

# Applied Statistics

Project objectives and evaluation points



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*"Statistics is merely a quantisation of common sense"*

# Project objective

The project in Applied Statistics is to **measure the gravitational acceleration,**

**g**

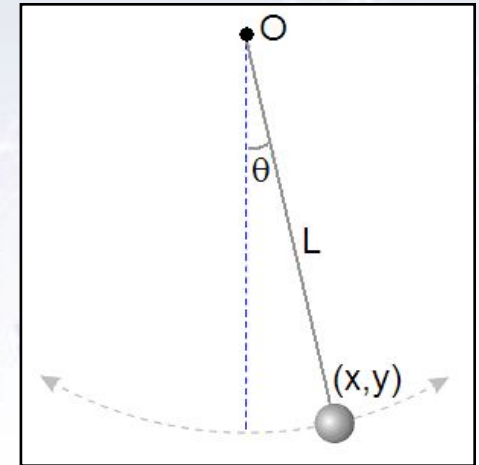
**with the greatest possible correct precision  
and the most possible cross checks,  
using two different experiments**

# Applied Statistics - Project

The project in Applied Statistics uses two different experiments:

## Simple pendulum:

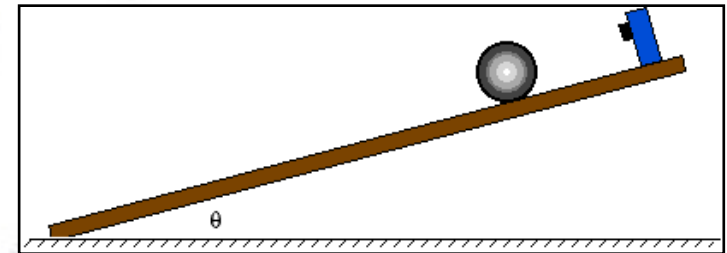
Measure **length** and **period** of the pendulum. Length is measured with a measuring band and a laser, and time by your hand.



## Ball rolling down incline:

Measure **incline angle**, **distance between gates**, **ball radius**, **rail distance** and **gate passage times**.

First four are measured by hand, while timing is extracted from data files.



**The project purpose is to learn how to extract, minimise and propagate errors.** Before doing the experiments, please consider through error propagation, which of the measurements are going to be most challenging/limiting.

For more information, please look at the [project webpage](#).

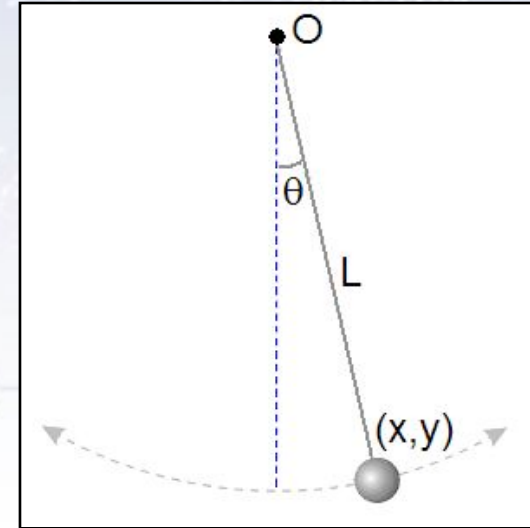
# Experiment formulae

The pendulum formula is well known:

$$g = L \left( \frac{2\pi}{T} \right)^2$$

The resulting error formula is easy:

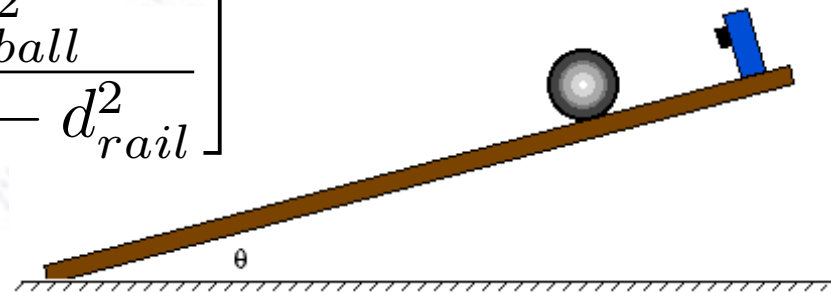
$$\sigma_g^2 = \left( \frac{2\pi}{T} \right)^4 \sigma_L^2 + \left( -2L \frac{(2\pi)^2}{T^3} \right)^2 \sigma_T^2$$



For the ball on incline, the formula is a bit more involved:

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{D_{ball}^2}{D_{ball}^2 - d_{rail}^2} \right]$$

The resulting error formula is in this case not that nice, but certainly doable.



This is a case, where the numerical solution is a good cross check!

# Measurement situation

There are four possible situations in experimental measurements of a quantity:

## One measurement, no error:

$$X = 3.14$$

Situation: You are f\*\*\*ed!

You have no clue about uncertainty, and you can not obtain it!

## Several measurements, no errors:

$$X_1 = 3.14$$

$$X_2 = 3.21$$

$$X_3 = \dots$$

Situation: You are OK

You can combine the measurements, and from RMS get error on mean.

## One measurement, with error:

$$X = 3.14 \pm 0.13$$

Situation: You are OK

You have a number with error, which you can continue with.

## Several measurements, with errors:

$$X_1 = 3.14 \pm 0.13$$

$$X_2 = 3.21 \pm 0.09$$

$$X_3 = \dots$$

Situation: You are on top of things!

You can both combine to a weighted, average and check with a chi-square.

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Situation: You are on top of things!

You can both combine to a weighted, average and check with a chi-square.

**Commit this line of thinking to memory!**

# Measurement situation

There are four possible situations in experimental measurement of a quantity:

One measurement, no error:

$$X = 3.14$$

Situation: You are frustrated!

You have no idea what you are doing, and you have no way to get error on mean.

Several measurements, with errors:

$$X_1 = 3.14 \pm 0.13$$

$$X_2 = 3.21 \pm 0.09$$

$$X_3 = \dots$$

Situation: You are on top of things!

You can both combine to a weighted average and check with a chi-square.

One measurement, with error:

You have a number with error, which you can continue with.

Situation: You are OK

You have a number with error, which you can continue with.

**Commit this line of thinking to memory!**  
**For project: Repeat measurements in an independent way to get uncertainties.**

# Pendulum objectives

What should you have measured in order to have everything needed for measuring  $g$ ?

$$g = L \left( \frac{2\pi}{T} \right)^2$$

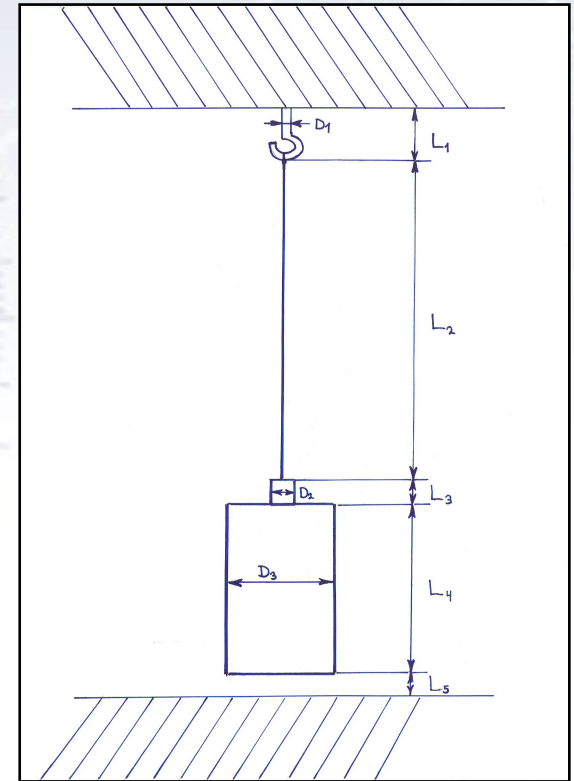
The answer is clear from the formula, but each measurement consists of several measurements!

It is generally worthwhile to make a good drawing ahead of doing the measurements.

Avoid bouncing pendulum, as it changes its length!

Make sure that you answer the following:

- What is the timing precision of **each person** in the group?
- What is the gravitational acceleration  $g$  and the errors from:
  - ♦ Length of pendulum.
  - ♦ Period of pendulum.





# Ball on incline objectives

What should you have measured in order to have everything needed for measuring  $g$ ?

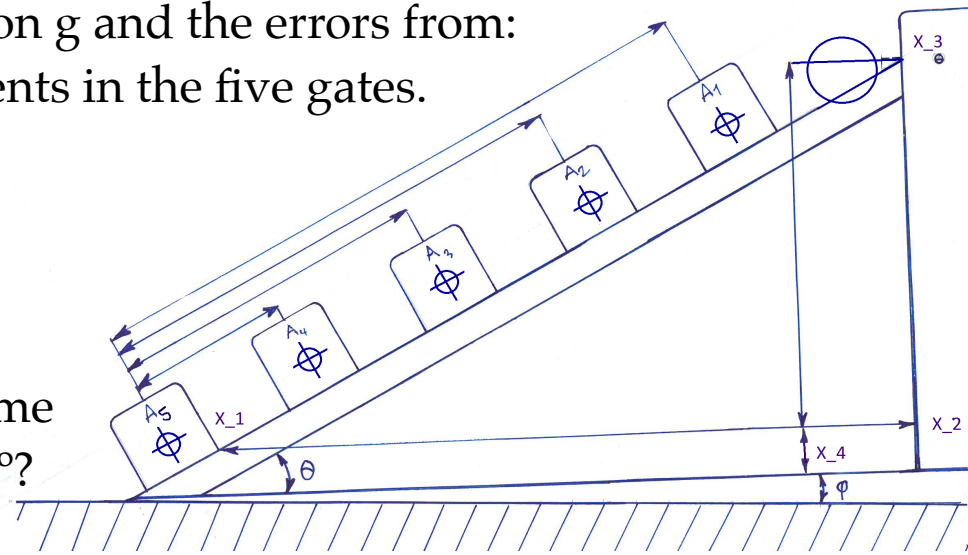
$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{D_{ball}^2}{D_{ball}^2 - d_{rail}^2} \right]$$

Ask yourself, what the critical measurements are. Where do you expect the largest impact on the result and uncertainty to come from?

Make sure that you answer the following:

- What is the angle of the rail  $\theta$ , and what is the angle of the table,  $\Delta\theta$ ?
- What is the gravitational acceleration  $g$  and the errors from:
  - ♦ Timing and distance measurements in the five gates.
  - ♦ Ball radius and rail distance.
  - ♦ Angle(s) of the rail.

Finally, perhaps you can eliminate some of your uncertainty by making  $\theta = 90^\circ$ ?



# Combining measurements

Given repeated measurements (by individual group members) of several quantities, that can be combined, what is the best way forward?

## Combine at the end of analysis:

### Measurements:

$$L1 = 3.543 \pm 0.002 \text{ m}$$

$$T1 = 3.942 \pm 0.002 \text{ s}$$

$$\Rightarrow \mathbf{g1 = 9.821 \pm 0.005 \text{ m/s}^2}$$

$$L2 = 3.545 \pm 0.003 \text{ m}$$

$$T2 = 3.940 \pm 0.003 \text{ s}$$

$$\Rightarrow \mathbf{g2 = 9.827 \pm 0.007 \text{ m/s}^2}$$

$$L3 = 3.523 \pm 0.002 \text{ m}$$

$$T3 = 3.944 \pm 0.003 \text{ s}$$

$$\Rightarrow \mathbf{g3 = 9.771 \pm 0.006 \text{ m/s}^2}$$

### Combination:

$$\mathbf{g = 9.806 \pm 0.004 \text{ m/s}^2}$$

$$\text{Chi2} = 28.3, \text{Ndof} = 2$$

$$\text{Prob}(\text{Chi2}, \text{Ndof}) = 7.5 \times 10^{-7}$$

## Combine each quantity first:

### Measurements:

$$L1 = 3.543 \pm 0.002 \text{ m}$$

$$L2 = 3.545 \pm 0.003 \text{ m}$$

$$L3 = 3.523 \pm 0.002 \text{ m}$$

$$\Rightarrow \mathbf{L = 3.537 \pm 0.002 \text{ m}}$$

$$\text{Chi2} = 30.8, \text{Ndof} = 2$$

$$\text{Prob}(\text{Chi2}, \text{Ndof}) = 2.1 \times 10^{-7}$$

$$T1 = 3.942 \pm 0.002 \text{ s}$$

$$T2 = 3.940 \pm 0.003 \text{ s}$$

$$T3 = 3.944 \pm 0.003 \text{ s}$$

$$\Rightarrow \mathbf{T = 3.942 \pm 0.002 \text{ s}}$$

$$\text{Chi2} = 1.3, \text{Ndof} = 2$$

$$\text{Prob}(\text{Chi2}, \text{Ndof}) = 0.52$$

### Combination:

$$\mathbf{g = 9.806 \pm 0.004 \text{ m/s}^2}$$

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Given repeated measurements (by individual group members) of several quantities, that can be combined, what is the best way forward?

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$$\Rightarrow \mathbf{g2 = 9.827 \pm 0.007 \text{ m/s}^2}$$

$$L3 = 3.523 \pm 0.002 \text{ m}$$

$$T3 = 3.944 \pm 0.003 \text{ s}$$

$$\Rightarrow \mathbf{g3 = 9.771 \pm 0.006 \text{ m/s}^2}$$

### Combination:

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$$T3 = 3.944 \pm 0.003 \text{ s}$$

$$\Rightarrow \mathbf{T = 3.942 \pm 0.002 \text{ s}}$$

$$\text{Chi2} = 1.3, \text{Ndof} = 2$$

$$\text{Prob}(\text{Chi2}, \text{Ndof}) = 0.52$$

### Combination:

$$\mathbf{g = 9.806 \pm 0.004 \text{ m/s}^2}$$

I would argue for combination within each quantity, to check for consistency.

# Cross checks are GOOD

The two experiments are relatively simple, but you should **imagine that they are more complicated** (and potentially ground breaking), and that you need to **convince others**, that what you're doing is **correct and accurate**.

Imagine the following question from a reviewer:

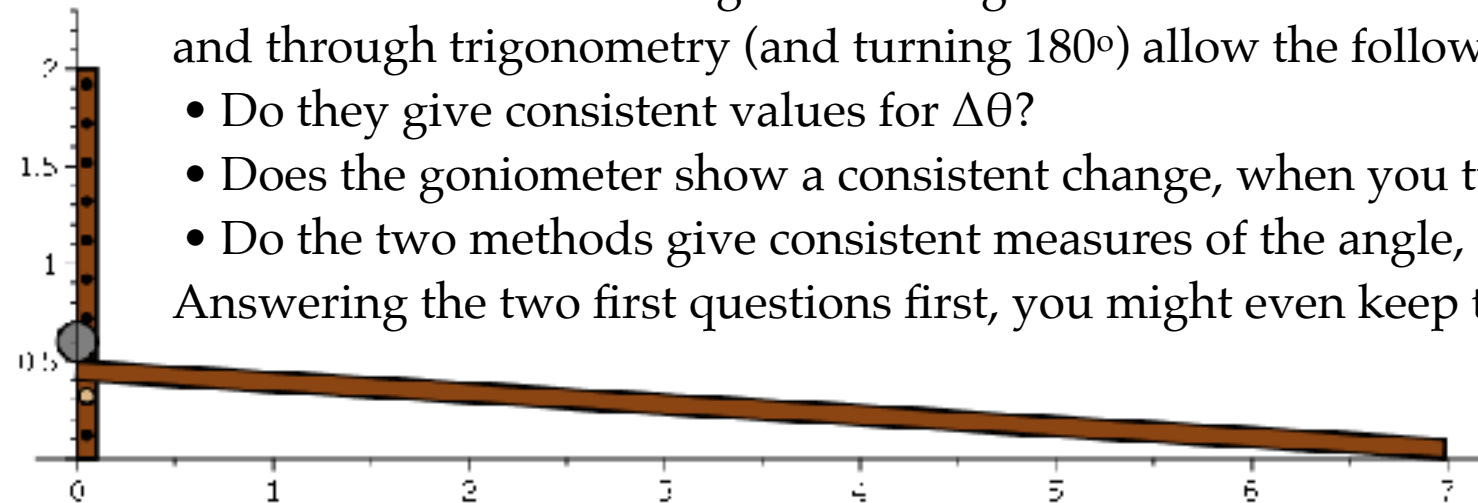
*"Do you know that you measure the angle correctly with the goniometer?"*

I know that it is unlikely, but in your next experiment, you'll be standing with a complicated Xmeter (which you build yourself?), and not being sure.

However, measuring things in **two independent ways** yielding consistent results is VERY convincing. For the angle measured both with the goniometer and through trigonometry (and turning  $180^\circ$ ) allow the following cross checks:

- Do they give consistent values for  $\Delta\theta$ ?
- Does the goniometer show a consistent change, when you turn it  $180^\circ$ ?
- Do the two methods give consistent measures of the angle,  $\theta$ ?

Answering the two first questions first, you might even keep the last blinded!



The background is a faded nautical chart. It features a compass rose with degree markings from 0 to 360. Concentric circles represent magnetic variation lines, with the word "MAGNETIC" printed in the center. The chart also shows latitude and longitude lines, and some text in the upper right corner, including "182 BITTER END" and "TACHT/ALUB".

# Notes on the angles

# Discussion of the angle $\theta$

The angle  $\theta$ , between the rail and the direction of gravity, can be measured in **two independent ways**, which allows for a vital cross check:

With the goniometer:  $\theta = \theta_{\text{gonio}}$

Using trigonometry and turning experiment:  $\theta = \theta_{\text{trig}} + \Delta\theta_{\text{turn}}$

You might think, that doing things in two independent ways is needless. But this is very important in experiments (which might be extremely complicated and rely on many assumptions!), as this ensures the correctness of the central value, and also tests if the uncertainties are realistic.

For this reason, the formula for  $g$  for the ball-on-incline experiment has two versions, depending on angular measurement, and with the above one has:

$$g = \frac{a}{\sin(\theta)} \left[ 1 + \frac{2}{5} \frac{D_{\text{ball}}^2}{D_{\text{ball}}^2 - d_{\text{rail}}^2} \right]$$

# Note on $\Delta\theta$

The angle of the table of the Ball-on-Incline (BoI) experiments - denoted  $\Delta\theta$  - can be determined in two ways (thus again allowing for cross check).

1. Using a goniometer before and after turning the experiment 180 degrees.
2. Measuring the acceleration before (normal direction, "norm") and after (reverse direction, "rev") turning the experiment 180 degrees, and equating the value for  $g$  between the two measurements:

$$\frac{a_{\text{norm}}}{\sin(\theta + \Delta\theta)} = g = \frac{a_{\text{rev}}}{\sin(\theta - \Delta\theta)}$$

As we can measure the acceleration in both configurations and also the angle  $\theta$ , we have one equation with one unknown, which happen to have an analytical solution:

$$\Delta\theta = \frac{(a_{\text{norm}} - a_{\text{rev}}) \sin(\theta)}{(a_{\text{norm}} + a_{\text{rev}}) \cos(\theta)}$$

The background is a faded nautical chart. It features a compass rose with degree markings from 0 to 360. Concentric circles represent magnetic isotherms, with values such as 0, 30, 60, 90, 120, 150, 180, 210, 240, 270, and 300. The word "MAGNETIC" is printed in the upper left quadrant. In the upper right, there is text that reads "THE BITTER END YACHT CLUB".

# Notes on your report



# Report content

Your report is intended for your fellow students, and you therefore do not need to make a long description of the experimental setup.

However, from your report, your fellow students (and we) should be able to **repeat/reproduce your experiment and subsequent data analysis**. Thus you have write what measurements you make (can be put in appendix, see next page), and exactly what you do with them.

Particularly important is, that you apply cross checks and Chi2 evaluations, whenever you can, and use these to evaluate uncertainties and possibly exclude measurements. This description is very important.

In the end, we simply want to see that you can get from raw data to final results, and that you can convince others (your peers and us), that what you have done is correct.

Therefore, make sure that you go through your numbers and errors and check that they are “reasonable”. If they are not, find and correct the error or at least comment.

# Example of appendix

APPENDICES

A Pendulum Experiment

A1 Experimental Setup

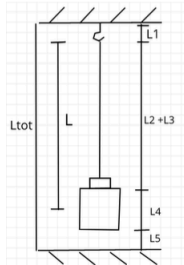


Fig. A1: Schematic representation of the pendulum experimental setup.

A2 Experimental Results

$L_1$ [cm]	$L_{(2+3)}$ [cm]	$L_4$ [cm]	$L_5$ [cm]	$L_{tot}$ [mm]
3.55	195.45	2.95	7.65	2099
3.65	195.25	3.05	7.71	2108
3.60	195.30	3.00	7.60	2108
3.50	195.30	3.10	7.70	2108

Table A1: Tape and laser measurements of different parts of the pendulum experimental setup. The laser measurement,  $L_{tot}$ , is an estimation of the distance from top (where the pendulum is hung), to the bottom of the floor (the laser's initial point).

$T_i$ [s]	$\sigma_T$ [s]	$\chi^2$	$P(\chi^2)$
2.8165	0.0009	24.7	0.37
2.8177	0.0015	32.0	0.10
2.818	0.003	14.3	0.92
2.8121	0.0012	19.0	0.70

Table A1II: Results of the second linear fit. A  $\chi^2$  test is performed. By construction a  $P(\chi^2) \sim 0.5$  was expected. The second test provides a really small  $P(\chi^2)$ , meaning that the fitting function does describe the histogrammed data very well (i.e., it is not gaussian), whereas the 3rd and 4th tests provide a rather large probability, that we could have got out of luck.

Oscillation number	Time [s]
1	2.7956
2	5.6492
3	8.4672
4	11.2882
5	14.0722
6	16.8781
7	19.7594
8	22.4898
9	25.3526
10	28.1713
11	30.9811
12	33.8216
13	36.6129
14	39.4555
15	42.2498
16	44.9828
17	47.9462
18	50.6521
19	53.5126
20	56.3769
21	59.1553
22	61.9397
23	64.7498
24	67.6178
25	70.4369

Table A1II: Obtained values for one measurement process, used in Fig. 1. The number of decimals is not significant, but corresponds to the digital precision of the script used to obtain the timing values.

$T$ [s]	$\sigma_T$ [s]	$t_0$ [s]	$\sigma_{t_0}$ [s]
2.82	0.03	0.0	0.4
2.82	0.03	-0.1	0.4
2.82	0.03	-0.1	0.4
2.81	0.03	0.0	0.4

Table A1IV: Results of the first linear fit, where  $t_0$  represents the offset parameter in the fit  $y = mx + b$ .  $\sigma_T$  represents the error on the slope estimation, which is not the actual error on  $T$  (that one is estimated and shown in Table A1II).

$\mu_{ts}$ [s]	$\sigma_{ts}$ [s]
0.003	0.03
-0.047	0.06
-0.088	0.11
0.006	0.04

Table A1V: Obtained means,  $\mu$ , and RMS,  $\sigma$ , from the gaussian fits to the binned time residuals.

$T$ [s]	$\sigma_T$ [s]	$\chi^2$	$P(\chi^2)$
2.8156	0.0006	11.89	0.008

Table A1VI: Values of resulting  $T$  and chi-squared test (see the Discussion section for an interpretation).

B Ball on Incline Experiment

B1 Experimental setup

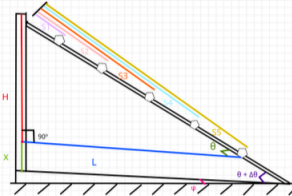


Fig. B1: Schematic representation of the experimental setup used to obtain  $g$  by throwing a ball down an inclined plane.

B2 Experimental Results

Gate 1[cm]	Gate 2[cm]	Gate 3[cm]	Gate 4[cm]	Gate 5[cm]
21.90	36.95	52.55	69.35	87.15
21.96	36.84	52.51	69.33	87.05
21.95	36.90	52.55	69.35	87.10
21.85	36.80	52.45	69.35	87.15

Table B1: Measurements of the position of each gate in the experimental setup.

$L$ [cm]	$x$ [cm]	$H$ [cm]
83.85	3.90	20.25
83.99	3.98	20.46
83.58	4.00	20.30
83.75	4.05	21.15

Table B1I: Measurements of the experimental setup used to obtain a trigonometric value of  $\theta$  (see Fig. B1).

$D_{small}$ [mm]	$D_{big}$ [mm]	$d_{rail}$ [mm]
10.98	12.70	5.95
10.88	12.70	6.18
10.84	12.67	5.98
10.82	12.70	6.10

Table B1II: Measurements obtained with a slide gauge in order to estimate the value of the gravitational constant.

Side A		Side B	
$\theta_{norm}$ [°]	$\theta_{180°}$ [°]	$\theta_{norm}$ [°]	$\theta_{180°}$ [°]
14.90	14.20	13.10	13.20
14.60	14.20	13.20	13.10
14.90	14.20	13.10	13.00
14.50	14.20	13.20	13.00

Table B1IV: Goniometer measurements of the angles.

Time <sub>1</sub> [s]	Time <sub>2</sub> [s]	Time <sub>3</sub> [s]	Time <sub>4</sub> [s]	Time <sub>5</sub> [s]
0.5197	0.7273	0.8831	1.0197	1.1435
0.1193	0.3305	0.4877	0.6248	0.749
0.0671	0.2848	0.4442	0.5829	0.7081
0.2966	0.521	0.6829	0.8227	0.9489

Table BV: Estimated passage times when the big ball is thrown and the experiment is facing side A.

$a$ [m/s <sup>2</sup> ]	$\sigma_a$ [m/s <sup>2</sup> ]	$v_0$ [m/s]	$\sigma_{v_0}$ [m/s]	$s_0$ [m]	$\sigma_{s_0}$ [m]
B.B norm. side					
1.560	0.011	-0.252	0.009	0.140	0.003
1.565	0.010	0.356	0.004	0.1655	0.0009
1.559	0.010	0.413	0.004	0.1879	0.0007
1.56	0.03	0.006	0.009	0.1416	0.0017
<b>Resulting <math>a</math> (1.56 ± 0.03) m/s<sup>2</sup></b>					

$a$ [m/s <sup>2</sup> ]	$\sigma_a$ [m/s <sup>2</sup> ]	$v_0$ [m/s]	$\sigma_{v_0}$ [m/s]	$s_0$ [m]	$\sigma_{s_0}$ [m]
B.B rev. side					
1.412	0.009	0.203	0.005	0.1438	0.0012
1.414	0.009	-0.259	0.008	0.156	0.003
1.412	0.09	-0.099	0.007	0.1325	0.0025
1.414	0.009	-1.225	0.015	0.656	0.011
<b>Resulting <math>a</math> (1.41 ± 0.03) m/s<sup>2</sup></b>					

$a$ [m/s <sup>2</sup> ]	$\sigma_a$ [m/s <sup>2</sup> ]	$v_0$ [m/s]	$\sigma_{v_0}$ [m/s]	$s_0$ [m]	$\sigma_{s_0}$ [m]
S.B norm. side					
1.503	0.000	-2.155	0.001	1.651	0.001
1.500	0.010	0.114	0.006	0.1186	0.0017
1.504	0.010	0.786	0.001	0.339	0.000
1.502	0.010	-0.354	0.009	0.157	0.004
<b>Resulting <math>a</math> (1.50 ± 0.03) m/s<sup>2</sup></b>					

$a$ [m/s <sup>2</sup> ]	$\sigma_a$ [m/s <sup>2</sup> ]	$v_0$ [m/s]	$\sigma_{v_0}$ [m/s]	$s_0$ [m]	$\sigma_{s_0}$ [m]
S.B rev. side					
1.360	0.009	-0.150	0.007	0.139	0.003
1.360	0.008	-1.599	0.015	1.097	0.013
1.363	0.009	-0.387	0.009	0.189	0.004
1.357	0.010	-0.283	0.009	0.126	0.004
<b>Resulting <math>a</math> (1.36 ± 0.04) m/s<sup>2</sup></b>					

Table BVI: Results obtained for the fit done to a parabola of the form  $y(x) = \frac{1}{2}ax^2 + v_0x + s_0$  for the values of the gate distances in function of the time elapse. Where  $a$ ,  $v_0$ , and  $s_0$  corresponds to the value of acceleration, the initial velocity and the initial position respectively. The resulting values of the acceleration is the average of the values obtained from the fit and the error is given by the RMS.

$\bar{a}$ [m/s <sup>2</sup> ]	$\sigma_a$ [m/s <sup>2</sup> ]	$\chi^2$	$P(\chi^2)$
1.56	0.03	0.01	0.99
1.41	0.03	0.001	0.99
1.50	0.03	0.005	0.99
1.36	0.04	0.002	0.99

Table BVII: Values of the resulting accelerations,  $\chi^2$ -test and Prob( $\chi^2$ ).

B3 Error propagation on g

$$\delta g = \frac{1}{\sin(\theta \pm \Delta\theta)} \left( 1 + \frac{2}{5} D_{ball}^2 \frac{\delta D_{ball}^2}{D_{ball}^2 - d_{rail}^2} \right) \quad (1.12)$$

Example of writing up "raw" measurements, making the analysis reproducible!



# Project evaluation

# Project evaluation

## Pendulum:

- Did you measure  $T \pm \sigma(T)$  correctly? Combine with Chi2 and comments?
- Did you measure  $L \pm \sigma(L)$  correctly? Combine and check correctly?
- Did you provide the individual T and L precisions/uncertainties on g?
- Did you measure each team members timing precision and submit these?

## Ball on incline:

- $T \pm \sigma(T)$
  - $L \pm \sigma(L)$
- }  $\Rightarrow a \pm \sigma(a)$ , with Chi2 and comments.
- $\theta, \Delta\theta$  obtained correctly and
  - d, R and errors propagated correctly?

## Generally:

- **Correctly propagated uncertainties, showing individual contributions.**
- **Using Chi2 and its probability, whenever possible.**
- All necessary figures and tables there? 2-3 essential figures needed.
- Text enough to understand results? Clear and fitting captions?
- Comment on result (especially inconsistencies) and correct significant digits.

Collect results: Pendulum (T, L, g) and Ball on Incline (T, L, a,  $\theta$ ,  $\Delta\theta$ , d, R)

# Project challenge

The project consist of experiments and data analysis, which well resembles those in real life.

There is TONS of experience to gather from these!!!

For this reason, we give 1-2 extra points to persons/ groups, who manage the following:

- Pendulum measurement better than  $1/1000$  with full and correct data analysis and error propagation consistent with  $g$ .
- Ball on incline measurement better than  $1/100$  with full and correct data analysis and error propagation consistent with  $g$ .

It is perfectly alright NOT to do this, and one is of course allowed to continue in person, and just submit a personal addition.



# Bonus Slides

# Different equation versions

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2} \right]$$

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{r_{ball}^2}{r_{ball}^2 - (d_{rail}/2)^2} \right]$$

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{D_{ball}^2}{D_{ball}^2 - d_{rail}^2} \right]$$