FACULTY OF SCIENCE
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**Master's Thesis**

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# Aspects of Lifshitz Holography

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**Abstract**

In recent years, extensions of the AdS/CFT correspondence have opened up the possibility of studying condensed matter systems using holography. In this thesis we will consider $z = 2$ Lifshitz spacetimes in 4 dimensions as possible duals of 3-dimensional non-relativistic condensed matter systems exhibiting anisotropic scaling, and construct a framework in which the sources can be identified and the vevs calculated. The basis for this construction will be the fact that $z = 2$ Lifshitz spacetimes in 4 dimensions are related to AlAdS spacetimes by Scherk-Schwarz reduction.

To calculate the vevs of the Lifshitz boundary theory, we show that a deformation of the Lifshitz spacetime is required. Various irrelevant deformations are considered and new classes of spacetimes, namely the generalized AlLif and Lifshitz UV, are defined. It is shown that the boundary geometry of these spacetimes is that of Newton–Cartan with torsion. Furthermore, we define ALif spacetimes whose boundary geometry is pure Newton–Cartan. Within the Lifshitz UV spacetime the sources of the 4-dimensional theory are analyzed. Due to the consistency of the reduction, the 4-dimensional vevs can be written entirely in terms of the 5-dimensional vevs. This fact is used to calculate the stress-energy tensor of the boundary theory. Furthermore, we find six Ward identities constraining the vevs of the boundary theory and use these to argue for a definition of a gauge invariant stress-energy tensor.

Relevant concepts, such as the geometry of Anti-de Sitter and Lifshitz spacetimes, dimensional reduction and holographic renormalization, are introduced along the way.

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# Contents

# Introduction

The advent of the AdS/CFT correspondence in 1997 [2] brought hope that an analytical description of strongly coupled systems was within reach. Since then, a lot of work has been done attempting to apply the correspondence to physical theories describing Nature, instead of the $\mathcal{N} = 4$ SYM theory that emerges from the original correspondence, which is theoretically appealing, but has far too much symmetry to match Standard Model physics. The fact that the duality (in its weak form) relates a strongly coupled field theory to a weakly coupled gravity theory has sparked hope that it might help shed light on some of the characteristic strongly coupled problems in physics, especially confinement and superconductivity. Such applications have led to an improved understanding of holography, although neither problem is fully understood.

In order to fully understand holographic dualities there are still many questions that need to be answered. The above applications are examples of phenomenological holography or bottom-up approaches. In such cases, the gravity theory is not constructed from a string theory, but instead from the demand that it should reproduce certain desired features of the field theory. Only in a few cases are the resulting gravity theories known to have string theory completions. Does it still make sense to discuss holographic dualities in such cases? Although the arguments by 't Hooft leading to the holographic principle [3] are rather general it is still an open question whether there exists a gravity dual to any field theory. According to the holographic principle any theory of quantum gravity is non-local in the sense that it is completely described by degrees of freedom on its boundary. However, the majority of the phenomenological models described above lack string theory completions, and are not necessarily consistent theories of quantum gravity. Will they still obey the holographic principle?

Another question pertains to the generality of holography. The original AdS/CFT correspondence is formulated on $\mathrm{AdS}_5 \times S^5$, but there are many solutions to string theory with completely different geometries. Due to the holographic principle, some of these geometries are believed to have a dual field theory as well, but even for the simple case of an asymptotically flat bulk spacetime very little is known of what type of theory this will be.

In the context of extensions of the Maldacena conjecture it is very interesting to consider the applicability of holography to the non-relativistic systems of condensed matter physics. The string theories used in the construction of a consistent theory of quantum gravity are naturally Lorentz invariant, but solutions exist which are anisotropic in space and time. It is plausible to consider such solutions as being the gravitational duals of non-relativistic theories of condensed matter physics, and this will be the focus of this thesis.

The solution to be considered exhibits an anisotropic scaling between time and space, with a dynamical critical exponent $z = 2$. The solution is called a $z = 2$ Lifshitz spacetime, and it is a 4-dimensional solution to the type IIB supergravity equations of motion after compactification [4]. This case is very interesting in the context of condensed matter applications, as many condensed matter systems exhibit such a non-relativistic scaling. However, very little is known about Lifshitz holography. In particular, the existence of a Fefferman-Graham expansion is not established and this, in turn, makes it rather difficult to perform holographic renormalization and to determine the appropriate sources and vevs. To circumvent these issues, the specific model considered is obtainable from a Scherk-Schwarz reduction of an axion-dilaton field theory living in Asymptotically locally AdS spacetime in 5 dimensions. This allows for a way of doing Lifshitz holography by Scherk-Schwarz reducing the results found in 5 dimensions, where the techniques required are just those from the ordinary AdS/CFT correspondence. This is the process which will be carried out in detail in this thesis, and it is based on work to be published in [1]. It should be noted that very little is known of the dual field theory at present, and investigating this theory is not the purpose of this thesis. Rather, we will present a framework in which to carry out calculations on the gravity side. In particular, the Lifshitz boundary stress-energy tensor will be calculated and Ward identities will be derived. To carry out these calculations it is necessary to define several deformations of Asymptotically locally Lifshitz spacetimes. We will show that the boundaries of these spacetimes possess a rich geometric structure, namely that of Newton–Cartan with torsion.

### Outline

This thesis will proceed by first introducing required concepts. Thus, chapter 1 opens with an introduction of the holographic principle which will serve as underlying motivation for the AdS/CFT correspondence, for which a heurestic derivation is presented in section 1.2. The chapter ends with a brief discussion of novel holographic dualities and phenomenological approaches to holography. Chapter 2 introduces necessary concepts from geometry. Hypersurfaces will play an important rôle and are described in detail in section 2.1. The geometry of Anti-de Sitter space is described in detail in section 2.2, while the extension of AdS to Asymptotically locally AdS is explained in section 2.3. Section 2.4 introduces the important Lifshitz and Asymptotically locally Lifshitz spacetimes. The Schrödinger spacetime is also briefly reviewed for completeness. In chapter 3 the construction of a well-posed variational problem for AlAdS spaces is discussed. This requires the introduction of the Gibbons-Hawking boundary term in section 3.1 as well as the procedure of holographic renormalization, described in section 3.2. The chapter is concluded with an investigation of pure AlAdS gravity. Finally, the concept of dimensional reduction is introduced in chapter 4. Section 4.1 discusses the Kaluza-Klein reduction as first described in 1926, and will serve as motivation and as a means of introducing the general ideas of dimensional reduction. In section 4.2 the general idea behind Scherk-Schwarz reduction is described. The Freund-Rubin compactification is introduced in section 4.3 where it is used to obtain an axion-dilaton theory on $\mathrm{AlAdS}_5$ from type IIB supergravity.

Having set the stage in chapters 1–4, we proceed by introducing the specific model of interest in chapter 5. The 5-dimensional theory is introduced in section 5.1, and in section 5.2 the Scherk-Schwarz reduction of the 5-dimensional theory is shown to yield a $z = 2$ Lifshitz theory. Section 5.3 briefly discusses the dual field theory. In chapter 6 the framework for doing Lifshitz holography is presented. In section 6.1 we introduce frame fields and discuss the boundary conditions that these are subjected to in order to reproduce a Lifshitz spacetime. The sources are also defined. In section 6.2 the boundary geometry is investigated and it is shown that the geometry on the boundary is, in a special case, Newton-Cartan. Calculations of the vevs and their associated Ward identities are presented in section 6.3, where also the conformal anomaly is investigated. We end with some concluding remarks.

Appendix A summarizes our conventions, while appendix B presents derivations of some important identities. Conformal field theory and $\mathcal{N} = 4$ SYM is briefly reviewed in appendix C.

# Chapter 1

# The Holographic Principle and the AdS/CFT Correspondence

The idea that our universe might obey a holographic principle was first put forth by 't Hooft in 1993 [3] and subsequently developed by Susskind [5]. The first realization of this principle came with the Maldacena conjecture on the AdS/CFT correspondence [2] in 1997. Today, the investigation of holographic theories is a very active area of research and Maldacena's paper is one of the most cited papers of theoretical physics.

In this chapter we will review these ideas as follows: In section 1.1 an overview of the original holographic principle is presented, along with some recent refinements of the idea. In section 1.2 the large $N$ limit of gauge theories is introduced and a heurestic derivation of the AdS/CFT correspondence is given. A brief discussion of the framework of AdS/CFT is also presented. Finally, in section 1.3, novel realizations of the holographic principle are discussed and a brief example of a phenomenological approach is given.

## 1.1 The Holographic Principle

In 1993 't Hooft showed that the number of degrees of freedom of a given gravitational system grows not with the volume of the system, as might be expected naïvely, but with the surface area of the system [3]. Recall that the entropy of a Schwarzschild black hole is given by $S_{\mathrm{BH}} = \frac{A}{4}$ [6]. It turns out that black holes are the most entropic objects which can exist within a given spherical surface [3]. If we were to attempt to estimate how many independent states that are required to completely describe the physics in a given system, we could do so knowing the entropy since $N = e^S$, with $N$ being the number of needed states[1]. Furthermore, since we are interested in some system in the thermodynamic limit, we could in principle be completely ignorant about the underlying microstates. The entropy of our system, which we can take to be spherical, will then be bounded above by the entropy of a black hole, $S \leq \frac{A}{4}$, since this is the most entropic configuration in existence. This is exactly the content of the spherical entropy bound, proposed by Susskind

[1]In this section only, we use $N$ for the number of states and $\mathcal{N}$ for the number of degrees of freedom. Otherwise they, as is standard, refer to the number of colours and the number of supersymmetry generators, respectively. Furthermore, all quantities are given in Planck units.

in [5],

$$S_{\text{Matter}} \leq \frac{A}{4}. \tag{1.1}$$

Furthermore, this means that the number of states required to completely describe the system is bounded by

$$N \leq e^{A/4}, \tag{1.2}$$

and the number of degrees of freedom is then

$$\mathcal{N} = \ln N = \frac{A}{4}. \tag{1.3}$$

Hence, it appears that the degrees of freedom required to describe the system scales as the area of the system.

A field theoretical approach to the same problem yields a different answer [7]. In quantum field theory one considers space to be filled by harmonic oscillators, such that a harmonic oscillator with an infinite dimensional Hilbert space sits at each point in spacetime. Thus there are an infinite number of harmonic oscillators, each with an infinite spectrum. However, once again our ignorance comes to the rescue. We should not expect a quantum field theory to resolve space at sizes comparable to the Planck length. Hence, we can consider there to be only one harmonic oscillator per Planck volume. Furthermore, in order to not form a black hole at the location of the oscillator, the energy should be bounded above by the Planck energy, since putting more energy into a Planck cube would form a black hole. The number of oscillators is then $V$ and the number of states of each oscillator is $n$. The total number of independent quantum states in the system is therefore

$$N \propto n^V, \tag{1.4}$$

meaning that the number of degrees of freedom is

$$\mathcal{N} \propto V, \tag{1.5}$$

scaling as the volume of the system, in contradiction to the argument above.

One of the above estimates are obviously wrong. Recall that for Schwarzschild black holes we have $R_{\text{Schw}} = 2M$, and the mass is therefore bounded by $M \leq \frac{R_{\text{Schw}}}{2}$. This is just the UV cut-off imposed in the field theoretical approach above. There, however, it was imposed on each Planck cube seperately, meaning $R = 1$. On larger scales, the UV cut-off would allow the formation of black holes as $M \propto R^3$, violating the bound imposed by the black hole itself. Thus, most of the states contributing to the entropy in the field theoretical approach are far too massive to be gravitationally stable and would form a black hole long before they would reach high energies. Demanding that the black hole should still be contained within the system of surface area $A$, we see that the spherical entropy bound can be saturated but never violated. The field theory estimate is therefore invalid, as many of the counted states cannot actually be used to store information. This lends support to an idea by 't Hooft, namely that black holes themselves represent a natural physical cut-off in field theories. This is in stark contrast to the fixed energy or distance cut-off usually applied and, as demonstrated above, a black hole cut-off scales with the size of the system. Note, however, that the introduction of black holes as a cut-off is not a confirmation that the maximal entropy is given by the area. In fact, the spherical entropy bound is exceeded in a number of spacetimes such as cosmological ones or spacetimes with no spherical symmetry. Hence it cannot be used to establish a general correspondence between the area of a system and the number of degrees of freedom. Fortunately, another bound exists, which is not known to be exceeded. This bound goes beyond the laws of black hole thermodynamics, hinting that it might be a consequence of a more fundamental theory. This is the covariant entropy bound [8], to which we now turn.

### 1.1.1 The covariant entropy bound

The covariant entropy bound was first conjectured by Bousso in 1999 [8] and later proved, in a limited setting, by Flanagan, Marolf and Wald [9]. In this section we present the construction and definition of the covariant entropy bound, but we will not present the details of the proof. The presentation will follow [7].

Consider any codimension 2 surface $B$. Construct light rays emanating from the surface $B$ and define a codimension 1 region $\mathcal{L}$ by following all non-expanding (in the sense of the Raychaudhuri equation, $\theta \leq 0$) light rays emanating from $B$. Hence $\mathcal{L}$ is a null hypersurface called a light-sheet. The construction is outlined in fig. 1.1. According to the Raychaudhuri equation, light rays with non-positive expansion will terminate at a caustic point, and the light-sheet can therefore be constructed by considering the family of all light rays emanating from $B$ and terminating at the caustic point. Due to the termination, the area of the light-sheet will be finite. In terms of quantities defined in figure 1.1, the surfaces $F_1$ and $F_3$ are light-sheets as these both terminate at a caustic point, while $F_2$ and $F_4$ are expanding indefinitely. The covariant entropy bound is the

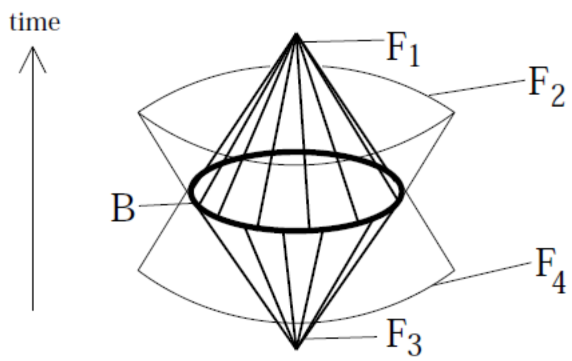


**Figure 1.1:** Pictorial description of light-sheets. The cones $\mathrm{F}_1$ and $\mathrm{F}_3$ have negative expansion and hence corresponds to light-sheets. The essence of the covariant entropy bound is that the entropy on each light-sheet will not exceed the area of B. The surfaces $\mathrm{F}_2$ and $\mathrm{F}_4$ are generated by light rays with non-negative expansion and do therefore not correspond to light-sheets. Figure from [7].

statement that for surfaces generated by non-expanding light rays,

$$\theta \leq 0\,, \tag{1.6}$$

the entropy is related to the area by

$$S[\mathcal{L}(B)] \leq \frac{A(B)}{4} \tag{1.7}$$

applied to each light-sheet of the non-expanding null hypersurfaces. The entropy on the light-sheet is to be thought of as follows: In the case of a thermodynamically isolated system, the past and future directed light-sheets can, to a good approximation, be thought of as light cones with the matter system completely enclosed within, in the same sense that a $t = \text{cst}$ surface contains the system. Light-sheets are just a different way of keeping the system fixed in time, only the time kept fixed is the light cone time. The entropy on the light-sheet is therefore just the entropy of the matter system contained. In fact, the covariant entropy bound contains the spherical entropy bound in cases where this applies. The situation is more complicated in the case where the light-sheets are allowed to vary, that is when the system is no longer thermally isolated. The calculation can still be performed in a static, asymptotically flat space using a black hole as a bound, however, in the most general case there is too much freedom in the choice of geometry to check the relation (1.7) explicitly.

One might worry that in the case where $\theta = 0$, the light-sheets become infinite in extent as they never terminate at a caustic point. This case was indeed examined in [10]. However, there the effects of quantum fluctuations were neglected. In any physically reasonable system the energy density of radiation will experience tiny quantum fluctuations which will drive the expansion parameter $\theta$ away from zero. If it becomes positive the light-sheet will terminate by the condition (1.6). If it is fluctuating but never positive, then it will be negative on average and the light rays will terminate at a caustic point.

As mentioned above, the covariant entropy bound was proved in two restricted cases by Flanagan, Marolf and Wald [9]. The assumptions made in the proofs can be used to remove a large class of counter-examples, thus improving the validity of the covariant entropy bound. The specific assumptions made will not concern us further and will play no rôle in the formulation of the holographic principle.

### 1.1.2 Formulation of the holographic principle

The holographic principle will play an essential rôle in the following chapters. It will serve as our main motivation for pursuing a gravitational description of field theory going beyond the well-established AdS/CFT correspondence. This formulation was first given by Bousso [8, 11]. The fact that the covariant entropy bound is not a consequence of known physical laws, but still has general validity, leads one to conclude that it must originate from some more fundamental theory. This leads to the holographic principle:

> *The covariant entropy bound must be manifest in any underlying theory unifying the quantum theory of matter with spacetime. From this theory the matter*

*content of Lorentzian geometries must emerge in a way such that the number of degrees of freedom, $\mathcal{N}$, involved in the description of the light-sheet $\mathcal{L}(B)$, satisfies*

$$\mathcal{N} \leq \frac{A}{4}.$$

This entails a dramatic reduction in the number of degrees of freedom required to give a sufficient description of Nature, implying that the current description of Nature through quantum field theory is highly redundant. There are two main approaches to constructing a theory which encorporates a holographic principle.

One approach is to insist on locality and hope to identify a certain explicit gauge invariance, which would leave the theory holographic by reducing the number of degrees of freedom such that they would satisfy the covariant entropy bound. Such an approach was taken up in [12], but an emergence of an area's worth of degrees of freedom has not yet been demonstrated.

The other approach is perhaps the more well-known. In this case locality is an emergent phenomenon and the fact that a theory can be completely described by degrees of freedom on the surface leads to interesting dualities, the most famous of them, the AdS/CFT correspondence, will be described in the next section. The holographic principle is thus taken very literally, and one takes the degrees of freedom of a theory containing gravity to be entirely determined by quantities on the boundary of spacetime. From this viewpoint, fields in the bulk spacetime will imprint data on the boundary and in this way act as sources for the operators of the theory living on the boundary. Hence, there exists a one-to-one map between the bulk and the boundary and the theories are said to be dual to each other, giving credence to the term holographic duality.

In this thesis we will take the second viewpoint and use our knowledge of the gravitational theory to attempt to build a field theory on the boundary.

## 1.2 The AdS/CFT Correspondence

The AdS/CFT correspondence was first formulated in 1997 by Maldacena [2], who considered the large $N$ limit of superconformal field theories and pointed out that they admit a dual description using supergravity on an Anti-de Sitter space times a compact manifold. Important details of the correspondence were later worked out by Witten [13] and Gubser, Klebanov and Polyakov [14]. In this section the large $N$ limit of gauge theories will be considered followed by a discussion of the arguments given in [2] leading to the AdS/CFT correspondence.

### 1.2.1 The Large $N$ limit of gauge field theories

It is well-known that the theory of quantum chromodynamics, based on the gauge group $SU(3)$, is asymptotically free. At energies above roughly 1 GeV [15] the theory therefore becomes amenable to a perturbative expansion in the coupling constant. However, at low energies, where interesting phenomena such as confinement occurs, the theory is strongly

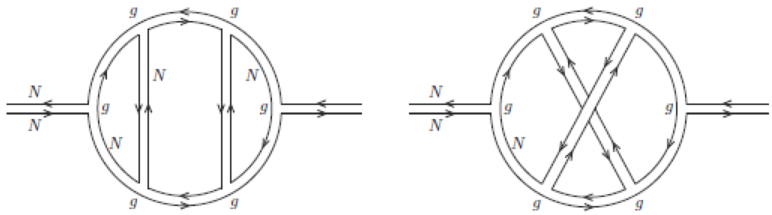


**Figure 1.2:** Two three-loop Feynman diagrams contributing to the gluon propagator of the theory (1.8). The first diagram can be drawn on the surface of a sphere and is called planar. The second must be drawn on the surface of a 2-torus and is therefore non-planar. In the large $N$ limit all non-planar diagrams are suppressed. Figure from [20].

coupled and a perturbative expansion is not an option. There have been several proposals for non-perturbative approaches to QCD, one of which is lattice QCD and another being chiral perturbation theory. Especially in the recent years, lattice QCD has been remarkably successful in making predictions that can be verified experimentally [16, 17], and with the advent of better algorithms and faster computers, it seems the 'Berlin wall' [18] may finally be surmounted. However, in this context we are interested in analytical approaches, one of which is offered by 't Hooft's large $N$ expansion [19].

The idea of the large $N$ expansion is to consider the number of colours, $N$, to be large. Thus, the gauge group becomes $SU(N)$. One might worry that increasing the number of colours would increase the complexity of the dynamics, however, in most cases the opposite turns out to be true. Consider the Yang-Mills Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_{f=1}^{n_f} \bar{\psi}_f \left(i\not{D} - m_f\right)\psi_f \,. \tag{1.8}$$

Here $D_\mu = \partial_\mu - igA^a_\mu T^a$ is the gauge covariant derivative of the gauge group $SU(N)$, the gauge field is in the adjoint representation, the $T^a$ are the (Hermitian and traceless) generators of the Lie algebra $\mathfrak{su}(N)$ and $g$ is the bare gauge coupling. The trace is over the gauge group. The quarks $\psi_f$ are in the fundamental representation of the gauge group and $f$ labels the number of flavours. In the following we will keep the number of flavours fixed. The properties of the $SU(N)$ gauge group implies that the propagators for the gauge bosons are [20]

$$\left\langle A^i_{\mu j}(x)A^k_{\nu l}(y)\right\rangle \propto \delta^i_l\delta^k_j - \frac{\delta^i_j\delta^k_l}{N}\,, \tag{1.9}$$

so Feynman graphs can be represented using the double line notation. Furthermore, in the large $N$ limit the second term on the right hand side can be ignored. Effectively this corresponds to considering a $U(N)$ gauge group instead, however, we are only interested in the leading order behaviour in the following, and the distinction between $U(N)$ and $SU(N)$ will not matter. The contributions to the gauge boson 2-point function can be sorted according to the number of vertices ($V$), loops ($I$) and propagators ($P$), see figure 1.2. Each diagram will contribute a factor [21]

$$g^{2P-2V}N^I = \left(g^2N\right)^{P-V} N^{I+V-P} \tag{1.10}$$

$$= \lambda^{P-V}N^{\chi} \tag{1.11}$$

to the amplitude. Here the 't Hooft coupling $\lambda = g^2N$ was defined and the Euler characteristic [22] was used. Recall that the Euler characteristic can also be written as $\chi = 2-2g$ where $g$ is the genus of the simplest Riemann surface on which the diagram can be drawn. For instance the diagram to the right in fig. 1.2 cannot be drawn on a sphere ($g = 0$) due to the two non-intersecting propagators, and must be drawn on a torus ($g = 1$). The amplitude for a generic $n$-point function can therefore be written as [23]

$$\mathcal{A} = \sum_{g,b=0}^{\infty} N^{2-2g-b-n}F_{g,n}(\lambda), \tag{1.12}$$

where we have added the possibility for the surfaces to have a boundary, $b$, as well [20]. The boundaries of the surfaces are related to quark loops and hence, in the large $N$ limit, the quenched approximation (neglecting virtual quark loops) is well justified. The 't Hooft limit is defined as taking $N \to \infty$ with $\lambda = g^2N$ kept fixed. Due to the factor of $N^{2-2g-b-n}$ in the amplitude, diagrams which can be drawn on the surface of a sphere will dominate in the 't Hooft limit. Such diagrams are often referred to as planar diagrams and the large $N$ limit as the planar limit. Furthermore, external propagators contributes a factor of $n$, so the most dominant part of 2-point functions will go as $\mathcal{O}(N^0)$, 3-point functions as $\mathcal{O}(N^{-1})$ and so forth. Hence $1/N$ acts as an effective coupling constant (and can be used as an expansion parameter), in addition to the 't Hooft coupling $\lambda$. This expansion is analagous to the topological expansion of perturbative string theory in which one is also summing over the genera of Riemann surfaces. This amusing fact was noted by 't Hooft already in 1974 when he first performed this calculation [19].

Below we will show that in a specific limit of string theory one recovers exactly the planar limit of a specific gauge theory, namely $\mathcal{N} = 4$ Super Yang-Mills. This will lead to the formulation of the Maldacena conjecture.

### 1.2.2 Motivating the AdS/CFT correspondence using D3-branes

There is no rigorous proof of the AdS/CFT conjecture. In this section an attempt will be made to motivate the conjecture using the open/closed string duality. We will consider a stack of $N$ coincident D3-branes from the point of view of both open and closed strings in type IIB string theory and apply the Maldacena limit, $\alpha' \to 0$, to both cases. In this limit the dynamics in the bulk will decouple from the dynamics on the brane and this will facilitate a comparison between the two systems. Imposing that the open and closed strings should describe the same physical theory will result in the AdS/CFT conjecture.

#### Open strings on D3-branes

In this section we will consider D3-branes from the point of view of open strings. In the Maldacena limit this will result in an $\mathcal{N} = 4$ $SU(N)$ SYM theory living on the world-volume of the D3-brane. Furthermore, we consider the bulk action and show that, in the

Maldacena limit, this reduces to 10 dimensional supergravity. The complete (low energy effective) action, with the massive modes integrated out, is

$$S_1 = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}}, \tag{1.13}$$

and for simplicity we will start by considering one D3-brane resulting in a $U(1)$ gauge theory. This situation will then be generalized to arbitrary $N$.

Consider one D3-brane in type IIB string theory. The mass spectrum of an open string will depend on the transverse distance between the D$p$-branes on which the string begins and ends as well as the excitation level. Thus, an open string which begins and ends on the same brane will have a massless ground state. As mentioned previously we consider the low energy limit such that only the massless modes are excited. The massless mode will induce a massless $U(1)$ gauge theory on the world-volume, which is effectively 4-dimensional. Furthermore, the brane will break half the supersymmetry, meaning it is 1/2 BPS, and the $U(1)$ gauge field will therefore have $\mathcal{N} = 4$ Poincaré supersymmetry. The argument will follow [24].

Consider an $\mathcal{N} = 4$ multiplet with a $U(1)$ gauge field and six scalars, as well as their fermionic superpartners (which we will for the most part ignore). The effective action for this theory on the D3-brane is the Dirac-Born-Infeld (DBI) action (here in flat space):

$$S_{\text{DBI}} = -T_{\text{D3}} \int \mathrm{d}^4\sigma \sqrt{-\det\left(\gamma_{ab} + 2\pi l_s^2 F_{ab}\right)} + \text{fermions}, \tag{1.14}$$

$$\gamma_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{1.15}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a, \tag{1.16}$$

with $\sigma^a$ $(a = 0, \ldots, 3)$ being the coordinates on the world-volume and $T_{\text{D3}}$ the tension of the D3-brane, $T_{\text{D3}} = \frac{1}{(2\pi)^3 g_s (\alpha')^2}$. The $X^\mu$ are the coordinates of the 10-dimensional embedding space, such that $\gamma_{ab}$ is the metric induced on the world-volume of the D3-brane. We can choose the embedding such that

$$X^a(\sigma) = \sigma^a\,, \tag{1.17}$$

while the remaining six coordinates, the tranverse position of the brane, can be parameterized by the six scalars:

$$X^{i+3}(\sigma) = 2\pi\alpha'\Phi_i(\sigma)\,, \qquad i = 1, \ldots, 6\,. \tag{1.18}$$

We can, without loss of generality, place the brane at $X^{i+3}(\sigma) = 0$ and therefore only consider fluctuations around this position. To leading order in $\alpha'$ the fluctuations of the position of the D3-brane is described by the scalar fields. The induced metric becomes

$$\gamma_{ab} = \eta_{ab} + (2\pi)^2(\alpha')^2 \sum_{i=1}^{6} \partial_a\Phi_i \partial_b\Phi_i \tag{1.19}$$

and in the low energy limit, the determinant in the DBI action can be expanded to lowest non-vanishing order in $\alpha'$. Such an expansion is most easily performed using the trace-log formula for the determinant (B.5) and yields

$$\sqrt{-\det\left(\eta_{ab}+2\pi\alpha' F_{ab}+(2\pi)^2(\alpha')^2\sum_{i=1}^{6}\partial_a\Phi_i\partial_b\Phi_i\right)} = \\ 1+\frac{1}{2}(2\pi)^2(\alpha')^2\left(F_{ab}F^{ab}+\sum_{i=1}^{6}\partial_a\Phi_i\partial^a\Phi_i\right)+\mathcal{O}((\alpha')^3), \quad (1.20)$$

where the field strength does not contribute at order $\alpha'$ since it is traceless. Taking the Maldecena limit, $\alpha \to 0$ keeping the energy fixed, while ignoring the constant contribution from the 1 in the DBI action, and applying the value of the brane-tension we find

$$S_{\mathrm{DBI}} = -\frac{1}{g_{YM}^2}\int \mathrm{d}^4\sigma\left(F_{ab}F^{ab}+\sum_{i=1}^{6}\partial_a\Phi_i\partial^a\Phi_i\right)+\text{fermions}, \quad (1.21)$$

where we identified $g_{YM}^2 = 4\pi g_s$. This is the action of $\mathcal{N}=4$ SYM with a $U(1)$ gauge group in 4 dimensions.

Extending the number of branes to $N$, the open strings can have endpoints on separate branes and will no longer be massless. However, in the limit of coincident branes, the string ground states will again be massless, but additional massless states will be present due to the strings being able to end on $N-1$ other D3-branes. The above argument can be generalized to this case and the induced theory will have a $U(N) = U(1)\times SU(N)$ gauge symmetry. The overall $U(1)$ symmetry corresponds to the position of the branes and can therefore be ignored when considering dynamics on the brane, making the theory an $SU(N)$ gauge theory containing an $SU(N)$ gauge field and six scalars in the adjoint representation of $SU(N)$, in addition to fermions. In the low energy limit the theory is therefore $\mathcal{N}=4$ $SU(N)$ SYM [25]. This is a superconformal theory with symmetry group $SU(2,2|4)$, which has, in addition to various fermionic symmetries which we will not be concerned with here, the bosonic symmetry group $SO(2,4)$ as well as an $SU(4)\cong SO(6)$ R-symmetry which rotates the six scalar fields into each other. We will later see that these symmetries match exactly with the ones on the AdS side. In the Maldacena limit the number of branes, $N$, and the string coupling, $g_s$, are kept fixed.

The supergravity action for the relevant bosonic fields (graviton and Ramond-Ramond field) is

$$S_{\mathrm{IIB}} = -\frac{1}{16\pi G_{10}}\int \mathrm{d}^{10}x\sqrt{-g}\left(\mathcal{R}+\frac{1}{4\cdot 5!}F^{\mu\nu\rho\sigma\lambda}F_{\mu\nu\rho\sigma\lambda}\right), \quad (1.22)$$

$$F_{(5)} = \mathrm{d}A_{(4)}, \quad (1.23)$$

with $F_{(5)}$ being self-dual, $F_{(5)} = \star F_{(5)}$. The Ramond-Ramond field is sourced by D3-branes at the origin and the supergravity solution describing this is

$$\mathrm{d}s^2 = H^{-1/2}\left(-\mathrm{d}t^2+\sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i\right)+H^{1/2}\left(\mathrm{d}r^2+r^2\mathrm{d}\Omega_5^2\right), \quad (1.24)$$

$$A_{0123} = H^{-1}-1, \quad (1.25)$$

with

$$H = 1 + \frac{l^4}{r^4}, \qquad l^4 = 4\pi g_s N(\alpha')^2. \tag{1.26}$$

Going to the limit $r \gg l$, $H \to 1$ and we recover flat Minkowski space and the Ramond-Ramond field vanishes. Expanding the action around flat space, such that $g = \eta + \kappa h$, we find schematically [26]

$$S_{\text{bulk}} = \frac{1}{2\kappa}\int \mathrm{d}^{10}x\sqrt{-g}\mathcal{R} \underset{r\gg l}{\longrightarrow} \int \mathrm{d}^{10}x\left((\partial h)^2 + \kappa(\partial h)^2 h\right), \tag{1.27}$$

with $\kappa \sim g_s(\alpha')^2$. The interaction between the bulk modes and the brane is also proportional to $\kappa$, so in the Maldacena limit the bulk dynamics decouple from the brane dynamics and gravity becomes free. The action can then be written as the sum of two completely decoupled systems [26]:

$$S_{\text{open}} \overset{\alpha'\to 0}{=} \text{SYM theory on the D3-brane} + \text{supergravity in flat space in the bulk},$$

concluding the treatment from the point of view of open strings.

### Closed strings in a background of $N$ coincident D3-branes

The D-branes also emit closed strings. To each D-brane solution there corresponds a black brane in supergravity sourced by the same Ramond-Ramond fields as the D-branes. The metric describing the black 3-brane is sourced by $N$ units of Ramond-Ramond flux is given by (1.24). We will now consider the Maldacena limit of a non-linear sigma model for string theory on a background given by this metric. First, let us define a new coordinate $z = l^2/r$ which will remain fixed in the Maldacena limit. This corresponds to keeping the mass of the stretched string states fixed. In the decoupling limit the branes are brought together, but the Higgs expectation value corresponding to the seperation in $z$ will remain fixed [2]. With this coordinate the metric becomes

$$G_{MN}\mathrm{d}x^M\mathrm{d}x^N = \frac{l^2}{z^2}\left[\widetilde{H}^{-1/2}\left(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i\right) + \widetilde{H}^{1/2}\left(\mathrm{d}z^2 + z^2\mathrm{d}\Omega_5^2\right)\right] \tag{1.28}$$

$$\widetilde{H} = 1 + \frac{l^4}{z^4}, \qquad l^4 = 4\pi g_s N(\alpha')^2, \tag{1.29}$$

where $M = 0,\ldots,9$. The Polyakov action for the non-linear sigma model on the worldsheet is

$$S_{\text{Polyakov}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-h}h^{ab}G_{MN}\partial_a X^M\partial_b X^N, \tag{1.30}$$

where $h_{ab}$ is the induced metric on the worldsheet. We are interested in the limit $\alpha' \to 0$ meaning $R \to 0$, however, we can rescale the metric such that $\widetilde{G}_{MN} = G_{MN}/l^2$ and the action becomes

$$S_{\text{Polyakov}} = \frac{l^2}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-h}h^{ab}\widetilde{G}_{MN}\partial_a X^M\partial_b X^N. \tag{1.31}$$

Recall that $\lambda = g_{YM}^2 N = 4\pi g_s N$ meaning

$$\frac{l^2}{4\pi\alpha'} = \sqrt{g_s N} = \sqrt{\frac{\lambda}{4\pi}}. \tag{1.32}$$

Hence $\lambda$ remains fixed in Maldacena limit. Since $z \neq 0$ this limit implies $\widetilde{H} \to 1$ and the metric reduces to

$$\widetilde{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N \overset{\alpha'\to 0}{=} \frac{1}{z^2}\left[\mathrm{d}z^2 - \mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i + z^2\mathrm{d}\Omega_5^2\right] \tag{1.33}$$

which is the metric of $\mathrm{AdS}_5 \times S^5$. The Polyakov action then describes a sigma model on $\mathrm{AdS}_5 \times S^5$ with string tension proportional to $\sqrt{\lambda}$. In the asymptotically flat region, $r \gg l$, nothing changes and supergravity is still a good approximation. It is important to note that, when the background is $\mathrm{AdS}_5 \times S^5$, $\alpha' \to 0$ is not a point particle limit as it is in the flat space case, instead it decouples string theory on $\mathrm{AdS}_5 \times S^5$ from supergravity in the asymptotically flat region. Hence we have an action which can be written as

$$S_{\text{closed}} \overset{\alpha'\to 0}{=} \text{Quantum string theory on } \mathrm{AdS}_5 \times S^5 + \text{supergravity in flat space}\,,$$

which is the theory seen from the point of view of the closed strings. The isometry group of $\mathrm{AdS}_5$ is $SO(2,4)$ while that of the 5-sphere is $SO(6) \cong SU(4)$ thus matching the bosonic symmetries on the field theory side of the duality.

To arrive at the action $S_{\text{closed}}$ we integrated out the massive string states. However, the string states in the throat region can have large energies $\sqrt{\alpha'}E_z = \text{cst}$ (with $E_z$ being the energy at a fixed radial position $z$), so why are these not integrated out as well? The argument follows [26]. The energy, $E_\infty$, measured by an observer at infinity is

$$E_\infty = \sqrt{-g_{00}}E_z = \left(1 + \frac{z^4}{l^4}\right)^{-1/4} E_z\,. \tag{1.34}$$

In the $\alpha' \to 0$ limit we have $l \to 0$. Imposing that the energy measured at infinity is finite we find

$$E_\infty \overset{\alpha'\to 0}{\to} \frac{l}{z}E_z \sim \frac{1}{z} \sim \frac{r}{\alpha'} \overset{!}{=} \text{cst}\,, \tag{1.35}$$

where we used (1.29) and the fact that $z = l^2/r$. This is only satisfied in the near-throat region $r \to 0$. Thus, we can have any string excitation close to $r = 0$ (corresponding to an excitation in string theory on $\mathrm{AdS}_5 \times S^5$) since their energy measured at infinity is zero. Therefore, these modes remain in the low energy approximation. This also demonstrates why $\alpha' \to 0$ is not a point particle limit on $\mathrm{AdS}_5 \times S^5$. Close to the throat any string excitation is allowed.

### The Maldacena conjecture

Above we considered a stack of $N$ coincident D3-branes from the point of view of both open and closed strings. Since these two situations are expected to describe the same physical system, we can compare the actions $S_{\text{open}}$ and $S_{\text{closed}}$. Since $S_{\text{open}} = S_{\text{closed}}$ and since the Maldacena limit ensures that the two systems are completely decoupled in both descriptions, we conclude that the supergravity description of flat space is identical in the two actions, and therefore we must have

$$\begin{array}{ccc} \mathcal{N}=4 \text{ SYM with gauge group } SU(N) & & \text{Full quantum type IIB string theory} \\ \text{all } N,\, g_{YM} & \Longleftrightarrow & \text{on } \mathrm{AdS}_5 \times S^5 \\ 4\pi g_s = g_{YM}^2 & & l^4 = 4\pi g_s N(\alpha')^2 \end{array}$$

implying that the correlation functions of the two theories will match. This is the strongest form of the Maldacena conjecture. This formulation of the duality is not very suited for calculations though, since the quantization of string theory on $\mathrm{AdS}_5 \times S^5$ is at present out of reach.

To make the duality more useful, we should therefore look for limits which simplify the string theory side. One such limit is the 't Hooft limit, described in sec. 1.2.1, where one takes $N \to \infty$ while keeping $\lambda = g_{YM}^2 N$ fixed. In this limit it is sufficient to consider only the planar diagrams of the $\mathcal{N} = 4$ $SU(N)$ SYM gauge theory. Recall that $g_{YM}^2 = 4\pi g_s$ meaning

$$\frac{\lambda}{N} = 4\pi g_s \to 0, \tag{1.36}$$

in the 't Hooft limit. Thus $g_s \to 0$ and on the string theory side we can neglect effects of quantum gravity and consider only classical string theory. In fact, this theory can be mapped to a Heisenberg spin chain and is therefore integrable [27]. An expansion in $1/N$ on the field theory side can be mapped to an expansion in the string coupling on the string theory side, although the various terms in the expansions do not necessarily match on either side of the duality. This moderate form of the duality can be formulated as

$$\begin{array}{ccc} \text{'t Hooft limit of } \mathcal{N}=4 \; SU(N) \text{ SYM} & & \text{Classical type IIB string theory} \\ \lambda = g_{YM}^2 N \text{ fixed, } N \to \infty & \Longleftrightarrow & \text{on } \mathrm{AdS}_5 \times S^5 \\ 1/N \text{ expansion} & & \text{Expansion in } g_s \end{array}$$

again an equivalence between the correlation functions of the two theories is implied.

There is yet another limit that can be taken, which will result in the weakest, yet most tractable version of the duality. The (classical) string theory on the worldsheet can be simplified further by taking the tension to be large. This implies that quantum fluctuations are exponentially suppressed in the path integral and the field theory on the worldsheet becomes classical, such that supergravity becomes a good description. The tension is proportional to $\sqrt{\lambda}$, so taking $\lambda \gg 1$ will result in the reduction of classical string theory to supergravity on $\mathrm{AdS}_5 \times S^5$. On the field theory side this limit implies that the gauge theory becomes strongly coupled. A perturbative expansion on the field theory side can be performed in the parameter $\lambda^{-1/2}$ while on the supergravity side loop corrections to the sigma model will be controlled by $\alpha'$. The result is a strong/weak duality which can be stated as

| | | |
|---|---|---|
| Large $\lambda$ limit of $\mathcal{N}=4$ $SU(N)$ SYM | | Classical type IIB supergravity |
| with $N\to\infty$ | $\Longleftrightarrow$ | on $\mathrm{AdS}_5\times S^5$ |
| $\frac{1}{\sqrt{\lambda}}$ expansion | | $\alpha'$ expansion |

The last of these three statements is the weakest form of the AdS/CFT correspondence. It has a very elegant formulation in terms of the on-shell value of the supergravity action, which we will consider below. There we will also consider how the symmetries of the two sides of the duality are related. It is worth remarking that the existence of an expansion parameter on either side of the duality does not necessarily imply that the expressions to each order will be equivalent, e.g. the third-order term in $1/\sqrt{\lambda}$ is not necessarily equivalent to the third-order term in $\alpha'$.

### 1.2.3 Correlation functions in the AdS/CFT correspondence

From the weakest version of the AdS/CFT correspondence we should be able to derive correlation functions in the ($N\to\infty$, $\lambda\gg 1$) field theory by doing calculations in the supergravity theory. We will now explore how this is done in detail. Recall that the partition function of a field theory encodes information about the correlation functions of the theory. A partition function for a generic CFT[2] in the presence of sources will schematically look like

$$Z[\phi_{\Delta_i}] = \left\langle \exp\left(-\int \mathrm{d}^d x \phi_{\Delta_i}(x)\mathcal{O}_{\Delta_i}(x)\right)\right\rangle_{\mathrm{CFT}}, \tag{1.37}$$

where a sum over $i$ is implied. Here $\mathcal{O}_{\Delta_i}(x)$ are primary operators with conformal weights $\Delta_i$. The classical fields $\phi_{\Delta_i}$ are said to source these operators in the sense that

$$\langle \mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}\cdots\rangle = (-1)^n \frac{1}{Z[\phi_{\Delta_i}]}\left.\frac{\delta^n Z[\phi_{\Delta_i}]}{\delta\phi_{\Delta_1}\delta\phi_{\Delta_2}\cdots}\right|_{\phi_{\Delta_i}=0}. \tag{1.38}$$

Attempting to directly apply this procedure to the AdS/CFT correspondence we immediately run into problems. One theory is formulated in 10 dimensions, and the other in only 4 dimensions. However, the dynamics on the 5-sphere can be captured by performing a Kaluza-Klein compactification over it, keeping all the massive modes. This leaves us with $\mathrm{AdS}_5$ with the added field content from the 5-sphere and the difference in dimensions is only one. We should therefore find a suitable 4-dimensional subspace of $\mathrm{AdS}_5$. The key to finding such a subspace lies in the special nature of the (conformal) boundary of $\mathrm{AdS}_5$ located at $z=0$ in Poincaré coordinates. The precise definition of conformal boundary is given somewhat later (see section 2.2.1). Suffice to say that the group of conformal isometries acts on the boundary of AdS in the same manner as the conformal group acts on 4-dimensional spacetime. The precise details of this is given in section 2.2.3. This means that the boundary values of fields ($\phi$ denotes a generic, possibly tensorial, field) $\lim_{z\to 0}\phi(z,x)$ on $\mathrm{AdS}_5$ transform as representations of the conformal group on 4-dimensional spacetime.

[2] In this section, Euclidean signature is used for convenience.

Consider the partition function of a theory on $\mathrm{AdS}_{d+1}$ with the boundary values, $\phi_{(0)}(x)$, of the fields kept fixed:

$$Z_{\mathrm{AdS}_{d+1}}[\phi_{(0)}] = \int_{\phi|_{z=0}=\phi_{(0)}} \mathcal{D}\phi e^{-S[\phi]}. \tag{1.39}$$

From the AdS/CFT correspondence this should resemble the partition function of the CFT. If the theory on $\mathrm{AdS}_{d+1}$ is a free scalar field theory, the action is

$$S[\phi] = \frac{1}{2}\int_{\mathrm{AdS}_{d+1}} \mathrm{d}^{d+1}x\sqrt{-g}\left((\partial\phi)^2 + m^2\phi^2\right) , \tag{1.40}$$

yielding the equations of motion

$$-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0\,. \tag{1.41}$$

This can be written as a modified Bessel equation using the field redefinition $\phi \equiv z^2\chi$ and Fourier transforming in the $x$-directions (yielding $\chi_k$):

$$z^2\partial_z^2\chi_k + z\partial_z\chi_k - (k^2z^2 + \nu^2)\chi_k = 0 \tag{1.42}$$

with

$$\nu = \pm\sqrt{4+m^2}\ . \tag{1.43}$$

The complete solution [28] is given by a superposition of the two independent solutions

$$\chi_k = K_\nu(kz) + I_\nu(kz)\ . \tag{1.44}$$

The asymptotic behaviour of the original field in real-space is then

$$\phi(z,x) \sim z^{d-\Delta}\phi_{(0)}(x) + \cdots + z^{\Delta}\phi_{(2\Delta-d)}(x) + \cdots\ . \tag{1.45}$$

Imposing that this expansion is a solution to the equation of motion implies that

$$\Delta(\Delta - d) = m^2, \tag{1.46}$$

which, from a CFT point of view, is the well-known relation between the mass of the source field and the conformal dimension, $\Delta$, of the dual operator. The asymptotic expansion (1.45) implies that the boundary value $\phi_{(0)}$ will transform as

$$\phi_{(0)}(x) \to \lambda^{d-\Delta}\phi_{(0)}(\lambda x)\,, \tag{1.47}$$

under a conformal rescaling $x \to \lambda x$ and $z \to \lambda z$. This is exactly the scaling of sources of (scalar) conformal primary operators found in equation (C.13), and we may identify $\phi_{(0)}$ as the source of such a scalar primary operator in the dual CFT. Furthermore, in every CFT there exists a local tensor $T^{ab}(x)$ with dimension $\Delta = d$. It is sourced by the boundary value of a spin-2 field $h_{ab}$ on AdS. The relation between the mass and conformal dimension for a symmetric spin-2 field is identical to (1.46) [29] and we see that $\Delta = d$ implies $m^2 = 0$, so $h_{ab}$ must be a massless, but dynamical spin-2 field. The only possibility is therefore a graviton. For more general gravity theories, where the bulk metric only asymptotes to AdS in a certain sense to be made precise in section 2.3, $h_{ab}$ will be the induced metric on the conformal boundary of the bulk spacetime. Note that it is the boundary values of the bulk fields which source the operators in the dual CFT. This ties in nicely with the holographic principle described above.

The relation (1.39) together with the fact that the boundary values of fields in the gravity theory act as sources for operators in the dual CFT indicates a relation between the partition functions of the two theories. In the strongest form of the conjecture, the CFT partition function with the boundary values as sources for primary operators is equivalent to the partition function of quantum type IIB string theory on $\mathrm{AdS}_5 \times S^5$ evaluated with certain boundary conditions. Hence,

$$Z_{\text{type IIB}}[\phi_{(0)}] = Z_{\text{CFT}}[\phi_{(0)}]. \tag{1.48}$$

However, as mentioned previously, the quantization of string theory on $\mathrm{AdS}_5 \times S^5$ is no easy task so some approximation is required. Fortunately, we already know that by going to the limit $N \to \infty$ and $\lambda \gg 1$ supergravity becomes a good approximation to string theory. Furthermore, setting the fermionic fields to zero is a consistent truncation and doing so we end up with Einstein gravity. Thus,

$$Z_{\text{type IIB}}[\phi_{(0)}] \approx \sum_{\{\phi_{\text{cl}}\}} e^{-S_{\text{Einstein}}[\phi_{\text{cl}}]}\,, \tag{1.49}$$

where $\{\phi_{\text{cl}}\}$ are the classical values of the fields, found by extremizing the action subject to the boundary conditions $\phi|_{z=0} = \phi_{(0)}$. The weakest form of the gauge/gravity duality is then

$$e^{-S^{\text{on-shell}}[\phi_{(0)}]} = Z_{\text{CFT}}[\phi_{(0)}]\,. \tag{1.50}$$

However, the on-shell value of the Einstein action will contain infrared divergences related to the infinite volume of AdS space, while, in the general case, the field theory side will contain UV divergences. This can be remedied by using the technique of holographic renormalization [30] to subtract divergences from the bulk gravity action and render it finite. By the relation (1.50) this renders the field theory finite as well. Holographic renormalization will be described in detail in section 3.2. When such a procedure has been carried out the correspondence can be written as

$$S^{\text{on-shell}}_{\text{ren}}[\phi_{(0)}] = W_{\text{CFT}}[\phi_{(0)}]\,, \tag{1.51}$$

where $W_{\text{CFT}}[\phi_{(0)}]$ is the generating functional of connected diagrams, familiar from ordinary quantum field theory. The connected correlation functions of the CFT can now be found using the formula (1.38):

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\cdots\rangle_{\text{connected}} = (-1)^n \frac{\delta^n S^{\text{on-shell}}_{\text{ren}}[\phi_{(0)}]}{\delta\phi_{(0)\Delta_1}(x_1)\delta\phi_{(0)\Delta_2}(x_2)\cdots}\,, \tag{1.52}$$

with the sign originating from the relation (1.37). Obviously the calculation of $n$-point functions with $n > 2$ requires the introduction of interaction terms in the action (1.40).

However, once this is taken care of, the formula (1.52) in principle determines the $n$-point functions of the dual field theory. It is found that these agree with the correlation functions calculated in $\mathcal{N} = 4$ SYM. There are many other checks of the correspondence, for instance agreement is found between the trace anomalies on either side [31]. On the gravity side this is a rather non-trivial calculation, and agreement is thus a convincing argument in favor of the duality.

## 1.3 Novel Realizations of Holography

Since the inception of the original AdS/CFT correspondence, a lot of research has been done to extend the conjecture and to find new realizations of the holographic principle. The approaches taken usually fall into two broad categories. One approach attempts to extend the Maldacena conjecture and go beyond the usual $\mathrm{AdS}_5 \times S^5$ geometry. This is typically done by considering a different string theory setup by the addition of e.g. fractional branes [32] or by considering wrapped branes [33]. By taking appropriate limits, the dual field theories can be determined. These will be different from the $\mathcal{N} = 4$ SYM resulting from the ordinary approach, and in most cases the theories fail to be conformal. For examples of this approach see for instance [32, 33]. The other approach is more phenomenological in nature. In this case one assumes that each gravity theory has a dual field theory. One then engineers a gravity theory in 4 or 5 dimensions (depending on the desired application) whose boundary exhibits the features one wishes to study in the field theory. However, in this approach one does not actually know the dual field theory, only that it exhibits certain features. In fact, one cannot even be certain that a dual field theory exists. Below, we briefly review a very interesting example of the second approach where subsequent research has resulted in string theory completions of many of the actions involved.

It should be stressed that the approach taken in this thesis resembles the first of the two, i.e. the spacetimes described in chapter 5 are consistent truncations of string theory.

### 1.3.1 Holographic superconductors

As an example of a phenomenological approach to holography, we briefly study holographic superconductors. A field theoretical description of the so-called high-$\mathrm{T}_c$ superconductors has long been sought by condensed matter theorists and it is hoped that the study of holographic superconductors will provide insights into this problem. This subsection will only provide a brief overview of this very active area of research.

To engineer a holographic superconductor, one should start with a gravity theory exhibiting the features of known superconductors. Thus, the gravity theory should include some notion of temperature. In addition the gravity theory should contain some operator which, below a certain temperature, obtains a vev breaking a $U(1)$ gauge symmetry. From applications of holography to condensed matter physics we know that to discuss charge transport on the boundary, the bulk theory should be an Einstein-Maxwell theory [34]. The notion of temperature is provided by placing a black hole in the bulk theory, as described in [35]. If we consider $s$-wave superconductors for simplicity, the condensate breaking $U(1)$ gauge invariance will not carry an angular momentum and can therefore be described by a scalar field. The action should therefore contain a complex scalar field with a potential such that, below a certain temperature, the scalar field obtains a non-vanishing vev. However, having a non-vanishing field, apart from the gravitational and electromagnetic ones, outside the event horizon of a black hole (referred to as the black hole having hair) is difficult, and is, in fact, prohibited in some cases. Fortunately, it was shown by Gubser [36] that a Schwarzschild black hole in AdS space can have scalar hair forming for a certain range of values of a parameter related to the temperature, with lower temperatures corresponding to an increasing condensate. This result allows one to write down a Lagrangian reproducing the desired features [37]:

$$\mathcal{L} = \frac{1}{2\kappa_5^2}\left(R + 2\Lambda\right) - \frac{1}{4g^2} F^{\mu\nu}F_{\mu\nu} - \left|\partial_\mu \phi - iqA_\mu \phi\right|^2 - m^2|\phi|^2 - V(|\phi|)\,. \tag{1.53}$$

In the superconducting phase of this model, the theory living on the boundary exhibits many of the phenomena of real superconductors, such as a second-order phase transition between the superconducting phase and the ordinary phase and dissipationless conductivity [37]. It should be noted though, that the field theory on the boundary is unknown and the agreement found is purely qualitative. However, the Lagrangian (1.53) can be embedded into string theory for specific values of the potential, so the dual field theory should be within reach.

Holographic superconductivity is a very active area of research and now includes, in addition to the $s$-wave superconductor described above, also $p$- and $d$-wave superconductors [38, 39, 40, 41, 42], as well as superconductors exhibiting Lifshitz scaling symmetry [43, 44]. Some of these references even approach the problem from a full string theoretic perspective.

# Chapter 2

# Spacetime Geometry

The structure of spacetime plays an important rôle in the understanding of the AdS/CFT correspondence. Here we consider the various geometric structures arising in gauge/gravity dualities.

The chapter opens with a description of hypersurfaces whose properties will be very important in the coming chapters. Anti-de Sitter space is then discussed in considerable detail, including the definition of conformal infinity and the isometry group of the boundary. A definition of the very important class of Asymptotically locally AdS spacetimes is then given, along with various properties of these. Finally we discuss the properties of Lifshitz and Asymptotically locally Lifshitz spacetimes, and, for completeness, we briefly review Schrödinger spacetimes.

## 2.1 Geometry of Hypersurfaces

Hypersurfaces play important rôles in gauge/gravity dualities. Apart from that, hypersurfaces are interesting objects in their own right, and a description of them serves as an opportunity to introduce other important geometric constructs such as foliations and projected derivatives. Hypersurfaces are required in order to understand the posing of a well-defined variational principle in general spacetimes and we will see when discussing the Lifshitz boundary geometry that a preferred foliation structure arises naturally.

### 2.1.1 Basic definitions

When discussing hypersurfaces one has to distinguish between two different kinds of geometries: The intrinsic geometry, which deals with the properties of the hypersurface inherited from a higher-dimensional space, and the extrinsic geometry, which is concerned with the embedding of the hypersurface into a higher-dimensional space.

A hypersurface can be defined in the following way [45]:

$$f(x^\mu) = \mathrm{cst}\,, \tag{2.1}$$

meaning we can denote a hypersurface, $\Sigma$, as

$$\Sigma = \{x^\mu \in \mathcal{M} | f(x^\mu) = \mathrm{cst}\}\,. \tag{2.2}$$

For example, a 2-sphere embedded in 3-dimensional flat space can be described by the equation

$$x^2 + y^2 + z^2 = R^2, \tag{2.3}$$

and by parameterizing $(x, y, z)$ by $(\theta, \phi)$ we find the usual relations describing a 2-sphere:

$$x = R \sin\theta \cos\phi \tag{2.4}$$
$$y = R \sin\theta \sin\phi \tag{2.5}$$
$$z = R \cos\theta. \tag{2.6}$$

Equation (2.1) can be used to define a normal vector field to the hypersurface by

$$\zeta^\mu = \nabla^\mu f. \tag{2.7}$$

If we consider any vector $X \in T_p\Sigma$ we have $X[f] = X^\mu \partial_\mu f = 0$ as $\partial_\mu f$ is non-zero only in the direction pointing away from the hypersurface. This also shows that normal vectors are unique up to scaling (we only consider non-null vectors and hypersurfaces here[1]) and we can therefore define a unit normal vector

$$n^\mu = \frac{\sigma \zeta^\mu}{|\zeta^\nu \zeta_\nu|^{1/2}}, \tag{2.8}$$

and demand that $n^\mu$ points in the direction of increasing $f$, $n^\mu \zeta_\mu > 0$. The sign must then be chosen such that

$$n^\mu n_\mu = \sigma \equiv \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike (timelike normal vector)}, \\ +1 & \text{if } \Sigma \text{ is timelike (spacelike normal vector)}. \end{cases} \tag{2.9}$$

A metric will be induced on each hypersurface through its embedding into a higher-dimensional manifold. This induced metric can be found by considering displacements confined to the hypersurface. Since a hypersurface is defined by (2.1) we can always eliminate one of the coordinates in favor of the others, meaning we can write

$$x^\mu = x^\mu(y^a), \tag{2.10}$$

and the vectors given by

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a} \tag{2.11}$$

will lie in $T_p\Sigma$ and thus satisfy $e^\mu_a n_\mu = 0$. Hence, considering only displacements on the hypersurface, we have [46]

$$\begin{aligned} \mathrm{d}s^2_\Sigma &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \mathrm{d}y^a \frac{\partial x^\nu}{\partial y^b} \mathrm{d}y^b \\ &= h_{ab} \mathrm{d}y^a \mathrm{d}y^b, \end{aligned} \tag{2.12}$$

[1] Null hypersurfaces are interesting objects in their own right. See for instance [46] for a brief overview.

where we defined the induced metric

$$h_{ab} = g_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b} \,. \tag{2.13}$$

It is worth noting that this object behaves like a tensor under transformations $y^a \to y^{a'}$, but as a scalar under transformations $x^\mu \to x^{\mu'}$. The induced metric on a hypersurface can also be defined by pulling back the metric on $\mathcal{M}$ to a submanifold $\Sigma$. If the submanifold is a hypersurface, the pullback of $g_{\mu\nu}$ is $h_{ab}$. Demanding that the equations

$$g^{\mu\nu} n_\mu n_\nu = \sigma \,, \qquad g^{\mu\nu} e_{\mu a} e_{\nu b} = h_{ab} \,, \qquad g^{\mu\nu} n_\mu e_{\nu a} = 0 \tag{2.14}$$

are satisfied we find the completeness relation for the inverse metric:

$$g^{\mu\nu} = \sigma n^\mu n^\nu + h^{ab} e^{\mu}_{a} e^{\nu}_{b} \,. \tag{2.15}$$

This allows us to define a projector onto the hypersurface $\Sigma$ by

$$h^{\mu\nu} \equiv h^{ab} e^{\mu}_{a} e^{\nu}_{b} = g^{\mu\nu} - \sigma n^\mu n^\nu . \tag{2.16}$$

Choosing $V^\mu$, $W^\mu \in T_p\Sigma$ and $Z^\mu \in T_p\mathcal{M}$ we have

$$h_{\mu\nu} V^\mu W^\nu = g_{\mu\nu} V^\mu W^\nu \,, \tag{2.17}$$

$$h_{\mu\nu} Z^\mu n^\nu = g_{\mu\nu} Z^\mu n^\nu - \sigma n_\mu n_\nu Z^\mu n^\nu = 0 \,, \tag{2.18}$$

$$h^\mu{}_\lambda h^\lambda{}_\nu = (\delta^\mu{}_\lambda + n^\mu n_\lambda)(\delta^\lambda{}_\nu + n^\lambda n_\nu) = h^\mu{}_\nu \,. \tag{2.19}$$

Hence $h_{\mu\nu}$ acts as a metric on the hypersurface, projects general vectors onto the hypersurface, and is idempotent. Note that we should be careful in distinguishing between the induced metric, $h_{ab}$, and the projection tensor, $h_{\mu\nu}$.

Returning to our example with the 2-sphere, we see that by embedding it into a flat 3-dimensional manifold we find the induced metric to be

$$h_{ab} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} , \tag{2.20}$$

which is indeed the well-known metric for the 2-sphere.

The existence of a metric allows us to define the intrinsic curvature of the hypersurface. However, as mentioned above, the hypersurface is embedded in a higher-dimensional manifold and therefore one needs to account for the extrinsic curvature of the hypersurface as it bends in the embedding manifold. The information about the curvature of a manifold is encoded in the Riemann curvature tensor. Below we will derive the Riemann curvature tensor for the hypersurface and show that, in addition to a term coming from the intrinsic curvature, there are terms related to how the hypersurface is embedded, that is how it is extrinsically curved. This calculation will be the basis for our definition of the extrinsic curvature tensor.

The derivation requires a notion of a covariant derivative on the hypersurface. We can define such an object by

$$D_\mu T^{\nu\cdots}{}_{\sigma\cdots} = h^\nu{}_\kappa \cdots h^\lambda{}_\sigma \cdots h^\rho{}_\mu \nabla_\rho T^{\kappa\cdots}{}_{\lambda\cdots} \,, \tag{2.21}$$

where $T^{\kappa\cdots}{}_{\lambda\cdots} \in \Sigma$. See [6] for proof that this indeed defines a derivative operator. Furthermore, it satisfies

$$D_\mu h_{\nu\sigma} = 0\,. \tag{2.22}$$

Equipped with a covariant derivative, we can define the Riemann tensor of the hypersurface as

$${}^{(\Sigma)}R_{\mu\nu\rho}{}^{\sigma}\omega_\sigma = D_\mu D_\nu \omega_\rho - D_\nu D_\mu \omega_\rho\,. \tag{2.23}$$

Applying the definition (2.21) we find

$$\begin{aligned} D_\mu D_\nu \omega_\rho = {}& h^\sigma{}_\mu h^\kappa{}_\nu h^\lambda{}_\rho \nabla_\sigma \nabla_\kappa \omega_\lambda \\ & + \sigma \left( h^\sigma{}_\mu h^\lambda{}_\rho h^\alpha{}_\nu n^\beta \nabla_\sigma n_\lambda + h^\sigma{}_\mu h^\kappa{}_\nu h^\beta{}_\rho n^\alpha \nabla_\sigma n_\kappa \right) \nabla_\alpha \omega_\beta\,. \end{aligned} \tag{2.24}$$

Defining the extrinsic curvature tensor as

$$K_{\mu\nu} \equiv h^\rho{}_\mu \nabla_\rho n_\nu \tag{2.25}$$

we see that

$$D_\mu D_\nu \omega_\rho = h^\sigma{}_\mu h^\kappa{}_\nu h^\lambda{}_\rho \nabla_\sigma \nabla_\kappa \omega_\lambda - \sigma K_{\mu\rho} K_\nu{}^\beta \omega_\beta + \sigma K_{\mu\nu} h^\beta{}_\rho n^\alpha \nabla_\alpha \omega_\beta, \tag{2.26}$$

where we used the fact that $K_{\mu\sigma} = h^\rho{}_\mu h^\kappa{}_\sigma \nabla_\rho n_\kappa$, as $K_{\mu\sigma}$ is tangential to the hypersurface. Furthermore, $h^\alpha{}_\nu n^\beta \nabla_\alpha \omega_\beta = -K_\nu{}^\beta \omega_\beta$, since $n^\beta \omega_\beta = 0$. It turns out that the extrinsic curvature tensor is symmetric, as argued in appendix B, so the last term in (2.26) vanishes when antisymmetrized over $\mu$ and $\nu$. The Riemann tensor of the hypersurface is therefore [6]

$${}^{(\Sigma)}R_{\mu\nu\rho}{}^{\sigma} = h^\alpha{}_\mu h^\kappa{}_\nu h^\lambda{}_\rho h^\sigma{}_\beta R_{\alpha\kappa\lambda}{}^\beta - \sigma K_{\mu\rho} K_\nu{}^\sigma + \sigma K_{\nu\rho} K_\mu{}^\sigma\,. \tag{2.27}$$

Thus, the curvature of a hypersurface will receive contributions from both the intrinsic curvature of the embedding space, corresponding to the first term in (2.27), and the extrinsic curvature related to how the hypersurface is embedded, corresponding to the other two terms in (2.27).

In appendix B it is shown that the extrinsic curvature tensor is symmetric, $K_{\mu\nu} = K_{\nu\mu}$. Using this property it is straightforward to show that $K_{\mu\nu}$ can be written as

$$K_{\mu\nu} = \frac{1}{2} h^\rho{}_\mu h^\sigma{}_\nu \mathcal{L}_n g_{\rho\sigma} \tag{2.28}$$

$$= \frac{1}{2} \mathcal{L}_n h_{\mu\nu}\,. \tag{2.29}$$

The details of these derivations are given in appendix B. A pictorial representation of what is meant by extrinsic curvature is given in figure 2.1.

The extrinsic curvature tensor plays an important rôle in posing a well-defined variational problem for the Einstein-Hilbert action. As we will see in chapter 3, it is needed to cancel boundary contributions coming from the variation of the Ricci tensor.

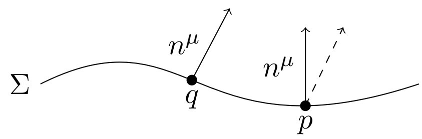


**Figure 2.1:** This figure illustrates the notion of extrinsic curvature based on the definition $K_{\mu\nu} = h^\rho{}_\mu \nabla_\rho n_\nu$. The failure of the normal vector $n^\mu$ at $q$ to coincide with the normal vector at $p$ is intuitively due to the bending of $\Sigma$ in the spacetime in which it is embedded.

### 2.1.2 Foliations

Foliations are interesting extensions of hypersurfaces. Equipping a manifold with a foliation structure is a way of singling out a preferred direction, in a way to be made precise below. The concept is used extensively in Hořava's proposal for a consistent theory of quantum gravity [47, 48], where it is used to pick a preferred time direction.

Foliations can be defined by first considering the concept of a foliation atlas [49]. Let $\mathcal{M}$ be a smooth manifold of dimension $n$. A foliation atlas of codimension $q$ of $\mathcal{M}$ is then an atlas of the form

$$\Big(\phi_i : \ U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q\Big)_{i\in I} \,. \tag{2.30}$$

If the change-of-charts diffeomorphisms $\phi_{ij}$ are locally of the form

$$\phi_{ij}(x,y) = \big(g_{ij}(x,y), h_{ij}(y)\big) \tag{2.31}$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$, with $i$ such that $\bigcup_{i\in I} U_i = \mathcal{M}$, the diffeomorphisms are said to be foliation-preserving[2]. This enables us to define a codimension $q$ foliation of $\mathcal{M}$ as a maximal foliation atlas of $\mathcal{M}$ of codimension $q$. A foliated maniold $(\mathcal{M}, \mathcal{F})$ is then a smooth manifold $\mathcal{M}$ with $\mathcal{F}$ being a foliation of $\mathcal{M}$.

Most interesting for our case will be the foliations of codimension 1. It can be shown [49] that if $\omega$ is a nowhere vanishing 1-form on $\mathcal{M}$ it defines a foliation of codimension 1 of $\mathcal{M}$ if and only if it is integrable, that is if it satisfies Frobenius' theorem [50]

$$\omega \wedge \mathrm{d}\omega = 0\,. \tag{2.32}$$

This condition is fulfilled by normal vectors to a hypersurface. Recall the definition of a hypersurface (2.1) allows us to define a normalized normal vector (2.8). It can be shown

[2]Some definitions of foliations include the fact that they should be preserved under the change-of-charts diffeomorphisms. We will always use the word foliation to mean a surface foliating spacetime without the constraint that the diffeomorphisms be foliation-preserving.

that this normal vector satisfies

$$n_{[\mu}\nabla_\nu n_{\rho]} = 0\,, \tag{2.33}$$

which follows simply from the definition of $n^\mu$. Using forms we can write this identity as

$$n \wedge \mathrm{d}n = 0\,. \tag{2.34}$$

Thus, the normal vectors define a foliation. If the diffeomorphisms are restricted to satisfy the specific form (2.31) we call them foliation-preserving.

To clarify these concepts we consider an example. Let $\mathcal{M}$ be the flat 3-dimensional space $\mathbb{R}^3$ and let us choose the coordinates on this space to be the usual spherical coordinates $(r, \phi, \theta)$. We choose the normal vector to be $\hat{r}$, such that the hypersurfaces are surfaces of constant $r$. This defines a foliation of the manifold $\mathcal{M}$. If, in addition, we restrict the change-of-charts diffeomorphisms to take the form

$$r = r(\tilde{r})\,, \tag{2.35}$$
$$\theta = \theta(\tilde{r}, \tilde{\theta}, \tilde{\phi})\,, \tag{2.36}$$
$$\phi = \phi(\tilde{r}, \tilde{\theta}, \tilde{\phi})\,, \tag{2.37}$$

this foliation will be preserved. This example demonstrates how a sphere can be foliated by surfaces of constant $r$, as illustrated in figure 2.2.

The reason for introducing foliations at this time might seem rather opaque. However, foliations play an important rôle in the context of AlLif spacetimes (section 2.4), where they arise naturally.

## 2.2 Anti-de Sitter space

The asymptotic structure of AdS spaces plays an essential rôle in the AdS/CFT correspondence. One talks about the field theory as living on the boundary of an AdS space, but AdS spaces are non-compact, so how is one to think of the boundary of a non-compact manifold? In this section we will attempt to answer this question.

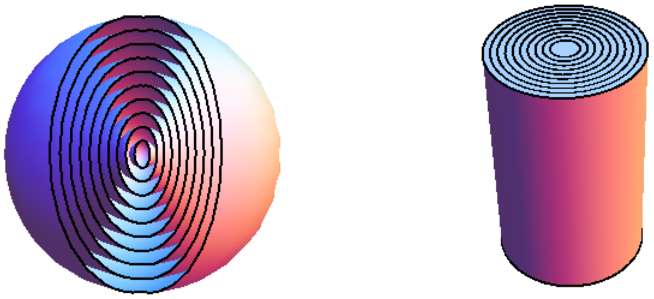

**Figure 2.2:** Two illustrations of foliations. On the left, a ball is represented using foliations of constant $r$, as described in detail in the text. On the right, a cylinder is foliated, also by surfaces of constant $r$, using a procedure analogous to the one described for the sphere.

Anti-de Sitter space is a solution of Einstein's field equations with negative cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \Lambda g_{\mu\nu} \,, \tag{2.38}$$

meaning that the Ricci tensor is proportional to the metric. $\Lambda$ is chosen to be negative. AdS space is often visualized by embedding a hyperboloid into a flat $(n+1)$-dimensional manifold with metric

$$\mathrm{d}s^2_{n+1} = -\mathrm{d}u^2 - \mathrm{d}v^2 + \sum_{i=1}^{n-1} \mathrm{d}x_i^2 \,. \tag{2.39}$$

The hypersurface defining the hyperboloid is then given by

$$-u^2 - v^2 + \sum_{i=1}^{n-1} x_i^2 = -\alpha^2 \,. \tag{2.40}$$

An advantage of this procedure is that the symmetry group of this space becomes apparent: The transformations leaving (2.40) invariant are precisely the transformations of the group $SO(2, n-1)$ which has dimension $\frac{1}{2} n(n+1)$. Below we will elaborate on how these symmetries are inherited by the boundary. A set of coordinates satisfying the constraint (2.40) is given by [45]

$$\begin{aligned} u &= \alpha \sin t \cosh \rho \,, \\ v &= \alpha \cos t \cosh \rho \,, \\ x_1 &= \alpha \sinh \rho \cos \theta_1 \,, \\ x_2 &= \alpha \sinh \rho \sin \theta_1 \cos \theta_2 \,, \\ &\vdots \\ x_{n-1} &= \alpha \sinh \rho \sin \theta_1 \cdots \sin \theta_{n-2} \,. \end{aligned} \tag{2.41}$$

The induced metric on the hypersurface, equation (2.13), is then given by

$$\mathrm{d}s^2 = \alpha^2 \left( -\cosh^2 \rho \; \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \; \mathrm{d}\Omega^2_{n-2} \right) , \tag{2.42}$$

which has the topology of $S^1 \times \mathbb{R}^{n-1}$. From the coordinates (2.41) it is seen that $t$ is periodic such that $t$ and $t + 2\pi$ refers to the same point on the hyperboloid. Since $\partial_t$ is everywhere timelike, the spacetime (2.42) contains closed timelike curves. However, since the spacetime is not simply connected, the $S^1$ can be unwrapped to obtain its universal covering $\mathbb{R}^1$, and we obtain the universal covering space of the metric (2.42) with $-\infty < t < \infty$ which has the topology of $\mathbb{R}^n$. This is what we will call AdS space. Performing the coordinate transformation

$$\cosh \rho = \frac{1}{\cos \chi}, \tag{2.43}$$

the metric is easily seen to become

$$\mathrm{d}s^2 = \frac{\alpha^2}{\cos^2 \chi} \left( -\mathrm{d}t^2 + \mathrm{d}\chi^2 + \sin^2 \chi \mathrm{d}\Omega^2_{n-2} \right) \,. \tag{2.44}$$

The Penrose diagram for this metric is shown in figure 2.3. Contrary to the case of Minkowski space, spacelike and null infinity is given by timelike hypersurfaces. Furthermore, from the metric (2.44) it is seen that AdS space is conformally flat, meaning it is equivalent to Minkowski space up to a conformal factor. Recall that the Weyl tensor (in dimensions $d \geq 4$) is given by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2}{n-2}\left(g_{\mu[\sigma}R_{\rho]\nu} + g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{2.45}$$

and is conformally invariant. Thus, the Weyl tensor of flat space is identical to the Weyl tensor of AdS space, but since the Riemann tensor, the Ricci tensor and Ricci scalar are all zero in flat space, the Weyl tensor must be zero as well. Combining this fact with the Einstein equations yields

$$R_{\mu\nu\rho\sigma} = \frac{2\Lambda}{(n-2)(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right), \tag{2.46}$$

and we customarily identify

$$\Lambda = -\frac{(n-2)(n-1)}{2l^2}, \tag{2.47}$$

with $l$ being the radius of AdS space. The form of (2.46) guarantees that the space in question is maximally symmetric, meaning it has $\frac{1}{2}n(n+1)$ independent Killing vectors and the isometry group is $\frac{1}{2}n(n+1)$-dimensional, as was to be expected from equation (2.40).

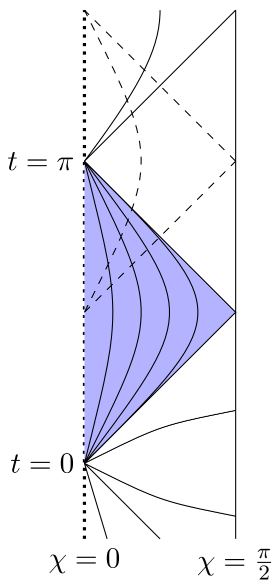


**Figure 2.3:** Penrose diagram depicting Anti-de Sitter space. Various timelike and spacelike paths are shown. Timelike paths remain within one copy of the space. Null and spacelike infinity is given by a timelike hypersurface at $\chi = \frac{\pi}{2}$. Each point in the diagram represents an $(n-2)$-sphere except for the points on the dotted line to the left which are single points at the origin.

### 2.2.1 Conformal infinity

In order to suitably define what is meant by the boundary of AdS spacetime, it is necessary to first make a detour to discuss how it is possible to even think of points at infinity as being part of spacetime. In a nutshell, the idea is to relate the physical metric of the non-compact spacetime to a conformally rescaled metric. This technique of 'making infinity finite' was first developed by Penrose, see for instance [51] and references therein.

To be more precise, consider a non-compact manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. Consider the metric to be a solution of Einstein's equations with a cosmological constant, equation (2.38). We are interested in solutions of this equation which have second order poles at infinity. This allows one to discuss the notion of a boundary metric and permits the definition of conformally compact manifolds. A rescaling of the metric can be performed with a suitably smooth function, $\Omega$, such that

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}\,, \tag{2.48}$$

where hats denote conformally rescaled quantities. If the defining function, $\Omega$, is chosen appropriately it is possible to adjoin to $\mathcal{M}$ a boundary surface, commonly denoted by $\mathscr{I}$. The defining function must also satisfy

$$\Omega(\mathscr{I}) = 0, \qquad d\Omega(\mathscr{I}) \neq 0, \qquad \Omega(\mathcal{M}) > 0\,. \tag{2.49}$$

See [52, 53] for details. These constraints follow from demanding that the spacetime be asymptotically simple. We will not be concerned further with the definition of this concept, but refer to [53] for details. Adjoining $\mathscr{I}$ to $\mathcal{M}$ results in a smooth manifold-with-boundary, $\overline{\mathcal{M}}$. A conformally compact manifold is thus a manifold with a second order pole at infinity, but with a defining function $\Omega$ chosen such that (2.48) smoothly extends to $\overline{\mathcal{M}}$, with $h_{(0)} = \Omega^2 g|_{\text{infinity}}$ being non-degenerate and the defining function satisfying (2.49). However, the metric $h_{(0)}$ is only defined up to conformal transformations, since for any defining function $\Omega$, $\widetilde{\Omega} = e^w \Omega$ is an equally good defining function. Thus, to be precise, only a conformal structure is induced on the boundary. This boundary is known as the conformal boundary of the metric $g$. Following this procedure, the points at infinity are now only a finite distance away as measured by the unphysical metric, $\hat{g}$. However, due to the second order pole at infinity, the physical metric is infinite on $\mathscr{I}$, and the points on $\mathscr{I}$ are thus infinitely far away from their neighbours, as seen from the perspective of the physical metric, $g$.

Note that this procedure only works for metrics which have poles at infinity. However, this is a coordinate dependent statement. Minkowski space can be written in a form exhibiting the same pole structure and analyzed in the same way, as discussed thoroughly in [51].

### 2.2.2 The conformal boundary of AdS space

Following the prescription in section 2.2.1, we can identify a defining function for the metric (2.44),

$$\Omega = \frac{\cos\chi}{\alpha}, \tag{2.50}$$

which clearly satisfies the constraints (2.49) and yields a conformal metric,

$$\mathrm{d}\hat{s}^2 = -\mathrm{d}t^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\Omega_{n-2}^2. \tag{2.51}$$

This allows one to identify a boundary of spacetime, namely the points for which $\chi = \frac{\pi}{2}$. The metric at the boundary is then

$$\mathrm{d}s_{\mathrm{bdry}}^2 = -\mathrm{d}t^2 + \mathrm{d}\Omega_{n-2}^2, \tag{2.52}$$

which is the Einstein static universe. It has the topology $\mathbb{R} \times S^{n-2}$. Minkowski space is conformally isometric to this space. Hence, a field theory on the boundary of AdS space can be said to live in Minkowski space. Note that the boundary metric is only defined up to conformal transformations.

Null and spacelike infinity is, in this case, given by a timelike hypersurface. This means that there do not exist any Cauchy surfaces in the spacetime. There exist families of spacelike hypersurfaces which cover the space, but one can find null geodesics which do not intersect any hypersurface in the family. Thus, a well-defined initial value problem does not exist for AdS space [53]. The evolution determined from data considered on a given spacelike hypersurface can be altered by information coming in through the timelike hypersurface at infinity.

However, the timelike hypersurface at infinity and the fact that AdS space has an associated scale (the AdS length, $l$) is exactly what permits one to write down a well-posed variational problem for the spacetime metric. The simplicity of the timelike hypersurface allows one to solve the Dirichlet problem and write the coefficients of the Fefferman-Graham expansion as local expressions of the boundary value. In the case of asymptotically flat spaces these terms would be non-local, meaning that there is no universal set of local counter-terms that can remove the divergences of the on-shell action for any solution to the equations of motion [30]. The scale is needed to organize the terms in the action as a derivative expansion. de Sitter space has such a scale but infinity is spacelike and the Dirichlet problem is therefore difficult to solve, although there are some ideas that one can rotate from AdS space to dS space and obtain sensible results [30]. Minkowski space has no associated scale and it is therefore difficult to arrange the terms in the action according to their degree of divergence. Hence only in the case of (Al)AdS spaces is it currently possible to write down a well-posed variational problem which gives meaningful results, even when spacetime is not confined to a hardwall box.

### 2.2.3 Isometries of the AdS boundary

As advertised above, the conformal group acts identically on the bulk of AdS space and on its boundary. This is an important fact in understanding the correspondence and we will now show the explicit details.

Consider the AdS metric (2.44) written in Poincaré coordinates

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{z^2}. \tag{2.53}$$

This is obviously Lorentz invariant, but it also has an additional symmetry given by conformal transformations, seen from the fact that scale transformations of the form $z \to \lambda z$, $x^\mu \to \lambda x^\mu$ leaves the metric invariant. This means, as was also seen above from a slightly different perspective, that the isometry group of AdS$_{d+1}$ is $SO(2,d)$. The conformal boundary of AdS$_{d+1}$ can be found by the procedure outlined above. As mentioned, this only defines a conformal structure at the boundary and not a metric. Hence, we are interested in the group which leaves this conformal structure invariant. In Poincaré coordinates the defining function is given by $\Omega = z$, which is easily seen to satisfy the constraints given in (2.49).

The arguments are clearest when presented in Euclidean signature. In this case, AdS space can be described by an open ball $B_{d+1}$ with isometry group $SO(1,d+1)$. The metric on $B_{d+1}$ can be written as [13]

$$\mathrm{d}s^2 = \frac{4\sum_{i=0}^{d} \mathrm{d}y_i^2}{\left(1-|y|^2\right)^2}, \tag{2.54}$$

and following the procedure described in section 2.2.1 the defining function can be identified as $\Omega = 1 - |y|^2$, which clearly vanishes at the boundary, $|y|^2 = 1$, and is positive inside the ball. As expected, the boundary of $B_{d+1}$ is the sphere $S^d$, defined by

$$\sum_{i=0}^{d} y_i^2 = 1. \tag{2.55}$$

However, this presents us with a problem, since the isometry group of the sphere $S^d$ is only $SO(d+1)$. Recall, that the defining function is only defined up to conformal transformations, since $\Omega = e^w\left(1-|y|^2\right)$ is an equally good defining function. Thus, the conformal structure defining $S^d$ is left invariant under the action of $SO(1,d+1)$.

By returning to Lorentzian signature, this argument demonstrates that the Minkowski space on the boundary of AdS space is left invariant under the action of $SO(1,d)$, while the conformal transformations act as conformal Killing vectors. Hence, the symmetry groups on either side of the duality match.

## 2.3 Asymptotically (locally) Anti-de Sitter spaces

The energy-momentum tensor of the dual theory is sourced by the boundary value of the bulk metric. Thus, to determine the energy-momentum tensor in a holographic setting, one would vary the boundary metric. For this to make sense the boundary metric must be arbitrary, and we must consider the field content of our theory to determine the metric dynamically. In order to accomodate this case we need to generalize the observations made above. An essential ingredient in identifying a boundary of spacetime was the fact that the metric had a second order pole at infinity such that a determination of the conformal boundary could be carried out. It turns out that any metric with a double pole at infinity that is a solution to Einstein's equations admits a conformal compactification [51]. In this section, two classes of spacetimes will be introduced, Asymptotically AdS (AAdS) and Asymptotically locally AdS (AlAdS), both of which generalize the concepts above.

By Asymptotically AdS we mean a spacetime whose boundary is equivalent to that of AdS. For instance, the Schwarzschild-AdS black hole in $d$ dimensions is AAdS. The metric is given by

$$\mathrm{d}s^2 = \frac{l^2}{r^2}\left(-f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) , \tag{2.56}$$

where

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^d , \tag{2.57}$$

and $r_h$ is the radius of the event horizon. Note that the singularity is at $r = \infty$ and the boundary is at $r = 0$. When the spacetime is AAdS the boundary is conformally flat [54].

A far more interesting case is that of an Asymptotically locally AdS spacetime. Recall that the relevant metrics will have second order poles at infinity, and hence the Riemann tensor for such a metric will look like

$$R_{\mu\nu\rho\sigma} = |\mathrm{d}\Omega|^2_{\hat{g}}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) + \mathcal{O}(r^{-3})\,, \tag{2.58}$$

with

$$|\mathrm{d}\Omega|^2_{\hat{g}} = \hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega\,. \tag{2.59}$$

Imposing Einstein's equations leads to

$$|\mathrm{d}\Omega|^2_{\hat{g}} = \frac{1}{l^2}\,. \tag{2.60}$$

Thus, we define AlAdS spaces as metrics satisfying the boundary conditions

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) + \mathcal{O}(r^{-3})\,, \tag{2.61}$$

as was done in [30][3]. This definition does not impose any constraints on the topology of the boundary and it includes the class of AAdS metrics discussed above. The definition of AlAdS metrics can be extended to situations where the spacetime includes matter by demanding that the matter fields solve the equations of motion and that the conformally rescaled energy-momentum tensor, $\Omega^2 T_{\mu\nu}$, analagously to the metric, admits a continuous limit to $\partial M$. A very important result which holds for the class of AlAdS metrics (with or without matter) was proved by Fefferman and Graham in 1985 [55]. They showed that any metric which satisfies (2.61) has an asymptotic expansion near the boundary of the form

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{\mathrm{d}r^2}{r^2} + h_{ab}\mathrm{d}x^a\mathrm{d}x^b\,. \tag{2.62}$$

The metric can always be put into this radial gauge by exploiting our freedom to do coordinate transformations. The remaining part is given by

$$h_{ab} = \frac{1}{r^2}\left[h_{(0)ab} + r^2 h_{(2)ab} + \cdots + r^d h_{(d)ab} + r^d \log r\, h_{(d,1)ab} + \mathcal{O}(r^{d+1})\right]. \tag{2.63}$$

[3]Note that this definition is what Skenderis calls AAdS in [30]. However, the definition was later changed to be that of AlAdS.

This is known as the Fefferman-Graham expansion. A more general version of this expansion is described in section 3.2. The expression (2.63) allows one to identify the boundary metric as the leading term of the expansion[4]. Furthermore, the expansion coefficients up to order $r^d \log r$ can be determined as local functions of the boundary metric by imposing the equations of motion [56]. The equations of motion will also constrain the trace and divergence of $h_{(d)ab}$, but the full expression is a non-local function of the boundary metric. The term $h_{(d)ab}$ will, in general, be related to the energy-momentum tensor of the dual theory, while the term $h_{(d,1)ab}$ is proportional the conformal anomaly, as described in [31].

In the cases above, the bulk metrics are constrained to be A(l)AdS and thus possess conformal boundaries, and the asymptotic boundary data for the metric is specified by the induced metric on this boundary. According to the Fefferman-Graham expansion, for an arbitrary boundary metric there exists a bulk solution in a neighbourhood of infinity, and for each bulk solution, there is a field theory living on the (possibly curved) boundary. It therefore makes sense to talk about the boundary metric as sourcing the energy-momentum tensor of the dual field theory.

### 2.3.1 Conformally flat bulk metrics

When the bulk spacetime is AdS the simplicity of the boundary structure allows for a complete determination of the coefficients of the expansion (2.63). In these cases the bulk Weyl tensor vanishes, the bulk metric satisfies the Einstein equations, and the boundary metric must be conformally flat [54]. The Einstein equations can then be integrated to all orders in $r$ and the Fefferman-Graham expansion truncates at order $r^4$ such that it now reads [54, 56]

$$h_{ab} = \frac{1}{r^2}\left[h_{(0)ab} + r^2 h_{(2)ab} + r^4 h_{(4)ab}\right] . \tag{2.64}$$

The logarithmic term vanishes as the boundary metric is conformally flat. In (boundary) dimensions $d > 2$ the coefficients are found to be [54, 56]

$$h_{(2)ab} = \frac{1}{d-2}\left(R_{(0)ab} - \frac{1}{2(d-1)} R_{(0)} h_{(0)ab}\right) , \qquad h_{(4)ab} = \frac{1}{4}\mathrm{Tr}\left(h_{(2)ac} h^{c}_{(2)b}\right) , \tag{2.65}$$

where $R_{(0)ab}$ is the Ricci tensor of the boundary metric $h_{(0)ab}$ and indices are raised and lowered with the boundary metric as well. This allows for an exact determination of the energy-momentum tensor of the dual theory, however, we shall not derive this here. Expressions for $d = 4$ and $d = 6$ are given in [54, 56].

### 2.3.2 Penrose-Brown-Henneaux transformations

Diffeomorphisms play an important rôle in physics. A cornerstone of gravitational physics is the covariance of the metric 2-tensor under such transformations. By equipping the manifold $\mathcal{M}$ with a conformal boundary $\partial\mathcal{M}$, we induce diffeomorphisms of $\partial\mathcal{M}$ by doing diffeomorphisms of $\mathcal{M}$. As usual, the metric tensor $g$ transforms correctly under

[4]As mentioned above, the metric $h_{(0)ab}$ is actually only a representative of the conformal structure at infinity. However, we will simply refer to it as the boundary metric.

diffeomorphisms, however, as will be shown below, the unphysical metric $\hat{g} = \Omega^2 g$ does not. The diffeomorphisms of the bulk induce an additional conformal transformation of the boundary metric. Such transformations are referred to as Penrose-Brown-Henneaux (PBH) transformations [51, 57].

We consider transformations which preserve the radial gauge (2.62). Thus, we demand

$$\delta g_{rr} = 0 = \delta g_{ar} \,, \tag{2.66}$$

with the variation given by

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \,, \tag{2.67}$$

where $\xi$ is the vector field generating the diffeomorphisms of $\mathcal{M}$, and the corresponding diffeomorphism on $\partial\mathcal{M}$ is generated by $\xi_{(0)}$ which is the restriction of $\xi$ to $\partial\mathcal{M}$. From (2.66) and (2.67) the transformations must satisfy

$$\mathcal{L}_\xi g_{rr} = \frac{2}{r}\partial_r\left(\frac{\xi^r}{r}\right) = 0 \,, \tag{2.68}$$

$$\mathcal{L}_\xi g_{ra} = \frac{1}{r^2}\partial_a\xi^r + h_{ab}\partial_r\xi^b = 0 \,. \tag{2.69}$$

These equations can be integrated using the constraints that $\xi$ should equal $\xi_{(0)}$ on the boundary. The first equation yields

$$\xi^r = r\xi^r_{(0)}(x^a), \tag{2.70}$$

and imposing this, the second equation gives

$$\xi^a = \xi^a_{(0)}(x^c) - \partial_b\xi^r_{(0)}(x^c)\int_0^r \frac{1}{r'}h^{ab}\mathrm{d}r'. \tag{2.71}$$

The remaining transformation is

$$\begin{aligned}\mathcal{L}_\xi g_{ab} &= \xi^r\partial_r g_{ab} + \xi^c\partial_c g_{ab} + g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c \\ &= \xi^r\partial_r h_{ab} + \mathcal{L}_{\xi^a}h_{ab} \,,\end{aligned} \tag{2.72}$$

since $g_{ab} = h_{ab}$. The first term is a scale transformation associated with coordinate transformations of the holographic direction in the bulk, while the second term describes the diffeomorphisms of the metric $h_{ab}$ on hypersurfaces of constant $r$. Going to the boundary, $r = 0$, we find that the boundary metric transforms as

$$\delta h_{(0)ab} = -2\xi^r_{(0)}h_{(0)ab} + \mathcal{L}_{\xi^a_{(0)}}h_{(0)ab} \,, \tag{2.73}$$

demonstrating that, apart from boundary diffeomorphisms, the bulk diffeomorphisms induce an additional conformal transformation, under which the boundary metric transforms as

$$h_{(0)ab} \to e^{-2\xi^r_{(0)}}h_{(0)ab} \,. \tag{2.74}$$

The fact that the Penrose prescription for defining a conformal boundary only induces a conformal structure at infinity is hence manifest in this conformal transformation of the boundary induced by the bulk diffeomorphisms. The PBH-transformations play an important rôle in deriving Ward identities for the boundary theory, as we will see in chapter 6.

## 2.4 Non-relativistic Spacetimes

A further class of metrics, of which AdS space is a special case, are the Lifshitz and Schrödinger metrics. These spacetimes are of interest when one desires to study theories at a quantum critical point which exhibits non-relativistic scale invariance

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}, \qquad r \to \lambda r, \tag{2.75}$$

where $z$ is the dynamical critical exponent, measuring the degree of anisotropy between space and time. Such theories recently attracted a lot of attention in connection with Hořava's proposal for a consistent theory of quantum gravity, in which Lorentz symmetry is broken at very high energies by introducing an anisotropy between space and time [47, 48]. Lorentz violating theories are also interesting when exploring gravity duals of condensed matter systems exhibiting such non-relativistic scaling [58, 59, 60]. The class of AlLif spacetimes introduced below will play an essential rôle in the remaining part of this thesis, while the Schrödinger spacetimes are merely discussed for completeness.

It is interesting to note that the AlLif spacetimes introduced below are intimately related to AlAdS spacetimes via dimensional reduction, in a manner to be made explicit in chapter 5. This will prove to be very convenient, as one can carry out complicated calculations in an AlAdS spacetime, and later dimensionally reduce to an AlLif spacetime. This was exploited in the holographic renormalization of $z = 2$ Lifshitz spacetimes in [61].

In a similar manner, the Schrödinger spacetimes discussed in section 2.4.2 are related to AAdS spacetimes through what is known as a TsT transformation [62].

### 2.4.1 Asymptotically (locally) Lifshitz spacetimes

An example of a system exhibiting the scale invariance (2.75) with $z = 2$ is the Lifshitz scalar field theory,

$$S = \int \mathrm{d}t\mathrm{d}^2x \left((\partial_t\phi)^2 - \kappa(\nabla^2\phi)^2\right), \tag{2.76}$$

which the non-relativistic Lifshitz spacetimes are named after. This theory arises in the description of phase diagrams for certain materials [63, 64]. Lifshitz metrics are invariant under the anisotropic scaling (2.75). It is therefore hoped that the gravity duals of condensed matter theories exhibiting such scale invariance lives on the boundaries of (Asymptotically locally) Lifshitz spacetimes. Indeed, it is one of the goals of this thesis to show how the calculation of vevs is performed in such a spacetime with $z = 2$, although in this section we will settle for introducing the concept of Lifshitz spacetimes. Furthermore, Asymptotically locally Lifshitz spacetimes will be defined. In addition the Lifshitz algebra will be discussed briefly.

For metrics with the scale invariance (2.75) to be stable points of the action in 4 dimensions, gravity must be coupled to a massive vector field [65]. Lifshitz metrics are therefore solutions to the equations of motion following from the action

$$S = \frac{1}{2\kappa_4^2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{2}A^\mu A_\mu\right], \tag{2.77}$$

where boundary terms have been omitted. The Lifshitz metric and the massive vector field is

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{r^{2z}} + \frac{\mathrm{d}\vec{x}^2}{r^2} + \frac{\mathrm{d}r^2}{r^2}, \tag{2.78}$$

$$A_\mu \mathrm{d}x^\mu = \frac{1}{r^z}\sqrt{\frac{2(z-1)}{z}}\mathrm{d}t, \tag{2.79}$$

with

$$m = \sqrt{2z}, \qquad \Lambda = -\frac{z^2+z+4}{2}, \tag{2.80}$$

where a length scale has been fixed, $l = 1$. See [66] for further details on these solutions. Black hole solutions in Lifshitz geometries were studied in [67]. Furthermore, the bulk theory should satisfy the null energy condition $T_{\mu\nu}n^\mu n^\nu \geq 0$ to preserve causality in the boundary theory [68]. Using the Einstein equations this leads to the demand that $z \geq 1$. In the context of gauge/gravity duality it is interesting to note that for $z = 2$ the action (2.77) can be obtained as a consistent truncation of 10- and 11-dimensional supergravity [4]. In the $z = 1$ case, the vector field vanishes and the metric is equivalent to the Poincaré patch of $\mathrm{AdS}_4$ with symmetry group $SO(2,3)$ (see subsection 2.2.3). In the general case the Killing vector fields of the metric (2.78) are

$$H = i\partial_t\,, \quad P_i = i\partial_i\,, \quad D = i\left(-zt\partial_t - \delta^{ij}x_i\partial_j\right)\,, \quad M_{ij} = i\left(x_i\partial_j - x_j\partial_i\right)\,, \tag{2.81}$$

and their commutation relations generate the Lifshitz algebra $\mathfrak{lif}_z(k)$:

$$[D, M_{ij}] = [H, M_{ij}] = 0, \quad [D, P_i] = iP_i, \quad [D, H] = izH, \quad [H, P_i] = 0, \tag{2.82}$$

as well as the well-known relations for rotations and spatial translations. Here $i = 1, 2, \ldots, k$ is the number of spatial dimensions (2 in the case above). The addition of boosts extends the Lifshitz algebra to the Galilean algebra. This algebra was recently studied in connection with the hydrodynamics of quantum field theories exhibiting Lifshitz scaling [69]. For Lifshitz spacetimes the dual field theory is not yet known, however, candidates should satisfy the algebra above. We will not have much to say about the dual field theory, instead we will focus on how to properly identify sources of dual operators and calculate expectation values of such sources. This requires the introduction of a broader class of metrics having the same asymptotic structure as (2.78).

One should mention that Lifshitz spacetimes contain a curvature singularity. In the deep bulk neighbouring geodesics will experience diverging tidal forces. However, since this is a deep bulk phenomena we will ignore it, as we are only interested in the behaviour near the boundary. In the $z = 1$ (AdS) case the curvature singularity is merely a coordinate singularity. The Lifshitz singularity is described in for instance [66, 70, 71, 72].

If we want to view the boundary of Lifshitz spacetime as some geometry sourcing an energy-momentum tensor of the dual field theory, we should consider what happens when one varies the asymptotic boundary data. Furthermore, in order to treat a change in the boundary data beyond the perturbative level, a definition of spacetimes which asymptotes to Lifshitz spacetimes is required. In the case of AlAdS spacetimes, the existence of a Feffermam-Graham expansion ensures that an arbitrary conformal class of metrics on the boundary generates a bulk solution. In the Lifshitz case this is different since the Lifshitz metric does not possess a conformal boundary in the way Penrose defined it. A definition of anisotropic conformal infinity was given in [73], but we shall not apply this definition here.

To define Asymptotically locally Lifshitz (AlLif) boundary conditions, one should start by considering spacetimes whose metric asymptotes to a metric locally written in the form (2.78). Since we are interested in a local object such spacetimes are most easily identified using frame fields in which the metric (2.78) takes the form [74]

$$\mathrm{d}s^2 = \eta_{\underline{\mu}\underline{\nu}} e^{\underline{\mu}}_{\mu} e^{\underline{\nu}}_{\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\underline{a}\underline{b}} e^{\underline{a}}_{a} e^{\underline{b}}_{b} \mathrm{d}x^a \mathrm{d}x^b + (e^{\underline{r}}_{r})^2 \mathrm{d}r^2, \tag{2.83}$$

where, in a manner similar to the gauge choice for the Fefferman-Graham expansion above, we choose the gauge such that $e^{\underline{r}}_{r} = r^{-1}$ and the $e^{\underline{a}}_{a}$ have no radial component. From the form of the metric (2.78) we see that the remaining non-zero frame fields are

$$e^{\underline{t}}_{t} = \frac{1}{r^z}, \qquad e^{\underline{i}}_{i} = \frac{1}{r}. \tag{2.84}$$

Since we want our spacetimes to approach such a metric asymptotically, we define AlLif boundary conditions [74] by the requirement that the spacetimes must admit a choice of frames such that, as $r \to 0$, the leading order of the frame fields are

$$e^{\underline{t}}_{a} = \frac{1}{r^z}\left(e^{\underline{t}}_{(0)a}(x^b) + \cdots\right), \qquad e^{\underline{i}}_{a} = \frac{1}{r}\left(e^{\underline{i}}_{(0)a}(x^b) + \cdots\right), \tag{2.85}$$

where the fall-off conditions at subleading order are dictated by the specific bulk theory. Below we will see that to obtain an AlLif spacetime we must set $e^{\underline{t}}_{(0)i} = 0$. In chapter 5 we will see that when the coefficient $e^{\underline{t}}_{(0)a}$ is constant or only depends on time, it makes sense to define the resulting space to be Asymptotically Lifshitz (ALif). Recall that the leading term in the Fefferman-Graham expansion, $h_{(0)ab}$, is the source for the energy-momentum tensor of the dual theory. Likewise, the leading terms of the expansion (2.85) are interpreted as sources for the so-called stress-energy tensor complex. This is analagous to the energy-momentum tensor of a relativistic theory.

As mentioned, we are interested in the case $z = 2$. The expansion of the curvature will involve explicit powers of $r$. The ability to renormalize relies on these powers being positive, such that the sources arising from considering arbitrary boundary data can be cancelled by adding further subleading terms, and the curvature of these terms will be further subleading. In this context it is relevant to consider the Ricci rotation coefficients $\Omega_{\underline{a}\underline{b}}{}^{\underline{c}}$. The majority of these contain positive powers for all values of $z$, however, one is given by [74]

$$\Omega_{\underline{i}\underline{j}}{}^{\underline{t}} = r^{2-z}\left(\mathrm{d}e^{\underline{t}}_{(0)}\right)_{ab} e^{[a}_{(0)\underline{i}} e^{b]}_{(0)\underline{j}}, \tag{2.86}$$

meaning that, in order to obtain a solution containing only positive powers for $z \geq 2$, we must choose $e^{\underline{t}}_{(0)i} = 0$ which ensures that $\Omega_{\underline{i}\underline{j}}{}^{\underline{t}}$ vanishes asymptotically. Hence, calculating the full stress tensor complex requires a deformation of the theory such that the AlLif boundary conditions are violated. Only after the relevant quantities have been calculated one would impose (2.85), thereby obtaining an AlLif spacetime. When calculating the stress-energy tensor complex, frame fields will turn out to provide the nicest framework in which to work. Using frame fields, the sources will arrange themselves as the leading terms in the asymptotic expansions, something which does not necessarily happen in a metric framework. In addition, it was argued in [75] that when a theory contains (non-scalar) boundary fields other than the metric, the stress-energy tensor should be given by a variation w.r.t. the inverse frame fields and not the inverse metric. Furthermore, this object will not be symmetric and it will fail to be covariantly conserved. We will describe this object in more detail in section 3.1.2. Therefore, when calculating the vevs of a theory in an AlLif spacetime, we will be working with frame fields. The statements made here will be corroborated in chapter 6 when such a calculation is performed.

We will introduce AlLif spaces in 4 dimensions as dimensionally reduced AlAdS spaces in 5 dimensions. The fact that the 4-dimensional metric should be AlLif therefore introduces constraints on the 5-dimensional theory, which should replicate the boundary conditions for the frame fields given above, after the dimensional reduction has been performed. This will be discussed further in chapter 5.

It is interesting to note that the condition $e_{(0)i}^{\underline{t}} = 0$ allows one to define a preferred foliation structure. Recall that a covector is hypersurface orthogonal if and only if it satisfies Frobenius' theorem, (2.34), which, in terms of frame fields, is

$$e_{(0)}^{\underline{t}} \wedge \mathrm{d}e_{(0)}^{\underline{t}} = 0\,. \tag{2.87}$$

Recall that for a torsion-free connection Cartan's first structure equation reads [22]

$$\mathrm{d}e^{\underline{a}} = \Omega_{\underline{bc}}{}^{\underline{a}} e^{\underline{b}} \wedge e^{\underline{c}}\,. \tag{2.88}$$

Plugging this relation into Frobenius' theorem leads to

$$\Omega_{\underline{ij}}{}^{\underline{t}} e_{(0)}^{\underline{t}} \wedge e_{(0)}^{\underline{i}} \wedge e_{(0)}^{\underline{j}} = 0\,, \tag{2.89}$$

which is automatically satisfied for AlLif spacetimes, where $\Omega_{\underline{ij}}{}^{\underline{t}}$ vanish asymptotically due to the choice of $e_{(0)i}^{\underline{t}} = 0$. Hence, the frame fields $e_{(0)}^{\underline{t}}$ are hypersurface orthogonal and define a preferred foliation of surfaces of constant $t$. We will take the hypersurface orthogonality of $e_{(0)}^{\underline{t}}$ as an indication of the spacetime being AlLif, in fact, we will see in chapter 6 that hypersurface orthogonality of $e_{(0)}^{\underline{t}}$ is natural when the spacetime is AlLif, even without imposing $e_{(0)i}^{\underline{t}} = 0$. The hypersurface orthogonality of $e_{(0)}^{\underline{t}}$ is indeed what allows one to choose coordinates such that $e_{(0)i}^{\underline{t}} = 0$.

### 2.4.2 Schrödinger spacetimes

The Schrödinger metric was first constructed, in the context of gauge/gravity dualities [58, 59], to replicate the symmetries of the Schrödinger equation, hence the name. However, it can also be obtained as solutions to truncations of supergravity. An important difference between the Schrödinger case and the Lifshitz case discussed above is the fact that the isometry group of the Schrödinger metric contains a generator related to the particle number. The Schrödinger metric is a solution of the 3-dimensional massive vector model [58, 59]. It can be generalized to dimensions $d > 3$ where it reads

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{r^{2z}} + \frac{\mathrm{d}\vec{x}^2 + 2\mathrm{d}t\mathrm{d}\xi}{r^2} + \frac{\mathrm{d}r^2}{r^2}, \tag{2.90}$$

with the constants and the massive vector field depending on the particular value of $z$, and $\vec{x}$ is a $(d-3)$–dimensional vector. The additional direction $\xi$ is a special feature of Schrödinger metrics. Note that this coordinate scales as

$$\xi \to \lambda^{2-z}\xi \tag{2.91}$$

under the scale transformation (2.75). The isometries of the metric (2.90) consists of dilatations, $D$, with the anisotropic scaling (2.75) and (2.91), as well as the usual rotations, $M_{ij}$, Galilean boosts, $K_i$, spatial translations, $P_i$, and time translations, $H$. Furthermore, there is an isometry related to translations in the $\xi$-direction which we denote $N$. It represents the particle number of the theory, so moving in the $\xi$-direction corresponds to increasing or decreasing the number of particles. The Killing vector fields correponding to these isometries constitute the Schrödinger group satisfying the algebra [59]

$$\begin{aligned} &[M_{ij}, N] = [M_{ij}, D] = 0\,, \quad [K_i, P_j] = i\delta_{ij}N\,, \quad [D, P_i] = iP_i\,, \\ &[D, K_i] = i(1-z)K_i\,, \quad [H, N] = [H, P_i] = [H, M_{ij}] = 0\,, \\ &[H, K_i] = -iP_i\,, \quad [D, H] = izH\,, \quad [D, N] = i(2-z)N\,, \end{aligned} \tag{2.92}$$

in addition to the well-known commutators between the generators of translations, boosts and rotations. Here $i = 1, 2, \ldots, k$ are the spatial dimensions. This is the Schrödinger algebra $\mathfrak{sch}_z(k)$. Furthermore, in the case $z = 2$ the metric has an additional isometry corresponding to special conformal transformations and the algebra is extended by the commutation relations

$$[M_{ij}, C] = [K_i, C] = 0\,, \quad [D, C] = -2iC\,, \quad [H, C] = -iD\,. \tag{2.93}$$

Since a wide variety of condensed matter systems at a quantum critical point exhibit the symmetries above, the metric (2.90) is a natural candidate for the gravity dual of such a system. This idea was first explored in [58, 59].

The existence of a Fefferman-Graham expansion is essential when doing calculations involving a non-trivial metric. In section 2.3 it was argued that such an expansion always exists in cases where the metric satisfies AlAdS boundary conditions. However, the theorem by Fefferman and Graham [55], which guarantees the existence of an asymptotic expansion, breaks down for unconstrained Schrödinger spacetimes, as explained in [76]. The meaning of Asymptotically locally Schrödinger is also defined in [76].

# Chapter 3

# Variational Principles in Gravity

The posing of a well-defined variational principle is an important ingredient in holography. The variation of an action will in general include boundary terms arising from the integrations by parts performed to make sure no derivatives act on field variations. In ordinary field theory it is generally sufficient to consider only variations of compact support meaning that all such boundary terms vanish trivially. In gravitational field theories, however, the boundary terms do not vanish unless also the first derivative of the variation is kept fixed, so care is needed when performing variations. In the context of holography the important quantity is the on-shell action, which in a general gravitational field theory contains infrared divergences due to the infinite volume of spacetime. Posing a well-defined variational problem in holography thus requires both the addition of counterterms to yield a finite on-shell value of the action, but also the addition of extra boundary terms to ensure that the variation of the action vanishes on-shell.

In the coming sections we will describe the various terms required. In section 3.1 the important Gibbons-Hawking boundary term is described, which ensures the vanishing of the derivative of the variation. In section 3.2 the procedure of holographic renormalization is introduced, which leaves the on-shell value of the action finite and ensures that the equations of motion are stationary points of the action. We conclude with an example of such a renormalization procedure in section 3.3, which also allows us to describe how a calculation of the stress-energy tensor takes place and how the Ward identities are derived.

## 3.1 The Gibbons-Hawking Boundary Term

In this section we start by considering the Einstein-Hilbert action and show that an additional boundary term is needed in order to make the variational problem well-posed. The Einstein-Hilbert action including boundary terms will play a very important part throughout this thesis, and the introduction of the Gibbons-Hawking boundary term will be motivated carefully here.

The Einstein-Hilbert action including a cosmological constant is

$$S = \frac{1}{2\kappa_{d+1}^2} \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{-g}\,(R - 2\Lambda)\ . \tag{3.1}$$

The field in this case is the metric and we assume that spacetime is confined to a hardwall box such that $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$. Later we will consider the more general case where $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = \delta h_{ab}$. The variation of the action is

$$\delta S = \frac{1}{2\kappa_{d+1}^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\left(R - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{1}{2\kappa_{d+1}^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,, \tag{3.2}$$

where the formula for the variation of the square root of the metric determinant, (B.12), was used. From (B.22) we have that

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu f^\mu\,, \tag{3.3}$$

with

$$f^\mu \equiv g^{\rho\nu}\delta\Gamma^\mu_{\rho\nu} - g^{\mu\nu}\delta\Gamma^\rho_{\rho\nu}\,. \tag{3.4}$$

Since this is a total derivative it will only contribute a boundary term and therefore it will not contribute to the equations of motion. However, since it depends on the derivative of the variation it does not vanish in cases where only the variation vanishes on the boundary, and to make the variational problem well-posed we should add some boundary term to the action (3.1) to cancel it. Using Stoke's theorem we find the boundary term to be

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\nabla^\mu f_\mu = \int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{-h}n^\mu f_\mu, \tag{3.5}$$

where $h_{ab}$ is the induced metric on the boundary. To cancel this term consider adding another boundary term $2\int_{\partial\mathcal{M}} K$. Recall the definition (2.25) of the extrinsic curvature tensor. Taking the trace of (2.25) we obtain

$$K^\mu{}_\mu = K = \nabla_\mu n^\mu. \tag{3.6}$$

Note that, due to the fact that the extrinsic curvature tensor is hypersurface tangential, this trace is

$$K = g^{\mu\nu}K_{\mu\nu} = h^{ab}e^\mu_a e^\nu_b K_{\mu\nu} = h^{ab}K_{ab}, \tag{3.7}$$

where $K_{ab}$ is understood as the extrinsic curvature induced on the boundary. However, the extrinsic curvature tensor is independent of the boundary metric, $\delta_h K = 0$. From (B.40) we have (since the metric variation vanishes on the boundary)

$$\delta K = -\frac{1}{2}n^\mu f_\mu\,. \tag{3.8}$$

The full details of this rather long derivation are given in appendix B. Hence, we see that the boundary term (3.5) vanishes if we add the term $2\int_{\partial\mathcal{M}} K$ to the action and instead of (3.1) take the fundamental action to be

$$S = \frac{1}{2\kappa_{d+1}^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\,(R - 2\Lambda) + \frac{1}{\kappa_{d+1}^2}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{-h}K. \tag{3.9}$$

The new term on the right hand side is the Gibbons-Hawking boundary term [77].

### 3.1.1 The Brown-York stress tensor

To the gravitational action (3.9) is associated a stress tensor related to the variation of the metric on the boundary. This stress tensor has a dual interpretation, both as the so-called Brown-York stress tensor [78] of the boundary of spacetime, and as the energy-momentum tensor of the dual field theory. In this section we derive the Brown-York stress tensor for a theory with only gravity. This will serve as a motivation for discussing the method of holographic renormalization, due to the resulting divergences of the Brown-York stress tensor.

Consider the total variation of the action (3.9),

$$\delta S = \frac{1}{2\kappa_{d+1}^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\left[\delta R_{\mu\nu}g^{\mu\nu} + \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\right] + \frac{1}{\kappa_{d+1}^2}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{-h}\left(\delta K - \frac{1}{2}K h_{ab}\delta h^{ab}\right). \tag{3.10}$$

The last term arises since we are now considering an arbitrary variation of the metric where we do not keep the variation fixed at the boundary. Thus, $\delta g^{\mu\nu}|_{\partial\mathcal{M}} = \delta h^{ab}$. This also means that the variation of the extrinsic curvature tensor is no longer given by (3.8) since the variation on the boundary is no longer zero. From (B.40) we have

$$\delta K = -\frac{1}{2}n_\mu f^\mu + \frac{1}{2}K_{\mu\nu}\delta g^{\mu\nu}, \tag{3.11}$$

where a total divergence can be dropped since the boundary of a boundary is the empty set. Furthermore, due to the hypersurface tangentiality of the extrinsic curvature tensor, we have

$$K_{\mu\nu}\delta g^{\mu\nu} = K_{\mu\nu}\delta h^{\mu\nu}\,, \tag{3.12}$$

and the integral is over the boundary so

$$K_{\mu\nu}\delta h^{\mu\nu}\big|_{\partial\mathcal{M}} = K_{ab}\delta h^{ab}\,. \tag{3.13}$$

Note that the extrinsic curvature is independent of the boundary metric $h_{ab}$ and only depends on the bulk metric $g_{\mu\nu}$. It is the variation of the bulk metric which contributes the term $K_{ab}$ to the variation. The only factor dependent on $h_{ab}$ in the Gibbons-Hawking term is the determinant $\sqrt{-h}$, and the variation of this contributes a factor of $Kh_{ab}$. The variation becomes

$$\delta S = \frac{1}{2\kappa_{d+1}^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} - \frac{1}{2\kappa_{d+1}^2}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{-h}\left(Kh_{ab} - K_{ab}\right)\delta h^{ab}. \tag{3.14}$$

Putting this on-shell a candidate for the Brown-York stress tensor can be identified:

$$\widehat{T}_{ab} = Kh_{ab} - K_{ab}\,, \tag{3.15}$$

where the hat denotes the fact that this object requires renormalization. Since this stress tensor has a dual interpretation as the energy-momentum tensor of the dual theory, developing a method of renormalization will be essential. A discussion of such a method will occupy us in the coming sections. However, we will first describe another definition of the stress-tensor, which will be important when the theory contains additional non-scalar boundary fields.

One should mention that the methods described in section 3.2 is not the only way to render the quantity in (3.15) finite. One can perform a background subtraction which even works for asymptotically flat spacetimes. In the case of the Schwarzschild black hole, for instance, it is easy to identify the background and perform the subtraction. However, for more complicated metrics it might not be possible, and holographic renormalization is required.

### 3.1.2 The Hollands-Ishibashi-Marolf stress-energy tensor

In the presence of additional non-scalar boundary fields an alternative definition of the boundary stress-energy tensor is required. In [75] such a modified stress-energy tensor, referred to as the HIM stress-energy tensor, was defined as[1]

$$S^{\underline{a}}{}_a = \frac{\delta S^{\text{on-shell}}_{\text{ren}}}{\delta e^a_{\underline{a}}}\,. \tag{3.16}$$

This definition clearly reduces to the Brown-York one when only scalar fields and the metric are present. However, when e.g. vector fields are present as is the case for Lifshitz spacetimes, the stress-energy tensor changes and is no longer symmetric. Since we can decompose the vector fields as $A^\mu = A^{\underline{\mu}} e^\mu_{\underline{\mu}}$, performing a functional differentiation w.r.t. the inverse frame fields will result in a contribution from the field $A^{\underline{\mu}}$. This result naturally generalizes to higher-spin fields, including fermionic ones. When such fields are present, the stress-energy tensor will in general fail to be covariantly conserved. However, it was shown in [75] that it will obey a conservation law of the form

$$\nabla_a \mathcal{S}^{ab} = -\sum_i \frac{\delta S}{\delta \phi^i_{(0)}} \nabla^b \phi^i_{(0)} - \mathcal{S}^c{}_a e^{\underline{a}}_c \nabla^b e^a_{\underline{a}}\,, \tag{3.17}$$

where the minus signs arise due to differing conventions. Here $\mathcal{S}_{ab} = S^{\underline{a}}{}_b e_{\underline{a}a}$. Since we are interested in holography for AlLif spacetimes in this thesis, it is the HIM stress-energy tensor that we will be calculating as Lifshitz spaces requires the presence of a massive vector field. In chapter 6 we will see that the Ward identity for the stress-energy tensor of the Lifshitz boundary follows the form (3.17), suitably generalized to account for the non-relativistic nature of the boundary.

Just as for the Brown-York stress-energy tensor described above, the evaluation of the HIM stress-energy tensor requires renormalizing the on-shell action, and this is the subject to which we now turn.

[1] Contrary to what was done in [75], we perform the variation w.r.t. the inverse frame field.

## 3.2 Holographic Renormalization

In the previous section it was seen that, even when spacetime is confined to a hardwall box, one is required to add a boundary term to have a well-posed variational problem. However, when attempting to calculate quantities which require putting the action on-shell, one encounters divergences. These divergences arise due to the infinite volume of AlAdS space and the addition of counter-terms to cancel these infinities is required, a procedure known as holographic renormalization. There are several ways to perform holographic renormalization, but we will only cover the Lagrangian method [30] in detail here. Alternative approaches are described in [79, 80].

The Lagrangian method revolves around introducing some cut-off hypersurface at finite radius. Identifying how different terms behave as a function of this cut-off then allows for a subtraction of the divergent terms from the action. However, there are some subtleties involved. The first step is to solve the bulk equations of motion with arbitrary Dirichlet boundary conditions. It is a central result by Fefferman and Graham [55] that for the class of AlAdS metrics discussed previously the boundary metric has an asymptotic expansion close to the boundary. In fact this result holds for any field in AlAdS space, be it scalar, spinor or tensor, resulting in the expansion (all indices are suppressed)

$$\mathcal{F}(x,r) = r^m \left( f_{(0)}(x) + r^2 f_{(2)}(x) + \cdots + r^{2n} \left( f_{(2n)}(x) + \log r\, f_{(2n,1)}(x) \right) + \cdots \right) , \quad (3.18)$$

where $r$ is the holographic direction or the radial coordinate of the AlAdS space. The specific value of $m$ is determined by demanding that the expansion is a solution to the equations of motion. The factor $f_{(0)}(x)$ is interpreted as a source in the dual field theory, while $f_{(2n)}(x)$ is related to the vev of the operator sourced by $f_{(0)}(x)$. The terms $f_{(2)}(x), \ldots, f_{(2n-2)}(x)$ and $f_{(2n,1)}(x)$ are uniquely determined in terms of $f_{(0)}(x)$ by the equations of motion. The importance of the Fefferman-Graham expansion in holography was first recognized by Witten [13]. In [56] the logarithmic term was noted to be required otherwise the determination of the coefficients $f_{(2)}, \ldots, f_{(2n-2)}$ from the equations of motion would result in constraints being imposed on the boundary geometry, which should remain arbitrary. $f_{(2n,1)}(x)$ is related to the anomalies in the theory. To be precise, the coefficient of the logarithmic term is precisely what one would get by varying the anomaly with respect to the appropriate field. For instance, varying the conformal anomaly with respect to the metric results in $h_{(d,1)ab}$. The term $f_{(2n)}(x)$ is undetermined by the asymptotic solution to the equations of motion. This is as it should be, since this term is the Dirichlet boundary condition for the solution linearly independent from the solution starting at order $r^m$ with Dirichlet boundary condition $f_{(0)}(x)$. However, in the case where $\mathcal{F}(x,r)$ is not a scalar field, the equations of motion will fix part of the coefficient $f_{(2n)}(x)$. In the case of a tensor field, for example, the trace and divergence of $f_{(2n)}(x)$ are determined by the equations of motion.

The asymptotic expansion allows a regularization to be performed by introducing some cut-off hypersurface at $r = \epsilon$, on which the boundary terms can be evaluated. Only a finite number of terms will diverge as $\epsilon \to 0$ and a counter-term action can be constructed. The regularized on-shell action takes the form [30]

$$S_{\text{reg}}^{\text{on-shell}}[f_{(0)},\epsilon] = \int_{r=\epsilon} \mathrm{d}^d x \sqrt{-h_{(0)}} \left( \epsilon^{-\nu} a_{(0)} + \epsilon^{-\nu+1} a_{(2)} + \cdots - \log\epsilon \; a_{(2\nu)} + \mathcal{O}(\epsilon^0) \right) , \tag{3.19}$$

where $\nu$ is a positive number depending on the scale dimension of the dual operator and all $a_{(2k)}$ are local functions of the $f_{(0)}$ and are independent of the term $f_{(2n)}$. We can now define a subtracted action as

$$S_{\text{sub}}[\mathcal{F}(x,\epsilon)] = S_{\text{reg}}^{\text{on-shell}}[f_{(0)},\epsilon] + S_{\text{ct}}[\mathcal{F}(x,\epsilon),\epsilon], \tag{3.20}$$

where $S_{\text{ct}}[\mathcal{F}(x,\epsilon),\epsilon]$ contains the divergent parts of $S_{\text{reg}}^{\text{on-shell}}[f_{(0)},\epsilon]$. In writing $S_{\text{ct}}[\mathcal{F}(x,\epsilon),\epsilon]$ an inversion of the expansion (3.18) is required in order to make the subtraction covariant, as it is the fields $\mathcal{F}(x,\epsilon)$ that transform covariantly under bulk diffeomorphisms. This inversion is the tedious step of the procedure as the equations of motion will, in general, couple the various fields of the action. The definition of $S_{\text{sub}}$ allows us to define a renormalized action as

$$S_{\text{ren}}^{\text{on-shell}}[f_{(0)}] = \lim_{\epsilon\to 0} S_{\text{sub}}[\mathcal{F}(x,\epsilon),\epsilon] \,. \tag{3.21}$$

The variation required to obtain correlation functions in the dual field theory is performed before the limit $\epsilon \to 0$ is taken. Hence, the distinction between $S_{\text{sub}}$ and $S_{\text{ren}}$. Having $S_{\text{ren}}$ we can define the one-point functions of the operators $\mathcal{O}_{\mathcal{F}}$ dual to the sources $f_{(0)}$ as

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s = \frac{1}{\sqrt{-h_{(0)}}} \frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta f_{(0)}} \,. \tag{3.22}$$

It is useful to rewrite this in terms of objects living on the regulated hypersurface:

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s = \lim_{\epsilon\to 0} \left( \frac{1}{\epsilon^{d-m}} \frac{1}{\sqrt{-h}} \frac{\delta S_{\text{sub}}}{\delta \mathcal{F}(x,\epsilon)} \right) . \tag{3.23}$$

This object is finite by construction. We will see that, in general, it is given by

$$\langle \mathcal{O}_{\mathcal{F}} \rangle_s \sim f_{(2n)} + C\left(f_{(0)}\right) , \tag{3.24}$$

where $C\left(f_{(0)}\right)$ is some local function of the sources and will be scheme dependent. One might wonder why this procedure differs from simply using the regulated action (3.19) written in terms of $f_{(0)}(x)$ to obtain one-point functions, while just throwing away any infinities. In order to make the subtraction covariant, one introduces finite counter-terms in the renormalized action, and these terms would not be there had one simply functionally differentiated (3.19) with respect to $f_{(0)}(x)$ and discarded the infinities.

To calculate $n$-point functions in an interacting theory, one would have to solve the equations of motion pertubatively and apply the result to the near-boundary expansion of the bulk-to-bulk propagator. This is covered in detail in [30].

## 3.3 Holographic Renormalization of Pure Gravity in AlAdS Space

In this section we will perform holographic renormalization of pure gravity in AlAdS space in order to clarify the details of the preceeding section. This procedure was also carried out in [56] using a slightly different approach. This example will also be relevant later when we consider AdS gravity coupled to an axion-dilaton field.

As we learned from section 3.1, the Gibbons-Hawking boundary term is required to make the variational problem for Einstein gravity well-posed. Thus, the relevant action is

$$S = \frac{1}{2\kappa_5^2}\left(\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\,(R+12) + 2\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h}K\right), \tag{3.25}$$

where we used the fact that, for negatively curved spaces, $\Lambda = -\frac{d(d-1)}{2l^2}$ and we take $l = 1$ (note that $d$ here refers to the dimension of the dual field theory). Factors of $l$ can later be reinstated by dimensional analysis. We take the metric to be in radial gauge

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{\mathrm{d}r^2}{r^2} + h_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{3.26}$$

with $h_{ab}$ given by a Feffermam-Graham expansion

$$h_{ab} = \frac{1}{r^2}\left[h_{(0)ab} + r^2h_{(2)ab} + r^4\log r\, h_{(4,1)ab} + r^4h_{(4)ab} + \mathcal{O}(r^6\log r)\right]. \tag{3.27}$$

Had we not included the logarithmic term the boundary metric would have been constrained to be conformally flat. In writing equation (3.26) we took advantage of two things. First, our freedom to do coordinate transformations was used to put the metric in radial gauge, using up our freedom to do coordinate transformations. Second, we used the result of Fefferman and Graham [55] to write the remaining part as an asymptotic expansion in the radial coordinate $r$. The boundary, where we assume the field theory is living, is therefore located at $r = 0$ and the metric on the boundary is $h_{(0)ab}$. Inverting (3.27) and expanding in $r$ yields the inverse metric

$$\begin{aligned} h^{ab} = r^2\Big[&h^{ab}_{(0)} - r^2h^{ac}_{(0)}h_{(2)cd}h^{db}_{(0)} - r^4\log r\, h^{ac}_{(0)}h_{(4,1)cd}h^{db}_{(0)} \\ &+ r^4\left(h^{ac}_{(0)}h_{(2)cd}h^{de}_{(0)}h_{(2)ef}h^{fb}_{(0)} - h^{ac}_{(0)}h_{(4)cd}h^{db}_{(0)}\right) + \mathcal{O}(r^6\log r)\Big], \end{aligned} \tag{3.28}$$

which is used in the coming calculations. The equations of motion can easily be found by varying the bulk metric, and demanding that $\delta S = 0$. We then find the familiar form

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} - 6g_{\mu\nu} = 0, \tag{3.29}$$

where $G_{\mu\nu}$ is the Einstein tensor. To find the coefficients of the expansion (3.27) we solve this order by order in $r$. In this case, the package xAct [81] for *Mathematica* was used to check the results. We find

$$h_{(2)ab} = -\frac{1}{2}R_{(0)ab} + \frac{1}{12}R_{(0)}h_{(0)ab} \tag{3.30}$$

at order $r^2$. All quantities are written with respect to the boundary metric $h_{(0)}$ and indices are raised and lowered with this object. At order $r^4 \log r$ the coefficient is

$$h_{(4,1)ab} = h_{(2)ac}h^{c}_{(2)b} + \frac{1}{4}\nabla^{(0)c}\left(\nabla^{(0)}_{a}h_{(2)bc} + \nabla^{(0)}_{b}h_{(2)ac} - \nabla^{(0)}_{c}h_{(2)ab}\right)$$
$$- \frac{1}{4}\nabla^{(0)}_{a}\nabla^{(0)}_{b}h^{c}_{(2)c} - \frac{1}{4}h_{(0)ab}h^{cd}_{(2)}h_{(2)cd}\,. \tag{3.31}$$

Note that this object is traceless. At order $r^4$ we find that $h_{(4)ab}$ is constrained by

$$h^{a}_{(4)a} = \frac{1}{4}h^{ab}_{(2)}h_{(2)ab}, \tag{3.32}$$

$$\nabla^{(0)b}h_{(4)ab} = \frac{1}{2}\left(h^{bc}_{(2)}\nabla^{(0)}_{b}h_{(2)ac} + h^{c}_{(2)a}\nabla^{(0)b}h_{(2)bc}\right) - \frac{1}{4}h^{bc}_{(2)}\nabla^{(0)}_{a}h_{(2)bc}$$
$$- \frac{1}{4}h_{(2)ac}\nabla^{(0)c}h^{b}_{(2)b}\,. \tag{3.33}$$

In order to calculate the boundary stress-energy tensor we need to regularize and renormalize the action (3.25). To regularize we introduce some cut-off hypersurface (with metric $h_{ab}$) at $r = \epsilon$ and in the end we take the limit $\epsilon \to 0$ which will take us to the boundary (with metric $h_{(0)ab}$). A renormalization procedure is required as solutions to (3.29) which are AlAdS will have second order poles at infinity ($r = 0$) and performing the integral over $r$ in (3.25) will lead to divergences.

To proceed we impose the (trace of the) bulk equations of motion, $R = -20$, in (3.25) and use the fact that the trace of the extrinsic curvature tensor is the divergence of the normal vector $K = \nabla_\mu n^\mu$. The action can then be written as

$$S^{\text{on-shell}}_{\text{reg}}[h,\epsilon] = -\frac{1}{2\kappa_5^2}\int \mathrm{d}^4x\left[\int_\epsilon \mathrm{d}r\frac{8}{r}\sqrt{-h} + \left(2r\partial_r\sqrt{-h}\right)\Big|_{r=\epsilon}\right]. \tag{3.34}$$

The metric determinant can be expanded in powers of $r$:

$$\sqrt{-h} = r^{-4}\sqrt{-h_{(0)}}\exp\left[\mathrm{tr}\log\left(\delta^{b}{}_{c} + r^2h^{b}_{(2)c} + r^4h^{b}_{(4)c} + r^4\log r\,h^{b}_{(4,1)c} + \mathcal{O}(r^6)\right)\right]$$
$$= r^{-4}\sqrt{-h_{(0)}}\left(1 + \frac{1}{2}r^2h^{a}_{(2)a} + \frac{1}{8}r^4\left(\left(h^{a}_{(2)a}\right)^2 - h^{ab}_{(2)}h_{(2)ab}\right) + \mathcal{O}(r^6)\right), \tag{3.35}$$

where (3.32) was used. The integration can then be performed and the divergences appear as poles in $\epsilon$:

$$S^{\text{on-shell}}_{\text{reg}}[h,\epsilon] = -\frac{1}{2\kappa_5^2}\int \mathrm{d}^4x\sqrt{-h_{(0)}}\left(\epsilon^{-4}a_{(0)} + \epsilon^{-2}a_{(2)} - \log\epsilon\, a_{(4)}\right) + \mathcal{O}(\epsilon^0), \tag{3.36}$$

where the coefficients $a_{(k)}$ are given by

$$a_{(0)} = -6\,, \qquad a_{(2)} = 0\,, \qquad a_{(4)} = \left(h^{a}_{(2)a}\right)^2 - h^{ab}_{(2)}h_{(2)ab}\,, \tag{3.37}$$

and the renormalized action is therefore

$$S^{\text{on-shell}}_{\text{ren}}[h_{(0)}] = \lim_{\epsilon\to 0}\frac{1}{2\kappa_5^2}\left[S^{\text{on-shell}}_{\text{reg}}[h,\epsilon] + \int \mathrm{d}^4x\sqrt{-h_{(0)}}\left(\epsilon^{-4}a_{(0)} + \epsilon^{-2}a_{(2)} - \log\epsilon\, a_{(4)}\right)\right]. \tag{3.38}$$

To make the counterterms covariant they should be written as functions of the induced metric on the cut-off hypersurface, $h_{ab}$. This entails an inversion of (3.35) and (3.30) to order $\epsilon^4$. Terms up to order $\epsilon^4$ are needed even though the counter-term action is defined as the divergent parts of the regularized action. This is because the logarithmic term will result in divergences when multiplying terms of order $\epsilon^0$. The result of the inversion is

$$\sqrt{-h_{(0)}} = \epsilon^4\sqrt{-h}\left(1 - \frac{1}{2}\epsilon^2 h^a_{(2)a} + \frac{1}{8}\epsilon^4\left(\left(h^a_{(2)a}\right)^2 - h^{ab}_{(2)}h_{(2)ab}\right)\right), \tag{3.39}$$

$$h^a_{(2)a} = \frac{1}{6\epsilon^2}\left(-R_{(h)} - \frac{1}{2}R_{(h)ab}R^{ab}_{(h)} + \frac{1}{12}R^2_{(h)} + \mathcal{O}(R^3_{(h)})\right), \tag{3.40}$$

$$h^{ab}_{(2)}h_{(2)ab} = \frac{1}{\epsilon^4}\left(\frac{1}{4}R_{(h)ab}R^{ab}_{(h)} - \frac{1}{18}R^2_{(h)} + \mathcal{O}(R^3_{(h)})\right), \tag{3.41}$$

where the terms cubic in curvature give vanishing contributions[2]. The first equation above follows trivially by inverting (3.35) and doing a series expansion in $r$. In writing the other two, we first note that the trace of $h_{(2)ab}$ can be found from (3.30) and is

$$h^a_{(2)a} = -\frac{1}{6}R_{(0)}\,. \tag{3.42}$$

The goal is to write this object in terms of quantities living on the hypersurface. The Ricci tensor on the hypersurface can be written as a series in $r$:

$$R_{(h)ab} = R_{(0)ab} + r^2\left(h_{(4,1)ab} - h_{(2)ac}h^c_{(2)b} + \frac{1}{4}h_{(0)ab}h^{cd}_{(2)}h_{(2)cd}\right) + \mathcal{O}(r^4)\,, \tag{3.43}$$

such that the Ricci scalar on the hypersurface, using the inverse metric expansion (3.28), can be written as

$$R_{(h)} = r^2 R_{(0)} + r^4 h^{ab}_{(2)}R_{(0)ab} + \mathcal{O}(r^6) \tag{3.44}$$

$$= r^2 R_{(0)} + r^4\left(\frac{1}{12}R^2_{(0)} - \frac{1}{2}R^{ab}_{(0)}R_{(0)ab}\right) + \mathcal{O}(r^6)\,, \tag{3.45}$$

where we used the tracelessness of $h_{(4,1)ab}$. Furthermore, using (3.28), we can raise the indices of the Ricci tensor and find that

$$R^{ab}_{(h)} = r^4 R^{ab}_{(0)} + \mathcal{O}(r^6), \tag{3.46}$$

and from (3.44) we also see that

$$R^2_{(h)} = r^4 R^2_{(0)} + \mathcal{O}(r^6). \tag{3.47}$$

Equation (3.40) now follows by combining (3.42), (3.43) and (3.45)–(3.47). Similarly, equation (3.41) can be shown by writing out $h^{ab}_{(2)}h_{(2)ab}$ and using (3.46) and (3.47). The counter-term action is therefore given by

$$S_{\rm ct}[h,\epsilon] = \frac{1}{\kappa_5^2}\int_{r=\epsilon} \mathrm{d}^4x\sqrt{-h}\left(-3 - \frac{1}{4}R_{(h)} + (\lambda + \log r)\left(\frac{1}{8}R^{ab}_{(h)}R_{(h)ab} - \frac{1}{24}R^2_{(h)}\right)\right), \tag{3.48}$$

[2]The terms quadratic in curvature in (3.40) will actually not contribute to the counter-term action either, as these are $\mathcal{O}(\epsilon^0)$. Furthermore, it is the square of $h^a_{(2)a}$ that appears in the anomaly term, so in this case only the first term is needed. We include them here for completeness.

where we added a scheme dependent parameter $\lambda$ (minimal subtraction is $\lambda = 0$). The full action is then

$$S_{\text{sub}}[h,\epsilon] = \frac{1}{2\kappa_5^2}\Bigg[\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\,(R+12) + 2\int_{r=\epsilon} \mathrm{d}^4x\sqrt{-h}\left(K - 3 - \frac{1}{4}R_{(h)} + (\log r + \lambda)\,\mathcal{A}\right)\Bigg], \tag{3.49}$$

where $\mathcal{A}$ is the conformal anomaly,

$$\mathcal{A} = \frac{1}{8}\left(R_{(h)ab}R^{ab}_{(h)} - \frac{1}{3}R^2_{(h)}\right). \tag{3.50}$$

From the action (3.49) we can calculate both the stress-energy tensor of the dual field theory and the 4-dimensional conformal anomaly.

### 3.3.1 The stress-energy tensor and the conformal anomaly

The stress-energy tensor of the dual field theory is obtained by varying the associated source in the renormalized on-shell action of the bulk theory. Thus, we write the variation of the action as

$$\delta S_{\text{sub}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\,(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu}) - \frac{1}{2\kappa_5^2}\int_{r=\epsilon} \mathrm{d}^4x\sqrt{-h}\left(T_{ab}\delta h^{ab}\right), \tag{3.51}$$

where $\mathcal{E}_{\mu\nu}$ is given by (3.29) and going on-shell imposes $\mathcal{E}_{\mu\nu} = 0$. Furthermore

$$T_{ab} = (K-3)h_{ab} - K_{ab} + \frac{1}{2}R_{(h)ab} - \frac{1}{4}R_{(h)}h_{ab} + (\lambda + \log r)\,T^{(A)}_{ab}, \tag{3.52}$$

with

$$T^{(A)}_{ab} = -\frac{2\kappa_5^2}{\sqrt{-h}}\frac{\delta A}{\delta h^{ab}}, \tag{3.53}$$

where $A$ is the integrated conformal anomaly

$$A = \frac{1}{\kappa_5^2}\int_{r=\epsilon} \mathrm{d}^4x\sqrt{-h}\mathcal{A}. \tag{3.54}$$

At this point $T_{ab}$ is just the stress-energy tensor of a hypersurface at constant $r = \epsilon$. To obtain the stress-energy tensor of the boundary we should take $\epsilon \to 0$. From (3.22) and (3.23) the expression for the stress energy tensor is

$$\left\langle T_{(0)ab}\right\rangle = -\frac{2\kappa_5^2}{\sqrt{-h_{(0)}}}\frac{\delta S^{\text{on-shell}}_{\text{ren}}}{\delta h^{ab}_{(0)}} = \lim_{\epsilon\to 0}\epsilon^{-2}T_{ab}. \tag{3.55}$$

Thus, all terms in the expression for $T_{ab}$ up to order $\epsilon^2$ should cancel, such that the resulting expression is finite. The renormalization procedure takes care of this by design, and an explicit calculation of the $\epsilon^2$-term in $T_{ab}$ can be performed using the fact that

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}, \tag{3.56}$$

with the normal to the hypersurface given by $n^\mu = -r\delta_r^\mu$, where the minus arises since the normal vector is taken to be inward pointing. Such a calculation is most easily carried out using xAct and yields

$$\left\langle T_{(0)ab}\right\rangle = 2h_{(4)ab} - h_{(2)ac}h^c_{(2)b} + \frac{1}{2}h_{(2)ab}h^c_{(2)c} + \frac{1}{2}h_{(0)ab}\mathcal{A}_{(0)} + \frac{1}{2}(3-4\lambda)h_{(4,1)ab}, \quad (3.57)$$

with $\mathcal{A}_{(0)}$ defined similarly to $\left\langle T_{(0)ab}\right\rangle$ as

$$\mathcal{A}_{(0)} = \lim_{\epsilon\to 0}\epsilon^{-4}\mathcal{A} = \frac{1}{2}\left(h^{ab}_{(2)}h_{(2)ab} - \left(h^a_{(2)a}\right)^2\right), \quad (3.58)$$

and can be found by a calculation resembling the one above. The term $\mathcal{A}_{(0)}$ is known as the conformal anomaly for reasons to become clear shortly.

One often defines

$$\left\langle T_{(0)ab}\right\rangle \equiv t_{ab} \quad (3.59)$$

such that $h_{(4)ab}$ can be written as $t_{ab}$ plus some correction given by the remaining terms in (3.57).

### 3.3.2 Ward identities

As is well-known from Noether's theorem any continuous symmetry gives rise to a conserved quantity. If the symmetries are local the conservation laws are referred to as Ward identities. Ward identities hold even off–shell. To see how this constrains the energy-momentum tensor consider

$$\delta S_{\text{sub}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2\kappa_5^2}\int_{r=\epsilon} \mathrm{d}^4x\sqrt{-h}T_{ab}\delta h^{ab}. \quad (3.60)$$

Recall from section 2.3.2 that bulk diffeomorphisms induce, in addition to local boundary diffeomorphisms, also a local conformal transformation of the boundary metric. Thus, there should be Ward identities associated to these two continuous symmetries. As shown in section 2.3.2 the bulk diffeomorphisms induce the following transformation of the boundary metric:

$$\delta h^{ab}_{(0)} = \mathcal{L}_\xi h^{ab}_{(0)} = -\nabla^a_{(0)}\xi^b_{(0)} - \nabla^b_{(0)}\xi^a_{(0)} + 2\xi^r_{(0)}h^{ab}_{(0)}\,, \quad (3.61)$$

where the diffeomorphisms arise from the 5-dimensional transformations

$$r = r' - \xi^r\,, \qquad x^a = x'^a - \xi^a\,. \quad (3.62)$$

Since we are now varying coordinates there will also be a contribution from the variation of the $\log r$ term in the action. Taking the action (3.60) on-shell on the boundary and applying the variation (3.61) results in

$$\delta S^{\text{on-shell}}_{\text{ren}} = -\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h_{(0)}}\left(t_{ab}\delta h^{ab}_{(0)} - 2\frac{\mathcal{A}_{(0)}}{r}\delta r\right). \quad (3.63)$$

Since $t_{ab}$ is symmetric and, by definition, $\delta r \equiv \xi^r = r\xi^r_{(0)}$, we can write this as

$$\delta S_{\text{ren}}^{\text{on-shell}} = -\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h_{(0)}}\left(2t_{ab}\nabla^a_{(0)}\xi^b_{(0)} + 2t^a{}_a\xi^r_{(0)} - 2\mathcal{A}_{(0)}\xi^r_{(0)}\right). \tag{3.64}$$

An application of Stoke's theorem (the terms that contribute total divergences on the boundary vanish as the boundary of a boundary is the empty set) yields

$$\delta S_{\text{ren}}^{\text{on-shell}} = -\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h_{(0)}}2\left(\nabla^a_{(0)}t_{ab}\xi^b_{(0)} + \left(t^a{}_a - \mathcal{A}_{(0)}\right)\xi^r_{(0)}\right), \tag{3.65}$$

and since the variation should vanish for arbitrary transformations we finally have that the energy-momentum tensor should satisfy the following Ward identities:

$$\nabla^a_{(0)}t_{ab} = 0, \tag{3.66}$$

$$t^a{}_a = \mathcal{A}_{(0)}. \tag{3.67}$$

Thus, the covariant conservation of the energy-momentum tensor is related to diffeomorphism invariance of the action, while the trace is proportional to the logarithmically divergent term in the action, hence the name conformal anomaly.

The lessons learned in this section will be used in the investigation of the $z = 2$ Lifshitz spacetime in 4 dimensions performed in chapter 5 and chapter 6.

# Chapter 4

# Dimensional Reduction

Dimensional reduction or compactification is a central concept in string theory today. Compactification is a very important tool as one is typically interested in what type of phenomena the theory predicts but in many cases the full string theory is too complicated to solve. The central assumption is that spacetime is given by $\mathcal{M}_m \times \mathcal{Q}_q$ where $\mathcal{M}$ is a manifold on which the reduced theory in $m$ dimensions lives, while $\mathcal{Q}$ is some compact manifold of dimension $q$. The fields on $\mathcal{M}$ are then taken to depend trivially on the coordinates of $\mathcal{Q}$. Typically some truncation of the field content is involved as well, otherwise the theory would not simplify and one would describe the full physics of $m+q$ dimensions using only $m$ dimensions but with numerous extra fields arising from the compactification of the $q$ directions. Such a truncation could happen by assuming that the extra dimensions are small compared to other relevant quantities in the theory. This is the central assumption in a Kaluza-Klein reduction and leads to a truncation of the massive modes arising from the reduction.

Kaluza-Klein reductions will be explored in further detail in section 4.1. A simple extension of the Kaluza-Klein reduction is the Scherk-Schwarz reduction where one gauges a specific global symmetry of the higher-dimensional theory, and this case is considered in section 4.2. The concept of Freund-Rubin (or flux) compactifications will also play a part in arriving at the theory considered in chapter 5. In this case one considers the $(p+2)$-forms sourcing the D$p$-branes to be proportional to (the Hodge dual of) the volume element of a $(p+2)$-sphere. This allows for a simple way of reducing the theory by $p+2$ dimensions and this method will be considered in section 4.3.

## 4.1 Kaluza-Klein Reduction

The idea of Kaluza-Klein reductions is to view one or more of the spatial coordinates as being compact. For instance one could perform the reduction over a circle. In this case one would identify the compact coordinate as $u \sim u + 2\pi L$ thus performing the reduction over the compact manifold $S^1$. We will therefore need $\partial_u$ to be a Killing vector of the metric. One could also extend the compactification to other more complicated compact manifolds. As an example of a compactification over $S^1$ consider a massless scalar field $\hat{\phi}$ and denote the compact direction by $u$. As $u$ is a compact direction we can decompose $\hat{\phi}$

as

$$\hat{\phi}(x,u) = \sum_n \phi_n(x) e^{\frac{inu}{L}}. \tag{4.1}$$

Since the massless scalar field satisfies

$$\hat{\Box}\hat{\phi} = 0, \tag{4.2}$$

the coefficients of the Fourier expansion will satisfy

$$\Box\phi_n - \frac{n^2}{L^2}\phi_n = 0, \tag{4.3}$$

and we see that the lower-dimensional scalar field has acquired a mass of $\frac{|n|}{L}$. Assuming that the compact dimension is comparable to the Planck length in size, we can neglect all the massive modes, and only worry about the $n = 0$ case. This is the aforementioned truncation of the massive modes. The fact that this truncation is consistent can be seen by considering the expansion

$$\hat{\phi}(x,u) = \sum_n \phi_n(x) e^{\frac{inu}{L}}. \tag{4.4}$$

There is a $U(1)$ symmetry rotating modes with $n = \pm m$ into each other. Thus all modes having non-zero value of $n$ transform as doublets under this $U(1)$ symmetry, while the mode with $n = 0$ transforms as a singlet. Since there are no rotations which will make a singlet transform as a doublet, neglecting all the doublets of the theory is a consistent truncation. Consistency is rather easy to show in this simple example, but as the compactifications increase in complexity, showing that the reduction is consistent becomes a more and more difficult task.

In the general case, the reduction ansatz will be more complicated. In the case of Einstein gravity this could for instance be taking $\hat{g}_{\hat{\mu}\hat{\nu}}(x,u)$ to be independent of $u$. The indices of the $(d+1)$-dimensional metric will split into values associated with either the $d$ lower dimensions, or with the compact dimension. The components of the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ can then be written as $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu u}$ and $\hat{g}_{uu}$, which, with some slight modifications, will be our relevant fields in $d$ dimensions. These fields can be interpreted as a metric, $\hat{g}_{\mu\nu}$, a one-form $\hat{g}_{\mu u}$ and a scalar, $\hat{g}_{uu}$. In order to make the $d$-dimensional equations of motion come out nice we consider the ansatz

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}\left(\mathrm{d}u + A\right)^2, \tag{4.5}$$

where $\alpha$ and $\beta$ are constants to be chosen for convenience and $A = A_\mu \mathrm{d}x^\mu$. The mapping between fields is then

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi}g_{\mu\nu} + e^{2\beta\phi}A_\mu A_\nu, \qquad \hat{g}_{\mu u} = e^{2\beta\phi}A_\mu, \qquad \hat{g}_{uu} = e^{2\beta\phi}. \tag{4.6}$$

In order to calculate the Ricci tensor and the Ricci scalar it is convenient to change to a frame field basis. From the above ansatz we see that such a choice of basis could be

$$\hat{e}^{\underline{a}} = e^{\alpha\phi}e^{\underline{a}}, \qquad \hat{e}^{\underline{u}} = e^{\beta\phi}\left(\mathrm{d}u + A\right), \tag{4.7}$$

where, as usual,

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\eta}_{\underline{\hat{a}\hat{b}}} \hat{e}^{\hat{\underline{a}}}_{\hat{\mu}} \hat{e}^{\hat{\underline{b}}}_{\hat{\nu}} \,. \tag{4.8}$$

Note that we can represent the frame field as a matrix

$$\hat{e}^{\hat{\underline{a}}}_{\hat{\mu}} = \begin{pmatrix} e^{\beta\phi} & e^{\beta\phi} A_\mu \\ 0 & e^{\alpha\phi} e^{\underline{a}}_{\mu} \end{pmatrix} , \tag{4.9}$$

making it straightforward to show that

$$\det(-\hat{g}) = \left(\det(\hat{e})\right)^2 = e^{2(d\alpha+\beta)\phi} \left(\det(e)\right)^2 = e^{2(d\alpha+\beta)\phi} \det(-g) \,, \tag{4.10}$$

since $\det(-\eta) = 1$. The requirements that we remain in Einstein frame and have a canonical normalization of the kinetic term of the scalar field can be used to fix the constants $\alpha$ and $\beta$ such that

$$\alpha^2 = \frac{1}{2(d-2)(d-1)} \,, \qquad\qquad \beta = -(d-2)\alpha \,, \tag{4.11}$$

and the determinant relation becomes

$$\det(-\hat{g}) = e^{4\alpha\phi} \det(-g). \tag{4.12}$$

The definition of the frame fields can be used to derive the Ricci scalar of the original theory in terms of dimensionally reduced quantities. The result is [82]

$$\hat{R} = e^{-2\alpha\phi} \left( R - \frac{1}{2} (\partial\phi)^2 + (d-3)\alpha\Box\phi \right) - \frac{1}{4} e^{-2d\alpha\phi} F_{\mu\nu} F^{\mu\nu}, \tag{4.13}$$

showing that the Kaluza-Klein reduction of pure gravity from $d+1$ dimensions to $d$ dimensions introduces two new fields into the theory. One is a $U(1)$ gauge field analagous to the Maxwell field, while the other is a scalar field which, due to its similarity to an analogous field in string theory, is called a dilaton. The $d$-dimensional equations of motion following from the dimensionally reduced action,

$$S = \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g} \left( R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2(d-1)\alpha\phi} F_{\mu\nu} F^{\mu\nu} \right) \,, \tag{4.14}$$

are

$$G_{\mu\nu} = \frac{1}{2} \left( \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right) + \frac{1}{2} e^{-2(d-1)\alpha\phi} \left( F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \right) \,, \tag{4.15}$$

$$0 = \nabla^\mu \left( e^{-2(d-1)\alpha\phi} F_{\mu\nu} \right) \,, \tag{4.16}$$

$$\Box\phi = -\frac{1}{2} (d-1)\alpha e^{-2(d-1)\alpha\phi} F_{\mu\nu} F^{\mu\nu} \,. \tag{4.17}$$

The dilaton field cannot in general be set to zero, since it needs to satisfy its equation of motion (4.17). This ties in with the fact that the reduction from $d+1$ dimensions to $d$ dimensions should be consistent. In the case of the free scalar field above, consistency was proved by showing that the truncated massive modes do not affect the massless mode. Here the situation is more complicated. If the reduction is to be consistent, any solution to the equations of motion in $d$ dimensions should also be a solution to the equations of motion in $d+1$ dimensions. Had the dilaton not been introduced in (4.5), i.e. had the ansatz had $\phi = 0$, the equation of motion for $\hat{R}_{uu}$ would not be satisfied and the reduction would be inconsistent. Proving consistency in complicated settings is, in general, a hard thing to do, but for simple compact manifolds it is possible. The Scherk-Schwarz circle reduction considered below is one of such cases.

The dimensional reduction of gravity will inevitably produce a non-trivial scalar field in addition to a vector field. Thus, it is not in general possible to compactify gravity in $d+1$ dimensions and end up with gravity coupled to a Maxwell field in $d$ dimensions and in this way unify gravity and electromagnetism. However, it is interesting to study such theories in a holographic setting. This was done in [83].

### 4.1.1 Symmetries of the reduced theory

The Einstein-Hilbert action is invariant under diffeomorphisms. Apart from diffeomorphisms the equations of motion have an additional global symmetry related to conformal rescalings of the higher-dimensional metric. The diffeomorphisms and conformal transformations can be represented infinitesimally as

$$\delta \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}} + \hat{g}_{\hat{\rho}\hat{\mu}} \partial_{\hat{\nu}} \hat{\xi}^{\hat{\rho}} + \hat{g}_{\hat{\rho}\hat{\nu}} \partial_{\hat{\mu}} \hat{\xi}^{\hat{\rho}} + 2a\delta \hat{g}_{\hat{\mu}\hat{\nu}}. \tag{4.18}$$

Thus the symmetries of the $d+1$ dimensional action is given by diffeomorphisms in $d+1$ dimensions, as the global conformal symmetry is only a symmetry of the Einstein equations in $d+1$ dimensions. In the reduced theory, the freedom to do diffeomorphisms in $d+1$ dimensions is reduced to $d$ dimensions, so something must have happened to the remaining symmetry. The existence of a global conformal transformation in $d+1$ dimensions allows one to consider global translations of the scalar field which, when combined with the global conformal transformation in $d$ dimensions, leave the action invariant. Hence our transformations should allow for general coordinate invariance in $d$ dimensions along with an expected local $U(1)$ gauge symmetry of the vector field, and, in addition, there should be a transformation which translates the scalar field. On these grounds we expect that the most general transformations allowed take the form

$$\hat{\xi}^{\hat{\mu}} = \xi^{\mu}(x^{\nu}) \,, \qquad \hat{\xi}^{\hat{u}} = cu + \lambda(x^{\mu}) \,, \tag{4.19}$$

where $c$ corresponds to a global translation and $\lambda$ to the local gauge transformation. We consider the transformations of the fields first under the local transformations $\xi^{\mu}$ and $\lambda$. Using (4.18) we investigate how the $d+1$ diffeomorphisms translate into symmetries of the $d$-dimensional system. We find

$$\delta \hat{g}_{uu} = \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{uu} + 2\hat{g}_{\hat{\rho}u} \partial_u \hat{\xi}^{\hat{\rho}} \,. \tag{4.20}$$

All fields are assumed to be independent of $u$ and at the moment we only consider the local transformations so $\hat{\xi}^u$ is independent of $u$. Hence

$$\delta \hat{g}_{uu} = \xi^{\rho} \partial_{\rho} \hat{g}_{uu} \,, \tag{4.21}$$

and from the ansatz (4.6) this means that

$$\delta\phi = \xi^\rho \partial_\rho \phi, \tag{4.22}$$

reproducing the expected result that the dilaton transforms as a scalar under coordinate transformations and is invariant under the local gauge transformations. Similarly,

$$\begin{aligned} \delta\hat{g}_{u\mu} &= \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{u\mu} + \hat{g}_{u\hat{\rho}} \partial_\mu \hat{\xi}^{\hat{\rho}} + \hat{g}_{\mu\hat{\rho}} \partial_u \hat{\xi}^{\hat{\rho}} \\ &= \xi^\rho \partial_\rho \hat{g}_{u\mu} + \hat{g}_{uu} \partial_\mu \hat{\xi}^u + \hat{g}_{u\rho} \partial_\mu \xi^\rho \,, \end{aligned} \tag{4.23}$$

and applying the ansatz (4.6) and using the transformation properties of the dilaton field leads to

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda. \tag{4.24}$$

Once again this result is expected. The field $A_\mu$ transforms as a 1-form under diffeomorphisms and as a $U(1)$ field under gauge transformations. The lower dimensional metric can be analyzed in the same manner, and using both the transformations of the vector field and of the dilaton we find that the lower-dimensional metric indeed does transform as a 2-tensor under coordinate transformations and is left unchanged by $U(1)$ gauge transformations:

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\rho\nu} \partial_\mu \xi^\rho \,. \tag{4.25}$$

A similar analysis for the global symmetries $a$ and $c$ yields

$$\delta\phi = \frac{c+a}{\beta}, \qquad \delta A_\mu = -cA_\mu, \qquad \delta g_{\mu\nu} = 2a g_{\mu\nu} - 2\alpha g_{\mu\nu} \frac{c+a}{\beta}. \tag{4.26}$$

It is clear from (4.26) that the relation between the constants $a$ and $c$, which parameterize the two global symmetries, can be chosen such that the metric is left unaltered. Recall that $\beta = -(d-2)\alpha$, so choosing

$$a = -\frac{c}{d-1} \tag{4.27}$$

implies that $\delta g_{\mu\nu} = 0$. Under the transformation (4.26) the fields then transform as

$$\delta\phi = -\frac{c}{\alpha(d-1)}, \qquad \delta A_\mu = -cA_\mu, \qquad \delta g_{\mu\nu} = 0 \tag{4.28}$$

and corresponds to the aforementioned combination of conformal scalings and translations which leave the $d$-dimensional equations of motion invariant. The vector field is scaled by a constant factor, while the dilaton field is shifted. One may wonder if we have not thrown away some symmetry by taking $a$ and $c$ to be related by (4.27). After all, we started with two symmetries parameterized by $a$ and $c$ and ended up with one, transforming the fields as (4.28). However, if any transformation by $a$ or $c$ is a symmetry then any linear combination of transformations by $a$ and $c$ is also a symmetry, and the relation (4.27) is just a particular linear combination. One can form another, independent, transformation by considering the choice

$$a = -c, \tag{4.29}$$

implying that $\delta\phi = 0$. The metric and vector field then scales as

$$\delta A_\mu = aA_\mu, \qquad \delta g_{\mu\nu} = 2ag_{\mu\nu}. \tag{4.30}$$

This can be generalized to saying that each field scales according to its number of indices. The choices (4.27) and (4.29) are just particular choices of linear combinations which make the fields transform in an especially desireable way. Other choices would be equally good, although they would not imply the nice facts that the metric is invariant under one choice of rescaling, while the scalar field is invariant under the other choice of rescaling.

### 4.1.2 Kaluza-Klein reductions over higher-dimensional manifolds

As mentioned, Kaluza-Klein reductions over more complicated manifolds are also possible. However, the complexity of the reduced theory increases rapidly as a vast number of fields is introduced. The reduction over a $q$-torus is relatively straightforward, but the reduction over $q$-spheres is still poorly understood, and, in addition, only a very limited number of sphere reductions are consistent [82]. These are the reductions of type IIB string theory over a 5-sphere to $\mathrm{AdS}_5$, and the reduction of M-theory over $S^4$ and $S^7$ down to $\mathrm{AdS}_7$ and $\mathrm{AdS}_4$, respectively.

## 4.2 Scherk-Schwarz Reduction

Scherk-Schwarz reductions [84] are, in a way, a special class of Kaluza-Klein reductions, which are possible when the action contains scalar fields which enter only through their derivatives. The defining assumption is still that the action be independent of the coordinate over which the compactification is performed. The compactification manifold will, in the case considered here, just be a circle $S^1$. Consider an action where the scalar field in question enters through a term

$$S = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-\hat{g}}\left(\hat{R} - \frac{1}{2}\hat{V}\left(\partial\hat{\chi}\right)^2\right), \tag{4.31}$$

where $\hat{V}$ is some function independent of $\hat{\chi}$ and $\hat{g}_{\hat{\mu}\hat{\nu}}$. The equations of motion are

$$\hat{R}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{R}\hat{g}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{V}\partial_{\hat{\mu}}\hat{\chi}\partial_{\hat{\nu}}\hat{\chi} + \frac{1}{4}\hat{V}\left(\partial\hat{\chi}\right)^2\hat{g}_{\hat{\mu}\hat{\nu}} = 0\,, \tag{4.32}$$

$$\hat{V}\hat{\Box}\hat{\chi} + \partial_{\hat{\mu}}\hat{V}\partial^{\hat{\mu}}\hat{\chi} = 0\,. \tag{4.33}$$

In a Scherk-Schwarz reduction, one makes an ansatz very similar to the one made in the Kaluza-Klein reduction but, in addition, the global symmetries of the theory are gauged. In the case considered here the theory has two global symmetries:

$$\hat{\chi} \to \hat{\chi} + a\,, \tag{4.34}$$

$$\hat{g}_{\hat{\mu}\hat{\nu}} \to e^{2b}\hat{g}_{\hat{\mu}\hat{\nu}}\,, \tag{4.35}$$

with $a$ and $b$ constants. However, the scaling symmetry of the metric is only a symmetry of the equations of motion, not of the full Lagrangian. If one wishes to include this symmetry, the reduction must be performed at the level of the equations of motion, since reducing the action leads to inconsistencies [85]. Here we are only interested in the global shift symmetry of the scalar field and the reduction can be performed at the level of the action. The existence of a global shift symmetry allows us to extend the Kaluza-Klein assumption a bit. Instead of simply assuming that the scalar field is independent of the compact direction $u$ (through some truncation of the massive modes), we can consider a scalar field linearly dependent on the $u$-direction:

$$\hat{\chi}(x^{\hat{\mu}}) = ku + \chi(x^{\mu}). \tag{4.36}$$

In a theory where such a scalar field appears alongside gravity in an action of the form (4.31), one can perform a dimensional compactification along the $u$-direction by doing a Kaluza-Klein reduction of gravity and a Scherk-Schwarz reduction of the scalar field. Recall that the Kaluza-Klein reduction results in a massless vector field. However, upon performing a Scherk-Schwarz reduction of the scalar field $\hat{\chi}$ in addition to a Kaluza-Klein reduction of gravity, the Kaluza-Klein vector will gauge the axion shift symmetry and the vector field acquires a mass proportional to $k^2$. A massive vector field is indeed required to support Lifshitz spacetimes as described in section 2.4. We will see in chapter 5 that a Scherk-Schwarz reduction of a specific AlAdS theory gives rise to a spacetime which, when certain conditions are imposed, is AlLif.

## 4.3 Freund-Rubin Compactification

Freund-Rubin compactification is a way of compactifying fluxes first described by Freund and Rubin in 1980 [86]. The central assumption is that spacetime is composed of a product manifold $\mathcal{M}_d \times S^{n-d}$. The inclusion of fluxes from the antisymmetric field strength tensor of rank $n-d$ stabilizes the sphere. Thus, one simply assumes a solution in which the field strength is proportional to the volume form of the sphere. Furthermore, since spacetime is a product manifold, the metric is block diagonal and the sphere directions are completely independent of the other spacetime directions. Any other fields are taken to be independent of the sphere directions. This allows one to integrate out the $S^{n-d}$ leaving an effective theory for the remaining $d$ dimensions.

Below we will consider an example of a Freund-Rubin compactification which will be very relevant for our setup. It will demonstrate that the theory we are considering can be obtained as a consistent truncation of string theory.

### 4.3.1 Freund-Rubin compactification of type IIB supergravity over a 5-sphere

We are interested in a theory of supergravity in 10 dimensions containing an axion-dilaton field, a 5-form field strength and gravity. We set the fermions and 2-forms to zero. Due to the self-duality constraint on the 5-form no consistent action exists and the reduction will have to take place at the level of the equations of motion. The procedure will be the following. We will state the relevant equations of motion and impose our solution.

This will result in a new set of equations of motion which will be shown to arise from a specific action in 5 dimensions. This will complete the reduction of 10-dimensional type IIB supergravity to a 5-dimensional theory.

The equations of motion are [87, 72]

$$\begin{aligned}\hat{R}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}g_{\hat{\mu}\hat{\nu}}\hat{R} = \frac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} + \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi}\partial_{\hat{\nu}}\hat{\chi} + \frac{1}{6}\hat{F}_{\hat{\lambda}_1\cdots\hat{\lambda}_4\hat{\mu}}\hat{F}^{\hat{\lambda}_1\cdots\hat{\lambda}_4}{}_{\hat{\nu}} \\ - \frac{1}{4}g_{\hat{\mu}\hat{\nu}}\left(\left(\partial\hat{\phi}\right)^2 + e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\right)\end{aligned} \tag{4.37}$$

$$\star\hat{F}_5 = F_5, \qquad \mathrm{d}\hat{F}_5 = 0, \tag{4.38}$$

in addition to the equations of motion for the scalar fields which reduce trivially. Any action containing gravity should also contain the Gibbons-Hawking boundary term. Here we assume it reduces trivially, $\hat{K} = K$. Due to the self-duality of the 5-form we can write the first equation as

$$\hat{R}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} + \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi}\partial_{\hat{\nu}}\hat{\chi} + \frac{1}{6}\hat{F}_{\hat{\lambda}_1\cdots\hat{\lambda}_4\hat{\mu}}\hat{F}^{\hat{\lambda}_1\cdots\hat{\lambda}_4}{}_{\hat{\nu}} \,. \tag{4.39}$$

We are interested in a solution where the 5-form is proportional to the volume-form on the 5-sphere[1]:

$$\hat{F}_{a_1\cdots a_5} = l^4\epsilon_{a_1\cdots a_5}, \tag{4.40}$$

however self-duality implies that we must also have

$$\hat{F}_{\mu_1\cdots\mu_5} = l^4\omega_{\mu_1\cdots\mu_5}, \tag{4.41}$$

with $\omega$ denoting the volume-form on the non-compact space. The fields are taken to be otherwise independent of the sphere directions. The ansatz (4.40) and (4.41) solves (4.38). The equation (4.39) splits into three seperate equations. The one with mixed indices is trivial,

$$R_{\mu a} = 0. \tag{4.42}$$

The one with only sphere indices yields (the metric is proportional to $l^{-1}$)

$$R_{ab} = \frac{1}{6}\epsilon_{c_1\cdots c_4 a}\epsilon^{c_1\cdots c_4}{}_b = 4g_{ab}, \tag{4.43}$$

while for the spacetime indices we have

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi - 4g_{\mu\nu}, \tag{4.44}$$

where the minus sign arises from the Lorentzian signature of spacetime. The two spaces therefore have identical radii but one is positively curved while the other is negatively curved. This indicates that the spacetime must contain a negative cosmological constant.

[1] Latin letters will denote sphere indices, while greek will denote spacetime indices

In 5 dimensions the action for gravity containing a negative cosmological constant and an axion-dilaton field is

$$S = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\left(R + 12 - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}e^{2\phi}\left(\partial\chi\right)^2\right) + \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h}K\,. \tag{4.45}$$

The equation of motion for the metric tensor is

$$G_{\mu\nu} - 6g_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi + \frac{1}{4}g_{\mu\nu}\left(\left(\partial\phi\right)^2 + e^{2\phi}\left(\partial\chi\right)^2\right) = 0. \tag{4.46}$$

Thus, the Ricci scalar is

$$R = \frac{1}{2}\left(\left(\partial\phi\right)^2 + e^{2\phi}\left(\partial\chi\right)^2\right) - 20, \tag{4.47}$$

and plugging this into the equation of motion results in

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi - 4g_{\mu\nu}, \tag{4.48}$$

confirming that this action correctly reproduces the 5-dimensional equations of motion. Hence 10-dimensional type IIB supergravity is shown to reduce to an $\mathrm{AdS}_5$ space with an axion-dilaton field.

# Chapter 5

# The Model: $z = 2$ Lifshitz$_4$ from AdS$_5$

In this chapter we will review the process by which one can obtain a $z = 2$ Lifshitz spacetime in 4 dimensions by dimensionally reducing an axion-dilaton theory including gravity with AlAdS boundary conditions. This will be used as a means of performing holographic renormalization of the $z = 2$ Lifshitz theory and will be the stepping stone for considering the holographic dual to Lifshitz spacetimes. Lifshitz holography is complicated by the fact that, in order to obtain a Lifshitz spacetime, many of the sources must be turned off. This was described in section 2.4. Below we will see that obtaining a 4-dimensional AlLif spacetime from a 5-dimensional AlAdS spacetime imposes certain constraints on the 5-dimensional theory. However, before these contraints are imposed the reduced theory is more general than AlLif, and in these spacetimes the vevs can be computed. Presenting a framework in which these calculations can take place is thus the main purpose of this chapter, however, the calculations themselves are postponed to the next chapter.

In section 5.1 the 5-dimensional model is presented and vevs and their associated Ward identities are computed. Then, in section 5.2, the reduction of the 5-dimensional theory is carried out. This allows us to define AlLif spacetimes from a 5-dimensional point of view. Several deformations of AlLif are then discussed. Finally, in section 5.3, we make some brief observations about the dual field theory.

## 5.1 Axion-Dilaton Gravity with AlAdS Boundary Conditions

In this section we review an axion-dilaton theory containing gravity obtained from 10-dimensional supergravity through a Freund-Rubin compactification. It turns out that such a theory is related to a 4-dimensional $z = 2$ Lifshitz theory through dimensional reduction [88, 89, 90]. Hence, knowledge of the 5-dimensional theory will aid us in analyzing the 4-dimensional Lifshitz theory, e.g. it will allow us to perform holographic renormalization of the Lifshitz theory, a fact which was utilized in [61].

The action for an axion-dilaton field theory containing gravity is

$$S = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-\hat{g}}\left[\hat{R} + 12 - \frac{1}{2}\left(\partial\hat{\phi}\right)^2 - \frac{1}{2}e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\right] + \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-\hat{h}}\hat{K} + S_{\mathrm{ct}}\,, \tag{5.1}$$

where the $\mathrm{AdS}_5$ length has been set to one and a counterterm action is added to make the variational problem well-posed. This is the action considered in section 3.3 except for the addition of the axion, $\hat{\chi}$, and the dilaton, $\hat{\phi}$. By varying the action we obtain the equations of motion:

$$\mathcal{E}_{\hat{\mu}\hat{\nu}} = \hat{G}_{\hat{\mu}\hat{\nu}} - 6\hat{g}_{\hat{\mu}\hat{\nu}} - \hat{T}^{\mathrm{bulk}}_{\hat{\mu}\hat{\nu}}\,, \tag{5.2}$$

$$\mathcal{E}_{\hat{\phi}} = \hat{\Box}\hat{\phi} - e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\,, \tag{5.3}$$

$$\mathcal{E}_{\hat{\chi}} = \hat{\Box}\hat{\chi} + 2\partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\chi}\,, \tag{5.4}$$

with the energy momentum tensor of the bulk spacetime given by

$$\hat{T}^{\mathrm{bulk}}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} + \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi}\partial_{\hat{\nu}}\hat{\chi} - \frac{1}{4}\hat{g}_{\hat{\mu}\hat{\nu}}\left(\left(\partial\hat{\phi}\right)^2 + e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\right)\,. \tag{5.5}$$

In a manner completely similar to what was described in section 3.3, imposing AlAdS boundary conditions imply that

$$\hat{g}_{\hat{\mu}\hat{\nu}}\mathrm{d}\hat{x}^{\hat{\mu}}\mathrm{d}\hat{x}^{\hat{\nu}} = \frac{\mathrm{d}r^2}{r^2} + \hat{h}_{\hat{a}\hat{b}}\mathrm{d}\hat{x}^{\hat{a}}\mathrm{d}\hat{x}^{\hat{b}}\,, \tag{5.6}$$

$$\hat{h}_{\hat{a}\hat{b}} = \frac{1}{r^2}\left[\hat{h}_{(0)\hat{a}\hat{b}} + r^2\hat{h}_{(2)\hat{a}\hat{b}} + r^4\log r\,\hat{h}_{(4,1)\hat{a}\hat{b}} + r^4\hat{h}_{(4)\hat{a}\hat{b}} + \mathcal{O}\left(r^6\log r\right)\right]\,, \tag{5.7}$$

$$\hat{\phi} = \hat{\phi}_{(0)} + r^2\hat{\phi}_{(2)} + r^4\log r\,\hat{\phi}_{(4,1)} + r^4\hat{\phi}_{(4)} + \mathcal{O}\left(r^6\log r\right)\,, \tag{5.8}$$

$$\hat{\chi} = \hat{\chi}_{(0)} + r^2\hat{\chi}_{(2)} + r^4\log r\,\hat{\chi}_{(4,1)} + r^4\hat{\chi}_{(4)} + \mathcal{O}\left(r^6\log r\right)\,. \tag{5.9}$$

From the discussion of the asymptotic behaviour of scalar fields in section 1.2.3 it is clear that the scalars will be dual to dimension 4 operators. As in the case of pure gravity, the coefficients of the asymptotic expansions can be determined as local functions of the boundary values. At second order the coefficients are

$$\begin{aligned}\hat{h}_{(2)\hat{a}\hat{b}} = &-\frac{1}{2}\left(\hat{R}_{(0)\hat{a}\hat{b}} - \frac{1}{2}\partial_{\hat{a}}\hat{\phi}_{(0)}\partial_{\hat{b}}\hat{\phi}_{(0)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\partial_{\hat{a}}\hat{\chi}_{(0)}\partial_{\hat{b}}\hat{\chi}_{(0)}\right)\\ &+ \frac{1}{12}\hat{h}_{(0)\hat{a}\hat{b}}\left(\hat{R}_{(0)} - \frac{1}{2}(\partial\hat{\phi}_{(0)})^2 - \frac{1}{2}e^{2\hat{\phi}_{(0)}}(\partial\hat{\chi}_{(0)})^2\right)\,,\end{aligned} \tag{5.10}$$

$$\hat{\phi}_{(2)} = \frac{1}{4}\left(\hat{\Box}^{(0)}\hat{\phi}_{(0)} - e^{2\hat{\phi}_{(0)}}\left(\partial\hat{\chi}_{(0)}\right)^2\right)\,, \tag{5.11}$$

$$\hat{\chi}_{(2)} = \frac{1}{4}\left(\hat{\Box}^{(0)}\hat{\chi}_{(0)} + 2\partial_{\hat{a}}\hat{\phi}_{(0)}\partial^{\hat{a}}\hat{\chi}_{(0)}\right)\,, \tag{5.12}$$

while at order $r^4 \log r$ they are

$$
\begin{aligned}
\hat{h}_{(4,1)\hat{a}\hat{b}} = {} & \hat{h}_{(2)\hat{a}\hat{c}}\hat{h}^{\hat{c}}_{(2)\hat{b}} + \frac{1}{4}\hat{\nabla}^{(0)\hat{c}}\left(\hat{\nabla}^{(0)}_{\hat{a}}\hat{h}_{(2)\hat{b}\hat{c}} + \hat{\nabla}^{(0)}_{\hat{b}}\hat{h}_{(2)\hat{a}\hat{c}} - \hat{\nabla}^{(0)}_{\hat{c}}\hat{h}_{(2)\hat{a}\hat{b}}\right) - \frac{1}{4}\hat{\nabla}^{(0)}_{\hat{a}}\hat{\nabla}^{(0)}_{\hat{b}}\hat{h}^{\hat{c}}_{(2)\hat{c}} \\
& - \frac{1}{2}\partial_{(\hat{a}}\hat{\phi}_{(0)}\hat{\nabla}^{(0)}_{\hat{b})}\hat{\phi}_{(2)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\partial_{(\hat{a}}\hat{\chi}_{(0)}\hat{\nabla}^{(0)}_{\hat{b})}\hat{\chi}_{(2)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\hat{\phi}_{(2)}\partial_{\hat{a}}\hat{\chi}_{(0)}\partial_{\hat{b}}\hat{\chi}_{(0)} \\
& - \hat{h}_{(0)\hat{a}\hat{b}}\left(\frac{1}{4}\hat{h}^{\hat{c}\hat{d}}_{(2)}\hat{h}_{(2)\hat{c}\hat{d}} + \frac{1}{2}\hat{\phi}^2_{(2)} + \frac{1}{2}e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)}\right), \qquad (5.13)\\
\hat{\phi}_{(4,1)} = {} & -\frac{1}{4}\Big[\hat{\Box}^{(0)}\hat{\phi}_{(2)} + 2\hat{\phi}_{(2)}\hat{h}^{\hat{a}}_{(2)\hat{a}} - 4e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)} + \frac{1}{2}\partial^{\hat{a}}\hat{\phi}_{(0)}\hat{\nabla}^{(0)}_{\hat{a}}\hat{h}^{\hat{b}}_{(2)\hat{b}} - \hat{h}^{\hat{a}\hat{b}}_{(2)}\hat{\nabla}^{(0)}_{\hat{a}}\partial_{\hat{b}}\hat{\phi}_{(0)} \\
& -\partial^{\hat{a}}\hat{\phi}_{(0)}\hat{\nabla}^{(0)\hat{b}}\hat{h}_{(2)\hat{a}\hat{b}} + e^{2\hat{\phi}_{(0)}}\partial_{\hat{a}}\hat{\chi}_{(0)}\left(\partial_{\hat{b}}\hat{\chi}_{(0)}\hat{h}^{\hat{a}\hat{b}}_{(2)} - 2\hat{\phi}_{(2)}\partial^{\hat{a}}\hat{\chi}_{(0)} - 2\hat{\nabla}^{(0)\hat{a}}\hat{\chi}_{(2)}\right)\Big], \\
& \qquad (5.14)\\
\hat{\chi}_{(4,1)} = {} & -\frac{1}{4}\Big[8\hat{\chi}_{(2)}\hat{\phi}_{(2)} + 2\hat{\chi}_{(2)}\hat{h}^{\hat{a}}_{(2)\hat{a}} + \hat{\Box}^{(0)}\hat{\chi}_{(2)} - \hat{h}^{\hat{a}\hat{b}}_{(2)}\hat{\nabla}^{(0)}_{\hat{a}}\partial_{\hat{b}}\hat{\chi}_{(0)} + 2\hat{\nabla}^{(0)}_{\hat{a}}\hat{\chi}_{(2)}\partial^{\hat{a}}\hat{\phi}_{(0)} \\
& +\partial^{\hat{a}}\hat{\chi}_{(0)}\left(\frac{1}{2}\hat{\nabla}^{(0)}_{\hat{a}}\hat{h}^{\hat{b}}_{(2)\hat{b}} - \hat{\nabla}^{(0)\hat{b}}\hat{h}_{(2)\hat{a}\hat{b}} - 2\partial^{\hat{b}}\hat{\phi}_{(0)}\hat{h}_{(2)\hat{a}\hat{b}} + 2\hat{\nabla}^{(0)}_{\hat{a}}\hat{\phi}_{(2)}\right)\Big]. \qquad (5.15)
\end{aligned}
$$

The metric coefficient at order $r^4$ is constrained by

$$
\begin{aligned}
\hat{h}^{\hat{a}}_{(4)\hat{a}} = {} & \frac{1}{4}\hat{h}_{(2)\hat{a}\hat{b}}\hat{h}^{\hat{a}\hat{b}}_{(2)} - \frac{1}{2}\phi^2_{(2)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)}, \qquad (5.16)\\
\hat{\nabla}^{(0)\hat{b}}\hat{h}_{(4)\hat{a}\hat{b}} = {} & -e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)}\partial_{\hat{a}}\hat{\phi}_{(0)} + \hat{\phi}_{(4)}\partial_{\hat{a}}\hat{\phi}_{(0)} + e^{2\hat{\phi}_{(0)}}\hat{\chi}_{(4)}\partial_{\hat{a}}\hat{\chi}_{(0)} + e^{2\hat{\phi}_{(0)}}\hat{\phi}_{(2)}\hat{\chi}_{(2)}\partial_{\hat{a}}\hat{\chi}_{(0)} \\
& - \frac{1}{2}\hat{\phi}_{(2)}\hat{\nabla}^{(0)}_{\hat{a}}\hat{\phi}_{(2)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\hat{\chi}_{(2)}\hat{\nabla}^{(0)}_{\hat{a}}\hat{\chi}_{(2)} - \frac{1}{4}\hat{h}^{\hat{b}\hat{c}}_{(2)}\hat{\nabla}^{(0)}_{\hat{a}}\hat{h}_{(2)\hat{b}\hat{c}} \\
& - \frac{1}{4}\hat{h}_{(2)\hat{a}\hat{c}}\hat{\nabla}^{(0)\hat{c}}\hat{h}^{\hat{b}}_{(2)\hat{b}} + \frac{1}{2}\hat{h}^{\hat{b}\hat{c}}_{(2)}\hat{\nabla}^{(0)}_{\hat{b}}\hat{h}_{(2)\hat{a}\hat{c}} + \frac{1}{2}\hat{h}^{\hat{c}}_{(2)\hat{a}}\hat{\nabla}^{(0)\hat{b}}\hat{h}_{(2)\hat{b}\hat{c}}. \qquad (5.17)
\end{aligned}
$$

The counterterm action for this theory was determined in [91], although it can also be inferred from the one given in section 3.3. However, while the general algorithmic procedure applied in section 3.3 works, it is very tedious when numerous coupled fields are involved. A cleaner approach is to simply guess the counterterm action by writing down all terms allowed by symmetries while keeping their coefficients arbitrary. Demanding that this cancels the infinities on-shell fixes the unknown coefficients. The counterterm action (which will not be derived here) is given by [91]

$$
S_{\text{ct}} = \int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-\hat{h}}\left[-3 - \frac{1}{4}\hat{Q} + \hat{\mathcal{A}}\left(\lambda + \log r\right)\right]. \qquad (5.18)
$$

Here, as before, $\lambda$ is a scheme dependent parameter with $\lambda = 0$ corresponding to minimal subtraction. Here

$$
\begin{aligned}
& \hat{Q} = \hat{h}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}}, \qquad \hat{Q}_{\hat{a}\hat{b}} = \hat{R}_{(\hat{h})\hat{a}\hat{b}} - \frac{1}{2}\partial_{\hat{a}}\hat{\phi}\partial_{\hat{b}}\hat{\phi} - \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{a}}\hat{\chi}\partial_{\hat{b}}\hat{\chi}, \qquad (5.19)\\
& \hat{\mathcal{A}} = \frac{1}{8}\left(\hat{Q}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}} - \frac{1}{3}\hat{Q}^2 + \frac{1}{2}\left(\Box_{(\hat{h})}\hat{\phi} - e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\right)^2 + \frac{1}{2}e^{2\hat{\phi}}\left(\Box_{(\hat{h})}\hat{\chi} + 2\partial_{\hat{a}}\hat{\phi}\partial^{\hat{a}}\hat{\chi}\right)^2\right). \qquad (5.20)
\end{aligned}
$$

Note that this agrees with the expression (3.48) when the scalar fields vanish. Knowledge of the counterterm action allows us to compute vevs in the dual field theory.

### 5.1.1 Vevs of the AdS$_5$ boundary theory

The vevs of operators dual to fields in the 5d theory will play a very important rôle later on. Hence, a derivation is presented in detail here. The total variation of $S_{\text{ren}} = S + S_{\text{ct}}$ can be written as

$$\begin{aligned} \delta S_{\text{ren}} = & \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-\hat{g}}\left(\hat{\mathcal{E}}_{\hat{\mu}\hat{\nu}}\delta\hat{g}^{\hat{\mu}\hat{\nu}} + \hat{\mathcal{E}}_{\hat{\phi}}\delta\hat{\phi} + \hat{\mathcal{E}}_{\hat{\chi}}\delta\hat{\chi}\right) \\ & - \frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-\hat{h}}\left(\hat{T}_{\hat{a}\hat{b}}\delta\hat{h}^{\hat{a}\hat{b}} + 2\hat{T}_{\hat{\phi}}\delta\hat{\phi} + 2\hat{T}_{\hat{\chi}}\delta\hat{\chi}\right), \end{aligned} \tag{5.21}$$

with the 2 in front of $T_{\hat{\chi}}$ and $T_{\hat{\phi}}$ coming from the fact that stress-energy tensor defined like so is symmetric. Here the $\hat{\mathcal{E}}_{\mu\nu}$, $\hat{\mathcal{E}}_{\phi}$ and $\hat{\mathcal{E}}_{\chi}$ are the equations of motion (5.2)–(5.4) and

$$\hat{T}_{\hat{a}\hat{b}} = (\hat{K}-3)\hat{h}_{\hat{a}\hat{b}} - \hat{K}_{\hat{a}\hat{b}} + \frac{1}{2}\hat{Q}_{\hat{a}\hat{b}} - \frac{1}{4}\hat{h}_{\hat{a}\hat{b}}\hat{Q} + (\lambda + \log r)\hat{T}^{(\hat{A})}_{\hat{a}\hat{b}}, \tag{5.22}$$

$$\hat{T}_{\hat{\phi}} = \frac{1}{2}\hat{n}^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\phi} + \frac{1}{4}\left(\Box_{(\hat{h})}\hat{\phi} - e^{2\hat{\phi}}\left(\partial\hat{\chi}\right)^2\right) + (\lambda + \log r)\hat{T}^{(\hat{A})}_{\hat{\phi}}, \tag{5.23}$$

$$\hat{T}_{\hat{\chi}} = \frac{1}{2}e^{2\hat{\phi}}\hat{n}^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\chi} + \frac{1}{4}e^{2\hat{\phi}}\left(\Box_{(\hat{h})}\hat{\chi} + 2\partial_{\hat{a}}\hat{\chi}\partial^{\hat{a}}\hat{\phi}\right) + (\lambda + \log r)\hat{T}^{(\hat{A})}_{\hat{\chi}}, \tag{5.24}$$

where we defined

$$\hat{T}^{(\hat{A})}_{\hat{a}\hat{b}} = -\frac{2\kappa_5^2}{\sqrt{-\hat{h}}}\frac{\delta\hat{A}}{\delta\hat{h}^{\hat{a}\hat{b}}}, \qquad \hat{T}^{(\hat{A})}_{\hat{\phi}} = -\frac{\kappa_5^2}{\sqrt{-\hat{h}}}\frac{\delta\hat{A}}{\delta\hat{\phi}}, \qquad \hat{T}^{(\hat{A})}_{\hat{\chi}} = -\frac{\kappa_5^2}{\sqrt{-\hat{h}}}\frac{\delta\hat{A}}{\delta\hat{\chi}}, \tag{5.25}$$

and $A$ is the integrated conformal anomaly

$$\hat{A} = \frac{1}{\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-h}\hat{\mathcal{A}}\,. \tag{5.26}$$

We then identify the leading components of the asymptotic expansions. Thus, $\sqrt{-\hat{h}} = r^{-4}\sqrt{-\hat{h}_{(0)}} + \mathcal{O}(r^{-2})$, $\delta\hat{h}^{\hat{a}\hat{b}} = r^2\delta\hat{h}^{\hat{a}\hat{b}}_{(0)} + \mathcal{O}(r^0)$, $\delta\hat{\phi} = \delta\hat{\phi}_{(0)} + \mathcal{O}(r^2)$ and $\delta\hat{\chi} = \delta\hat{\chi}_{(0)} + \mathcal{O}(r^2)$, and the vevs are

$$\langle\hat{T}_{(0)\hat{a}\hat{b}}\rangle = -\frac{2\kappa_5^2}{\sqrt{-\hat{h}_{(0)}}}\frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta\hat{h}^{\hat{a}\hat{b}}_{(0)}} = \lim_{\epsilon\to 0}\epsilon^{-2}\hat{T}_{\hat{a}\hat{b}} = 2\hat{h}_{(4)\hat{a}\hat{b}} - 2\hat{X}_{\hat{a}\hat{b}} = \hat{t}_{\hat{a}\hat{b}}\,, \tag{5.27}$$

$$\begin{aligned} \langle\mathcal{O}_{\hat{\phi}}\rangle &= -\frac{\kappa_5^2}{\sqrt{-\hat{h}_{(0)}}}\frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta\hat{\phi}_{(0)}} = \lim_{\epsilon\to 0}\epsilon^{-4}\hat{T}_{\hat{\phi}} \\ &= -2\hat{\phi}_{(4)} - \frac{1}{2}\hat{\phi}_{(2)}\hat{h}^{\hat{a}}_{(2)\hat{a}} + e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)} - \frac{1}{2}\left(3-4\lambda\right)\hat{\phi}_{(4,1)}\,, \end{aligned} \tag{5.28}$$

$$\begin{aligned} \langle\mathcal{O}_{\hat{\chi}}\rangle &= -\frac{\kappa_5^2}{\sqrt{-\hat{h}_{(0)}}}\frac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta\hat{\chi}_{(0)}} = \lim_{\epsilon\to 0}\epsilon^{-4}\hat{T}_{\hat{\chi}} \\ &= -2e^{2\hat{\phi}_{(0)}}\hat{\chi}_{(4)} - \frac{1}{2}e^{2\hat{\phi}_{(0)}}\left(\hat{\chi}_{(2)}\hat{h}^{\hat{a}}_{(2)\hat{a}} + 4\hat{\chi}_{(2)}\hat{\phi}_{(2)} + (3-4\lambda)\hat{\chi}_{(4,1)}\right), \end{aligned} \tag{5.29}$$

where

$$\hat{X}_{\hat{a}\hat{b}} = \frac{1}{2}\hat{h}_{(2)\hat{a}\hat{c}}\hat{h}^{\hat{c}}_{(2)\hat{b}} - \frac{1}{4}\hat{h}^{\hat{c}}_{(2)\hat{c}}\hat{h}_{(2)\hat{a}\hat{b}} - \frac{1}{4}\hat{h}_{(0)\hat{a}\hat{b}}\hat{\mathcal{A}}_{(0)} - \frac{1}{4}\left(3-4\lambda\right)\hat{h}_{(4,1)\hat{a}\hat{b}}\,, \tag{5.30}$$

and

$$\hat{\mathcal{A}}_{(0)} = \lim_{\epsilon\to 0} \epsilon^{-4}\hat{\mathcal{A}} = \frac{1}{2}\left(\hat{h}^{\hat{a}\hat{b}}_{(2)}\hat{h}_{(2)\hat{a}\hat{b}} - (\hat{h}^{\hat{a}}_{(2)\hat{a}})^2\right) + \hat{\phi}^2_{(2)} + e^{2\hat{\phi}_{(0)}}\hat{\chi}^2_{(2)}\,. \tag{5.31}$$

Choosing $\lambda = \frac{3}{4}$ removes any contribution from the $r^4 \log r$ terms in the Fefferman-Graham expansions.

### 5.1.2 Ward identities for the vevs

The Ward identities can be found following the procedure detailed in section 3.3: Varying the bulk coordinates the action changes according to

$$\delta S^{\text{on-shell}}_{\text{ren}} = -\frac{1}{2\kappa_5^2}\int_{\partial\mathcal{M}} \mathrm{d}^4\hat{x}\sqrt{-\hat{h}_{(0)}}\left(\hat{t}_{\hat{a}\hat{b}}\delta\hat{h}^{\hat{a}\hat{b}}_{(0)} + 2\langle\mathcal{O}_{\hat{\phi}}\rangle\delta\hat{\phi}_{(0)} + 2\langle\mathcal{O}_{\hat{\chi}}\rangle\delta\hat{\chi}_{(0)} - 2\hat{\mathcal{A}}_{(0)}\frac{\delta r}{r}\right)\,, \tag{5.32}$$

with the boundary metric transforming under bulk diffeomorphisms as

$$\delta\hat{h}^{\hat{a}\hat{b}}_{(0)} = -\hat{\nabla}^{\hat{a}}_{(0)}\hat{\xi}^{\hat{b}}_{(0)} - \hat{\nabla}^{\hat{b}}_{(0)}\hat{\xi}^{\hat{a}}_{(0)} + 2\hat{\xi}^{r}_{(0)}\hat{h}^{\hat{a}\hat{b}}_{(0)}\,, \tag{5.33}$$

with $\delta r = \hat{\xi}^r = r\hat{\xi}^r_{(0)}$. The remaining fields transform as scalars:

$$\delta\hat{\phi}_{(0)} = \hat{\xi}^{\hat{a}}_{(0)}\partial_{\hat{a}}\hat{\phi}_{(0)}\,, \tag{5.34}$$

$$\delta\hat{\chi}_{(0)} = \hat{\xi}^{\hat{a}}_{(0)}\partial_{\hat{a}}\hat{\chi}_{(0)}\,, \tag{5.35}$$

such that the Ward identity associated to boundary diffeomorphisms is

$$\hat{\nabla}^{(0)}_{\hat{a}} t^{\hat{a}}{}_{\hat{b}} = -\langle\mathcal{O}_{\hat{\phi}}\rangle\partial_{\hat{b}}\hat{\phi}_{(0)} - \langle\mathcal{O}_{\hat{\chi}}\rangle\partial_{\hat{b}}\hat{\chi}_{(0)}\,. \tag{5.36}$$

Thus, the presence of scalar fields breaks the covariant conservation of the stress energy tensor. As in section 3.3, the Ward identity associated to scale transformations is

$$\hat{t}^{\hat{a}}{}_{\hat{a}} = \hat{\mathcal{A}}_{(0)}\,. \tag{5.37}$$

Both the 5d vevs (5.27)–(5.29) and the 5d Ward identities can be related to 4d vevs and 4d Ward identities through dimensional reduction (as long as the reduction is consistent). The expressions given above will therefore play crucial rôles in the calculation of the Lifshitz boundary vevs. Note, however, that the 4d boundary stress-energy tensor is not simply given by dimensionally reducing (5.27), since, as explained in section 3.1.2, the introduction of non-scalar boundary fields will alter the form of the stress-energy tensor. The correct 4d stress-energy tensor will therefore be a linear combination of the components of the 5d stress-energy tensor, the specific form of which will be given below.

## 5.2 Obtaining $z = 2$ Lifshitz$_4$ from AdS$_5$

In this section we will present a detailed description of the Scherk-Schwarz reduction of the axion-dilaton theory described above. This will enable us to define AlLif from a 5d perspective and to discuss various deformations of AlLif.

Since the action (5.1) is invariant under translations of the field $\hat{\chi}$ a Scherk-Schwarz reduction over a circle can be performed. This will introduce a massive vector field which is required to support Lifshitz spacetimes. Note that the ansatz below is slightly different from the one given in eq. (4.5), in particular, the constraint that the kinetic term for $\Phi$ is canonically normalized was dropped. This ansatz ensures that we remain in Einstein frame in 4 dimensions. Thus, the 5-dimensional coordinates are split as $(x^\mu, u)$ and $u$ is periodically identified, $u \sim u + 2\pi L$, where $L$ is the radius of the compactifying circle, taken to be small. The reduction ansatz is

$$\hat{g}_{\hat{\mu}\hat{\nu}}\mathrm{d}\hat{x}^{\hat{\mu}}\mathrm{d}\hat{x}^{\hat{\nu}} = e^{-\Phi}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2\Phi}\left(\mathrm{d}u + A_\mu\mathrm{d}x^\mu\right)^2 \tag{5.38}$$

$$= \frac{\mathrm{d}r^2}{r^2} + e^{-\Phi}h_{ab}\mathrm{d}x^a\mathrm{d}x^b + e^{2\Phi}\left(\mathrm{d}u + A_a\mathrm{d}x^a\right)^2 , \tag{5.39}$$

$$\hat{\chi} = \chi + ku \,, \tag{5.40}$$

$$\hat{\phi} = \phi \,, \tag{5.41}$$

where all unhatted fields are now independent of the compact coordinate $u$. In performing the reduction, the following relations between the 5d and 4d quantities are useful:

$$\sqrt{-\hat{g}} = e^{-\Phi}\sqrt{-g_E} \,, \tag{5.42}$$

$$\sqrt{-\hat{h}} = e^{-\Phi/2}\sqrt{-h_E} \,, \tag{5.43}$$

$$\hat{n}^\mu = e^{\Phi/2}n_E^\mu \,, \tag{5.44}$$

$$\hat{R} = e^{\Phi}\left(R_E + \Box_E\Phi - \frac{3}{2}\left(\partial\Phi\right)_E^2 - \frac{1}{4}e^{3\Phi}F_E^2\right) , \tag{5.45}$$

$$\hat{K} = e^{\Phi/2}\left(K_E - \frac{1}{2}n_E^\mu\partial_\mu\Phi\right) , \tag{5.46}$$

with $E$ signifying the fact that the objects are in Einstein frame (or contracted with an Einstein frame metric). Note that the boundary term arising in the extrinsic curvature term cancels exactly the boundary term from the Ricci scalar due to the overall factor of 2 in front of $\hat{K}$ in the action. From now on everything will be in Einstein frame and the subscripts will be dropped. The action in 4 dimensions is

$$S = \frac{2\pi L}{2\kappa_5^2}\int d^4x\sqrt{-g}\left[R - \frac{3}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}e^{3\Phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}D_\mu\chi D^\mu\chi - V\right] + \frac{2\pi L}{\kappa_5^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K + S_{\text{ct}} \,, \tag{5.47}$$

$$S_{\text{ct}} = \frac{2\pi L}{\kappa_5^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Big[ - 3e^{-\Phi/2} - \frac{1}{4}e^{\Phi/2}\Big(R_{(h)} - \frac{3}{2}\partial_a\Phi\partial^a\Phi - \frac{1}{4}e^{3\Phi}F_{ab}F^{ab} - \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}e^{2\phi}D_a\chi D^a\chi - \frac{k^2}{2}e^{2\phi-3\Phi}\Big)\Big] + \log r\frac{2\pi L}{\kappa_5^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}e^{-\Phi/2}\mathcal{A} \,, \tag{5.48}$$

with

$$D_\mu\chi = \partial_\mu\chi - kA_\mu\,, \tag{5.49}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{5.50}$$
$$V = \frac{k^2}{2}e^{-3\Phi+2\phi} - 12e^{-\Phi}\,, \tag{5.51}$$

and $k \neq 0$. $A_\mu$ is a massless vector field. We reserve the notation $B_\mu$ for the massive one, with

$$B_\mu = A_\mu - \frac{1}{k}\partial_\mu\chi. \tag{5.52}$$

A thorough analysis of the 4-dimensional anomaly is postponed to chapter 6. In doing Lifshitz holography, it turns out to be convenient to stick to the massless vector field using $A_\mu$ and $\chi$ as independent fields, and we shall do so from now on (with a few exceptions). The equations of motion are

$$\begin{aligned}\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + \frac{1}{8}e^{3\Phi}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{2}e^{3\Phi}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4}e^{2\phi}g_{\mu\nu}D_\rho\chi D^\rho\chi - \frac{1}{2}e^{2\phi}D_\mu\chi D_\nu\chi \\ &\quad + \frac{3}{4}g_{\mu\nu}\partial_\rho\Phi\partial^\rho\Phi - \frac{3}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{4}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}V\,, \end{aligned}\tag{5.53}$$
$$\mathcal{E}_\Phi = 3\Box\Phi - \frac{3}{4}e^{3\Phi}F_{\mu\nu}F^{\mu\nu} + \frac{3}{2}k^2e^{-3\Phi+2\phi} - 12e^{-\Phi}\,, \tag{5.54}$$
$$\mathcal{E}_\phi = \Box\phi - e^{2\phi}D_\mu\chi D^\mu\chi - k^2e^{-3\Phi+2\phi}\,, \tag{5.55}$$
$$\mathcal{E}^\nu = \nabla_\mu\left(e^{3\Phi}F^{\mu\nu}\right) + ke^{2\phi}D^\nu\chi\,, \tag{5.56}$$

and a solution is given by a pure $z = 2$ Lifshitz spacetime [61]:

$$\mathrm{d}s^2 = e^{\Phi_{(0)}}\left(\frac{\mathrm{d}r^2}{r^2} - e^{-2\Phi_{(0)}}\frac{\mathrm{d}t^2}{r^4} + \frac{1}{r^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\right)\,, \tag{5.57}$$
$$A = e^{-2\Phi_{(0)}}\frac{\mathrm{d}t}{r^2}\,, \tag{5.58}$$
$$\Phi = \Phi_{(0)} = \phi_{(0)} + \log\frac{k}{2}\,, \tag{5.59}$$
$$\phi = \phi_{(0)} = \mathrm{cst}\,. \tag{5.60}$$

From a 5d perspective this is a $z = 0$ Schrödinger spacetime [61]:

$$\mathrm{d}\hat{s}^2 = \frac{\mathrm{d}r^2}{r^2} + \frac{1}{r^2}\left(2\mathrm{d}t\mathrm{d}u + \mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{k^2}{4}g_s^2\mathrm{d}u^2\,, \tag{5.61}$$
$$\hat{\phi} = \hat{\phi}_{(0)} = \phi_{(0)} = \log g_s = \mathrm{cst}\,, \tag{5.62}$$
$$\hat{\chi} = ku + \mathrm{cst}\,. \tag{5.63}$$

This fact will be important momentarily when we discuss how to deform the geometry to obtain an AlLif spacetime. The Fefferman-Graham expansions (5.7)–(5.9) of the 5d fields are related to an expansion of the 4d fields through a map induced by the dimensional reduction ansatz. From (5.39) we infer that the 4d fields can be written as functions of the 5d fields:

$$h_{ab} = e^{\Phi}\left(\hat{h}_{ab} - e^{-2\Phi}\hat{h}_{au}\hat{h}_{ub}\right) , \tag{5.64}$$

$$A_a = e^{-2\Phi}\hat{h}_{au} , \tag{5.65}$$

$$\Phi = \frac{1}{2}\log \hat{h}_{uu} . \tag{5.66}$$

It should be noted that these expressions are not in radial gauge.

Before we proceed we should note that the reduction is consistent since we are reducing over a circle [82].

### 5.2.1 Asymptotically locally Lifshitz spacetimes from AdS$_5$

The dimensional reduction of AdS$_5$ leading to a $z = 2$ Lifshitz spacetime allows for a definition of AlLif as seen from a 5d perspective. A natural choice is to define an AlLif spacetime as a solution to the equations of motion (5.53)–(5.56) whose 5d uplift is AlAdS. Hence, the 5d uplift should satisfy the following properties [61]:

$$\hat{\phi}_{(0)} = \text{cst} , \tag{5.67}$$

$$\hat{h}_{(0)\hat{a}\hat{b}} \text{ must admit a hypersurface orthogonal null Killing vector } \partial_u . \tag{5.68}$$

We will take these conditions as implying that the reduced spacetime is AlLif. Let us consider the motivations leading to this definition.

In the previous section it was seen that the 4d pure Lifshitz spacetime is a $z = 0$ Schrödinger spacetime in 5d. This allows for a study of the pure Lifshitz case from a 5d perspective using an arbitrary Fefferman-Graham coordinate system. For the 5d Schrödinger spacetime to correctly reduce to a pure $z = 2$ Lifshitz spacetime the following conditions on the fields must be satisfied:

$$\hat{\phi}_{(0)} = \text{cst} , \tag{5.69}$$

$$\hat{\phi}_{(4)} = 0 , \tag{5.70}$$

$$\hat{\chi}_{(0)} = ku + \text{cst} , \tag{5.71}$$

$$\hat{\chi}_{(4)} = 0 , \tag{5.72}$$

$$\hat{h}_{(0)\hat{a}\hat{b}} \text{ is conformally flat and admits a hypersurface orthogonal NKV } \partial_u , \tag{5.73}$$

$$\hat{t}_{\hat{a}\hat{b}} = 0 . \tag{5.74}$$

The demands on the leading components follow straightforwardly from the $z = 0$ Schrödinger solution, as does the demands on the subleading terms of the Fefferman-Graham expansion. The conformal flatness is a consequence of the fact that the solution (5.61) is AAdS and hence the boundary metric is conformally flat. The remaining constraints on $\hat{h}_{(0)\hat{a}\hat{b}}$ can be understood as follows. From the reduction ansatz (5.39) we find that

$$e^{2\Phi} = \hat{h}_{uu} , \tag{5.75}$$

and if $\Phi$ is to be a constant it is necessary that

$$\hat{h}_{(0)uu} = 0 , \tag{5.76}$$

otherwise it would depend on $r$. We will always impose (5.76). In order to do the reduction in the first place, it is necessary that $\partial_u$ is a Killing vector of the 5-dimensional metric and since $u$ is a boundary coordinate, $\partial_u$ will be a null Killing vector (null because $\hat{h}_{(0)uu} = 0$). Additionally, the boundary value of $\Phi$ should be fixed by (5.59) which means that

$$e^{2\Phi_0} = \frac{k^2}{4} e^{2\phi_{(0)}} = \hat{h}_{(2)uu} \,. \tag{5.77}$$

Due to the asymptotic solution to the equations of motion (5.10) this is only possible if

$$\hat{R}_{(0)uu} = 0 \,. \tag{5.78}$$

$\partial_u$ is then hypersurface orthogonal, since it is a null Killing vector and thus tangent to a null geodesic congruence. The full proof of this statement is given in [61].

The deformation to an AlLif spacetime can proceed as follows. For the pure Lifshitz case the value of $\Phi - \phi$ is not a free parameter, but equivalent to $\log \frac{k}{2}$. In the AlLif case this should still hold. Asymptotically this means that

$$\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2} \,. \tag{5.79}$$

This again implies that $\hat{h}_{(0)uu}$ is zero, such that $\Phi$ and $\phi$ start at the same order, and as the boundary value of $\Phi$ shold be determined from (5.79), we must again require that $\hat{R}_{(0)uu}$ vanishes. In addition, $\partial_u$ should of course still be a Killing vector of the boundary metric. These were the conditions leading to the fact that $\hat{h}_{(0)\hat{a}\hat{b}}$ should admit a hypersurface orthogonal null Killing vector. Later we will show that the condition $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ is equivalent to hypersurface orthogonality of $\partial_u$, and one should therefore keep in mind that having $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ puts a constraint on the metric. Thus, the only property of the metric not implied by the requirement (5.79) is the conformal flatness. Note that we did not impose any constraints on the subleading terms in the Fefferman-Graham expansion. The condition that a spacetime be AlLif is a statement only about the asymptotic structure of said spacetime.

To show that $\phi_{(0)}$ must be constant in AlLif spacetimes, we write the 4d Einstein frame metric in radial gauge, i.e. as

$$\mathrm{d}s^2 = e^{\Phi} \frac{\mathrm{d}r^2}{r^2} + h_{ab}\mathrm{d}x^a\mathrm{d}x^b = l_{\mathrm{Lif}}^2 \left( \frac{\mathrm{d}r'^2}{r'^2} + h'_{ab}\mathrm{d}x'^a\mathrm{d}x'^b \right) , \tag{5.80}$$

with $l_{\mathrm{Lif}}^2$ the Lifshitz radius. We then perform a coordinate transformation using

$$r = r' - \xi^r , \qquad x^a = x'^a - \xi^a . \tag{5.81}$$

Hence, we obtain the following equations for the generators

$$l_{\mathrm{Lif}}^2 = e^{\Phi} \left( 1 - \xi^\mu \partial_\mu \Phi + 2r^{-1}\xi^r - 2\partial_r \xi^r \right) , \tag{5.82}$$

$$0 = \partial_r \xi^a + r^{-2} e^{\Phi} h^{ab} \partial_b \xi^r , \tag{5.83}$$

$$l_{\mathrm{Lif}}^2 h'_{ab} = h_{ab} - \xi^\mu \partial_\mu h_{ab} - h_{cb}\partial_a \xi^c - h_{ac}\partial_b \xi^c \,. \tag{5.84}$$

The most general solution is given by

$$\xi^r = r\left[\log r \xi^r_{(0,1)} + \xi^r_{(0)} + \mathcal{O}(r^2 \log^2 r)\right] , \quad (5.85)$$

$$\xi^a = \xi^a_{(0)} + \mathcal{O}(r^2 \log^2 r) , \quad (5.86)$$

with

$$\xi^r_{(0,1)} = \frac{1}{2}\left(1 - l^2_{\mathrm{Lif}} e^{-\Phi_{(0)}} - \xi^a_{(0)} \partial_a \Phi_{(0)}\right) . \quad (5.87)$$

Assuming $\Phi_{(0)}$ is infinitesimal with $\Phi_{(0)} = 2\log l_{\mathrm{Lif}} + \delta\Phi_{(0)}$, this becomes

$$\xi^r_{(0,1)} = -\xi^a_{(0)} \partial_a \Phi_{(0)} , \quad (5.88)$$

thus reproducing the expression given in [61]. Plugging this into equation (5.84) we find (now dropping the primes)

$$h_{ab} = \frac{1}{r^4}\left[\log r h_{(0,1)ab} + h_{(0)ab} + \mathcal{O}(r^2 \log^2 r)\right] , \quad (5.89)$$

with

$$h_{(0,1)ab} = 4\xi^r_{(0,1)} \gamma_{(0)ab} , \quad (5.90)$$

$$h_{(0)ab} = \gamma_{(0)ab} + 4\gamma_{(0)ab}\xi^r_{(0)} - \gamma_{(0)bc}\partial_a \xi^c_{(0)} - \gamma_{(0)ac}\partial_b \xi^c_{(0)} - \xi^c_{(0)}\partial_c \gamma_{(0)ab} . \quad (5.91)$$

Here $\gamma_{(0)ab}$ is the leading component in the expansion of the boundary metric in the original gauge. In order to keep the $z = 2$ Lifshitz behaviour of the metric in radial gauge and not have the $r^{-4}\log r$-term which violates this, we must impose $\Phi_{(0)} = \mathrm{cst}$. Since the leading behaviour of the two dilaton fields, $\Phi$ and $\phi$, are related by $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ in AlLif spacetimes, having $\Phi_{(0)} = \mathrm{cst}$ implies having $\phi_{(0)} = \mathrm{cst}$. Hence, we make the condition that $\phi_{(0)} = \mathrm{cst}$ part of the definition of an AlLif spacetime from a 5d point of view.

The conditions for the 5d solution to reduce to an AlLif spacetime are therefore those given in (5.67) and (5.68). A 4d AlLif spacetime thus uplifts to a 5d AlAdS spacetime. Obtaining an AlLif spacetime entails fixing the boundary value of both dilaton fields, thus preventing us from calculating their associated vevs. Therefore we need to consider deformations of this spacetime where these constraints are relaxed.

We still need to make contact with the vielbein-based definition of AlLif given in [74] and reviewed in section 2.4. From the reduction ansatz (5.39) we can write the metric as

$$ds^2 = e^{\Phi}\frac{dr^2}{r^2} + h_{ab}dx^a dx^b , \quad (5.92)$$

where $r$ is the 5-dimensional radial gauge coordinate and

$$h_{ab} = \left(\hat{h}_{uu}\right)^{1/2}\left(\hat{h}_{ab} - \frac{\hat{h}_{au}\hat{h}_{ub}}{\hat{h}_{uu}}\right) . \quad (5.93)$$

Since we impose (5.76) the second term will start at order $r^{-4}$, and the frame fields given by

$$h_{ab}dx^a dx^b = -e^{\underline{t}}e^{\underline{t}} + \delta_{\underline{ij}}e^{\underline{i}}e^{\underline{j}} , \quad (5.94)$$

will have the expansions

$$e^{\underline{t}} = r^{-2}\tau_{(0)a}\mathrm{d}x^a + \cdots\,, \tag{5.95}$$

$$e^{\underline{i}} = r^{-1}e^{\underline{i}}_{(0)a}\mathrm{d}x^a + \cdots\,. \tag{5.96}$$

The $uu$-component of the Ricci tensor in these coordinates is given by

$$\hat{R}_{(0)uu} = \frac{1}{2}e^{3\Phi_{(0)}}\left(\varepsilon^{abc}_{(0)}\tau_{(0)a}\partial_b\tau_{(0)c}\right)^2\,, \tag{5.97}$$

where

$$\varepsilon^{abc}_{(0)} = \epsilon^{\underline{abc}}e^{a}_{(0)\underline{a}}e^{b}_{(0)\underline{b}}e^{c}_{(0)\underline{c}}\,, \tag{5.98}$$

with $\epsilon^{\underline{tij}} = -\epsilon^{\underline{ij}}$. Recall that to have $\Phi_{(0)} - \phi_{(0)} = \log\frac{k}{2}$ we had to take $\hat{R}_{(0)uu} = 0$. By (5.97) this is equivalent to taking $\varepsilon^{abc}_{(0)}\tau_{(0)a}\partial_b\tau_{(0)c} = 0$, meaning that $\tau_{(0)a}$ is hypersurface orthogonal. This is in agreement with the statement made in section 2.4 about what was there called $e^{\underline{t}}_{(0)}$. Furthermore, for hypersurface orthogonal $\tau_{(0)a}$, we can always choose coordinates such that $\tau_{(0)i} = 0$. These are given by $\tau_{(0)a} = \tau_{(0)t}\partial_a t$ and results in an ADM decomposition of the spacetime in which surfaces of constant $t$ describe absolute simultaneity. By considering

$$\hat{h}_{(0)\hat{a}\hat{b}}\left(\partial_u\right)^{\hat{b}} = \delta^a_{\hat{a}}\hat{h}_{(0)au} \sim \delta^a_{\hat{a}}\tau_{(0)a}\,, \tag{5.99}$$

it is apparent that hypersurface orthogonality of $\tau_{(0)a}$ implies hypersurface orthogonality of $\partial_u$ and vice versa. Here the last step follows from the relation (6.59) given in the next chapter.

As was demonstrated above, the constraint that $\phi_{(0)} = \mathrm{cst}$ allows us to write the metric in a radial gauge form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = l^2_{\mathrm{Lif}}\left(\frac{\mathrm{d}r^2}{r^2} - \tilde{e}^{\underline{t}}\tilde{e}^{\underline{t}} + \delta_{\underline{ij}}\tilde{e}^{\underline{i}}\tilde{e}^{\underline{j}}\right)\,, \tag{5.100}$$

where $l^2_{\mathrm{Lif}} = e^{\Phi_{(0)}}$ and

$$\tilde{e}^{\underline{t}} = r^{-2}\tau_{(0)t}\mathrm{d}t + \cdots\,, \tag{5.101}$$

$$\tilde{e}^{\underline{i}} = r^{-1}e^{\underline{i}}_{(0)t}\mathrm{d}t + r^{-1}e^{\underline{i}}_{(0)i}\mathrm{d}x^i + \cdots\,. \tag{5.102}$$

Hence, the condition that $\Phi_{(0)} - \phi_{(0)} = \log\frac{k}{2}$ with $\phi_{(0)} = \mathrm{cst}$ implies the vielbein based definition by Ross [74].

The class of Asymptotically Lifshitz (ALif) spacetimes introduced in section 2.4 had $\tau_{(0)t} = \mathrm{cst}$ (up to a possible time dependence). This implies that $\tau_{(0)a} = \partial_a t$, since we can always absorb an overall constant in a redefinition of the time coordinate. In addition to hypersurface orthogonality, this also satisfies the much stronger constraint

$$\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0\,. \tag{5.103}$$

It will be shown in the next chapter that this condition gives rise to the boundary geometry being Newton-Cartan.

Let us recapitulate the findings of this section. As a definition of AlLif spaces we postulated (5.67)–(5.68). It was shown that imposing $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ is equivalent to (5.68). The fact that transforming to radial gauge introduced $r^{-4} \log r$-terms in the metric expansion motivated us to take $\phi_{(0)} = \text{cst}$. Furthermore, we had $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ which implied that $\hat{R}_{(0)uu} = 0$, resulting in $\tau_{(0)a}$ being hypersurface orthogonal. This fact, combined with $\phi_{(0)}$ being constant, allowed us to write down the frame field expansions (5.101) and (5.102), reproducing the original definition of AlLif given in section 2.4. This shows that a minimal definition of AlLif is to take $\Phi_{(0)} - \phi_{(0)} = \log \frac{k}{2}$ with $\phi_{(0)} = \text{cst}$, and this is what we will be referring to as an AlLif spacetime in the rest of this thesis. Furthermore, we saw that there is a natural definition of ALif spacetimes which results in the boundary geometry being Newton-Cartan.

Having a definition of AlLif should enable us to calculate vevs of operators in the dual theory. However, as was explained above, the definition of an AlLif spacetime requires setting some of the sources to zero. From the 5d perspective this is seen as the fixing of the sources of the scalar fields, $\phi_{(0)}$ and $\Phi_{(0)}$. The hypersurface orthogonality of $\tau_{(0)a}$ allowed us to choose coordinates in which $\tau_{(0)i} = 0$, thus making this source term constrained as well. Calculating the vevs therefore requires deforming the AlLif case, and only after the vevs have been calculated, specifying to the AlLif case. A discussion of such deformations will occupy us in the next subsection. Furthermore, frame fields should be used to calculate the stress-energy tensor complex. Not only do frame fields arise naturally in the definition of an AlLif space, the existence of a vector field indicates the use of frame fields, as argued by Hollands, Ishibashi and Marolf in [75], see section 3.1.2.

### 5.2.2 Deformations of AlLif

To be able to calculate expectation values of operators in the dual field theory, the sources should be kept arbitrary in the gravity theory and the constraints (5.67)–(5.68) should be relaxed. Recall that having

$$\hat{h}_{(2)uu} = \frac{k^2}{4} e^{2\phi_{(0)}} \tag{5.104}$$

required $\hat{R}_{(0)uu} = 0$. This led to $\partial_u$ being hypersurface orthogonal. If we drop this constraint and only impose (5.76) we find from the expansion of

$$e^{2\Phi} = \hat{h}_{uu} \tag{5.105}$$

that

$$e^{2\Phi_{(0)}} = -\frac{1}{2}\hat{R}_{(0)uu} + \frac{k^2}{4} e^{2\phi_{(0)}} \,. \tag{5.106}$$

The boundary fields $\phi_{(0)}$ and $\Phi_{(0)}$ can therefore not fluctuate arbitrarily[1]. Above we saw that when the only constraint imposed on the reduction ansatz (5.39) is (5.76), the frame fields can be written as (5.95) and (5.96). Recall that the source term $\tau_{(0)i}$ appearing in

[1] We will remark on this constraint again later and show that in the case of AlLif it will not cause contributions to the boundary stress-energy tensor.

these expressions could be turned off if $\tau_{(0)a}$ is hypersurface orthogonal. As we saw above, hypersurface orthogonality of $\tau_{(0)a}$ is equivalent to the $uu$-component of the Ricci tensor being zero. Hence, in the general case considered here, the source term $\tau_{(0)i}$ cannot be turned off since $\tau_{(0)a}$ is not hypersurface orthogonal.

Relaxing the fact that $\phi_{(0)} = \text{cst}$ leads to logarithmic violations of the characteristic $r^{-4}$ behaviour of Lifshitz spacetimes. This was demonstrated above. However, this will not affect whether $\tau_{(0)a}$ is hypersurface orthogonal.

This motivates defining three types of asymptotic structures depending on how many of the above conditions are satisfied. The first condition is

$$\text{(I)} \qquad \tau_{(0)[a}\partial_b\tau_{(0)c]} = 0\,, \tag{5.107}$$

which follows from demanding that $\Phi_{(0)} - \phi_{(0)} = \log\frac{k}{2}$. In this case, as we will see in chapter 6, the boundary geometry will be torsional Newton-Cartan, with torsion proportional to $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a}$. The second condition is

$$\text{(II)} \qquad \Phi_{(0)} = \text{cst}\,, \tag{5.108}$$

which was imposed as to not introduce logarithmic violations of the $r^{-4}$ behaviour of Lifshitz spacetimes. We will refer to spacetimes which satisfies I, but not II, as generalized AlLif spacetimes. The fact that the boundary geometry of spacetimes satisfying I is Newton-Cartan with torsion motivates defining a third condition, namely the vanishing of this torsion. Hence,

$$\text{(III)} \qquad \partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0\,. \tag{5.109}$$

As expected, this will give rise to Newton-Cartan without torsion, as we will see in chapter 6. We summarize the various asymptotic structures in table 5.1. The Lifshitz UV spacetime is the most general spacetime we will consider and only satisfies the boundary condition (5.76). The last column in the table indicates the type of boundary geometry arising from imposing the various boundary conditions. From the table it is clear that the motivation for defining ALif as above is that the boundary geometry becomes pure Newton-Cartan. This is contrary to the case of AlLif where the boundary geometry is twistless torsional Newton-Cartan (TTNC). For the case of generalized AlLif (gen. AlLif)

| Asymptotics | I | II | III | Boundary Geometry |
|---|---|---|---|---|
| ALif | Y | Y | Y | NC |
| AlLif | Y | Y | N | TTNC |
| gen. AlLif | Y | N | Y/N | TTNC |
| Lif UV | N | N | – | TNC |

**Table 5.1:** Indicated are the four different boundary conditions discussed in the text depending on whether (Y) or not (N) they satisfy conditions I, II and III, as given in equations (5.107), (5.108) and (5.109), respectively. The last column indicates the type of boundary geometry.

we do not distinguish between whether or not condition III is satisfied. The concepts introduced here will be properly defined in chapter 6.

There are two additional sources which will play a rôle later on and we include them now for completeness. These are the sources for the vector field and the axion. The Kaluza-Klein reduction ansatz for the vector field is

$$A_r = 0\,, \tag{5.110}$$

$$A_a = \frac{\hat{h}_{au}}{\hat{h}_{uu}}\,, \tag{5.111}$$

such that we find the expansion

$$A_a = \frac{1}{r^2} e^{-3\Phi_{(0)}/2} \tau_{(0)a}\,. \tag{5.112}$$

The sources can then be written as

$$A_a - e^{-3\Phi/2} e^t_a = A_{(0)a} + \cdots\,, \tag{5.113}$$

$$\chi = \chi_{(0)} + \cdots\,, \tag{5.114}$$

with $A_{(0)a}$ the boundary gauge field and $\chi_{(0)}$ the boundary axion. The deformations of AlLif are independent of these two sources.

In chapter 6 we will use the most general boundary conditions, what is here called Lifshitz UV. As we saw above these include all irrelevant[2] deformations that take one away from AlLif boundary conditions. The goal will be to study the boundary geometry and compute the vevs and their Ward identities for this most general case. In particular, the focus will be on computing the stress-energy tensor of the boundary theory as defined in [75, 92] and its associated Ward identities. We will then look at what happens for the special case of AlLif boundary conditions.

## 5.3 Comments on the Dual Field Theory

In this section we will briefly consider the approximations involved in performing the dimensional reduction of $AdS_5$ and what effect they have on the dual field theory. A similar discussion appears in [1]. For this we reintroduce the $AdS_5$ length parameter, $l$. The weak version of the AdS/CFT correspondence is motivated in the supergravity approximation, meaning small curvature and weak string coupling,

$$\frac{l}{l_s} \gg 1\,, \qquad g_s \ll 1\,. \tag{5.115}$$

The first condition corresponds to the large 't Hooft coupling, $\lambda^{1/4} = \frac{l}{l_s}$, approximation. The radius of the compactifying circle, $L_{\text{phys}}$, is, in units of the string length, given by

$$\frac{2\pi L_{\text{phys}}}{l_s} = \frac{1}{l_s}\int_0^{2\pi L} \mathrm{d}u \sqrt{g_{uu}} = \frac{1}{l_s}\left(2\pi L\right)\frac{lkg_s}{2}\,, \tag{5.116}$$

[2]This terminology is borrowed from condensed matter physics, such that irrelevant operators refer to operators which leave the IR behaviour of the theory invariant, but deform the UV.

where we used the fact that

$$g_{uu} = e^{2\Phi_{(0)}} = \frac{(lkg_s)^2}{4} \,. \tag{5.117}$$

In order to avoid having light string winding modes we demand

$$\frac{L_{\text{phys}}}{l_s} = \frac{l}{l_s}\frac{Lkg_s}{2} \gg 1 \,. \tag{5.118}$$

Since we want to remain in a regime where supergravity on the background of a circle is a good approximation, we will always assume that (5.115) and (5.118) holds. Hence, the circle is everywhere space-like in the bulk and can be taken to be large in units of the string length, so as to satisfy (5.118). However, the circle is null on the boundary, as can be seen from the conformal compactification of the $z = 0$ Schrödinger metric (2.90), obtained by rescaling it with $r^2$ and setting $r = 0$.

We can relate the size of the compactification circle to the AdS length:

$$\frac{L_{\text{phys}}}{l} = L\frac{kg_s}{2} \,. \tag{5.119}$$

When we have

$$\frac{L_{\text{phys}}}{l} \gg 1 \,, \tag{5.120}$$

neglecting the massive Kaluza-Klein modes is no longer a good approximation and the theory decompactifies. However, we are interested in the opposite regime where

$$\frac{L_{\text{phys}}}{l} \ll 1 \,. \tag{5.121}$$

We still consider $\frac{l}{l_s}$ to be sufficiently large such that (5.115) remains satisfied. We will see that it is in this regime that the boundary theory becomes 3-dimensional. Consider a probe scalar field on the 5-dimensional background (5.61) described by

$$\left(\hat{\Box} - m^2\right)\hat{\varphi} = 0 \,. \tag{5.122}$$

Following the usual Kaluza-Klein procedure we decompose $\hat{\varphi}$ in Fourier modes and write

$$\hat{\varphi} = \sum_n e^{\frac{inu}{L}}\varphi_n \,, \tag{5.123}$$

where each $\varphi_n$ satisfies the equation

$$\left(D_\mu D^\mu - m_{\text{Lif}}^2\right)\varphi_n = 0 \,, \tag{5.124}$$

with

$$D_\mu = \partial_\mu - i\frac{n}{L}A_\mu \,, \tag{5.125}$$

$$m_{\text{Lif}}^2 = e^{-\Phi_{(0)}}\left(m^2 + e^{-2\Phi_{(0)}}\frac{n^2}{L^2}\right) \,. \tag{5.126}$$

Since the Lifshitz radius is

$$l_{\mathrm{Lif}}^2 = l^2 e^{\Phi_{(0)}} , \tag{5.127}$$

we have

$$m_{\mathrm{Lif}}^2 l_{\mathrm{Lif}}^2 = m^2 l^2 + \frac{4n^2}{k^2 g_s^2 L^2} = m^2 l^2 + \frac{l^2 n^2}{L_{\mathrm{phys}}^2} , \tag{5.128}$$

and to stay well below the Kaluza-Klein mass scale we therefore need

$$\frac{L_{\mathrm{phys}}}{l} \ll 1 . \tag{5.129}$$

Thus, in the regime we consider, the Kaluza-Klein truncation of massive modes is well justified. Going above the decompactification scale, $kL \gg g_s^{-1}$, the theory is the expected 4-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM well-known from the ordinary AdS/CFT correspondence, in this case in the background of a non-trivial theta-angle sourced by the axion. For $kL \sim g_s^{-1}$ the Kaluza-Klein modes cannot be ignored and the dual field theory is a DLCQ of $\mathcal{N} = 4$ SYM but with the DLCQ deformed by the axion flux. Finally, in the case where $kL \ll g_s^{-1}$, the boundary theory is a 3-dimensional Lifshitz-Chern-Simons non-Abelian gauge theory [93, 94]. The Chern-Simons part arises due to the presence of the axion. One might worry about the fact that the circle becomes null on the boundary [95]. However, the reduction performed is not the equivalent of a standard DLCQ. In the case discussed here, there is a well-defined parameter regime in which we can truncate the tower of massive Kaluza-Klein modes and the boundary theory is described by the Lifshitz-Chern-Simons theory mentioned above. The correspondence therefore yields sensible results both far above and far below the decompactification scale. This indicates that the region between might be trustworthy as well. This can be explained by the fact that the null circle is deformed by an axion, and the usual arguments for inconsistency of DLCQ do not carry through.

# Chapter 6

# Sources and Vevs in Lifshitz Holography

In this chapter we will take advantage of the framework discussed in chapter 5 to calculate the vevs and their associated Ward identities. This requires that we work in the Lifshitz UV spacetime such that all sources are turned on. Here, as in the previous chapter, we will rely heavily on the fact that the Lifshitz spacetimes are consistent reductions of AdS spacetimes. This reduction will be used to relate the sources in 5 dimensions, written in terms of frame fields, to the sources in 4 dimensions. As we will see, imposing the constraint (5.76) leads to a natural choice of boundary conditions for the 4-dimensional frame fields. These boundary conditions allows one to study the boundary geometry. The boundary geometry will be shown, under some additional assumptions, to be Newton–Cartan, thus agreeing nicely with the expectation that the dual field theory lives on some non-relativistic space. We will also derive the object referred to by Ross as the stress-energy tensor complex [74] and show that the associated Ward identities have the predicted structure [75]. The contents of this chapter will match the contents of [1] which was written concurrently with this thesis.

The chapter is organized as follows: In section 6.1 we describe the UV completion of the Lifshitz theory. This will allow us to discuss the various sources of the 4-dimensional theory. In section 6.2 we show that our choice of boundary conditions results in a contraction of the Lorentz group. Furthermore, we discuss a choice of vielbein postulate on the boundary and show that the boundary geometry becomes a generalized version of Newton–Cartan. Then, in section 6.3, we calculate the vevs along with their associated Ward identities. We also discuss the appearence of a free function which is not a vev and does not show up in any Ward identities. We briefly remark on the Ward identities in the case of AlLif spacetimes, and, additionally, we compute the anomaly for the special case of ALif spacetimes.

## 6.1 The Lifshitz UV Completion

Since we will be working with frame fields it will be very useful to derive a relation between the 5- and 4-dimensional ones. The frame fields are very well suited for describing the boundary geometry, the boundary conditions, and for the computation of the boundary stress-energy tensor. We will see that it is only in terms of frame fields that the sources are always the leading component in the expansion.

### 6.1.1 Frame fields

Consider writing the 5d metric in terms of frame fields using a null-bein tangent space metric:

$$\mathrm{d}\hat{s}^2 = \frac{\mathrm{d}r^2}{r^2} + \left(-\hat{e}^+_{\hat{a}}\hat{e}^-_{\hat{b}} - \hat{e}^+_{\hat{b}}\hat{e}^-_{\hat{a}} + \delta_{\underline{ij}}\hat{e}^{\underline{i}}_{\hat{a}}\hat{e}^{\underline{j}}_{\hat{b}}\right) dx^{\hat{a}}dx^{\hat{b}}\,, \tag{6.1}$$

where $\underline{i} = 1, 2$. The choice of a null-bein frame is convenient since we want to choose the most general frame fields compatible with the condition $\hat{h}_{(0)uu} = 0$. The 5d expression can be related to the 4d expression

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + \left(-e^{\underline{t}}_a e^{\underline{t}}_b + \delta_{\underline{ij}} e^{\underline{i}}_a e^{\underline{i}}_b\right) \mathrm{d}x^a \mathrm{d}x^b\,, \tag{6.2}$$

using the Kaluza-Klein reduction ansatz (5.64)–(5.66). The frame fields are then related by

$$\hat{e}^+_u = -\hat{e}^-_u = \frac{1}{\sqrt{2}} e^{\Phi}\,, \tag{6.3}$$

$$\hat{e}^{\underline{i}}_u = 0\,, \tag{6.4}$$

$$\hat{e}^{\underline{i}}_a = e^{-\Phi/2} e^{\underline{i}}_a\,, \tag{6.5}$$

$$\hat{e}^+_a = \frac{1}{\sqrt{2}}\left(e^{-\Phi/2} e^{\underline{t}}_a + e^{\Phi} A_a\right)\,, \tag{6.6}$$

$$\hat{e}^-_a = \frac{1}{\sqrt{2}}\left(e^{-\Phi/2} e^{\underline{t}}_a - e^{\Phi} A_a\right)\,. \tag{6.7}$$

To derive a map between the inverse frame fields we need the Kaluza-Klein ansatz for the inverse metric. Inverting the relations (5.64)–(5.66) results in

$$\hat{h}^{ab} = e^{\Phi} h^{ab}\,, \tag{6.8}$$

$$\hat{h}^{uu} = e^{-2\Phi} + e^{\Phi} A^a A_a\,, \tag{6.9}$$

$$\hat{h}^{au} = -e^{\Phi} A^a\,. \tag{6.10}$$

Using the frame field decompositions of the inverse metrics we find the relations between the inverse frame fields to be

$$\hat{e}^u_+ = \frac{1}{\sqrt{2}}\left(-e^{\Phi/2} A_a e^a_{\underline{t}} + e^{-\Phi}\right)\,, \tag{6.11}$$

$$\hat{e}^u_- = \frac{1}{\sqrt{2}}\left(-e^{\Phi/2} A_a e^a_{\underline{t}} - e^{-\Phi}\right)\,, \tag{6.12}$$

$$\hat{e}^u_{\underline{i}} = -e^{\Phi/2} A_a e^a_{\underline{i}}\,, \tag{6.13}$$

$$\hat{e}^a_+ = \hat{e}^a_- = \frac{1}{\sqrt{2}} e^{\Phi/2} e^a_{\underline{t}}\,, \tag{6.14}$$

$$\hat{e}^a_{\underline{i}} = e^{\Phi/2} e^a_{\underline{i}}\,. \tag{6.15}$$

Because of our choice of frame (6.3) and (6.4) we have

$$\hat{h}_{ab} = -\hat{e}^+_a\hat{e}^-_b - \hat{e}^+_b\hat{e}^-_a + \delta_{\underline{ij}}\hat{e}^i_a\hat{e}^j_b\,, \tag{6.16}$$

$$\hat{h}_{au} = \hat{e}^+_u\left(\hat{e}^+_a - \hat{e}^-_a\right)\,, \tag{6.17}$$

$$\hat{h}_{uu} = 2\hat{e}^+_u\hat{e}^+_u\,. \tag{6.18}$$

These expressions will be required when we determine the $r$-dependence of the frame fields below.

### 6.1.2 Boundary conditions

To investigate the appropriate boundary conditions for the frame fields consider the Lifshitz metric, equation (2.78). We are free to choose any boundary condition we want, as long as we can continuously deform our spacetime to be AlLif. A choice which naturally leads to the fact that $\hat{h}_{(0)uu} = 0$ is

$$\hat{e}^+_a = \frac{1}{r^2}\hat{e}^+_{(0)a} + \cdots\,. \tag{6.19}$$

Then we must take

$$\hat{e}^-_a = \hat{e}^-_{(0)a} + \cdots\,, \tag{6.20}$$

in order that $\hat{h}_{ab}$ in (6.16) is $\mathcal{O}(r^{-2})$. It also implies that we must take

$$\hat{e}^+_u = -\hat{e}^-_u = \hat{e}^+_{(0)u} + \cdots\,, \tag{6.21}$$

in order that $\hat{h}_{au}$ in (6.17) is $\mathcal{O}(r^{-2})$. From (6.18) we then see that $\hat{h}_{uu} = \mathcal{O}(1)$ so that

$$\hat{h}_{(0)uu} = 0\,. \tag{6.22}$$

The remaining frame fields are unconstrained by the choice (6.19) so we can choose

$$\hat{e}^i_a = \frac{1}{r}\hat{e}^i_{(0)a} + \cdots\,, \tag{6.23}$$

to preserve manifest tangent space $SO(2)$ rotation invariance at leading order. Hence, we see that the boundary condition (6.19) is well suited for arbitrary boundary metrics obeying (6.22). Recall from equation (5.106) that (6.22) implies the following constraint on the sources

$$2\hat{e}^+_{(0)u}\hat{e}^+_{(0)u} = \hat{h}_{(2)uu} = -\frac{1}{2}\hat{R}_{(0)uu} + \frac{k^2}{4}e^{2\hat{\phi}_{(0)}}\,. \tag{6.24}$$

We will assume that

$$\hat{h}_{(2)uu} > 0\,, \tag{6.25}$$

so that $\hat{e}^+_{(0)u} \neq 0$ and the reduction circle remains spacelike in the bulk. The expansions of the 5d frame fields including subleading terms are then

$$\hat{e}^+_u = \hat{e}^+_{(0)u} + r^2\log r\,\hat{e}^+_{(2,1)u} + r^2\hat{e}^+_{(2)u} + O(r^4\log^2 r)\,, \tag{6.26}$$

$$\hat{e}^+_a = \frac{1}{r^2}\hat{e}^+_{(0)a} + \log r\,\hat{e}^+_{(2,1)a} + \hat{e}^+_{(2)a} + O(r^2\log^2 r)\,, \tag{6.27}$$

$$\hat{e}^-_a = \hat{e}^-_{(0)a} + r^2\log r\,\hat{e}^-_{(2,1)a} + r^2\hat{e}^-_{(2)a} + O(r^4\log^2 r)\,, \tag{6.28}$$

$$\hat{e}^i_a = \frac{1}{r}\hat{e}^i_{(0)a} + r\hat{e}^i_{(2)a} + O(r^3\log r)\,, \tag{6.29}$$

where the coefficients can be computed using (6.16)–(6.18) and the expansions given in section 5.1. The expansion of the inverse frame fields starts as

$$\begin{aligned}
\hat{e}^u_+ &= r^2\hat{e}^u_{(0)+} + \cdots\,, && (6.30)\\
\hat{e}^a_+ &= \hat{e}^a_- = r^2\hat{e}^a_{(0)+} + \cdots\,, && (6.31)\\
\hat{e}^u_- &= \hat{e}^u_{(0)-} + \cdots\,, && (6.32)\\
\hat{e}^u_{\underline{i}} &= r\hat{e}^u_{(0)\underline{i}} + \cdots\,, && (6.33)\\
\hat{e}^a_{\underline{i}} &= r\hat{e}^a_{(0)\underline{i}} + \cdots\,, && (6.34)
\end{aligned}$$

subject to the completeness relations

$$\begin{aligned}
\hat{e}^u_{(0)-} &= -(\hat{e}^+_{(0)u})^{-1}\,, && (6.35)\\
\hat{e}^u_{(0)+} &= (\hat{e}^+_{(0)u})^{-1}\hat{e}^a_{(0)+}\hat{e}^-_{(0)a}\,, && (6.36)\\
\hat{e}^u_{(0)\underline{i}} &= (\hat{e}^+_{(0)u})^{-1}\hat{e}^a_{(0)\underline{i}}\hat{e}^-_{(0)a}\,, && (6.37)\\
\hat{e}^a_{(0)+}\hat{e}^+_{(0)a} &= 1\,, && (6.38)\\
\hat{e}^a_{(0)\underline{i}}\hat{e}^+_{(0)a} &= 0\,, && (6.39)\\
\hat{e}^a_{(0)+}\hat{e}^{\underline{i}}_{(0)a} &= 0\,, && (6.40)\\
\hat{e}^a_{(0)\underline{j}}\hat{e}^{\underline{i}}_{(0)a} &= \delta^{\underline{i}}_{\underline{j}}\,. && (6.41)
\end{aligned}$$

### 6.1.3 The 4D sources

We define the following 4d sources $\tau^a_{(0)}$, $e^{ia}_{(0)}$, $\Phi_{(0)}$, $A_{(0)\underline{t}}$ and $A_{(0)\underline{i}}$ by writing the leading components of the 5d vielbeins and inverse vielbeins as follows

$$\begin{aligned}
\hat{e}^+_{(0)u} &= \frac{1}{\sqrt{2}}e^{\Phi_{(0)}}\,, && (6.42)\\
\hat{e}^+_{(0)a} &= \sqrt{2}e^{-\Phi_{(0)}/2}\tau_{(0)a}\,, && (6.43)\\
\hat{e}^-_{(0)a} &= -\frac{1}{\sqrt{2}}e^{\Phi_{(0)}}A_{(0)a}\,, && (6.44)\\
\hat{e}^{\underline{i}}_{(0)a} &= e^{-\Phi_{(0)}/2}e^{\underline{i}}_{(0)a}\,, && (6.45)\\
\hat{e}^u_{(0)+} &= -\frac{1}{\sqrt{2}}e^{\Phi_{(0)}/2}A_{(0)\underline{t}}\,, && (6.46)\\
\hat{e}^u_{(0)-} &= -\sqrt{2}e^{-\Phi_{(0)}}\,, && (6.47)\\
\hat{e}^u_{(0)\underline{i}} &= -e^{\Phi_{(0)}/2}A_{(0)\underline{i}}\,, && (6.48)\\
\hat{e}^a_{(0)+} &= -\frac{1}{\sqrt{2}}e^{\Phi_{(0)}/2}\tau^a_{(0)}\,, && (6.49)\\
\hat{e}^a_{(0)\underline{i}} &= e^{\Phi_{(0)}/2}e^a_{(0)\underline{i}}\,, && (6.50)
\end{aligned}$$

where[1]

$$\begin{aligned}
\tau_{(0)a}\tau^a_{(0)} &= -1\,, &(6.53)\\
\tau_{(0)a}e^a_{(0)\underline{i}} &= 0\,, &(6.54)\\
e^{\underline{i}}_{(0)a}\tau^a_{(0)} &= 0\,, &(6.55)\\
e^{\underline{i}}_{(0)a}e^a_{(0)\underline{j}} &= \delta^{\underline{i}}_{\underline{j}}\,, &(6.56)\\
A_{(0)a} &= A_{(0)\underline{t}}\tau_{(0)a} + A_{(0)\underline{i}}e^{\underline{i}}_{(0)a}\,. &(6.57)
\end{aligned}$$

Note the definition of $A_{(0)a}$. It is *not* the leading component of the expansion of $A_a$, rather, it is given by equation (5.113). This turns out to be a convenient choice as it is exactly the combination (5.113) which acts as a source for the vector field. Equations (6.46)–(6.50) then constitute 15 sources. However, these are constrained by the local symmetries. The frame fields $\tau^a_{(0)}$ and $e^a_{(0)\underline{i}}$ are invariant under local Lorentz transformations, reducing the number of independent sources by three. They are also subject to general coordinate transformations removing an additional three sources. As we saw in chapter 4, the Kaluza-Klein vector field inherits a $U(1)$ gauge symmetry which can be used to remove an additional source. Furthermore, it was shown in chapter 2 that bulk diffeomorphims induce boundary diffeomorphisms in addition to a scale transformation. This scale transformation can be used to remove one further source, bringing the total number of independent sources to 7. However, there is a further constraint among the sources, namely the fact that $\hat{h}_{(0)uu} = 0$. This constraint relates the sources $\phi_{(0)}$ and $\Phi_{(0)}$ by equation (5.106). Hence, there are 6 independent sources in our theory, and we should expect 6 independent vevs. In section 6.3.2 we will confirm this expectation.

The relations (6.42)–(6.50) results in the following map between the 5-dimensional boundary metric and the 4-dimensional sources:

$$\begin{aligned}
\hat{h}_{(0)ab} &= e^{\Phi_{(0)}/2}\left(\tau_{(0)a}A_{(0)b} + \tau_{(0)b}A_{(0)a}\right) + e^{-\Phi_{(0)}}\Pi_{(0)ab}\,, &(6.58)\\
\hat{h}_{(0)au} &= e^{\Phi_{(0)}/2}\tau_{(0)a}\,, &(6.59)\\
\hat{h}_{(0)uu} &= 0\,, &(6.60)\\
\hat{h}^{ab}_{(0)} &= e^{\Phi_{(0)}}\Pi^{ab}_{(0)}\,, &(6.61)\\
\hat{h}^{au}_{(0)} &= -e^{-\Phi_{(0)}/2}\tau^a_{(0)} - e^{\Phi_{(0)}}\delta^{\underline{ij}}e^a_{(0)\underline{i}}A_{(0)\underline{j}}\,, &(6.62)\\
\hat{h}^{uu}_{(0)} &= -2e^{-\Phi_{(0)}/2}A_{(0)\underline{t}} + e^{\Phi_{(0)}}\delta^{\underline{ij}}A_{(0)\underline{i}}A_{(0)\underline{j}}\,, &(6.63)
\end{aligned}$$

where we defined

$$\begin{aligned}
\Pi_{(0)ab} &\equiv \delta_{\underline{ij}}e^{\underline{i}}_{(0)a}e^{\underline{j}}_{(0)b}\,, &(6.64)\\
\Pi^{ab}_{(0)} &\equiv \delta^{\underline{ij}}e^a_{(0)\underline{i}}e^b_{(0)\underline{j}}\,. &(6.65)
\end{aligned}$$

[1] For boundary vectors and frame field components we use the notation

$$X_{(0)\underline{t}} = -X_{(0)a}\tau^a_{(0)}\,, \qquad X_{(0)\underline{i}} = X_{(0)a}e^a_{(0)\underline{i}}\,,$$

and

$$X_{(0)a} = X_{(0)\underline{t}}\tau_{(0)a} + X_{(0)\underline{i}}e^{\underline{i}}_{(0)a}\,.$$

We have identified the most general boundary conditions compatible with the condition $\hat{h}_{(0)uu} = 0$. Using the relation between the 4- and 5-dimensional frame fields given in section 6.1.1 we have thus obtained the most general 4-dimensional boundary conditions corresponding to the Lifshitz UV spacetime. The leading components of the expansions of the 4-dimensional frame fields are the sources defined in section 6.1.3. These are all the 4-dimensional sources compatible with the Lifshitz UV.

## 6.2 The Boundary Geometry

With the definition of the Lifshitz UV boundary conditions and the knowledge of the 4-dimensional sources acquired in the previous section it is possible to compute the variation of the on-shell action (using the reduced counterterms of section 5.2) and study the Ward identities. However, before we study these quantities we should first consider what kind of geometry arises on the boundary. We will see that in the case of a hypersurface orthogonal $\tau_{(0)a}$, i.e. for the generalized AlLif boundary conditions of subsection 5.2.2, the boundary geometry is Newton–Cartan. This is to be expected as it follows from a reduction along a null circle with $\partial_u$ Killing and hypersurface orthogonal. This relation between Newton–Cartan and Lorentzian geometries has been observed in [96]. We will also study the more general case where $\tau_{(0)a}$, or equivalently $\partial_u$, is not hypersurface orthogonal.

### 6.2.1 Contraction of the local Lorentz group

To get an idea about the boundary geometry described by $\tau_{(0)a}$ and $e^{\underline{i}}_{(0)a}$ we study how bulk local Lorentz transformations act on the leading components of the frame fields. To this end we consider local Lorentz transformations transforming the $e^{\underline{a}}_a$ amongst each other, i.e. the group of tangent space $SO(1,2)$ rotations leaving $e^3 = e^{\Phi/2}\frac{\mathrm{d}r}{r}$ invariant. Here $e^3$ is the radial part of (5.39). These transformations leave invariant the metric

$$\mathrm{d}s^2 = e^{\Phi}\frac{\mathrm{d}r^2}{r^2} - e^{\underline{t}}_a e^{\underline{t}}_b + \delta_{\underline{i}\underline{j}} e^{\underline{i}}_a e^{\underline{j}}_b\,, \tag{6.66}$$

but due to the non-trivial $r$-dependence of the frame fields, constraints are imposed on the transformations. We have

$$\begin{aligned} e^{\underline{t}}_a &= \Lambda^{\underline{t}}{}_{\underline{t}'} e^{\underline{t}'}_a + \Lambda^{\underline{t}}{}_{\underline{i}'} e^{\underline{i}'}_a\,, &(6.67)\\ e^{\underline{i}}_a &= \Lambda^{\underline{i}}{}_{\underline{t}'} e^{\underline{t}'}_a + \Lambda^{\underline{i}}{}_{\underline{i}'} e^{\underline{i}'}_a\,, &(6.68)\end{aligned}$$

where

$$\begin{aligned} -\Lambda^{\underline{t}}{}_{\underline{t}'}\Lambda^{\underline{t}}{}_{\underline{t}'} + \delta_{\underline{i}\underline{j}}\Lambda^{\underline{i}}{}_{\underline{t}'}\Lambda^{\underline{j}}{}_{\underline{t}'} &= -1\,, &(6.69)\\ -\Lambda^{\underline{t}}{}_{\underline{t}'}\Lambda^{\underline{t}}{}_{\underline{i}'} + \delta_{\underline{i}\underline{j}}\Lambda^{\underline{i}}{}_{\underline{t}'}\Lambda^{\underline{j}}{}_{\underline{i}'} &= 0\,, &(6.70)\\ -\Lambda^{\underline{t}}{}_{\underline{i}'}\Lambda^{\underline{t}}{}_{\underline{j}'} + \delta_{\underline{i}\underline{j}}\Lambda^{\underline{i}}{}_{\underline{i}'}\Lambda^{\underline{j}}{}_{\underline{j}'} &= \delta_{\underline{i}'\underline{j}'}\,. &(6.71)\end{aligned}$$

Since the transformations leave $e^3$ invariant the frame fields can be expanded in $r$ in the same way before and after the local Lorentz transformations, namely

$$
\begin{aligned}
e_a^{\underline{t}'} &= r^{-2}\tau'_{(0)a} + \cdots\,, && (6.72)\\
e_a^{\underline{i}'} &= r^{-1}e^{\underline{i}'}_{(0)a} + \cdots\,, && (6.73)\\
e_a^{\underline{t}} &= r^{-2}\tau_{(0)a} + \cdots\,, && (6.74)\\
e_a^{\underline{i}} &= r^{-1}e^{\underline{i}}_{(0)a} + \cdots\,. && (6.75)
\end{aligned}
$$

This requires that

$$
\begin{aligned}
\Lambda^{\underline{t}}{}_{\underline{t}'} &= \Lambda^{\underline{t}}_{(0)\underline{t}'} + \cdots\,, && (6.76)\\
\Lambda^{\underline{t}}{}_{\underline{i}'} &= r^{-1}\Lambda^{\underline{t}}_{(0)\underline{i}'} + \cdots\,, && (6.77)\\
\Lambda^{\underline{i}}{}_{\underline{t}'} &= r\Lambda^{\underline{i}}_{(0)\underline{t}'} + \cdots\,, && (6.78)\\
\Lambda^{\underline{i}}{}_{\underline{i}'} &= \Lambda^{\underline{i}}_{(0)\underline{i}'} + \cdots\,. && (6.79)
\end{aligned}
$$

Plugging this into (6.69)–(6.71) we get

$$
\begin{aligned}
\Lambda^{\underline{t}}_{(0)\underline{t}'}\Lambda^{\underline{t}}_{(0)\underline{t}'} &= 1\,, && (6.80)\\
\Lambda^{\underline{t}}_{(0)\underline{i}'} &= 0\,, && (6.81)\\
\delta_{\underline{i}\underline{j}}\Lambda^{\underline{i}}_{(0)\underline{i}'}\Lambda^{\underline{j}}_{(0)\underline{j}'} &= \delta_{\underline{i}'\underline{j}'}\,. && (6.82)
\end{aligned}
$$

We will choose

$$
\Lambda^{\underline{t}}_{(0)\underline{t}'} = 1\,, \qquad (6.83)
$$

so that we can recover the identity. Hence, we find the following transformation of the leading components of the frame field expansions

$$
\begin{aligned}
\tau_{(0)a} &= \tau'_{(0)a}\,, && (6.84)\\
e^{\underline{i}}_{(0)a} &= \Lambda^{\underline{i}}_{(0)\underline{t}'}\tau'_{(0)a} + \Lambda^{\underline{i}}_{(0)\underline{i}'}e^{\underline{i}'}_{(0)a}\,, && (6.85)
\end{aligned}
$$

where $\Lambda^{\underline{i}}_{(0)\underline{t}'}$ are two free parameters[2]. The transformation acting on the leading components of the inverse frame fields reads

$$
\begin{aligned}
\tau^a_{(0)} &= \tau'^a_{(0)} + e^a_{(0)\underline{i}'}\Lambda^{\underline{i}'}_{(0)\underline{i}}\Lambda^{\underline{i}}_{(0)\underline{t}'}\,, && (6.86)\\
e^a_{(0)\underline{i}} &= e^a_{(0)\underline{i}'}\Lambda^{\underline{i}'}_{(0)\underline{i}}\,. && (6.87)
\end{aligned}
$$

Infinitesimally the transformations (6.84)–(6.87) become

$$
\begin{aligned}
\delta\tau_{(0)a} &= 0\,, && (6.88)\\
\delta e^{\underline{i}}_{(0)a} &= \xi^b_{(0)}\omega_{(0)b}{}^{\underline{i}}{}_{\underline{t}}\tau_{(0)a} + \xi^b_{(0)}\omega_{(0)b}{}^{\underline{i}}{}_{\underline{j}}e^{\underline{j}}_{(0)a}\,, && (6.89)\\
\delta\tau^a_{(0)} &= \xi^b_{(0)}\omega_{(0)b}{}^{\underline{i}}{}_{\underline{t}}e^a_{(0)\underline{i}}\,, && (6.90)\\
\delta e^a_{(0)\underline{i}} &= -\xi^b_{(0)}\omega_{(0)b}{}^{\underline{j}}{}_{\underline{i}}e^a_{(0)\underline{j}}\,, && (6.91)
\end{aligned}
$$

[2] The three generators of these transformations are $J, G_1, G_2$ whose nonzero commutators are $[J, G_1] = G_2$ and $[J, G_2] = -G_1$. We can think of this as the contraction of the Lorentz group $SO(1, 2)$ in which the $G_i$ play the rôle of Galilean boost generators.

where

$$\delta_{\underline{ik}}\omega_{(0)a}{}^{\underline{k}}{}_{\underline{j}} = -\delta_{\underline{jk}}\omega_{(0)a}{}^{\underline{k}}{}_{\underline{i}} \,. \tag{6.92}$$

Thus, the boundary geometry is not invariant under local Lorentz transformations. However, the explicit structure of the infinitesimal transformations (6.88)–(6.91) allows for a more thorough study of the boundary geometry.

### 6.2.2 The vielbein postulate

From the relations (6.88)–(6.91) it is seen how the frame fields transform under local tangent space transformations. This information can be used to define covariant derivatives by introducing the connections $\omega_{(0)b}{}^{\underline{i}}{}_{\underline{t}}$ and $\omega_{(0)b}{}^{\underline{i}}{}_{\underline{j}}$ as well as the symmetric connection $\Gamma^c_{(0)ab}$. Demanding covariance under both general coordinate transformations and local tangent space transformations results in the following covariant derivatives:

$$\mathcal{D}_{(0)a}\tau_{(0)b} = \partial_a\tau_{(0)b} - \Gamma^c_{(0)ab}\tau_{(0)c} \,, \tag{6.93}$$

$$\mathcal{D}_{(0)a}e^{\underline{i}}_{(0)b} = \partial_a e^{\underline{i}}_{(0)b} - \Gamma^c_{(0)ab}e^{\underline{i}}_{(0)c} + \omega_{(0)a}{}^{\underline{i}}{}_{\underline{t}}\tau_{(0)b} + \omega_{(0)a}{}^{\underline{i}}{}_{\underline{j}}e^{\underline{j}}_{(0)b} \,, \tag{6.94}$$

$$\mathcal{D}_{(0)a}e^{b}_{(0)\underline{i}} = \partial_a e^{b}_{(0)\underline{i}} + \Gamma^b_{(0)ac}e^{c}_{(0)\underline{i}} - \omega_{(0)a}{}^{\underline{j}}{}_{\underline{i}}e^{b}_{(0)\underline{j}} \,, \tag{6.95}$$

$$\mathcal{D}_{(0)a}\tau^b_{(0)} = \partial_a\tau^b_{(0)} + \Gamma^b_{(0)ac}\tau^c_{(0)} + \omega_{(0)a}{}^{\underline{i}}{}_{\underline{t}}e^{b}_{(0)\underline{i}} \,. \tag{6.96}$$

This only defines covariant derivatives. We have not yet chosen the connections by imposing a vielbein postulate. Recall that, ordinarily, the vielbein postulate allows one to relate the spin coefficients to the Christoffel symbols and is typically chosen such that the frame fields are covariantly constant [45]:

$$\mathcal{D}_a e^{\underline{a}}_b = 0 \,. \tag{6.97}$$

This will clearly not work in our case. The metric on the boundary is not invariant under local Lorentz transformations, but instead the frame fields transform according to the rules given in (6.88)–(6.91). Below, we will motivate a different choice of vielbein postulate. We will denote the covariant derivative containing only the $\Gamma^c_{(0)ab}$ connection by $\nabla_{(0)a}$. Using the relations (6.93)–(6.96) we can write for example ($e_{(0)} = \det e^{\underline{a}}_{(0)a}$)

$$\begin{aligned}\frac{1}{e_{(0)}}\partial_a e_{(0)} &= \Gamma^b_{(0)ab} + \tau_{(0)b}\mathcal{D}_{(0)a}\tau^b_{(0)} + e^b_{(0)\underline{i}}\mathcal{D}_{(0)a}e^{\underline{i}}_{(0)b} \\ &= \Gamma^b_{(0)ab} - \tau^b_{(0)}\nabla_{(0)a}\tau_{(0)b} - \frac{1}{2}\Pi_{(0)bc}\nabla_{(0)a}\Pi^{bc}_{(0)} \,.\end{aligned} \tag{6.98}$$

This allows us to consider a variety of vielbein postulates. Clearly something like $\mathcal{D}_{(0)a}\tau_{(0)b} = \nabla_{(0)a}\tau_{(0)b} = 0$ (as is done for Newton–Cartan [97]) is too restrictive as it implies that $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0$. This is easily seen by writing

$$\mathcal{D}_{(0)a}\tau_{(0)b} = \partial_a\tau_{(0)b} - \Gamma^c_{(0)ab}\tau_{(0)c} = 0 \Rightarrow \tag{6.99}$$

$$\partial_a\tau_{(0)b} = \Gamma^c_{(0)ab}\tau_{(0)c} \,, \tag{6.100}$$

implying that

$$\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = \Gamma^c_{(0)ab}\tau_{(0)c} - \Gamma^c_{(0)ba}\tau_{(0)c} = 0 \,, \tag{6.101}$$

as follows from the Christoffel connection being torsionless. This implies that $\tau_{(0)a}$ is hypersurface orthogonal which is true for AlLif space-times (see section 5.2.1) but not for our more general boundary conditions. Also the condition $\mathcal{D}_{(0)a}\tau_{(0)b} = 0$ would not fix any $\omega_{(0)a}{}^{\underline{a}}{}_{\underline{b}}$ connection coefficients as seen from equation (6.93). If instead we impose $\mathcal{D}_{(0)a}e^b_{(0)\underline{i}} = 0$ then it follows that $\nabla_{(0)a}\Pi^{bc}_{(0)} = 0$ and using (6.95) we can only relate $\omega_{(0)a}{}^{\underline{i}}{}_{\underline{j}}$ to the $\Gamma^c_{(0)ab}$ connection. Similarly, imposing $\mathcal{D}_{(0)a}\tau^b_{(0)} = 0$ would only fix $\omega_{(0)a}{}^{\underline{i}}{}_{\underline{t}}$. The only vielbein postulate relating all components of $\omega_{(0)a}{}^{\underline{a}}{}_{\underline{b}}$ to the $\Gamma^c_{(0)ab}$ connection while allowing maximal freedom regarding the choice of $\Gamma^c_{(0)ab}$ is given by

$$\mathcal{D}_{(0)a}e^{\underline{i}}_{(0)b} = 0\,. \tag{6.102}$$

Using this in equation (6.94) implies that

$$\begin{aligned} \omega_{(0)a}{}^{\underline{i}}{}_{\underline{t}} &= -e^{\underline{i}}_{(0)b}\nabla_{(0)a}\tau^b_{(0)}\,, &(6.103)\\ \omega_{(0)a}{}^{\underline{i}}{}_{\underline{j}} &= e^{\underline{i}}_{(0)b}\nabla_{(0)a}e^b_{(0)\underline{j}}\,, &(6.104)\end{aligned}$$

where we used the completeness relations (6.55) and (6.56). Using another completeness relation, namely

$$-\tau^a_{(0)}\tau_{(0)b} + e^a_{(0)\underline{i}}e^{\underline{i}}_{(0)b} = \delta^a_b\,, \tag{6.105}$$

we can write

$$e^{\underline{i}}_{(0)b}e^d_{(0)\underline{i}}e^{\underline{j}}_{(0)c}e^e_{(0)\underline{j}}\nabla_{(0)a}\Pi_{(0)de} = \left(\delta^d_b + \tau_{(0)b}\tau^d_{(0)}\right)\left(\delta^e_c + \tau_{(0)c}\tau^e_{(0)}\right)\nabla_{(0)a}\Pi_{(0)de}\,. \tag{6.106}$$

Using the relation (6.104) and writing out the $\Pi_{(0)ab}$ we find that the $\Gamma^c_{(0)ab}$ connection must satisfy

$$\left(\delta^d_b + \tau_{(0)b}\tau^d_{(0)}\right)\left(\delta^e_c + \tau_{(0)c}\tau^e_{(0)}\right)\nabla_{(0)a}\Pi_{(0)de} = 0\,. \tag{6.107}$$

By differentiating the completeness relation

$$\Pi_{(0)ab}\Pi^{bc}_{(0)} - \tau_{(0)a}\tau^c_{(0)} = \delta^c_a\,, \tag{6.108}$$

we obtain a completely analogous expression for the object $\Pi^{ab}_{(0)}$:

$$\left(\delta^b_d + \tau_{(0)d}\tau^b_{(0)}\right)\left(\delta^c_e + \tau_{(0)e}\tau^c_{(0)}\right)\nabla_{(0)a}\Pi^{de}_{(0)} = 0\,. \tag{6.109}$$

With the choice (6.102) for the connection $\omega_{(0)a}{}^{\underline{a}}{}_{\underline{b}}$ we find

$$\begin{aligned} \mathcal{D}_{(0)a}\tau^b_{(0)} &= \tau^b_{(0)}\tau^c_{(0)}\nabla_{(0)a}\tau_{(0)c}\,, &(6.110)\\ \mathcal{D}_{(0)a}e^b_{(0)\underline{i}} &= \tau^b_{(0)}e^c_{(0)\underline{i}}\nabla_{(0)a}\tau_{(0)c}\,, &(6.111)\end{aligned}$$

from (6.95) and (6.96).

### 6.2.3 The choice of $\Gamma^c_{(0)ab}$

Our choice of $\Gamma^c_{(0)ab}$ will be inspired by the null dimensional reduction of the boundary geometry. Consider the Christoffel connection of the non-degenerate 5-dimensional boundary metric $\hat{h}_{(0)\hat{a}\hat{b}}$ possessing a null Killing vector $\partial_u$ and take all its legs in the directions of the three non-compact directions. Using (6.58)–(6.63) we decompose this quantity as follows

$$\begin{aligned}
\hat{\Gamma}^a_{(0)bc} &= \Gamma^a_{(0)bc} - \frac{1}{2}e^{3\Phi_{(0)}/2}\Pi^{ad}_{(0)}\left[\left(\partial_d\tau_{(0)b} - \partial_b\tau_{(0)d}\right)A_{(0)c} + \left(\partial_d\tau_{(0)c} - \partial_c\tau_{(0)d}\right)A_{(0)b}\right] \\
&\quad -\frac{1}{2}\delta^a_b\partial_c\Phi_{(0)} - \frac{1}{2}\delta^a_c\partial_b\Phi_{(0)} + \frac{1}{2}\Pi_{(0)bc}\Pi^{ad}_{(0)}\partial_d\Phi_{(0)} - \frac{3}{4}\tau^a_{(0)}\tau_{(0)b}\partial_c\Phi_{(0)} \\
&\quad -\frac{3}{4}\tau^a_{(0)}\tau_{(0)c}\partial_b\Phi_{(0)} - \frac{1}{4}e^{3\Phi_{(0)}/2}\left(A_{(0)b}\tau_{(0)c} + A_{(0)c}\tau_{(0)b}\right)\Pi^{ad}_{(0)}\partial_d\Phi_{(0)}\,, \qquad (6.112)
\end{aligned}$$

where $\Gamma^a_{(0)bc}$ is given by

$$\begin{aligned}
\Gamma^a_{(0)bc} &= -\frac{1}{2}\tau^a_{(0)}\left(\partial_b\tau_{(0)c} + \partial_c\tau_{(0)b}\right) + \frac{1}{2}\Pi^{ad}_{(0)}\left(\partial_b\Pi_{(0)cd} + \partial_c\Pi_{(0)bd} - \partial_d\Pi_{(0)bc}\right) \\
&\quad -\frac{1}{2}e^{3\Phi_{(0)}/2}\Pi^{ad}_{(0)}\left(F_{(0)db}\tau_{(0)c} + F_{(0)dc}\tau_{(0)b}\right)\,, \qquad (6.113)
\end{aligned}$$

with $F_{(0)ab} = \partial_a A_{(0)b} - \partial_b A_{(0)a}$. The connection $\Gamma^a_{(0)bc}$ satisfies the following properties

$$\begin{aligned}
\Gamma^a_{(0)ac} &= e^{-1}_{(0)}\partial_c e_{(0)} - \frac{1}{2}\tau^a_{(0)}\left(\partial_a\tau_{(0)c} - \partial_c\tau_{(0)a}\right)\,, \qquad (6.114) \\
\nabla_{(0)a}\tau_{(0)b} &= \frac{1}{2}\left(\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a}\right)\,, \qquad (6.115) \\
\nabla_{(0)a}\tau^b_{(0)} &= \frac{1}{2}\tau^b_{(0)}\tau^c_{(0)}\left(\partial_a\tau_{(0)c} - \partial_c\tau_{(0)a}\right) + \frac{1}{2}\Pi^{bc}_{(0)}\mathcal{L}_{\tau_{(0)}}\Pi_{(0)ac} \\
&\quad -\frac{1}{2}e^{3\Phi_{(0)}/2}h^{bc}_{(0)}\tau^d_{(0)}\left(F_{(0)ca}\tau_{(0)d} + F_{(0)cd}\tau_{(0)a}\right)\,, \qquad (6.116) \\
\nabla_{(0)a}\Pi^{bc}_{(0)} &= \frac{1}{2}\left(\partial_a\tau_{(0)d} - \partial_d\tau_{(0)a}\right)\left(\Pi^{bd}_{(0)}\tau^c_{(0)} + \Pi^{cd}_{(0)}\tau^b_{(0)}\right)\,, \qquad (6.117) \\
\nabla_{(0)a}\Pi_{(0)bc} &= \frac{1}{2}\tau_{(0)b}\mathcal{L}_{\tau_{(0)}}\Pi_{(0)ac} + \frac{1}{2}\tau_{(0)c}\mathcal{L}_{\tau_{(0)}}\Pi_{(0)ab} \\
&\quad +\frac{1}{2}e^{3\Phi_{(0)}/2}\left[\left(\delta^e_c + \tau_{(0)c}\tau^e_{(0)}\right)F_{(0)ea}\tau_{(0)b} + \left(\delta^e_b + \tau_{(0)b}\tau^e_{(0)}\right)F_{(0)ea}\tau_{(0)c}\right. \\
&\quad \left. + \left(\delta^e_c + \tau_{(0)c}\tau^e_{(0)}\right)F_{(0)eb}\tau_{(0)a} + \left(\delta^e_b + \tau_{(0)b}\tau^e_{(0)}\right)F_{(0)ec}\tau_{(0)a}\right] \\
&= \left(\tau_{(0)b}\Pi_{(0)cd} + \tau_{(0)c}\Pi_{(0)bd}\right)\nabla_{(0)a}\tau^d_{(0)}\,, \qquad (6.118)
\end{aligned}$$

so that $\Gamma^c_{(0)ab}$ is compatible with (6.107).

### 6.2.4 Newton–Cartan

Since the boundary values of the frame fields transform according to (6.88)–(6.91) the boundary geometry becomes rather special. In fact, it becomes Newton-Cartan [98, 99, 100] (see also [101]) if and only if $\tau_{(0)a}$ additionally satisfies (5.109). With this additional assumption on $\tau_{(0)a}$ we find from the vielbein postulate (6.102) as well as $\Gamma^a_{(0)bc}$ that

$$\begin{aligned}\mathcal{D}_{(0)a}\tau_{(0)b} &= 0\,, &(6.119)\\ \mathcal{D}_{(0)a}\tau^b_{(0)} &= 0\,, &(6.120)\\ \mathcal{D}_{(0)a}e^{\underline{i}}_{(0)b} &= 0\,. &(6.121)\end{aligned}$$

This implies

$$\begin{aligned}\nabla_{(0)a}\Pi^{bc}_{(0)} &= 0\,, &(6.122)\\ \nabla_{(0)a}\tau_{(0)b} &= 0\,. &(6.123)\end{aligned}$$

Provided we have $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0$ the $\Gamma^a_{(0)bc}$ is of the form given in [102, 97] and of the form used in [103] if furthermore $F_{(0)ab} = 0$. We therefore have a Newton–Cartan boundary geometry for the class of ALif space-times introduced in section 5.2.2.

### 6.2.5 Torsional Newton–Cartan

We define a torsion tensor $T^c_{(0)ab}$ as

$$T^c_{(0)ab} = -\frac{1}{2}\tau^c_{(0)}\left(\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a}\right)\,. \tag{6.124}$$

Next consider a covariant derivative $\nabla^T_{(0)a}$ defined as

$$\begin{aligned}\nabla^T_{(0)a}X^b_{(0)} &= \nabla_{(0)a}X^b_{(0)} + T^b_{(0)ac}X^c_{(0)}\,, &(6.125)\\ \nabla^T_{(0)a}X_{(0)b} &= \nabla_{(0)a}X_{(0)b} - T^c_{(0)ab}X_{(0)c}\,. &(6.126)\end{aligned}$$

The relations of section 6.2.3 can then be written as

$$\begin{aligned}\nabla^T_{(0)a}\tau_{(0)b} &= 0\,, &(6.127)\\ \nabla^T_{(0)a}\tau^b_{(0)} &= \frac{1}{2}\Pi^{bc}_{(0)}\mathcal{L}_{\tau_{(0)}}\Pi_{(0)ac}\\ &\quad -\frac{1}{2}e^{3\Phi_{(0)}/2}\Pi^{bc}_{(0)}\tau^d_{(0)}\left(F_{(0)ca}\tau_{(0)d} + F_{(0)cd}\tau_{(0)a}\right)\,, &(6.128)\\ \nabla^T_{(0)a}\Pi^{bc}_{(0)} &= 0\,, &(6.129)\\ \nabla^T_{(0)a}\Pi_{(0)bc} &= \left(\tau_{(0)b}\Pi_{(0)cd} + \tau_{(0)c}\Pi_{(0)bd}\right)\nabla^T_{(0)a}\tau^d_{(0)}\,. &(6.130)\end{aligned}$$

The vielbein postulate (6.102) is also

$$\mathcal{D}^T_{(0)a}e^{\underline{i}}_{(0)b} = 0\,, \tag{6.131}$$

where we have replaced the covariant derivative $\nabla_{(0)a}$ that is contained in $\mathcal{D}_{(0)a}$ by $\nabla^T_{(0)a}$. Equations (6.110) and (6.111) can then be written as

$$\begin{aligned}\mathcal{D}^T_{(0)a}\tau^b_{(0)} &= 0\,, &(6.132)\\ \mathcal{D}^T_{(0)a}e^b_{(0)\underline{i}} &= 0\,. &(6.133)\end{aligned}$$

An important special case of torsional Newton–Cartan (TNC) geometry is obtained when we impose (5.107) but not (5.109). This allows us to write

$$\partial_a \tau_{(0)b} - \partial_b \tau_{(0)a} = \tau_{(0)a}\sigma_{(0)b} - \tau_{(0)b}\sigma_{(0)a}\,, \tag{6.134}$$

where

$$\sigma_{(0)a} = -\tau^c_{(0)}\left(\partial_c \tau_{(0)a} - \partial_a \tau_{(0)c}\right)\,. \tag{6.135}$$

We define the twist tensor $\omega_{(0)ab}$ as

$$\omega_{(0)ab} = \left(\delta^c_a + \tau_{(0)a}\tau^c_{(0)}\right)\left(\delta^d_b + \tau_{(0)b}\tau^d_{(0)}\right)\left(\partial_c \tau_{(0)d} - \partial_d \tau_{(0)c}\right)\,. \tag{6.136}$$

This quantity vanishes for the case where we impose (5.107). We will refer to this as twistless torsional Newton–Cartan geometry (TTNC). This explains the last column in table 5.1.

## 6.3 Sources and Vevs

As we saw in chapter 5, one can obtain $z = 2$ Lifshitz spacetimes in 4 dimensions from AlAdS spacetimes in 5 dimensions via Scherk-Schwarz reduction. Since the reduction is consistent, this allows for a determination of the vevs of the Lifshitz theory by considering the dimensional reduction of the vevs of the 5-dimensional theory. To keep all sources turned on, we will work in the Lifshitz UV spacetime throughout this section, unless otherwise stated. Working with the frame fields introduced in section 6.1 we determine the sources of the reduced theory and the Ward identities satisfied by the vevs.

### 6.3.1 Variation of the renormalized on-shell action

The action for the 4-dimensional theory is given in (5.47). Following the spirit of section 3.3, we can write the total variation as

$$\begin{aligned}\delta S_{\text{ren}} \;=\;& \frac{2\pi L}{2\kappa_5^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}^{\mu}\delta A_\mu + \mathcal{E}_\Phi \delta\Phi + \mathcal{E}_\phi \delta\phi + \mathcal{E}_\chi \delta\chi\right)\\ & -\frac{2\pi L}{2\kappa_5^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Bigg(T_{ab}\delta h^{ab} + 2T^a\delta A_a + 2T_\chi \delta\chi + 2T_\Phi \delta\Phi\\ & +2T_\phi\delta\phi - e^{-\Phi/2}\mathcal{A}\frac{\delta r}{r}\Bigg)\,,\end{aligned} \tag{6.137}$$

where the equations of motion are given by (5.53)–(5.56) (although in this case there will be an equation of motion for both $A_\mu$ and $\chi$) and

$$\begin{aligned} T_{ab} &= Kh_{ab} - K_{ab} + \frac{1}{2}e^{\Phi/2}G_{(h)ab} - 3e^{-\Phi/2}h_{ab} + \frac{k^2}{8}e^{2\phi-5\Phi/2}h_{ab} \\ &\quad -\frac{1}{4}e^{7\Phi/2}F_{ac}F_b{}^c + \frac{1}{16}e^{7\Phi/2}h_{ab}F_{cd}F^{cd} + \frac{1}{8}e^{2\phi+\Phi/2}h_{ab}D_c\chi D^c\chi \\ &\quad -\frac{1}{4}e^{\Phi/2}\left(\nabla_{(h)a}\partial_b\Phi - h_{ab}\Box_{(h)}\Phi\right) - \frac{7}{8}e^{\Phi/2}\partial_a\Phi\partial_b\Phi + \frac{1}{2}e^{\Phi/2}h_{ab}\partial_c\Phi\partial^c\Phi \\ &\quad +\frac{1}{8}e^{\Phi/2}h_{ab}\partial_c\phi\partial^c\phi - \frac{1}{4}e^{\Phi/2}\partial_a\phi\partial_b\phi - \frac{1}{4}e^{2\phi+\Phi/2}D_a\chi D_b\chi \,, \qquad (6.138) \\ T^a &= -\frac{1}{2}e^{3\Phi}n_\mu F^{a\mu} - \frac{1}{4}\nabla_{(h)b}\left(e^{7\Phi/2}F^{ab}\right) + \frac{k}{4}e^{2\phi+\Phi/2}D^a\chi \,, \qquad (6.139) \\ T_\chi &= \frac{1}{4}\nabla_{(h)a}\left(e^{2\phi+\Phi/2}D^a\chi\right) + \frac{1}{2}e^{2\phi}n^\mu D_\mu\chi \,, \qquad (6.140) \\ T_\Phi &= \frac{3}{2}n^\mu\partial_\mu\Phi + \frac{1}{8}e^{\Phi/2}R_{(h)} - \frac{1}{16}e^{\Phi/2}\partial_a\phi\partial^a\phi + \frac{3}{16}e^{\Phi/2}\partial_a\Phi\partial^a\Phi + \frac{3}{4}e^{\Phi/2}\Box_{(h)}\Phi \\ &\quad -\frac{7}{32}e^{7\Phi/2}F_{ab}F^{ab} - \frac{1}{16}e^{2\phi+\Phi/2}D_a\chi D^a\chi - \frac{3}{2}e^{-\Phi/2} + \frac{5k^2}{16}e^{2\phi-5\Phi/2} \,, \qquad (6.141) \\ T_\phi &= \frac{1}{2}n^\mu\partial_\mu\phi + \frac{1}{4}e^{\Phi/2}\Box_{(h)}\phi + \frac{1}{8}e^{\Phi/2}\partial_a\Phi\partial^a\phi - \frac{1}{4}e^{2\phi+\Phi/2}D_a\chi D^a\chi \\ &\quad -\frac{k^2}{4}e^{2\phi-5\Phi/2} \,. \qquad (6.142) \end{aligned}$$

Usually, one considers the stress-energy tensor to be the object sourced by the inverse boundary metric. However, due to the anisotropic nature of the Lifshitz spacetime and the fact that the theory contains a vector field, frame fields present a more natural framework. The stress-energy tensor derived below will therefore be the HIM boundary stress-energy tensor [75], which we will denote $S^{\underline{a}}{}_a$, in contrast to the Brown-York boundary stress-energy tensor, $T_{ab}$, which results from varying the inverse boundary metric. Thus, we vary the action with respect to the inverse frame field $e^a_{\underline{a}}$, defined via

$$h^{ab} = \eta^{\underline{ab}}e^a_{\underline{a}}e^b_{\underline{b}} \,. \qquad (6.143)$$

This implies the following relation between the Brown-York tensor and the HIM tensor

$$T_{ab}\delta h^{ab} + 2T^a\delta A_a = 2S^{\underline{a}}{}_a\delta e^a_{\underline{a}} + 2T^{\underline{a}}\delta A_{\underline{a}} \,, \qquad (6.144)$$

where $A_{\underline{a}} = A_a e^a_{\underline{a}}$ and

$$S^{\underline{a}}{}_a = \left(T_{ab} - T_b A_a\right)e^{b\underline{a}} \,. \qquad (6.145)$$

The relations (6.138)–(6.142) are quite unwieldy. Fortunately, we can relate the 4d expressions for $T_{ab}$, $T^a$ and $T_\Phi$ to the 5d ones given in (5.22)–(5.24) by dimensionally reducing (5.21). In this way we avoid using the large expressions (6.138)–(6.142). We write

$$\sqrt{-\hat{h}}\hat{T}_{\hat{a}\hat{b}}\delta\hat{h}^{\hat{a}\hat{b}} = \sqrt{-h}\left(T_{ab}\delta h^{ab} + 2T^a\delta A_a + 2T_\Phi\delta\Phi\right) \,. \qquad (6.146)$$

Comparing terms on either side we obtain

$$\begin{aligned} T_{ab} &= (\hat{h}_{uu})^{-7/4}\left[(\hat{h}_{uu})^2\hat{T}_{ab} - \hat{h}_{uu}\hat{h}_{au}\hat{T}_{bu} - \hat{h}_{uu}\hat{h}_{bu}\hat{T}_{au} + \hat{h}_{au}\hat{h}_{bu}\hat{T}_{uu}\right] \,, \qquad (6.147) \\ T^a &= -(\hat{h}_{uu})^{-1/4}\hat{h}^{a\hat{b}}\hat{T}_{\hat{b}u} \,, \qquad (6.148) \\ T_\Phi &= \frac{1}{2}(\hat{h}_{uu})^{-1/4}\hat{h}^{\hat{a}\hat{b}}\hat{T}_{\hat{a}\hat{b}} - \frac{3}{2}(\hat{h}_{uu})^{-5/4}\hat{T}_{uu} \,. \qquad (6.149) \end{aligned}$$

According to the definition (6.145) it is a certain linear combination of the 4d quantities which makes up the HIM stress-energy tensor. In terms of 5d quantities these are

$$\begin{aligned} T_a &= (\hat{h}_{uu})^{-3/4}\left(\hat{h}_{au}\hat{T}_{uu} - \hat{h}_{uu}\hat{T}_{au}\right) , & (6.150)\\ T_{ab} - T_a A_b &= (\hat{h}_{uu})^{-3/4}\left(\hat{h}_{uu}\hat{T}_{ab} - \hat{h}_{au}\hat{T}_{bu}\right) . & (6.151)\end{aligned}$$

The expressions for $T_\phi$ and $T_\chi$ reduce trivially to 4d

$$\begin{aligned} T_\phi &= (\hat{h}_{uu})^{-1/4}\hat{T}_{\hat{\phi}} , & (6.152)\\ T_\chi &= (\hat{h}_{uu})^{-1/4}\hat{T}_{\hat{\chi}} . & (6.153)\end{aligned}$$

In the variation of the 5d axion $\delta\hat{\chi} = \delta\chi + k\delta u$, where the right hand side follows from the Scherk–Schwarz reduction ansatz, we have absorbed the gauge transformation $k\delta u$ into the variation of the 4d axion.

The expressions derived above enables us to identify the order at which the vevs appear. To this end we write the variation of the action using the sources defined in (6.46)–(6.50) as

$$\begin{aligned} \delta S^{\text{on-shell}}_{\text{ren}} &= -\frac{2\pi L}{\kappa_5^2}\int_{\partial\mathcal{M}} d^3x e\left(S^{\underline{t}}_{\underline{a}}\delta e^a_{\underline{t}} + S^{\underline{i}}_{\underline{a}}\delta e^a_{\underline{i}} + T^{\underline{i}}\delta A_{\underline{i}}\right.\\ &\qquad \left. + T_\varphi\delta\varphi + T_\psi\delta\psi + T_\phi\delta\phi + T_\chi\delta\chi - e^{-\Phi/2}\mathcal{A}\frac{\delta r}{r}\right) . & (6.154)\end{aligned}$$

The field redefinitions, of which (6.46)–(6.47) are the leading components, are given by[3]

$$\begin{aligned} \varphi &= A_{\underline{t}} - e^{-3\Phi/2} = r^2 A_{(0)\underline{t}} + \dots , & (6.155)\\ \psi &= A_{\underline{t}} + e^{-3\Phi/2} = 2e^{-3\Phi_{(0)}/2} + \dots , & (6.156)\end{aligned}$$

and they source the objects

$$\begin{aligned} T_\varphi &= \frac{1}{2}T^{\underline{t}} + \frac{1}{3}e^{3\Phi/2}T_\Phi , & (6.157)\\ T_\psi &= \frac{1}{2}T^{\underline{t}} - \frac{1}{3}e^{3\Phi/2}T_\Phi . & (6.158)\end{aligned}$$

To investigate the $r$-dependence we note that the metric determinant is

$$e = \sqrt{-h} = r^{-4}e_{(0)} + \cdots . \qquad (6.159)$$

Using this fact in (6.154) along with equations (6.145), (6.147)–(6.153), (6.157), (6.158) and the results of section 5.1 we have

$$\begin{aligned} S^{\underline{t}}_{\underline{a}} &= r^2 S^{\underline{t}}_{(0)a} + \cdots , & (6.160)\\ S^{\underline{i}}_{\underline{a}} &= r^3 S^{\underline{i}}_{(0)a} + \cdots , & (6.161)\\ T^{\underline{i}} &= r^3 T^{\underline{i}}_{(0)} + \cdots , & (6.162)\\ T_\varphi &= r^2 T^{\underline{t}}_{(0)} + \cdots , & (6.163)\\ T_\psi &= -\frac{1}{3}r^4 e^{3\Phi_{(0)}/2}\langle\mathcal{O}_\Phi\rangle + \cdots , & (6.164)\\ T_\phi &= r^4\langle\mathcal{O}_\phi\rangle + \cdots , & (6.165)\\ T_\chi &= r^4\langle\mathcal{O}_\chi\rangle + \cdots . & (6.166)\end{aligned}$$

[3]Note that $A_{(0)\underline{t}}$ is not the leading component of $A_{\underline{t}} = A_a e^a_{\underline{t}}$.

Inserting these expansions in the variation of the on-shell action and going to the boundary, the relation (6.154) reads

$$
\begin{aligned}
\delta S^{\text{on-shell}}_{\text{ren}} &= -\frac{2\pi L}{\kappa_5^2}\int_{\partial\mathcal{M}} d^3x e_{(0)}\Big(-S^{\underline{t}}_{(0)a}\delta\tau^a_{(0)} + S^{\underline{i}}_{(0)a}\delta e^a_{(0)\underline{i}} + T^{\underline{t}}_{(0)}\delta A_{(0)\underline{t}} + T^{\underline{i}}_{(0)}\delta A_{(0)\underline{i}} \\
&\quad + \langle\mathcal{O}_\Phi\rangle\delta\Phi_{(0)} + \langle\mathcal{O}_\phi\rangle\delta\phi_{(0)} + \langle\mathcal{O}_\chi\rangle\delta\chi_{(0)} - e^{-\Phi_{(0)}/2}\mathcal{A}_{(0)}\frac{\delta r}{r}\Big)\,. \qquad (6.167)
\end{aligned}
$$

It is now clear that the stress-energy tensor of the boundary theory is really composed of two objects $S^{\underline{t}}_{(0)a}$ and $S^{\underline{i}}_{(0)a}$, sourced by $\tau^a_{(0)}$ respectively $e^a_{(0)\underline{i}}$. Furthermore, these expressions start at different orders, a sign of the anisotropy between space and time present in Lifshitz theories. This anisotropy will also appear in the Ward identities below.

### 6.3.2 Ward identities

The Ward identities are found by considering the local symmetries of the 5d bulk theory. Demanding that the local symmetries preserve the gauge means that the transformations to be considered are the PBH-transformations. For the 5d frame fields these are

$$
\begin{aligned}
\delta\hat{e}^u_{(0)+} &= 2\hat{\xi}^r_{(0)}\hat{e}^u_{(0)+} + \hat{\xi}^a_{(0)}\partial_a\hat{e}^u_{(0)+} - \hat{e}^a_{(0)+}\partial_a\hat{\xi}^u_{(0)}\,, && (6.168)\\
\delta\hat{e}^a_{(0)-} = \delta\hat{e}^a_{(0)+} &= 2\hat{\xi}^r_{(0)}\hat{e}^a_{(0)+} + \hat{\xi}^b_{(0)}\partial_b\hat{e}^a_{(0)+} - \hat{e}^b_{(0)+}\partial_b\hat{\xi}^a_{(0)}\,, && (6.169)\\
\delta\hat{e}^u_{(0)-} &= \hat{\xi}^a_{(0)}\partial_a\hat{e}^u_{(0)-}\,, && (6.170)\\
\delta\hat{e}^u_{(0)\underline{i}} &= \hat{\xi}^r_{(0)}\hat{e}^u_{(0)\underline{i}} + \hat{\xi}^a_{(0)}\partial_a\hat{e}^u_{(0)\underline{i}} - \hat{e}^a_{(0)\underline{i}}\partial_a\hat{\xi}^u_{(0)}\,, && (6.171)\\
\delta\hat{e}^a_{(0)\underline{i}} &= \hat{\xi}^r_{(0)}\hat{e}^a_{(0)\underline{i}} + \hat{\xi}^b_{(0)}\partial_b\hat{e}^a_{(0)\underline{i}} - \hat{e}^b_{(0)\underline{i}}\partial_b\hat{\xi}^a_{(0)}\,. && (6.172)
\end{aligned}
$$

Using the map between the 4d and 5d frame fields given in section 6.1.3 and the equations above, the PBH-transformations of the 4d frame fields are

$$
\begin{aligned}
\delta\tau^a_{(0)} &= 2\hat{\xi}^r_{(0)}\tau^a_{(0)} + \hat{\xi}^b_{(0)}\partial_b\tau^a_{(0)} - \tau^b_{(0)}\partial_b\hat{\xi}^a_{(0)}\,, && (6.173)\\
\delta e^a_{(0)\underline{i}} &= \hat{\xi}^r_{(0)}e^a_{(0)\underline{i}} + \hat{\xi}^b_{(0)}\partial_b e^a_{(0)\underline{i}} - e^b_{(0)\underline{i}}\partial_b\hat{\xi}^a_{(0)}\,, && (6.174)\\
\delta A_{(0)\underline{t}} &= 2\hat{\xi}^r_{(0)}A_{(0)\underline{t}} + \hat{\xi}^a_{(0)}\partial_a A_{(0)\underline{t}} - \tau^a_{(0)}\partial_a\hat{\xi}^u_{(0)}\,, && (6.175)\\
\delta A_{(0)\underline{i}} &= \hat{\xi}^r_{(0)}A_{(0)\underline{i}} + \hat{\xi}^a_{(0)}\partial_a A_{(0)\underline{i}} + e^a_{(0)\underline{i}}\partial_a\hat{\xi}^u_{(0)}\,, && (6.176)\\
\delta\Phi_{(0)} &= \hat{\xi}^a_{(0)}\partial_a\Phi_{(0)}\,, && (6.177)\\
\delta\phi_{(0)} &= \hat{\xi}^a_{(0)}\partial_a\phi_{(0)}\,, && (6.178)\\
\delta\chi_{(0)} &= \hat{\xi}^a_{(0)}\partial_a\chi_{(0)} + k\hat{\xi}^u_{(0)}\,. && (6.179)
\end{aligned}
$$

Using these expressions in (6.167) results in three 4d Ward identities, scaling, $\hat{\xi}^r_{(0)}$, 3d diffeomorphisms, $\hat{\xi}^a_{(0)}$ and gauge transformations, $\hat{\xi}^u_{(0)}$. These read

$$
\begin{aligned}
0 &= 2S^{\underline{t}}_{(0)\underline{t}} + 2T^{\underline{t}}_{(0)}A_{(0)\underline{t}} + S^{\underline{i}}_{(0)\underline{i}} + T^{\underline{i}}_{(0)}A_{(0)\underline{i}} - e^{-\Phi_{(0)}/2}\mathcal{A}_{(0)} && (6.180)\\
0 &= -\frac{1}{e_{(0)}}\partial_a\left(e_{(0)}T^a_{(0)}\right) + k\left\langle\mathcal{O}_\chi\right\rangle && (6.181)\\
0 &= -S^{\underline{t}}_{(0)b}\partial_a\tau^b_{(0)} + S^{\underline{i}}_{(0)b}\partial_a e^b_{(0)\underline{i}} + \frac{1}{e_{(0)}}\partial_b\left(e_{(0)}S^b_{(0)a}\right) + T^{\underline{t}}_{(0)}\partial_a A_{(0)\underline{t}} \\
&\quad + T^{\underline{i}}_{(0)}\partial_a A_{(0)\underline{i}} + \left\langle\mathcal{O}_\Phi\right\rangle\partial_a\Phi_{(0)} + \langle\mathcal{O}_\phi\rangle\partial_a\phi_{(0)}\left\langle\mathcal{O}_\chi\right\rangle\partial_a\chi_{(0)}\,, && (6.182)
\end{aligned}
$$

where $T^a_{(0)} = -T^{\underline{t}}_{(0)}\tau^a_{(0)} + T^{\underline{i}}_{(0)}e^a_{(0)\underline{i}}$ and $S^b_{(0)a} = -S^{\underline{t}}_{(0)a}\tau^b_{(0)} + S^{\underline{i}}_{(0)a}e^b_{(0)\underline{i}}$. Comparing the trace Ward identity (6.180) with the 5d one (5.37) the consequence of the anisotropic scaling between time and space in Lifshitz spacetimes is apparent. The 2 in the first two terms in (6.180) is exactly the dynamic critical exponent $z = 2$ of our model. From the equations (6.160)–(6.166) we see that we have 15 vevs. The Ward identities above reduce this number by five, thus resulting in 10 independent vevs. However, from our discussion in section 6.1.3 we expect there to be only 6 independent vevs, indicating that we are missing some Ward identities.

To solve this apparent contradiction we consider the 4d vevs written in terms of the 5d vevs. This will also allow us to write the expressions (6.181) and (6.182) in a covariant and gauge invariant manner and make contact with the stress-energy tensor complex defined in [74]. We have

$$
\begin{aligned}
S^{\underline{t}}_{(0)a} &= -\frac{1}{\sqrt{2}}\hat{e}^{u}_{(0)-}\hat{t}_{au} = e^{-\Phi_{(0)}}\hat{t}_{au}\,, && (6.183)\\
S^{\underline{i}}_{(0)a} &= \hat{e}^{b\underline{i}}_{(0)}\hat{t}_{ab} + \hat{e}^{u\underline{i}}_{(0)}\hat{t}_{au} = e^{\Phi_{(0)}/2}e^{\underline{i}\,b}_{(0)}\hat{t}_{ab} - e^{\Phi_{(0)}/2}A^{\underline{i}}_{(0)}\hat{t}_{au}\,, && (6.184)\\
T^{\underline{t}}_{(0)} &= \frac{1}{\sqrt{2}}\hat{e}^{u}_{(0)-}\hat{t}_{uu} = -e^{-\Phi_{(0)}}\hat{t}_{uu}\,, && (6.185)\\
T^{\underline{i}}_{(0)} &= -\hat{e}^{u\underline{i}}_{(0)}\hat{t}_{uu} - \hat{e}^{a\underline{i}}_{(0)}\hat{t}_{au} = e^{\Phi_{(0)}/2}A^{\underline{i}}_{(0)}\hat{t}_{uu} - e^{\Phi_{(0)}/2}e^{\underline{i}\,a}_{(0)}\hat{t}_{au}\,, && (6.186)\\
-\frac{1}{3}e^{3\Phi_{(0)}/2}\langle\mathcal{O}_\Phi\rangle &= -\frac{\sqrt{2}}{6}\hat{e}^{+}_{(0)u}\hat{t}^{\hat{a}}{}_{\hat{a}} + \frac{1}{\sqrt{2}}\hat{e}^{u}_{(0)+}\hat{t}_{uu} + \frac{1}{\sqrt{2}}\hat{e}^{a}_{(0)+}\hat{t}_{au}\\
&= -\frac{1}{6}e^{\Phi_{(0)}}\mathcal{A}_{(0)} - \frac{1}{2}e^{\Phi_{(0)}/2}A_{(0)\underline{t}}\hat{t}_{uu} - \frac{1}{2}e^{\Phi_{(0)}/2}\tau^a_{(0)}\hat{t}_{au}\,. && (6.187)
\end{aligned}
$$

From (6.183)–(6.185) it follows that

$$
\begin{aligned}
\hat{t}_{au} &= e^{\Phi_{(0)}}S^{\underline{t}}_{(0)a}\,, && (6.188)\\
e^{\underline{i}\,b}_{(0)}\hat{t}_{ab} &= e^{-\Phi_{(0)}/2}S^{\underline{i}}_{(0)a} + e^{\Phi_{(0)}}A^{\underline{i}}_{(0)}S^{\underline{t}}_{(0)a}\,, && (6.189)\\
\hat{t}_{uu} &= -e^{\Phi_{(0)}}T^{\underline{t}}_{(0)}\,. && (6.190)
\end{aligned}
$$

Substituting these relations in (6.186) and (6.187) we obtain

$$
\begin{aligned}
0 &= A^{\underline{i}}_{(0)}T^{\underline{t}}_{(0)} + e^{\underline{i}\,a}_{(0)}S^{\underline{t}}_{(0)a} + \mathrm{e}^{-3\Phi_{(0)}/2}\,T^{\underline{i}}_{(0)}\,, && (6.191)\\
0 &= S^{\underline{t}}_{(0)\underline{t}} + S^{\underline{i}}_{(0)\underline{i}} + A_{(0)\underline{t}}T^{\underline{t}}_{(0)} - A_{(0)\underline{i}}T^{\underline{i}}_{(0)} + 2\langle\mathcal{O}_\Phi\rangle\,, && (6.192)
\end{aligned}
$$

where we used (6.180) to remove $\mathcal{A}_{(0)}$ from (6.187). Further by contracting (6.189) with $e^{\underline{j}\,a}_{(0)}$ and antisymmetrizing in $(\underline{i},\underline{j})$ we obtain the relation

$$
0 = e^{-3\Phi_{(0)}/2}e^{\underline{i}\,a}_{(0)}S^{\underline{j}}_{(0)a} + e^{\underline{i}\,a}_{(0)}A^{\underline{j}}_{(0)}S^{\underline{t}}_{(0)a} - (\underline{i}\leftrightarrow\underline{j})\,. \qquad (6.193)
$$

The result is three additional Ward identities (6.191)–(6.193) not found by considering the PBH-transformations above. To see where these relations come from consider the inverse boundary metric $\hat{h}^{\hat{a}\hat{b}}_{(0)}$ written in terms of the 4D sources, equations (6.61)–(6.63). We now look for transformations of the sources that leave these expressions invariant. The first such transformation is

$$\begin{aligned}
\delta\tau^a_{(0)} &= \omega^{\underline{i}}_{(0)} e^a_{(0)\underline{i}}\,, &(6.194)\\
\delta A_{(0)\underline{i}} &= -\omega_{(0)\underline{i}} e^{-3\Phi_{(0)}/2}\,, &(6.195)\\
\delta A_{(0)\underline{t}} &= -\omega^{\underline{i}}_{(0)} A_{(0)\underline{i}}\,. &(6.196)
\end{aligned}$$

Using (6.167) the associated Ward identity is (6.191). The next symmetry leaving $\hat{h}^{\hat{a}\hat{b}}_{(0)}$ invariant is given by

$$\begin{aligned}
\delta e^a_{(0)\underline{i}} &= -\omega^{\underline{j}}_{(0)\underline{i}} e^a_{(0)\underline{j}}\,, &(6.197)\\
\delta A_{(0)\underline{i}} &= -\omega^{\underline{j}}_{(0)\underline{i}} A_{(0)\underline{j}}\,, &(6.198)
\end{aligned}$$

with $\omega_{(0)\underline{i}\underline{j}} = -\omega_{(0)\underline{j}\underline{i}}$ giving rise to the Ward identity (6.193). In section 6.2.1 we have shown that the transformations with parameters $\omega^{\underline{i}}_{(0)}$ and $\omega^{\underline{j}}_{(0)\underline{i}}$ are induced by bulk local $SO(1,2)$ Lorentz transformations with $\omega_{(0)}{}^{\underline{j}}{}_{\underline{i}} = \xi^b_{(0)}\omega_{(0)b}{}^{\underline{j}}{}_{\underline{i}}$ and $\omega^{\underline{i}}_{(0)} = \xi^b_{(0)}\omega_{(0)b}{}^{\underline{i}}{}_{\underline{t}}$ acting on the frame fields $e^{\underline{a}}_a$ and therefore correspond to local symmetries of the full frame field decomposition of the metric. There is one more local transformation leaving $\hat{h}^{\hat{a}\hat{b}}_{(0)}$ invariant which is given by

$$\begin{aligned}
\delta\tau^a_{(0)} &= -\omega_{(0)}\tau^a_{(0)}\,, &(6.199)\\
\delta e^a_{(0)\underline{i}} &= \omega_{(0)} e^a_{(0)\underline{i}}\,, &(6.200)\\
\delta A_{(0)\underline{t}} &= -\omega_{(0)} A_{(0)\underline{t}}\,, &(6.201)\\
\delta A_{(0)\underline{i}} &= \omega_{(0)} A_{(0)\underline{i}}\,, &(6.202)\\
\delta\Phi_{(0)} &= -2\omega_{(0)}\,, &(6.203)
\end{aligned}$$

leading to the relation (6.192). Different from the other two local symmetries, this symmetry is only there at leading order. It leaves $\hat{h}_{(0)\hat{a}\hat{b}}$ invariant but not for example $\hat{h}_{(2)uu}$ which is given in (6.24). The relations (6.191)–(6.193) constitute an additional three Ward identities reducing the number of independent vevs by an additional four. Recall that the first set of Ward identities reduced the number of independet vevs to 10. With these extra Ward identities the number of independent vevs is reduced to 6, thus matching the number of independent sources.

To write the Ward identities in a covariant and gauge invariant manner we can use the expressions (6.183) and (6.184) to derive the transformation rules under the $\omega^{\underline{i}}_{(0)}$, $\omega^{\underline{j}}_{(0)\underline{i}}$ and $\omega_{(0)}$ transformations

$$\begin{aligned}
\delta S^{\underline{t}}_{(0)a} &= 2\omega_{(0)} S^{\underline{t}}_{(0)a}\,, &(6.204)\\
\delta S^{\underline{i}}_{(0)a} &= \omega^{\underline{i}}_{(0)} S^{\underline{t}}_{(0)a} - \omega_{(0)\underline{j}}{}^{\underline{i}} S^{\underline{j}}_{(0)a}\,. &(6.205)
\end{aligned}$$

Using the above transformations of $\tau^a_{(0)}$ and $e^a_{(0)\underline{i}}$ we find that the quantity $S^b_{(0)a} = -S^{\underline{t}}_{(0)a}\tau^b_{(0)} + S^{\underline{i}}_{(0)a} e^b_{(0)\underline{i}}$ appearing prominently in (6.182) transforms as

$$\delta S^b_{(0)a} = \omega_{(0)} S^b_{(0)a}\,. \tag{6.206}$$

We conclude that $S^b_{(0)a}$ is invariant under the local tangent space rotations and scales under the $\omega_{(0)}$ transformation. This fact, along with the vielbein postulate of section 6.2.2 (eq. (6.102)) and our choice of $\Gamma^c_{(0)ab}$, allows us to write the Ward identities (6.181) and (6.182) as

$$\begin{aligned} k\left\langle\mathcal{O}_\chi\right\rangle &= \nabla_{(0)a}T^a_{(0)} - \frac{1}{2}\tau^b_{(0)}\left(\partial_b\tau_{(0)a}-\partial_a\tau_{(0)b}\right)T^a_{(0)}\,, \qquad (6.207)\\ 0 &= \nabla_{(0)b}S^b_{(0)a} + \frac{1}{2}S^b_{(0)a}\tau^c_{(0)}\left(\partial_c\tau_{(0)b}-\partial_b\tau_{(0)c}\right) - S^{\underline{t}}_{(0)b}\nabla_{(0)a}\tau^b_{(0)} + S^{\underline{i}}_{(0)b}\nabla_{(0)a}e^b_{(0)\underline{i}}\\ &\quad + T^{\underline{t}}_{(0)}\partial_a A_{(0)\underline{t}} + T^{\underline{i}}_{(0)}\partial_a A_{(0)\underline{i}} + \left\langle\mathcal{O}_\Phi\right\rangle\partial_a\Phi_{(0)}\\ &\quad + \langle\mathcal{O}_\phi\rangle\partial_a\phi_{(0)} + \left\langle\mathcal{O}_\chi\right\rangle\partial_a\chi_{(0)}\,. \qquad (6.208)\end{aligned}$$

To figure out how the object $S^b_{(0)a}$ transforms under gauge transformation, we consider how the sources and vevs transform under these. To work out the gauge transformations of the sources we use the action of the PBH-transformations given in (6.173) and (6.176) with $\hat\xi^{\hat a}_{(0)} = \delta^{\hat a}_u\hat\xi^u_{(0)}$ and $\hat\xi^r_{(0)} = 0$. To find the action of gauge transformations on the vevs induced by the PBH-transformations we use that

$$\delta\hat t_{\hat a\hat b} = \hat\xi^{\hat c}_{(0)}\partial_{\hat c}\hat t_{\hat a\hat b} + \hat t_{\hat c\hat b}\partial_{\hat a}\hat\xi^{\hat c}_{(0)} + \hat t_{\hat a\hat c}\partial_{\hat b}\hat\xi^{\hat c}_{(0)} + \delta_{\hat\xi^r_{(0)}}\hat t_{\hat a\hat b}\,. \qquad (6.209)$$

Taking $\hat\xi^{\hat a}_{(0)} = \delta^{\hat a}_u\hat\xi^u_{(0)}$ and $\hat\xi^r_{(0)} = 0$ and using (6.183)–(6.187) we obtain the following gauge transformations of the vevs

$$\begin{aligned} \delta S^{\underline{t}}_{(0)a} &= -T^{\underline{t}}_{(0)}\partial_a\hat\xi^u_{(0)}\,, \qquad (6.210)\\ \delta S^{\underline{i}}_{(0)a} &= e^{3\Phi_{(0)}/2}\left(e^{\underline{i}\,b}_{(0)}S^{\underline{t}}_{(0)b} + A^{\underline{i}}_{(0)}T^{\underline{t}}_{(0)}\right)\partial_a\hat\xi^u_{(0)} = -T^{\underline{i}}_{(0)}\partial_a\hat\xi^u_{(0)}\,, \qquad (6.211)\\ \delta T^{\underline{i}}_{(0)} &= 0\,, \qquad (6.212)\\ \delta T^{\underline{t}}_{(0)} &= 0\,, \qquad (6.213)\\ \delta\left\langle O_\Phi\right\rangle &= 0\,, \qquad (6.214)\\ \delta\left\langle O_\chi\right\rangle &= 0\,, \qquad (6.215)\\ \delta\left\langle O_\phi\right\rangle &= 0\,, \qquad (6.216)\end{aligned}$$

where we used (6.191) in (6.211). Using these transformations one can show

$$\delta S^b_{(0)a} = -T^b_{(0)}\partial_a\hat\xi^u_{(0)}\,, \qquad (6.217)$$

so that $S^b_{(0)a}$ is not gauge invariant. This motivates defining an object $\mathcal{T}^b_{(0)a}$ as

$$\mathcal{T}^b_{(0)a} = S^b_{(0)a} + T^b_{(0)}\frac{1}{k}\partial_a\chi_{(0)}\,. \qquad (6.218)$$

This object is both gauge invariant as well as invariant under local tangent space transformations. It is the object referred to by Ross [74] as the stress-energy tensor complex.

We are now in a position to express the Ward identities (6.180)–(6.182) and (6.191)–

(6.193) in a more natural way. Using the gauge invariant vevs we find

$$\begin{aligned}
e^{-\Phi_{(0)}/2}\mathcal{A}_{(0)} &= 2\mathcal{T}^{\underline{t}}_{(0)\underline{t}} + \mathcal{T}^{\underline{i}}_{(0)\underline{i}} + 2B_{(0)\underline{t}}T^{\underline{t}}_{(0)} + B_{(0)\underline{i}}T^{\underline{i}}_{(0)}\,, &(6.219)\\
k\left<O_\chi\right> &= \nabla_{(0)a}T^a_{(0)} - \frac{1}{2}\tau^b_{(0)}\left(\partial_b\tau_{(0)a} - \partial_a\tau_{(0)b}\right)T^a_{(0)}\,, &(6.220)\\
\nabla_{(0)b}\mathcal{T}^b_{(0)a} &= -\mathcal{T}^c_{(0)b}\left(-\tau_{(0)c}\nabla_{(0)a}\tau^b_{(0)} + e^{\underline{i}}_{(0)c}\nabla_{(0)a}e^b_{(0)\underline{i}}\right)\\
&\quad +\frac{1}{2}\mathcal{T}^b_{(0)a}\tau^c_{(0)}\left(\partial_c\tau_{(0)b} - \partial_b\tau_{(0)c}\right) - T^{\underline{t}}_{(0)}\partial_a B_{(0)\underline{t}} - T^{\underline{i}}_{(0)}\partial_a B_{(0)\underline{i}}\\
&\quad -\left<O_\Phi\right>\partial_a\Phi_{(0)} - \langle O_\phi\rangle\partial_a\phi_{(0)}\,, &(6.221)\\
\mathcal{T}^{\underline{t}\underline{i}}_{(0)} &= -e^{-3\Phi_{(0)}/2}T^{\underline{i}}_{(0)} - B^{\underline{i}}_{(0)}T^{\underline{t}}_{(0)}\,, &(6.222)\\
0 &= \mathcal{T}^{\underline{i}\underline{j}}_{(0)} - B^{\underline{i}}_{(0)}T^{\underline{j}}_{(0)} - (\underline{i}\leftrightarrow\underline{j})\,, &(6.223)\\
-2\left<O_\Phi\right> &= \mathcal{T}^{\underline{t}}_{(0)\underline{t}} - \mathcal{T}^{\underline{i}}_{(0)\underline{i}} + B_{(0)\underline{t}}T^{\underline{t}}_{(0)} - B_{(0)\underline{i}}T^{\underline{i}}_{(0)}\,, &(6.224)
\end{aligned}$$

where we wrote

$$\begin{aligned}
B_{(0)\underline{t}} &= A_{(0)\underline{t}} + \frac{1}{k}\tau^a_{(0)}\partial_a\chi_{(0)}\,, &(6.225)\\
B_{(0)\underline{i}} &= A_{(0)\underline{i}} - \frac{1}{k}e^a_{(0)\underline{i}}\partial_a\chi_{(0)}\,. &(6.226)
\end{aligned}$$

Using the torsional covariant derivatives defined in section 6.2.5 we find that the Ward identities (6.220) and (6.221) can be written as

$$\begin{aligned}
&k\left<\mathcal{O}_\chi\right> = \nabla^T_{(0)a}T^a_{(0)}\,, &(6.227)\\
&\nabla^T_{(0)b}\mathcal{T}^b_{(0)a} = -\mathcal{T}^c_{(0)b}\left(-\tau_{(0)c}\nabla^T_{(0)a}\tau^b_{(0)} + e^{\underline{i}}_{(0)c}\nabla^T_{(0)a}e^b_{(0)\underline{i}}\right) + 2\mathcal{T}^c_{(0)b}T^b_{(0)ac}\\
&\qquad - T^{\underline{t}}_{(0)}\partial_a B_{(0)\underline{t}} - T^{\underline{i}}_{(0)}\partial_a B_{(0)\underline{i}} - \left<\mathcal{O}_\Phi\right>\partial_a\Phi_{(0)} - \langle\mathcal{O}\rangle\partial_a\phi_{(0)}\,. &(6.228)
\end{aligned}$$

The form of the diffeomorphism Ward identity (6.228) is similar to the one given in [75] (see equation (3.17)) except that here the vielbeins do not transform under the Lorentz group and we have a torsion term $T^a_{(0)bc}$.

### 6.3.3 An extra free function

Information about the 5-dimensional stress-energy tensor is contained in the expressions (6.188)–(6.190). However, there is a problem since the expression for $\hat{t}_{ab}$ is contracted with $e^{ib}_{(0)}$. To untangle the expression we contract with $e_{(0)\underline{i}c}$ and use the completeness relation (6.105). This results in

$$\hat{t}_{ab} = e^{-\Phi_{(0)}/2}e_{(0)\underline{i}a}S^{\underline{i}}_{(0)b} + e^{\Phi_{(0)}}e_{(0)\underline{i}a}A^{\underline{i}}_{(0)}S^{\underline{t}}_{(0)b} - \tau_{(0)a}\tau^c_{(0)}\hat{t}_{bc}\,. \tag{6.229}$$

Using this expression to replace $\hat{t}_{bc}$ on the right-hand-side yields

$$\begin{aligned}
\hat{t}_{ab} &= \tau_{(0)a}\tau_{(0)b}\tau^c_{(0)}\tau^d_{(0)}\hat{t}_{cd} + e^{-\Phi_{(0)}/2}e_{(0)\underline{i}a}S^{\underline{i}}_{(0)b} + e^{\Phi_{(0)}}e_{(0)\underline{i}a}A^{\underline{i}}_{(0)}S^{\underline{t}}_{(0)b}\\
&\quad - \tau_{(0)a}\tau^c_{(0)}\left(e^{-\Phi_{(0)}/2}e_{(0)\underline{i}b}S^{\underline{i}}_{(0)c} + e^{\Phi_{(0)}}e_{(0)\underline{i}b}A^{\underline{i}}_{(0)}S^{\underline{t}}_{(0)c}\right)\,. &(6.230)
\end{aligned}$$

The quantity $\tau^c_{(0)}\tau^d_{(0)}\hat{t}_{cd}$ appearing here is not related to any source and is therefore not a vev. Nor does it appear in any of the Ward identities. It is a completely free function.

Furthermore, when writing down a Fefferman-Graham expansion for the 4-dimensional fields the object $\hat{t}_{ab}$ appears at order $r^2$. The Fefferman-Graham expansion therefore contains 6 sources with 6 corresponding vevs and one free function, given by $\tau^c_{(0)}\tau^d_{(0)}\hat{t}_{cd}$.

The fact that the function decouples in the Ward identities can be seen from the demand that $\hat{h}_{(0)uu} = 0$. This condition implies that the vev $\hat{t}^{uu}$ decouples completely from the 5d Ward identities (5.37) and (5.36). We have

$$\hat{t}^{\hat{a}}{}_{\hat{b}} = \hat{h}_{(0)\hat{b}\hat{c}}\hat{t}^{\hat{a}\hat{c}}\,, \tag{6.231}$$

$$\hat{t}^{\hat{a}}{}_{\hat{a}} = \hat{h}_{(0)\hat{a}\hat{b}}\hat{t}^{\hat{a}\hat{b}}\,, \tag{6.232}$$

and imposing $\hat{h}_{(0)uu}$ means that any reference to $\hat{t}^{uu}$ disappears. This is expected from the general structure of Ward identities which is that sources are multiplying the partial derivatives of vevs and vice versa. Thus, turning off $\hat{h}_{(0)uu}$ corresponds to turning off $\hat{t}^{uu}$. Now writing $\hat{t}^{uu}$ as

$$\begin{aligned}\hat{t}^{uu} &= \hat{h}^{u\hat{a}}_{(0)}\hat{h}^{u\hat{b}}_{(0)}\hat{t}_{\hat{a}\hat{b}} &(6.233)\\ &= \hat{h}^{uu}_{(0)}\hat{h}^{uu}_{(0)}\hat{t}_{uu} + 2\hat{h}^{au}_{(0)}\hat{h}^{uu}_{(0)}\hat{t}_{au} + \hat{h}^{au}_{(0)}\hat{h}^{bu}_{(0)}\hat{t}_{ab}\,, &(6.234)\end{aligned}$$

and using the map (6.61)–(6.63) we see that in $\hat{h}^{au}_{(0)}\hat{h}^{bu}_{(0)}\hat{t}_{ab}$ there will appear a term of the form $\tau^a_{(0)}\tau^b_{(0)}\hat{t}_{ab}$ which is exactly the free function we are looking for. Furthermore, seeing as this object will only appear through a term of the form $\hat{h}^{au}_{(0)}\hat{h}^{bu}_{(0)}\hat{t}_{ab}$, it will only appear in $\hat{t}^{uu}$ and not in any of the other components of $\hat{t}^{\hat{a}\hat{b}}$. Since $\hat{t}^{uu}$ decouples from the Ward identity this function will make no appearence. However, it is not clear how this extra function should be interpreted physically.

### 6.3.4 Constraint on the sources

Due to the relation (5.106) there is a constraint on the sources of the Lifshitz theory. This results in some ambiguity when defining what exactly is meant by a vev of the boundary theory. Here we briefly explore the consequences of the constraint coming from (5.106) and explain what is meant by a vev.

Recall that the $uu$-component of the Ricci tensor is

$$\hat{R}_{(0)uu} = \frac{1}{2}e^{3\Phi_{(0)}}\left(\varepsilon^{abc}_{(0)}\tau_{(0)a}\partial_b\tau_{(0)c}\right)^2\,, \tag{6.235}$$

and the constraint (6.24) can be written as

$$1 = -\frac{1}{4}e^{\Phi_{(0)}}\left(\varepsilon^{abc}_{(0)}\tau_{(0)a}\partial_b\tau_{(0)c}\right)^2 + \frac{k^2}{4}e^{2(\phi_{(0)}-\Phi_{(0)})}\,. \tag{6.236}$$

From a geometric point of view this constraint arises due to the fact that we are reducing over a null circle on the boundary. Performing a variation we obtain

$$\begin{aligned}0 \;=\;& \frac{k^2}{2}e^{2(\phi_{(0)}-\Phi_{(0)})}\delta\phi_{(0)} + \left(1-\frac{3k^2}{4}e^{2(\phi_{(0)}-\Phi_{(0)})}\right)\delta\Phi_{(0)} + \Bigg[2e^{\underline{i}}_{(0)a} - \frac{k^2}{2}e^{2(\phi_{(0)}-\Phi_{(0)})}e^{\underline{i}}_{(0)a}\\ &+\frac{1}{2}e^{\Phi_{(0)}}\left(\varepsilon^{bcd}_{(0)}\tau_{(0)b}\partial_c\tau_{(0)d}\right)\left(\varepsilon^{efg}_{(0)}e^{\underline{i}}_{(0)e}\partial_f\tau_{(0)g}\right)\tau_{(0)a}\Bigg]\delta e^a_{(0)\underline{i}}\\ &-\frac{1}{2}e^{\Phi_{(0)}}\left(\varepsilon^{def}_{(0)}\tau_{(0)d}\partial_e\tau_{(0)f}\right)\varepsilon^{abc}_{(0)}\tau_{(0)a}\partial_b\delta\tau_{(0)c}\,.\end{aligned} \tag{6.237}$$

This can be solved for either $\delta\phi_{(0)}$ or $\delta\Phi_{(0)}$ and substituted into the variation of the on-shell action (6.167). For instance, solving for $\delta\phi_{(0)}$ and performing a partial integration we obtain

$$\begin{aligned}
&\int_{\partial\mathcal{M}} d^3x e_{(0)} \langle O_\phi\rangle \delta\phi_{(0)} = \int_{\partial\mathcal{M}} d^3x e_{(0)} \Bigg( \langle O_\phi\rangle \left(\frac{3}{2} - \frac{2}{k^2} e^{2(\Phi_{(0)}-\phi_{(0)})}\right) \delta\Phi_{(0)} \\
&+\langle O_\phi\rangle \Bigg[ \left(1 - \frac{4}{k^2} e^{2(\Phi_{(0)}-\phi_{(0)})}\right) e_{(0)a}^{\underline{i}} \\
&-\frac{1}{k^2} e^{3\Phi_{(0)}-2\phi_{(0)}} \left(\varepsilon_{(0)}^{bcd} \tau_{(0)b} \partial_c \tau_{(0)d}\right) \left(\varepsilon_{(0)}^{efg} e_{(0)e}^{\underline{i}} \partial_f \tau_{(0)g}\right) \tau_{(0)a} \Bigg] \delta e_{(0)\underline{i}}^{a} \\
&-\frac{1}{k^2} \tau_{(0)a} \varepsilon_{(0)}^{bcd} e_{(0)b}^{\underline{a}} \partial_c \left[ e^{3\Phi_{(0)}-2\phi_{(0)}} \langle O_\phi\rangle \tau_{(0)d} \varepsilon_{(0)}^{efg} \tau_{(0)e} \partial_f \tau_{(0)g} \right] \delta e_{(0)\underline{a}}^{a} \Bigg) . \qquad (6.238)
\end{aligned}$$

Note that the terms proportional to the variations of the vielbein are all proportional to $\varepsilon_{(0)}^{abc}\tau_{(0)a}\partial_b\tau_{(0)c}$ as can be seen from (6.236). Hence, for AlLif space-times, there are no contributions to the boundary stress-energy tensor that originate from the constraint on the sources. Yet, even in that case the variations of $\Phi_{(0)}$ and $\phi_{(0)}$ are related. Despite of this fact, we call the terms in front of the source variations in (6.167) vevs. Furthermore, the constraint has no consequences for the Ward identities because the local symmetries acting on the various sources are not affected by it.

### 6.3.5 AlLif revisited

Generalizing to Lifshitz UV spacetimes allowed us to compute vevs and their associated Ward identities. To make contact with AlLif spacetimes again we can impose the constraints I and II discussed in section 5.2.2. Imposing these properties, the Ward identities (6.219)–(6.224) reduce to

$$\begin{aligned}
e^{-\Phi_{(0)}/2}\mathcal{A}_{(0)} &= 2\mathcal{T}_{(0)\underline{t}}^{\underline{t}} + \mathcal{T}_{(0)\underline{i}}^{\underline{i}} + 2B_{(0)\underline{t}}T_{(0)}^{\underline{t}} + B_{(0)\underline{i}}T_{(0)}^{\underline{i}} , && (6.239)\\
k\langle O_\chi\rangle &= \nabla_{(0)a}T_{(0)}^{a} - \frac{1}{2}\tau_{(0)}^{b}\left(\partial_b\tau_{(0)a} - \partial_a\tau_{(0)b}\right)T_{(0)}^{a} , && (6.240)\\
\nabla_{(0)b}\mathcal{T}_{(0)a}^{b} &= -\mathcal{T}_{(0)b}^{c}\left(-\tau_{(0)c}\nabla_{(0)a}\tau_{(0)}^{b} + e_{(0)c}^{\underline{i}}\nabla_{(0)a}e_{(0)\underline{i}}^{b}\right) \\
&\quad +\frac{1}{2}\mathcal{T}_{(0)a}^{b}\tau_{(0)}^{c}\left(\partial_c\tau_{(0)b} - \partial_b\tau_{(0)c}\right) - T_{(0)}^{\underline{t}}\partial_a B_{(0)\underline{t}} - T_{(0)}^{\underline{i}}\partial_a B_{(0)\underline{i}} , && (6.241)\\
\mathcal{T}_{(0)}^{\underline{t}\underline{i}} &= -e^{-3\Phi_{(0)}/2}T_{(0)}^{\underline{i}} - B_{(0)}^{\underline{i}}T_{(0)}^{\underline{t}} , && (6.242)\\
0 &= \mathcal{T}_{(0)}^{\underline{i}\underline{j}} - B_{(0)}^{\underline{i}}T_{(0)}^{\underline{j}} - (\underline{i}\leftrightarrow\underline{j}) , && (6.243)\\
-2\langle O_\Phi\rangle &= \mathcal{T}_{(0)\underline{t}}^{\underline{t}} - \mathcal{T}_{(0)\underline{i}}^{\underline{i}} + B_{(0)\underline{t}}T_{(0)}^{\underline{t}} - B_{(0)\underline{i}}T_{(0)}^{\underline{i}} , && (6.244)
\end{aligned}$$

These are the Ward identities satisfied by the vevs in an AlLif spacetime. One should note that, had we specialized to ALif spacetimes, the simplification would have been much greater. In this case, all terms proportional to $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a}$ would vanish.

### 6.3.6 The anomaly

Let us now return to the anomaly term of equation (6.154). In analogy with what was found in [61] we will see that the anomaly term is equivalent to a Hořava-Lifshitz type action [47, 48]. Here we will restrict to the simple case of ALif spacetimes. The anomaly will be evaluated by reducing the expression (5.31) for the 5d anomaly. To this end we write the boundary metric and its inverse, equations (6.58)–(6.63), as

$$\hat{h}_{(0)ab} = A_{(0)a}\tau_{(0)b} + A_{(0)b}\tau_{(0)a} + \Pi_{(0)ab}\,, \tag{6.245}$$
$$\hat{h}_{(0)au} = \tau_{(0)a}\,, \tag{6.246}$$
$$\hat{h}_{(0)uu} = 0\,, \tag{6.247}$$
$$\hat{h}^{ab}_{(0)} = \Pi^{ab}_{(0)}\,, \tag{6.248}$$
$$\hat{h}^{au}_{(0)} = -\tau^a_{(0)} - \Pi^{ab}_{(0)}A_{(0)b}\,, \tag{6.249}$$
$$\hat{h}^{uu}_{(0)} = 2A_{(0)a}\tau^a_{(0)} + \Pi^{ab}_{(0)}A_{(0)a}A_{(0)b}\,, \tag{6.250}$$

where we have temporarily rescaled $e^{\Phi_{(0)}}$ away to make the expressions tractable. In the following expressions, partial derivatives will be replaced by covariant derivatives using the connection $\Gamma_{(0)}$ of section 6.2.3. In addition, we define

$$K_{(0)ab} = \frac{1}{2}\mathcal{L}_{\tau_{(0)}}\Pi_{(0)ab}\,, \tag{6.251}$$
$$F_{(0)ab} = \partial_{(0)a}A_{(0)b} - \partial_{(0)b}A_{(0)a}\,, \tag{6.252}$$

with $K_{(0)} = \Pi^{ab}_{(0)}K_{(0)ab}$. For simplicity, we restrict to the ALif case where $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0$ and $\phi_{(0)} = \text{cst}$ meaning that the connection of section 6.2.3 is

$$\hat{\Gamma}^a_{(0)bc} = \Gamma^a_{(0)bc}\,, \tag{6.253}$$
$$\hat{\Gamma}^u_{(0)bc} = \frac{1}{2}\Big(\nabla_{(0)b}A_{(0)c} + \nabla_{(0)c}A_{(0)b} + 2K_{(0)bc} + F_{(0)ab}\tau_{(0)c}\tau^a_{(0)} + F_{(0)ac}\tau_{(0)b}\tau^a_{(0)}\Big)\,, \tag{6.254}$$
$$\hat{\Gamma}^a_{(0)bu} = \hat{\Gamma}^a_{(0)uu} = \hat{\Gamma}^u_{(0)ua} = \hat{\Gamma}^u_{(0)uu} = 0\,, \tag{6.255}$$

where $\Gamma^a_{(0)bc}$ is given by (6.113). This means that

$$\hat{R}_{(0)ab} = R_{(0)ab}\,, \tag{6.256}$$
$$\hat{R}_{(0)au} = 0\,, \tag{6.257}$$
$$\hat{R}_{(0)uu} = 0\,. \tag{6.258}$$

From eqs. (5.10)–(5.12) we then find

$$\hat{h}_{(2)ab} = -\frac{1}{2}\left(R_{(0)ab} - \frac{1}{2}e^{2\phi_{(0)}}\partial_a\chi_{(0)}\partial_b\chi_{(0)}\right) + \frac{1}{12}\left(Q_{(0)} + ke^{2\phi_{(0)}}\tau^c_{(0)}\mathcal{D}_c\chi_{(0)}\right)\left(A_{(0)a}\tau_{(0)b} + A_{(0)b}\tau_{(0)a} + \Pi_{(0)ab}\right)\,, \tag{6.259}$$
$$\hat{h}_{(2)au} = \frac{1}{4}ke^{2\phi_{(0)}}\partial_a\chi_{(0)} + \frac{1}{12}\left(Q_{(0)} + ke^{2\phi_{(0)}}\tau^c_{(0)}\mathcal{D}_c\chi_{(0)}\right)\tau_{(0)a}\,, \tag{6.260}$$
$$\hat{h}_{(2)uu} = \frac{1}{4}k^2e^{2\phi_{(0)}}\,, \tag{6.261}$$
$$\hat{\phi}_{(2)} = \frac{1}{4}\left(-e^{2\phi_{(0)}}\mathcal{D}_a\chi_{(0)}\mathcal{D}_b\chi_{(0)}\right)\Pi^{ab}_{(0)} + \frac{1}{2}ke^{2\phi_{(0)}}\tau^a_{(0)}\mathcal{D}_a\chi_{(0)}\,, \tag{6.262}$$
$$\hat{\chi}_{(2)} = \frac{1}{4}\left(\nabla_{(0)a}\mathcal{D}_b\chi_{(0)} - kK_{(0)ab}\right)\Pi^{ab}_{(0)}\,, \tag{6.263}$$

where

$$Q_{(0)ab} = R_{(0)ab} - \frac{1}{2}e^{2\phi_{(0)}}\mathcal{D}_a\chi_{(0)}\mathcal{D}_b\chi_{(0)}\,, \tag{6.264}$$

$$Q_{(0)} = \Pi^{ab}_{(0)}Q_{(0)ab}\,, \tag{6.265}$$

$$\mathcal{D}_a\chi_{(0)} = \partial_a\chi_{(0)} - kA_{(0)a}\,. \tag{6.266}$$

The full anomaly is therefore given by

$$\begin{aligned}\hat{\mathcal{A}}_{(0)} \;=\;& \frac{1}{8}Q_{(0)ab}Q_{(0)cd}\left(\Pi^{ac}_{(0)}\Pi^{bd}_{(0)} - \Pi^{ab}_{(0)}\Pi^{cd}_{(0)}\right) + \frac{1}{48}\left(ke^{2\phi_{(0)}}\tau^a_{(0)}\mathcal{D}_a\chi_{(0)} - 2Q_{(0)}\right)^2 \\ & +\frac{1}{4}ke^{2\phi_{(0)}}\mathcal{D}_a\chi_{(0)}Q_{(0)bc}\Pi^{ab}_{(0)}\tau^c_{(0)} - \frac{1}{8}k^2e^{2\phi_{(0)}}Q_{(0)ab}\tau^a_{(0)}\tau^b_{(0)} \\ & +\hat{\phi}^2_{(2)} + e^{2\phi_{(0)}}\hat{\chi}^2_{(2)}\,. \end{aligned} \tag{6.267}$$

Assuming, for simplicity, that $\mathcal{D}_a\chi_{(0)} = 0$ such that $F_{(0)ab} = 0$, we obtain

$$\begin{aligned}\hat{\mathcal{A}}_{(0)} \;=\;& -\frac{1}{24}\left(\Pi^{ab}_{(0)}R_{(0)ab}\right)^2 + \frac{1}{8}R_{(0)ab}R_{(0)cd}\Pi^{ac}_{(0)}\Pi^{bd}_{(0)} - \frac{k^2}{8}e^{2\phi_{(0)}}\tau^a_{(0)}\tau^b_{(0)}R_{(0)ab} \\ & +\frac{k^2}{16}e^{2\phi_{(0)}}\left(\Pi^{ab}_{(0)}K_{(0)ab}\right)^2\,. \end{aligned} \tag{6.268}$$

For the case $F_{(0)ab} = 0$ and $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0$ the Ricci tensor contracted with $\tau^a_{(0)}\tau^b_{(0)}$ is

$$\begin{aligned}e^{2\phi_{(0)}}\tau^a_{(0)}\tau^b_{(0)}R_{(0)ab} \;=\;& e^{2\phi_{(0)}}\left(\Pi^{ab}_{(0)}K_{(0)ab}\right)^2 - e^{2\phi_{(0)}}\Pi^{bc}_{(0)}\Pi^{ad}_{(0)}K_{(0)ac}K_{(0)bd} \\ & -\nabla_{(0)a}\left[e^{2\phi_{(0)}}\tau^a_{(0)}\Pi^{bc}_{(0)}K_{(0)bc}\right]\,. \end{aligned} \tag{6.269}$$

To investigate in what sense the anomaly is related to a Hořava-Lifshitz type action, we define the projector $\Pi^b_{(0)a}$ via

$$\Pi^b_{(0)a} = \delta^b_a + \tau_{(0)a}\tau^b_{(0)}\,. \tag{6.270}$$

The projected covariant derivative is defined by [6]

$$D_{(0)a}X^b_{(0)} = \Pi^c_{(0)a}\Pi^b_{(0)d}\nabla_{(0)c}X^d_{(0)}\,, \tag{6.271}$$

where $X^d_{(0)} = \Pi^d_{(0)c}Y^c_{(0)}$. The associated Riemann tensor is

$$[D_{(0)a}, D_{(0)b}]X^c_{(0)} = \mathcal{R}^c_{(0)dab}X^d_{(0)}\,. \tag{6.272}$$

Using that $\nabla_{(0)a}\tau_{(0)b} = 0$ we obtain

$$\mathcal{R}_{(0)ab} = \Pi^c_{(0)a}\Pi^d_{(0)b}R_{(0)cd}\,. \tag{6.273}$$

Note that this is a non-trivial consequence of the fact that $\nabla_{(0)a}\tau_{(0)b} = 0$. In the general case more terms would appear in this relation. In two dimensions the Einstein tensor vanishes meaning $\mathcal{R}_{(0)ab}$ satisfies the identity

$$\mathcal{R}_{(0)ab} = \frac{1}{2}\mathcal{R}_{(0)}\Pi_{(0)ab}\,. \tag{6.274}$$

From this we conclude that

$$\Pi^{ac}_{(0)}\Pi^{bd}_{(0)}R_{(0)ab}R_{(0)cd}-\frac{1}{3}\left(\Pi^{ab}_{(0)}R_{(0)ab}\right)^2=\frac{1}{6}\mathcal{R}^2_{(0)}\,. \tag{6.275}$$

Putting it all together we get

$$\begin{aligned}\hat{\mathcal{A}}_{(0)} &= \frac{k^2}{8}e^{2\phi_{(0)}}\left(\Pi^{bc}_{(0)}\Pi^{ad}_{(0)}K_{(0)ac}K_{(0)bd}-\frac{1}{2}\left(\Pi^{ab}_{(0)}K_{(0)ab}\right)^2\right)\\ &\quad+\frac{1}{48}\mathcal{R}^2_{(0)}+\frac{k^2}{8}\nabla_{(0)a}\left(e^{2\phi_{(0)}}\tau^a_{(0)}\Pi^{bc}_{(0)}K_{(0)bc}\right)\,.\end{aligned} \tag{6.276}$$

Hence the on-shell expression forms a Hořava-Lifshitz type action. It exhibits the expected $z=2$ Lifshitz scaling by having a $\frac{1}{2}$ in front of the term $\left(\Pi^{ab}_{(0)}K_{(0)ab}\right)^2$. The last term is a boundary term.

# Concluding Remarks

The purpose of this thesis was to study extensions of holography to non-relativistic systems. To this end we considered the ordinary AdS/CFT correspondence in 5 dimensions, but assumed that the AdS space contained a compact dimension. The existence of this compact direction allowed us to perform dimensional reduction, and thereby end up with a theory in 4 dimensions. In particular, we showed that if the reduction is a Scherk-Schwarz reduction and the AdS theory contains an axion-dilaton field, the resulting 4-dimensional theory is a $z = 2$ Lifshitz theory.

We considered such Lifshitz spacetimes as possible gravity duals of non-relativistic field theories. To this end, we relied heavily on the fact that Lifshitz spacetimes and AlAdS spacetimes are connected by dimensional reduction. Using this reduction we described the holographic renormalization of a $z = 2$ Lifshitz spacetime. The reduction to AlLif spacetimes placed certain constraints on the 5-dimensional theory and we gave a definition of AlLif from a 5-dimensional perspective and showed that it matched the definition given by Ross in [74]. In addition, we also defined three other types of spacetimes. The ALif spacetimes are specializations of AlLif which have the special property that their boundary geometry is pure Newton–Cartan. Generalized AlLif spacetimes had a non–constant $\phi_{(0)}$ which introduced $r^{-4}\log r$-deformations in the Fefferman-Graham expansion. The Lifshitz UV spacetime is the most general spacetime we can define which respects the boundary condition (5.76). Furthermore, it is possible to continuously deform the Lifshitz UV spacetime to obtain AlLif.

The boundary conditions of the 4-dimensional frame fields allowed us to investigate the action of bulk local Lorentz transformations on the boundary frame fields. This analysis paved the way for a determination of the boundary geometry. It was seen that when the bulk spacetime is ALif, the boundary geometry is pure Newton–Cartan. In addition, two extensions of Newton–Cartan spacetimes were defined, corresponding to the cases where the bulk geometry is either AlLif or Lifshitz UV. This provides a very interesting alternative to the usual AdS/CFT correspondence, in which the dual field theory lives on Minkowski space.

In the Lifshitz UV spacetime it is possible to calculate vevs and their associated Ward identities. Particular focus was on calculating the stress-energy tensor. Following the prescription outlined by Hollands, Ishibashi and Marolf in [75] we considered the inverse frame fields as sources of the boundary stress-energy tensor, which is required since a Lifshitz theory necessarily contains a massive vector field. The stress-energy tensor found by this procedure is equivalent to the one proposed by Ross in [74]. The associated Ward identity arising from diffeomorphisms is shown to have the form predicted in [75] except for the fact that the frame fields do not transform under the Lorentz group and there is a torsion term $T^{a}_{(0)bc}$. Furthermore, an extra free function worth of degrees of freedom is observed to appear in the Fefferman-Graham expansion of the 4-dimensional metric. The physical interpretation of this function is not yet clear.

To compare with other approaches to holography, we briefly review how the Lifshitz case stands in relation to the three questions posed in the Introduction. As illustrated, the Lifshitz theory is deriveable from string theory and thus not an example of a phenomological model. However, the techniques developed here will have applications to phenomological models with possible string theory completions. The question on the generality of holography is a bit less straightforward. Although the Lifshitz geometry is completely different from the well-studied AdS cases, the case considered here is, by the dimensional reduction, just a low-energy limit of the AdS cases and does not really present a novel realization of holography, but merely a new take on an old one. This new perspective was taken to investigate non-relativistic applications of holography, manifesting itself in the explicit anisotropy between space and time in the Lifshitz metric. When investigating the boundary geometry it is found that this is indeed also non-relativistic and an example of a Newton–Cartan geometry. This fact is very interesting from a dual field theory point of view, and a Newton–Cartan geometry was recently investigated as the geometry underlying a theory describing the quantum Hall effect [103].

## Outlook

The work summarized here, to be published in [1], paves the way for applications of holography to non-relativistic systems. However, there are still some unanswered questions. To gain a full understanding of the case discussed here, the physical meaning of the extra function should be made apparent. Furthermore, the rôle of the scalar fields should be investigated further. They play an important part in the constraints imposed on the sources, but is it possible to construct models where they are absent? Additionally, it would be interesting to solve the equations of motion in 5 dimensions in non-radial gauge and, upon dimensional reduction, end up with an Einstein frame metric in radial gauge.

To test the correspondence it would be interesting to find an exact solution of the 4-dimensional equations of motion and compare the predictions made by the Lifshitz theory discussed here with objects calculated in the dual field theory.

# Appendix A

# Conventions

In this thesis we will abide to the following conventions. We will be working in Planck units unless otherwise indicated. We use Lorentzian signature throughout the thesis, except for subsections 1.2.3 and 2.2.3 where Euclidean signature is used for convenience. The metric signature is mostly plus $(-,+,\cdots,+)$ and the curvature conventions are those of Wald [6]. The curvature tensor is thus

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}_{\rho\nu} + \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu}\,, \tag{A.1}$$

while the Ricci tensor is

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}\,. \tag{A.2}$$

Throughout we will take the action of gravity containing a cosmological constant to be

$$S = \frac{1}{2\kappa_d^2}\int \mathrm{d}^d x \sqrt{-g}\,(R - 2\Lambda)\,, \tag{A.3}$$

and AdS-space, being negatively curved, will have a negative cosmological constant.

Spacetime indices will be denoted with greek letters $\mu, \nu, \ldots$ while boundary quantities will carry indices denoted by latin letters $a, b, \ldots$. In the context of dimensional reductions, the higher dimensional objects will always carry hats as will any indices enumerating the higher dimensions. Unhatted indices on a hatted object will therefore refer to the non-compact part of the higher dimensional object (except in section 4.3 where compact directions are denoted by $a, b, \ldots$. No confusion with boundary coordinates is possible). Spatial indices will be denoted by $i, j, \ldots$. Furthermore, tangent space indices will be underlined.

# Appendix B

# Useful Identities

In this chapter brief derivations of several useful identities are given. The material presented in the main text is entirely self-contained, the derivations are included her for completeness.

**The trace-log formula for the determinant**

We here consider a derivation of the extremely useful formula relating the determinant of a matrix to the trace of the logarithm of that matrix. Let $M$ be a symmetric[1] (non-degenerate) $n \times n$ matrix with eigenvalues $\lambda_i$. The determinant of the exponent of this matrix can be written as

$$\det\, e^M = \exp\left(\sum_{i=1}^{n} \lambda_i\right), \tag{B.1}$$

while the trace is

$$\mathrm{tr}\; M = \sum_{i=1}^{n} \lambda_i\,. \tag{B.2}$$

Exponentiating this leads to

$$\exp \mathrm{tr}\; M = \exp\left(\sum_{i=1}^{n} \lambda_i\right), \tag{B.3}$$

thus demonstrating that

$$\det\, e^M = \exp\left(\mathrm{tr}\; M\right)\,. \tag{B.4}$$

Assuming, in addition, that the trace is non-vanishing, we can write $M = \log N$ and it follows that

$$\boxed{\det\; N = \exp\left(\mathrm{tr}\log N\right)\;.} \tag{B.5}$$

[1]This constraint can be relaxed but we will keep it here since we are only interested in applications to metrics.

### Variation of the inverse metric

It is useful to know how the variation of the metric is related to the variation of the inverse metric. Consider

$$\begin{aligned}\delta\left(g^{\mu\lambda}g_{\nu\lambda}\right) &= \delta\left(\delta^{\mu}_{\nu}\right)\\ &= 0\,. \end{aligned} \tag{B.6}$$

However, writing out the left hand side yields

$$\delta g^{\mu\lambda}g_{\nu\lambda} + g^{\mu\lambda}\delta g_{\nu\lambda} = 0\,, \tag{B.7}$$

meaning

$$\boxed{\delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\rho}\delta g_{\lambda\rho}\,.} \tag{B.8}$$

Raising the indices on the variation of the metric therefore results in an extra minus sign.

### Variation of the metric determinant

Here we briefly show the result of varying the metric determinant. Consider the relation (B.5). Performing a variation on both sides we find

$$\delta\left(\log\det\, g\right) = \delta\left(\mathrm{tr}\;\log\;g_{\mu\nu}\right)\,. \tag{B.9}$$

Using the chain rule results in

$$\frac{\delta\left(\det\, g\right)}{\det\, g} = g^{\mu\nu}\delta g_{\mu\nu}\,, \tag{B.10}$$

and the variation of the determinant is

$$\delta\left(\det\, g\right) = \det\, g\;g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{B.11}$$

Hence, the variation of the square root of the metric determinant is

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,,} \tag{B.12}$$

where equation (B.8) was used.

### Properties of the extrinsic curvature tensor

In this paragraph we present derivations of some important properties of the extrinsic curvature tensor. The fact that $K_{\mu\nu}$, defined by

$$K_{\mu\nu} = h^{\sigma}{}_{\mu}\nabla_{\sigma}n_{\nu}\,, \tag{B.13}$$

is symmetric follows from the fact that the $n_\mu$ is hypersurface orthogonal, $n_\mu = \partial_\mu f$ and also satisfies $n^\mu \nabla_\nu n_\mu = -n^\mu \nabla_\nu n_\mu = 0$. Using eq. (2.16) we can write

$$\begin{aligned}
K_{\mu\nu} &= h^\sigma{}_\mu \nabla_\sigma n_\nu \\
&= h^\sigma{}_\mu \delta^\rho{}_\nu \nabla_\sigma n_\rho \\
&= h^\sigma{}_\mu h^\rho{}_\nu \nabla_\sigma n_\rho + \sigma h^\sigma{}_\mu n^\rho n_\nu \nabla_\sigma n_\rho \\
&= h^\sigma{}_\mu h^\rho{}_\nu \nabla_\sigma n_\rho \\
&= h^\sigma{}_\mu h^\rho{}_\nu \left(\partial_\sigma \partial_\rho f - \Gamma^\kappa_{\rho\sigma} \partial_\kappa f\right) \\
&= h^\sigma{}_\mu h^\rho{}_\nu \left(\partial_\rho \partial_\sigma f - \Gamma^\kappa_{\sigma\rho} \partial_\kappa f\right) \\
&= h^\sigma{}_\mu h^\rho{}_\nu \nabla_\rho n_\sigma \\
&= h^\rho{}_\nu \nabla_\rho n_\mu \,,
\end{aligned} \tag{B.14}$$

and it follows that

$$\boxed{K_{\mu\nu} = K_{\nu\mu}\,.} \tag{B.15}$$

To show that the extrinsic curvature tensor is the Lie derivative of the induced metric, and the projected Lie derivative of the full metric we again consider the expression

$$\begin{aligned}
K_{\mu\nu} &= h^\sigma{}_\mu h^\rho{}_\nu \nabla_\sigma n_\rho \\
&= \frac{1}{2}\left(h^\sigma{}_\mu h^\rho{}_\nu \nabla_\sigma n_\rho + h^\sigma{}_\nu h^\rho{}_\mu \nabla_\sigma n_\rho\right) \\
&= \frac{1}{2} h^\sigma{}_\mu h^\rho{}_\nu \left(\nabla_\sigma n_\rho + \nabla_\rho n_\sigma\right),
\end{aligned} \tag{B.16}$$

showing that indeed

$$\boxed{K_{\mu\nu} = \frac{1}{2} h^\sigma{}_\mu h^\rho{}_\nu \mathcal{L}_n g_{\rho\sigma}\,.} \tag{B.17}$$

From the definition of the projected derivative, eq. (2.21), the expression (B.16) can also be written as

$$K_{\mu\nu} = \frac{1}{2}\left(D_\mu n_\nu + D_\nu n_\mu\right), \tag{B.18}$$

demonstrating that

$$\boxed{K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}\,.} \tag{B.19}$$

### Variation of the Ricci tensor and the extrinsic curvature tensor

Lets us first consider the variation of the Ricci tensor which arises as a boundary term in the variation of the Einstein-Hilbert action. The Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\kappa}\Gamma^\kappa_{\mu\nu} - \Gamma^\lambda_{\nu\kappa}\Gamma^\kappa_{\mu\lambda}\,. \tag{B.20}$$

Performing a variation we find

$$\begin{aligned}\delta R_{\mu\nu} &= \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\lambda\kappa}\Gamma^\kappa_{\mu\nu} + \Gamma^\lambda_{\lambda\kappa}\delta\Gamma^\kappa_{\mu\nu} - \delta\Gamma^\lambda_{\nu\kappa}\Gamma^\kappa_{\mu\lambda} - \Gamma^\lambda_{\nu\kappa}\delta\Gamma^\kappa_{\mu\lambda} \\ &= \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu},\end{aligned} \tag{B.21}$$

due to a cancellation between two of the terms when writing out the covariant derivative. Hence

$$\boxed{g^{\mu\nu}\delta R_{\mu\nu} \equiv \nabla_\lambda f^\lambda\,,} \tag{B.22}$$

where

$$f^\lambda \equiv g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\lambda\mu}\delta\Gamma^\rho_{\rho\mu}\,, \tag{B.23}$$

as also defined in chapter 3.

Consider now the variation of the extrinsic curvature. Throughout this derivation we will keep the variation of the boundary metric arbitrary. The extrinsic curvature tensor can be written as

$$K_{\mu\nu} = \nabla_\mu n_\nu - \sigma n_\mu n^\lambda \nabla_\lambda n_\nu \tag{B.24}$$

which follows from the definitions of the extrinsic curvature tensor and the definition of the induced metric, eqs. (2.16) and (2.25). The trace is then

$$K = \nabla_\mu n^\mu\,, \tag{B.25}$$

since $n^\mu n^\lambda \nabla_\lambda n_\mu = 0$, as argued above. The following identity will also be useful

$$\begin{aligned}\delta\left(\nabla_\mu n_\nu\right) &= \partial_\mu n_\nu - \delta\Gamma^\lambda_{\mu\nu} n_\lambda - \Gamma^\lambda_{\mu\nu}\delta n_\lambda \\ &= \nabla_\mu \delta n_\nu - \delta\Gamma^\lambda_{\mu\nu} n_\lambda\,.\end{aligned} \tag{B.26}$$

The variation of the trace of the extrinsic curvature is then

$$\begin{aligned}\delta K &= \delta\left(\nabla_\mu n^\mu\right) \\ &= \delta g^{\mu\nu}\nabla_\mu n_\nu + g^{\mu\nu}\nabla_\mu \delta n_\nu - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} n_\lambda \\ &= \nabla^\mu \delta n_\mu - \frac{1}{2}\nabla^\mu n^\nu \delta g_{\mu\nu} - \frac{1}{2}\nabla^\mu n^\nu \delta g_{\mu\nu} - \frac{1}{2} g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} n_\lambda - \frac{1}{2} g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} n_\lambda \\ &\quad + \frac{1}{2}\delta\Gamma^\nu_{\mu\nu} n^\mu - \frac{1}{2}\delta\Gamma^\nu_{\mu\nu} n^\mu \\ &= \nabla^\mu \delta n_\mu - \frac{1}{2}\nabla^\mu n^\nu \delta g_{\mu\nu} - \frac{1}{2}\nabla^\mu n^\nu \delta g_{\mu\nu} - \frac{1}{2} f^\mu n_\mu - \frac{1}{2} g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} n_\lambda - \frac{1}{2}\delta\Gamma^\nu_{\nu\mu} n^\mu\,,\end{aligned} \tag{B.27}$$

where we used eqs. (B.8) and (B.23), split up two terms, and added and subtracted the same term for convenience. Furthermore, we can use the chain rule in reverse and write the expression as

$$\begin{aligned}\delta K = &-\frac{1}{2} f^\mu n_\mu + \nabla^\mu \delta n_\mu - \frac{1}{2}\nabla^\mu n^\nu \delta g_{\mu\nu} - \frac{1}{2}\nabla^\mu\left(n^\nu \delta g_{\mu\nu}\right) + \frac{1}{2} n^\nu \nabla^\mu \delta g_{\mu\nu} \\ &- \frac{1}{2} g^{\mu\nu} g_{\lambda\rho}\delta\Gamma^\lambda_{\mu\nu} n^\rho - \frac{1}{2}\delta\Gamma^\nu_{\nu\mu} n^\mu\,.\end{aligned} \tag{B.28}$$

Several of the remaining terms can be made to cancel. Consider the following variation

$$\begin{aligned} \delta\left(\nabla^{\mu} g_{\mu\nu}\right) &= 0 \\ &= g^{\mu\rho}\nabla_{\rho}\delta g_{\mu\nu} - g^{\mu\rho}\delta\Gamma^{\lambda}_{\rho\mu}g_{\lambda\nu} - \delta\Gamma^{\lambda}_{\lambda\nu} \Rightarrow \\ \nabla^{\mu}\delta g_{\mu\nu} &= g^{\mu\rho}\delta\Gamma^{\lambda}_{\rho\mu}g_{\lambda\nu} + \delta\Gamma^{\lambda}_{\lambda\nu}\,, \end{aligned} \tag{B.29}$$

where terms have been combined in the covariant derivative of the variation of the metric and metric compatibility was used. Using the relation (B.29) in (B.28) results in

$$\delta K = -\frac{1}{2}f^{\mu}n_{\mu} + \nabla^{\mu}\delta n_{\mu} - \frac{1}{2}\nabla^{\mu}n^{\nu}\delta g_{\mu\nu} - \frac{1}{2}\nabla^{\mu}\left(n^{\nu}\delta g_{\mu\nu}\right)\,. \tag{B.30}$$

The relation (B.24) implies that

$$\nabla^{\mu}n^{\nu}\delta g_{\mu\nu} = K^{\mu\nu}\delta g_{\mu\nu} + \sigma n^{\mu}n^{\rho}\nabla_{\rho}n^{\nu}\delta g_{\mu\nu}\,, \tag{B.31}$$

and the covariant derivative of the variation of the normal vector can be found by writing

$$n_{\mu} = N\partial_{\mu}f\,, \tag{B.32}$$

where $N$ normalizes the normal vector, thus

$$N = \sqrt{\frac{\sigma}{g^{\rho\sigma}\partial_{\rho}f\partial_{\sigma}f}}\,. \tag{B.33}$$

The variation of the normal vector is then

$$\begin{aligned} \delta n_{\mu} &= \delta N\partial_{\mu}f \\ &= \frac{\delta N}{N}n_{\mu}, \end{aligned} \tag{B.34}$$

since the function $f$ is not varied. Writing it out we find

$$\begin{aligned} \frac{\delta N}{N} &= -\frac{1}{2}\left(g^{\rho\sigma}\partial_{\rho}f\partial_{\sigma}f\right)^{-1}\delta g^{\rho\sigma}\partial_{\rho}f\partial_{\sigma}f \\ &= \frac{1}{2}\sigma n^{\rho}n^{\sigma}\delta g_{\rho\sigma} \Rightarrow \\ \delta n_{\mu} &= \frac{1}{2}\sigma n_{\mu}n^{\rho}n^{\sigma}\delta g_{\sigma\rho}\,, \end{aligned} \tag{B.35}$$

where we used $\frac{1}{\sigma} = \sigma$, as follows from the definition. Applying the results from eqs. (B.31) and (B.35) the variation of $K$ is

$$\begin{aligned} \delta K =& -\frac{1}{2}n_{\mu}f^{\mu} - \frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\sigma\nabla^{\mu}\left(n_{\mu}n^{\rho}n^{\sigma}\delta g_{\sigma\rho}\right) - \frac{1}{2}\sigma n^{\mu}n^{\rho}\nabla_{\rho}n^{\nu}\delta g_{\mu\nu} - \frac{1}{2}\nabla^{\mu}\left(n^{\nu}\delta g_{\mu\nu}\right) \\ =& -\frac{1}{2}n_{\mu}f^{\mu} - \frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\sigma n^{\mu}n^{\nu}\nabla_{\rho}n^{\rho}\delta g_{\mu\nu} + \frac{1}{2}\sigma n^{\rho}n^{\mu}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) \\ & - \frac{1}{2}g^{\mu\rho}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) + \frac{1}{2}\sigma n^{\rho}n^{\mu}\nabla_{\rho}n^{\nu}\delta g_{\mu\nu} - \frac{1}{2}\sigma n^{\rho}n^{\mu}\nabla_{\rho}n^{\nu}\delta g_{\mu\nu} \\ =& -\frac{1}{2}n_{\mu}f^{\mu} - \frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\sigma n^{\mu}n^{\nu}\nabla_{\rho}n^{\rho}\delta g_{\mu\nu} + \frac{1}{2}\sigma n^{\rho}n^{\mu}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) \\ & - \frac{1}{2}h^{\mu\rho}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) - \frac{1}{2}\sigma n^{\rho}n^{\mu}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) \\ =& -\frac{1}{2}n_{\mu}f^{\mu} - \frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\sigma n^{\mu}n^{\nu}\nabla_{\rho}n^{\rho}\delta g_{\mu\nu} - \frac{1}{2}h^{\mu\rho}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right)\,. \end{aligned} \tag{B.36}$$

Where we used the definition of the induced metric in terms of normal vectors and the full metric, equation (2.16). This can be reduced further using the fact that

$$\begin{aligned}\frac{1}{2}\sigma n^{\mu}n^{\nu}\nabla_{\rho}n^{\rho}\delta g_{\mu\nu} &= \frac{1}{2}\sigma n^{\mu}n^{\nu}g^{\rho\sigma}\nabla_{\rho}n_{\sigma}\delta g_{\mu\nu} \\ &= \frac{1}{2}\sigma n^{\mu}n^{\nu}h^{\rho\sigma}\nabla_{\rho}n_{\sigma}\delta g_{\mu\nu} + \frac{\sigma^2}{2}n^{\mu}n^{\nu}n^{\rho}n^{\sigma}\nabla_{\rho}n_{\sigma}\delta g_{\mu\nu} \\ &= \frac{1}{2}\sigma n^{\mu}n^{\nu}h^{\rho\sigma}\nabla_{\rho}n_{\sigma}\delta g_{\mu\nu}\,. \end{aligned} \tag{B.37}$$

The last two terms in (B.36) can be combined and the result is

$$\frac{1}{2}\sigma n^{\mu}n^{\nu}h^{\rho\sigma}\nabla_{\rho}n_{\sigma}\delta g_{\mu\nu} - \frac{1}{2}h^{\mu\rho}\nabla_{\rho}\left(n^{\nu}\delta g_{\mu\nu}\right) = \frac{1}{2}h^{\rho\sigma}\nabla_{\rho}\left(\sigma n^{\mu}n^{\nu}n_{\sigma}\delta g_{\mu\nu} - \delta^{\mu}_{\sigma}n^{\nu}\delta g_{\mu\nu}\right)\,, \tag{B.38}$$

which follows from the fact that $h^{\rho\sigma}n_{\sigma} = 0$. Using (2.16) to rewrite the Kronecker delta we find

$$\begin{aligned}\frac{1}{2}h^{\rho\sigma}\nabla_{\rho}\left(\sigma n^{\mu}n^{\nu}n_{\sigma}\delta g_{\mu\nu} - \delta^{\mu}_{\sigma}n^{\nu}\delta g_{\mu\nu}\right) &= \frac{1}{2}h^{\rho\sigma}\nabla_{\rho}\left(\sigma n^{\mu}n^{\nu}n_{\sigma}\delta g_{\mu\nu} - h^{\mu}{}_{\sigma}n^{\nu}\delta g_{\mu\nu} - \sigma n^{\mu}n^{\nu}n_{\sigma}\delta g_{\mu\nu}\right) \\ &= -\frac{1}{2}h^{\rho}{}_{\sigma}\nabla_{\rho}\left(h^{\mu\sigma}n^{\nu}\delta g_{\mu\nu}\right)\,, \end{aligned} \tag{B.39}$$

and the final result for the variation of the trace of the extrinsic curvature is

$$\boxed{\delta K = -\frac{1}{2}n_{\mu}f^{\mu} - \frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} - \frac{1}{2}h^{\rho}{}_{\sigma}\nabla_{\rho}\left(h^{\mu\sigma}n^{\nu}\delta g_{\mu\nu}\right)\,.} \tag{B.40}$$

Recall that the Gibbons-Hawking boundary term is $2K$, so when evaluated on the boundary, the first term in the variation serves to cancel the variation of the Ricci tensor. The third term is a total divergence on the boundary, and will therefore never contribute as the boundary of a boundary is the empty set. The term in the middle makes no contribution in the case where the variation of the boundary metric is kept fixed $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0$. However, in applications to AdS/CFT, the variation of the boundary metric is kept arbitrary, $\delta g_{\mu\nu}|_{\partial\mathcal{M}} = \delta h_{ab}$. Recall that the extrinsic curvature is hypersurface tangential, $K_{\mu\nu}n^{\mu} = 0$, meaning that we can write the variation as

$$-\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}K_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}K_{\mu\nu}h^{\mu\nu}\,, \tag{B.41}$$

and evaluating this at the boundary $\partial\mathcal{M}$ yields

$$-\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu}\bigg|_{\partial\mathcal{M}} = \frac{1}{2}K_{ab}\delta h^{ab}\,. \tag{B.42}$$

Hence, the variation of the Gibbons-Hawking boundary term will contribute a term $K_{ab}$ to the boundary stress-energy tensor.

# Appendix C

# Conformal Field Theory

Conformal field theories are realized at fixed points of the renormalization group flows of cut-off effective field theories. Here the theory exhibits a scale invariance such that the theory is invariant under the transformation $x \to \lambda x$. For instance, asymptotically free Yang-Mills theories in 4 spacetime dimensions flow to scale invariant theories in the UV.

The conformal group is the group of reparameterizations leaving the spacetime metric invariant up to a local scale factor,

$$g_{\mu\nu} \to e^{2\Omega(x)} g_{\mu\nu}. \tag{C.1}$$

The generators of the conformal group for $d > 2$ are the Lorentz rotations and translations in addition to the dilatation and special conformal transformations, which act infinitesimally as

$$D : x^\mu \to (1+\epsilon)x^\mu \tag{C.2}$$

$$K_\nu : x^\mu \to x^\mu + \epsilon_\nu \left(g^{\mu\nu}x^2 - 2x^\mu x^\nu\right) \tag{C.3}$$

and they obey the algebra [104]

$$[D, K_\mu] = iK_\mu, \qquad [D, P_\mu] = -iP_\mu, \qquad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2ig_{\mu\nu}D, \tag{C.4}$$

with the rest either vanishing or following from rotational invariance. One might wonder why this extension of the Poincaré group is allowed by the Coleman-Mandula theorem. A conformal field theory possesses no S-matrix, which is an assumption of the Coleman-Mandula theorem, so it does not apply to CFT's. In a way analogous to how boosts are included in the Lorentz rotations, one can define additional rotation generators by including the $P_\mu$, $K_\mu$ and $D$:

$$M_{\mu\ d+1} \equiv \frac{K_\mu - P_\mu}{2}, \qquad M_{\mu\ d+2} \equiv \frac{K_\mu + P_\mu}{2}, \qquad M_{d+1\ d+2} \equiv D, \tag{C.5}$$

demonstrating that the conformal algebra is equivalent to that of $SO(2, d)$.

Following the same spirit as in ordinary quantum field theory, we are interested in the behaviour of local operators $\mathcal{O}(x^\mu)$ under transformations of the little group $SO(2) \times$

$SO(d) \subset SO(2,d)$ of the conformal group. The charge under the infinite cover of the $SO(2)$ is the scaling dimension $\Delta$ of the field operator, meaning

$$\mathcal{O}_\Delta(\lambda x^\mu) = \lambda^{-\Delta}\mathcal{O}_\Delta(x^\mu) \qquad \Leftrightarrow \qquad [D, \mathcal{O}_\Delta(0)] = -i\Delta\mathcal{O}_\Delta(0). \tag{C.6}$$

The action of $D$ on an operator at an arbitrary position is

$$[D, \mathcal{O}_\Delta(x)] = i\left(x^\mu\partial_\mu - \Delta\right)\mathcal{O}_\Delta(x). \tag{C.7}$$

Acting with $D$ on $[K_\mu, \mathcal{O}_\Delta(0)]$ and using the Jacobi identity we find

$$[D, [K_\mu, \mathcal{O}_\Delta(0)]] = i\left(-\Delta + 1\right)[K_\mu, \mathcal{O}_\Delta(0)]. \tag{C.8}$$

Thus, $K_\mu$ acts as a lowering operator which lowers the conformal dimension by 1 (recall the minus sign in (C.6)). We can therefore define a highest weight state as the state annihilated by $K_\mu$. The operator associated with such a state is called a primary operator. The creation operator is given by $P_\mu$ since

$$[D, [P_\mu, \mathcal{O}_\Delta(0)]] = i\left(-\Delta - 1\right)[P_\mu, \mathcal{O}_\Delta(0)]. \tag{C.9}$$

Descendants of a primary operators are defined as the operators obtained by acting on the primary operator with $P_\mu$. All local operators in the conformal field theory can be obtained by first identifying the primary operators, and then building up a ladder of descendants by acting with $P_\mu$. Unitarity puts a lower bound on the conformal dimension of primary operators, for instance a scalar field must obey $\Delta \geq (d-2)/2$ with equality only for the free scalar field.

Conformal invariance determines the correlation functions of primary operators in terms of their scaling dimensions and spin, up to an undertermined function of their insertion points. Thus for a scalar primary operator the 2-, 3- and 4-point functions have the form [23]

$$\langle \mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}\rangle = \delta_{\Delta_1\Delta_2}\prod_{i<j}^{2}|x_i - x_j|^{-\Delta}\,, \tag{C.10}$$

$$\langle \mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}\mathcal{O}_{\Delta_3}\rangle = c_{\Delta_1\Delta_2\Delta_3}\prod_{i<j}^{3}|x_i - x_j|^{\Delta-2\Delta_i-2\Delta_j}\,, \tag{C.11}$$

$$\langle \mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}\mathcal{O}_{\Delta_3}\mathcal{O}_{\Delta_4}\rangle = c_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)\prod_{i<j}^{4}|x_i - x_j|^{\frac{1}{3}\Delta-\Delta_i-\Delta_j}\,, \tag{C.12}$$

with $\Delta \equiv \sum_i \Delta_i$. The $c_{\Delta_1\Delta_2\Delta_3}$ is not determined by conformal invariance, while the $c_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$ is an undetermined function of the two independent conformally invariant harmonic ratios $u$ and $v$. Higher-point functions will quickly grow very complicated as the number of harmonic ratios is $\frac{n(n-3)}{2}$.

Since the correlation functions of a CFT are conformally invariant the partition function should also be conformally invariant. Hence, under a scaling of $x \to \lambda x$, the partition function should be invariant:

$$\begin{aligned}\int \mathrm{d}^d x \phi_{\Delta_i}(x)\mathcal{O}_{\Delta_i}(x) &= \int \mathrm{d}^d x \lambda^d \phi_{\Delta_i}(\lambda x)\mathcal{O}_{\Delta_i}(\lambda x) \\ &= \lambda^{d-\Delta}\int \mathrm{d}^d x \phi_{\Delta_i}(\lambda x)\mathcal{O}_{\Delta_i}(x)\,,\end{aligned} \tag{C.13}$$

where the scaling relation for a conformal primary operator,

$$\mathcal{O}_\Delta(\lambda x) = \lambda^{-\Delta}\mathcal{O}_\Delta(x)\,, \tag{C.14}$$

was used. The relation (C.13) implies that the field $\phi_\Delta$ behaves as

$$\phi_\Delta(x) \to \lambda^{d-\Delta}\phi_\Delta(\lambda x) \tag{C.15}$$

under the scale transformation $x \to \lambda x$.

## C.1 $\mathcal{N}=4$ Super Yang-Mills theory

In the original AdS/CFT correspondence the CFT refers to $\mathcal{N}=4$ SYM. Therefore, we present a brief overview of some of the primary features of this theory here. $\mathcal{N}=4$ SYM is the unique maximally supersymmetric theory in 4 spacetime dimensions. By unique we mean that the Lagrangian is entirely determined from symmetries. Furthermore, it is maximal since it has 4 supersymmetry generators, and this is the maximum number allowed if the spin is to be limited to lie between $-1$ and 1. Since the theory is maximally supersymmetric, all fields are superpartners of one another, and the theory is therefore massless. The $\mathcal{N}=4$ vector multiplet can be decomposed into three $\mathcal{N}=1$ chiral multiplets and one $\mathcal{N}=1$ vector multiplet. The field content is then a gauge field, $A_\mu$, four complex Weyl spinors, $\lambda_A$, and six real scalars, $X^I$. All the fields transform in the adjoint of the gauge group $SU(N)$, and the Lagrangian can be written as [26]

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg(&\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \sum_A i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A - \sum_I D_\mu X^I D^\mu X^I \\ &+ \sum_{A,B,I} gC_I^{AB}\lambda_A[X^I,\lambda_B] + \sum_{A,B,I} gC_{IAB}\lambda^A[X^I,\lambda^B] + \sum_{I,J}\frac{g^2}{2}[X^I,X^J][X_I,X_J]\Bigg)\,.\end{aligned} \tag{C.16}$$

This Lagrangian is invariant under the superconformal group $SU(2,2|4)$ which consists of the conformal group $SO(2,4)$ in addition to the fermionic symmetries of the generators $Q^A_\alpha$ and $\bar{Q}_{\dot{\alpha}A}$ and superconformal generators $S^A_\alpha$ and $\bar{S}_{\dot{\alpha}A}$. Furthermore, there is also an R-symmetry of the $\mathcal{N}=4$ under which the scalar fields transform in the **6** representation, the $\lambda^A$ ($\lambda_A$) in the **4** ($\bar{\mathbf{4}}$) representation while the $A_\mu$ is a singlet, **1**. The bosonic parts of the superconformal group are the conformal group $SO(2,4)$ and the group of R-symmetries $SU(4) \cong SO(6)$. In the AdS/CFT correspondence, one usually ignores the fermions and the symmetries on the field theory side are then exactly the isometry group of AdS space, $SO(2,4)$, and the isometry group of the 5-sphere, $SO(6) \cong SU(4)$. It has been shown to all orders in the coupling constant [26] that the $\beta$-function of $\mathcal{N}=4$ SYM vanishes, indicating that the conformal symmetry persists at a quantum level.

The vast amount of symmetry of $\mathcal{N}=4$ SYM makes it a particularly well-suited framework for calculations. Although there are no indications that Nature contains this much supersymmetry, results found from $\mathcal{N}=4$ SYM calculations can still be used as guidelines for theories with less symmetry. For instance, a theory with less supersymmetry can be obtained from $\mathcal{N} = 4$ SYM by adding to the Lagrangian a relevant operator, or by giving an operator a vev. This will break some of the supersymmetry in addition to conformal invariance. An example of this is the GPPZ-flow, investigated in [105].

# Appendix D

# Errata

Here follows a list of errata which has been corrected since the thesis was handed in.

**Page 18:** Changed the dimension from 5 to $d + 1$ in the first part of the calculation.
**Page 21:** Added reference to [43].
**Page 26:** Specified condition on $T$.
**Page 26:** Applied $n^\beta \omega_\beta = 0$.
**Page 26:** Clarified statement concerning curvature of hypersurface.
**Page 30:** Fixed confusing sentence in figure text.
**Page 31:** Clarified statement concerning conformal infinity of Minkowski space.
**Page 32:** Changed Minkowski space $\to$ Einstein static universe.
**Page 32:** Removed confusing statement concerning the lack of a hardwall box.
**Page 33:** Clarified how symmetries act on the boundary of AdS space in Lorentzian signature.
**Page 35:** Corrected AAdS $\to$ AdS.
**Page 38:** Added reference to [67].
**Page 39:** Removed confusing statement concerning tensorial properties of the stress–energy tensor in Lifshitz spacetimes.
**Page 40:** The stress–energy tensor fails to be conserved even without extra background fields.
**Page 41:** Clarified dimensions of Schrödinger spaces.
**Page 44:** Clarified statement concerning the variation of the metric.
**Page 49:** Corrected 4 transformations to all transformations.
**Page 49:** Removed confusing statement.
**Page 53:** Clarified statement concerning Ward identities.
**Page 55:** Fixed a typo.
**Sec. 5.1:** Added missing hats to expressions.
**Page 115:** Fixed a typo.

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