(2)\nu u} + h_{(2,1)\mu\nu}. \tag{6.6.23}$$

Writing

$$h_{\mu\nu} = -E_\mu^0 E_\nu^0 + \delta_{ab}E_\mu^a E_\nu^b, \qquad E_\mu^0 \sim r^{-2}\tau_\mu,\ E_\mu^a \sim r^{-1}e_\mu^a. \tag{6.6.24}$$

we infer from the second term in (6.6.22) that

$$E_\mu^0 = r^{-2}\alpha_{(0)}^{1/3}\tau_\mu + \log r Y_\mu - \alpha_{(0)}^{-1}\left(M_\mu - I\tau_\mu\right) + \cdots, \tag{6.6.25}$$

where we find that

$$Y_\mu = \frac{1}{2}\alpha_{(0)}^{1/3}\Phi_{(2,1)}\tau_\mu, \tag{6.6.26}$$

whereas M_μ emerges from $h_{(2)\mu\nu}$ in (6.6.23) as the combination involving only a single factor of τ_μ. Since there is no term of order $r^{-1}\log r$, we obtain

$$E_\mu^a = r^{-1}\alpha_{(0)}^{-1/3} + \mathcal{O}(r^0). \tag{6.6.27}$$

The behaviour of the new fields follows straightforwardly from their FG expansions and the expansions of the four-dimensional fields that we have worked out above,

$$\Upsilon = \vartheta + r^2\log r\Upsilon_{(2,1)} + r^2\Upsilon_{(2)} + \cdots, \tag{6.6.28}$$

$$C_\mu = C_{(0)\mu}r^{-2} + \log r C_{(2,1)\mu} + C_{(2)\mu} + r^2\log r C_{(4,1)\mu} + r^2 C_{(4)\mu} + \cdots \tag{6.6.29}$$

where

$$\Upsilon_{(2,1)} = e^{-1}\partial_\mu(e\partial^\mu\vartheta), \qquad \Upsilon_{(2)} = \frac{1}{2}\left(\mathcal{J}_{(0)u} - e^{-1}\partial_\mu(e\partial^\mu\vartheta)\right), \tag{6.6.30}$$

$$C_{(0)\mu} = -\vartheta\alpha_{(0)}^{4/3}\tau_\mu, \qquad C_{(2,1)} = -\alpha_{(0)}^{4/3}\left(\Upsilon_{(2,1)} - 2\vartheta\Phi_{(2,1)}\right)\tau_\mu, \qquad C_{(2)\mu} = a_\mu - \vartheta m_\mu - \tilde{\varphi}\tau_\mu - \vartheta A_{(2)\mu}. \tag{6.6.31}$$

6.7 THE VEVS

In this section, we consider the VEVs corresponding to the sources introduced in section 6.3.1. We consider both the relation to the four-dimensional responses determined in section 6.2.3 and the relation to the VEVs in the electromagnetic uplift.

6.7.1 *Definition & Near-Boundary Behaviour*

The goal of this section is to identify the VEVs corresponding to the sources in (6.3.19)–(6.3.28). These can be obtained in terms of the responses (6.2.35)–(6.2.42) via the following identity (which is understood to be sitting inside an integral),

$$\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^\nu\delta B_\nu + T_\Phi\delta\Phi + \mathfrak{T}^\nu\delta C_\nu + \mathcal{T}_\Upsilon\delta\Upsilon = \tag{6.7.1}$$

$$\mathcal{S}_\mu^0\delta E_\mu^0 + \mathcal{S}_\mu^a\delta E_a^\mu + \mathcal{T}_\varphi\delta\varphi + \mathcal{T}^a\delta A_a + \mathcal{T}_\Xi\delta\Xi + \mathcal{T}_\Phi\delta\Phi + \mathfrak{T}^a\delta C_a + \mathcal{T}_\Omega\delta\Omega + \mathcal{T}_\Upsilon\delta\Upsilon, \tag{6.7.2}$$

where we used that $B_\nu = A_\nu - \partial_\nu \Xi$ (which, due to the dependence of the derivative, implies that the identity above holds up to a total derivative), $A_a = E_a^\mu A_\mu$. In the above, we have used in particular

$$\varphi = E_0^\nu A_\nu - \alpha(\Phi), \tag{6.7.3}$$

$$\Omega = E_0^\nu C_\nu + Y\alpha(\Phi), \tag{6.7.4}$$

In order to identify the VEVs appearing in (6.7.2), we need to relate the variations of the sources to the variations of $(h_{\mu\nu}, B_\nu, \Phi)$. For $h_{\mu\nu}$ and Φ this is straightforward, but for the massive vector and the Maxwell field a few subtleties arise, so we perform that calculation in detail. We start by noting that

$$\mathcal{T}^\nu \delta B_\nu = \mathcal{T}^\nu \delta A_\nu + e^{-1}\partial_\nu(e\mathcal{T}^\nu) + \text{total derivative}, \tag{6.7.5}$$

due to the Stückelberg decomposition of the massive vector. Further, we see that the flat version satisfies

$$\mathcal{T}^0 \delta A_0 + \mathcal{T}^a \delta A_a = E_\mu^0 \mathcal{T}^\mu \delta\left(E_0^\nu A_\nu\right) + E_\mu^a \mathcal{T}^\mu \delta\left(E_a^\nu A_\nu\right) \tag{6.7.6}$$

$$= E_\mu^0 \mathcal{T}^\mu A_\nu \delta E_a^\nu + E_\mu^a \mathcal{T}^\mu A_\nu \delta E_a^\nu + \overbrace{\left(E_\mu^0 \mathcal{T}^\mu E_0^\nu + E_\mu^a \mathcal{T}^\mu E_a^\nu\right) \delta A_\nu}^{=\mathcal{T}^\nu \delta A_\nu}, \tag{6.7.7}$$

so that

$$\mathcal{T}^\nu \delta A_\nu = \mathcal{T}^0 \delta A_0 + \mathcal{T}^a \delta A_a - E_\mu^0 \mathcal{T}^\mu A_\nu \delta E_a^\nu - E_\mu^a \mathcal{T}^\mu A_\nu \delta E_a^\nu. \tag{6.7.8}$$

Observe further that since—by (6.7.3)—$A_0 = \varphi + \alpha(\Phi)$,

$$\mathcal{T}^0 \delta A_0 = E_\mu^0 \mathcal{T}^\mu \left(\underbrace{\frac{\delta A_0}{\delta\varphi}}_{=1} \delta\varphi + \underbrace{\frac{\delta A_0}{\delta\Phi}}_{=\frac{d\alpha(\Phi)}{d\Phi}} \delta\Phi \right). \tag{6.7.9}$$

Repeating the analysis above for the Maxwell field, we find that

$$\mathfrak{T}^\nu \delta C_\nu = \mathfrak{T}^0 \delta C_0 + \mathfrak{T}^a \delta C_a - E_\mu^0 \mathfrak{T}^\mu C_\nu \delta E_a^\nu - E_\mu^a \mathfrak{T}^\mu C_\nu \delta E_a^\nu, \tag{6.7.10}$$

where, since $C_0 = \Omega - Y\alpha$

$$\mathfrak{T}^0 \delta C_0 = E_\mu^0 \mathfrak{T}^\mu \left(\delta\Omega - Y \frac{d\alpha}{d\Phi} \delta\Phi - \alpha \delta Y \right). \tag{6.7.11}$$

Combining our findings, the responses can be written in terms of the quantities of (6.2.35)–(6.2.42)

$$\mathcal{S}_\mu^0 = -\left(T_{\mu\nu} E_0^\nu + \mathcal{T}^\rho E_\rho^0 A_\mu + \mathfrak{T}^\rho E_\rho^0 C_\mu \right), \tag{6.7.12}$$

$$\mathcal{S}_\mu^a = \left(T_{\mu\nu} E^{\nu a} - \mathcal{T}^\rho E_\rho^a A_\mu - \mathfrak{T}^\rho E_\rho^a C_\mu \right), \tag{6.7.13}$$

$$\mathcal{T}_\varphi = \mathcal{T}^\nu E_\nu^0, \tag{6.7.14}$$

$$\mathcal{T}_\Phi = T_\Phi + \mathcal{T}^\nu E_\nu^0 \frac{d\alpha(\Phi)}{d\Phi} + Y E_\mu^0 \mathfrak{T}^\mu \frac{d\alpha}{d\Phi}, \tag{6.7.15}$$

$$\mathcal{T}^a = \mathcal{T}^\nu E_\nu^a, \tag{6.7.16}$$

$$\mathcal{T}_\Xi = e^{-1}\partial_\mu\left(e\mathcal{T}^\mu\right), \tag{6.7.17}$$

$$\mathfrak{T}^a = \mathfrak{T}^\nu E_\nu^a, \tag{6.7.18}$$

$$\mathcal{T}_\Omega = \mathfrak{T}^\nu E_\nu^0, \tag{6.7.19}$$

$$\mathcal{T}_Y = T_Y + \mathfrak{T}^\mu E_\mu^0 \alpha \tag{6.7.20}$$

Below we list the responses and their near-boundary behaviours and thus identify the VEVs. Note that we have to take into account the near-boundary behaviour of the measure $\sqrt{-\gamma}$ as was the case in section 5.3.1, where we performed the same analysis for general values of z. Doing this, we obtain

$$\mathcal{S}^0_\mu \simeq r^2\alpha_{(0)}^{2/3}S^0_\mu, \tag{6.7.21}$$

$$\mathcal{S}^a_\mu \simeq r^3 S^a_\mu, \tag{6.7.22}$$

$$\mathcal{T}_\varphi \simeq -r^2\alpha_{(0)}^{2/3}T^0, \tag{6.7.23}$$

$$\mathcal{T}^a \simeq -r^3 T^a, \tag{6.7.24}$$

$$\mathcal{T}_\Phi \simeq r^4\alpha_{(0)}^{1/3}\langle O_\phi\rangle, \tag{6.7.25}$$

$$\mathcal{T}_\Xi \simeq -r^4\alpha_{(0)}^{1/3}\langle O_\chi\rangle, \tag{6.7.26}$$

$$\mathscr{T}^a \simeq r^3 j^a, \tag{6.7.27}$$

$$\mathcal{T}_\Omega \simeq r^2\alpha_{(0)}^{2/3}j^0, \tag{6.7.28}$$

$$\mathcal{T}_\Upsilon \simeq r^4\alpha_{(0)}^{1/3}\langle O_\vartheta\rangle. \tag{6.7.29}$$

From these, we may immediately read off their scaling weights:

VEV	S^0_μ	S^a_μ	T^0	T^a	$\langle O_\phi\rangle$	$\langle O_\chi\rangle$	j^a	j^0	$\langle O_\vartheta\rangle$
scaling dim.	2	3	2	3	4	4	3	2	4

Table 6.7: The VEVs and their scaling dimensions.

We note that, as was the case with the sources, the scaling dimensions of the new VEVs correspond to the results in the field theory (cf. table 6.4) when setting $z = 2$.

6.7.2 *Relation to Five-Dimensional VEVs Via Null Reduction*

We now proceed to relate the five-dimensional VEVs to the four-dimensional VEVs and perform a general analysis of these. In the uplift, we have that

$$\delta S_{\text{ren}} = \lim_{r\to 0}\frac{1}{2\kappa_5^2}\int_{\Sigma_r} \mathrm{d}^4x\sqrt{-\gamma}\left(\frac{1}{2}T_{AB}\delta\gamma^{AB} + T_\psi\delta\psi + \mathcal{J}^A\delta\mathcal{A}_A\right) \tag{6.7.30}$$

$$= \int_{\partial\mathcal{M}} \mathrm{d}^3x\, e\Bigg(\underbrace{\frac{1}{2}t_{AB}\delta\gamma^{AB}_{(0)}}_{=:(*)} + \underbrace{\langle O_\psi\rangle\,\delta\psi_{(0)}}_{=:(\dagger)} + \underbrace{\mathcal{J}^A_{(0)}\delta\mathcal{A}_{(0)A}}_{=:(\ddagger)}\Bigg), \tag{6.7.31}$$

where we have performed the u-integral and used that $\sqrt{-\gamma_{(0)}} =: e = \det(\tau_\mu, e^a_\mu)$, which follows from (**??**). In the four-dimensional theory, we then have

$$\delta S_{\text{ren}} = \lim_{r\to\infty}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\sqrt{-h}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + \mathcal{T}^\nu\delta B_\nu + T_\Phi\delta\Phi + \mathscr{T}^\nu\delta C_\nu + \mathcal{T}_\Upsilon\delta\Upsilon\right) \tag{6.7.32}$$

$$= \int_{\partial\mathcal{M}} \mathrm{d}^3x\, e\left(-S^0_\mu\delta v^\mu + S^a_\mu\delta e^\mu_a + T^0\delta m_0 + T^a\delta m_a + \langle O_\chi\rangle\,\delta\chi + \langle\tilde{O}_\phi\rangle\,\delta\phi + j^a\delta a_a - j^0\delta\tilde{\varphi} + \langle O_\vartheta\rangle\,\delta\vartheta\right), \tag{6.7.33}$$

where $m_0 = -v^\mu m_\mu$, $m_a = e^\mu_a m_\mu$, and where we have used

$$\psi_{(0)} = 2u - 2\chi, \qquad \langle O_\psi\rangle = -\frac{1}{2}\langle O_\chi\rangle. \tag{6.7.34}$$

This implies that $\delta\psi_{(0)} = -2\delta\chi$. The VEV $\langle\tilde{O}_\phi\rangle$ is given by

$$\langle\tilde{O}_\phi\rangle = \langle O_\phi\rangle - \frac{1}{2}\left[v^\mu(S^0_\mu + T^0 m_\mu) + e^\mu_a(S^a_\mu + T^a m_\mu) + j^a a_a - j^0\tilde{\varphi}\right] = 0, \tag{6.7.35}$$

and vanishes, as we will explain shortly. The extra terms in $\langle \tilde{O}_\phi \rangle$ come from the presence of $\alpha_{(0)}(\phi)$-terms in the responses (6.3.2)–(6.3.8) considered from the perspective of the right-hand side of the identity (6.7.2), e.g.

$$\sqrt{-\gamma}S^0_\mu \delta E^\mu_0 \simeq -eS^0_\mu \delta v^\mu - e\alpha^{1/3}_{(0)} S^0_\mu v^\mu \frac{\mathrm{d}\left(\alpha^{-1/3}_{(0)}\right)}{\mathrm{d}\phi}\delta\phi = -eS^0_\mu\delta v^\mu - \frac{1}{2}eS^0_\mu v^\mu \delta\phi. \tag{6.7.36}$$

The new terms will contribute

$$\sqrt{-\gamma}\mathcal{T}^a\delta C_a \simeq ej^a\delta a_a + e\alpha^{-1/3}_{(0)} j^a a_a \frac{\mathrm{d}\alpha^{1/3}_{(0)}}{\mathrm{d}\phi}\delta\phi = ej^a\delta a_a - \frac{1}{2}ej^a a_a\delta\phi, \tag{6.7.37}$$

as well as (using $\Omega \simeq -r^2\alpha^{-1/3}_{(0)}\tilde{\varphi}$)

$$\sqrt{-\gamma}\mathcal{T}_\Omega\delta\Omega \simeq -ej^0\delta\tilde{\varphi} - \alpha^{1/3}_{(0)} ej^0\tilde{\varphi}\frac{\mathrm{d}\alpha^{-1/3}_{(0)}}{\mathrm{d}\phi}\delta\phi = -ej^0\delta\tilde{\varphi} - \frac{1}{2}ej^0\tilde{\varphi}\delta\phi. \tag{6.7.38}$$

Since none of the bulk fields have a near-boundary behaviour that involves the source ϑ, the same *does not happen* to $\langle O_\vartheta \rangle$,

$$\langle \tilde{O}_\vartheta \rangle = \langle O_\vartheta \rangle\,, \tag{6.7.39}$$

i.e. there will be no additional terms in front of $\delta\vartheta$ as was the case for $\delta\phi$.

Now, recall from our analysis of the FG expansions in section 6.6 that

$$e^{2\phi} = -\frac{1}{4}\varpi^2 + 1, \tag{6.7.40}$$

where ϖ is the twist. This implies that the source ϕ is not independent and has the consequence that the variation of the renormalized on-shell action with respect to ϕ gives zero. This can also be seen from the shift transformation (the δ_c transformation, also known as local dilatations in [24]) we considered in section 6.3.2.4, which directly implies $\langle \tilde{O}_\phi \rangle = 0$. Similarly, from the gauge shift transformation δ_ℓ, we get the relation

$$\langle O_\vartheta \rangle = 0. \tag{6.7.41}$$

Now, as we demonstrated in section 6.4.1, the Scherk-Schwarz reduction employed becomes a null reduction on the boundary. In order to relate the VEVs appearing on the right-hand side in (6.7.21)–(6.7.29) to the VEVs in the electromagnetic uplift, we perform a null reduction of the expression in (6.7.31), which is then equated with the same expression from a lower-dimensional perspective (6.7.33). In components, the null reduction (6.4.18) can be written as

$$\gamma^{\mu\nu}_{(0)} = h^{\mu\nu} = e^{\mu a}e^\nu_a, \qquad \gamma^{u\mu}_{(0)} = -\hat{v}^\mu = e^{\mu a}m_a - v^\mu, \qquad \gamma^{uu}_{(0)} = 2\tilde{\Phi} = m^a m_a + 2m_0, \tag{6.7.42}$$

which means that $\delta h^{\mu\nu} = 2e^{(\mu}_a\delta e^{\nu)a}$. From the expansion of C_μ in (6.3.18), we note that

$$\mathcal{A}_{(0)\mu} = a_\mu - \vartheta m_\mu - \tilde{\varphi}\tau_\mu, \qquad \mathcal{A}_{(0)u} = \vartheta, \tag{6.7.43}$$

which in particular means that we can rewrite

$$\mathcal{A}_{(0)\mu} = e^a_\mu\left(a_a - \vartheta m_a\right) - \tau_\mu\left(\vartheta m_0 + \tilde{\varphi}\right). \tag{6.7.44}$$

Omitting the integral[4] and applying the null reduction of (6.7.42) to each term in (6.7.31), we obtain:

$$(*) = t_{\mu\nu}e^\mu_a\delta e^{\nu a} + t_{u\mu}\left(e^{\mu a}\delta m_a + m_a\delta e^{\mu a} - \delta v^\mu\right) + t_{uu}\left(m^a\delta m_a + \delta m_0\right), \tag{6.7.45}$$

$$(\dagger) = \langle O_\chi \rangle\,\delta\chi \qquad \text{(follows immediately from (6.7.34))}, \tag{6.7.46}$$

$$(\ddagger) = \mathcal{J}^u_{(0)}\delta\vartheta + \mathcal{J}^\mu_{(0)}\left[e^a_\mu\left(\delta a_a - \vartheta\delta m_a - m_a\delta\vartheta\right) - \tau_\mu\left(\vartheta\delta m_0 + m_0\delta\vartheta + \delta\tilde{\varphi}\right)\right] + \mathcal{J}^\mu_{(0)}(a_a - \vartheta m_a)\delta e^a_\mu - \mathcal{J}^\mu_{(0)}(\vartheta m_0 + \tilde{\varphi})\delta\tau_\mu. \tag{6.7.47}$$

4 When omitting the integral, we must recall our ability to integrate by parts. Equalities therefore only hold up to total derivatives.

We observe that the expression above involves two additional variations, δe^a_μ and $\delta\tau_\mu$, in the last two terms, and that these are not present in the expression for δS_{ren} in (6.7.33) which we'll eventually want to compare with. Therefore, we use the second identity in (6.1.10) to write, first off

$$\mathcal{J}^\mu_{(0)} = \left(e^\mu_a e^a_\nu - \tau_\nu v^\mu\right)\mathcal{J}^\nu_{(0)}. \tag{6.7.48}$$

Now, using the first, third, fourth and fifth identity of (6.1.10), we find that

$$\tau_\mu \delta v^\mu = -v^\mu \delta\tau_\mu, \qquad e^a_\mu \delta v^\mu = -v^\mu \delta e^a_\mu, \qquad \tau_\mu \delta e^\mu_a = -e^\mu_a \delta\tau_\mu, \qquad e^a_\mu \delta e^\mu_b = -e^\mu_b \delta e^a_\mu, \tag{6.7.49}$$

and so we can rewrite the last two terms in the expression for the term (‡) (6.7.47) in the following manner:

$$\mathcal{J}^\mu_{(0)}(a_a - \vartheta m_a)\delta e^a_\mu = \mathcal{J}^\nu_{(0)}(a_a - \vartheta m_a) e^a_\mu \left[\tau_\nu \delta v^\mu - e^b_\nu \delta e^\mu_b\right], \tag{6.7.50}$$

$$\mathcal{J}^\mu_{(0)}(m_0 + \tilde{\varphi})\delta\tau_\mu = \mathcal{J}^\nu_{(0)}(\vartheta m_0 + \tilde{\varphi})\tau_\mu \left[\tau_\nu \delta v^\mu - e^b_\nu \delta e^\mu_b\right]. \tag{6.7.51}$$

Combining our findings, we obtain

$$\frac{1}{2} t_{AB}\delta\gamma^{AB}_{(0)} + \langle O_\psi\rangle \delta\psi_{(0)} + \mathcal{J}^A_{(0)}\delta\mathcal{A}_{(0)A} = -\left[t_{u\mu} - \mathcal{J}^\nu_{(0)}\tau_\nu\left\{(a_a - \vartheta m_a)e^a_\mu - (\vartheta m_0 + \tilde{\varphi})\,\tau_\mu\right\}\right]\delta v^\mu \tag{6.7.52}$$

$$+\left[t_{\nu\mu}e^{\nu a} + t_{u\mu}m^a - \mathcal{J}^\nu_{(0)}e^a_\nu\left\{(a_b - \vartheta m_b)e^b_\mu - (\vartheta m_0 + \tilde{\varphi})\tau_\mu\right\}\right]\delta e^\mu_a + \left[t_{uu} - \vartheta\mathcal{J}^\mu_{(0)}\tau_\mu\right]\delta m_0 \tag{6.7.53}$$

$$+\left[t_{u\mu}e^{\mu a} + t_{uu}m^a - \mathcal{J}^\mu_{(0)}e^a_\mu\vartheta\right]\delta m_a + \left[-\mathcal{J}^\mu_{(0)}(\tau_\mu m_0 + e^a_\mu m_a) + \mathcal{J}^u_{(0)}\right]\delta\vartheta + \mathcal{J}^\mu_{(0)}e^a_\mu\delta a_a + \langle O_\chi\rangle\delta\chi - \mathcal{J}^\mu_{(0)}\tau_\mu\delta\tilde{\varphi}, \tag{6.7.54}$$

giving us the result

$$S^0_\mu = t_{u\mu} - \mathcal{J}^\nu_{(0)}\tau_\nu\left\{(a_a - \vartheta m_a)e^a_\mu + (\vartheta m_0 - \tilde{\varphi})\,\tau_\mu\right\} = t_{u\mu} - \mathcal{J}^\nu_{(0)}\tau_\nu\hat{a}_\mu, \tag{6.7.55}$$

$$S^a_\mu = t_{\nu\mu}e^{\nu a} + t_{u\mu}m^a - \mathcal{J}^\nu_{(0)}e^a_\nu\left\{(a_b - \vartheta m_b)e^b_\mu - (\vartheta m_0 + \tilde{\varphi})\tau_\mu\right\} = t_{\nu\mu}e^{\nu a} + t_{u\mu}m^a - \mathcal{J}^\nu_{(0)}e^a_\nu\hat{a}_\mu, \tag{6.7.56}$$

$$T^0 = t_{uu} - \mathcal{J}^\mu_{(0)}\vartheta\tau_\mu, \tag{6.7.57}$$

$$T^a = t_{u\mu}e^{\mu a} + t_{uu}m^a - \mathcal{J}^\mu_{(0)}e^a_\mu\vartheta, \tag{6.7.58}$$

$$\langle O_\vartheta\rangle = -\mathcal{J}^\mu_{(0)}(\tau_\mu m_0 + e^a_\mu m_a) + \mathcal{J}^u_{(0)} = \mathcal{J}^u_{(0)} - \mathcal{J}^\mu_{(0)}m_\mu = 0, \tag{6.7.59}$$

$$\langle O_\chi\rangle = -2\langle O_\psi\rangle, \tag{6.7.60}$$

$$j^a = \mathcal{J}^\mu_{(0)}e^a_\mu, \tag{6.7.61}$$

$$j^0 = \mathcal{J}^\mu_{(0)}\tau_\mu, \tag{6.7.62}$$

where we have introduced the notation

$$\hat{a}_\mu = a_\mu - \vartheta m_\mu - \tilde{\varphi}\tau_\mu, \tag{6.7.63}$$

which is the same GED field combination we considered in section 6.1. These allow is to determine the transformation properties of the VEVs, which we will consider in section 6.7.3. Now, we invert the relations above to find the expressions for the higher dimensional VEVs in terms of the lower dimensional ones, which will allow for a straightforward translation of the higher-dimensional Ward identities:

$$\mathcal{J}^\mu_{(0)} = j^\mu, \qquad t_{u\mu} = S^0_\mu + j^0\hat{a}_\mu, \qquad e^{\nu a}t_{\nu\mu} = S^a_\mu + j^a\hat{a}_\mu - (S^0_\mu + j^0\hat{a}_\mu)m^a, \qquad t_{uu} = T^0 + \vartheta j^0. \tag{6.7.64}$$

6.7.3 *Local Transformations of the VEVs*

Having obtained expressions for the four-dimensional VEVs in terms of the VEVs in the electromagnetic uplift in (6.7.55)–(6.7.62), we can use our knowledge of the transformation properties of t_{AB} to obtain the transformations of the VEVs, just like in section 6.3.2.3. Using

$$\pounds_\xi t_{AB} = \xi^C\partial_C t_{AB} + t_{CB}\partial_A\xi^C + t_{AC}\partial_B\xi^C + \delta_D t_{AB}, \tag{6.7.65}$$

with

$$\hat{\xi}^u = -\sigma, \tag{6.7.66}$$

we find

$$\begin{aligned}
\delta S^0_\mu &= \pounds_\xi S^0_\mu - T^0\partial_\mu\sigma + j^0\vartheta\partial_\mu\sigma - j^0\partial_\mu\Gamma_{(0)} - 2\Lambda_D S^0_\mu, && (6.7.67)\\
\delta S^a_\mu &= \pounds_\xi S^a_\mu + \lambda^a S^0_\mu + \lambda^a{}_b S^b_\mu - T^a\partial_\mu\sigma + j^a\vartheta\partial_\mu\sigma - j^a\partial_\mu\Gamma_{(0)} - 3\Lambda_D S^a_\mu, && (6.7.68)\\
\delta T^0 &= \pounds_\xi T^0 - 2\Lambda_D T^0, && (6.7.69)\\
\delta T^a &= \pounds_\xi T^a + \lambda^a T^0 + \lambda^a{}_b T^b - 3\Lambda_D T^a, && (6.7.70)\\
\delta\langle O_\chi\rangle &= \pounds_\xi\langle O_\chi\rangle - 4\Lambda_D\langle O_\chi\rangle, && (6.7.71)\\
\delta j^0 &= \pounds_\xi j^0 - 2\Lambda_D j^0, && (6.7.72)\\
\delta j^a &= \pounds_\xi j^a + \lambda^a j^0 + \lambda^a{}_b j^b - 3\Lambda_D j^a. && (6.7.73)
\end{aligned}$$

Thus, we see that—just like the sources—the VEVs transform under the symmetry group

$$Diff(\mathcal{M}) \times Sch_2(d,1) \times U(1). \tag{6.7.74}$$

6.8 THE WARD IDENTITIES & BOUNDARY STRESS TENSOR

6.8.1 *The HIM Boundary Energy-Momentum Tensor & Augmentations*

The HIM stress tensor,

$$T^\mu{}_\nu = -v^\mu S^0_\nu + e^\mu_a S^a_\nu, \tag{6.8.1}$$

transforms under gauge transformations ($U_\sigma(1)$ and $U_\Gamma(1)$) as

$$\delta_{\sigma,\Gamma} T^\mu{}_\nu = -\left(T^\mu + j^\mu\vartheta\right)\partial_\nu\sigma - j^\mu\partial_\nu\Gamma_{(0)}. \tag{6.8.2}$$

This leads us to define a totally gauge invariant augmentation,

$$T_{(\chi,\hat{a})}{}^\mu{}_\nu = T^\mu{}_\nu + T^\mu\partial_\nu\chi + j^\mu\hat{a}_{(\chi)\nu}, \tag{6.8.3}$$

where

$$\hat{a}_{(\chi)\mu} = a_\mu - \vartheta M_\mu - \tilde{\varphi}\tau_\mu. \tag{6.8.4}$$

Note in particular that $T_{(\chi,\hat{a})}{}^\mu{}_\nu$ is also invariant under δ_ℓ. The combination $T^\mu + j^\mu\vartheta$ featuring prominently above is δ_ℓ invariant, and we write it as

$$T^\mu_{(\hat{a})} = T^\mu + j^\mu\vartheta. \tag{6.8.5}$$

Note that this implies the relation

$$T_{(\chi,\hat{a})}{}^\mu{}_\nu = T^\mu_{(\hat{a})\nu} + T^\mu_{(\hat{a})}\partial_\nu\chi, \tag{6.8.6}$$

where

$$T_{(\hat{a})}{}^\mu{}_\nu = T^\mu{}_\nu + j^\mu\hat{a}_\nu \tag{6.8.7}$$

is the $U_\Gamma(1)$ gauge invariant HIM tensor. We will eventually want to write Ward identities in terms of quantities carrying space-time indices, so it is useful to note that

$$T^\mu = -T^0 v^\mu + T^a e^\mu_a, \qquad t^\mu = -j^0 v^\mu + j^a e^\mu_a. \tag{6.8.8}$$

We also have augmented Galilean boost invariant objects,

$$\begin{aligned}
\tilde{\Phi}_{(\chi)} &= -v^\rho M_\rho + \frac{1}{2}h^{\rho\sigma}M_\rho M_\sigma, && (6.8.9)\\
\hat{v}^\mu_{(\chi)} &= v^\mu - h^{\mu\nu}M_\nu, && (6.8.10)\\
\hat{e}^a_{(\chi)\mu} &= e^a_\mu - M_\nu e^{\nu a}\tau_\mu, && (6.8.11)\\
\hat{h}_{(\chi)\mu\nu} &= \delta_{ab}\hat{e}^a_{(\chi)\mu}\hat{e}^b_{(\chi)\nu}, && (6.8.12)
\end{aligned}$$

where we have replaced m_μ with the gauge invariant version M_μ. Note that we can relate the lower dimensional quantities to higher dimensional fields in a more covariant manner. For example, observe that

$$T^\mu{}_\nu = -v^\mu S^0_\nu + e^\mu_a S^a_\nu = -v^\mu t_{u\nu} + e^\mu_a \left(t_{\rho\nu}e^{\rho a} + t_{u\nu}m^a\right) + v^\mu j^0 \hat{a}_\nu - e^\mu_a j^a \hat{a}_\nu \tag{6.8.13}$$

$$= \gamma^{\mu A}_{(0)} t_{A\nu} - j^\mu \hat{a}_\nu = t^\mu{}_\nu - j^\mu \hat{a}_\nu. \tag{6.8.14}$$

A similar analysis shows that

$$T^\mu = t^\mu{}_u - j^\mu \vartheta. \tag{6.8.15}$$

Continuing this procedure, we find

$$t^{\mu u} = 2\tilde{\Phi}\left(T^\mu + \vartheta j^\mu\right) - \hat{v}^\sigma \left(T^\mu{}_\sigma + j^\mu \hat{a}_\sigma\right), \qquad t^{\mu\nu} = -\hat{v}^\mu \left(T^\nu + \vartheta j^\nu\right) + h^{\mu\rho}\left(T^\nu{}_\rho + j^\mu \hat{a}_\rho\right). \tag{6.8.16}$$

This implies further that

$$t^u{}_\nu = 2\tilde{\Phi} T^\mu_{(\hat{a})} \bar{h}_{\mu\nu} - \hat{v}^\sigma T^\mu_{(\hat{a})\sigma} \bar{h}_{\mu\nu}, \tag{6.8.17}$$

where we have used that $\gamma_{(0)uu} = 0$.

6.8.2 *Ward Identities*

These Ward identities are obtained by requiring invariance of the renormalized on-shell action under the respective symmetries, that is to say, the expression

$$\delta S_{\text{ren}} = \int_{\partial\mathcal{M}} d^3x\, e \left(-S^0_\mu \delta v^\mu + S^a_\mu \delta e^\mu_a + T^0 \delta m_0 + T^a \delta m_a + \langle O_\chi \rangle \delta\chi + j^a \delta a_a - j^0 \delta\tilde{\varphi} + \langle O_\vartheta \rangle \delta\vartheta - \mathcal{A}^{(0)}_{\text{red}} \frac{\delta r}{r}\right), \tag{6.8.18}$$

should be invariant under $Diff(\mathcal{M}) \times Sch_2(d,1) \times U(1)$. Note also that $\langle O_\vartheta \rangle = 0$ above.

6.8.2.1 *Galilean Boosts*

We start with the boost Ward identity; the fields transforming under Galilean boosts G are

$$\delta_G v^\mu = \lambda^a e^\mu_a, \qquad \delta_G m_a = \lambda_a, \qquad \delta_G m_0 = -\lambda^a m_a, \qquad \delta_G \tilde{\varphi} = a_a \lambda^a, \qquad \delta_G a_a = \vartheta \lambda_a. \tag{6.8.19}$$

Requiring invariance of (6.8.18) under Galilean boosts, we obtain the Ward identity

$$-S^0_\mu e^\mu_a + T^b \delta_{ba} - T^0 m_a - j^0 a_a + j^b \delta_{ba} \vartheta = 0. \tag{6.8.20}$$

Employing techniques identical to those in section 5.3.3, we may covariantize the boost Ward identity above. Using our result for the pure Lifshitz boost Ward identity (5.3.34), we note that (6.8.20) can be written as

$$\hat{e}^a_{(\chi)\mu} T^\mu - j^0 a^a + j^a \vartheta = \tau_\nu e^{\mu a}\left(T^\nu{}_\mu + T^\nu \partial_\mu \chi\right). \tag{6.8.21}$$

To covariantize the new part, observe that

$$j^0 = \tau_\mu j^\mu, \qquad j^b = e^b_\mu j^\mu, \qquad a_a = e^\mu_a a_\mu, \tag{6.8.22}$$

implying that

$$-j^0 a_a + j^b \delta_{ba} \vartheta = j^\mu \left(-\tau_\mu e^\nu_a a_\nu + \vartheta e_{\mu a}\right) = j^\mu \left(-\tau_\mu e^\nu_a \left(a_\nu - \vartheta M_\nu\right) + \vartheta \hat{e}_{(\chi)\mu a}\right). \tag{6.8.23}$$

We can replace $a_\nu - \vartheta M_\nu$ with a Stückelberg invariant extension of $\hat{a}_\mu$,

$$\hat{a}_{(\chi)\mu} = a_\mu - \vartheta M_\mu - \tilde{\varphi}\tau_\mu, \tag{6.8.24}$$

due to the orthogonality property $\tau_\mu e^\mu_a = 0$. All in all, this means that the boost Ward identity becomes

$$\hat{e}^a_{(\chi)\mu} T^\mu + j^\mu \left(\vartheta \hat{e}^a_{(\chi)\mu} - \tau_\mu e^{\nu a} \hat{a}_{(\chi)\nu}\right) = \tau_\nu e^{\mu a}\left(T^\nu_\mu + T^\nu \partial_\mu \chi\right), \tag{6.8.25}$$

which can also be written as

$$\hat{e}^a_{(\chi)\mu} T^\mu_{(\hat{a})} = \tau_\nu e^{\mu a} T_{(\chi,\hat{a})}{}^\nu{}_\mu. \tag{6.8.26}$$

Thus, we see that the boost Ward identity takes a form identical to what we find in pure Lifshitz holography in chapter 5, but instead in written in terms of the GED-augmented quantities defined in section 6.8.1.

6.8.2.2 *Rotations*

Next, we turn to rotations J, where the relevant transformations are

$$\delta_J a_a = \lambda_a{}^b a_b, \qquad \delta_J m_a = \lambda_a{}^b m_b, \qquad \delta_J e^\mu_a = \lambda_a{}^b e^\mu_b, \tag{6.8.27}$$

which means that the rotation Ward identity becomes

$$S^{[a}_\mu e^{\mu b]} + T^{[a} m^{b]} + j^{[a} a^{b]} = 0. \tag{6.8.28}$$

Multiplying by m^b and antisymmetrizing, this becomes

$$S^0_\mu e^{\mu[a} m^{b]} + t^0 a^{[a} m^{b]} - t^{[a} m^{b]} \vartheta = T^{[a} m^{b]}. \tag{6.8.29}$$

Following the insight from section 5.3.3, we expect that in order to obtain the rotation Ward identity, we need to apply the boost Ward identity in the form

$$-S^0_\mu e^{\mu a} + T^a - T^0 m^a - j^0 a^a + j^a \vartheta = 0, \tag{6.8.30}$$

so that our rotation Ward identity (6.8.28) can be recast in the form

$$S^{[a}_\mu e^{\mu b]} + S^0_\mu e^{\mu[a} m^{b]} + j^0 a^{[a} m^{b]} - j^{[a} m^{b]} \vartheta + j^{[a} a^{b]} = 0. \tag{6.8.31}$$

Now, observe that we can rewrite the left-hand side of the expression above in the following manner

$$\hat{e}^{[a}_{(\chi)\nu} e^{b]\mu} T_{(\chi,\hat{a})}{}^\nu{}_\mu - \partial_\mu \chi e^{\mu b} \left(\vartheta t^a - j^0 a^a + T^a - m^a T^0 - e^{\rho a} S^0_\rho \right) \tag{6.8.32}$$

$$+ \partial_\mu \chi e^{\mu a} \left(\vartheta j^b - j^0 a^b + T^b - m^b T^0 - e^{\rho b} S^0_\rho \right). \tag{6.8.33}$$

Here we recognize the two last terms as two copies of the boost Ward identity[5] (6.8.30), implying that the rotation Ward identity can be written covariantly as

$$\hat{e}^{[a}_{(\chi)\nu} e^{b]\mu} T_{(\chi,\hat{a})}{}^\nu{}_\mu = 0. \tag{6.8.34}$$

This statement is the non-relativistic analogue of symmetry of the energy-momentum tensor, and we see again that the Ward identity can be expressed compactly in terms of the GED-augmented HIM stress tensor.

6.8.2.3 *Gauge Transformations*

For gauge transformations δ_Γ, the relevant transformations are

$$\delta_\Gamma \tilde{\varphi} = v^\mu \partial_\mu \Gamma_{(0)}, \qquad \delta_\Gamma a_a = e^\mu_a \partial_\mu \Gamma_{(0)}, \tag{6.8.35}$$

which means that we get

$$0 = \int d^3x\, e \left(-j^0 v^\mu \partial_\mu \Gamma_{(0)} + j^a e^\mu_a \partial_\mu \Gamma_{(0)} \right) = - \int d^3x\, e \frac{\Gamma_{(0)}}{e} \partial_\mu \left[e \left(-j^0 v^\mu + j^a e^\mu_a \right) \right] = - \int d^3x\, e \frac{\Gamma_{(0)}}{e} \partial_\mu \left(e j^\mu \right), \tag{6.8.36}$$

that is to say, the $U_\Gamma(1)$ Ward identity is

$$\frac{1}{e} \partial_\mu \left(e j^\mu \right) = 0. \tag{6.8.37}$$

This expresses the TNC analogue of conservation of the GED current. Note that we can also write this in the form

$$0 = \mathring{\nabla}_\mu j^\mu = \nabla_\mu j^\mu + \bar{\Gamma}^\nu_{[\nu\mu]} j^\mu. \tag{6.8.38}$$

We note that [36] finds an identical Ward identity.

5 Recall that the same happened in our treatment of pure Lifshitz holography, cf. section 5.3.3.

6.8.2.4 *Stückelberg Gauge Transformations*

In this case, the relevant field transformations are

$$\delta_N \chi = \sigma, \qquad \delta_N m_\mu = \partial_\mu \sigma \Rightarrow \delta_N m_0 = -v^\mu \partial_\mu \sigma, \quad \delta_N m_a = e_a^\mu \partial_\mu \sigma. \tag{6.8.39}$$

Plugging this into the expression for δS in (6.8.18), we get

$$0 = \int d^3x \, e \left(T^\mu \partial_\mu \sigma + \langle O_\chi \rangle \sigma \right) \tag{6.8.40}$$

$$= \int d^3x \, e \left(-\frac{\sigma}{e} \partial_\mu \left(e T^\mu \right) + \langle O_\chi \rangle \right), \tag{6.8.41}$$

which implies the Ward identity

$$\langle O_\chi \rangle = \frac{1}{e} \partial_\mu (e T^\mu). \tag{6.8.42}$$

Since the left-hand side is generically different from zero, this breaks mass current conservation.

6.8.2.5 *Weyl Transformations*

The relevant transformations read in this case

$$\delta_D v^\mu = -2\Lambda_D v^\mu, \qquad \delta_D e_a^\mu = -\Lambda_D e_a^\mu, \qquad \delta_D m_0 = -2\Lambda_D m_0, \qquad \delta_D m_a = -\Lambda_D m_a \tag{6.8.43}$$

$$\delta_D \tilde\varphi = -2\Lambda_D \tilde\varphi, \qquad \delta_D a_a = -\Lambda_D a_a, \qquad \delta_D r = -\Lambda_D r. \tag{6.8.44}$$

This produces the Weyl Ward Identity

$$\mathcal{A}^{(0)}_{\text{red}} = -2S^0_\mu v^\mu + S^a_\mu e^\mu_a + 2T^0 m_0 + T^a m_a - 2j^0 \tilde\varphi + j^a a_a. \tag{6.8.45}$$

The right-hand side can be rewritten in the following manner:

$$-2v^\mu \tau_\nu \left(T^\nu{}_\mu + j^\nu \hat a_\mu \right) + e^\mu_a e^a_\nu \left(T^\mu{}_\nu + j^\nu \hat a_\mu \right) + 2\tilde\Phi \tau_\mu \left(T^\mu + j^\mu \vartheta \right) + m_a \left(-m^a T^0 + T^a - m^a j^0 \vartheta + \vartheta j^a \right). \tag{6.8.46}$$

Using the boost Ward identity in the form (6.8.20), the part proportional to m_a takes the form

$$m_a \left(S^0_\mu e^{\mu a} + j^0 \left(a^a - m^a \vartheta \right) \right) = m_a \tau_\nu e^{\mu a} \left[T^\nu{}_\mu + j^\nu \hat a_\mu \right], \tag{6.8.47}$$

which means that we can further rewrite the Weyl Ward identity (6.8.45) as

$$\mathcal{A}^{(0)}_{\text{red}} = -2\hat v^\mu \tau_\nu \left(T^\nu{}_\mu + j^\nu \hat a_\mu \right) + e^\mu_a \hat e^a_\nu \left(T^\mu{}_\nu + j^\nu \hat a_\mu \right) + 2\tilde\Phi \tau_\mu \left(T^\mu + j^\mu \vartheta \right) \tag{6.8.48}$$

$$= \left(e^\mu_a \hat e^a_\nu - 2\hat v^\mu \tau_\nu \right) T_{(\hat a)}{}^\mu{}_\nu + 2\tilde\Phi \tau_\mu T^\mu_{(\hat a)}. \tag{6.8.49}$$

The first term in (6.8.49) is the $z = 2$ deformed trace, while the second can be interpreted as a form of potential energy since T^0 is roughly the mass, while $\tilde\Phi$ plays the rôle of the Newtonian potential.

6.8.2.6 *Diffeomorphisms*

All the fields transform under infinitesimal boundary diffeomorphisms generated by ξ, but there are only two types of transformations depending on the number of space-time indices:

$$\delta_\xi X = \xi^\nu \partial_\nu X, \qquad \delta_\xi V^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu. \tag{6.8.50}$$

In particular, the only fields carrying space-time indices are v^μ and e^μ_a, and we note that due their transformation properties it will be necessary to integrate by parts in order to obtain the Ward identity. This procedure is straightforward and gives the Ward identity

$$0 = \frac{1}{e} \partial_\mu \left(e T^\mu{}_\nu \right) - S^0_\mu \partial_\nu v^\mu + S^a_\mu \partial_\nu e^\mu_a + T^0 \partial_\nu m_0 + T^a \partial_\nu m_a + \langle O_\chi \rangle \partial_\nu \chi + j^a \partial_\nu a_a - j^0 \partial_\nu \tilde\varphi + \langle O_\vartheta \rangle \partial_\nu \vartheta \tag{6.8.51}$$

$$= \frac{1}{e} \partial_\mu \left(e \left(T^\mu{}_\nu + T^\mu \partial_\nu \chi \right) \right) - S^0_\mu \partial_\nu v^\mu + S^a_\mu \partial_\nu e^\mu_a + T^0 \partial_\nu m_0 + T^a \partial_\nu m_a + j^a \partial_\nu a_a - j^0 \partial_\nu \tilde\varphi. \tag{6.8.52}$$

We will consider the covariant version of this result from the perspective of null reduction in the next section.

6.8.3 *Ward Identities From Null Reduction*

The five-dimensional Ward identities read

$$t^A{}_A = \mathcal{A}^{(0)}_{5\mathrm{D}}, \qquad \mathring{\nabla}_A t^A{}_B + \mathcal{F}_{(0)BA}\mathcal{J}^A_{(0)} + \langle O_\psi \rangle \partial_B \psi_{(0)} = 0, \qquad \mathring{\nabla}_A \mathcal{J}^A_{(0)} = 0, \tag{6.8.53}$$

where $\mathcal{A}^{(0)}_{5\mathrm{D}}$ is the boundary value of the Weyl anomaly, which we emphasize has nothing to with the five-dimensional gauge field $\mathcal{A}_A$, and $\mathring{\nabla}$ is the covariant derivative of $\gamma_{(0)}$. Reducing the last of these gives us

$$\partial_\mu \left(e j^\mu\right) = 0, \tag{6.8.54}$$

as we found in (6.8.37). Note that this does not put any constraints on $\mathcal{J}^u_{(0)}$. Now, recall that

$$t^\mu{}_\nu = T^\mu{}_\nu + j^\mu \hat{a}_\nu, \qquad t^\mu{}_u = T^\mu + j^\mu \vartheta. \tag{6.8.55}$$

The u-component of the higher-dimensional diffeomorphism Ward identity then takes the form

$$0 = \frac{1}{e}\partial_\mu \left(e t^\mu{}_u\right) + \mathcal{F}_{(0)u\mu}\mathcal{J}^\mu_{(0)} + \langle O_\psi \rangle \partial_u \psi_{(0)} = \frac{1}{e}\partial_\mu \left(e\left[T^\mu + j^\mu \vartheta\right]\right) - j^\mu \partial_\mu \vartheta - \langle O_\chi \rangle \tag{6.8.56}$$

$$= \frac{1}{e}\partial_\mu \left(e T^\mu\right) + \cancel{j^\mu \partial_\mu \vartheta} + \frac{\vartheta}{e}\underbrace{\partial_\mu \left(e j^\mu\right)}_{=0} - \cancel{j^\mu \partial_\mu \vartheta} - \langle O_\chi \rangle\,, \tag{6.8.57}$$

leading to the Stückelberg Ward identity (6.8.42) found previously.

We now begin our reduction of the Weyl Ward identity (using $\mathcal{A}^{(0)}_{5\mathrm{D}} = \mathcal{A}^{(0)}_{\mathrm{red}}$):

$$\mathcal{A}^{(0)}_{\mathrm{red}} = \gamma^{AB}_{(0)} t_{AB} = \gamma^{\mu\nu}_{(0)} t_{\mu\nu} + 2\gamma^{u\mu}_{(0)} t_{u\mu} + \gamma^{uu}_{(0)} t_{uu} \tag{6.8.58}$$

$$= h^{\mu\nu} t_{\mu\nu} - 2\hat{\upsilon}^\mu t_{u\mu} + 2\tilde{\Phi} t_{uu} \tag{6.8.59}$$

$$= e^\mu_a S^a_\mu + j^a \hat{a}_\mu e^\mu_a - \left(S^0_\mu + j^0 \hat{a}_\mu\right) m^a e^\mu_a - 2\hat{\upsilon}^\mu \left(S^0_\mu + j^0 \hat{a}_\mu\right) + 2\tilde{\Phi}\left(T^0 + \vartheta j^0\right) \tag{6.8.60}$$

$$= \left(T^\mu{}_\nu + j^\mu \hat{a}_\nu\right)\left[e^\nu_a \hat{e}^a_\mu - 2\hat{\upsilon}^\nu \tau_\mu\right] + 2\tilde{\Phi}\tau_\mu \left(T^\mu + \vartheta j^\mu\right) \tag{6.8.61}$$

$$= T^\mu_{(\hat{a})\nu}\left[e^\nu_a \hat{e}^a_\mu - 2\hat{\upsilon}^\nu \tau_\mu\right] + 2\tilde{\Phi}\tau_\mu T^\mu_{(\hat{a})}, \tag{6.8.62}$$

in agreement with our previous result (6.8.49).

Turning to the diffeomorphism Ward identity, we observe

$$\mathring{\nabla}_A t^A{}_\mu = \partial_\nu t^\nu{}_\mu + \mathring{\Gamma}^\nu_{\nu\lambda} t^\lambda{}_\mu - \mathring{\Gamma}^\lambda_{\nu\mu} t^\nu{}_\lambda + \mathring{\Gamma}^u_{u\lambda} t^\lambda{}_\mu - \mathring{\Gamma}^\lambda_{u\mu} t^u{}_\lambda \tag{6.8.63}$$

$$= \nabla_\nu T^\nu_{(\hat{a})\mu} + \bar{\Gamma}^\lambda_{[\nu\mu]} T^\nu_{(\hat{a})\lambda} - 2\bar{\Gamma}^\nu_{[\nu\lambda]} T^\lambda_{(\hat{a})\mu} + h^{\lambda\rho}\partial_{(\mu}\tau_{\rho)}\left[2\tilde{\Phi} T^\sigma_{(\hat{a})}\bar{h}_{\sigma\lambda} - \hat{\upsilon}^\eta T^\sigma_{(\hat{a})\eta}\bar{h}_{\sigma\lambda}\right]. \tag{6.8.64}$$

Now, defining $\hat{f}_{\mu\nu} = 2\partial_{(\mu}\hat{a}_{\nu)}$ and using the relation (6.7.59) for $\mathcal{J}^u_{(0)}$, the rest of the diffeomorphism Ward identity can be written as

$$\mathcal{F}_{(0)\mu A}\mathcal{J}^A_{(0)} + \langle O_\psi \rangle \partial_\mu \psi_{(0)} = \hat{f}_{\mu\nu} j^\nu + j^\nu m_\nu \partial_\mu \vartheta + \langle O_\chi \rangle \partial_\mu \chi \tag{6.8.65}$$

$$= \hat{f}_{\mu\nu} j^\nu + j^\nu m_\nu \partial_\mu \vartheta + \mathring{\nabla}_\nu T^\nu \partial_\mu \chi. \tag{6.8.66}$$

Results from [57] indicate that the diffeomorphism Ward identity can also be covariantly expressed in the form

$$0 = e^{-1}\left[\partial_\nu \left(e T^\nu_{(\hat{a})\mu}\right) + \partial_\nu \left(e T^\nu\right)\partial_\mu \chi\right] + T^\rho_{(\hat{a}\nu)}\left(\hat{\upsilon}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu_{(\hat{a})}\partial_\mu \tilde{\Phi} + j^\nu \left(\hat{f}_{\mu\nu} + m_\nu \partial_\mu \vartheta\right). \tag{6.8.67}$$

6.9 GENERAL–z CONJECTURE

Based on the analysis for $z = 2$ above and the dimensional analysis of section 6.1.3, we can put forward a general conjecture for charged Lifshitz holography with $z > 1$.

6.9.1 *The Action*

Amalgamating our starting point for pure Lifshitz holography (5.2.1) in chapter (5) with our specific upliftable model (6.2.21), we take as our starting point the action

$$S = \int \mathrm{d}^4x\, \sqrt{-g}\bigg(R - \frac{1}{4}Z_{(F)}(\Phi,\Upsilon)F^2 - \frac{1}{2}W(\Phi,\Upsilon)B^2 - \frac{x}{2}(\partial\Phi)^2 - \frac{1}{4}Z_{(H)}(\Phi,\Upsilon)H^2 \tag{6.9.1}$$

$$- \frac{1}{2}Z_{(F,H)}(\Phi,\Upsilon)F_{MN}H^{MN} - \frac{1}{2}Z_{(\Upsilon)}(\partial\Upsilon)^2 + V(\Phi,\Upsilon)\bigg). \tag{6.9.2}$$

We have kept the dependency on Υ general in the action above. The functions $Z_{(i)}$ and W are restricted to be positive and to be of such a form that the function α does not depend on Υ. If we did not do that, we would no longer find that $\alpha \simeq \alpha_{(0)} + \cdots$, since, based on our dimensional analysis in section 6.1.3, this field behaves as $\Upsilon \simeq r^{z/2-1}\vartheta$, so for $1 < z < 2$, the leading component would no longer be $\mathcal{O}(r^0)$. Whenever $z \geq 2$, α can depend on Υ, but we will not explicitly consider this. The potential V is negative close to a Lifshitz solution.

We expect in analogy with our analysis in this chapter that this action admits solutions of the form

$$ds^2 = -\frac{1}{r^{2z}}\mathrm{d}t^2 + \frac{1}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{6.9.3}$$

$$B = A_0\frac{1}{r^z}\mathrm{d}t, \qquad C = -\frac{\beta(\Phi,\Upsilon)}{\alpha(\Phi)}A_0\frac{1}{r^z}\mathrm{d}t, \qquad \Upsilon = r^{z/2-1}\Upsilon_\star, \qquad \Phi = \Phi_\star, \tag{6.9.4}$$

where $\beta(\Phi,\Upsilon)$ is a suitable generalization of the combination $\Upsilon\alpha(\Phi)$, which appears in the upliftable model.

6.9.2 *Sources & VEVs*

The metric and Proca field are expressed as in chapter 5,

$$ds^2 = \frac{\mathrm{d}r^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b, \qquad B_M = A_M - \partial_M\Xi, \tag{6.9.5}$$

Invoking the equivalence principle, we can translate the results of section 6.1.3 for the scaling weights of the GED fields on a flat background to the scaling weights of the frame components of the sources, which we saw matched our holographic results for $z = 2$. Our boundary conditions, suitably generalized, can therefore be expressed in the form (where $\Delta \geq 0$)

$$E_0^\mu \simeq -r^z\alpha_{(0)}^{-1/3}v^\mu, \tag{6.9.6}$$

$$E_a^\mu \simeq r\alpha_{(0)}^{1/3}e_a^\mu, \tag{6.9.7}$$

$$A_\mu - \alpha(\Phi)E_\mu^0 \simeq -r^{z-2}\tilde{m}_\mu, \tag{6.9.8}$$

$$\Phi \simeq r^\Delta\phi, \tag{6.9.9}$$

$$\Xi \simeq -r^{z-2}\chi, \tag{6.9.10}$$

$$A_r \simeq -(z-2)r^{z-3}\chi, \tag{6.9.11}$$

$$\Upsilon \simeq r^{z/2-1}\vartheta \tag{6.9.12}$$

$$E_0^\mu C_\mu + \Upsilon\alpha(\Phi) \simeq -r^{z/2+1}\alpha_{(0)}^{-1/3}\tilde{\varphi}, \tag{6.9.13}$$

$$E_a^\mu C_\mu \simeq r^{z/2}\alpha_{(0)}^{1/3}a_a. \tag{6.9.14}$$

Note that, as was the case in chapter 5, $\alpha_{(0)}$ is only a function of ϕ when $\Delta = 0$, which, as explicitly demonstrated in section 6.7.2, leads to additional contributions to the VEV $\langle\tilde{O}_\phi\rangle$.

These generalized sources transform according to

$$\delta e^a_\mu = \pounds_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu. \tag{6.9.15}$$

$$\delta \tau_\mu = \pounds_\xi \tau_\mu + z\Lambda_D \tau_\mu, \tag{6.9.16}$$

$$\delta \chi = \pounds_\xi \chi + \sigma - (z-2)\Lambda_D \chi, \tag{6.9.17}$$

$$\delta v^\mu = \pounds_\xi v^\mu + \lambda^a e^\mu_a - z\Lambda_D v^\mu, \tag{6.9.18}$$

$$\delta e^\mu_a = \pounds_\xi e^\mu_a + \lambda_a{}^b e^\mu_b - \Lambda_D, \tag{6.9.19}$$

$$\delta \phi = \pounds_\xi \phi - \Delta\Lambda_D \phi, \tag{6.9.20}$$

$$\delta \tilde{m}_\mu = \pounds_\xi m_\mu + e^a_\mu \lambda_a + \partial_\mu \sigma - (z-2)\Lambda_D \tilde{m}_\mu, \tag{6.9.21}$$

$$\delta \vartheta = \pounds_\xi \vartheta - (z/2-1)\Lambda_D \vartheta, \tag{6.9.22}$$

$$\delta \tilde{\varphi} = \pounds_\xi \tilde{\varphi} + a_a \lambda^a + v^\mu \partial_\mu \Gamma_{(0)} - (z/2+1)\Lambda_D \tilde{\varphi}, \tag{6.9.23}$$

$$\delta a_\mu = \pounds_\xi a_\mu + \vartheta \lambda_a e^a_\mu + \tau_\mu a_\nu e^\nu_a \lambda^a + \partial_\mu \Gamma_{(0)} + \tau_\mu v^\nu \partial_\nu \Gamma_{(0)} - (z/2-1)\Lambda_D a_\mu, \tag{6.9.24}$$

$$\delta a_a = \pounds_\xi a_a + \vartheta \lambda_a + \lambda_a{}^b a_b - \Lambda_D a_a + e^\mu_a \partial_\mu \Gamma_{(0)} - z/2\Lambda_D a_a. \tag{6.9.25}$$

Similarly, as we did in chapter 5, we can assume the existence of a renormalized bulk action write the variation of the on-shell renormalized action as $\delta S_{\text{ren}} = -\int_{\partial\mathcal{M}} \mathrm{d}^3x\, e\, \left(\boldsymbol{\mathcal{V}}\delta\boldsymbol{\mathcal{X}} - \mathcal{A}\frac{\delta r}{r}\right)$, where the source complex is given by $\boldsymbol{\mathcal{X}} = \{E^0_\mu, E^\mu_a, \varphi, A_a, \Xi, \Phi, \Omega, C_a, Y\}$, and the responses are collected in the complex $\boldsymbol{\mathcal{V}} = \{\mathcal{S}^0_\mu, \mathcal{S}^a_\mu, \mathcal{T}_\varphi, \mathcal{T}^a, \mathcal{T}_\Xi, \mathcal{T}_\Phi, \mathcal{T}_\Omega, \mathcal{J}^a, \mathcal{T}_Y\}$, whose leading terms are the VEVs:

$$\mathcal{S}^0_\mu \simeq r^2 \alpha_{(0)}^{2/3} S^0_\mu, \tag{6.9.26}$$

$$\mathcal{S}^a_\mu \simeq r^{z+1} S^a_\mu, \tag{6.9.27}$$

$$\mathcal{T}_\varphi \simeq r^{4-z} \alpha_{(0)}^{2/3} T^0, \tag{6.9.28}$$

$$\mathcal{T}^a \simeq r^3 T^a, \tag{6.9.29}$$

$$\mathcal{T}_\Xi \simeq r^4 \alpha_{(0)}^{1/3} \langle O_\chi \rangle, \tag{6.9.30}$$

$$\mathcal{T}_\Phi \simeq r^{z+2-\Delta} \alpha_{(0)}^{1/3} \langle O_\phi \rangle, \tag{6.9.31}$$

$$\mathcal{T}_\Omega \simeq r^{z/2+1} \alpha_{(0)}^{2/3} j^0, \tag{6.9.32}$$

$$\mathcal{J}^a \simeq r^{z/2+2} j^a, \tag{6.9.33}$$

$$\mathcal{T}_Y \simeq r^{z/2+3} \alpha_{(0)}^{1/3} \langle O_\vartheta \rangle, \tag{6.9.34}$$

where we have, once more, used the results of the field theory dimensional analysis of section 6.1.3. Thus, the variation of the renormalized on-shell action becomes[6]:

$$\delta S_{\text{ren}} = \int \mathrm{d}^3x\, e\, \left[-S^0_\mu \delta v^\mu + S^a_\mu \delta e^\mu_a + T^0 \delta \tilde{m}_0 + T^a \delta \tilde{m}_a + \langle O_\chi \rangle \delta\chi + \langle \tilde{O}_\phi \rangle \delta\phi - j^0 \delta\tilde{\varphi} + j^a \delta a_a + \langle O_\vartheta \rangle \delta\vartheta - \mathcal{A}_{(0)} \frac{\delta r}{r}\right], \tag{6.9.35}$$

where

$$\langle \tilde{O}_\phi \rangle = \langle O_\phi \rangle + \delta_{\Delta,0} \frac{1}{3} \left[v^\mu (S^0_\mu + T^0 m_\mu) + e^\mu_a (S^a_\mu + T^a m_\mu) + j^a a_a - j^0 \tilde{\varphi}\right] \frac{\mathrm{d}\log\alpha_{(0)}}{\mathrm{d}\phi} = 0. \tag{6.9.36}$$

6 In this section, we use the symbol $\mathcal{A}_{(0)}$ for the anomaly, which in our $z = 2$ was the leading part of the five-dimensional gauge field. We hope that this does not cause confusion.

The VEVs transform under the Schrödinger group in the following manner

$$\delta S^0_\mu = \pounds_\xi S^0_\mu - T^0\partial_\mu\sigma + j^0\vartheta\partial_\mu\sigma - j^0\partial_\mu\Gamma_{(0)} - 2\Lambda_D S^0_\mu + \cdots , \tag{6.9.37}$$

$$\delta S^a_\mu = \pounds_\xi S^a_\mu + \lambda^a S^0_\mu + \lambda^a{}_b S^b_\mu - T^a\partial_\mu\sigma + j^a\vartheta\partial_\mu\sigma - j^a\partial_\mu\Gamma_{(0)} - (z+1)\Lambda_D S^a_\mu + \cdots , \tag{6.9.38}$$

$$\delta T^0 = \pounds_\xi T^0 - (4-z)\Lambda_D T^0 + \cdots , \tag{6.9.39}$$

$$\delta T^a = \pounds_\xi T^a + \lambda^a T^0 + \lambda^a{}_b T^b - 3\Lambda_D T^a + \cdots , \tag{6.9.40}$$

$$\delta \langle O_\chi \rangle = \pounds_\xi \langle O_\chi \rangle - 4\Lambda_D \langle O_\chi \rangle + \cdots , \tag{6.9.41}$$

$$\delta \langle O_\phi \rangle = \pounds_\xi \langle O_\phi \rangle + \delta_{\Delta,0}\frac{\mathrm{d}\log\alpha_{(0)}}{\mathrm{d}\phi}(\lambda_a T^a + \vartheta j^a\lambda_a) - (z+2-\Delta)\Lambda_D \langle O_\phi \rangle + \cdots , \tag{6.9.42}$$

$$\delta \langle O_\vartheta \rangle = \pounds_\xi \langle O_\vartheta \rangle - (z/2+3)\Lambda_D \langle O_\vartheta \rangle + \cdots , \tag{6.9.43}$$

$$\delta j^0 = \pounds_\xi j^0 - (z/2+2)\Lambda_D j^0 + \cdots , \tag{6.9.44}$$

$$\delta j^a = \pounds_\xi j^a + \lambda^a j^0 + \lambda^a{}_b j^b - (z/2+2)\Lambda_D j^a + \cdots , \tag{6.9.45}$$

where the dots denote possible terms involving derivatives of Λ_D.

6.9.3 *Ward Identities*

As demonstrated in [29, 32] and discussed in section 5.3, the Ward identities in pure Lifshitz holography are the same in all the four cases (5.2.28), although their derivation are differ from case to case. Assuming this also applies for charged Lifshitz holography, we restrict our attention to the case $1 < z < 2$ with $\Delta > 0$. In this case, the analysis of section 6.8.2 is nearly unchanged: the main difference is that now $\langle O \rangle_\vartheta$ and $\langle \tilde{O}_\phi \rangle$ are no longer zero. Therefore, we conjecture the following Ward identities

$$\hat{e}^a_{(\chi)\mu} T^\mu_{(\hat{a})} = \tau_\nu e^{\mu a} T^\nu_{(\chi,\hat{a})\mu} \qquad \text{(boosts)}, \tag{6.9.46}$$

$$0 = \hat{e}^{[a}_{(\chi)\nu} e^{b]\mu} T^\nu_{(\chi,\hat{a})\mu} \qquad \text{(rotations)}, \tag{6.9.47}$$

$$0 = \frac{1}{e}\partial_\mu \left(e j^\mu \right) \qquad (U_\Gamma(1) \text{ gauge transformations}), \tag{6.9.48}$$

$$\langle O_\chi \rangle = \frac{1}{e}\partial_\mu (e T^\mu) \qquad \text{(Stückelberg gauge transformations)}, \tag{6.9.49}$$

$$\mathcal{A}_{(0)} = -zS^0_\mu v^\mu + S^a_\mu e^\mu_a + (2z-2)T^0 m_0 + (z-1)T^a m_a + (z-2)\langle O_\chi \rangle \chi \tag{6.9.50}$$

$$+ \Delta \langle \tilde{O}_\phi \rangle \phi - (z/2+1) j^0 \tilde{\varphi} + \frac{z}{2} j^a a_a + (z/2-1)\langle O_\vartheta \rangle \vartheta \qquad \text{(Weyl transformations)}, \tag{6.9.51}$$

$$0 = \frac{1}{e}\partial_\mu \left(e T^\mu_{(\chi)\nu} \right) - S^0_\mu \partial_\nu v^\mu + S^a_\mu \partial_\nu e^\mu_a + T^0 \partial_\nu m_0 \tag{6.9.52}$$

$$+ T^a \partial_\nu m_a + j^a \partial_\nu a_a - j^0 \partial_\nu \tilde{\varphi} + \langle O_\vartheta \rangle \partial_\nu \vartheta + \langle \tilde{O}_\phi \rangle \partial_\nu \phi \qquad \text{(Diffeomorphisms)}. \tag{6.9.53}$$

7 CONCLUSION & OUTLOOK

In this thesis, we have developed charged Lifshitz holography for $z = 2$ by Scherk-Schwarz reducing the electromagnetic uplift, which we holographically renormalized to obtain a novel counterterm. We have shown that the new sources transform as the fields of Galilean electrodynamics, and we have worked out the VEVs and the corresponding Ward identities. We have shown that the boundary geometry becomes TNC and that fixing the boundary value of the dilaton $\phi = 0$ turns the boundary geometry into TTNC, and we have commented on the relation to the gauging of the Schrödinger algebra.

Our results open up a number of very interesting avenues for future research: one immediate thing to do would be the explicit verification of the general-z charged Lifshitz holography conjecture made in section 6.9 using the methods of [32]: once it has been verified that (6.9.3)–(6.9.4) provides a solution to the general model (6.9.2), the identification of sources and VEVs as well as the computation of the Ward identities will match our results.

With an understanding of charged Lifshitz holography in conjunction with a suitably generalized hydrodynamic analysis à la [57], we would be able to undertake a promising first foray into what we may dub *Lif/CMT*. In particular, given the link between TNC geometry and certain condensed matter problems, this for example paves the way for a Lifshitz holographic realization of the fractional quantum hall effect (FQHE).

Another possibly worthwhile path of exploration—also related to hydrodynamics—would be to investigate if a hydrodynamical analysis of charged $z = 0$ Schrödinger black branes in the uplift can be null reduced to give an interesting perspective on existing results [156] on non-relativistic hydrodynamics where the boost symmetry is intact.

In another direction, it would be interesting to develop a notion of holography with HL gravity in the bulk, especially given the connection between dynamical TNC geometry and HL gravity [53] and to see if the boundary geometry in this scenario is also described by TNC geometry. This again has ties to the FQHE [38]. In a similar vein, it would be interesting the explore the holographic implications of bulk HL gravity coupled to GED on a TNC geometry. As we have demonstrated, such an action arises from the Weyl anomaly of our $z = 2$ charged Lifshitz model.

It would also be fruitful to explore the holographic rôle of other non-Lorentzian geometries. In particular it was recently discovered in [157] that by null reducing the Polyakov action in target space and sending the string tension to zero, a novel type of non-Lorentzian geometry called $U(1)$-Galilean geometry emerges. Whereas NC geometry arises by gauging the Galilei algebra which arises as a $c \to \infty$ contraction of the Poincaré algebra, as discussed in chapter 4, one may instead take the limit $c \to 0$ to obtain the Carroll algebra, which was gauged in [121] to obtain Carrollian geometry. It was shown in [158] that the BMS group is the conformal extension of the Carroll group, and the relation to flat space holography was explored in [159]. The boundary geometry for Lifshitz space-times with $z < 1$ would be Carroll geometry, but such space-times violate the null energy condition and as such are not usually considered.

Extending the analysis of the holographic superconductor (cf. appendix E) using the version of Lifshitz holography developed in this thesis would also be worthwhile. In this context, the results of [160] could be useful.

There are other models admitting Lifshitz solutions. In particular, setting $W = 0$ in the EPD action (5.2.1) results in an EMD model, where the Lifshitz solutions are supported by a Maxwell field and a logarithmically running dilaton [22]. Lifshitz hydrodynamics in this theory was explored in [148, 149], and it would be interesting to also add additional charged fields to this analysis and working out the consequences.

In regards to holographic renormalization, as we described in chapter 3, it would be interesting to work out the properties of p-form fields in general and verify the conjecture put forward in section 3.3.

We also point out that in the Lifshitz holography literature, there is a disagreement about the scaling weight of a scalar ψ that appears in metric analyses of Einstein-Proca models [27] as well as of EPD models [28, 106]. This discrepancy carries over to the charged Lifshitz holography that we have developed in this thesis.

In particular, [27, 28, 106] argue that ψ has scaling weight Δ_- given by

$$\frac{1}{2}\left(z+2-\beta_z\right), \qquad \beta_z^2 = (z+2)^2 + 8(z-1)(z-2). \tag{7.0.1}$$

This source is related to $\tilde{\Phi}_{(\chi)} = -v^\mu M_\mu + \frac{1}{2}h^{\mu\nu}M_\mu M_\nu$ [33, 57], which, from our analysis in chapter 5, we know has scaling weight $2(z-1)$, which only agrees with

$$\delta_D M_\mu = -(z-2)\Lambda_D M_\mu, \qquad \delta_D v^\mu = -z\Lambda_D v^\mu, \tag{7.0.2}$$

implying that $\tilde{\Phi}$ has scaling weight $2(z-1)$. Thus, the scaling weight of ψ in (7.0.1) only agrees with that of $\tilde{\Phi}_{(\chi)}$ when $z=1$. It would therefore be interesting to work out the precise relation between $\tilde{\Phi}$ and ψ. In particular, a possible resolution could involve the scalar field $\Phi \simeq r^\Delta \phi$, where the scaling weight Δ can in principle be computed, and it could be that ψ is related to a combination of ϕ and $\tilde{\Phi}_{(\chi)}$.

APPENDIX

A GEOMETRY OF ADS SPACE-TIMES

In this appendix, we describe anti de-Sitter (AdS) spaces to provide some background for the material discussed in chapter 2. What follows is based on [42, 43, 161].

A.1 ADS SPACE-TIMES

A.1.1 *Definition and Useful Coordinates*

$(d+1)$-dimensional anti de-Sitter space, AdS_{d+1}, is the unique maximally symmetric Einstein space[1] with constant negative curvature. AdS_{d+1} can be embedded into $\mathbb{R}^{d,2}$ with metric $ds^2 = -(\mathrm{d}X^0)^2 - (\mathrm{d}X^{d+1})^2 + \sum_{i=1}^{d}(\mathrm{d}X^i)^2$ as the quadric,

$$-(X^0)^2 - (X^{d+1})^2 + \sum_{i=1}^{d}(X^i)^2 = -L^2, \tag{A.1.1}$$

where L is the radius of curvature of AdS_{d+1}. The embedding is clearly invariant under the "Lorentz group" for $\mathbb{R}^{d,2}$, $SO(d,2)$, which has dimension $\frac{1}{2}(d+1)(d+2)$—this is the number of Killing vectors associated to AdS_{d+1}, leading us to conclude that AdS_{d+1} is maximally symmetric. $SO(d,2)$ is the conformal group (see B) of d-dimensional Minkowski space. We now turn to describe some explicit parametrizations satisfying the AdS quadric (A.1.1); first, introduce $t \in \mathbb{R}$, $\vec{x} = (x^1, \dots, x^{d-1}) \in \mathbb{R}^{d-1}$ and, finally, $r \in \mathbb{R}^+$, in terms of which we have

$$X^0 = \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 + L^2\right)\right), \tag{A.1.2}$$

$$X^i = \frac{rx^i}{L}, \quad i \in \{1, \dots, d-1\}, \tag{A.1.3}$$

$$X^d = \frac{L^2}{2r}\left(1 + \frac{r^2}{L^2}\left(\vec{x}^2 - t^2 - L^2\right)\right), \tag{A.1.4}$$

$$X^{d+1} = \frac{rt}{L}. \tag{A.1.5}$$

Since $r > 0$, we only cover half of AdS_{d+1}—the local coordinates $t, r, \vec{x}$ define the *Poincaré patch* (see also figure A.1) coordinates, where the induced metric becomes[2],

$$ds^2_{\mathrm{AdS}_{d+1}} = \frac{L^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{L^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) = \frac{L^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{L^2}\underbrace{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}_{=ds^2_{\mathbb{R}^{d-1,1}}}, \tag{A.1.7}$$

where we have recognized the metric of d-dimensional Minkowski space. Using this metric, we can calculate the Ricci scalar, which becomes $R = -\frac{d(d+1)}{L^2}$, implying that L^2 is indeed the radius of curvature. Another useful form of the Poincaré metric is obtained by inverting the radial coordinate, $z = L^2/r$, thus yielding the metric in Poincaré z-coordinates,

$$ds^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right). \tag{A.1.8}$$

Note that the boundary in these coordinates is located at $z = 0$. When doing holographic renormalization, we use another set of coordinates known as *domain-wall coordinates*, which are nothing but

1 That is, it solves the Einstein equations.

2 For example, any one of the xx-components (to be understood as any of $d-1$ x^i's) of the induced metric, γ, is calculated as follows:

$$\gamma_{xx} = \eta_{MN}\frac{\partial X^M}{\partial x}\frac{\partial X^N}{\partial x} = \cancel{-\frac{\partial X^0}{\partial x}\frac{\partial X^0}{\partial x} + \frac{\partial X^d}{\partial x}\frac{\partial X^d}{\partial x}} + \underbrace{\frac{\partial X^i}{\partial x}\frac{\partial X^i}{\partial x}}_{\text{only one } X^i \text{ depends on } x} = \frac{r^2}{L^2}, \tag{A.1.6}$$

where η_{MN} is the $d+2$ dimensional Minkowski metric of signature $(-,+,\dots,+,-)$.

Gaussian normal coordinates. They are obtained from the Poincaré coordinates of (A.1.8) by setting $z = e^{-r}$—which means that the boundary will now be located at $r \to \infty$—and they read

$$ds^2 = \mathrm{d}r^2 + e^{2r}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{A.1.9}$$

It is also possible to introduce *global coordinates* $\{\tau, \rho, \{\theta_i\}_{i\in\{1,\dots,d-1\}}\}$ for AdS_{d+1}: the following parametrization satisfies the AdS_{d+1} quadric:

$$X^0 = L\cosh\rho\cos\tau, \tag{A.1.10}$$
$$X^{d+1} = L\cosh\rho\sin\tau, \tag{A.1.11}$$
$$X^1 = L\sinh\rho\ \cos\theta_1, \tag{A.1.12}$$
$$X^2 = L\sinh\rho\ \sin\theta_1\cos\theta_2, \tag{A.1.13}$$
$$\vdots \tag{A.1.14}$$
$$X^{d-1} = L\sinh\rho\ \sin\theta_1\cdots\sin\theta_{d-2}\cos\theta_{d-1} \tag{A.1.15}$$
$$X^{d} = L\sinh\rho\ \sin\theta_1\cdots\sin\theta_{d-2}\sin\theta_{d-1}, \tag{A.1.16}$$

where the $\{\theta_i\}_{i\in\{1,\dots,d-1\}}$ precisely parametrize S^{d-1}. The induced metric becomes,

$$ds^2_{\mathrm{AdS}_{d+1}} = L^2\left(-\cosh^2\rho\ \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\ \mathrm{d}\Omega^2_{S^{d-1}}\right), \tag{A.1.17}$$

where $\mathrm{d}\Omega^2_{S^{d-1}}$ is the metric of the $(d-1)$-sphere. Since the metric above does not depend on τ, we infer the existence of a timelike killing vector[3] ∂_τ, and since this killing vector is defined globally on the manifold, τ acts as a sensible global time coordinate. Near the center of AdS_{d+1}, $\rho = 0$, the metric assumes the form $ds^2_{\mathrm{AdS}_{d+1}} \overset{\rho\sim 0}{\simeq} L^2(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \rho^2\ \mathrm{d}\Omega^2_{S^{d-1}})$, implying that the space-time described by (A.1.17) has topology—since τ is periodic—of $S^1 \times \mathbb{R}^d$, where S^1 is the periodic time; in particular, since ∂_τ is everywhere timelike, keeping ρ and θ_i fixed while varying τ will produce closed time-like curves. This is, however, not an intrinsic property of the space-time—merely an artefact of our embedding: $\mathbb{R}^{d,2}$ has two timelike directions, so the appearance of closed timelike curves is not so surprising after all. To get rid these, we may instead choose to define [44, 161] (although some authors, e.g. [43], do not!) AdS_{d+1} as the *universal covering space*, where we decompactify so that τ assumes all real values with no periodicity constraints, which makes AdS_{d+1} a causal space-time. Since the universal covering[4] of S^1 is $\mathbb{R}$, the universal covering of AdS_{d+1} will have topology $\mathbb{R}^{d+1}$. We also want to draw the conformal diagram for AdS_{d+1}, and for this, we change coordinates again:

$$\cosh\rho = \frac{1}{\cos\phi}, \tag{A.1.18}$$

implying that $\sinh\rho = \tan\phi$. In this manner, the metric takes the form

$$ds^2_{\mathrm{AdS}_{d+1}} = \frac{L^2}{\cos^2\phi}\left(-\mathrm{d}\tau^2 + \mathrm{d}\phi^2 + \sin^2\phi\ \mathrm{d}\Omega^2_{S^{d-1}}\right), \tag{A.1.19}$$

which we see implies that AdS_{d+1} is *conformally flat*[5]. The range of the new radial coordinate is $0 \le \phi \le \frac{\pi}{2}$.

3 Timelike since the $\tau\tau$-component of the metric is less than zero.
4 To see this [162], one can check that $p : \mathbb{R} \to S^1$ with $p(t) = (\cos t, \sin t)$ satisfies the properties of a covering map.
5 A manifold (M, g) is conformally flat if for each $x \in M$, there exists a neighborhood U of x as well as a smooth function f on U such that $(U, e^{2f}g)$ is flat. In our case, we can choose for all $x \in \mathrm{AdS}_{d+1}$ the function, $e^{2f} = \frac{L^2}{\cos^2\phi}$, or $f = \log L - \log\cos\phi$, which is smooth.

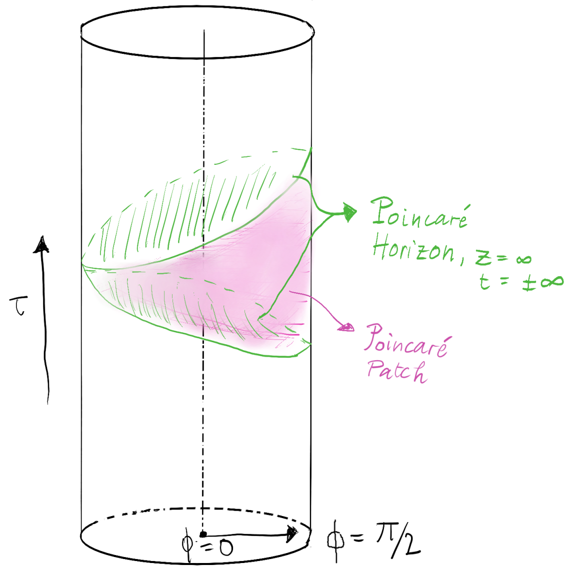


Figure A.1: Conformal diagram of global AdS$_{d+1}$ (i.e. we drop the conformal factor $L^2/\cos\phi$ in the global AdS metric (A.1.17)) with radial direction ϕ and global time τ. In the figure, every point (τ, ϕ) contains a S^{d-1}. and the boundary topology is (set $\phi = \pi/2$ in (A.1.17)) that of a cylinder, $\mathbb{R} \times S^{d-1}$. The Poincaré patch, described by the metric (A.1.8), covers only part of global AdS (the pink area), and its boundary is conformal to $\mathbb{R}^{d-1,1}$, i.e. $(d-1)$-dimensional Minkowski space. See appendix A of [163] for more details.

A.1.2 *Conformally Compact Einstein Manifolds & The Boundary*

We now turn to examining the boundary of AdS$_{d+1}$. In particular, the metric in Poincaré coordinates, (A.1.7), we have in the $r \to 0$ limit a degenerate *Killing horizon*[6], known as the Poincaré horizon. On the other hand, in the limit $r \to \infty$, the metric has a pole of second order. As described in [79, 93, 164]—and in the more rigorous [165]—we now turn to describing a general method for taming the conformal boundary of AdS. Therefore, we let M be the interior[7] of a compact $(d+1)$-dimensional manifold $\overline{M}$ with boundary ∂M (sometimes denoted $\mathscr{I}$ [166]). A metric $g_{\mu\nu}$ on M is *conformally compact* if it has second order pole at the boundary ∂M and there exists a *defining function* ρ on $\overline{M}$ such that the conformally equivalent metric,

$$\tilde{g}_{\mu\nu} = \rho^2 g_{\mu\nu}, \tag{A.1.20}$$

smoothly extends to a metric on the compactification[8] $\overline{M}$. The defining function is a smooth non-negative satisfying $\rho|_{\partial M} = 0, \ \ \partial_\mu \rho|_{\partial M} \neq 0$. The induced metric $\gamma_{\mu\nu} = \tilde{g}_{\mu\nu}|_{\partial M}$ is the boundary metric associated to the specific conformal compactification of (A.1.20)—there are generally many possible defining functions, and so many conformal compactifications of a given metric $g_{\mu\nu}$, which induces a conformal (equivalence) class of metrics $[\gamma]$ on ∂M, known as conformal infinity. A *conformally compact Einstein manifold* is a conformally compact manifold that also solves Einstein's equation. We shall have more to say about this when considering the generalized spaces AAdS and AlAdS spaces. We can now apply this analysis the AdS$_{d+1}$ metric in the coordinates of (A.1.19)—which indeed has a second order pole in the radial coordinate ϕ at $\phi = \pi/2$. Taking $\rho = \frac{\cos\phi}{L}$, we see that it satisfies all the conditions for a defining function, and thus the boundary metric becomes $\eta_{\mu\nu}$. Now, let's discuss how conformal transformations act on the boundary. From the quadric, (A.1.1), the full conformal symmetry $SO(d,2)$ is clear, whereas from the Poincaré metric (A.1.7), only two subgroups of $SO(d,2)$ are manifest: the set of Poincaré transformations acting on $t, \vec{x}$, $ISO(d-1,1)$, and $SO(1,1)$, which has the action,

$$(t, \vec{x}, r) \mapsto (\lambda t, \lambda \vec{x}, r/\lambda). \tag{A.1.21}$$

6 A Killing horizon is a null hypersurface defined by vanishing norm of a Killing vector k_μ. That it is degenerate just means that the surface gravity vanishes on the hypersurface.
7 In particular, M will be non-compact.
8 Note that we now use the word in the mathematical sense: it simply refers to the process of making a topological space into a compact space.

Using a light-cone formalism for the conformal algebra and the isomorphism $SO(d,2) \simeq ISO(d,1) \times SO(1,1)$, one can make explicit exactly how the symmetries are inherited by the boundary, see [11] for details.

A.1.3 *The Boundary: There and Back Again*

In this section, we consider the behaviour of radial geodesics and investigate if they can reach the boundary. Being interested in radial rays, we may set $\mathrm{d}\Omega_{d-1}^2 = 0$. Starting light rays, we immediately set $\mathrm{d}s^2 = 0$, allowing us to determine the coordinate time it takes to go to the boundary. Using global coordinates (A.1.17), we find

$$0 = \mathrm{d}s^2 = L^2(-\cosh^2(\rho)\mathrm{d}\tau^2 + \mathrm{d}\rho^2) \therefore \mathrm{d}\rho^2 = \cosh^2(\rho)\mathrm{d}\tau^2. \tag{A.1.22}$$

This is a differential equation that we can solve:

$$\int_{\rho_0}^{\rho} \frac{1}{\cosh(\rho')}\mathrm{d}\rho' = \tau(\rho) - \tau(\rho_0) = \tau(\rho), \tag{A.1.23}$$

i.e.

$$\tau(\rho) = 2\tan^{-1}[\tanh(\rho/2)] - 2\tan^{-1}[\tanh(\rho_0/2)]. \tag{A.1.24}$$

The time for a light ray to go to the boundary and back must be $2t(\infty)$, which is then

$$2\tau(\infty) = \lim_{\rho\to\infty} 4(\tan^{-1}[\tanh(\rho/2)] - \tan^{-1}[\tanh(\rho_0/2)]) = 4(\pi/4 - \tan^{-1}[\tanh(\rho_0/2)]), \tag{A.1.25}$$

so for $\rho_0 = 0$ we find, specifically, that $2\tau(\infty) = \pi$.

Note that if we had instead been interested in how much *affine time* it would take the light ray, we would find that the answer diverges.

Now, turning to massive geodesics, we could in principle go on to compute the Christoffel symbols and look at the geodesic equation, but we can obtain some interesting qualitative features of the behaviour of a massive, radially-directed particle just from the normalization of 4-velocity, that is $u^\mu u_\mu = -1$, with $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$, where λ is the proper time. We write

$$-1/L^2 = \frac{1}{L^2} g_{\mu\nu}u^\mu u^\nu = -\cosh^2(\rho)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}\right)^2.$$

Note that ∂_τ is a (time-like) Killing vector of AdS, with components which we will denote $K^\mu = \delta^\mu_\tau$, meaning that $p^\mu \nabla_\mu (K_\nu p^\nu) = 0$. In other words, $K_\nu p^\nu = g_{\nu\rho} K^\rho \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = g_{\nu\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = -\cosh^2(\rho)\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}$ is conserved along the trajectory, i.e. $\cosh^2(\rho)\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = C$, which means that

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}\right)^2 = C^2/\cosh^2(\rho) - 1/L^2.$$

Notice that $\frac{\mathrm{d}\rho}{\mathrm{d}\lambda} = 0$ has a solution $\rho = \rho^*$ given by

$$\cosh^2(\rho^*) = (LC)^2,$$

which means that there is always a turning point beyond which the massive particle cannot probe. The position of this turning point is "energy-dependent" (that is, it is C-dependent—recall that C is the conserved quantity corresponding to the time-translation symmetry). We find

$$\frac{\mathrm{d}\rho}{\mathrm{d}\lambda} = \pm\frac{1}{L}\sqrt{\frac{(LC)^2}{\cosh^2(\rho)} - 1},$$

and choosing the positive sign, we obtain

$$L\int_{\rho_0}^{\rho}\left(\frac{(LC)^2}{\cosh^2(\rho')} - 1\right)^{-1/2}\mathrm{d}\rho' = \lambda(\rho) - \lambda(\rho_0).$$

If we let $\rho_0 = 0$ and compute $\lambda(\rho^*) - \lambda(0)$, we find

$$\lambda(\rho^*) - \lambda(0) = L \int_0^{\rho^*} \left(\frac{(LC)^2}{\cosh^2(\rho')} - 1 \right)^{-1/2} \mathrm{d}\rho',$$

so, since $\rho^* = \cosh^{-1}(LC)$, we obtain (using Mathematica)

$$\lambda(\cosh^{-1}(LC)) - \lambda(0) = L \int_0^{\cosh^{-1}(LC)} \left(\frac{(LC)^2}{\cosh^2(\rho')} - 1 \right)^{-1/2} \mathrm{d}\rho' = \pi/2,$$

implying that the turning point ρ^* for massive particles is analogous to the boundary for massless particles, i.e. it takes proper time π to go from $\rho = 0$ to the turning point and back again.

A.2 ALADS SPACE-TIMES & THE FEFFERMAN-GRAHAM THEOREM

Holographically, as we discussed in chapter 2, the energy-momentum tensor of the dual field theory is sourced by the boundary value of the bulk metric. Consequently, a dynamical boundary metric is required in order for the field theory energy-momentum tensor to be well defined as the usual variational derivative. This can be achieved via so-called *asymptotically locally AdS* space-times [48]:

Definition (AldS): *An asymptotically locally AdS space-time is a conformally compact Einstein manifold.*

An important subset of AldS space-times are the asymptotically AdS (AAdS) space-times, the boundaries of which are equal to that of AdS; a nice example is the $(d+1)$-dimensional AdS-Schwarzschild (static and spherically symmetric) black hole with metric [167]

$$ds^2 = -f_d(r)\mathrm{d}t^2 + f_d(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{S^{d-1}}, \tag{A.2.1}$$

where the blackening factor is $f_d(r) = \frac{r^2}{L^2} + 1 - \left(\frac{r_h}{r}\right)^{d-3}\left(\frac{r_h^2}{L^2}+1\right) \overset{r\to\infty}{\longrightarrow} \frac{r^2}{L^2} + 1$, which thus reduces to the global AdS metric of (A.1.17) in the limit $r \to \infty$ upon doing the coordinate transformation $r \mapsto L\sinh\rho$ and $t \mapsto L\tau$, since

$$\begin{aligned} ds^2_{r\gg 1} &\sim -\left(\frac{r^2}{L^2}+1\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{L^2}+1} + r^2\mathrm{d}\Omega^2_{S^{d-1}} && \text{(A.2.2)}\\ &\overset{\substack{r\mapsto L\sinh\rho\\ t\mapsto L\tau}}{\longrightarrow} -L^2(\sinh^2\rho+1)\mathrm{d}\tau^2 + L^2\frac{\cosh^2\rho}{\sinh^2\rho+1}\mathrm{d}\rho^2 + L^2\sinh^2\rho\mathrm{d}\Omega^2_{S^{d-1}} && \text{(A.2.3)}\\ &= L^2\left(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega^2_{S^{d-1}}\right). && \text{(A.2.4)} \end{aligned}$$

A concrete condition, which we can take as the defining property of AlAdS space-times was also given in [48], and reads

$$R_{\mu\nu\rho\sigma} = \frac{1}{L^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) + \mathcal{O}(r^{-3}), \tag{A.2.5}$$

in Poincaré coordinates. The class of AlAdS metrics are subject to the Fefferman-Graham theorem [87], and a generic AlAdS metric can be written in Fefferman-Graham coordinates as follows[9] [48, 168],

$$ds^2 = L^2\left(\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}g_{\mu\nu}(\rho,x)\mathrm{d}x^\mu\mathrm{d}x^\nu\right), \tag{A.2.6}$$

where $g_{\mu\nu}(x,\rho)$ admits an expansion which depends on whether d is even or odd (that is, whether $d+1$ is odd or even); if d is odd, the expansion takes the form (where the subscript in parentheses indicates the number of derivatives involved in the term under scrutiny) [169]

$$g_{\mu\nu}(x,\rho) = g_{(0)\mu\nu}(x) + \rho g_{(2)\mu\nu}(x) + \rho^2 g_{(4)\mu\nu}(x) + \ldots, \tag{A.2.7}$$

9 In the case of pure AdS, where $g_{\mu\nu} = \eta_{\mu\nu}$, the Fefferman-Graham metric can be obtained from the z-metric (A.1.8) by defining $\rho = z^2$.

while for even d, the expansion involves logarithmic terns

$$g_{\mu\nu}(x,\rho) = g_{(0)\mu\nu}(x) + \rho g_{(2)\mu\nu}(x) + \ldots \rho^{d/2} g_{(d)\mu\nu}(x) + \rho^{d/2}\log(\rho) h_{(d)\mu\nu}(x) + \mathcal{O}(\rho^{\frac{d}{2}+1}). \tag{A.2.8}$$

The coefficients $g_{(0)\mu\nu}, \ldots, g_{(d-2)\mu\nu}$ are determined by solving the Einstein equations order by order in ρ. In a holographic setting, the term $h_{(d)\mu\nu}$ turns out to be equal to the metric variation of the holographic conformal anomaly [169], while $g_{(d)\mu\nu}$ is related to the one-point function of the energy-momentum tensor of the dual field theory. Note that the solution obtained in this manner is only valid near the boundary; more powerful techniques are required if one is interested in solutions that extend into the deep interior. An important ingredient of the method of holographic renormalization is a Fefferman-Graham-like expansion for generic bulk fields (see chapter 3 and in particular appendix **??**)—this was first realized by Witten in [79].

B

CONFORMAL FIELD THEORY

B.1 CFT AND THE CONFORMAL ALGEBRA

Conformal invariance (see [42, 43, 170, 171] for more comprehensive reviews) generalizes Poincaré invariance by imposing scale invariance. This circumvents the Coleman-Mandula no-go theorem, since scale invariance does not allow the existence of a non-trivial S-matrix due to the fact that asymptotic states cannot be defined in the usual manner[1]. Although no formal proof exists in dimensions different from two, evidence suggests that any unitary scale invariant theory is invariant under the full conformal group.

The conformal group consists of diffeomorphisms which preserve the form of the metric up to an arbitrary scale factor[2],

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x), \tag{B.1.1}$$

which is the smallest group containing the Poincaré group ($\Omega = 1$) as well as inversion symmetry, $x \to x^\mu/x^2$. In Minkowski space, the conformal group is generated by the usual Poincaré transformations as well as the scale transformation,

$$x^\mu \to \lambda x^\mu, \tag{B.1.2}$$

and the special conformal transformations,

$$x^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}. \tag{B.1.3}$$

The generators of the corresponding Lie algebra are $J_{\mu\nu}$ for Lorentz transformations, P_μ for translations, D for the scaling transformation (dilatation), and K_μ for the special conformal transformation, and the new—i.e. those different from Poincaré commutation relations (see (4.1.1))—non-vanishing commutation relations of the conformal algebra read

$$[D, P_\mu] = -iP_\mu, \quad [J_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \quad [D, K_\mu] = iK_\mu, \quad [P_\mu, K_\nu] = 2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D. \tag{B.1.4}$$

The conformal algebra is isomorphic to $\mathfrak{so}(d,2)$ (with signature $(-,+,\cdots,+,-)$); the isomorphism is constructed by enhancing the J's in the following manner:

$$J_{\mu d} = \frac{K_\mu - P_\mu}{2}, \quad J_{\mu(d+1)} = \frac{K_\mu + P_\mu}{2}, \quad J_{(d+1)d} = D. \tag{B.1.5}$$

The physically interesting representations of the conformal algebra give rise to operators satisfying $[D, \phi(0)] = -i\Delta\phi(0)$, which—since $\phi(x) = e^{ix^\mu P_\mu}\phi(0)$—implies that,

$$[D, \phi(x)] = [D, e^{ix^\mu P_\mu}\phi(0)] = \left([D, e^{ix^\mu P_\mu}] + e^{ix^\mu P_\mu} D\right)\phi(0) + e^{ix^\mu P_\mu}\phi(0)D, \tag{B.1.6}$$

where

$$[D, e^{ix^\mu P_\mu}] = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^{\mu_1} \cdots x^{\mu_n} \underbrace{[D, P_{\mu_1} \cdots P_{\mu_n}]}_{=:[D,P^n]}. \tag{B.1.7}$$

Then, since $[D, P_\mu] = iP_\mu$, we postulate the commutation relation

$$[D, P^n] = inP^n, \tag{B.1.8}$$

1 In fact, the only allowed asymptotic state is $|0\rangle$. One way to see this is to note that since the theory is scale invariant, the interactions are always *long-range forces*, so asymptotic in and out states cannot be defined in the usual manner. One would think that this voids the entire amplitude program which uses (primarily) $\mathcal{N} = 4$ SYM in the planar limit, which is, as we have seen, conformal. However, mirror symmetry relates the conformal Coulomb branch of the theory to the Higgs branch, which breaks conformal invariance and thus possesses an S-matrix [172].

2 We discuss conformal symmetry in dimensions $d > 2$; in two dimensions, the conformal group is infinite dimensional, and the corresponding algebra is the familiar Witt algebra.

which can easily be proved by induction: assume (B.1.8) to be true and consider the commutator,

$$[D, P^{n+1}] = [D, P^n]P + P^n[D, P] = inP^{n+1} + iP^{n+1} = i(n+1)P^{n+1}. \tag{B.1.9}$$

Therefore, we find that

$$[D, \phi(x)] = \sum_{n=0}^{\infty} \frac{i^{n+1}n}{n!}(x^\mu P_\mu)^n \phi(0) + e^{ix^\mu P^\mu}[D, \phi(0)] \tag{B.1.10}$$

$$= i^2 x^\mu P_\mu \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!}(x^\mu P_\mu)^{n-1}\phi(0) - i\Delta\phi(x) \tag{B.1.11}$$

$$= i\left(x^\mu \partial_\mu - \Delta\right)\phi(x), \tag{B.1.12}$$

where we have used the explicit form of the momentum operator, $P_\mu = -i\partial_\mu$. This relation can be used to fix the two-point function for scalar operators up to a constant; by rotation and translation invariance, we have

$$\langle \phi_1(x)\phi_2(y)\rangle = f(|x-y|), \tag{B.1.13}$$

where f is an as of yet undetermined function. We now need the Ward identity corresponding to dilatation: this may be found in the "usual" Schwinger-Dyson manner, or by noting that $D\,|0\rangle = 0$, implying that, for two-point functions[3],

$$0 = \langle 0|[D, \phi_1(x)\phi_2(y)]|0\rangle = \langle 0|\phi_1(x)[D, \phi_2(y)] + [D, \phi_1(x)]\phi_2(y)|0\rangle = (x^\mu\partial_\mu^{(x)} - \Delta_1 + y^\mu\partial_\mu^{(y)} - \Delta_2)\,\langle\phi_1(x)\phi_2(y)\rangle\,, \tag{B.1.15}$$

where the superscript on the derivative refers to the variable that is differentiated. The differential equation $0 = (x^\mu\partial_\mu^{(x)} - \Delta_1 + y^\mu\partial_\mu^{(y)} - \Delta_2)f(|x-y|)$ is solved by

$$f(|x-y|) = \frac{C}{|x-y|^{\Delta_1+\Delta_2}}, \tag{B.1.16}$$

where C is a constant. Invariance under special conformal transformations further fixes $\Delta_1 = \Delta_2$. Conformal symmetry also fixes the tree-point function uniquely (up to a constant); this takes the form

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{B.1.17}$$

where f_{123} is a constant and $x_{ij} = |x_i - x_j|$. We now derive (B.1.17) from conformal symmetry, using a slightly different approach than the one we used to obtain (B.1.16) (but the method can easily be "degeneralized" to also give the two-point function). A spin-less quasi-primary field ϕ of scaling dimension Δ transforms—by definition—under general conformal transformations as

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x), \tag{B.1.18}$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the coordinate transformation and d the space-time dimension. For conformal rescalings, $x \to x' = \lambda x$, (B.1.18) implies that

$$\phi'(\lambda x) = \lambda^{-\Delta}\phi(x). \tag{B.1.19}$$

Now, translation invariance restricts the form of the three-point function

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = f(|x_{12}|\,, |x_{13}|\,, |x_{23}|), \tag{B.1.20}$$

where $x_{ij} = x_i - x_j$. The behaviour of the fields under conformal rescalings (B.1.19) implies that

$$f(x, y, z) = \lambda^{\Delta_1+\Delta_2+\Delta_3} f(\lambda x, \lambda y, \lambda z). \tag{B.1.21}$$

3 The calculation below is easily generalized to an arbitrary number n of scalar fields, in which case one obtains the Ward identity

$$0 = \sum_{j=1}^{n}\left(x_j^\mu \frac{\partial}{\partial x_j^\mu} + \Delta_j\right)\langle\phi(x_1)\ldots\phi_j(x_j)\ldots\phi_n(x_n)\rangle\,. \tag{B.1.14}$$

Introducing $a+b+c=\Delta_1+\Delta_2+\Delta_3$, the above implies that

$$x^a y^b z^c f(x,y,z) = x^a y^b z^c \lambda^{\Delta_1+\Delta_2+\Delta_3} f(\lambda x, \lambda y, \lambda z) = (\lambda x)^a (\lambda y)^b (\lambda z)^c f(\lambda x, \lambda y, \lambda z), \quad \text{(B.1.22)}$$

i.e. it transforms as a constant under conformal rescalings, implying that it is independent of x, z and z. Thus, we are led to conclude that

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}^{abc}}{|x_{12}|^a |x_{13}|^b |x_{23}|^c}, \quad \text{(B.1.23)}$$

where $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle$ is a (structure) constant. Now, special conformal transformations with parameter b can be shown[4] to have Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2b\cdot x+b^2x^2)^d}. \quad \text{(B.1.24)}$$

Defining $\gamma_i := 1-2b\cdot x_i + b^2 x_i^2$, the distance $|x_{ij}| = \sqrt{(x_i-x_j)\cdot(x_i-x_j)}$ transforms under special conformal transformations as

$$|x_{ij}| \to |x'_{ij}| = \frac{|x_{ij}|}{\sqrt{\gamma_i\gamma_j}}. \quad \text{(B.1.25)}$$

Transforming both sides of (B.1.23) using (B.1.18) and (B.1.25), we see that

$$\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = (\gamma_1\gamma_2)^{a/2}(\gamma_1\gamma_3)^{b/2}(\gamma_2\gamma_3)^{c/2}\frac{C_{123}^{abc}}{|x_{12}|^a |x_{13}|^b |x_{23}|^c}, \quad \text{(B.1.26)}$$

which implies that

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \gamma_1^{-\Delta_1+\frac{a+b}{2}}\gamma_2^{-\Delta_2+\frac{a+c}{2}}\gamma_3^{-\Delta_3+\frac{b+c}{2}}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle, \quad \text{(B.1.27)}$$

which is true for arbitrary γ_i, which gives us the equations

$$2\Delta_1 = a+b, \quad 2\Delta_2 = a+c, \quad 2\Delta_3 = b+c, \quad \text{(B.1.28)}$$

the solution to which is

$$a = \Delta_1+\Delta_2, \quad b = \Delta_1+\Delta_3-\Delta_2, \quad c = \Delta_2+\Delta_3-\Delta_1, \quad \text{(B.1.29)}$$

which gives (B.1.17).

Now, an infinitesimal conformal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ satisfies:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\epsilon^\mu = \left(1+\frac{1}{d}\partial_\rho\epsilon^\rho\right)\left(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu\epsilon^\mu) - \partial^\mu\epsilon_\nu\right), \quad \text{(B.1.30)}$$

where we have used the conformal Killing equation in the last equality. This we see to be an infinitesimal rescaling times and infinitesimal rotation, which, upon exponentiating, gives us the finite transformation

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega'(x)R^\mu{}_\nu(x), \quad R \in O(d), \quad \text{(B.1.31)}$$

where the parameter Ω' is equal to the Ω of (B.1.1), since

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Omega'^2\eta_{\mu\nu}, \quad \text{(B.1.32)}$$

where we have used that $R \in O(d)$. Next, we need to find out how these guys act on the Hilbert space, and, by extension, on the primary fields $\phi(0)$. To this end, we construct unitary operators which are representations of the conformal generators, $U = e^{iQ_\epsilon}$, where the Q_ϵ are the charges associated with

4 One approach to showing this is to use the homomorphism property of determinants and to use that a special conformal transformation is equivalent to an inversion followed by a translation followed by another inversion; see e.g. [43].

infinitesimal conformal transformations [43, 171], which allows us to write the general transformation of a field ϕ^a transforming in the $SO(d)$ representation D as,

$$U\phi^a(x)U^{-1} = \Omega(x')^{\Delta} D(R(x'))_b{}^a \phi^b(x'), \tag{B.1.33}$$

where, as we have seen, $\frac{\partial x'^\mu}{\partial x^\nu} = \Omega'(x) R^\mu{}_\nu(x)$. For the scalar representation, $D(R) = 1$, while for the vector representation, we have $D(R)_\mu{}^\nu = R_\mu{}^\nu$. For a scalar operator, as claimed, this reduces to the relation

$$\phi(\lambda x) = \lambda^{-\Delta}\phi(x), \tag{B.1.34}$$

where we have renamed $\Omega \to \lambda$.

Now, the commutation relations of (B.1.4) imply that K_μ is a lowering operator for the conformal weight,

$$[D, K_\mu\phi(0)] = [D, K_\mu]\phi(0) + K_\mu[D, \phi(0)] = -i(\Delta - 1)K_\mu\phi(0), \tag{B.1.35}$$

implying that K_μ reduces the conformal weight by 1. Note that $K\phi(0)$ above is shorthand for $[K, \phi(0)]$—this notation is valid due to the Jacobi identity and simplifies the notation (see also [171]); we will use it in the rest of this section.

RepresentationNow, s of the four-dimensional conformal group can be labeled by (Δ, j_L, j_R), corresponding to the quantum numbers of the subgroup $SO(1,1) \times SO(3,1)$, where the j's are the usual quantum numbers associated with the (algebra) isomorphism $SO(3,1) = SU_L(2) \times SU(2)_R$, which, since the quantum theory is required to be unitary, imposes the bounds $\Delta \geq 1 + j_L$ for $j_R = 0$ and the same with $L \leftrightarrow R$, as well as $\Delta \geq 2 + j_L + j_R$ for both $j_L, j_R \neq 0$. In four dimensions, therefore, scalars[5] $(j_L = 0 = j_R)$ satisfy $\Delta \geq 1$, while vectors $(j_L = 1/2 = j_R)$ satisfy $\Delta \geq 3$ and so forth. The unitarity bound implies the existence of fields with lowest possible conformal weight, which are consequently annihilated by K_μ—these are known as primary fields,

$$K_\mu\phi(0) = 0 \overset{\text{DEF}}{\iff} \phi \text{ a primary operator.} \tag{B.1.36}$$

where we remind the reader that $K\phi(0)$ is short-hand for $[K, \phi(0)]$. From such a primary operator, we can construct the conformal descendants by acting with the momentum operator, which acts as a raising operator, since

$$[D, P_\mu\phi(0)] = -i(\Delta + 1)P_\mu\phi(0), \tag{B.1.37}$$

so the operator $P_{\mu_1} \cdots P_{\mu_n}\phi(0)$ has conformal weight $\Delta + n$ (provided that $\phi(0)$ has weight Δ). It's amusing to note that $\phi(x) = e^{ix^\mu P_\mu}\phi(0)$, which we used previously, is an infinite linear combinations of descendant operators. Finally, the primary operator satisfies the relation $[J_{\mu\nu}, \phi(0)] = -\mathcal{J}_{\mu\nu}\phi(0)$, where $\mathcal{J}_{\mu\nu}$ is a finite-dimensional representation of the Lorentz group, implying that a primary conformal field satisfies the relations:

$$[D, \phi(0)] = -i\Delta\phi(0), \quad [J_{\mu\nu}, \phi(0)] = \mathcal{J}_{\mu\nu}\phi(0), \quad [K_\mu, \phi(0)] = 0. \tag{B.1.38}$$

This allows us to construct a representation of the conformal algebra out of the primary $\phi(0)$ and its descendants[6].

B.2 THE SUPERCONFORMAL ALGEBRA

The four-dimensional superconformal group $PSU(2,2|\mathcal{N})$, which we will focus on, is the symmetry group of SYM with $\mathcal{N}$ supersymmetries. The corresponding algebra, $\mathfrak{psu}(2,2|\mathcal{N})$, consists of the conformal generators, $J_{\mu\nu}, P_\mu, D, K_\mu$ as well as the Poincaré supercharges $\mathcal{Q}^a_\alpha$, $\bar{\mathcal{Q}}^a_{\dot{\alpha}}$—in addition to these, closure of the superconformal algebra requires further supercharges $S^a_\alpha, \bar{S}^a_{\dot{\alpha}}$, which are the fermionic superpartners of K_μ (the Poincaré supercharges are superpartners of P_μ). Furthermore, the internal $\mathcal{R}$-symmetry group $U(\mathcal{N}) = U(1) \times SU(\mathcal{N}) \subset SU(2,2|\mathcal{N})$ is generated by T (the $U(1)$) and T^i (the $SU(\mathcal{N})$) for $i = 1, \ldots, \mathcal{N}^2 - 1$.

5 In d dimensions, this generalizes to $\Delta \geq \frac{d-2}{2}$.

6 This is accomplished by using the technique of induced representations: we pass from a representation of the subalgebra generated by $D, J_{\mu\nu}, K$ at $x = 0$ to a representation of the full conformal algebra by shifting the position using P_μ. Mathematicians like to call such a construction a parabolic Verma module.

In order to find representations of the superconformal algebra, we do the same as for the conformal algebra above: we identity the superconformal primary operators[7] $\mathcal{O}$, which in addition to the requirement (B.1.36) has to satisfy[8]

$$[S_{a\alpha}, \mathcal{O}\} = 0 = [\bar{S}^a_{\dot{\alpha}}, \mathcal{O}\}, \tag{B.2.1}$$

for all $a = 1, \ldots, \mathcal{N}$ and $\alpha, \dot{\alpha} = 1, 2$, and where $[\cdot, \cdot\}$ is a commutator if $\mathcal{O}$ is bosonic and an anti-commutator if $\mathcal{O}$ is fermionic. This is because the $S, \bar{S}$—just like K_μ—lowers the conformal dimension by one half, since $[D, S_{a\alpha}] = \frac{i}{2} S_{a\alpha}$, implying that, if ϕ is a fermionic primary:

$$[D, S_{a\alpha}\phi(0)] = -i(\Delta - 1/2) S_{a\alpha}\phi(0), \tag{B.2.2}$$

and similarly for $\bar{S}^a_{\dot{\alpha}}$. In particlar, superconformal primary operators are conformal primaries, and the formalism developed in the previous section carries over; in particular P_μ acts as a raising operator for the conformal weight.

A new feature, however, is the existence of superdescendants: the commutation relation $[D, \mathcal{Q}^a_\alpha] = -\frac{i}{2}\mathcal{Q}^a_\alpha$ implies that $\mathcal{Q}^a_\alpha$ (and its conjugate) raise the conformal dimension by one half, i.e. the superdescendant $\mathcal{O}' = [\mathcal{Q}, \mathcal{O}\}$ has conformal dimension $\Delta' = \Delta + 1/2$. Note that superdescandants of superconformal primaries are conformal primary operators, since (we take $\mathcal{O}$ to be bosonic here since it makes the computation easier)

$$[K_\mu, \mathcal{O}'] = [K_\mu, [Q, \mathcal{O}]] = [Q, \overbrace{[K_\mu, \mathcal{O}]}^{=0}] + [[K_\mu, Q], \mathcal{O}] \sim [S, \mathcal{O}] = 0, \tag{B.2.3}$$

where we have used the Jacobi identity and the condition (B.2.1).

A class of particularly interesting operators are the so-called chiral primary operators that are annihilated by at least one of the $\mathcal{Q}^a_\alpha$,

$$[\mathcal{Q}^a_\alpha, \mathcal{O}\} = 0, \tag{B.2.4}$$

which are 1/2 BPS operators, which, as we now show, means that the scaling dimension Δ is protected. Let $\mathcal{O}_\Delta$ be a chiral primary operator of conformal dimension Δ and spin $\mathcal{J}_{\mu\nu}$, i.e.

$$[D, \mathcal{O}_\delta(0)] = -i\Delta\mathcal{O}_\Delta(0), \quad [J_{\mu\nu}, \mathcal{O}_\Delta(0)] = -\mathcal{J}_{\mu\nu}\mathcal{O}_\Delta(0). \tag{B.2.5}$$

Superconformal primarity implies that

$$[S^a_\alpha, \mathcal{O}_\Delta(x)\} = 0 = [\bar{S}_{a\dot{\alpha}}, \mathcal{O}_\Delta(x)\}, \tag{B.2.6}$$

for all $a \in \{1, \ldots, \mathcal{N}\}$ and $\alpha, \dot{\alpha} \in \{1, 2\}$, while the condition that $\mathcal{O}_\Delta$ be chiral primary amounts to the requirement

$$[Q^a_\alpha, \mathcal{O}_\Delta(x)\} = 0, \tag{B.2.7}$$

for at least one of the Q^a_α's, which we will denote $Q^{\bar{a}}_{\bar{\alpha}}$. Now, recall that the anti-commutation relation between the special conformal supercharges and the Poincaré supercharges has the form

$$\left\{Q^a_\alpha, S_{\beta b}\right\} = \epsilon_{\alpha\beta}\,(\delta^a_b D + R^a{}_b) + \frac{1}{2}\delta^a_b J_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}, \tag{B.2.8}$$

where $R^a{}_b \in \mathfrak{u}_R(\mathcal{N})$, i.e. the automorphism subalgebra (R-symmetry) of the full $\mathfrak{su}(2,2|\mathcal{N})$. Using (B.2.6) and (B.2.7), we see that (schematically)

$$0 = [\{S, Q\}, \mathcal{O}_\Delta(0)] = [J + D + R, \mathcal{O}_\Delta(0)] \sim (\Delta + \mathcal{R} + \mathcal{J})\mathcal{O}_\Delta(0). \tag{B.2.9}$$

So, the dimension is given in terms of R-charge and spin, implying that it is indeed protected from quantum corrections.

In four-dimensional $\mathcal{N} = 4$ SYM, the elementary fields are the fermions, $\bar{\psi}_{a\dot{\alpha}}, \psi^a_\alpha$, the six scalars, ϕ^i and the gauge field A_μ, all transforming in the adjoint of the gauge group, $SU(N)$. Representations of the superalgebra $\mathfrak{psu}(2,2|4)$ are labeled by the quantum numbers of the bosonic subgroup (just as we had for the conformal group above, but with more structure)

$$\overbrace{\mathfrak{so}(3,1)}^{(j_L, j_R)} \times \overbrace{\mathfrak{so}(1,1)}^{\Delta} \times \overbrace{\mathfrak{su}(4)}^{[r_1, r_2, r_3]}, \tag{B.2.10}$$

7 We consider only local gauge invariant operators constructed from the elementary fields of the CFT.
8 The latin index structure of $\mathcal{Q}^a_\alpha$ and $S_{a\alpha}$ comes from the fact they transform in conjugated representations of $U(\mathcal{N})$.

where $[r_1, r_2, r_3]$ are the Dynkin labels of the representations of $\mathfrak{su}(4)$, labelling a given irreducible representation. The dimension of such a representation is given in terms of the Dynkin labels by [173]

$$\mathrm{Dim}(r_1, r_2, r_3) = (r_1+1)(r_2+1)(r_3+1)\left(1+\frac{r_1+r_2}{2}\right)\left(1+\frac{r_1+r_2+r_3}{3}\right), \tag{B.2.11}$$

and gives the degeneracy of the state transforming in the representation under scrutiny. In order to construct gauge-invariant local operators, we must use only gauge-invariant operators such as the fermions, the scalars and the field strength tensor evaluated at the same space-time point x. A local single-trace operator involving scalars, for example, becomes

$$\mathcal{O}(x) = \mathrm{Tr}\left[\phi^i \cdots \phi^j\right](x), \tag{B.2.12}$$

where the trace is over the adjoint representation of $SU(N)$; i.e. we write $\phi^i = \phi^i_a T^a$ and take the matrix trace of the generators. In general, such single-trace scalar operators have the form [45],

$$\mathcal{O}(x) = \mathrm{Str}\left[\phi^{\{i_1} \cdots \phi^{i_k\}}\right], \tag{B.2.13}$$

where the $\{i_1 \dots i_k\}$ stands for the traceless part and the Str stands for symmetrized trace over the gauge algebra, given by

$$\mathrm{Str}\left[T_{a_1} \cdots T_{a_n}\right] = \sum_{\sigma \in S_n} \mathrm{Tr}\left[T_{\sigma(a_1)} \cdots T_{\sigma_{a_n}}\right]. \tag{B.2.14}$$

These operators can be shown to be 1/2 BPS and chiral primary operators with conformal dimension $\Delta = k$; in particular, in the large-N limit of AdS/CFT, the single-trace operators are leading, so these are the most important to us. For example, the $\Delta = 2$ single trace operator is given by Str $\left[\phi^{\{i}\phi^{j\}}\right] = \mathrm{Tr}\left[\phi^i\phi^j\right] - \frac{1}{6}\delta^{ij}\mathrm{Tr}\left[\phi^k\phi^k\right]$, where we have used that the $\mathcal{N} = 4$ multiplet contains six scalars as well as the cyclic property of the trace. A more complete treatment [45] reveals that various BPS multiplets and non-BPS multiplets satisfy the properties listed in the table below (the column #Q lists the number of Poincaré supercharges left invariant by the primary operator in question):

Operator type	#Q	spin range	$\mathfrak{su}(4)$ rep.	conformal dim. Δ
identity	16	0	$[0,0,0]$	0
1/2 BPS	8	2	$[0,k,0],\quad k>2$	k
1/4 BPS	4	3	$[\ell,k,\ell],\quad \ell \geq 1$	$k+2\ell$
1/8 BPS	2	7/2	$[\ell,k,\ell+2m],\quad m \geq 1$	$k+2\ell+3m$
non-BPS	0	4	any	unprotected

Table B.1: Properties of superconformal BPS multiplets.

The Konishi operator is an example of a non-BPS operator; it is given by [43]

$$K = \mathrm{Tr}\left[\phi^i\phi^i\right]. \tag{B.2.15}$$

C

$\mathcal{N} = 4$ SUPER YANG-MILLS THEORY & IIB SUPERGRAVITY

C.1 SUPERSYMMETRIC GAUGE THEORY

In 1967, Coleman and Mandula proved a *no-go* theorem [174] for the S-matrix of quantum field theory: given certain reasonable assumptions, the only possible symmetry group is a direct product of the Poincaré group and an internal symmetry group[1]. Another nifty albeit rather long proof of the theorem using Lie algebras directly is presented by Weinberg in [176]. It is, however, possible to circumvent the Coleman-Mandula theorem in two ways: if a theory contains only massless particles, one may extend the Poincaré group to the conformal group; or, one may introduce supersymmetry [177] by enhancing the Poincaré algebra to a superalgebra (graded Lie algebra) via the introduction of a set of spinorial[2] supercharges $\mathcal{Q}^a$, $a = 1, \ldots, \mathcal{N}$, where $\mathcal{N}$ is the number of independent supersymmetries in the system—see appendix B and also e.g. [170, 178]. A particularly interesting supersymmetric theory is (planar) $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four spacetime dimensions. It is the CFT of AdS/CFT, and it has garnered much interest as a more tractable toy model of QCD: understanding the perturbative behavior of $\mathcal{N} = 4$ has lead to great advances in the field of scattering amplitudes, e.g. it has been realized that the off-shell gauge freedoms in Feynman diagrams in general QFTs are hugely redundant [172], and new recursion relations for tree amplitudes [179]. Great progress at loop level has also been achieved—recently, the planar $\mathcal{N} = 4$ four-point gluon amplitude has been computed to no less than ten loops [180].

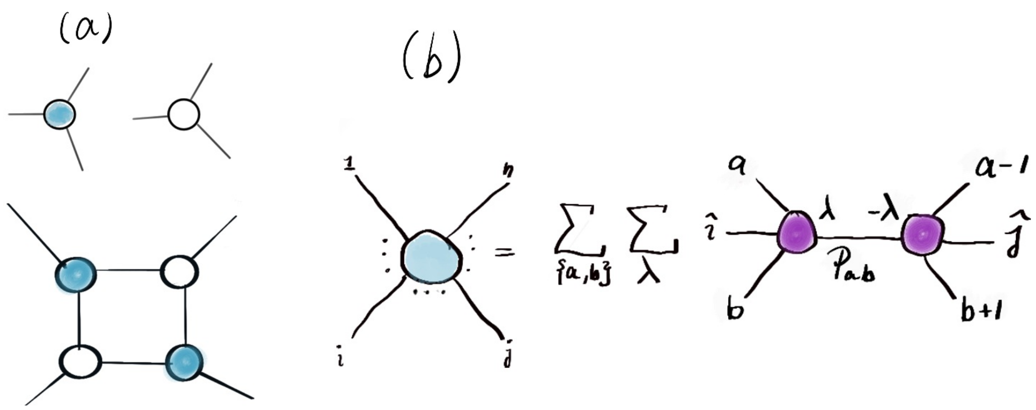


Figure C.1: (a): This particular on-shell diagram corresponds to the four-gluon tree amplitude of $\mathcal{N} = 4$ SYM. The vertices are equivalent to three-particle on-shell amplitudes which are fixed uniquely by little group scaling—specifically, the filled vertex corresponds to the MHV three-point amplitude, given by the Parke-Taylor expression $\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\delta^{2\times 2}(\lambda\cdot\tilde{\lambda})$, where δ-function imposes momentum conservation expressed in spinor-helicity variables. Similarly, the $\overline{\text{MHV}}$ vertex (the non-filled), is given by $\frac{[23]^3}{[12][31]}\delta^{2\times 2}(\lambda\cdot\tilde{\lambda})$. It is worth noting that this one on-shell diagram corresponds to the sum of three Feynman diagrams. (b): Illustration of BCFW recursion: an n-point tree amplitude $\mathcal{A}_n(\{h,p\})$ (LHS) is obtained by shifting the momentum of two lines, i and j, say, by two $\mathbb{C}$-valued vectors zr_i^μ, zr_j^μ, for a complex parameter z in such a way that the shifted lines $\hat{\imath}, \hat{\jmath}$ are on-shell. Setting $z = 0$, the shifted amplitude, $\hat{\mathcal{A}}_n(z)$ reduces to $\mathcal{A}_n$, so considering $\hat{\mathcal{A}}_n/z$, we can use Cauchy's theorem (disregarding the possibility of a pole at infinity) to relate $\mathcal{A}_n = \sum \mathrm{Res}_{z\neq 0}\frac{\hat{\mathcal{A}}_n(z)}{z}$. Now, since we are considering tree-level diagrams, the poles occur when propagators between legs a, b, say, go on-shell, for $z = z_{ab}$, which results in a residue of the form $\hat{\mathcal{A}}_L(z_{ab})\frac{1}{p_{ab}^2}\hat{\mathcal{A}}_R(z_{ab})$, where the shifted left/right amplitudes are on-shell evaluated at $z = z_{ab}$. Summing over all possible lines that can go on shell and all helicity configurations λ then gives the total sum of residues, which is the sought-after tree amplitude.

1 In [175], Witten presents a delightful argument as to why the Coleman-Mandula theorem should be true: it turns out that extending the Poincaré symmetry *overconstrains* scattering amplitudes, which can be non-vanishing only for a discrete set of scattering angles.

2 That is, they transform as spinors under Lorentz transformations and they are an odd element of a $\mathbb{Z}_2$-graded symmetry algebra.

Roughly following [181], we now describe $\mathcal{N} = 4$ in four spacetime dimensions. The spinorial supercharges—in four dimensions—are Weyl spinors, consisting of two complex components, $\mathcal{Q}^a = \begin{pmatrix} \mathcal{Q}^a_\alpha \\ \bar{\mathcal{Q}}^{a\dot{\alpha}} \end{pmatrix}$, implying that $\mathcal{N} = 4$ SYM has 16 real supercharges, which is also the dimension of a Majorana-Weyl spinor in ten dimensions, which, as we shall see, is quite significant. $\mathcal{N} = 4$ is the maximum number of supercharges we can have, since they raise/lower the helicity by $\Delta\lambda = \frac{1}{2}$ when acting on the fields, so—since we require that no particles have spin greater than one (otherwise we'd have a theory of gravity)—the $\mathcal{N} = 4$ is the maximal degree of SUSY possible. The supersymmetry algebra has a global $U(\mathcal{N})$ symmetry rotating the supercharges into each other: this is known as $\mathcal{R}$-symmetry. The extended supersymmetry algebra read [43],

$$[\mathcal{Q}^a_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta \mathcal{Q}^a_\beta, \qquad [\bar{\mathcal{Q}}^a_{\dot{\alpha}}, J^{\mu\nu}] = \epsilon_{\dot{\alpha}\dot{\beta}}(\sigma^{\mu\nu})^{\dot{\beta}}{}_{\dot{\gamma}}\bar{\mathcal{Q}}^{a\dot{\gamma}}, \tag{C.1.1}$$

$$[\mathcal{Q}^a_\alpha, P^\mu] = 0, \qquad [\bar{\mathcal{Q}}^a_{\dot{\alpha}}, P^\mu] = 0, \tag{C.1.2}$$

$$\{\mathcal{Q}^a_\alpha, \bar{\mathcal{Q}}_{b\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \tag{C.1.3}$$

$$\{\mathcal{Q}^a_\alpha, \mathcal{Q}^b_\beta\} = \epsilon_{\alpha\beta} Z^{ab} \tag{C.1.4}$$

$$\{\bar{\mathcal{Q}}_{a\dot{\alpha}}, \bar{\mathcal{Q}}_{b\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ab}, \tag{C.1.5}$$

where the anti-symmetric Z^{ab} and its barred cousin are the central charges of the SUSY algebra, which generate its center (i.e., they commute with all other generators of the algebra). They also satisfy $Z^{ab} = (\bar{Z}^\dagger)_{ab}$, since $\bar{\mathcal{Q}}_{a\dot{\alpha}} = (\mathcal{Q}^a_\alpha)^*$. We can now explicitly see the emergence of the $\mathcal{R}$-symmetry: under global phase rotations ("reshufflings") of the supercharges,

$$\mathcal{Q}^a_\alpha \mapsto (\mathcal{Q}^a)'_\alpha = R^a{}_b \mathcal{Q}^b_\alpha, \qquad \bar{\mathcal{Q}}_{a\dot{\alpha}} \mapsto \bar{\mathcal{Q}}'_{a\dot{\alpha}} = \bar{\mathcal{Q}}'_{b\dot{\alpha}} (R^\dagger)^b{}_a. \tag{C.1.6}$$

Under $\mathcal{R}$-symmetry, the first two lines of the SUSY algebra are clearly unchanged, while invariance of (C.1.3) implies unitarity of $R^a{}_b$—to see this, take the trace of (C.1.3) and $\mathcal{R}$-transform the resulting expression,

$$\{\mathcal{Q}^a_\alpha, \bar{\mathcal{Q}}_{a\dot{\beta}}\} = 2\mathcal{N}\sigma^\mu_{\alpha\dot{\beta}} P_\mu \mapsto 2\mathcal{N}\sigma^\mu_{\alpha\dot{\beta}} P_\mu \overset{(!)}{=} (R^\dagger)^c{}_a R^a{}_b \{\mathcal{Q}^b_\alpha, \bar{\mathcal{Q}}_{c\dot{\beta}}\} \tag{C.1.7}$$

$$\therefore (R^\dagger)^c{}_a R^a{}_b = \delta^c_b, \tag{C.1.8}$$

implying unitarity of $R^a{}_b$. Thus—as long as the central charges vanish—we conclude that the $\mathcal{R}$-symmetry group is $U(\mathcal{N})$. It turns that when the central charges *do not vanish*, the $\mathcal{R}$-symmetry group (which is of course still required to be unitary by the argument just presented) is given by the compact symplectic unitary group, $PSU(\mathcal{N})$. It turns out that for $D = 4$ the central charges *do* vanish, and for $\mathcal{N} = 4$ SYM in four space-time dimensions, the $\mathcal{R}$-symmetry group is $U(4)$—however, for $\mathcal{N} = 4$ SYM, the superconformal algebra (as we shall see) breaks this to $SU(4)$. We now turn to the classification of massless particle representations of the SUSY algebra in four spacetime dimensions. Since the particles are massless, we choose a Lorentz frame in which the momentum takes the form $P^\mu = (E, 0, 0, E)$ for $E > 0$, and so the relation (C.1.3) takes the explicit form[3]—where the $\mathcal{Q}$ are unitary reps of the SUSY algebra on the relevant Hilbert space $\mathcal{H}$:

$$\{\mathcal{Q}^a_\alpha, \bar{\mathcal{Q}}_{b\dot{\beta}}\} = 2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta^a_b = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^a_b. \tag{C.1.10}$$

Setting $\alpha = \dot{\beta} = 2$ and $a = b$, we get (no sum on a) $\{\mathcal{Q}^a_2, \bar{\mathcal{Q}}_{a\dot{2}}\} = \{\mathcal{Q}^a_2, (\mathcal{Q}^b_2)^\dagger\} = 0$, so for any (normalized) $|\phi\rangle \in \mathcal{H}$, we have that

$$0 = \langle\phi|\, \{\mathcal{Q}^a_2, (\mathcal{Q}^a_2)^\dagger\}\, |\phi\rangle = 2\,||\mathcal{Q}^a_2||^2 \quad \text{(no sum on } a\text{)}, \tag{C.1.11}$$

implying that $\mathcal{Q}^a_2 = 0$. Furthermore, using the relation $\{\mathcal{Q}^a_\alpha, \mathcal{Q}^b_\beta\} = \epsilon_{\alpha\beta} Z^{ab}$ with $\alpha = 1$ and $\beta = 2$ gives us that $0 = Z^{ab}$ for all a, b since $\epsilon_{12} = 1$ and $\mathcal{Q}^b_2$ vanishes for all b; that is, central charges are absent

3 Note that for any vector v^μ, we have that

$$v_\mu \sigma^\mu_{\alpha\dot{\beta}} = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}. \tag{C.1.9}$$

from $\mathcal{N}=4$ SYM. The commutation relations also imply that $\{\mathcal{Q}^a_1, \mathcal{Q}_{b\dot{1}}\} = 4E\delta^a_b$, which allows us to define creation and annihilation operators realising the fermionic harmonic oscillator algebra:

$$a^b = \frac{\mathcal{Q}^b_1}{2\sqrt{E}}, \qquad a^\dagger_b = \frac{\bar{\mathcal{Q}}_{b\dot{1}}}{2\sqrt{E}}, \tag{C.1.12}$$

satisfying the anti-commutation relations,

$$\{a^b, a^\dagger_c\} = \delta^b_c, \qquad \{a^b, a^c\} = 0 = \{a^\dagger_b, a^\dagger_c\}. \tag{C.1.13}$$

Now, the helicity λ of a given single particle state $|p^\mu, \lambda\rangle \in \mathcal{H}^{\otimes 1}$ is the eigenvalue[4] of J_{12}, and thus, using (C.1.1), we find that (taking $\alpha = 1$)

$$[\mathcal{Q}^a_1, J_{12}] = (\sigma_{12})_1{}^1 \mathcal{Q}^a_1 = \frac{1}{2}(\sigma_3)_1{}^1 \mathcal{Q}^a_1 = \frac{1}{2}\mathcal{Q}^a_1, \tag{C.1.16}$$

and thus,

$$J_{12}\mathcal{Q}^a_1 |p^\mu, \lambda\rangle = \lambda \mathcal{Q}^a_1 |p^\mu, \lambda\rangle - [\mathcal{Q}^a_1, J_{12}] |p^\mu, \lambda\rangle \tag{C.1.17}$$

$$= (\lambda - 1/2)\mathcal{Q}^a_1 |p^\mu, \lambda\rangle\,, \tag{C.1.18}$$

implying that $\mathcal{Q}^a_1$—and by extension a^b—lowers the helicity by $1/2$. A completely analogous argument can be invoked to see that $\bar{\mathcal{Q}}_{\dot{1}b}$ and $a^\dagger_b$ increases the helicity by $1/2$. Now, take $\mathcal{N}=1$ and note that the spinorial nature of $\bar{\mathcal{Q}}_{\dot{1}}$ implies its nilpotency. To form a supermultiplet, we take a vacuum state $|\Omega\rangle$—which by definition is annihilated by a, i.e. $a|\Omega\rangle = 0$— of lowest helicity, λ_0, and act with $a^\dagger$, and so, by nilpotency of $a^\dagger$, the supermultiplet (since we require that no helicity be greater than one) consists only of two states, $|\Omega\rangle = |p^\mu, \lambda_0\rangle$ and $a^\dagger|\Omega\rangle = |p^\mu, \lambda_0 + 1/2\rangle$. However, in order to make the multiplet CPT self-conjugate, we have—in general—to add the states with opposite chirality, implying that the full multiplet contains the states $|p^\mu, \pm\lambda_0\rangle$, $|p^\mu, \pm(\lambda_0 + 1/2)\rangle$. This makes possible only two kinds of multiplets: starting from $\lambda_0 = 0$, we get helicities $\{-1/2, 0, 0, +1/2\}$—this is the chiral multiplet, consisting of two fermionic degrees of freedom and a complex scalar (two real scalar fields). Taking $\lambda_0 = 1/2$, we obtain the vector multiplet with helicities $\{-1, -1/2, +1/2, +1\}$, the on-shell field content of which consists of two spinors and one vector.

Now, for $\mathcal{N}=4$, however, starting with the lowest helicity state $\lambda_0 = -1$, we get a supermultiplet of $16 = 2^{\mathcal{N}}$ states which is already CPT self-conjugate[5]. For the excited states (obtained by acting with a combination of the four creation operators on the chosen minimal helicity state) themselves—which translates to the field content of the supermultiplet—this produces a *Pascal's triangle structure*: There are $\binom{4}{4} = 1$ ways of of selecting four operators out of four, $\binom{4}{3} = 4$ of selecting three out of four operators etc. All in all, this produces a vector multiplet containing a gauge boson A_μ, which takes values in $\mathfrak{su}(N)$, four left-handed fermions, ψ_A, $A \in \{1, \ldots, 4\}$, and six real scalars, which we can group together to form three complex scalars, Φ_k, $k \in \{1,2,3\}$. Further, since A_μ transforms in the adjoint representation of $SU(N)$, the fact that all the other fields are related to A_μ by supersymmetry implies that they also transform in the adjoint of $SU(N)$. Under the $SU(4)$ $\mathcal{R}$-symmetry, the fieds transform in more interesting ways: the gauge field A_μ is a singlet, **1**; scalars are in the **6**, whereas the fermions transform in **4**.

The Lagrangian for $\mathcal{N}=4$ can be obtained by dimensional reduction: starting from the unique $\mathcal{N}=1$ SYM in ten dimensions,

$$S^{\mathcal{N}=1}_{10D} = \int \mathrm{d}^{10}x\ \mathrm{Tr}\left[-\frac{1}{2}F_{MN}F^{MN} + \frac{i}{2}\bar{\Psi}\Gamma^M D_M \Psi\right], \quad M, N \in \{0, \ldots, 9\} \tag{C.1.19}$$

4 To see this, recall that helicity is defined as the projection of angular momentum along the direction of the momentum,

$$\lambda = \frac{\vec{J}\cdot\vec{P}}{\left|\vec{P}\right|}. \tag{C.1.14}$$

For our concrete massless momentum, we have $\left|\vec{P}\right| = \sqrt{2}E$, and since $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$,, this implies that

$$\lambda = \frac{1}{\sqrt{2}}J^{12} = \frac{1}{\sqrt{2}}J_{12}. \tag{C.1.15}$$

5 One can do the same exercise for $\mathcal{N}=3$, but to ensure CPT self-conjugacy, one has to double the field content which makes it identical to $\mathcal{N}=4$.

where Γ^M are the 32×32 (since $2^{\lfloor 5 \rfloor} = 32$) Dirac matrices in ten spacetime dimensions, and the Ψ's are Majorana-Weyl spinors. The field strength is given by $F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N]$ whereas the covariant derivative is given by $D_M \Psi = \partial_M \Psi + ig[A_M, \Psi]$. The actual procedure, which is done in some detail in [44], involves dimensional reduction on a T^6. Explicitly, the ten-dimensional gauge field decomposes in the following manner,

$$A_M = (A_\mu(x^\nu), \phi_i(x^\nu)), \quad \mu, \nu \in \{0, \dots, 3\}, \quad i \in \{1, \dots, 6\}, \tag{C.1.20}$$

where—by assumption—nothing depends on the compactified directions. So, for example, $F_{\mu i} = \partial_\mu \phi_i + ig[A_\mu, \phi_i] = D_\mu \phi_i$, where we have recognized the covariant derivative. Similarly, $F_{ij} = ig[\phi_i, \phi_j]$, so for the dimensionally reduced $F_{MN}F^{MN}$, we obtain,

$$\mathrm{Tr}\left[F_{MN}F^{MN}\right] = \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu} + F_{\mu i}F^{\mu i} + F_{ij}F^{ij}\right] = \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_i D^\mu \phi_i - g^2[\phi_i, \phi_j][\phi_i, \phi_j]\right]. \tag{C.1.21}$$

The full Lagrangian reads[6]:

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g_{\mathrm{YM}}^2}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}^a \bar{\sigma}^\mu D_\mu \psi_a - D_\mu \phi_i D^\mu \phi_i + g_{\mathrm{YM}}\left(C^{ab}{}_i \psi_a[\phi_i, \psi_b] + \text{h.c.}\right) + \frac{g_{\mathrm{YM}}^2}{2}[\phi_i, \phi_j]^2\right], \tag{C.1.22}$$

where the $C^{ab}{}_i$ are the Clebsch-Gordan coefficients coupling two $\mathbf{4}$ representations of $SO(6)$ to a $\mathbf{6}$. Now, we see that the reduction of (C.1.20) breaks the original $SO(9,1)$ Lorentz invariance: it gets broken to $SO(3,1) \times SO(6)$, where the six scalars ϕ_i transform as a vector under $SO(6)$, whereas the eight chiral spinors ψ_a and $\bar{\psi}_a$ transform in $\mathbf{4}$ and $\bar{\mathbf{4}}$, respectively, of $SO(6)$. Recall that the $\mathcal{R}$-symmetry group of $\mathcal{N} = 4$ is $SU(4)$, which is isomorphic to $SO(6)$—so this is nothing but the $\mathcal{R}$-symmetry group.

Further, one can show that the β-function of $\mathcal{N} = 4$ vanishes, implying conformal invariance of the theory, which enhances the Lorentz group $SO(3,1)$ to the conformal group, $SO(4,2)$. We now see the first glimpse of the AdS/CFT correspondence: as we shall see, the isometry group of $\mathrm{AdS}_5 \times S^5$ is *precisely* $SO(4,2) \times SO(6)$, so the global symmetries on either side of the correspondence match. In fact, the full symmetry group of $\mathcal{N} = 4$ is obtained by combining the conformal symmetry with supersymmetry: in this manner, one obtains the supergroup $PSU(2,2|4)$. The corresponding Lie superalgebra $\mathfrak{psu}(2,2|4)$ contains, somewhat surprisingly, two extra superconformal spinorial generators, S^a_α and $\bar{S}^a_{\dot{\alpha}}$, which are needed for the algebra to close. The total set of generators of $\mathfrak{psu}(2,2|4)$ consists of the bosonic generators from the conformal part, $D, J_{\mu\nu}, P_\mu$ and K_μ, and the generators T and T^j from the $\mathfrak{u}(\mathcal{N}) = \mathfrak{u}(1) \times \mathfrak{su}(\mathcal{N})$ subalgebra, where T generates $\mathfrak{u}(1)$ and the T^j generate $\mathfrak{su}(\mathcal{N})$. The fermionic generators are $\mathcal{Q}^a_\alpha, \bar{\mathcal{Q}}_{a\dot{\alpha}}, S^a_\alpha, \bar{S}^a_{\dot{\alpha}}$. The full algebra is somewhat formidable, but it turns out that for $\mathcal{N} = 4$, the $\mathfrak{u}(1)$ generator T commutes with all other generators, and so, by Schur's lemma, is a multiple of the identity. It can be made to vanish [43], and thus, the $\mathcal{R}$-symmetry group is not $U(4)$ but rather $SU(4)$.

C.1.1 *The Planar Limit of $SU(N)$ Gauge Theories*

In particular, the prototypical gauge theory is $SU(N)$ Yang-Mills theory, which has the Lagrangian,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2g^2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right], \tag{C.1.23}$$

which is identical to the gluonic part of $\mathcal{N} = 4$ SYM we considered previously. The $\mathfrak{su}(N)$-valued gauge field $A_\mu = A^a_\mu T_a$ is written in terms of the generators $\{T^a\}_{a \in \{1, \dots, N^2-1\}}$ of $\mathfrak{su}(N)$ satisfying $[T_a, T_b] = i f_{ab}{}^c T_c$ and $\mathrm{Tr}\left[T_a T_b\right] = C(\mathbf{R})\delta^{ab}$, where $C(\mathbf{R})$ is a number that depends on the representation that turns out to be $1/2$. Thus. there are two "perspectives" on A_μ: it can either be considered a N-vector, A^a_μ, *or* it can be written in terms of the generators $\{T_a\}$. Taking the gauge group to be $U(N)$ rather than $SU(N)$ (which in the limit of large N give the same theory) and thinking of A_μ as a

6 We exclude the topological θ-term, proportional *in the action* to $\int F \wedge F$, which breaks CP invariance.

$N \times N$ matrix $(A_\mu)^i{}_j = A^a_\mu (T_a)^i{}_j$, we can write the (Lorenz gauge) Feynman rules of the theory in the following manner [11],

$$\langle (A_\mu)^i{}_j(p)(A_\nu)^k{}_l(k)\rangle = g^2 \delta^i_j \delta^k_l \frac{\eta_{\mu\nu}}{p^2}(2\pi)^d \delta^d(p+k), \tag{C.1.24}$$

$$\langle (A_\mu)^i{}_j(p)(A_\nu)^k{}_l(k)(A_\rho)^m{}_n(q)\rangle = \frac{1}{g^2}\delta^i_n \delta^k_j \delta^m_l (p_\mu + k_\nu + q_\rho)(2\pi)^d \delta^d(p+k+q), \tag{C.1.25}$$

$$\langle (A_\mu)^i{}_j(p)(A_\nu)^k{}_l(k)(A_\rho)^m{}_n(q)(A_\sigma)^h{}_g(r)\rangle = \frac{1}{g^2}\delta^i_g \delta^k_j \delta^m_l \delta^h_n \eta_{\mu\nu}\eta_{\rho\sigma}(2\pi)^d \delta^d(p+k+q+r), \tag{C.1.26}$$

which can represented by the 't Hooft the double line notation diagrams [182] of figure C.2.

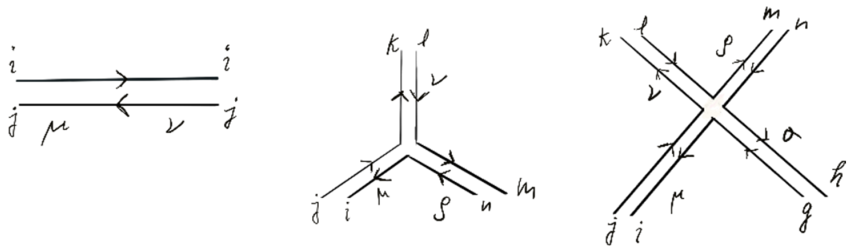


Figure C.2: The propagator and the vertices for $U(N)$ Yang-Mills theory. Their values are given in eqs. (C.1.24) to (C.1.26).

Clearly, each closed loop contributes a factor N, since $\delta^{ij}\delta^{ji} = \mathrm{Tr}\left[\mathbb{1}_{N\times N}\right] = N$; thus, a Feynman diagram with F loops, or *faces*, will carry a factor of N^F, whereas for each propagator (edge) (P), there will be a factor g^2, while each vertex (whether it be cubic or quartic) contributes a factor g^{-2}, implying that a generic Feynman diagram will contribute a factor,

$$\text{diagram} \sim g^{2P-2V}N^F = (g^2N)^{P-V}N^{F-P+V} = \lambda^{P-V}N^\chi, \tag{C.1.27}$$

where we have defined the 't Hooft coupling $\lambda = g^2 N$—which becomes the effective coupling of the theory at large N—and identified the Euler characteristic χ. which is also related to the genus g of the surface—in the absence of boundaries—via the relation $\chi = 2 - 2g$. Thus, in the 't Hooft (or *planar*) limit, where $N \to \infty$ with λ held fixed, only planar diagrams, that is, diagrams that can be drawn on S^2 contribute. It is also worth noting that taking the limit $N \to \infty$ naïvely leads to a divergent β-function, while taking the limit with fixed 't Hooft coupling leads to a renormalization equation with finite coefficients [43].

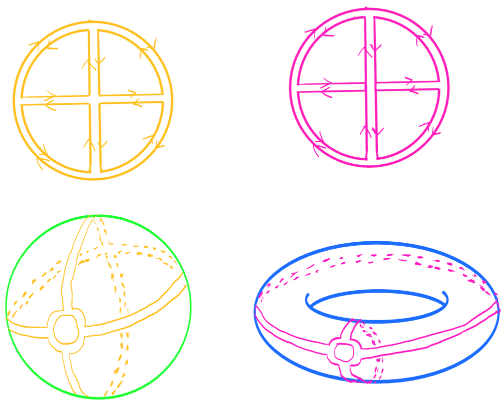


Figure C.3: Two vacuum diagrams: the left is planar; it can be *folded* over the sphere S^2, whereas the other cannot: it has genus one, and thus maps to the 2-torus, T^2. Figure inspired by [183].

Crucially, the large N expansion of our field theory is formally the same as the topological expansion of strings with string coupling $1/N$, which is also a sum over genera. For $\mathcal{N} = 4$, the string theory that leads to the correct expansion is, as we explore in appendix D, ten-dimensional type IIB string theory on $\mathrm{AdS}_5 \times S^5$, which we now briefly describe.

C.2 TYPE IIB SUPERGRAVITY

Following [184] (see also [45]), we take the limit $\alpha' \to 0$, thereby retaining only the massless modes of the type IIB superstring—this is precisely type IIB supergravity. The bosonic spectrum from the NS-NS sector thus consists of the dilaton, Φ, the graviton G_{MN} and the Kalb-Ramond two-form, B_2, and its associated field strength $H_3 = \mathrm{d}B_2$. The R-R sector contributes form fields, C_0, C_2 and C_5, with their associated odd-form field strength, of which $\tilde{F}_5$ is particularly interesting since it is self-dual. Finally, the fermionic part of the spectrum is provided by the NS-R (and its cousin) sector, which includes two left-handed Majorana-Weyl gravitinos and two right-handed Majorana-Weyl gravitinos.

However, it turns out that the self-duality of $\tilde{F}_5$ makes writing down a classical action for type IIB SUGRA quite complicated—the issue is that the 'usual' F^2 action, i.e. something like $\int F_5 \wedge \star F_5$, does not incorporate the self-duality constraint, and thus describes twice the number of degrees of freedom. One solution to this problem is the PST approach [185] makes use of an auxiliary scalar field and an extra gauge symmetry, which allows the scalar field to be set equal to one of the space-time coordinates (this results in a loss of general covariance in the chosen direction).

Another way to deal with this is to produce an action which gives the correct equations of motion when one imposes the self-duality constrain. The action—which is not supersymmetric due to the overcounting that has to be fixed by the self-duality constraint—turns out to be,

$$S_{\text{IIB SUGRA}} = \frac{1}{2\kappa^2}\left(\int \mathrm{d}^{10}x\,\sqrt{-G}\left\{e^{-2\Phi}\left[R + 4\partial_M\Phi\partial^M\Phi - \frac{1}{2}\left|H_3\right|^2\right] - \frac{1}{2}\left|F_1\right|^2 - \frac{1}{2}\left|\tilde{F}_3\right|^2 + \frac{1}{2}\left|\tilde{F}_5\right|^2\right\}\right) \tag{C.2.1}$$

$$-\frac{1}{\kappa^2}\int C_4 \wedge H_3 \wedge F_3, \tag{C.2.2}$$

where $F_{n+1} = \mathrm{d}C_n$, $H_3 = \mathrm{d}B_2$, $\tilde{F}_3 = F_3 - C_0 H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. Here, the R-R fields differ by field redefinitions from the ones that couple simply to the the D-brane world volumes. The self-duality constraint that makes the action above "correct" reads

$$\tilde{F}_5 = \star\tilde{F}_5. \tag{C.2.3}$$

Furthermore, the ten-dimensional gravitational constant is given by.

$$2\kappa = (2\pi)^7\alpha'^4. \tag{C.2.4}$$

Interestingly, type IIB SUGRA has a global $SL(2,\mathbb{R})$ (Möbius) symmetry which is not manifest in the action (C.2.2); in order to identify it, one has to rewrite to Einstein frame and perform a couple of field redefinitions (see [45, 184]). This $SL(2,\mathbb{R})$ invariance is holographically related to the Montonen-Olive duality of $\mathcal{N} = 4$ SYM [186].

D THE ADS/CFT CORRESPONDENCE FROM STRING THEORY

In this appendix, which serves as supplementary material to chapter 2, we provide a "derivation" of the AdS/CFT correspondence from string theory by considering a stack of N D3 branes from the perspectives of open and closed strings, following [42–44].

D.1 D3 DUALITY AND ADS5/CFT4

Consider a stack of N D3 branes in $\mathbb{R}^{9,1}$, which extend along a $(3+1)$-dimensional hyperplane. Perturbative string theory on this background contains both open and closed strings: the closed strings enter as excitations of the vacuum, while the open strings end on the D3 branes and, as such, act as excitations of the branes. At low energies, $E \ll l_s^{-1}$, only massless string modes are excited. When the string coupling is small, $g_s \ll 1$, the Dirac-Born-Infeld (DBI) action describes the string dynamics accurately: the dynamics of open strings are then described by a supersymmetric gauge theory living on the worldvolume of the D branes, where the gauge field A_μ corresponds to open string excitations parallel to the D brane, while excitations transverse to the D brane—which are interpreted as fluctuations in the position of the D-brane—behave as scalar fields. If we consider only a single D brane, we get a $U(1)$ gauge theory (see e.g. [187]), but if we consider N coincident branes, we have to introduce Chan-Paton factors, which are non-dynamical degrees of freedom assigned to the endpoints of the string: the Chan-Paton factor λ_{ij} labels strings that go between the i'th and the j'th brane. One can show that the λ_{ij} form the Lie algebra $U(N)$, resulting in $U(N)$ gauge theory with effective coupling $g_s N$. On the other hand, D-branes are also solitonic solutions to supergravity, where they act as sources of the gravitational field. In order for supergravity to be a good description, we require that the characteristic scale L of the spacetime under scrutiny to be large implying weak curvature and, by extension, low energy, so that supergravity is a good description of the dynamics. For N coincident D-branes, we have $\frac{L^4}{\alpha'^2} \propto g_s N$, leading us to conclude that the closed string perspective is valid for $g_s N \gg 1$. In summary, the (schematic) action describing the configuration of N D3 branes is given by

$$S = S_{\text{closed}} + S_{\text{open}} + S_{\text{int}}, \tag{D.1.1}$$

where S_{closed} is the the action of type IIB supergravity (SUGRA) with some additional corrections—since we're in the low-energy limit—and S_{open} describes the dynamics on the (3+1)-dimensional worldvolume constituted by the brane-stack, and S_{int} describes the interactions (which we will largely ignore; see [42] for details). The (effective) action above involves only the massless modes, but in general takes into account the effects of integrating out the massive modes. The action is not renormalizable.

We will consider the action above *from two different points of view*, and—from the physical equivalence of these perspectives—arrive at the AdS/CFT correspondence. Note that in both perspectives, we will be in the low energy limit, $E \ll \frac{1}{l_s}$, while the two perspectives will be valid for opposite extremes of the string coupling.

D.1.1 *Open Strings*

We start by taking the point of view of the open strings; valid in the regime $g_s N \ll 1$. While we should really consider a stack of N coincident D3-branes in $\mathbb{R}^{1,9}$ extending into the directions X^μ for $\mu \in \{0,1,2,3\}$ with $X^I = 0$ for $I \in \{4,\ldots,9\}$, we will, as is standard, focus on the case $N = 1$ and then hand-wavily generalize our results. The dynamics on the D3 brane is described by the DBI action,

$$S_{\text{DBI}} = -\frac{1}{(2\pi)^3 l_s g_s} \int d^4\sigma \sqrt{-\det\left[\gamma_{ab} + 2\pi\alpha' F_{ab}\right]} + \text{fermions}, \tag{D.1.2}$$

where $\gamma_{ab} = \eta_{MN}\partial_a X^M \partial_b X^N$ is the induced metric, and $F = dA$ is the field strength two-form of the $U(1)$ gauge field. The σ^a with $a \in \{0,\ldots,3\}$ are the coordinates of the wold-volume, which we can choose by specifying our embedding such that $X^a(\sigma) = \sigma^a$, whereas the directions transverse to the brane—which we center at the origin—are described in terms of six scalar fields, which are essentially fluctuations in brane position,

$$X^{i+3}(\sigma) = 2\pi\alpha'\phi_i(\sigma), \quad i \in \{1, \ldots, 6\}. \tag{D.1.3}$$

Hence, the induced metric takes the form,

$$\gamma_{ab} = \eta_{ab} + (2\pi\alpha')^2 \partial_a\phi^i\partial_b\phi_i, \tag{D.1.4}$$

which means that the determinant featuring in the DBI action (D.1.2) takes the form:

$$\det := \det\left[\eta_{ab} + 2\pi\alpha' F_{ab} + (2\pi\alpha')^2\partial_a\phi^i\partial_b\phi_i\right]. \tag{D.1.5}$$

The low-energy limit $l_s E \to 0$ implies $\alpha' \to 0$ and allows us to expand the determinant in the by-now small parameter α', which we will do shortly. Consider first the general expression $\eta_{ab} + \epsilon B_{ab}$ where ϵ is some small parameter and perform the rewriting

$$\eta_{ab} + \epsilon B_{ab} = \eta_{ac}\left(\delta^c_b + \epsilon B^c{}_b\right). \tag{D.1.6}$$

The determinant (D.1.6) can now be computed by using the homomorphism property of the determinant,

$$\det\left(\eta_{ac}\left(\delta^c_b + \epsilon B^c{}_b\right)\right) = \det\left(\eta_{ac}\right)\det\left(\delta^c_b + \epsilon B^c{}_b\right) = -\det\left(\delta^c_b + \epsilon B^c{}_b\right). \tag{D.1.7}$$

Now, since $\det \mathbf{A} = \exp\left(\mathrm{Tr}\left[\log \mathbf{A}\right]\right)$ for $\mathbf{A}$ an $n \times n$ matrix, setting $\mathbf{A} = \mathbb{1} + \epsilon\mathbf{B}$, we get

$$\det(\mathbb{1} + \epsilon\mathbf{B}) = \exp\left(\mathrm{Tr}\left[\log(\mathbb{1} + \epsilon\mathbf{B})\right]\right) = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon^n \mathrm{Tr}\left[\mathbf{B}^n\right]\right) \tag{D.1.8}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon^n\mathrm{Tr}\left[\mathbf{B}^n\right] + \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon^n\mathrm{Tr}\left[\mathbf{B}^n\right]\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\epsilon^m\mathrm{Tr}\left[\mathbf{B}^m\right]\right) + \ldots \tag{D.1.9}$$

$$= 1 + \epsilon\mathrm{Tr}\left[\mathbf{B}\right] - \frac{1}{2}\epsilon^2\mathrm{Tr}\left[\mathbf{B}^2\right] + \frac{1}{2}\epsilon^2\mathrm{Tr}\left[\mathbf{B}\right]^2. \tag{D.1.10}$$

Combining this with the expansion $\sqrt{1+x} = 1 + x/2 + \mathcal{O}(x^2)$, we find that the expansion of the DBI determinant (D.1.5) becomes

$$\sqrt{-\det} = 1 + (2\pi\alpha')^2\left(-\frac{1}{2}F^a{}_bF^b{}_a + \frac{1}{2}\partial_a\phi^i\partial^a\phi_i\right) + \mathcal{O}(\alpha'^3) \tag{D.1.11}$$

$$= 1 + (2\pi\alpha')^2\left(\frac{1}{2}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^i\partial^a\phi_i\right) + \mathcal{O}(\alpha'^3), \tag{D.1.12}$$

where the "matrix index structure" (i.e. one raised and one lowered index) we impose in determinant relation (D.1.7) incurs an additional minus due to antisymmetry of the field strength. This means that the DBI action assumes the form[1]

$$S_{\mathrm{DBI}} = -\frac{1}{(4\pi)g_s}\int \mathrm{d}^4\sigma\,\left(\frac{1}{4}F_{ab}F^{ab} + \partial_a\phi^i\partial^a\phi_i\right) + \text{fermions}, \tag{D.1.13}$$

which is the action for four-dimensional $\mathcal{N} = 4$ SYM with gauge group $U(1)$ provided that we identify (compare (C.1.23))

$$g^2_{\mathrm{YM}} = 4\pi g_s. \tag{D.1.14}$$

Generalizing the above to a stack of N coincident D3-branes, the gauge fields and scalars are valued in $\mathfrak{u}(N)$, $\phi^i = \phi^{ia}T_a$, $A_\mu = A^a_\mu T_a$, where the T_a are the $N^2 - 1$ generators of the Lie algebra satisfying $[T_a, T_b] = f_{ab}{}^cT_c$. This complicates matters somewhat: the partial derivatives turn into covariant derivatives, but the end result is the same: the identification of (D.1.14) makes equivalent the DBI action and that of four-dimensional $U(N)$ SYM theory [43]. Furthermore, the limit $\alpha' \to 0$ reduces the type IIB action—i.e. S_{closed} of the full action (D.1.1)—to free supergravity in the $\mathbb{R}^{1,9}$ bulk. Similarly,

1 We ignore the constant contribution from the 1, which will just integrate to the world-volume.

we see that $S_{\text{int}} \xrightarrow{\alpha' \to 0} 0$, so the $\mathcal{N} = 4$ $U(N)$ gauge theory living on the brane world-volume decouples (hence the $\alpha' \to 0$ is sometimes called the *decoupling limit*) from the free SUGRA theory in the flat bulk. Now, let's examine the limit that we're taking in some detail: starting with $N+1$ D3-branes in $\mathbb{R}^{1,10}$, we take N of them to be coincident with $X^{i+3} = 0$ for $i \in \{1, \ldots, 6\}$, while we take the last brane to be separated from the stack in the X^9-direction by a distance r; i.e. the last brane has $X^9 = r$. Considering only the massless modes, the fields living on the brane are now described by a $U(N) \times U(1)$ gauge theory. This separation is characterized by a Higgs expectation value $\langle X^9 \rangle = \frac{r}{2\pi\alpha'}$, which has to be kept fixed when bringing together the stack and the rogue brane. This results in the *Maldacena limit* [1],

$$\alpha' \to 0, \quad U := \frac{r}{\alpha'} = \text{fixed}. \tag{D.1.15}$$

In particular, keeping U fixed implies that the mass of the stretched strings remains fixed. Seemingly, then, we get $U(N)$ four-dimensional $\mathcal{N} = 4$ SYM living in the branes; it turns out, however, that the $U(1) \subset U(N)$ corresponds to singleton fields living on the boundary in the gravity theory that cannot propagate into the bulk and thus decouple, leaving us with $SU(N)$ four-dimensional $\mathcal{N} = 4$ SYM. Note also that what we have done so far is valid for any N!

D.1.2 *Closed Strings*

From the point of view of closed strings, a stack of N D3-branes in the strongly coupled limit[2], $g_s N \gg 1$, arise as a BPS solution to type IIB supergravity preserving $SO(3,1) \times SO(6)$ isometries of $\mathbb{R}^{9,1}$. Explicitly, this solution is given by [43]

$$ds^2 = H^{-1/2}\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu + H^{1/2}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{S^5}), \tag{D.1.16}$$

$$H = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2, \tag{D.1.17}$$

$$C_4 = (1/H - 1)\,\mathrm{d}X^0 \wedge \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 + \text{ terms that ensure self-duality of } F_{(5)}, \tag{D.1.18}$$

$$e^\phi = g_s, \quad B_{MN} = 0, \tag{D.1.19}$$

where $r^2 = X_i X^i$ and C_4 is the four-form gauge field, which is related to the self-dual five-form R-R field via $F_{(5)} = \mathrm{d}C_{(4)}$. Further, since g_{00} is non-constant, the energy E_r of a string excitation as measured by an observer at a constant position r and the energy E_∞ measured by an observer at infinity are related by [42]

$$E_\infty = H^{-1/4}E_r. \tag{D.1.20}$$

The D3-branes act as sources for the self-dual five-form $F_{(5)}$, which has a flux on the five-sphere. The background of the D3 SUGRA solution has two distinct regions: for large $r \gg L$, we have $H \sim 1$, and we're left with ten-dimensional Minkowski space, while $r \ll L$ corresponds to the near-horizon or throat region, where the 1 in the harmonic function H can be neglected, leaving us with

$$ds^2_{\text{throat}} = \frac{r^2}{L^2}\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu + \frac{L^2}{r^2}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{S^5}) \tag{D.1.21}$$

$$= \underbrace{\left(\frac{r^2}{L^2}\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu + \frac{L^2}{r^2}\mathrm{d}r^2\right)}_{\text{AdS}_5 \text{ with radius } L} + \underbrace{L^2\mathrm{d}\Omega^2_{S^5}}_{S^5 \text{ with radius } L} \tag{D.1.22}$$

which we recognize as the metric of $\text{AdS}_5 \times S^5$ in Poincaré patch coordinates (cf. eq. (A.1.7)). This leads us to conclude that we have two different types of closed strings; closed strings propagating in the ten-dimensional Minkowski space (the asymptotically flat region) far away from the horizon and closed strings propagating in the near-horizon geometry $\text{AdS}_5 \times S^5$. In the Maldacena limit, however, these regions decouple, as we shall now argue, following [42]: consider a string state near the throat

2 Note that $g_s N$, while large, is kept *fixed*, which in particular implies that the 't Hooft coupling $\lambda = g^2_{\text{YM}} N \sim g_s N$ is kept fixed from the open string perspective. Note further that this assumes that $g_s < 1$; if $g_s > 1$, we'd have instead $N/g_s \gg 1$—that is, we need large N, *not* large g. This requirement arises because SUGRA is only a good description when the background curvature is much larger than the string length, $L \gg l_s$, because we need to avoid string worldsheet quantum corrections. Further, quantum string corrections are avoided when $g_s \to 0$.

with (fixed) energy $E_r \gg 1$ (at fixed radial position r). The energy E_∞ of this string state as measured by an observer at infinity is then given by (D.1.20)

$$E_\infty \sim \frac{r}{L} E_r, \tag{D.1.23}$$

where we have used that r is in the throat region, i.e. $r \sim 0$, implying that E_∞ is negligible since $\frac{r}{L} E_r \to 0$ for E_r fixed and $r \ll L$. Thus, the observer at infinity sees two different low-energy contributions: closed strings in the asymptotically flat region described by type IIB SUGRA in $\mathbb{R}^{1,9}$, while the closed strings in the throat region are described by fluctuations around the $\mathrm{AdS}_5 \times S^5$ solution of IIB SUGRA, and since these fluctuations can be arbitrarily large even in the low energy limit, we are dealing with *the full quantum string theory* on $\mathrm{AdS}_5 \times S^5$. When taking the low energy limit, the two regions decouple from each other[3]. To summarize, we have found that in the low-energy large-N limit, the open and closed string perspectives, which are equivalent, give rise to two different decoupled effective theories:

- Open string perspective: four-dimensional $SU(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R}^{1,3}$ and type IIB SUGRA on $\mathbb{R}^{1,9}$.
- Closed string perspective: Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and type IIB SUGRA $\mathbb{R}^{1,9}$.

This leads us to conclude that four-dimensional $SU(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R}^{1,3}$ is dual to type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with the identification:

$$g_{\mathrm{YM}}^2 = 4\pi g_s, \quad L^4 = 4\pi g_s N(\alpha')^2, \quad \lambda = g_{\mathrm{YM}}^2 N = 2\pi g_s N = \frac{L^4}{2(\alpha')^2}, \tag{D.1.26}$$

where λ is the 't Hooft coupling that we introduced previously. Of course, our derivation is valid only for large N and fixed λ, but it is natural to conjecture that the duality holds for any values of λ and N.

3 When writing the throat metric (D.1.22), we have used the coordinate r, which is *not* fixed in the Maldacena limit (D.1.15). Instead, we should replace our radial coordinate with U, which *is* kept fixed in the Maldacena limit, $r = U\alpha'$, yielding the near-horizon metric:

$$ds^2_{\text{throat}} = \alpha'^2 \frac{U^2}{L^2} \eta_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu + \frac{L^2}{U^2}(\mathrm{d}U^2 + U^2 \mathrm{d}\Omega_{S^5}^2) \tag{D.1.24}$$

$$= \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} \eta_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu + \frac{\sqrt{4\pi g_s N}}{U^2} \mathrm{d}U^2 + \sqrt{4\pi g_s N} \mathrm{d}\Omega_{S^5}^2 \right], \tag{D.1.25}$$

which—up to an overall factor of α'—again produces the metric of $\mathrm{AdS}_5 \times S^5$ in Poincaré patch coordinates. This is the procedure of the original paper [1].

THE HOLOGRAPHIC SUPERCONDUCTOR

E.1 HOLOGRAPHY AT FINITE TEMPERATURE, DENSITY & CHEMICAL POTENTIAL

E.1.1 *Gravity Dual of Finite Temperature Field Theory I: Black D3-Branes*

The gravity dual of a finite temperature field theory is obtained by considering a black brane[1] in AdS space (cf. the dictionary in 2.3)—thus, in this case, making the spacetime AAdS; see the metric in (E.1.1). These black branes radiate and have a temperature, T_H—the Hawking temperature—associated to them. This identification was first realized by Witten in [188], where he considers the AdS Schwarzschild black hole; we shall take this approach shortly. Our starting point for now—to make contact with our heuristic derivation of the AdS/CFT correspondence—will be a black D3-brane, which is a non-extremal D3-brane (see [189, 190]) described by the metric (compare (D.1.16))

$$ds^2 = H(r)^{-1/2}\left(-f(r)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + H(r)^{1/2}\left(\frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{S^5}\right), \tag{E.1.1}$$

with blackening factor $f(r) = 1 - \left(\frac{r_h}{r}\right)^4$ and harmonic function $H(r) = 1 + \left(\frac{L}{r}\right)^4$. The black brane horizon r_h is related to the Hawking temperature of the black brane—we will have more to say about this connection later; in particular, $r_h = 0$ reduces the black D3-brane to an ordinary extremal D3-brane with metric (D.1.16). Proceeding in the same manner as we did when motivating the AdS/CFT correspondence in appendix D, we—following [43]—focus on the near-horizon (throat) region of the black D3-brane, where $r/L \ll 1$. Upon introducing the coordinate $z = L^2/r$, the black D3-brane in the throat limit takes the form

$$\begin{aligned} ds^2_{\text{throat}} &= \frac{r^2}{L^2}\left(-\left[1-\left(\frac{r_h}{r}\right)^4\right]\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{L^2}{r^2\left(1-\left(\frac{r_h}{r}\right)^4\right)}\mathrm{d}r^2 + L^2\mathrm{d}\Omega^2_{S^5} && \text{(E.1.2)}\\ &= \frac{L^2}{z^2}\left[-\left(1-\left(\frac{z}{z_h}\right)^4\right)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \frac{1}{1-\left(\frac{z}{z_h}\right)^4}\mathrm{d}z^2\right] + L^2\mathrm{d}\Omega^2_{S^5} && \text{(E.1.3)}\\ &= \frac{L^2}{z^2}\left[\left(1-\left(\frac{z}{z_h}\right)^4\right)\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2 + \frac{1}{1-\left(\frac{z}{z_h}\right)^4}\mathrm{d}z^2\right] + L^2\mathrm{d}\Omega^2_{S^5} && \text{(E.1.4)}\end{aligned}$$

where $z_h = L^2/r_h$, and where, in the last equality, we have Wick rotated to Euclidean time, $\tau = it$. Further, we define a radial variable $\rho = L\sqrt{1-\frac{z}{z_h}}$, which measures the distance from the black brane horizon at z_h, in terms of which the Euclidean metric close to the black brane horizon (that is, to lowest order in ρ) takes the form

$$ds^2_{\text{throat–NBBH}} = \frac{4\rho^2}{z_h^2}\mathrm{d}\tau^2 + \frac{L^2}{z_h^2}\mathrm{d}\vec{x}^2 + \mathrm{d}\rho^2 + L^2\mathrm{d}\Omega^2_{S^5}, \tag{E.1.5}$$

where NBBH stands for *near black brane horizon*. Next, we focus on the (τ, ρ) plane and rescale the Euclidean time coordinate, $\phi = 2\tau/z_h$, which yields the (ϕ, ρ) plane metric

$$ds^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2, \tag{E.1.6}$$

and thus, in order to avoid a conical singularity, we need periodicity; that is, we must identify $\phi \sim \phi + 2\pi$ (the $\sim$ is to be read as *identified with*), implying that $\tau \sim \tau + z_h\pi$. The periodicity of the Euclidean time is precisely the inverse temperature (see table 2.3), leading us to conclude that

$$T = \frac{1}{z_h\pi}. \tag{E.1.7}$$

1 In contradistinction to black holes, the spatial topology of a black brane is not compact: this is required in order for the spatial geometry of the dual field theory to be $\mathbb{R}^3$. In particular, the black D3-brane has horizon topology $\mathbb{R}^3 \times S^5$, i.e. planar with five-spheres at every point.

Note that we have to use the periodicity of the time coordinate τ and *not* its diffeomorphic cousin ϕ, since we identify τ as the (Euclidean) time coordinate in the dual $\mathcal{N} = 4$ theory. We now want to apply the Bekenstein-Hawking formula for the black brane entropy, $S = \frac{A}{4G}$. First, we determine the horizon area A of the black D3-brane, for which we introduce the metric $\tilde{g}_{ij}$— derived from (E.1.5)—describing a hyperplane with $\rho = 0$ (equivalently, $z = z_h$) and τ fixed, but arbitrary. The determinant, owing to the diagonality of the metric (E.1.5), is consequently given by $\tilde{g} = \frac{L^6}{z_h^6} \times L^{10} g_{S^5}$, with g_{S^5} the determinant of the metric of S^5, implying that the area becomes[2],

$$A = \int_{\mathbb{R}^3 \times S^5} \mathrm{d}^8x \sqrt{\tilde{g}} = \frac{L^8}{z_h^3} \int_{\mathbb{R}^3} \mathrm{d}^3x \times \int_{S^5} \mathrm{d}^5x \sqrt{g_{S^5}} = \frac{L^8}{z_h^3} \mathrm{Vol}(\mathbb{R}^3)\mathrm{Vol}(S^5) = \frac{\pi^3 L^8}{z_h^3} \mathrm{Vol}(\mathbb{R}^3) \tag{E.1.8}$$

$$= \pi^6 T^3 L^8 \mathrm{Vol}(\mathbb{R}^3), \tag{E.1.9}$$

where we have used (E.1.7). Now, a nice holographic calculation [43] shows that the ten-dimensional Newton constant G is given by

$$G = \frac{\pi^4 L^8}{2N^2}, \tag{E.1.10}$$

implying that the entropy of the black D3-brane, and—by extension—the field theory, takes the form[3]

$$S = \frac{A}{4G} = \frac{\pi^2}{2} N^2 T^3 \mathrm{Vol}(\mathbb{R}^3). \tag{E.1.12}$$

This implies that the free energy density takes the form $f = -\int \mathrm{d}T\, S(T)/\mathrm{Vol}(\mathbb{R}^3) = -\frac{1}{8} N^2 \pi^2 T^4$, which, as noted in [12], is precisely the Stefan-Boltzmann law. Finally, note that (E.1.12) is a highly non-trivial result: it is impossible to calculate the entropy $\mathcal{N} = 4$ SYM via the usual route—attempting to do so would require summing infinitely many Feynman diagrams to arbitrarily high loop order, which is not presently feasible. It is, however, instructive to compare this with the entropy of free $\mathcal{N} = 4$ SYM. This is given by [191]

$$S_{\text{Free}} = \frac{2\pi^2}{3} N^2 T^3 \mathrm{Vol}(\mathbb{R}^3), \tag{E.1.13}$$

implying that the entropy varies extremely slowly with coupling (see also [11]).

E.1.2 *Gravity Dual of Finite Temperature Field Theory II: AdS Schwarzschild Black Hole*

Rather than starting from the black D3-brane solution of type IIB SUGRA, we can instead, as Witten does in [188]—and as is done in [11–13]—focus on the ordinary planar AdS-Schwarzschild Black hole, which in $(d+1)$ dimensions[4] has the form

$$ds^2 = \frac{L^2}{z^2} \left(-f(z)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + f(z)^{-1}\mathrm{d}z^2 \right), \tag{E.1.14}$$

in Poincaré-like coordinates, where $f(z) = 1 - \frac{z^d}{z_h^d}$. For $d = 4$, this metric is identical to the black D3-brane throat metric, and we may apply the same general method to compute the Hawking temperature of the Ads$_{d+1}$ Schwarzschild black hole; in particular, we apply the coordinate transformation $\rho = \frac{2L}{\sqrt{d}} \sqrt{1 - \frac{z}{z_h}}$ and Wick rotate, $\tau = it$, which puts the metric into the form $ds^2 = \frac{d^2 \rho^2}{4z_h^2} \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \frac{L^2}{z_h^2} \mathrm{d}\vec{x}^2$. Again, we focus on the (τ, ρ) plane and rescale Euclidean time, $\phi = d/(2z_h)\tau$, to bring the (τ, ρ) plane metric to the form $ds^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$ which, to avoid a conical singularity, requires $\phi \sim \phi + 2\pi$, or, equivalently, $\tau \sim \tau + \frac{4z_h \pi}{d}$, implying that the Hawking temperature is

$$T = \frac{d}{4z_h \pi}. \tag{E.1.15}$$

2 The volume of S^d is given by $\mathrm{Vol}(S^d) = \frac{2\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}$.

3 We note in passing that for a conformal theory, which possesses no internal energy scales, the temperature dependence of thermodynamic variables is entirely determined by scaling,

$$F = \alpha \frac{1}{d+1}, \quad U = \alpha \frac{d}{d+1}, \quad S = \alpha T^d. \tag{E.1.11}$$

in d dimensions, where α is a constant. Our findings above, e.g. (E.1.12), confirm this behavior. For more details see [11].

4 Since we forget about the supergravity origins of the objects under scrutiny, we can keep the dimensionality general.

It is useful to generalize this argument to generic AAdS manifolds, which in particular include the static black holes that we are currently dealing with. These have the form of (E.1.14) for some function $f(z)$ with a first order zero at the black hole horizon z_h. Now, we can ignore the conformal factor L^2/z^2, since the Hawking temperature is invariant under conformal transformations (see [192]), so, after Wick rotating, we introduce $\rho^2 = \frac{4}{f'(z_h)}(z - z_h)$ and $\frac{1}{2}f'(z_h)$, which brings the (conformal) metric—to lowest order in ρ—into the form in the (ϕ, ρ) plane, $ds^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$, so, to avoid the usual conical singularity, we must identify $\phi \sim \phi + 2\pi$, which leads to the periodicity $\tau \sim \tau + \frac{4\pi}{|f'(z_h)|}$, so that the Hawking temperature takes the form:

$$T = \frac{|f'(z_h)|}{4\pi}. \tag{E.1.16}$$

Thus, considering a planar Schwarzschild black hole reproduces the results obtained from the black D3-brane (for appropriate values of the dimension). This represents a more heuristic approach to the AdS/CFT correspondence: rather than referring to the supergravity roots of the correspondence, we can instead take the correspondence for granted and consider various field configurations in AdS to see what consequences they have in the field theory. This *bottom-up* approach is very useful, but lacking an explicit string theory embedding comes at a cost: determining relations between certain parameters (such as the Newton constant in (E.1.10)) is no longer possible.

E.1.3 *The D3/D7 Brane Configuration and the AdS Reissner-Nordström Black Hole*

Referring to the holographic dictionary of 2.3, we observe that a finite chemical potential μ in the field theory requires an electric monopole in the bulk: the time component of the bulk $U(1)$ gauge field is related to μ. To motivate this, we consider a theory with global $U(1)$ gauge containing a massless complex scalar and a massless Dirac fermion charged under the gauge symmetry. The Lagrangian of such a theory reads

$$\mathcal{L} = -(D_\mu \varphi)^* D^\mu \varphi + i\bar{\psi} \not{D}_\mu \psi - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu}, \tag{E.1.17}$$

where the covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$. Now, introduce a non-vanishing background field $\mu \in \mathbb{R}$ for the time component of the gauge field around which the dynamic field fluctuates, $\tilde{A}_0 = \mu + A_0$. This constant shift will not affect the field strengths and thus generates an extra potential $V = \mu^2 \varphi^* \varphi - \mu \psi^\dagger \psi$, where we have used the fact that $\bar{\psi}\gamma^0\psi = \psi^\dagger \beta \gamma^0 \psi = \psi^\dagger \psi$, since the β-matrix and γ^0 are numerically equivalent (though they differ in van der Waerden indices) and square to one (this is why we need the time component—none of the other γ-matrices would produce the pure number operator). The modified Lagrangian becomes $\tilde{\mathcal{L}} = \mathcal{L} + V$, and we recognize $\psi^\dagger\psi$ as the number density operator. Further, we see that the scalar acquires a negative mass squared, $-\mu^2$, which is allowed by the BF bound (2.4.10) as long as it isn't too negative. This analysis naturally leads us to propose that a finite chemical potential is holographically encoded in a bulk one-form $A = A_t(r)\mathrm{d}t$ with near-boundary $(r \to \infty)$

$$A_t(r) \sim \mu + \frac{\tilde{\rho}}{r^{d-2}}, \tag{E.1.18}$$

in d dimensions, where $\tilde{\rho}$ is related to the density ρ. There is one obstacle, however: there is no one-form field in type IIB SUGRA! There are two ways to get around this, which we now describe.

E.1.3.1 *D3/D7 Brane Configuration*

As described in [43, 193], one can introduce a stack of N_f (with f standing for *flavor*) D7-branes in addition to the usual stack of N D3-branes. One can show that the theory dual to this configuration is $\mathcal{N} = 4$ SYM coupled to N_f multiplets of $\mathcal{N} = 2$ with gauge group $U(N_f)$. In particular, the $\mathcal{N} = 2$ multiplets transform in the fundamental representation of the gauge group—just as quarks in ordinary QCD. Collectively, we dub the fields in such a multiplet *flavor fields*. The mass of the flavor fields turns out to be given by the separation ℓ between the brane-stacks, $m_f = \frac{\ell}{2\pi\alpha'}$. To get the desired $U(1)$ symmetry of the field theory, we need just a single flavor multiplet, N_f, which is well within the so-called *probe limit*, where any back-reaction to the supergravity solution can be neglected; that is: the $\mathrm{AdS}_5 \times S^5$ geometry of the D3-branes will still be the near-horizon geometry. In this probe limit, the combined action becomes $S_{\text{probe}} = S_{\text{IIB–SUGRA}} + S_{\text{D7}}$, where S_{D7} at $T = 0$ reduces to the DBI action[5]. In this case, we have

$$S_{\text{D7}} = -\tau_7 \int \mathrm{d}^8\xi \sqrt{-\det\left(\gamma_{ab} + (2\pi\alpha')F_{ab}\right)}, \tag{E.1.19}$$

where τ_7 is the D7-brane tension. Now, the rest of the argument proceeds as follows: the field theory has, by Noether's theorem, a conserved current J^μ corresponding to the global $U(1)$ symmetry, the (expectation value of) time component of which is precisely the charge density, $\rho = \langle J^0 \rangle$. Obtaining and solving the equations of motion of (E.1.19), we can find the on-shell action $S_{\text{D7}}^{\text{on-shell}}$, which we may then renormalize and identify with the grand canonical potential, $\Omega \sim S_{\text{D7,ren}}^{\text{on-shell}}$. Noting that the density satisfies $\langle J^0 \rangle \sim \frac{\partial \Omega}{\partial \mu}$, we obtain

$$\langle J^0 \rangle = \frac{\delta S_{\text{D7,ren}}^{\text{on-shell}}}{\delta A_t(r = \infty)}. \tag{E.1.20}$$

Performing this variation, we obtain

$$\langle J^0 \rangle \sim \tilde{\rho}, \tag{E.1.21}$$

up to string-theoretical factors.

E.1.3.2 *Bottoms Up & The AdS Reissner-Nordström Black Hole*

As described in [11–13, 194], another—perhaps more useful method—of obtaining a finite chemical potential and a global $U(1)$ symmetry in the field theory is to simply add a Maxwell gauge field in the bulk, in a "bottom-up approach" to holography. The bulk is consequently described by the negative cosmological constant Einstein-Maxwell action in $(d+1)$ dimensions,

$$S_{\text{EM}} = \int \mathrm{d}^d x \, \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \frac{d(d+1)}{L^2} \right) - \frac{1}{4g_F^2} F_{\mu\nu}F^{\mu\nu} \right], \tag{E.1.22}$$

where g_F is the electromagnetic coupling and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Varying this action produces the equations of motion,

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \frac{d(d-1)}{L^2} g_{\mu\nu} = \frac{\kappa^2}{2g^2} \left(2F_{\mu\sigma}F_{\rho\nu}g^{\sigma\rho} - \frac{1}{2} g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} \right), \quad \nabla_\mu F^{\mu\nu} = 0, \tag{E.1.23}$$

which are solved by the AdS Reissner-Nordström planar black hole,

$$ds^2 = \frac{L^2}{z^2} \left(-f(z)\mathrm{d}t^2 + f(z)^{-1}\mathrm{d}z^2 + \mathrm{d}\vec{x}^2 \right), \tag{E.1.24}$$

where $f(z) = 1 - Mz^d + Q^2 z^{2(d-1)}$. The horizon, determined by the condition $f(z_0) = 0$, has two solutions; we identify the larger of the two, located at z_+, with the horizon. The parameters involved in $f(z)$ are explicitly given by [13]

$$M = z_+^{-d} + \frac{z_+^{2-d}\mu^2}{\gamma^2}, \quad Q^2 = \frac{z_+^{4-2d}\mu^2}{\gamma^2}, \tag{E.1.25}$$

where

$$\gamma^2 = \frac{(d-1)g_F^2 L^2}{(d-2)\kappa^2}, \tag{E.1.26}$$

Finally, the Maxwell one-form becomes

$$A = \mu \left(1 - \left(\frac{z}{z_+} \right)^{d-2} \right) \mathrm{d}t. \tag{E.1.27}$$

5 In principle, there is also a contribution from a Chern-Simons term, but since it vanishes at zero temperature we'll ignore it.

We now proceed to compute the Hawking temperature of the RN-AdS black hole:

$$T = \frac{|f'(z_h)|}{4\pi} = \frac{\left|(2d-2)Q^2 z_+^{2d-3} - dMz_+^{d-1}\right|}{4\pi} = \frac{1}{4\pi z_+}\left(d - \frac{(d-2)z_+^2\mu^2}{\gamma^2}\right). \tag{E.1.28}$$

We may then determine $\rho = \langle J^t \rangle$ in the same way as previously outlined: we compute the grand canonical ensemble potential at the boundary as $\Omega = -T\log \mathcal{Z}$, where $\mathcal{Z}$ is the gravity partition function in the saddle-point approximation, $\mathcal{Z} = \exp\left[-S_E^{\text{on-shell}}\right]$ with E standing for Euclidean. We may then identify $\Omega = TS_E^{\text{on-shell}}$, and the density relation becomes $\rho = \langle J^t \rangle = -\frac{1}{\text{Vol}(\mathbb{R}^{d-1})}\frac{\partial \Omega}{\partial \mu}$.

E.2 THE s-WAVE HOLOGRAPHIC SUPERCONDUCTOR

First proposed in [5], where the "vanilla" s-wave superconductor was constructed in a bottom-up manner, holographic superconductors has since grown into a large area of research, with the more complicated varieties, the p- and d-wave superconductors, which carry angular momenta of $\ell = 1$ and $\ell = 2$, respectively, now having been embedded in a holographic framework [195–197]. To get to grips with the formalism, we now describe the simplest holographic superconductor: the s-wave holographic superconductor, following the reviews [11, 13, 198–200].

As we have seen, systems with finite $U(1)$ charge density are gravitationally encoded in AdS-Reissner-Nordström black holes, which arise as solutions to the Einstein-Maxwell action with negative cosmological constant. Superconductivity is a consequence of spontaneous breaking of a (gauged) $U(1)$ symmetry to $\mathbb{Z}_2$. For our purposes, however, we focus on *global* $U(1)$ symmetry, since we cannot holographically obtain a gauged $U(1)$ symmetry on the boundary. This implies that what we are *really* describing is a holographic superfluid.

In this scenario, photons are effectively treated as an external electromagnetic field, corresponding to non-dynamical photons[6]. This global symmetry is then taken to be spontaneously broken: a charged operator—in this case, the Cooper pair order parameter—acquires a finite VEV.

The holographic dictionary (cf. 2.3) tells us that such a charged operator is itself dual to charged operators (with gauged $U(1)$ symmetry) in the bulk—in particular, this implies that a superfluid on the boundary is dual to a form of superconductivity in the bulk. In order to simplify the analysis, we take the charged field to be a complex scalar (Higgs field) φ coupled to a gauge field A_μ, so the bulk action describing the holographic s-wave superconductor takes the form

$$S = \int \mathrm{d}^{d+1}x\sqrt{-g}\left[\frac{1}{2\kappa_2}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{4g_F}F_{\mu\nu}F^{\mu\nu} - \left|D_\mu\varphi\right|^2 - V(|\varphi|)\right], \tag{E.2.1}$$

where $D_\mu = \partial_\mu - iqA_\mu$ is the gauge covariant derivative and q the charge of the scalar field. In what follows, we'll specialize to $d = 3$ and take the potential to be a mass term, $V(|\varphi|) = m^2|\varphi|^2$. Interestingly, this model has been realized as a Freund-Rubin compactification if M-theory in [8]. Furthermore, before the advent of holographic superconductors, this model was studied quite thoroughly by Gubser in [201]. To proceed, we note that the normal—non-superconducting—state of the theory, which satisfies $\varphi = 0$, is given AdS-RN planar black hole. We now wish to lower the temperature see if we can obtain a transition to a charged condensate at a critical temperature, T_c.

In order to obtain such a charged condensate, we need a finite VEV, $\langle \mathcal{O} \rangle$, where $\mathcal{O}$ is the operator dual to φ. In particular, this implies the existence of a stability of the AdS-RN black hole with respect to perturbations of φ. Now, the equation of motion for φ is obtained by varying the action with respect to φ^*, $\frac{\delta S}{\delta\varphi^*} = 0$, leading to

$$\delta_{\varphi^*}S = -\int \mathrm{d}^4x\,\sqrt{-g}\left[\left(\partial_\mu\delta\varphi^* + iqA_\mu\delta\varphi^*\right)D^\mu\varphi - m^2\varphi\delta\varphi^*\right] \tag{E.2.2}$$

$$\overset{\text{IBP}}{=} \int \mathrm{d}^4x\,\left[\partial_\mu\left(\sqrt{-g}D^\mu\varphi\right) - iq\sqrt{-g}A_\mu D^\mu\varphi - \sqrt{-g}m^2\varphi\right]\delta\varphi^*, \tag{E.2.3}$$

implying that the equations of motion take the form

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}D^\mu\varphi\right) - iqA_\mu D^\mu\varphi - m^2\varphi = \nabla_\mu(D^\mu\varphi) - iqA_\mu D^\mu\varphi - m^2\varphi. \tag{E.2.4}$$

6 This neglecting of effectively virtual photons is also made in BCS theory, so saying that we are describing a superconductor is less of a stretch than it may seem at first.

where we have used that $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right)$. Further, using $\nabla_\mu\varphi = D_\mu\varphi$, the above can be recast in the form

$$(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\varphi - m^2\varphi = 0. \tag{E.2.5}$$

Alternatively, the equations of motion can be obtained from doing the minimal coupling procedure twice: starting from the flat space non-charged coupled scalar Lagrangian,

$$\mathcal{L} = -\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi, \tag{E.2.6}$$

the Euler-Lagrange equations of motion $\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$ lead to the Klein-Gordon equation,

$$\partial_\mu\partial^\mu\varphi - m^2\varphi = 0, \tag{E.2.7}$$

which we can "gauge minimally couple", $\partial_\mu \to D_\mu$

$$D_\mu D^\mu\varphi - m^2\varphi = 0, \tag{E.2.8}$$

which we can again "gravitationally minimally couple", where the ∂_μ that sits inside D_μ is sent to ∇_μ, in which case we end up with

$$(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\varphi - m^2\varphi = 0. \tag{E.2.9}$$

Moving on, we see that this induces a mass correction to the scalar:

$$m^2_{\text{eff}} = m^2 + q^2A_\mu A^\mu, \tag{E.2.10}$$

which in the presence of an exclusively electrostatic gauge field, $A_i = 0$, becomes,

$$m^2_{\text{eff}} = m^2 - |g^{tt}|q^2A_t^2. \tag{E.2.11}$$

E.2.1 *Holographic Superconductivity in the Probe Limit* $(d = 3)$

We now turn to the simplest model for holographic superconductivity first proposed in the seminal paper [5]. The key simplifying assumption is to ignore backreaction on the metric, that is, we take $\kappa^2 \ll g^2L^2$, which decouples the Maxwell-Higgs sector from the gravity sector. This results in an AdS-Schwarzschild background described by

$$ds^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{E.2.12}$$

where

$$f = \frac{r^2}{L^2} - \frac{M}{r}. \tag{E.2.13}$$

The Hawking temperature is determined in the usual fashion (cf. (E.1.16)),

$$T = \frac{|f'(r_h)|}{4\pi}, \tag{E.2.14}$$

where r_h is determined by the condition $f(r_h) = 0$, that is, $r_h = L^{2/3}M^{1/3}$, implying that

$$T = \frac{3M^{1/3}}{4L^{4/3}\pi}, \tag{E.2.15}$$

taking the potential to be $V(|\varphi|) = -\frac{2|\varphi|^2}{L^2}$, and employing a radial ansatz, $\varphi = \phi(r)$, we find that the equations of motion take the form—assuming $q = 1$ and that the spatial part of A vanishes,

$$0 = \nabla_\mu\nabla^\mu\phi - g^{tt}A_t^2\phi + \frac{2}{L^2}\phi = \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\phi\right) + \frac{A_t^2}{f} + \frac{2}{L^2}\phi \tag{E.2.16}$$

$$= \frac{1}{r^2}\partial_r\left(r^2f\phi'\right) + \frac{A_t^2}{f}\phi + \frac{2}{L^2}\phi \tag{E.2.17}$$

$$= f\phi'' + \left(f' + \frac{2f}{r}\right)\phi' + \frac{A_t^2}{f}\phi + \frac{2}{L^2}\phi \tag{E.2.18}$$

where we have used that $\sqrt{-g} = r^2$. Dividing by f yields the equation of motion:

$$\phi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\phi' + \frac{A_t^2}{f^2}\phi + \frac{2}{fL^2}\phi = 0, \tag{E.2.19}$$

in agreement with [5].

Doing the same for A_t, we find

$$A_t'' + \frac{2}{r}A_t' - \frac{2\phi^2}{f}A_t = 0. \tag{E.2.20}$$

while the equation of motion for ϕ implies that it is a constant phase—these findings are in agreement with those presented in the original article, [5]. We note that ϕ provides a mass to the bulk theory gauge field via a Higgs mechanism.

Let's now discuss boundary conditions. *A priori*, we have a four-parameter family of solutions; in order to remedy this, we impose boundary conditions. First, requiring A_t to have finite norm at the horizon, r_0, defined as the value of r satisfying $f(r_0) = 0 \Rightarrow r_0 = \left(ML^2\right)^{1/3}$, we see that $g^{tt}(r_0)A_t(r_0) < 0$, implying that $A_t(r_0) = 0$ (since $\lim_{r\to r_0} g^{tt} = \infty$). In this way, we infer that $A_t' \sim A_t''$ at the horizon; that is, they are not independent. Plugging $A_t(r = r_0) = 0$ into (E.2.19), which we then proceed to multiply by f, gives us,

$$0 = f'(r_0)\phi'(r_0)(r_0)\phi'(r_0) + \frac{2}{L^2}\phi(r_0) = \frac{3r_0}{L^2}\phi'(r_0) + \frac{2}{L^2}\phi, \tag{E.2.21}$$

where we have used that $M = \frac{r_0^3}{L^2}$. This can be rewritten as,

$$\phi(r_0) = -\frac{3}{2}r_0\phi'(r_0), \tag{E.2.22}$$

thus leading us to conclude that at the horizon $r = r_0$, both ϕ and A_t are described by one-parameter families of solutions. A similar asymptotic analysis can be carried out for the boundary: assuming the leading behaviour (at $r \to \infty$) for the fields to be of the general form,

$$\phi(r) = K_\phi r^{\alpha_\phi} + \cdots, \quad A_t(r) = K_{A_t} r^{\alpha_{A_t}} + \cdots, \tag{E.2.23}$$

where K_ϕ, K_{A_t} are constants. We then insert the above into the appropriate equations of motion; for ϕ, (E.2.19) gives us (where we have expanded the coefficients around $r = \infty$),

$$\alpha_\phi(\alpha_\phi - 1)K_\phi r^{\alpha_\phi - 2} + \alpha_\phi K_\phi r^{\alpha_\phi - 1}\left(\frac{4}{r} + \cdots\right) + r^{\alpha_\phi + 2\alpha_{A_t}}\left(\frac{K_\phi K_{A_t}^2 L^4}{r^4} + \cdots\right) + K_\phi r^{\alpha_\phi}\left(\frac{2}{r^2} + \cdots\right) = 0, \tag{E.2.24}$$

so, to leading order, we obtain

$$\alpha_\phi(\alpha_\phi - 1) + 4\alpha_\phi + K_{A_t}^2 L^4 r^{2\alpha_{A_t} - 2} + 2 = 0, \tag{E.2.25}$$

which means that $\alpha_\phi = \frac{1}{2}\left(-3 \pm \sqrt{1 - 4B^2L^4r^{2\alpha_{A_t} - 2}}\right)$. Since we have constrained $\phi \in \mathbb{R}$, we must demand that $\alpha_{A_t} < 1$, implying that (E.2.25) becomes

$$\alpha_\phi(\alpha_\phi - 1) + 4\alpha_\phi + 2 = 0. \tag{E.2.26}$$

which means that

$$\alpha_\phi = -1, -2, \tag{E.2.27}$$

so, since the equations of motion are linear, we can write the solution as:

$$\phi = \frac{\phi^{(1)}}{r} + \frac{\phi^{(2)}}{r^2} + \ldots. \tag{E.2.28}$$

Similarly, the equation (E.2.20) implies (after plugging in the solution (E.2.28) and getting rid of sub-leading terms)

$$\alpha_{A_t}(\alpha_{A_t} - 1) + 2\alpha_{A_t} = 0, \tag{E.2.29}$$

so that $\alpha_{A_t} = 0, -1$, and so

$$A_t = \mu - \frac{\rho}{r} + \cdots, \tag{E.2.30}$$

where the density is determined by the condition $A_t(r_0) = 0$, so that the charge density is given by $\rho = \mu r_0$. Both terms in ϕ above are normalizable[7]. Now, recall that (cf. 2.2) $m^2L^2 = \Delta(\Delta - 3)$. Since we have chosen $m^2L^2 = -2$, this corresponds to either $\Delta = 1$ or $\Delta = 2$. Recalling our discussion in section 2.4, we see that if $\Delta = 2$, $\phi^{(1)}$ is interpreted as a source for the dual operator in the field theory, whereas $\phi^{(2)} \sim \langle \mathcal{O}_2 \rangle$ is the VEV.

Similarly, if $\Delta = 1$, the discussion of section 2.4 implies that the rôles are switched: now $\phi^{(1)} \sim \langle \mathcal{O}_1 \rangle$ is the VEV, whereas $\phi^{(2)}$ acts as a source. In order to study phase transitions, we turn off the source and look for solutions where the VEV acquires a non-zero value at some critical temperature—this leaves a one-parameter family of solutions for both ϕ and A_t. In summary, then, we have

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\phi^{(i)}, \quad i = 1, 2, \quad \varepsilon_{ij}\phi^{(j)} = 0. \tag{E.2.31}$$

where, following [5], we have chosen the $\sqrt{2}$-normalization, which turns out to simplify subsequent calculations. From now on, we also set $M = 1 = L$. Next, note that $[\rho] = 2$ and $[T] = 1$, implying that, first off, $\frac{\langle \mathcal{O}_i \rangle}{T^i}$ are dimensionless quantities, and that $T_c \sim \rho^{1/2}$. Solving the equations numerically[8] yields the curves of figure E.1.

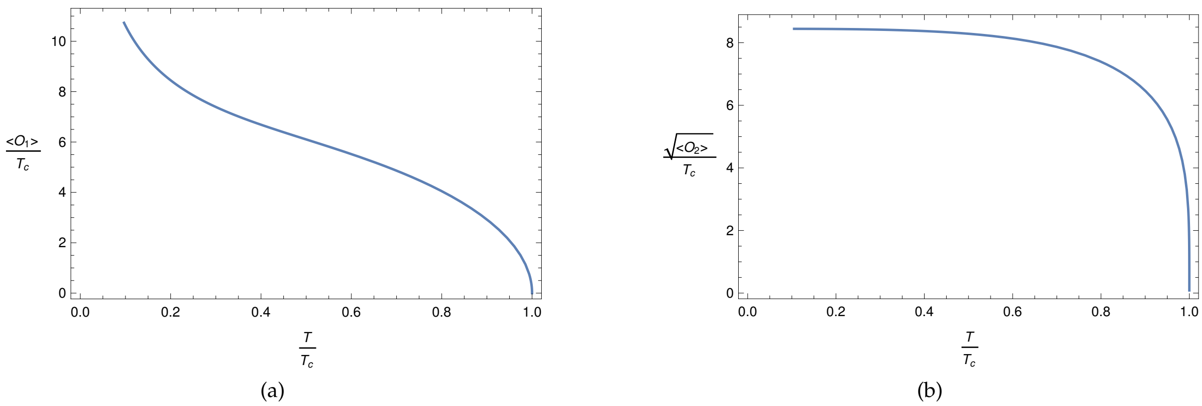


Figure E.1: The condensate as a function of $\mathcal{O}_i$ exhibiting the usual second order Ginzburg-Landau behaviour near the phase transition, $\langle O \rangle_i \sim (1 - T/T_c)^{1/2}$.

Fitting the curves of figure E.1, we get:

$$T_c^{(1)} \simeq 0.2255\rho^{1/2}, \quad T_c^{(2)} \simeq 0.1184\rho^{1/2}. \tag{E.2.32}$$

Note further that at low temperatures, the condensate VEV $\langle \mathcal{O}_1 \rangle$ begins to diverge. This is an artifact of the probe approximation that we're employing: for large values of the condensate, we can no longer ignore backreaction to the bulk metric, so our scheme is invalid in this region. We shall see, by studying the conductivity of the system, that $\langle \mathcal{O}_1 \rangle$ and $\sqrt{\langle \mathcal{O}_2 \rangle}$ can be interpreted as twice the superconducting gap (which is conventionally denoted Δ, but since this clashes with the notation we use for the conformal dimension, we'll simply write it as "gap"). BCS theory predicts that at $T = 0$, we have $2 \times \text{gap} = 3.54T_c$ [202]; for our holographic superconductor, twice the gap is either infinite for $\Delta = 1$ or $2 \times \text{gap} \simeq 8T_c$ for $\Delta = 2$, which is a characteristic of strongly interacting high-T_c superconductors (see [202] for more details).

7 That is, their action is finite.
8 Christopher Herzog has provided a nice Mathematica document that does this on his web-page: click here.

E.2.2 *Conductivity in the Probe Limit*

E.2.2.1 *Conductivity and the Drude Model*

In this section we discuss the physics of conductivity as it appears in condensed matter systems, following [76, 203]. Consider a spatially constant electric field varying in time. After Fourier transforming, Ohm's law reads

$$\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega), \tag{E.2.33}$$

where $\sigma(\omega)$, which in general will be complex, is the optical conductivity. Note in particular that if we shake the electric field at some frequency ω, the "system" responds at the same frequency: this where linear response can be applied. The real part of the optical conductivity, $\mathrm{Re}[\sigma(\omega)]$ describes the dissipation of the current; in other words, it behaves as an actual conductivity, whereas the imaginary part, $\mathrm{Im}[\sigma(\omega)]$, describes the so-called reactive part. Next, let us briefly discuss the Drude model: consider, therefore, a particle of mass m, charge q and velocity $\mathbf{v}$, described by Newton's second law with a friction term linear in the velocity,

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{m}{\tau}\mathbf{v} = q\mathbf{E}, \tag{E.2.34}$$

where τ is the scattering time (or mean free path); the average time the particle travels unhindered before bumping into something. If we then turn on an AC electric field with frequency ω, $\mathbf{E}(t) = \mathrm{Re}\left[\mathbf{E}_0 e^{-i\omega t}\right]$, the solution to the Drude differential equation (E.2.34) becomes

$$\mathbf{v}(t) = \frac{q\tau}{m}\frac{\mathbf{E}(t)}{1-i\omega\tau} + \overbrace{\text{Const.} \times e^{-t/\tau}}^{\text{Negligible in steady-state regime}}, \tag{E.2.35}$$

so, assuming we're in the steady state the regime where the second term can be safely neglected, the optical conductivity becomes; according to (E.2.33)

$$\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}, \quad \sigma_0 = \frac{nq^2\tau}{m}, \tag{E.2.36}$$

where we have used that $\mathbf{j} = nq\mathbf{v}$, where n is the electron density and defined the DC ($\omega = 0$) conductivity, σ_0. At high frequencies, particle-anti-particle creation contributes to the conductivity, but for low frequencies, the Drude model captures the physics nicely. In the superconducting limit, where the mean free path is infinite, $\tau \to \infty$, we may use L'Hôpital's rule to obtain:

$$\lim_{\tau\to\infty}\sigma(\omega) = \lim_{\tau\to\infty}\frac{nq^2}{-i\omega m} = i\frac{nq^2}{m\omega}, \tag{E.2.37}$$

but since the conductivity is analytic, the real part can be determined[9] via the Kramers-Kronig relation,

$$\mathrm{Re}\left[\sigma(\omega)\right] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\mathrm{Im}\left[\sigma(\omega')\right]}{\omega'-\omega} \tag{E.2.38}$$

$$= \frac{nq^2}{m\pi}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{1}{\omega'\left(\omega'-\omega\right)} \tag{E.2.39}$$

$$= \frac{\pi n e^2}{m}\delta(\omega). \tag{E.2.40}$$

Let's quickly prove this: consider the original integral

$$I = \mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{1}{\omega'(\omega'-\omega)}, \tag{E.2.41}$$

and define for infinitesimal ϵ and η

$$I_{\epsilon,\eta} = \overbrace{\int_{-\infty}^{-\eta}\mathrm{d}\omega'\frac{1}{\omega'(\omega'-\omega+i\epsilon)}}^{=:(*)} + \int_{\eta}^{\infty}\mathrm{d}\omega'\frac{1}{\omega'(\omega'-\omega+i\epsilon)}, \tag{E.2.42}$$

9 Alternatively, one can observe that the integral $\int_{-\infty}^{\infty}\mathrm{Re}\left[\sigma(\omega)\right] = \frac{\pi n q^2}{m}$ is independent of τ, and thus infer the presence of the δ-function in the real part of the optical conductivity in the superconducting limit. This is an example of a sum rule; this particular one is known as the Ferrell-Glover-Tinkham sum rule.

which we have chosen such that $I = \lim_{\epsilon,\eta\downarrow 0} \mathrm{Re}[I_{\epsilon,\eta}]$. Next, note that under a change of variables $\chi' = -\omega'$, we have

$$(*) = \int_\eta^\infty \mathrm{d}\chi' \, \frac{1}{\chi'(\chi' + \omega - i\epsilon)}, \tag{E.2.43}$$

so that

$$\begin{aligned} I_{\epsilon,\eta} &= \int_\eta^\infty \mathrm{d}\omega' \left[\frac{1}{\omega'(\omega' - \omega + i\epsilon)} + \frac{1}{\omega'(\omega' + \omega - i\epsilon)} \right] && \text{(E.2.44)} \\ &= \int_\eta^\infty \mathrm{d}\omega' \, \frac{2}{\omega'^2 - (\omega - i\epsilon)^2} && \text{(E.2.45)} \\ &= - \left. \frac{\tanh^{-1}\left(\frac{\omega'}{\omega - i\epsilon}\right)}{\omega - i\epsilon} \right|_{\omega'=\eta}^{\omega'=\infty} . && \text{(E.2.46)} \end{aligned}$$

Taking the limit $\eta \to 0$ and using $\lim_{x\to\infty} \tanh^{-1}(x) = -\frac{i\pi}{2}$, we find that

$$I_\epsilon := \lim_{\eta\downarrow 0} I_{\epsilon,\eta} = \frac{\pi}{i\,(\omega - i\epsilon)}, \tag{E.2.47}$$

the real part of which is

$$\mathrm{Re}\,[I_\epsilon] = \frac{\pi\epsilon}{\omega^2 + \epsilon^2}, \tag{E.2.48}$$

which, as $\epsilon \to 0$, converges to $I = \pi^2\delta(\omega)$—this can be seen as follows for some smooth test function f,

$$\begin{aligned} \int_{-\infty}^\infty \mathrm{d}t f(t) \frac{\epsilon}{t^2 + \epsilon^2} &\overset{t=\epsilon u}{=} \int_{-\infty}^\infty \mathrm{d}x f(\epsilon x) \frac{1}{1+x^2} && \text{(E.2.49)} \\ &\overset{\epsilon\to 0}{\longrightarrow} f(0) \int_{-\infty}^\infty \mathrm{d}x \, \frac{1}{1+x^2} && \text{(E.2.50)} \\ &= \pi f(0). && \text{(E.2.51)} \end{aligned}$$

This characteristic δ-function behaviour of the conductivity in the superconducting limit is something we will see reproduced by our holographic model.

E.2.2.2 *Maxwell Perturbations: Fluctuations of A_x in the Bulk*

Consulting again the dictionary of 2.3, we see that if we want a current in our system, we need to introduce a spatial component of our gauge field with some finite frequency ω; following [5] we pick A_x—this is of no consequence due to the symmetry of the problem. We are interested in the zero-momentum (in the AdS-Sch geometry, the zero-momentum assumption decouples A_x from the other polarizations; if we did not make this assumption there would be interactions between the fluctuations that would have to be diagonalized, see [15]) conductivity and as such we take $A_x(t, r) = a_x(r)e^{-i\omega t}$, implying that the Maxwell equation for A_x reads:

$$\nabla_\mu F^{\mu x} = 2g^{xx} A_x \phi^2 = \frac{2A_x\phi^2}{r^2}, \tag{E.2.52}$$

where

$$\begin{aligned} \nabla_\mu F^{\mu x} &= \frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} g^{\sigma x} F_{\nu\sigma} \right] && \text{(E.2.53)} \\ &= \frac{1}{r^2} \partial_\mu \left[r^2 g^{\mu\nu} g^{xx} F_{\nu x} \right] && \text{(E.2.54)} \\ &= \frac{1}{r^2} \partial_r (f A_x') + \frac{\omega^2 A_x}{r^2 f}, && \text{(E.2.55)} \end{aligned}$$

so the full equation of motion becomes (where we have divided out the common exponential)

$$a_x'' + \frac{f'}{f} a_x' + \left(\frac{\omega^2}{f^2} - \frac{2\phi^2}{f} \right) a_x = 0. \tag{E.2.56}$$

To ensure causality, we must impose in-falling boundary conditions at the horizon, $r = 1$; this imposition of boundary conditions at the horizon implies that at the horizon, we have $a_x \sim f^{-i\omega/3}$ (see [43, 204]). Similarly, performing the near-boundary analysis of a_x, we obtain the equation $(\alpha_{A_x} + 1)\alpha_{A_x} = 0$ implying that $\alpha_{A_x} = 0, 1$, and thus,

$$a_x = A^{(0)} + \frac{A^{(1)}}{r} + \dots, \tag{E.2.57}$$

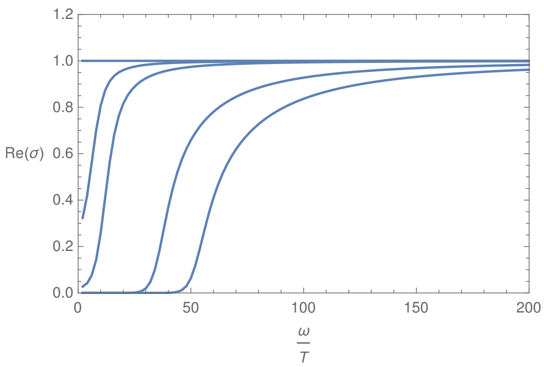
where $A_x^{\text{bd}} = A_x^{(0)}$ is the gauge field on the boundary (the source), satisfying $\partial_t A_x^{\text{bd}} = -E_x = -i\omega A_x^{\text{bd}}$, where E_x is the electric field on the boundary. The other term, $\langle J_x \rangle = A_x^{(1)}$, represents the expectation value of the current. From Ohm's law, then, we obtain the following optical conductivitys

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -\frac{iA_x^{(1)}}{\omega A_x^{(0)}}. \tag{E.2.58}$$

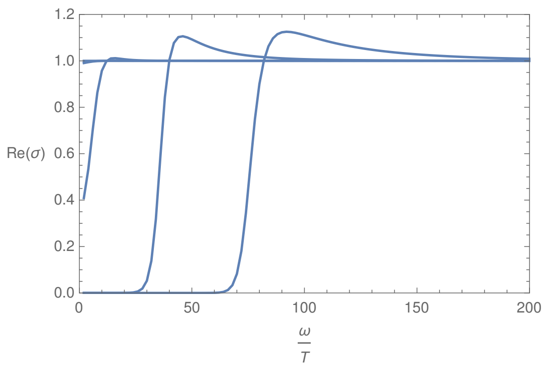
Below, we plot the conductivity as a function of the frequency. At $\omega = 0$, a δ-function appears when $T < T_c$: although it cannot be seen by the numerics, its existence may again be inferred by use of the Kramers-Kronig relation (or, alternatively—as before—by the Ferrell-Glover-Tinkham sum rule),

$$\text{Im}\left[\sigma(\omega)\right] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{\text{Re}\left[\sigma(\omega')\right]}{\omega' - \omega}, \tag{E.2.59}$$

from which we infer, from our previous considerations, that the real part of the conductivity contains a δ-function, $\text{Re}[\sigma(\omega)] \sim \pi\delta\omega$, provided that $\text{Im}[\sigma(\omega)]$ has a simple pole, $\text{Im}[\sigma(\omega)] \sim -\frac{1}{\omega}$; this is indeed the case [5, 6]. It is interesting to note that had we *not* simplified our problem by considering the probe limit but instead considered the full AdS-RN spacetime, the δ-function is present even when $T > T_c$, which seemingly presents a problem: the DC conductivity is infinite even in the normal phase! However, this infinite conductivity is not true superconductivity but rather an artefact of the translation invariance of the system; fixing the background the AdS-Schw in the probe limit implicitly breaks translation invariance by decoupling electric and energy currents [6]. It is, however, possible to break the translation invariance of the AdS-RN setup by introducing impurities—this was considered in [205].



(a)



(b)

Figure E.2: Conductivities for $\mathcal{O}_1$ (left) and $\mathcal{O}_2$ (right). At the origin, there is a δ-function unseen by the numerics, hile the horizontal line is the conductivity for $T = T_c$.

The horizontal line in both figures represents the conductivity in the case $T \geq T_c$—this is interesting because it was shown in [15] the conductivity is independent of frequency in the normal phase is characteristic of field theories with AdS_4 duals. The following curves describe successively lower values of the temperature with fixed charge density.

F

HOLOGRAPHIC RENORMALIZATION: SUPPLEMENTARY MATERIAL

This appendix deals with additional approaches to holographic renormalization, as advertised in the main text. We begin by reviewing the original FG approach in section F.1 and then proceed to discuss the dBVV ansatz method in section F.2.

F.1 FEFFERMAN-GRAHAM HOLOGRAPHIC RENORMALIZATION

Following the review article [48], we describe the FG approach to holographic renormalizatoin. In this section, we will use Euclidean signature AlAdS space (see appendix A), where the Fefferman-Graham metric (A.2.6) assumes the form

$$ds^2 = L^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j \right), \tag{F.1.1}$$

with an expansion for g_{ij} entirely analogous to (A.2.8). Suppressing all spacetime and internal indices, a generic bulk field $\mathcal{F}(\rho, x)$ has a near-boundary Fefferman-Graham asymptotic expansion of the form

$$\mathcal{F}(x,\rho) = \rho^m \left(f_{(0)}(x) + \rho f_{(2)}(x) + \cdots + \rho^n \left[f_{(2n)} + \log(\rho) \tilde{f}_{(2n)}(x) \right] + \cdots \right). \tag{F.1.2}$$

Generally, the equations of motion for $\mathcal{F}(x,\rho)$ are second order differential equations, which gives two independent solutions with asymptotic behaviors ρ^m and ρ^{m+n}—this is just a generalization of what we saw in (2.4.9), where a similar solution arises for the scalar. Just as for the scalar case, we identify $f_{(0)}(x)$, which multiplies the leading behavior, with the source for the dual operator. By solving the equations of motion order by order in ρ (which is treated as a small parameter in the near-boundary analysis), one obtains algebraic equations for $f_{(2k)}(x)$ for $k < n$ in terms of $f_{(0)}(x)$ and its derivatives up to order $2k$. This procedure, however, does not determine $f_{(2n)}$, which is associated with the VEV of the dual operator. In summary, the equations of motion imply that [48]

- $f_{(0)}(x)$ is the source of the dual field theory operator,
- $f_{(2)}, \ldots, f_{(2n-2)}$ and $\tilde{f}_{(2n)}$ are uniquely determined by the equations of motion and are local functions of $f_{(0)}$,
- $\tilde{f}_{(2n)}$ is related to conformal anomalies,
- $f_{(2n)}$, which is related to the VEV of the dual field theory operator, is undetermined by the near-boundary analysis.

Thus equipped with a generic asymptotic solution to the given field equations of motion, we proceed to calculate the on-shell value of the bulk action. In order to do so, we need to regularize: this is implemented by restricting the range of the ρ integration, $\rho \geq \epsilon$ for $\epsilon \ll 1$. We then evaluate the boundary terms at $\rho = \epsilon$, a finite number of which will diverge in the limit $\epsilon \to 0$—thus, we can write the regulated on-shell action in the following manner:

$$S_{\text{reg}}^{\text{on-shell}}[f_{(0)}, \epsilon] = \int_{\rho=\epsilon} d^d x \, \sqrt{g_{(0)}} \left[\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu+1)} a_{(2)} + \cdots - \log(\epsilon) \, a_{(2\nu)} + \mathcal{O}(\epsilon^0) \right], \tag{F.1.3}$$

where $\nu > 0$ depends only on the conformal dimension on the dual operator, and the coefficients $a_{(k)}$ are local functions of the source $f_{(0)}$, whereas $a_{(2\nu)}$ is directly related to the conformal anomaly [169]. The counterterm action is then defined as the divergent (in the $\epsilon \to 0$ limit) part of the regularized action,

$$S_{\text{ct}}[\mathcal{F}(x,\epsilon); \epsilon] = -\text{divergent terms of } S_{\text{reg}}^{\text{on-shell}}[f_{(0)}(\mathcal{F}(x,\epsilon)), \epsilon], \tag{F.1.4}$$

where the divergent terms are expressed in terms of the fields $\mathcal{F}(x,\epsilon)$ living at the regulated hypersurface $\rho = \epsilon$. This is required for covariance of the counterterm action, and consequently it is necessary to invert the relation (F.1.2) up to the required order—this inversion is precisely what makes the FG approach so cumbersome. When the counterterm has been covariantized, we may define the subtracted action,

$$S_{\text{sub}}[\mathcal{F}(x,\epsilon);\epsilon] = S_{\text{reg}}^{\text{on-shell}}[f_{(0)},\epsilon] + S_{\text{ct}}[\mathcal{F}(x,\epsilon);\epsilon], \tag{F.1.5}$$

in terms of which the renomarlized on-shell action is obtained as the limit $\epsilon \to 0$,

$$S_{\text{ren}}^{\text{on-shell}}[f_{(0)}] = \lim_{\epsilon\to 0} S_{\text{sub}}[\mathcal{F}(x,\epsilon);\epsilon]. \tag{F.1.6}$$

Using the renormalized on-shell action, we can define exact one-point functions (VEVs) of the field theory operator $\mathcal{O}_{\mathcal{F}}$ dual to the bulk field $\mathcal{F}$ as the variation of $S_{\text{ren}}^{\text{on-shell}}$ with respect to the boundary source, and since $g_{(0)ij}$ is in general non-trivial, we obtain (the subscript s stands for source)

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}^{\text{on-shell}}[f_{(0)}]}{\delta f_{(0)}}. \tag{F.1.7}$$

It can also be computed in terms of fields living at the regulated boundary,

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-m}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\text{sub}}[\mathcal{F}(x,\epsilon);\epsilon]}{\delta \mathcal{F}(x,\epsilon)}\right), \tag{F.1.8}$$

where $\gamma_{ij} = g_{ij}(x,\epsilon)/\epsilon$ is the induced metric on the regulated surface $\rho = \epsilon$. One can explicitly evaluate this limit, which gives the result [48]

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s \sim f_{2n} + \mathcal{C}(f_{(0)}), \tag{F.1.9}$$

where the scheme-dependent[1] function $\mathcal{C}(f_{(0)})$ depends locally on $f_{(0)}$, while the coefficient in front of $f_{(2n)}$ is scheme-independent.

F.1.1 *FG Holographic Renormalization of Pure Gravity in AlAdS$_5$*

To illustrate the application of the method described above, The Einstein-Hilbert action with the Gibbons-Hawking term[2] for $\mathcal{M} = \text{AlAdS}_5$ with $\Lambda = -\frac{d(d-1)}{2} = -6$,

$$S = \int_{\mathcal{M}} \mathrm{d}^5x\,\sqrt{-G}\left(R^{(G)} + 12\right) + 2\oint_{\partial\mathcal{M}} \mathrm{d}^4x\,\sqrt{-\gamma}K, \tag{F.1.10}$$

where $\gamma_{\mu\nu}$ is the metric induced on the boundary, while K is the trace of the extrinsic curvature, $K = \nabla_m n^m$, where n^m is a unit normal vector to the boundary $\partial\mathcal{M}$. The AlAdS$_5$ (with $L = 1$) metric has the Fefferman-Graham form of (A.2.6),

$$G_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{\mathrm{d}\hat{\rho}^2}{4\hat{\rho}^2} + \frac{1}{\hat{\rho}}g_{\mu\nu}(\hat{\rho},x)\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{F.1.11}$$

where for $d = 4$, the expansion (A.2.8) reads

$$g_{\mu\nu}(x,\hat{\rho}) = g_{(0)\mu\nu}(x) + \hat{\rho}g_{(2)\mu\nu}(x) + \hat{\rho}^2 g_{(4)\mu\nu} + \hat{\rho}^2\log(\hat{\rho})h_{(4)\mu\nu}(x) + \mathcal{O}(\hat{\rho}^3). \tag{F.1.12}$$

The boundary is located at $\hat{\rho} = 0$, and the corresponding boundary metric[3], therefore, is $g_{(0)\mu\nu}(x)$. The reason why we can't use this metric immediately is, of course, that in the process of holographic renormalization we need to introduce a cut-off hypersurface at $\hat{\rho} = \epsilon$, so we need the full expansion. The equations of motion from varying the action (F.1.10) produces the usual Einstein equation,

$$R_{mn} - \frac{1}{2}RG_{mn} - 6G_{mn} = 0. \tag{F.1.13}$$

In our calculations, we also require the inverse metric. In order to obtain it, we start with the ansatz

$$g^{\mu\nu} = \tilde{g}^{\mu\nu}_{(0)} + \hat{\rho}\tilde{g}^{\mu\nu}_{(2)} + \hat{\rho}^2\tilde{g}^{\mu\nu}_{(4)} + \cdots, \tag{F.1.14}$$

1 Since we have only subtracted the divergences, the scheme we have used to obtain (F.1.6) is, by analogy to the procedure in quantum field theory, known as minimal subtraction.

2 The Gibbons-Hawking term is required to make the variational problem for gravity well posed, i.e. to ensure that we can impose Dirichlet boundary conditions consistently; see [206].

3 Setting $g_{(0)\mu\nu} = \eta_{\mu\nu}$ corresponds to imposing AAdS$_5$ boundary conditions.

where $\tilde{g}^{\mu\nu}_{(4)}$ contains logarithmic terms in $\hat{\rho}$. We have that G_{mn} is block diagonal

$$G_{mn} = \begin{pmatrix} \frac{1}{4\hat{\rho}^2} & 0 \\ 0 & \frac{1}{\hat{\rho}} g_{\mu\nu} \end{pmatrix}, \tag{F.1.15}$$

so its inverse is

$$G^{mn} = \begin{pmatrix} 4\hat{\rho}^2 & 0 \\ 0 & \hat{\rho} g^{\mu\nu} \end{pmatrix}, \tag{F.1.16}$$

which shows that $g_{\mu\nu}(\hat{\rho} = \text{cst.})$ acts as a metric on slices of constant $\hat{\rho}$. In particular, it works for $\hat{\rho} = 0$, implying that the expansion component $g_{(0)\mu\nu}$ is the metric on the boundary. Therefore, we can write

$$g^{\mu\nu} = g^{\mu\lambda} g^{\nu\sigma} g_{\lambda\sigma}, \tag{F.1.17}$$

we—by inserting the metric expansion (F.1.12) and the ansatz (F.1.14)—get to lowest order in $\hat{\rho}$:

$$\mathcal{O}(\hat{\rho}^0) : \tilde{g}^{\mu\nu}_{(0)} = \tilde{g}^{\mu\lambda}_{(0)} \tilde{g}^{\nu\sigma}_{(0)} g_{(0)\lambda\sigma}, \tag{F.1.18}$$

implying that $\tilde{g}^{\nu\sigma}_{(0)} g_{(0)\lambda\sigma} = \delta^\nu_\lambda$; that is, $\tilde{g}^{\mu\nu}_{(0)}$ is the inverse of $g_{(0)\mu\nu}$ (as expected, since we argued that it works as proper metric), so we'll drop the tilde on this component. Similarly, to first order in $\hat{\rho}$, we get

$$\mathcal{O}(\hat{\rho}) : \tilde{g}^{\mu\nu}_{(2)} = \tilde{g}^{\mu\lambda}_{(2)} \overbrace{g^{\nu\sigma}_{(0)} g_{(0)\lambda\sigma}}^{=\delta^\nu_\lambda} + \tilde{g}^{\nu\sigma}_{(2)} \overbrace{g^{\mu\lambda}_{(0)} g_{(0)\lambda\sigma}}^{\delta^\mu_\sigma} + g^{\mu\lambda}_{(0)} g^{\nu\sigma}_{(0)} g_{(2)\lambda\sigma}, \tag{F.1.19}$$

so that

$$\tilde{g}^{\mu\nu}_{(2)} = -g^{\mu\lambda}_{(0)} g_{(2)\lambda\sigma} g^{\sigma\nu}_{(0)}, \tag{F.1.20}$$

and finally,

$$\mathcal{O}(\hat{\rho}^2) : \tilde{g}^{\mu\nu}_{(4)} = 2\tilde{g}^{\mu\nu}_{(4)} + g^{\mu\lambda}_{(0)} g_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)} + g^{\mu\lambda}_{(0)} g^{\nu\sigma}_{(0)} \log(\hat{\rho}) h_{(4)\lambda\sigma} + \tilde{g}^{\mu\lambda}_{(2)} \tilde{g}^{\nu\sigma}_{(2)} g_{(0)\lambda\sigma} + g^{\mu\lambda}_{(0)} \tilde{g}^{\nu\sigma}_{(2)} g_{(2)\lambda\sigma} + \tilde{g}^{\mu\sigma}_{(2)} g^{\nu\sigma} g_{(2)\lambda\sigma}, \tag{F.1.21}$$

which, upon insertion of the previous result for $\tilde{g}^{\mu\nu}_{(2)}$, gives:

$$\tilde{g}^{\mu\nu}_{(4)} = -g^{\mu\lambda}_{(0)} g^{\nu\sigma}_{(0)} \log(\hat{\rho}) h_{(4)\lambda\sigma} - g^{\mu\lambda}_{(0)} g_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)} - \cancel{g^{\mu\alpha}_{(0)} g_{(2)\alpha\beta} g^{\beta\lambda}_{(0)} g^{\nu\gamma}_{(0)} g_{(2)\gamma\delta} g^{\delta\sigma}_{(0)} g_{(0)\lambda\sigma}} \tag{F.1.22}$$

$$+ g^{\mu\lambda}_{(0)} g^{\nu\alpha}_{(0)} g_{(2)\alpha\beta} g^{\beta\sigma}_{(0)} g_{(2)\lambda\sigma} + \cancel{g^{\mu\alpha}_{(0)} g_{(2)\alpha\beta} g^{\beta\lambda}_{(0)} g^{\nu\sigma}_{(0)} g_{(2)\lambda\sigma}} \tag{F.1.23}$$

$$= -\log(\hat{\rho}) g^{\mu\lambda}_{(0)} h_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)} + g^{\mu\lambda}_{(0)} g_{(2)\lambda\sigma} g^{\sigma\beta}_{(0)} g_{(2)\beta\alpha} g^{\alpha\nu}_{(0)} - g^{\mu\lambda}_{(0)} g_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)}, \tag{F.1.24}$$

and so the inverse metric takes the full form:

$$g^{\mu\nu} = g^{\mu\nu}_{(0)} - \hat{\rho} g^{\mu\lambda}_{(0)} g_{(2)\lambda\sigma} g^{\sigma\nu}_{(0)} - \hat{\rho}^2 \log(\hat{\rho}) g^{\mu\lambda}_{(0)} h_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)} + \hat{\rho}^2 \left(g^{\mu\lambda}_{(0)} g_{(2)\lambda\sigma} g^{\sigma\beta}_{(0)} g_{(2)\beta\alpha} g^{\alpha\nu}_{(0)} - g^{\mu\lambda}_{(0)} g_{(4)\lambda\sigma} g^{\sigma\nu}_{(0)} \right) + \mathcal{O}(\hat{\rho}^4). \tag{F.1.25}$$

Using the trace-reversed form of Einstein's equations, the equations of motion take the form

$$R^{(G)}_{mn} = -4G_{mn}. \tag{F.1.26}$$

We begin by computing the Christoffel symbols Γ^ℓ_{mn}; there are six classes of these distinguished by their index structure

$$\Gamma^\lambda_{\mu\nu}, \quad \Gamma^{\hat{\rho}}_{\mu\nu}, \quad \Gamma^\lambda_{\hat{\rho}\nu}, \quad \Gamma^{\hat{\rho}}_{\hat{\rho}\nu}, \quad \Gamma^\lambda_{\hat{\rho}\hat{\rho}}, \quad \Gamma^{\hat{\rho}}_{\hat{\rho}\hat{\rho}}. \tag{F.1.27}$$

Now, since G is block diagonal, $\Gamma^\lambda_{\mu\nu} = \Gamma^{(g)\lambda}_{\mu\nu}$, where $\Gamma^{(g)\lambda}_{\mu\nu}$ is given entirely in terms of g. Block diagonality also implies that $\Gamma^{\hat{\rho}}_{\hat{\rho}\mu} = 0, \Gamma^\lambda_{\hat{\rho}\hat{\rho}} = 0$. The others evaluate to

$$\Gamma^{\hat{\rho}}_{\mu\nu} = 2g_{\mu\nu} - 2\hat{\rho} g'_{\mu\nu}, \quad \Gamma^\lambda_{\hat{\rho}\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g'_{\sigma\nu} - \frac{1}{\hat{\rho}} g_{\mu\sigma} \right), \quad \Gamma^{\hat{\rho}}_{\hat{\rho}\hat{\rho}} = -\frac{1}{\hat{\rho}}, \tag{F.1.28}$$

where we have defined $g'_{\mu\nu} =: \frac{\partial}{\partial\hat{\rho}} g_{\mu\nu}$. The $(\mu\nu)$-component of the Ricci tensor (for G) then assumes the form

$$R^{(G)}_{\mu\nu} = \partial_m \Gamma^m_{\mu\nu} - \partial_\nu \Gamma^\ell_{\mu\ell} + \Gamma^m_{\mu\nu}\Gamma^\ell_{\ell m} - \Gamma^m_{\mu\ell}\Gamma^\ell_{\nu m} \tag{F.1.29}$$

$$= \partial_{\hat{\rho}} \Gamma^{\hat{\rho}}_{\mu\nu} - \partial_\nu \overbrace{\Gamma^{\hat{\rho}}_{\mu\hat{\rho}}}^{=0} + \Gamma^{\hat{\rho}}_{\mu\nu}\Gamma^\ell_{\ell\hat{\rho}} + \Gamma^\sigma_{\mu\nu} \overbrace{\Gamma^{\hat{\rho}}_{\hat{\rho}\sigma}}^{=0} - \Gamma^{\hat{\rho}}_{\mu\ell}\Gamma^\ell_{\nu\hat{\rho}} - \Gamma^\sigma_{\mu\hat{\rho}}\Gamma^{\hat{\rho}}_{\nu\sigma} + R^{(g)}_{\mu\nu} \tag{F.1.30}$$

$$= g_{\mu\nu} g^{\sigma\lambda} g'_{\lambda\sigma} - 2g'_{\mu\nu} + \hat{\rho}\left(2g^{\lambda\sigma} g'_{\sigma\mu} g'_{\lambda\nu} - 2g''_{\mu\nu} - g'_{\mu\nu} g^{\sigma\lambda} g'_{\lambda\sigma}\right) + R^{(g)}_{\mu\nu} - 4\hat{\rho}^{-1} g_{\mu\nu}. \tag{F.1.31}$$

This must be set equal to $-4G_{\mu\nu}$, which will just eat the last term, producing the equation

$$g_{\mu\nu} g^{\sigma\lambda} g'_{\lambda\sigma} + 2g'_{\mu\nu} + \hat{\rho}\left(2g^{\lambda\sigma} g'_{\sigma\mu} g'_{\lambda\nu} - 2g''_{\mu\nu} - g'_{\mu\nu} g^{\sigma\lambda} g'_{\lambda\sigma}\right) + R^{(g)}_{\mu\nu} = 0, \tag{F.1.32}$$

There are two other equations corresponding to the $R^{(G)}_{\hat{\rho}\hat{\rho}}$ and $R^{(G)}_{\hat{\rho}\mu}$ components. They read

$$g^{\mu\nu} g''_{\mu\nu} - \frac{1}{2} g^{\mu\lambda} g'_{\lambda\sigma} g^{\sigma\nu} g'_{\nu\mu} = 0 \text{ (from } R^{(G)}_{\hat{\rho}\hat{\rho}}\text{)}, \tag{F.1.33}$$

$$g^{\nu\lambda}\left(\nabla^{(g)}_\mu g'_{\nu\lambda} - \nabla^{(g)}_\lambda g'_{\mu\nu}\right) = 0 \text{ (from } R^{(G)}_{\hat{\rho}\mu}\text{)}. \tag{F.1.34}$$

Considering the Fefferman-Graham expansion to be a perturbation in the small parameter (we're eventually interested in the limit $\hat{\rho} \to 0$) $\hat{\rho}$ around $g_{(0)\mu\nu}$, all indices are raised and lowered with $g_{(0)}$. Plugging the Fefferman-Graham expansion into the equations of motion using the *Mathematica* package *xAct* [88] with the subroutine *xTras* [108], the coefficients work out to be

$$g_{(2)\mu\nu} = \frac{1}{12} R_{(0)} g_{(0)\mu\nu} - \frac{1}{2} R_{(0)\mu\nu}, \tag{F.1.35}$$

where $R_{(0)\mu\nu}$ is the Ricci tensor of the boundary metric $g_{(0)\mu\nu}$. The precise form of $g_{(4)\mu\nu}$ is undetermined by this perturbative analysis, but we *can* say something about the trace and the divergence using the equation of motion (F.1.34); the full expressions can be found in [86]. The trace, however, will be important later—it is given by

$$g^{\mu\nu}_{(0)} g_{(4)\mu\nu} = g^\mu_{(4)\mu} = \frac{1}{4} g_{(2)\mu\nu} g^{\mu\nu}_{(2)}. \tag{F.1.36}$$

The coefficient $h_{(4)\mu\nu}$ satisfies

$$g^{\mu\nu}_{(0)} h_{(4)\mu\nu} = 0, \tag{F.1.37}$$

$$\nabla^{(0)\mu} h_{(4)\mu\nu} = 0. \tag{F.1.38}$$

The next step in the renormalization procedure is to compute the regularized on-shell action; imposing $R^{(G)} = -20$ and $K = \nabla_m n^m$ with $n^\mu = n\delta^m_{\hat{\rho}}$, so normality requires $n = 2\hat{\rho}$, so that $\nabla_m n^m = \frac{2}{\sqrt{-G}} \partial_{\hat{\rho}}\left(\hat{\rho}\sqrt{-G}\right)$. Further, block diagonality of G implies that $\sqrt{-G} = \frac{1}{2\hat{\rho}^3}\sqrt{-g}$, so that the regularized Gibbons-Hawking term becomes

$$2\sqrt{-\gamma}K\Big|_{\hat{\rho}=\epsilon} = 2\hat{\rho}^{-2}\sqrt{-g}K\Big|_{\hat{\rho}=\epsilon} = \sqrt{-g}\frac{4\hat{\rho}^{-2}}{\sqrt{-G}}\partial_{\hat{\rho}}\left(\hat{\rho}\sqrt{-G}\right)\Big|_{\hat{\rho}=\epsilon} \tag{F.1.39}$$

$$= 4\hat{\rho}\partial_{\hat{\rho}}\left(\frac{\sqrt{-g}}{\hat{\rho}^2}\right)\Big|_{\hat{\rho}=\epsilon} \tag{F.1.40}$$

$$= 4\hat{\rho}^{-1}\partial_{\hat{\rho}}\left(\sqrt{-g}\right)\Big|_{\hat{\rho}=\epsilon} - 4\hat{\rho}^{-2}\sqrt{-g}\Big|_{\hat{\rho}=\epsilon}, \tag{F.1.41}$$

which produces the on-shell action

$$S^{\text{on-shell}}_{\text{reg}}[g,\epsilon] = \oint_{\partial\mathcal{M}_\epsilon} \mathrm{d}^4x \left[-\int_\infty^\epsilon \mathrm{d}\hat{\rho}\, 4\hat{\rho}^{-3}\sqrt{-g} + 4\hat{\rho}^{-1}\partial_{\hat{\rho}}\left(\sqrt{-g}\right)\Big|_{\hat{\rho}=\epsilon} - 8\hat{\rho}^{-2}\sqrt{-g}\Big|_{\hat{\rho}=\epsilon}\right]. \tag{F.1.42}$$

In order to identify the divergent terms, we need to expand the determinant $\sqrt{-g}$, something we already did for the DBI action when "deriving" the AdS/CFT correspondence in section 2.3; replicating our method, we obtain

$$g_{\mu\nu} = g_{(0)\mu\lambda}\left[\delta^\lambda_\nu + \hat{\rho} g^\lambda_{(2)\nu} + \hat{\rho}^2 g^\lambda_{(4)\nu} + \hat{\rho}^2\log(\hat{\rho}) h^\lambda_{(4)\nu}\right], \tag{F.1.43}$$

implying that

$$\sqrt{-g} = \sqrt{-g_{(0)}}\sqrt{1+\hat{\rho} g^{\lambda}_{(2)\lambda} + \hat{\rho}^2\left[g^{\lambda}_{(4)\lambda} + \frac{1}{2}\left\{g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} - g^{\lambda}_{(2)\nu}g^{\nu}_{(2)\lambda}\right\}\right] + \hat{\rho}^2\log(\hat{\rho})h^{\lambda}_{(4)\lambda} + \mathcal{O}(\hat{\rho}^4)} \tag{F.1.44}$$

$$= \sqrt{-g_{(0)}}\sqrt{1+\hat{\rho} g^{\lambda}_{(2)\lambda} + \hat{\rho}^2\left(\frac{1}{2}g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} - \frac{1}{4}g_{(2)\lambda\nu}g^{\lambda\nu}_{(2)}\right) + \mathcal{O}(\hat{\rho}^4)} \tag{F.1.45}$$

$$= \sqrt{-g_{(0)}}\left(1+\frac{\hat{\rho}}{2}g^{\lambda}_{(2)\lambda} + \underbrace{\frac{\hat{\rho}^2}{8}\left[g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} - g^{\lambda\nu}_{(2)}g_{(2)\lambda\nu}\right]}_{(*)}\right) + \mathcal{O}(\hat{\rho}^4) \tag{F.1.46}$$

where we have used the relations (F.1.37) and (F.1.36) for $h_{(4)\mu\nu}$ as well as the series expansion $\sqrt{1+x} = 1 + x/2 - x^2/8 + \mathcal{O}(x^3)$. The integration over $\hat{\rho}$ can now be performed, which leads to the structure of the generic regularized action (F.1.3),

$$S^{\text{on-shell}}_{\text{reg}}[g_{(0)},\epsilon] = \int_{\partial\mathcal{M}_\epsilon} \mathrm{d}^4x\,\sqrt{g_{(0)}}\left[\epsilon^{-2}a_{(0)} + \epsilon^{-1}a_{(2)} - \log(\epsilon)\,a_{(4)} + \mathcal{O}(\epsilon^0)\right]. \tag{F.1.47}$$

In particular, the $\log(\epsilon)$ contribution comes from the term $(*)$ when plugged into the on-shell action (F.1.42), from which we infer that

$$a_{(4)} = \frac{1}{2}\left(g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} - g^{\lambda\nu}_{(2)}g_{(2)\lambda\nu}\right). \tag{F.1.48}$$

Performing the remaining integrations and derivatives and collecting orders of ϵ, we find that

$$a_{(0)} = -6, \quad a_{(2)} = 0. \tag{F.1.49}$$

This is in agreement with [86]. Next, we determine the counterterm action, which, in order to ensure covariance, requires us to express quantities at the boundary in terms of quantities on the regularized hypersurface $\hat{\rho} = \epsilon$, which has metric $\gamma_{\mu\nu} = \epsilon^{-1}g_{\mu\nu}(x,\epsilon)$. Inverting the expression for $\sqrt{-g}$ (F.1.46), we find that

$$\sqrt{-g_{(0)}} = \sqrt{-g}\left(1 - \frac{\epsilon}{2}g^{\lambda}_{(2)\lambda} + \frac{\epsilon^2}{8}\left(g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} + g^{\lambda}_{(2)\nu}g^{\nu}_{(2)\lambda}\right)\right) + \mathcal{O}(\epsilon^4) \tag{F.1.50}$$

$$= \sqrt{-\gamma}\epsilon^2\left(1 - \frac{\epsilon}{2}g^{\lambda}_{(2)\lambda} + \frac{\epsilon^2}{8}\left(g^{\lambda}_{(2)\lambda}g^{\nu}_{(2)\nu} + g^{\lambda}_{(2)\nu}g^{\nu}_{(2)\lambda}\right)\right) + \mathcal{O}(\epsilon^4). \tag{F.1.51}$$

Furthermore, taking the trace of (F.1.35) gives us the relation

$$g^{\lambda}_{(2)\lambda} = -\frac{1}{6}R_{(0)}, \tag{F.1.52}$$

so we now need to express $R_{(0)}$ in terms of $R^{(\gamma)}$. We can now do a series expansion in $\hat{\rho}$, and contract the resulting expression using the inverse hypersurface metric (F.1.25), in which case we find that

$$g^{\lambda}_{(2)\lambda} = \frac{1}{6}\frac{1}{\epsilon}\left(R_{(\gamma)} + \frac{1}{2}\left[R^{(\gamma)}_{\mu\nu}R^{(\gamma)\mu\nu} - \frac{1}{6}\left(R^{(\gamma)}\right)^2\right]\right) + \mathcal{O}\left(\left(R^{(\gamma)}\right)^3\right), \tag{F.1.53}$$

where terms cubic in curvatures do not contribute[4]. A similar analysis reveals that

$$g^{\lambda}_{(2)\nu}g^{\nu}_{(2)\lambda} = \frac{1}{\epsilon^2}\frac{1}{4}\left(R^{(\gamma)}_{\mu\nu}R^{(\gamma)\mu\nu} - \frac{2}{9}\left(R^{(\gamma)}\right)^2\right) + \mathcal{O}\left(\left(R^{(\gamma)}\right)^3\right). \tag{F.1.54}$$

Using this, the counterterm action assumes the form

$$S_{\text{ct}}[g(x,\epsilon);\epsilon] = -\int_{\partial\mathcal{M}_\epsilon}\mathrm{d}^4x\,\sqrt{-\gamma}\left[-6 + \frac{1}{2}R^{(\gamma)} - \log(\epsilon)\left(\frac{R^{(\gamma)2}}{12} - \frac{R^{(\gamma)\mu\nu}R^{(\gamma)}_{\mu\nu}}{4}\right)\right]. \tag{F.1.55}$$

This allows us to immediately compute various quantities such as one-point functions (VEVs) and Ward identities (see e.g. [95] for details on how to do this in a FG framework).

4 This statement is valid in dimensions below six.

F.2 THE DBVV ANSATZ APPROACH

Developed in [49] by de Boer and the Verlinde twins (hence the name, *dBVV*) and reviewed in e.g. [89, 207, 208], this method is the precursor of the approach discussed in chapter 3. It also makes use of the fact that the radial direction of AlAdS normal to the boundary allows us to foliate the space into slices of constant radius (at least near the boundary), which can be used to set up a radial analogue of the ADM formalism [98, 209]. This method can be extended to Lifshitz space-times: starting from a Einstein-Proca model, this was pursued in [25]. The drawback of this method, however, is the need to specify a counterterm ansatz, and it is generically very difficult to specify a sufficiently general ansatz.

F.2.1 *Setting up the ADM Hamiltonian and the Hamilton-Jacobi Equation*

For AlAdS manifolds (see appendix A, in particular (F.2.14)), we use the radial coordinate r, which is normal to the boundary, to foliate the manifold into radial hypersurfaces (leaves) Σ_r defined by $r =$ const.; these slices are diffeomorphic to the conformal boundary. Disregarding pathological topological complications, this is always possible (at least in the neighbourhood of the conformal boundary). The normal vector to Σ_r, which points in the direction of increasing r, can be written as $n^m = n\delta^m_r$, where the normalization is fixed by requiring $g_{mn}n^m n^n = 1$, allowing us to define a hypersurface projector $h_{mn} = g_{mn} - n_m n_n$ projecting onto Σ_r. Restricting the indices on the projector also gives us the induced metric, we may decompose the (Euclidean) metric as

$$ds^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = N^2\mathrm{d}r^2 + h_{\mu\nu}\left(\mathrm{d}x^\mu + N^\mu\mathrm{d}r\right)\left(\mathrm{d}x^\nu + N^\nu\mathrm{d}r\right), \tag{F.2.1}$$

where N is the shift and N^μ the lapse. The (Euclidean) Einstein-Hilbert action on a $(d+1)$-dimensional manifold $\mathcal{M}$ (with AlAdS boundary condition) and boundary $\Sigma_r = \partial\mathcal{M}|_r$ is given by,

$$S[g,h] = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\,\sqrt{g}\left(R^{(g)} - 2\Lambda\right) - 2\int_{\Sigma_r}\mathrm{d}^d x\,\sqrt{h}K, \tag{F.2.2}$$

where K is the trace of the extrinsic curvature on Σ_r. The ADM-parametrized version of this reads,

$$S[h,N] = \int_r^\infty \mathrm{d}r'\int\mathrm{d}^d x\,\sqrt{h}N\left(R^{(h)} - 2\Lambda + K_{\mu\nu}K^{\mu\nu} - K^2\right), \tag{F.2.3}$$

where the Gibbons-Hawking cancels out by partial integration of second derivative terms in the expansion of the Ricci scalar. Classically, the action above is equivalent to the following action,

$$S[\pi^{\mu\nu},N,N^\mu] = \int\mathrm{d}r\int\mathrm{d}^d x\,\sqrt{h}\left[\pi^{\mu\nu}\partial_r h_{\mu\nu} + N\mathcal{H} - N^\mu\mathcal{H}_\mu\right], \tag{F.2.4}$$

where $\sqrt{h}\pi^{\mu\nu}$ is the canonical momentum conjugate to the induced metric $h_{\mu\nu}$ on Σ_r, whereas

$$\mathcal{H} = R^{(h)} - 2\Lambda + \pi^{\mu\nu}\pi_{\mu\nu} - \frac{1}{d-1}\pi^2,\quad \mathcal{H}^\mu = 2D_\nu\pi^{\mu\nu}, \tag{F.2.5}$$

where D_ν is the covariant derivative on the hypersurface Σ_r. On a general hypersurface Σ, such a derivative is defined via $D_m T^{n\dots}{}_{\ell\dots} = h^n{}_k\cdots h^j{}_\ell\cdots h^i{}_m\nabla_i T^{k\dots}{}_{j\dots}$ for $T^{k\dots}{}_{j\dots}\in\Sigma$ and satisfies, in particular, metric compatibility with the hypersurface projector, $D_m h_{n\ell} = 0$. Upon noting that the Wick rotated Legendre transformation changes sign—schematically $H = \sum_i p_i\dot{q}_i + L$—we may construct the radial Hamiltonian on an equal-r slice Σ_r as follows,

$$H = \int_{\Sigma_r}\mathrm{d}^d x\,\sqrt{h}\left(N\mathcal{H} + N^\mu\mathcal{H}_\mu\right), \tag{F.2.6}$$

The radial Hamilton-Jacobi equation then takes the form

$$\partial_r S + H = 0,\quad \pi^{\mu\nu} = \frac{1}{\sqrt{h}}\frac{\delta S}{\delta h_{\mu\nu}}, \tag{F.2.7}$$

where S is the on-shell action—which is known as Hamilton's principal function in analytical mechanics (see [91])—where it is understood that the value of $\pi^{\mu\nu}$ given above is inserted into the HJ equation. Since S cannot depend on r (see [207]), the HJ equation reduces to

$$H = 0, \tag{F.2.8}$$

implying the *Hamiltonian constraint and the momentum constraint*

$$\mathcal{H} = 0, \quad \mathcal{H}_\mu = 0. \tag{F.2.9}$$

Alternatively, these are the equations of motion obtained from (F.2.4) by varying with respect the Lagrange multiplicators[5] N and N^μ—a point of view favoured in [51]. Note that the momentum $\pi^{\mu\nu}$ is directly related to the Brown-York stress tensor (see [210]) on Σ_r,

$$\hat{T}^{\mu\nu} = \pi^{\mu\nu}. \tag{F.2.10}$$

F.2.2 *Counterterm Action Ansatz*

As in the FG approach, we write the renormalized on-shell action as $S_{\text{ren}} = S + S_{\text{ct}}$, where the counterterm action is specified by the following locally covariant ansatz[6]

$$S_{\text{ct}} = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{h} \Big[c_0 + c_1 R^{(h)} + c_2^{(1)} {R^{(h)}}^2 + c_2^{(2)} R^{(h)}_{\mu\nu} R^{(h)\mu\nu} \tag{F.2.11}$$

$$+ c_2^{(3)} R^{(h)}_{\mu\nu\rho\sigma} R^{(h)\mu\nu\rho\sigma} + \mathcal{O}(R^3) \Big], \tag{F.2.12}$$

which, by (F.2.7), implies that the momentum splits up as $\pi^{\mu\nu} = \pi^{\mu\nu}_{\text{ren}} - \pi^{\mu\nu}_{\text{ct}}$, which is useful since covariance of our ansatz means that the momentum constraint associated with the counterterm vanishes identically by covariant conservation of the associated Brown-York (quasi.local) stress tensor, $D_\mu \pi^{\mu\nu}_{\text{ct}} = D_\mu \hat{T}^{\mu\nu}_{\text{ct}}$. Thus, the momentum constraint reduces to the d-dimensional diffeomorphism Ward identity on Σ_r,

$$0 = \mathcal{H}^\mu = D_\mu \hat{T}^{\mu\nu}_{\text{ren}}. \tag{F.2.13}$$

Employing domain-wall coordinates[7], where the pure AdS metric reads $ds^2 = \mathrm{d}r^2 + e^{2r}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)$, we see that the AlAdS boundary condition on the induced metric reads

$$\partial_r h_{\mu\nu} = 2h_{\mu\nu} + \mathcal{O}\left(e^{-r}\right). \tag{F.2.14}$$

Now, following [49], we use the diffeomorphism invariance to gauge fix $N = 1$ and $N^\mu = 0$; when this is done, the imposition of the asymptotic behaviour (F.2.14) is achieved via the Hamilton equation,

$$\partial_r h_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta H}{\delta \pi^{\mu\nu}} = 2\pi_{\mu\nu} - \frac{2}{d-1} \pi h_{\mu\nu}. \tag{F.2.15}$$

Now, want to impose (F.2.14), which is, of course, an asymptotic condition. Near the boundary, therefore—by assumption—only the counterterm action gives a contribution to $\pi_{\mu\nu}$; in particular, by varying (F.2.12), we obtain:

$$\pi_{\mu\nu} = -\frac{1}{\sqrt{h}} \frac{\delta S_{\text{ren}}}{\delta h^{\mu\nu}} + \mathcal{O}\left(e^{-r}\right) \tag{F.2.16}$$

$$= \frac{1}{2} h_{\mu\nu} c_0 + \mathcal{O}\left(e^{-r}\right), \tag{F.2.17}$$

where none of the curvatures contribute, that is, we effectively only vary $\int \mathrm{d}^d x \, \sqrt{h} c_0$. This means that

$$\partial_r h_{\mu\nu} = h_{\mu\nu} c_0 - \frac{c_0}{d-1} h_{\lambda\rho} h^{\lambda\rho} h_{\mu\nu} = h_{\mu\nu} c_0 \left(\frac{d-1-d}{d-1} \right) = -\frac{c_0}{d-1} h_{\mu\nu}. \tag{F.2.18}$$

The asymptotic boundary condition of (F.2.14) thus requires that:

$$c_0 = -2(d-1). \tag{F.2.19}$$

Let's just check the variation of the next term, $\int \mathrm{d}^d x \, \sqrt{h} R$, which becomes

$$\delta \left(\sqrt{h} R^{(h)} \right) = -\frac{1}{2} \sqrt{h} h_{\mu\nu} \delta h^{\mu\nu} R^{(h)} + R^{(h)}_{\mu\nu} \delta h^{\mu\nu} \sqrt{g}, \tag{F.2.20}$$

implying that the contribution to $\pi^{\mu\nu}$ takes the form

$$\pi^{(R)}_{\mu\nu} = -\frac{1}{2} R^{(h)} h_{\mu\nu} + R^{(h)}_{\mu\nu} = (R + \mathcal{K}) \, h_{\mu\nu} \sim e^{-2r} h_{\mu\nu}, \tag{F.2.21}$$

where $\mathcal{K}$ is a constant, so this will not contribute. The same goes for the higher order curvatures.

5 That is, non-dynamical (no kinetic terms).
6 This expression becomes more involved if we were to include more fields in our model.
7 The boundary in domain-wall coordinates is located at $r \to \infty$ since it is obtained from the usual Poincaré patch metric by $r \to e^{-r}$; in Poincaré coordinates, the boundary is located at $r = 0$.

F.2.3 *The DeWitt Bracket*

Having fixed the asymptotic boundary condition, which determined c_0, we're now ready to solve the Hamiltonian constraint, $\mathcal{H} = 0$. In order to accomplish this, it is convenient to introduce the following piece of notation—which we shall call the *DeWitt bracket* which is symmetric and bilinear—for the kinetic part of $\mathcal{H}$,

$$\{S, S\}_{\mathrm{DW}} := G_{\mu\nu\rho\sigma} \frac{\delta S}{\delta h_{\mu\nu}} \frac{\delta S}{\delta h_{\rho\sigma}}, \tag{F.2.22}$$

where $G_{\mu\nu\rho\sigma}$ is the DeWitt metric, given by

$$G_{\mu\nu\rho\sigma} = \frac{1}{h}\left(h_{\mu(\rho}h_{\sigma)\nu} - \frac{1}{d-1} h_{\mu\nu}h_{\rho\sigma}\right). \tag{F.2.23}$$

In terms of the DeWitt bracket, the Hamiltonian constraint in our gauge takes the form

$$\mathcal{H} = \{S, S\}_{\mathrm{DW}} + R^{(h)} - 2\Lambda. \tag{F.2.24}$$

Bilinearity and symmetry of the DeWitt bracket implies that upon writing $S = S_{\mathrm{ren}} + S_{\mathrm{ct}}$, the Hamiltonian constraint becomes

$$\mathcal{H} = \{S_{\mathrm{ren}}, S_{\mathrm{ren}}\}_{\mathrm{DW}} + \{S_{\mathrm{ct}}, S_{\mathrm{ct}}\}_{\mathrm{DW}} + 2\,\{S_{\mathrm{ct}}, S_{\mathrm{ren}}\}_{\mathrm{DW}} + R^{(h)} - 2\Lambda = 0. \tag{F.2.25}$$

The next step is to sort the Hamiltonian $\mathcal{H}$ by the number of radial derivatives n; schematically

$$\mathcal{H} = \sum_{n\geq 0} \mathcal{H}^{(2n)}. \tag{F.2.26}$$

As before, this means that each $\mathcal{H}^{(n)}$ vanishes in isolation. An analogous derivative expansion for $\sqrt{g}\mathcal{L}_{\mathrm{ct}}$ gives us that, in domain-wall coordinates, the asymptotic radial scaling for AlAdS geometries depends on the number of derivatives n as

$$\sqrt{g}\mathcal{L}_{\mathrm{ct}}^{(n)} \sim e^{(d-n)r}. \tag{F.2.27}$$

As before—by assumption—the renormalized action S_{ren} has a finite limit as $r \to \infty$; thus the relevant DeWitt brackets scale asymptotically as

$$\left\{S_{\mathrm{ct}}^{(m)}, S_{\mathrm{ct}}^{(n)}\right\}_{\mathrm{DW}} \sim e^{-(m+n)r}, \tag{F.2.28}$$

$$\left\{S_{\mathrm{ct}}^{(n)}, S_{\mathrm{ren}}\right\}_{\mathrm{DW}} \sim e^{-(d+n)r}, \tag{F.2.29}$$

$$\{S_{\mathrm{ren}}, S_{\mathrm{ren}}\}_{\mathrm{DW}} \sim e^{-2dr}. \tag{F.2.30}$$

The lowest-order Hamiltonian constraint, therefore, takes the form

$$0 = \mathcal{H}^{(0)} = \left\{S_{\mathrm{ct}}^{(0)}, S_{\mathrm{ct}}^{(0)}\right\}_{\mathrm{DW}} - 2\Lambda. \tag{F.2.31}$$

In particular, we find that

$$\left\{S_{\mathrm{ct}}^{(0)}, S_{\mathrm{ct}}^{(0)}\right\}_{\mathrm{DW}} = \frac{c_0^2}{h}\left(h_{\mu(\rho}h_{\sigma)\nu} - \frac{1}{d-1}h_{\mu\nu}h_{\rho\sigma}\right)\frac{1}{4}hh^{\mu\nu}h^{\rho\sigma} \tag{F.2.32}$$

$$= \frac{c_0^2}{4}\left(h_{\mu(\rho}h_{\sigma)\nu} - \frac{1}{d-1}h_{\mu\nu}h_{\rho\sigma}\right)h^{\mu\nu}h^{\rho\sigma} \tag{F.2.33}$$

$$= \frac{c_0^2}{4}\left(\frac{1}{2}\left(h_{\mu\rho}h_{\sigma\nu} + h_{\mu\sigma}h_{\rho\nu}\right)h^{\mu\nu}h^{\rho\sigma} - \frac{d^2}{d-1}\right) \tag{F.2.34}$$

$$= \frac{c_0^2}{4}\left(d - \frac{d^2}{d-1}\right) \tag{F.2.35}$$

$$= \frac{c_0^2}{4}\left(\frac{d^2-d}{d-1} - \frac{d^2}{d-1}\right) \tag{F.2.36}$$

$$= -\frac{c_0^2 d}{4(d-1)}. \tag{F.2.37}$$

This should be equal to

$$2\Lambda = -d(d-1), \tag{F.2.38}$$

implying that

$$c_0 = \pm 2(d-1), \tag{F.2.39}$$

which agrees with our previous boundary condition analysis, provided we choose the negative solution. It is amusing to note that this only fixes the square of c_0; we need the boundary conditions to fix the sign. The next Hamiltonian constraint (assuming $d > 2$—if we did not assume this, there'd be additional contributions from $\left\{S_{\text{ct}}^{(0)}, S_{\text{ren}}\right\}_{\text{DW}}$; we'll have more to say about this later when discussing the holographic Weyl anomaly), which has two derivatives, consequently reads

$$0 = \mathcal{H}^{(2)} = 2\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(2)}\right\}_{\text{DW}} + R^{(h)}, \tag{F.2.40}$$

where[8]

$$2\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(2)}\right\}_{\text{DW}} = -\frac{c_0 c_1 (d-2) R^{(h)}}{2(d-1)} = c_1(d-2)R^{(h)}, \tag{F.2.41}$$

implying that

$$c_1 = \frac{1}{d-2}. \tag{F.2.42}$$

The next constraint reads:

$$0 = \mathcal{H}^{(4)} = 2\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(4)}\right\}_{\text{DW}} + \left\{S_{\text{ct}}^{(2)}, S_{\text{ct}}^{(2)}\right\}_{\text{DW}}. \tag{F.2.43}$$

The first of these brackets work out to be,

$$2\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(4)}\right\}_{\text{DW}} {}^{9} = c_0 \frac{(4-d)}{2(d-1)} \left(c_2 R^{(h)2} + c_3 R_{\mu\nu}^{(h)} R^{(h)\mu\nu} + c_4 R_{\mu\nu\rho\sigma}^{(h)} R^{(h)\mu\nu\rho\sigma}\right) \tag{F.2.44}$$

$$+ \frac{c_0}{d-1}\left([2(1+d)c_2 + c_3]\, \nabla_\mu^{(h)} \nabla^{(h)\mu} R^{(h)} + [4c_4 + c_3(d-2)]\, \nabla_\mu^{(h)} \nabla_\nu^{(h)} R^{(h)\mu\nu}\right), \tag{F.2.45}$$

while the second takes the form:

$$\left\{S_{\text{ct}}^{(2)}, S_{\text{ct}}^{(2)}\right\}_{\text{DW}} = c_1^2 \left(R_{\mu\nu}^{(h)} R^{(h)\mu\nu} - \frac{dR^{(h)2}}{4(d-1)}\right). \tag{F.2.46}$$

From these, we immediately infer the equations

$$c_2 = \frac{d}{4(d-4)(d-1)(d-2)^2}, \quad c_3 = -\frac{1}{(d-4)(d-2)^2}, \quad c_4 = 0. \tag{F.2.47}$$

Setting $d = 4$, we reassuringly obtain the result we also got the the FG approach to holographic renormalization. It is worth noting, however, that the above is *not* the whole story. We have ignored certain additional terms involving covariant derivatives (produced by *xAct*), which do not cancel, suggesting that the ansatz employed is not sufficiently general. Curiously, the authors of [207] find similar terms, but claim that they cancel precisely.

F.2.4 *Holographic Weyl Anomalies*

In the Hamilton-Jacobi approach to holographic renormalization, a d-dimensional Weyl anomaly[10] appears when $\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(d)}\right\}_{\text{DW}}$ vanishes identically, which is the case when d is even. However, since

8 These variations are now somewhat tedious to perform, but they can be evaluated efficiently in *xAct*, which turns out to be admirably suited for this.

10 For a cool cohomological approach to Weyl anomalies, see [211, 212].

$\left\{S_{\text{ct}}^{(0)}, S_{\text{ren}}\right\}_{\text{DW}} \sim e^{-dr}$, the Hamiltonian constraint $\mathcal{H}^{(d)} = 0$ is non-trivial, and, in particular, involves the Weyl anomaly via

$$-2\left\{S_{\text{ct}}^{(0)}, S_{\text{ct}}^{(d)}\right\}_{\text{DW}} = \frac{2}{\sqrt{h}}\frac{\delta S_{\text{ren}}}{\delta h_{\mu\nu}} h_{\mu\nu} = h_{\mu\nu} T^{\mu\nu}_{\text{ren}}. \tag{F.2.48}$$

In two and four dimensions, the Hamiltonian constraints read

$$0 = \mathcal{H}^{(2)} = R - 2\left\{S_{\text{ct}}^{(0)}, S_{\text{ren}}\right\}_{\text{DW}} \qquad (d = 2) \tag{F.2.49}$$

$$0 = \mathcal{H}^{(4)} = \left\{S_{\text{ct}}^{(2)}, S_{\text{ct}}^{(2)}\right\}_{\text{DW}} - 2\left\{S_{\text{ct}}^{(0)}, S_{\text{ren}}\right\}_{\text{DW}} \qquad (d = 4), \tag{F.2.50}$$

which gives us, straightforwardly, the Weyl anomaly in the dimensions considered above,

$$h_{\mu\nu} T^{\mu\nu}_{\text{ren}} = -R^{(h)} \qquad (d = 2) \tag{F.2.51}$$

$$h_{\mu\nu} T^{\mu\nu}_{\text{ren}} = -\frac{1}{4}\left(R^{(h)}_{\mu\nu} R^{(h)\mu\nu} - \frac{{R^{(h)}}^2}{3}\right) \qquad (d = 4). \tag{F.2.52}$$

$$\tag{F.2.53}$$

G HOLOGRAPHIC RENORMALIZATION OF LIFSHITZ SPACE-TIMES

G.1 HOLOGRAPHIC RENORMALIZATION OF EINSTEIN-PROCA THEORY WITH LIFSHITZ BOUNDARY CONDITIONS

It is the purpose of this appendix to describe the process of holographic renormalization applied to Lifshitz space-times arising from Einstein-Proca models. The notation differs slightly from the one used in the main text (cf. chapter 3); we hope this will not cause any confusion.

G.1.1 *The Einstein-Proca ADM Hamiltonian*

The Lifshitz space-time arises as a solution to the Einstein-Proca action with a Gibbons-Hawking term,

$$S = \int \mathrm{d}^{d+1}x\, \sqrt{g}\left(\hat{R} - 2\Lambda - \frac{1}{4}\hat{F}^2 - \frac{1}{2}M^2\hat{A}^2\right) - 2\int_{\Sigma_r} \mathrm{d}^d x\, \sqrt{h}K, \tag{G.1.1}$$

where we have used hats to distinguish $(d+1)$-dimensional quantities from d-dimensional ones. In domain-wall coordinates, the pure Lifshitz solution has the form

$$ds^2 = \mathrm{d}r'^2 - e^{2zr'}\mathrm{d}t^2 + e^{2r'}\mathrm{d}\vec{x}^2. \tag{G.1.2}$$

Since the FG theorem does not apply to asymptotically locally Lifshitz, we must use either (i) the dBVV-deWitt method, which was done in [25], or (ii) the HJ approach described in chapter 3—this was done in [27, 96]. It is to this second approach that we will now turn our attention.

First, we sketch the derivation of the Einstein-Proca ADM Hamiltonian, starting from (G.1.1). First, we'll need to set up a codimension-1 foliation of $\mathcal{M}$ = AlLif in terms of radial hypersurfaces (as we pointed out when renormalizing AlAdS, it is sufficient to assume this can be done in a neighbourhood of the boundary). In general [213, 214], a non-zero one-form ω on $\mathcal{M}$ generates a codimension-1 foliation of $\mathcal{M}$ if and only if it is integrable, that is, it satisfies the Frobenius condition,

$$\omega \wedge \mathrm{d}\omega = 0. \tag{G.1.3}$$

which, in particular is satisfied by vectors n normal to some hyperplane, $n \wedge \mathrm{d}n = 0$. In our case, the vector normal to Σ_r—which is defined by r = const.—is[1] $n^{\hat{\mu}} = \frac{\hat{\nabla}^{\hat{\mu}}r}{|\hat{\nabla}^{\hat{\nu}}r\hat{\nabla}_{\hat{\nu}}r|^{1/2}}$, and the foliation one-form constructed from the normal vector is

$$n_{\hat{\mu}} = g_{\hat{\mu}\hat{\nu}}n^{\hat{\nu}} = N\hat{\nabla}_{\hat{\mu}}r, \tag{G.1.4}$$

where the normalization N is the lapse function. The hypersurface projector, which pulls back tensors on $\mathcal{M}$ onto Σ_r, is given by,

$$h_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} - n_{\hat{\mu}}n_{\hat{\nu}}, \tag{G.1.5}$$

whereas $n_{\hat{\mu}}n_{\hat{\nu}}$ projects onto $\Sigma_r^{\perp}$. The Proca field is thus decomposable in the following manner,

$$\hat{A}_{\hat{\mu}} = \Phi n_{\hat{\mu}} + A_{\hat{\mu}}, \quad \Phi = n^{\hat{\nu}}\hat{A}_{\hat{\nu}}, \quad A_{\hat{\mu}} = h_{\hat{\mu}}{}^{\hat{\nu}}\hat{A}_{\hat{\nu}}. \tag{G.1.6}$$

Next, we introduce the flow vector $r^{\hat{\mu}}$ defined implicitly via $r^{\hat{\mu}}n_{\hat{\mu}} = N$. The hypersurface tangential part of the flow vector is called the shift function, $N^{\hat{\mu}} = h^{\hat{\mu}}{}_{\hat{\nu}}r^{\hat{\nu}}$, which means that the flow vector is decomposable as

$$r^{\hat{\mu}} = Nn^{\hat{\mu}} + N^{\hat{\mu}}. \tag{G.1.7}$$

Eventually, we'll want to express the action (G.1.1) in terms of canonical velocities. We define these as the Lie derivative along $r^{\hat{\mu}}$,

$$\dot{h}_{\hat{\mu}\hat{\nu}} = \pounds_r h_{\hat{\mu}\hat{\nu}}, \qquad \dot{A}_{\hat{\mu}} = \pounds_r A_{\hat{\mu}}. \tag{G.1.8}$$

1 In principle, one has to be careful about the sign, but Σ_r is time-like, i.e. $n_m n^m = +1$.

If $r^{\hat{\mu}} = \delta_r^{\hat{\mu}}$, as will be the case eventually when imposing radial gauge on the shift and lapse, the Lie derivative reduces to the radial derivative. The extrinsic curvature $K_{\hat{\mu}\hat{\nu}}$ of Σ_r is given by

$$K_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\pounds_n h_{\hat{\mu}\hat{\nu}} = h_{\hat{\mu}}{}^{\hat{\lambda}}\hat{\nabla}_{\hat{\lambda}} n_{\hat{\nu}}. \tag{G.1.9}$$

Carrying on these calculations (see Appendix A of [25]) to the end and imposing the Gauss-Codazzi equations, we get the familiar expression

$$H = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{h} \left(N\mathcal{H} + N^\mu \mathcal{H}_\mu \right). \tag{G.1.10}$$

Diffeomorphism invariance allows us to gauge fix $\mathcal{N} = 1, \mathcal{N}^\mu = 0$, leading to the result

$$H = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{h} \left[\left(K_{\mu\nu}\pi^{\mu\nu} + \frac{1}{2}Q_\mu Q^\mu \right) + R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}\left(\Phi^2 - A_\mu A^\mu \right) \right], \tag{G.1.11}$$

where Φ is the radial component of the Proca one-form, $\hat{A} = \Phi \mathrm{d}r + A_\mu \mathrm{d}x^\mu$ and is constrained by the Proca constraint,

$$\Phi = -\frac{1}{M^2} D_\mu Q^\mu. \tag{G.1.12}$$

Note that in domain-wall coordinates, the Proca field supporting the vanilla Lifshitz solution has the form [22]

$$\hat{A} = \alpha e^{zr} \mathrm{d}t, \tag{G.1.13}$$

for some constant α, which depends on both d and z. When considering the Proca field on a radial hypersurface Σ_r, the hypersurface tangential part of has the form $A = \tilde{\alpha}\mathrm{d}t$. Restricting our attention to timelike Proca fields, we introduce an auxiliary field ψ, which measures the deformation away from the pure Lifshitz Proca field value, that is,

$$A = (\tilde{\alpha} + \psi)\, \mathrm{d}t. \tag{G.1.14}$$

ψ corresponds to the source of a relevant operator for $1 < z < d_s$; for $d_s = z$, it becomes marginally relevant [215]. The canonical momenta are given by the usual relations:

$$\pi^{\mu\nu} = \frac{1}{\sqrt{h}}\frac{\delta S}{\delta h_{\mu\nu}}, \qquad Q^\mu = \frac{1}{\sqrt{h}}\frac{\delta S}{\delta A_\mu}. \tag{G.1.15}$$

Note that any AlLif metric may be written as (in domain-wall coordinates)

$$ds^2 = \mathrm{d}r^2 + h_{\mu\nu}(r,x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{G.1.16}$$

which, in order to set up the ADM Hamiltonian (G.1.11), was decomposed in a radial ADM-manner, i.e. by writing

$$ds^2 = \mathcal{N}^2 \mathrm{d}r^2 + h_{\mu\nu}(r,x)\left(\mathrm{d}x^\mu + \mathcal{N}^\mu \mathrm{d}r \right)\left(\mathrm{d}x^\nu + \mathcal{N}^\nu \mathrm{d}r \right), \tag{G.1.17}$$

where $\mathcal{N}$ is the radial lapse function, whereas $\mathcal{N}^\mu$ is the radial shift. We shall as in the case of AlAdS, we will gauge-fix—by means of foliation preserving diffeomorphisms—the shift and set the lapse equal to one. On the leaves Σ_r of the radial foliation, we introduce a temporal foliation, allowing us to write the induced metric on Σ_r as

$$h_{\mu\nu}(r,x)\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2 \mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i \mathrm{d}t \right)\left(\mathrm{d}x^j + N^j \mathrm{d}t \right), \tag{G.1.18}$$

where N is the temporal lapse on Σ_r and N^i the temporal shift. We use indices from the middle of the Latin alphabet, i, j, k, l, m, n to denote spatial directions on Σ_r. The validity of such a foliation in terms of leaves of constant (absolute) time ties in nicely with our non-relativistic interpretation of the boundary theory. The gravity field multiplet then consists of the fields,

$$N, \;\; N^i, \;\; \gamma_{ij}, \tag{G.1.19}$$

More information about this double-foliation scheme may be found in [96]. Note that foliation preserving diffeomorphisms on Σ_r allows us to set $N^i = 0$, while we choose to keep N general. This, then, allows us to introduce the following vielbeine, where we use lower-case Latin letters from the beginning of the alphabet to denote flat indices, i.e. a, b, c, d. Since we'll be working with vielbeine for the most part, it is convenient to introduce the following objects

$$T_a^\mu = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta e_\mu^a}, \quad \pi_\psi = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \psi}. \tag{G.1.20}$$

We now proceed to derive a couple of useful relations involving the objects above. Writing $e = \sqrt{h}$, we see that

$$T^a{}_b = e_\mu^a T^\mu{}_a = e_\mu^a \frac{1}{e} \frac{\delta S}{\delta e_\mu^b}, \tag{G.1.21}$$

so, writing

$$A_\rho = e_\rho^c A_c, \tag{G.1.22}$$

we find by virtue of the functional chain rule (note that r is fixed!),

$$e_\mu^a \frac{1}{e} \frac{\delta S}{\delta e_\mu^b} = e_\mu^a(x,r) \frac{1}{e(x,r)} \int \mathrm{d}^d x' \left(\frac{\delta S}{\delta h_{\rho\sigma}(x')} \frac{\delta h_{\rho\sigma}(x')}{\delta e_\mu^b(x)} + \frac{\delta S}{A_\rho(x')} \frac{A_\rho(x')}{\delta e_\mu^b(x)} \right) \tag{G.1.23}$$

$$= \int \mathrm{d}^d x' \delta^{(d)}(x'-x) \left(e_\mu^a(x) \pi^{\rho\sigma}(x') \eta_{cd} \left(\delta_b^c \delta_\rho^\mu e_\sigma^d(x') + \delta_b^d \delta_\sigma^\mu e_\rho^c \right) + e_\mu^a(x) Q^\rho(x') A_c(x') \delta_b^c \delta_\rho^\mu \right) \tag{G.1.24}$$

$$= 2\pi^{ad} \eta_{db} + Q^a A_b \tag{G.1.25}$$

$$= 2\pi^a{}_b + Q^a A_b, \tag{G.1.26}$$

so that,

$$T^a{}_b = 2\pi^a{}_b + Q^a A_b. \tag{G.1.27}$$

Now, since $\delta A_\mu = \delta\psi \delta_\mu^0$, we get

$$Q^\mu = \pi_\psi \delta_0^\mu, \tag{G.1.28}$$

which means that

$$Q^a = \pi_\psi e_\mu^a \delta_0^\mu = \pi_\psi e_0^a. \tag{G.1.29}$$

G.1.2 *Boundary Conditions & the δ_D Expansions*

We now introduce spatial flat indices as $\bar{a}$. There are now two ways of imposing boundary conditions. Either we linearise the field equations as done in [27, 131], in which case we find the asymptotic behaviour for the vielbeine and the Proca deformation field[2],

$$e_\mu^0 = e^{zr} e_\mu^{(0)0} + \cdots, \tag{G.1.30}$$

$$e_\mu^{\bar{a}} = e^r e_\mu^{(0)\bar{a}} + \cdots, \tag{G.1.31}$$

$$\psi = e^{-\Delta_- r} \psi^{(0)} + \cdots, \tag{G.1.32}$$

where

$$\Delta_- = \frac{1}{2} \left(z + D - \sqrt{(z+d-1)^2 + 8(z-1)(z-d+1)} \right). \tag{G.1.33}$$

Using the chain rule, we find that

$$\delta_r = \int_{\Sigma_r} \mathrm{d}^d x \left(\dot{e}_\mu^0 \frac{\delta}{\delta e_\mu^0} + \dot{e}_\mu^{\bar{a}} \frac{\delta}{\delta e_\mu^{\bar{a}}} + \dot{\psi} \frac{\delta}{\delta \psi} \right), \tag{G.1.34}$$

2 Note that these conditions differ from those in [27, 96] due to our usage of domain-wall coordinates.

where a dot denotes radial differentiation. Similarly, the dilatation operator becomes

$$\delta_D = \int_{\Sigma_r} \mathrm{d}^d x \left(\lambda_1 e^0_\mu \frac{\delta}{\delta e^0_\mu} + \lambda_2 e^{\bar{a}}_\mu \frac{\delta}{\delta e^{\bar{a}}_\mu} - \Delta_- \psi \frac{\delta}{\delta \psi} \right), \tag{G.1.35}$$

where the coefficients in front of the various terms in δ_D are the leading scaling behaviour of the field, and will be fixed by imposing the HJ equation. Now we'll want to impose the asymptotic boundary condition $\delta_r = \delta_D$, but before we do that we'll rewrite both δ_r and δ_D, using that $2K_{\mu\nu} = \frac{1}{2}\partial_r h_{\mu\nu}$

$$\delta_r = \int_{\Sigma_r} \mathrm{d}^d x \sqrt{h} \left(2K_{ab} \frac{e^a_\rho e^b_\sigma}{\sqrt{h}} \frac{\delta}{\delta h_{\rho\sigma}} + Q_a \frac{e^a_\rho}{\sqrt{h}} \frac{\delta}{\delta A_\rho} \right), \tag{G.1.36}$$

which is identical to (G.1.34), since

$$\begin{aligned} 2K_{ab} e^a_\rho e^b_\sigma \frac{\delta}{\delta h_{\rho\sigma}} &= \left(e^\mu_a \dot{e}_{\mu b} + e^\mu_b \dot{e}_{\mu a} \right) e^a_\rho e^b_\sigma \frac{\delta}{\delta h_{\rho\sigma}} = \left(\delta^\mu_\rho \dot{e}_{\mu b} e^b_\sigma + \delta^\mu_\sigma \dot{e}_{\mu a} e^a_\rho \right) \frac{\delta}{\delta h_{\rho\sigma}} && \text{(G.1.37)} \\ &= 2\dot{e}_{\mu a} e^a_\rho \frac{\delta}{\delta h_{\mu\rho}} = \eta_{ab} \dot{e}^a_\mu 2 e^b_\rho \frac{\delta}{\delta h_{\mu\rho}} = \dot{e}^a_\mu(x) \int_{\Sigma_r} \mathrm{d}^d x' \frac{\delta h_{\nu\rho}(x')}{\delta e^a_\mu(x)} \frac{\delta}{\delta h_{\nu\rho}(x')} && \text{(G.1.38)} \\ &= \dot{e}^a_\mu \frac{\delta}{\delta e^a_\mu}. && \text{(G.1.39)} \end{aligned}$$

Note also that

$$2\dot{e}_{\mu a} e^a_\rho \frac{\delta}{\delta h_{\mu\rho}} = \dot{h}_{\mu\rho} \frac{\delta}{\delta h_{\mu\rho}}, \tag{G.1.40}$$

which shows how δ_r in (G.1.34) is obtained from the corresponding expression involving the metric rather than vielbeine in the first place. Now, the relation

$$A_a = (\alpha + \psi)\, \delta^0_a, \tag{G.1.41}$$

implies that the implicit radial derivative reads

$$\begin{aligned} \delta_r &= \int_{\Sigma_r} \mathrm{d}^d x \sqrt{h} \left(2K_{ab} e^a_\rho e^b_\sigma \frac{1}{\sqrt{h}} \frac{\delta}{\delta h_{\rho\sigma}} + \dot{A}_\mu \frac{1}{\sqrt{h}} \frac{\delta}{\delta A_\mu} \right) && \text{(G.1.42)} \\ &= \int_{\Sigma_r} \mathrm{d}^d x\, N\sqrt{\gamma} \left(2K_{ab} e^a_\rho e^b_\sigma \frac{1}{\sqrt{h}} \frac{\delta}{\delta h_{\rho\sigma}} + \frac{1}{\sqrt{h}} Q_\mu \frac{\delta}{\delta A_\mu} \right) && \text{(G.1.43)} \\ &= \int_{\Sigma_r} \mathrm{d}^d x\, N\sqrt{\gamma} \left(2K_{ab} e^a_\rho e^b_\sigma \frac{1}{\sqrt{h}} \frac{\delta}{\delta h_{\rho\sigma}} + \frac{1}{\sqrt{h}} Q_a \frac{\delta}{\delta A_a} \right) && \text{(G.1.44)} \\ &= \int_{\Sigma_r} \mathrm{d}^d x\, N\sqrt{\gamma} \left(2K_{ab} e^a_\rho e^b_\sigma \frac{1}{\sqrt{h}} \frac{\delta}{\delta h_{\rho\sigma}} + Q_0 \frac{1}{\sqrt{h}} \frac{\delta}{\delta \psi} \right) && \text{(G.1.45)} \\ &= \int_{\Sigma_r} \mathrm{d}^d x\, N\sqrt{\gamma} \left(2K_{ab} \hat{\pi}^{ab} + Q_0 \hat{\pi}_\psi \right), && \text{(G.1.46)} \end{aligned}$$

and thus,

$$\begin{aligned} \hat{\pi}^{ab} &= \frac{1}{2} \left(\hat{T}^{ab} - A^b \hat{Q}^a \right) = \frac{1}{2} \left(\hat{T}^{ab} - \eta^{bc} A_c \delta^a_0 \hat{\pi}_\psi \right) && \text{(G.1.47)} \\ &= \frac{1}{2} \left(\hat{T}^{ab} - \eta^{bc} (\alpha + \psi) \delta^0_c \delta^a_0 \hat{\pi}_\psi \right) && \text{(G.1.48)} \\ &= \frac{1}{2} \left(\hat{T}^{ab} + (\alpha + \psi) \delta^b_0 \delta^a_0 \hat{\pi}_\psi \right). && \text{(G.1.49)} \end{aligned}$$

With this, we obtain the result with all indices *flat*:

$$\delta_r = \int_{\Sigma_r} \mathrm{d}^d x\, N\sqrt{\gamma} \left(K_{ab} \hat{T}^{ab} + [Q_0 + (\alpha + \psi) K_{00}]\, \hat{\pi}_\psi \right). \tag{G.1.50}$$

Now, the asymptotic boundary conditions are implemented as

$$\delta_r = \delta_D. \tag{G.1.51}$$

The non-derivative part of the counterterm action must [25], due to general covariance, necessarily be of the form

$$S = \int_{\Sigma_r} \mathrm{d}^d x \, \sqrt{h} U(\psi) + \text{derivative terms.} \tag{G.1.52}$$

Now, the next thing we require is a way to relate the quantities above to the extrinsic curvature. This is achieved through the relation

$$\pi_{\mu\nu} = K_{\mu\nu} - h_{\mu\nu} K, \tag{G.1.53}$$

the flat version of which simply reads

$$\pi_{ab} = K_{ab} - \eta_{ab} K. \tag{G.1.54}$$

In particular, we have that

$$\pi = K(1 - (d_s + 1)) = d_s K, \tag{G.1.55}$$

and thus

$$\pi^{ab} = K^{ab} - \eta^{ab} \frac{\pi}{d_s} \Rightarrow K^{ab} = \pi^{ab} + \eta^{ab} \frac{\pi}{d_s}. \tag{G.1.56}$$

Now, we use that

$$\pi^{ab} = \frac{1}{2} \left(T^{ab} - Q^a A^b \right) = \frac{1}{2} \left(T^{ab} + U'(\psi)(\alpha + \psi) \delta_0^a \delta_0^b \right). \tag{G.1.57}$$

In particular, we have that

$$T^{\mu}{}_{a} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta e^a_\mu} = e^\mu_a U(\psi), \tag{G.1.58}$$

where we have used that $\delta e = e e^\mu_a \delta e^a_\mu$, implying that

$$T^{ab} = \eta^{ab} U(\psi), \tag{G.1.59}$$

and thus

$$\pi^{ab} = \frac{1}{2} \left(\eta^{ab} U(\psi) + U'(\psi)(\alpha + \psi) \delta_0^a \delta_0^b \right). \tag{G.1.60}$$

Now, we can find the extrinsic curvature,

$$\pi^{ab} = K^{ab} + \eta^{ab} \frac{\pi}{d_s}, \tag{G.1.61}$$

where

$$\pi = \frac{1}{2} \left((d_s + 1) U(\psi) - U'(\psi)(\alpha + \psi) \right), \tag{G.1.62}$$

so that

$$K^{ab} = \pi^{ab} - \eta^{ab} \frac{\pi}{d_s} = -\frac{\eta^{ab}}{2 d_s} U(\psi) + \frac{U'(\psi)(\alpha + \psi)}{2} \left(\delta_0^a \delta_0^b + \frac{\eta^{ab}}{d_s} \right) \tag{G.1.63}$$

$$= \frac{U(\psi) + (d_s - 1) U'(\psi) \, (\alpha + \psi)}{2 d_s} \delta_0^a \delta_0^b + \frac{-U(\psi) + (\alpha + \psi) U'(\psi)}{2 d_s} \delta^{\bar{a}\bar{b}} \delta^a_{\bar{a}} \delta^b_{\bar{b}}. \tag{G.1.64}$$

Next, note that

$$Q^a = \frac{1}{\sqrt{h}} \frac{\delta S}{\psi} \delta_0^a = U'(\psi) \delta_0^a \Rightarrow Q_0 = -U'(\psi). \tag{G.1.65}$$

We now want impose the boundary conditions, which implies that

$$\overbrace{K_{ab} \hat{T}^{ab} + (Q_0 + (\alpha + \psi) K_{00}) \hat{\pi}_\psi}^{r \to \infty} = z \hat{T}^0{}_0 + \hat{T}^{\bar{a}}{}_{\bar{a}} - \Delta_- \psi \hat{\pi}_\psi. \tag{G.1.66}$$

Comparing scaling weights and writing $K_{ab}\hat{T}^{ab} = K^b{}_a \hat{T}^a{}_b$ writing out scaling expansions in terms of dilatation weights, e.g. $Q_0 = \sum_w Q_0^{(w)}$ with $\delta_D Q_0^{(w)} = -w Q_0^{(w)}$. This procedure immediately gives us

$$K^{(0)b}{}_a \hat{T}^a{}_b = z\hat{T}^0{}_0 + \hat{T}^{\bar{a}}{}_{\bar{a}} \Rightarrow K_{ab}^{(0)} = -z\delta_a^0\delta_b^0 + \delta_{\bar{a}\bar{b}}\delta_a^{\bar{a}}\delta_b^{\bar{b}}, \tag{G.1.67}$$

and, similarly,

$$Q_0^{(0)} + \alpha K_{00}^{(0)} = 0, \tag{G.1.68}$$
$$\psi K_{00}^{(0)} + \alpha K_{00}^{(\Delta_-)} + Q_0^{(\Delta_-)} = -\Delta_- \psi. \tag{G.1.69}$$

To proceed, we write $U(\psi)$ as a series in ψ:

$$U(\psi) = \sum_{n=0}^{n_{\max}} u_n \psi^n, \tag{G.1.70}$$

and thus we find for the quantities above, using (G.1.64)

$$K_{00}^{(0)} = \frac{u_0 + \alpha(d_s - 1)u_1}{2d_s}, \quad K_{00}^{(\Delta_-)} = \psi \frac{u_1 + (d_s - 1)(2\alpha u_2 + u_1)}{2d_s} \tag{G.1.71}$$
$$K_{\bar{a}\bar{b}}^{(0)} = \frac{-u_0 + \alpha u_1}{2d_s}\delta_{\bar{a}\bar{b}}, \quad Q_0^{(0)} = -u_1, \quad Q_0^{(\Delta_-)} = -2u_2\psi. \tag{G.1.72}$$

Combining the results above with eqs. (G.1.67)–(G.1.69) then gives us—since $K_{00}^{(0)} = -z$

$$Q_0^{(0)} + \alpha K_{00}^{(0)} = 0 \Rightarrow u_1 = -\alpha z, \tag{G.1.73}$$

which we can plug into the relation $K_{\bar{a}\bar{b}}^{(0)} = \delta_{\bar{a}\bar{b}}$ to obtain for u_0 the expression

$$u_0 = -\alpha^2 z - 2d_s. \tag{G.1.74}$$

Similarly, plugging the value of u_1 into the relation $K_{00}^{(0)} = -z$ gives us the equality $u_0 = -2zd_s + \alpha^2(d_s - 1)z$. Equating the two expression for u_0 gives us the following result for α:

$$\alpha = \sqrt{\frac{2(z-1)}{z}}, \tag{G.1.75}$$

which we in turn can plug into (G.1.74) to obtain:

$$u_0 = -2(z + d_s - 1). \tag{G.1.76}$$

Finally, plugging all the above results into (G.1.69), we obtain

$$u_2 = -\frac{zd_s\,(2z - 1 - \Delta_-)}{2\,(z + d_s - 1)}, \tag{G.1.77}$$

which implies that the first couple of (non-derivative) counterterms are given by

$$S_{\text{ct}} = \int_{\Sigma_r} \mathrm{d}^d x \sqrt{h} \left(-2(z + d_s - 1) - z\alpha\psi - \frac{zd_s\,(2z - 1 - \Delta_-)}{2\,(z + d_s - 1)}\psi^2 + \cdots \right). \tag{G.1.78}$$

Expanding the Hamiltonian density $\mathcal{H}$ extracted from (G.1.11), we can expand $\mathcal{H}$ in scaling weights, $\mathcal{H} = \sum_w \mathcal{H}^{(w)}$ with $\delta_D \mathcal{H}^{(w)} = -w\mathcal{H}^{(w)}$. Note the symmetry property (which we derived in chapter 3)

$$K_{ab}^{(n)} \pi^{(m)ab} = K_{ab}^{(m)} \pi^{(n)ab}, \tag{G.1.79}$$

which has to be taken into account when writing

$$\mathcal{H}^{(w)} = \sum_{n+m=w} \left(K_{ab}^{(m)} \pi^{(n)ab} + \frac{1}{2} Q_a^{(n)} Q^{(m)a} - \frac{M^2}{2} \Phi^{(n)} \Phi^{(m)} \right) + \mathcal{V}^{(w)}, \tag{G.1.80}$$

where $\mathcal{V}^{(w)}$ is the part of

$$\mathcal{V} = R - 2\Lambda - \frac{1}{2} F_{ab} F^{ab} - \frac{M^2}{2} A_a A^a \tag{G.1.81}$$

with dilatonic scaling weight w. With this at hand, we can solve the Hamiltonian constraint recursively as we did for AlAdS boundary conditions[3]. The next step will be to determine a *useful relation*, which allows us to find the terms of the on-shell Lagrangian $\mathcal{L}$. The idea is to do exactly the same as we did for AlAdS boundary conditions in chapter 3: we expand the (unknown) counterterm Lagrangian in eigenmodes of δ_D and relate the terms to the Hamiltonian constraint. We follow the approaches taken by [27, 96]. By the same arguments as for AlAdS boundary conditions, we must now include a linear term (logarithmic in Poincaré-type coordinates), i.e.

$$\mathcal{L}_{\text{ct}} = -\sum_{0\leq\lambda<z+d_s} \mathcal{L}^{(\lambda)} - \widetilde{\mathcal{L}}^{(z+d_s)} r. \tag{G.1.83}$$

As before, the individual terms satisfy the scaling relations:

$$\delta_D \mathcal{L}^{(w)} = -w\mathcal{L}^{(w)}, \quad w \neq z + d_s, \tag{G.1.84}$$
$$\delta_D \mathcal{L}^{(z+d_s)} = -(z+d_s)\mathcal{L}^{(z+d_s)} + \widetilde{\mathcal{L}}^{(z+d_s)}, \tag{G.1.85}$$
$$\delta_D \widetilde{\mathcal{L}}^{(z+d_s)} = -(z+d_s)\widetilde{\mathcal{L}}^{(z+d_s)}. \tag{G.1.86}$$

As we have seen:

$$(z + d_s + \delta_D)\mathcal{L} = zT^0{}_0 + T^{\bar{a}}{}_{\bar{a}} - \Delta_-\psi\pi_\psi, \tag{G.1.87}$$

which we can expand in scaling weights to get

$$(z + d_s - w)\mathcal{L}^{(w)} = zT^{(w)0}_{0} + T^{(w)\bar{a}}_{\bar{a}} - \Delta_-\psi\pi_\psi^{(w-\Delta_-)}, \quad w \neq z + d_s, \tag{G.1.88}$$

while for $w = z + d_s$, we get instead

$$(z + d_s + \delta_D)\,\mathcal{L}^{(z+d_s)} = \widetilde{\mathcal{L}}^{(z+d_s)} = zT^{(z+d_s)0}_{0} + T^{(z+d_s)\bar{a}}_{\bar{a}} - \Delta_-\psi\pi_\psi^{(z+d_s-\Delta_-)}. \tag{G.1.89}$$

Now the Hamiltonian constraint, which we have now seen multiple times, is obtained by varying the bulk action with respect to $\mathcal{N}$ (before setting it equal to one). In particular, using $\pi_{ab} = K_{ab} - \eta_{ab}K$ and the Proca constraint for Φ (recall $\hat{A} = \Phi \mathrm{d}r + A_\mu \mathrm{d}x^\mu$), which reads $\Phi = -\frac{1}{M^2}\nabla_\mu Q^\mu$, we can write rewrite the Hamiltonian constraint as

$$K^2 - K_{ab}K^{ab} - \frac{1}{2}Q_aQ^a - \frac{1}{2M^2}\left(\nabla_a Q^a\right)^2 = R - 2\Lambda - \frac{1}{2}F_{ab}F^{ab} - \frac{M^2}{2}A_aA^a, \tag{G.1.90}$$

which we can expand dilatation weights and plus into (G.1.89), culminating in the following expression, valid for all $w \notin \{0, \Delta_-, 2\Delta_-\}$

$$(z + d_s - w)\mathcal{L}^{(w)} = \mathfrak{Q}^{(w)} + \mathfrak{S}^{(w)}, \tag{G.1.91}$$

where the quadratic term $\mathfrak{Q}^{(w)}$ has the form

$$\mathfrak{Q}^{(w)} = \sum_{\substack{0<\lambda<w/2 \\ w\notin\{\Delta_-,2\Delta_-\}}} \left[2K^{(\lambda)}_{ab}\pi^{(w-\lambda)ab} + Q^{(\lambda)}_a Q^{(w-\lambda)a} + \frac{1}{M^2}\left(\nabla_a Q^a\right)^{(\lambda)}\left(\nabla_a Q^a\right)^{(w-\lambda)}\right] \tag{G.1.92}$$
$$+\left[K^{(\Delta_-)}_{ab}T^{(w-\Delta_-)ab} + K^{(\Delta_-)}_{00}Q^{(w-2\Delta_-)0}\psi + Q^{(\Delta_-)}_{\bar{a}}Q^{(w-\Delta_-)\bar{a}}\right] \tag{G.1.93}$$
$$+\left[K^{(w/2)}_{ab}\pi^{(w/2)ab} + \frac{1}{2}Q^{(w/2)}_a Q^{(w/2)a} + \frac{1}{2M^2}\left(\nabla_a Q^a\right)^{(w/2)}\left(\nabla_a Q^a\right)^{(w/2)}\right] \tag{G.1.94}$$

where we have conveniently grouped terms in accordance to general scaling behaviour; i.e. the terms in the first bracket involve λ but not Δ_-, the next involves Δ_- but not λ, while the last involves neither. The contribution from the *source* term is the part of

$$\mathfrak{S} = R - 2\Lambda - \frac{1}{2}F_{ab}F^{ab} - \frac{M^2}{2}A_aA^a, \tag{G.1.95}$$

3 A good consistency would have been to take $\Delta_- = \Delta$ to be general and then combining $\delta_r = \delta_D$ along with the Hamiltonian constraint. Doing this, one finds that $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\Delta)}$ vanish identically, while $\mathcal{H}^{2\Delta}$ only vanishes when

$$\Delta_\pm = \frac{1}{2}\left(z + d_s \pm \sqrt{(z+d_s)^2 + 8(z-1)(z-d_s)}\right), \tag{G.1.82}$$

where we must pick $\max(-\Delta_\pm) = \Delta_-$, since it will be leading.

with scaling weight w. For $w \in \{0, \Delta_-, 2\Delta_-\}$, we find instead

$$(z+d_s)\mathcal{L}^{(0)} = 2\mathcal{L}^{(0)}, \tag{G.1.96}$$

$$(z+d_s-\Delta_-)\mathcal{L}^{(\Delta_-)} = (\Delta_- - z)\psi\pi_\psi^{(0)} + \mathfrak{S}^{(\Delta_-)}, \tag{G.1.97}$$

$$(z+d_s-2\Delta_-)\mathcal{L}^{(2\Delta_-)} = (\Delta_- - z)\psi\pi_\psi^{(\Delta_-)} + K_{ab}^{(\Delta_-)}\pi^{(\Delta_-)ab} + \frac{1}{2}Q_a^{(\Delta_-)}Q^{(\Delta_-)a} + \mathfrak{S}^{(2\Delta_-)}. \tag{G.1.98}$$

Now, expanding $\mathfrak{S}$ in dilatation weights, we find that R has terms of weight 2 and $2z$, $F_{ab}F^{ab}$ has terms of order $2, 2+\Delta_-, 2+2\Delta_-$, whereas A_aA^a has components of order $0, \Delta_-, 2\Delta_-$:

$$\mathfrak{S}^{(0)} = -2\Lambda + \frac{M^2}{2}\alpha^2 = (z+d_s)(z+d_s-1), \tag{G.1.99}$$

$$\mathfrak{S}^{(2)} = R^{(2)} - \frac{1}{4}\left(F_{ab}F^{ab}\right)^{(2)} = R^{(\gamma)} - \frac{2\nabla^2 N}{N} + \frac{\alpha^2}{2}\left(\frac{\nabla N}{N}\right)^2, \tag{G.1.100}$$

$$\mathfrak{S}^{(2z)} = R^{(2z)} = K_{ij}\,K^{ij} - (K)^2 + \text{total derivatives}, \tag{G.1.101}$$

$$\mathfrak{S}^{(\Delta_-)} = M^2\alpha\psi = d_s z\alpha\psi, \tag{G.1.102}$$

$$\mathfrak{S}^{(2\Delta_-)} = \frac{M^2}{2}\psi^2 = \frac{d_s z}{2}\psi^2, \tag{G.1.103}$$

$$\mathfrak{S}^{(2+\Delta_-)} = -\frac{1}{4}\left(F_{ab}F^{ab}\right)^{(2+\Delta_-)} = \frac{\alpha\nabla^i N\nabla_i(N\psi)}{N}, \tag{G.1.104}$$

$$\mathfrak{S}^{(2+2\Delta_-)} = -\frac{1}{4}\left(F_{ab}F^{ab}\right)^{(2+2\Delta_-)} = \frac{\nabla^i(N\psi)\nabla_i(N\psi)}{N}. \tag{G.1.105}$$

Note that we have already determined $\mathcal{L}^{(\Delta_-)}$ and $\mathcal{L}^{(2\Delta_-)}$ by imposing the asymptotic equality $\delta_r = \delta_D$.

G.1.3 *Finding the Counterterms*

The considerations above have enabled us to finally begin extracting the actual counterterms. In particular, at the non-derivative level at zero weight, we get

$$\mathcal{L}^{(0)} = \frac{2\mathfrak{S}^{(0)}}{z+d_s} = 2(z+d_s-1), \tag{G.1.106}$$

in agreement with our previous findings. Reassuringly, the terms $\mathcal{L}^{(\Delta_-)}$ and $\mathcal{L}^{(2\Delta_-)}$ also give what we found by imposing the asymptotic equality $\delta_r = \delta_D$, although this way of determining them is rather more cumbersome. Noting that a weight two, the quadratic term vanishes, i.e. $\mathfrak{Q}^{(0)} = 0$, we find that

$$\mathcal{L}^{(2)} = \frac{1}{z+d_s-2}\left(R^{(\gamma)} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}\right). \tag{G.1.107}$$

In this manner, all the counterterms can be calculated. For details, see the appendices of [96], which presents extensive calculations of the counterterms.

H DIMENSIONAL REDUCTION

In this appendix, we describe Kaluza-Klein reduction and Scherk-Schwarz reduction. We then apply the Scherk-Schwarz reduction to the *uplift*: a five-dimensional Einstein-Dilaton theory that reduces to a four-dimensional EPD model admitting $z = 2$ Lifshitz solutions. The uplift is the precursor of the electromagnetic uplift discussed in chapter (6).

H.1 KALUZA-KLEIN REDUCTION

Kaluza-Klein (KK) reduction was originally introduced by Kaluza and Klein [216, 217] as a way to unify gravity and electromagnetism. They showed that by compactifying a fifth dimension, it was possible to obtain gravity coupled to electromagnetism with a small caveat: this produced an additional scalar particle, the dilaton. In this section, we describe the generalities of the method, following [218–220].

Starting from a generic theory containing gravity $\hat{g}_{\hat{\mu}\hat{\nu}}$ (indices $\hat{\mu}, \hat{\nu} = 1, \ldots, D+1$) and matter fields collectively denoted $\hat{\Phi}$, described by the Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} + \cdots , \tag{H.1.1}$$

we assume that $(D+1)$-dimensional spacetime can be written as a direct product $\mathcal{M}_{D+1} = \mathcal{Q}_{d+1} \times \mathcal{C}_{D-d}$ with $\mathcal{C}_{D-d}$ a compact manifold. This product manifold will thus be a solution to the equations of motion obtained from (H.1.1), exhibiting what [220] terms *spontaneous compactification*, which just means that the metric respects the product structure

$$\hat{g}(x,y) = \begin{pmatrix} \mathring{g}_{\mu\nu}(x) & 0 \\ 0 & \mathring{g}_{mn}(y), \end{pmatrix} \tag{H.1.2}$$

where x^μ are the coordinates on the Lorentzian $\mathcal{Q}_{d+1}$ and y^m the coordinates on the compact Riemannian manifold $\mathcal{C}_{D-d}$. The metric is naturally dynamical, so the metric above is the background *ground state* solution of the theory. Generally, the fields on $\mathcal{Q}_{d+1}$ are taken to depend trivially on the coordinates of $\mathcal{C}_{D-d}$, as we see happens automatically in (H.1.2). The fields on $\mathcal{C}_{D-d}$, on the other hand, must be truncated in a consistent manner. Typically, this involves keeping only certain modes of a Fourier-like expansion. To be specific, let the compact manifold be S^1, in which case we have the decomposition $\mathcal{M}_{D+1} = \mathcal{Q}_D \times S^1$. Let y be the compactified coordinate, which is thus periodically identified,

$$y = y + 2\pi L, \tag{H.1.3}$$

with L the radius of the circle, which is usually assumed to be small. In this case, a generic field (including the metric) of our theory can be expanded in Fourier-like modes on S^1,

$$\hat{\Phi}(x,y) = \sum_n \Phi^{(n)}(x)e^{iny/L}. \tag{H.1.4}$$

The modes $n \neq 0$ correspond to massive fields, while the mode $n = 0$ represents a massless field, as can be realized by considering a massless scalar $\hat{\phi}$ on *flat* $(D+1)$-dimensional space, which satisfies the Klein-Gordon equation,

$$\hat{\Box}\hat{\phi} = 0, \tag{H.1.5}$$

where $\hat{\Box} = \partial^{\hat{\mu}}\partial_{\hat{\mu}}$, which, when compactified on S^1, admits an expansion of the form (H.1.4), $\hat{\phi} = \sum_n \phi^{(n)}e^{iny/L}$, so the Klein-Gordon equation implies that

$$0 = \partial^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\phi} = \sum_n \left(\Box\phi^{(n)} - \frac{n^2}{L^2} \right) e^{iny/L}, \tag{H.1.6}$$

so linear independence of the exponentials implies that the mass of the n'th mode is

$$m_n = \frac{|n|}{L}, \tag{H.1.7}$$

which, due to the infinity of every increasing masses, is known as the Kaluza-Klein tower of states. Thus, when $L \ll 1$ (in practice, one takes L to be of the order of the Planck length), the non-zero modes will be immensely heavy and can be safely neglected. This corresponds, as we claimed, to only keeping the zero-mode of the generalized Fourier expansion and thus truncates the Kaluza-Klein spectrum. This is sometimes known as the Kaluza-Klein reduction ansatz, since, generally[1], we have to worry about the consistency of our truncation, that is, whether ours setting all massive modes equal to zero solves the $(D+1)$-dimensional equations of motion. In general, if the fields in the Kaluza-Klein expansion (H.1.4) have a global symmetry group G, then the truncation will be consistent if we keep all G-singlets (that is, modes that do not transform under G). Clearly, in the case of the massless field on flat space, the expansion $\hat{\phi} = \sum_n \phi^{(n)} e^{iny/L}$ and spectrum (H.1.7) has a $G = U(1)$ symmetry rotating modes $n = \pm m$ into each other, implying that only the $n = 0$ mode is invariant under G, and thus our truncation is consistent. See [44] for more details.

H.1.1 *Kaluza-Klein Reduction of Pure Gravity & the EMD Model*

Now, let's try our hand at an explicit example: let's compactify $(d+1)$-dimensional pure gravity over S^1, following the analysis in [218]. Heuristically, the reduction of $\hat{g}_{\hat{\mu}\hat{\nu}}$ results in the following fields

- $\hat{g}_{\mu\nu}$, which is the metric of the reduced theory,
- $\hat{g}_{\mu y}$, a one-form, which seems likely to become a $U(1)$ gauge field,
- a scalar $\hat{g}_{yy}$.

In order to perform the reduction, we employ the following KK ansatz for the $(D+1)$-dimensional metric,

$$d\hat{s}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi}(dy + \mathcal{A})^2, \tag{H.1.8}$$

where α, β are constants that we'll choose in a convenient matter shortly, and $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. The reduced fields we discussed earlier may then be expressed in terms of the quantities appearing in (H.1.8) in the following manner,

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu, \quad \hat{g}_{\mu y} = e^{2\beta\phi}\mathcal{A}_\mu, \quad \hat{g}_{yy} = e^{2\beta\phi}. \tag{H.1.9}$$

The metric (H.1.8) is admits the following description in terms of vielbeine,

$$\hat{e}^a = \hat{e}^a_\mu dx^\mu = e^{\alpha\phi} e^a, \qquad \hat{e}^z = e^{\beta\phi}(dy + \mathcal{A}). \tag{H.1.10}$$

where Latin letters represent D-dimensional flat indices. Introducing capital Latin letters for $(D+1)$-dimensional flat indices, we can write our vielbeine in matrix form,

$$\hat{e}^A_{\hat{\mu}} = \begin{pmatrix} e^{\beta\phi} & e^{\beta\phi}A_\mu \\ 0 & e^{\alpha\phi}e^a_\mu \end{pmatrix}. \tag{H.1.11}$$

Noting that this matrix is triangular, the determinant is just the product of the diagonal elements, i.e. $\sqrt{-\hat{g}} = \hat{e} = e^{(D\alpha+\beta)}e$. Requiring that we be in Einstein-Frame (i.e., that the Einstein-Hilbert action retains its canonical form, $S \sim \int \sqrt{-g}R$), we should choose $\beta = -\alpha(D-2)$, since, as we shall shortly, the Ricci scalar carries a factor $e^{-2\alpha\phi}$. Similarly, by requiring that the kinetic scalar term, which comes from the reduced Ricci scalar, is canonically normalized, we get

$$\alpha^2 = \frac{1}{2(D-1)(D-1)}, \quad \beta = -(D-2)\alpha. \tag{H.1.12}$$

Now we proceed to calculate the Ricci scalar. Since our connection is symmetric, the spin connection is determined via

$$d\hat{e}^A + \hat{\omega}^A{}_B \wedge \hat{e}^B = 0, \tag{H.1.13}$$

1 Except for the case of compactification over tori, where keeping only the massless mode is always consistent, note in particular the (trivial) isomorphism $T^1 \simeq S^1$, so that keeping only massless modes

which gives us

$$\hat{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha\phi}\left(\partial^b\phi\hat{e}^a - \partial^a\phi\hat{e}^b\right) - \frac{1}{2}\mathcal{F}^{ab}e^{(\beta-2\alpha)\phi}\hat{e}^y, \tag{H.1.14}$$

$$\hat{\omega}^{ay} = -\beta e^{-\alpha\phi}e^{-\alpha\phi}\partial^a\phi\hat{e}^y - \frac{1}{2}\mathcal{F}^a{}_b e^{(\beta-2\alpha)\phi}\hat{e}^b, \tag{H.1.15}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$. Now, the curvature two form is given by

$$\hat{R}_{AB} = \mathrm{d}\hat{\omega}_{AB} + \hat{\omega}_{AC}\wedge\hat{\omega}^C{}_B, \tag{H.1.16}$$

the trace of which can be computed using the explicit results for the spin connections (H.1.14)–(H.1.15); the result is [218]

$$\hat{R} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi\right] - \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2, \tag{H.1.17}$$

and thus, combining our findings, we obtain the reduced action[2]

$$S = \int \mathrm{d}^D\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(d-1)\alpha\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right), \tag{H.1.18}$$

for which we can readily determine the equations of motion; they read

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{-2(D-1)\alpha\phi}\left(\mathcal{F}_{\mu\lambda}\mathcal{F}_{\nu\rho}g^{\rho\lambda} - \frac{1}{2(D-2)}\mathcal{F}^2 g_{\mu\nu}\right), \tag{H.1.19}$$

$$0 = \nabla^\mu\left(e^{-2(D-1)\alpha\phi}\mathcal{F}_{\mu\nu}\right), \tag{H.1.20}$$

$$\Box\phi = -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}\mathcal{F}^2. \tag{H.1.21}$$

In particular, we note that the equation of motion for ϕ, (H.1.21) implies that setting $\phi = 0$ would be an inconsistent truncation.

Thus, reducing pure gravity in $(D+1)$ dimensions results in D-dimensional Einstein-Maxwell-Dilaton theory.

H.2 SCHERK-SCHWARZ REDUCTION

The Scherk-Schwarz reduction[3], which was introduced in [221, 222], is a generalization of the KK method. The main idea of Scherk-Schwarz reduction is use certain symmetries of the higher-dimensional theory to generate masses in the reduced theory.

This is achieved by allowing the higher-dimensional fields to depend on the compact direction[4] u in a manner consistent with the symmetries of the higher-dimensional action, which ensures that the u–dependence drops out of the equations of motion of the reduced action. These symmetries fall into two categories

- Global/internal symmetries: phase, scale and shift symmetries etc.
- Local/external symmetries: space-time symmetries, e.g. translations or rotations in the compact manifold.

We will deal only with symmetries of the first kind.

Consider a global $U(1)$ symmetry acting on the fields as $\hat{\Phi} \to e^{i\Lambda}\hat{\Phi}$ for some $\Lambda \in \mathbb{R}$. The generalization of the periodicity condition (H.1.3) then reads

$$\hat{\Phi}(x, u + 2\pi L) = \overbrace{e^{2\pi i m L}}^{\in U(1)}\hat{\Phi}(x,u), \quad m \in \mathbb{R}, \tag{H.2.1}$$

i.e. we identify the two fields up to a global phase transformation, which is sometimes known as a *twist*. By using this twisted periodicity, we may obtain the following Fourier decomposition

$$\hat{\Phi}(x,u) = e^{imu}\sum_n \Phi_n(x)e^{inu/L}. \tag{H.2.2}$$

2 We ignore the extra factor of $2\pi L$ arising from the integeation over the compact dimension.
3 Sometimes the procedure is known as generalized dimensional reduction or *twisted reduction*, for reasons we will see shortly.
4 We change our notation in this section so as to match the one used in chapter 6—we hope that this will not cause any confusion.

In the limit $L \to 0$—just as for the KK case—the massive modes decouple (i.e. become infinitely massive and can be ignored), leaving us with the ansatz,

$$\hat{\Phi}(x,u) = e^{imu}\Phi_0(x), \tag{H.2.3}$$

which is a *local* $U(1)$ transformation ($\Lambda = mu$) acting on the zero-mode; in this way, the global $U(1)$ has been gauged. This allows us to unravel the general structure (see also [223]): if the higher-dimensional action has a global symmetry group G acting on the fields as $\hat{\Phi} \to g(\hat{\Phi})$ for $g \in G$, The generalization of the reduction ansatz (H.2.3) then involves an element $g_u = g(u) \in G$ with an explicit dependence on the compact direction u, i.e., schematically,

$$\hat{\Phi}(x,u) = g_z(\Phi(x)), \tag{H.2.4}$$

where it can (and usually will; cf. (H.2.3)) happen that only certain modes are included on the right-hand side. This results in a reduced theory independent of u and, as we saw in the case of global $U(1)$ symmetry, gauges the group G. Now, since $g(u)$ is not periodic, going around the compact coordinate results in a twist, also in this context known as the *monodromy*,

$$G \ni \mathcal{M}(g_u) := g(2\pi L) \tag{H.2.5}$$

which is just the generalization of the twist in (H.2.2). The exponential map allows us to express g_u in terms an element M of the Lie algebra $\mathfrak{g}$ of G,

$$g(u) = e^{\frac{Mu}{2\pi L}}, \qquad M \in \mathfrak{g}. \tag{H.2.6}$$

$M \in \mathfrak{g}$ spans a one-dimensional subgroup H of G. These considerations imply that the monodromy may be expressed as

$$\mathcal{M}(g) = e^M, \tag{H.2.7}$$

whereas M, which is in fact the mass matrix of the reduced theory, can be written as

$$M = 2\pi L g^{-1}\partial_u g. \tag{H.2.8}$$

By demanding that M does not depend on u, it is possible to determine the function g.

H.2.1 *Scherk-Schwarz Reduction of the Uplift*

In this section, we give the details of the reduction alluded to in section 6.2.1. Consider a five-dimensional Einstein-Dilaton model of the form

$$S_{5\mathrm{d}} = \frac{1}{2\kappa_5^2}\int \mathrm{d}^5x\sqrt{-\mathcal{G}}\left(R^{(\mathcal{G})} + 12 - \frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi\right), \tag{H.2.9}$$

with $\kappa_5^2 = 8\pi G_5$ and $\mathcal{M} = (u, M)$. The reduction ansatz is the one we used in chapter 6; it reads

$$ds_5^2 = \mathcal{G}_{\mathcal{M}\mathcal{N}}\mathrm{d}x^{\mathcal{M}}\mathrm{d}x^{\mathcal{N}} = \frac{\mathrm{d}r^2}{r^2} + \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B = \overbrace{e^{-\Phi}g_{MN}\mathrm{d}x^M\mathrm{d}x^N + e^{2\Phi}\left(\mathrm{d}u + A_M\mathrm{d}x^M\right)^2}^{\text{Form we'll be using for the reduction}} \tag{H.2.10}$$

$$= e^{-\Phi}\left(e^{\Phi}\frac{\mathrm{d}r^2}{r^2} + h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right) + e^{2\Phi}\left(\mathrm{d}u + A_\mu\mathrm{d}x^\mu\right)^2, \tag{H.2.11}$$

$$\psi = 2u + 2\Xi, \tag{H.2.12}$$

where the four dimensional fields g_{MN}, A_M, Ξ and Φ do not depend on the compactified u–direction, which—since we're reducing on a circle of radius L—is periodically identified, $u \sim u + 2\pi L$. Note also that since our normalization is such that $\frac{1}{16\pi G_4} = 1$, the five-dimensional Newton constant satisfies $\frac{2\pi L}{16\pi G_5} = 1$.

The renormalized on-shell four-dimensional EPD $z = 2$ action has the schematic form

$$S_{\mathrm{ren}} = S + S_{\mathrm{gh}} + S_{\mathrm{ct}}, \tag{H.2.13}$$

where $S_{\text{gh}} = 2\int \mathrm{d}^3x\,\sqrt{-h}K$ is the Gibbons-Hawking boundary, which, as per usual, is required to make the variational problem well posed, while the counterterm is the dimensionally reduction of the expression found in section 3.2.1 with the additional Maxwell field set to zero.

The reduction ansatz reads in components:

$$\mathcal{G}_{MN} = e^{-\Phi}g_{MN} + e^{2\Phi}A_M A_N, \quad \mathcal{G}_{Mu} = e^{2\Phi}A_M, \quad \mathcal{G}_{uu} = e^{2\Phi}, \tag{H.2.14}$$

while the inverse becomes

$$\mathcal{G}^{MN} = e^{\Phi}g^{MN}, \quad \mathcal{G}^{Mu} = -e^{\Phi}A^M, \quad \mathcal{G}^{uu} = e^{-2\Phi} + e^{\Phi}A^M A_M, \tag{H.2.15}$$

as we now demonstrate. The requirement $\mathcal{G}_{\mathcal{MR}}\mathcal{G}^{\mathcal{RN}} = \delta^{\mathcal{N}}_{\mathcal{M}}$ is equivalent to

$$\mathcal{G}_{M\mathcal{M}}\mathcal{G}^{\mathcal{M}N} = \delta^N_M, \qquad \mathcal{G}_{u\mathcal{M}}\mathcal{G}^{\mathcal{M}u} = 1, \qquad \mathcal{G}_{u\mathcal{M}}\mathcal{G}^{\mathcal{M}M} = 0. \tag{H.2.16}$$

To check this, we observe that

$$\begin{aligned}
\mathcal{G}_{M\mathcal{M}}\mathcal{G}^{\mathcal{M}N} &= \mathcal{G}_{MR}\mathcal{G}^{RN} + \mathcal{G}_{Mu}\mathcal{G}^{uN} && \text{(H.2.17)}\\
&= \left(e^{-\Phi}g_{MR} + e^{2\Phi}A_M A_R\right)e^{\Phi}g^{RN} - e^{3\Phi}A_N A^N && \text{(H.2.18)}\\
&= g_{MR}g^{RN} && \text{(H.2.19)}\\
&= \delta^N_M, && \text{(H.2.20)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{u\mathcal{M}}\mathcal{G}^{\mathcal{M}u} &= \mathcal{G}_{uu}\mathcal{G}^{uu} + \mathcal{G}_{uM}\mathcal{G}^{Mu} && \text{(H.2.21)}\\
&= e^{2\Phi}\left(e^{-2\Phi} + \cancel{e^{\Phi}A_M A^N}\right) - \cancel{e^{3\Phi}A_M A^M} && \text{(H.2.22)}\\
&= 1, && \text{(H.2.23)}
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathcal{G}_{u\mathcal{M}}\mathcal{G}^{\mathcal{M}M} &= \mathcal{G}_{uu}\mathcal{G}^{uM} + \mathcal{G}_{uN}\mathcal{G}^{NM} && \text{(H.2.24)}\\
&= -e^{3\Phi}A^M + e^{3\Phi}A_N g^{MN} && \text{(H.2.25)}\\
&= 0. && \text{(H.2.26)}
\end{aligned}$$

So what we have found is indeed the inverse. Alternatively, we could have used the block matrix inversion identity

$$\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}^{-1} = \begin{pmatrix}\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1} & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\end{pmatrix}. \tag{H.2.27}$$

In a similar vein, 2×2 block matrices for invertible $\mathbf{D}$ satisfy the following determinant relation,

$$\det\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \det(\mathbf{D})\det\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right), \tag{H.2.28}$$

implying that

$$\det(-\mathcal{G}) = e^{2\Phi}\det\left(-e^{-\Phi}g_{MN}\cancel{-e^{2\Phi}A_M A_N + e^{2\Phi}A_M A_N}\right) = e^{-2\Phi}\det(-g), \tag{H.2.29}$$

i.e.

$$\sqrt{-\mathcal{G}} = e^{-\Phi}\sqrt{-g}. \tag{H.2.30}$$

From this, we may immediately identify the KK scalar k (in the language of [219]),

$$k = e^{-\Phi}. \tag{H.2.31}$$

Following [219], we now proceed with the reduction of the Ricci scalar; starting off with the Christoffel symbols, we find that

$$\Gamma^{\mathcal{R}}_{\mathcal{MN}} = \frac{1}{2}\mathcal{G}^{\mathcal{RS}}\left(\partial_{\mathcal{M}}\mathcal{G}_{\mathcal{NS}} + \partial_{\mathcal{N}}\mathcal{G}_{\mathcal{MS}} - \partial_{\mathcal{S}}\mathcal{G}_{\mathcal{MN}}\right), \tag{H.2.32}$$

so

$$\Gamma^R_{MN} = \frac{1}{2}\mathcal{G}^{RS}\left(\partial_M \mathcal{G}_{NS} + \partial_N \mathcal{G}_{MS} - \partial_S \mathcal{G}_{MN}\right) \tag{H.2.33}$$

$$= \frac{1}{2}\mathcal{G}^{RS}\left(\partial_M \mathcal{G}_{NS} + \partial_N \mathcal{G}_{MS} - \partial_S \mathcal{G}_{MN}\right) + \frac{1}{2}\mathcal{G}^{Ru}\left(\partial_M \mathcal{G}_{Nu} + \partial_N \mathcal{G}_{Mu}\right). \tag{H.2.34}$$

This means that the five-dimensional Ricci can be decomposed in the following manner [219],

$$R^{(\mathcal{G})}_{uu} = -e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}, \tag{H.2.35}$$

$$R^{(\mathcal{G})}_{uM} = R^{(\mathcal{G})}_{uu}A_M + \frac{1}{2}\nabla^N\left(e^{3\Phi}F_{MN}\right), \tag{H.2.36}$$

$$R^{(\mathcal{G})}_{MN} = A_M R^{(\mathcal{G})}_{uN} + A_N R^{(\mathcal{G})}_{uM} - A_M A_N R^{(\mathcal{G})}_{uu} + R^{(g)}_{MN} - \frac{3}{2}\partial_M\Phi\partial_N\Phi + \frac{1}{2}g_{MN}\Box_{(g)}\Phi - \frac{1}{2}e^{3\Phi}F_{MP}F_N{}^P. \tag{H.2.37}$$

Hence, the five-dimensional Ricci scalar can be written as

$$R^{(\mathcal{G})} = \mathcal{G}^{\mathcal{MN}}R^{(\mathcal{G})}_{\mathcal{MN}} = \mathcal{G}^{uu}R^{(\mathcal{G})}_{uu} + 2\mathcal{G}^{uM}R^{(\mathcal{G})}_{uM} + \mathcal{G}^{MN}R^{(\mathcal{G})}_{MN}. \tag{H.2.38}$$

We compute these contributions individually. First up is

$$\mathcal{G}^{uu}R^{(\mathcal{G})}_{uu} = \left(e^{-2\Phi} + e^{\Phi}A^M A_M\right)\left(-e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}\right), \tag{H.2.39}$$

whereas

$$2\mathcal{G}^{uM}R^{(\mathcal{G})}_{uM} = -2e^{\Phi}A^M A_M\left(-e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}\right) - e^{\Phi}A^M\nabla^N\left(e^{3\Phi}F_{MN}\right), \tag{H.2.40}$$

and, finally,

$$\mathcal{G}^{MN}R^{(\mathcal{G})}_{MN} = \cancel{2}e^{\Phi}\left(-e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}\right)A^M A_M + e^{\Phi}A^M\nabla^N\left(e^{3\Phi}F_{MN}\right) \tag{H.2.41}$$

$$\cancel{- e^{\Phi}A^M A_M\left(-e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}\right)} + e^{\Phi}R^{(g)} - \frac{3}{2}e^{\Phi}\partial^M\Phi\partial_M\Phi \tag{H.2.42}$$

$$+ 2e^{\Phi}\Box_{(g)}\Phi - \frac{1}{2}e^{4\Phi}F_{MN}F^{MN} \tag{H.2.43}$$

$$= e^{\Phi}\left(-e^{3\Phi}\Box_{(g)}\Phi + \frac{1}{4}e^{6\Phi}F_{MN}F^{MN}\right)A^M A_M + e^{\Phi}A^M\nabla^N\left(e^{3\Phi}F_{MN}\right) \tag{H.2.44}$$

$$+ e^{\Phi}R^{(g)} - \frac{3}{2}e^{\Phi}\partial^M\Phi\partial_M\Phi + 2e^{\Phi}\Box_{(g)}\Phi - \frac{1}{2}e^{4\Phi}F_{MN}F^{MN}. \tag{H.2.45}$$

Combining everything, we obtain

$$R^{(\mathcal{G})} = e^{\Phi}R^{(g)} - \frac{3}{2}e^{\Phi}\partial^M\Phi\partial_M\Phi - \frac{1}{4}e^{4\Phi}F_{MN}F^{MN} + e^{\Phi}\Box_{(g)}\Phi. \tag{H.2.46}$$

Next, we consider the reduction of the scalar term, for which the ansatz (H.2.12) gives us

$$-\frac{1}{2}\partial_{\mathcal{M}}\psi\partial^{\mathcal{M}}\psi = -\frac{1}{2}\mathcal{G}^{uu}\partial_u\psi\partial^u\psi - \mathcal{G}^{uM}\partial_u\psi\partial_M\psi - \frac{1}{2}\mathcal{G}^{MN}\partial_M\psi\partial_N\psi \tag{H.2.47}$$

$$= -2\mathcal{G}^{uu} - 4\mathcal{G}^{uM}\partial_M\psi - 2\mathcal{G}^{MN}\partial_M\psi\partial_N\psi \tag{H.2.48}$$

$$= -2\left(e^{-2\Phi} + 4e^{\Phi}A^M A_M\right) + 4e^{\Phi}A^M\partial_M\Xi - 2e^{\Phi}\partial^M\Xi\partial_M\Xi \tag{H.2.49}$$

$$= -2B^M B_M - 2e^{-2\Phi}, \tag{H.2.50}$$

where $B_M = A_M - \partial_M\Xi$.

Turning our attention to the Gibbons-Hawking term, we note that, first of all

$$K_{AB} = -\frac{1}{2}\pounds_n\gamma_{AB}, \tag{H.2.51}$$

where $n^{\mathcal{M}} = -r\delta_r^{\mathcal{M}}$ is the normal vector to radial slices. From the definition of the Lie derivative, this gives us

$$K_{AB} = \frac{1}{2} n^{\mathcal{M}} \partial_{\mathcal{M}} \gamma_{AB} = \frac{r}{2} \partial_r \gamma_{AB}. \tag{H.2.52}$$

The trace of the extrinsic curvature is given by

$$K = \gamma^{AB} K_{AB}. \tag{H.2.53}$$

To determine the inverse metric γ^{AB}, we apply the inversion identity (H.2.27): since

$$\gamma_{AB} = \begin{pmatrix} e^{-\Phi} h_{\mu\nu} + e^{2\Phi} A_\mu A_\nu & e^{2\Phi} A_\nu \\ e^{2\Phi} A_\mu & e^{2\Phi} \end{pmatrix}, \tag{H.2.54}$$

we get

$$\gamma^{AB} = \begin{pmatrix} e^{\Phi} h^{\mu\nu} & -e^{\Phi} A^\nu \\ -e^{\Phi} A^\mu & e^{-2\Phi} + e^{\Phi} A^\mu A_\mu \end{pmatrix}. \tag{H.2.55}$$

Therefore, we find that

$$K = \frac{r}{2} \gamma^{AB} \partial_r \gamma_{AB} = \left[\frac{r}{2} \gamma^{\mu\nu} \partial_r \gamma_{\mu\nu} \right] + \left\{ r\gamma^{\mu u} \partial_r \gamma_{\mu u} \right\} + \left(\frac{r}{2} \gamma^{uu} \partial_r \gamma_{uu} \right) \tag{H.2.56}$$

$$= \left[-\frac{3r}{2} \partial_r \Phi + \frac{r}{2} h^{\mu\nu} \partial_r h_{\mu\nu} + \cancel{r e^{3\Phi} A^\mu A_\mu \partial_r \Phi} + \cancel{r e^{3\Phi} A^\mu \partial_r A_\mu} \right] \tag{H.2.57}$$

$$- \left\{ \cancel{2 r e^{3\Phi} A^\mu A_\mu \partial_r \Phi} + \cancel{r e^{3\Phi} A^\mu \partial_r A_\mu} \right\} \tag{H.2.58}$$

$$+ \left(r \partial_r \Phi + \cancel{r e^{3\Phi} A^\mu A_\mu \partial_r \Phi} \right) \tag{H.2.59}$$

$$= -\frac{r}{2} \partial_r \Phi + \frac{r}{2} h^{\mu\nu} \partial_r h_{\mu\nu}, \tag{H.2.60}$$

and so

$$2\sqrt{-\gamma} K = \sqrt{-h} \left(-n^M \partial_M \Phi + 2K^{(h)} \right), \tag{H.2.61}$$

where $n^M = e^{-\Phi/2} \delta_r^M$. We note that the first term precisely cancels out out the boundary term produced by the Laplacian term present in the expression for the reduced Ricci scalar (H.2.46).

The reduction of the counterterm proceeds completely analogously, and gives the result

$$S_{\mathrm{ct}} = 2 \int_{\partial \mathcal{M}} \mathrm{d}^3 x \, \sqrt{-h} \left(-\frac{1}{4} e^{\Phi/2} \left[R[h] - \frac{3}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{4} e^{3\Phi} F_{\mu\nu} F^{\mu\nu} - 2 B_\mu B^\mu + 10 e^{-\Phi} \right] \right) \tag{H.2.62}$$

$$- \log r \int_{\partial \mathcal{M}} \mathrm{d}^3 x \, \sqrt{-h} e^{-\Phi/2} \mathcal{A}. \tag{H.2.63}$$

Combining our findings, the reduced renormalized action becomes—after integration over the compact direction u and using $\frac{2\pi L}{2\kappa_5^2} = 1$

$$S_{\mathrm{ren}} = \int \mathrm{d}^4 x \, \sqrt{-g} \left(R[g] - \frac{3}{2} \partial^M \Phi \partial_M \Phi - \frac{1}{4} e^{3\Phi} F_{MN} F^{MN} - 2 B^M B_M - V(\Phi) \right) \tag{H.2.64}$$

$$+ 2 \int \mathrm{d}^3 x \, \sqrt{-h} K + S_{\mathrm{ct}}, \tag{H.2.65}$$

where

$$V(\Phi) = 2 e^{-3\Phi} - 12 e^{-\Phi}. \tag{H.2.66}$$

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