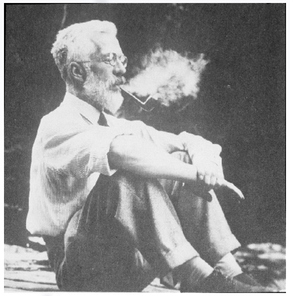
Applied Statistics

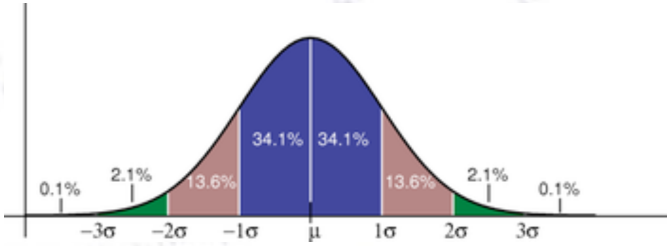
Mean and Width





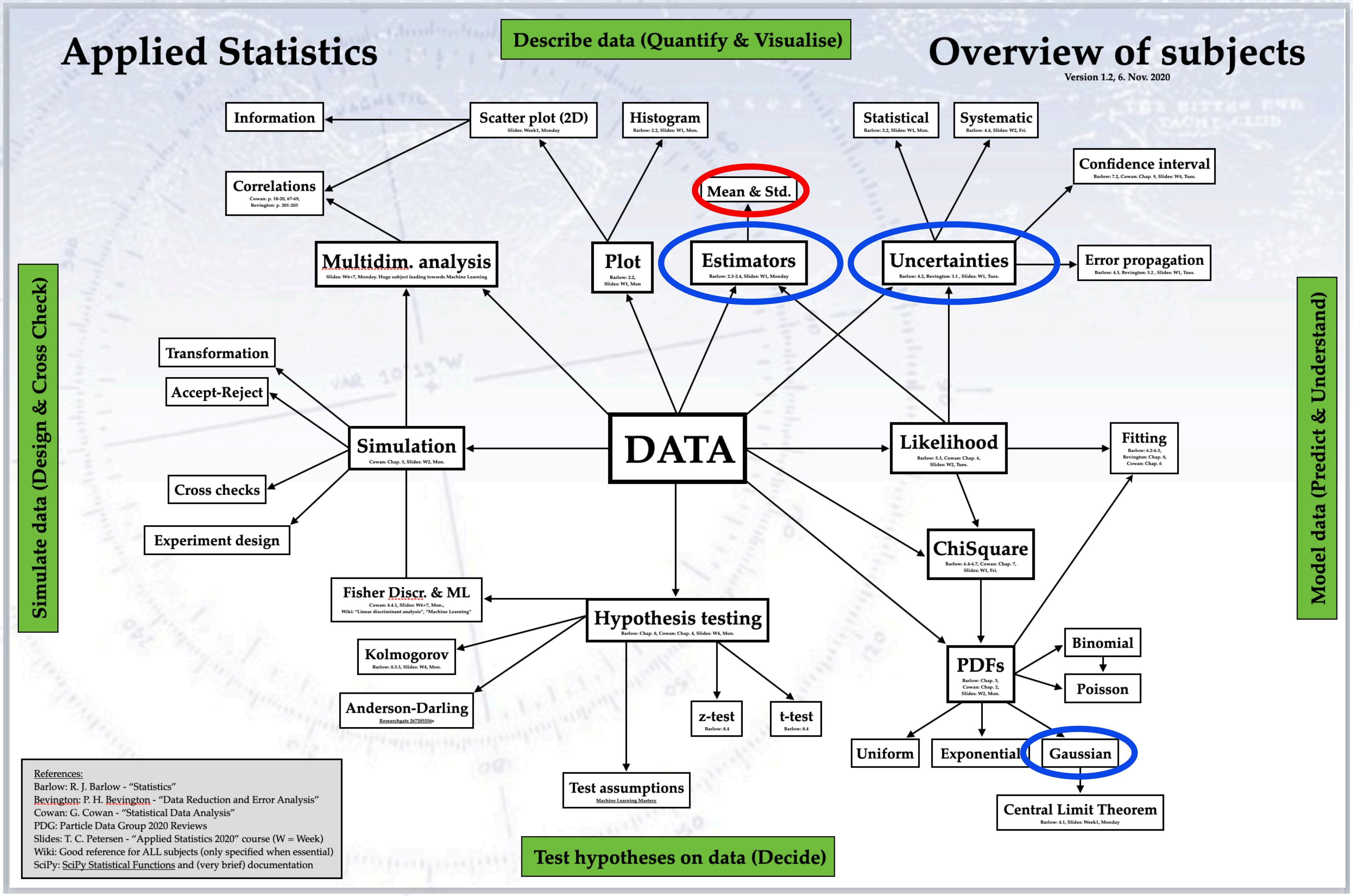



Troels C. Petersen (NBI)



"Statistics is merely a quantisation of common sense"

Mean & Width



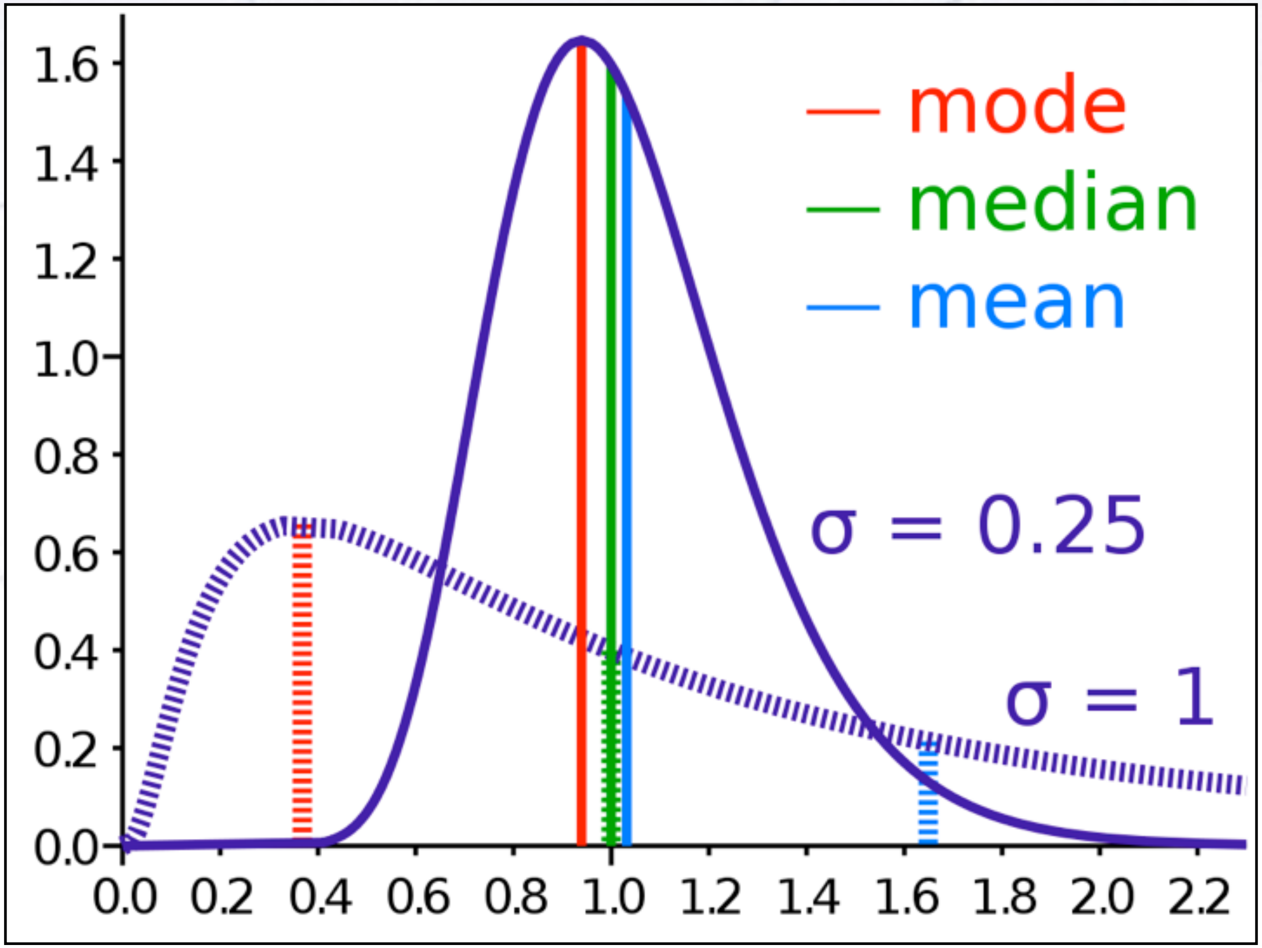
Applied Statistics
Describe data (Quantify & Visualise)
Overview of subjects
Version 1.2, 6. Nov. 2020
Information
Scatter plot (2D)
Histogram
Statistical
Systematic
Correlations
Mean & Std.
Confidence interval
Multidim. analysis
Plot
Estimators
Uncertainties
Error propagation
Simulate data (Design & Cross Check)
Model data (Predict & Understand)
Transformation
Accept-Reject
Simulation
DATA
Likelihood
Fitting
Cross checks
Experiment design
ChiSquare
Fisher Discr. & ML
Hypothesis testing
Kolmogorov
Anderson-Darling
PDFs
Binomial
Poisson
z-test
t-test
Uniform
Exponential
Gaussian
Test assumptions
Central Limit Theorem
Test hypotheses on data (Decide)
References:
Barlow: R. J. Barlow - "Statistics"
Bevington: P. H. Bevington - "Data Reduction and Error Analysis"
Cowan: G. Cowan - "Statistical Data Analysis"
PDG: Particle Data Group 2020 Reviews
Slides: T. C. Petersen - "Applied Statistics 2020" course (W = Week)
Wiki: Good reference for ALL subjects (only specified when essential)
SciPy: SciPy Statistical Functions and (very brief) documentation

Defining the mean

There are several ways of defining "a typical" value from a dataset:

a) Arithmetic mean   b) Mode (most probably)   c) Median (half below, half above)

d) Geometric mean   e) Harmonic mean   f) Truncated mean (robustness)



Mean and Width

It turns out, that the best estimator for the **mean** is (as you all know):

$$\hat{\mu} = \frac{1}{N} \sum_i x_i = \bar{x}$$

The second (central) moment of the data is called the **variance**, defined as:

$$\hat{V} = \frac{1}{N} \sum_i (x_i - \mu)^2$$

Note the "hat", which means "estimator". It is sometimes dropped...

Mean and Width

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For the **standard deviation (Std)**, a.k.a. **width** or **RMSE**, it is:

$$\hat{\sigma} = \sqrt{\frac{1}{N} \sum_i (x_i - \mu)^2}$$

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$$\hat{s} = \sqrt{\frac{1}{N-1} \sum_i (x_i - \bar{x})^2}$$

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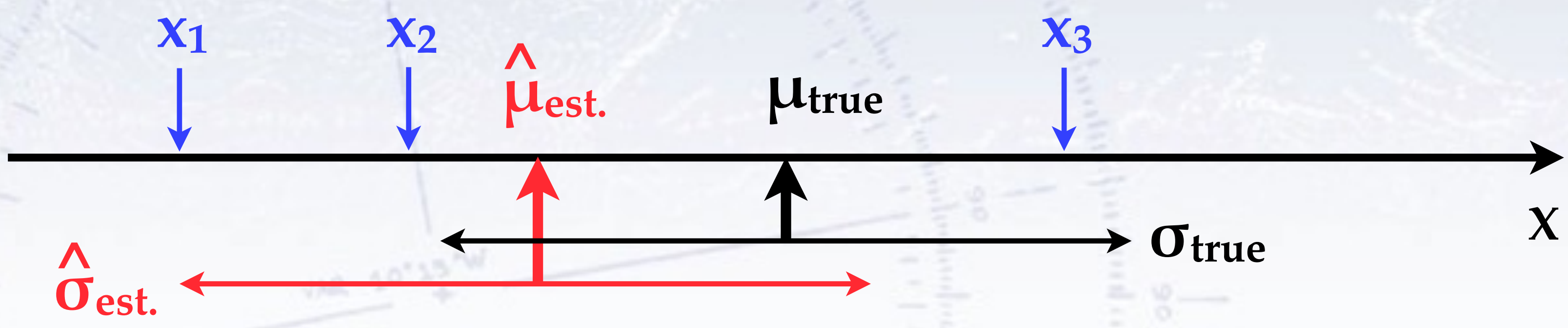
Why not "just" the naive SD?

Imagine taking 3 independent measurements, and then the mean and SD:



Above, all went well, because measurements were nicely distributed on both sides of the mean, and spread out according to SD.

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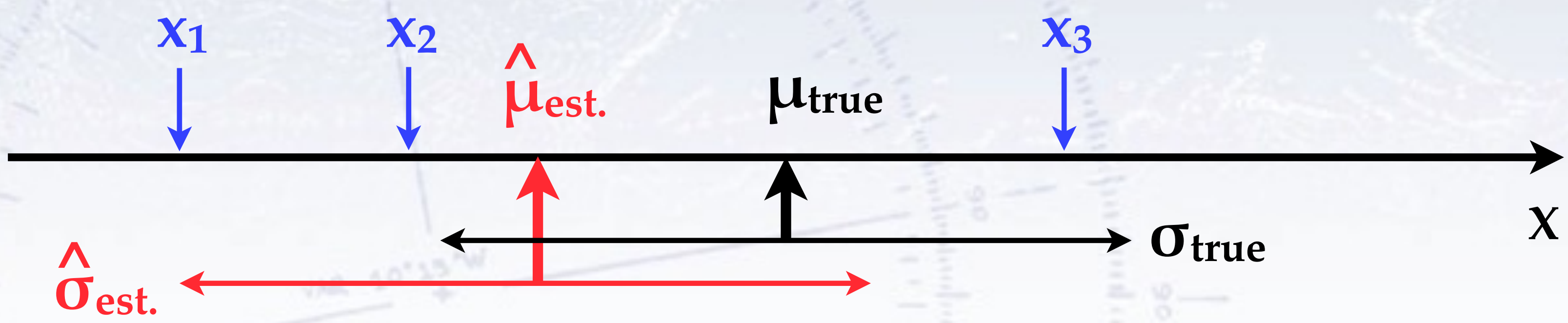
Imagine taking 3 independent measurements, and then the mean and RMSE:



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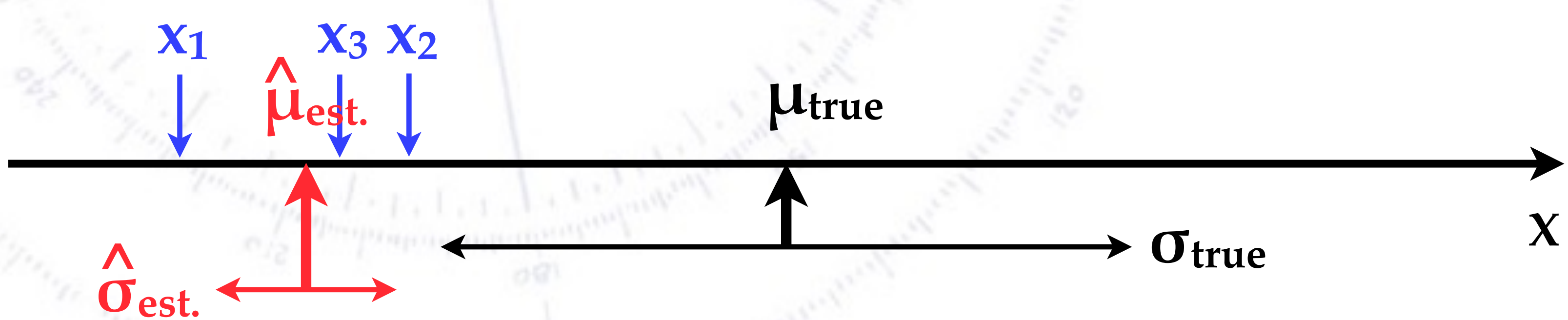
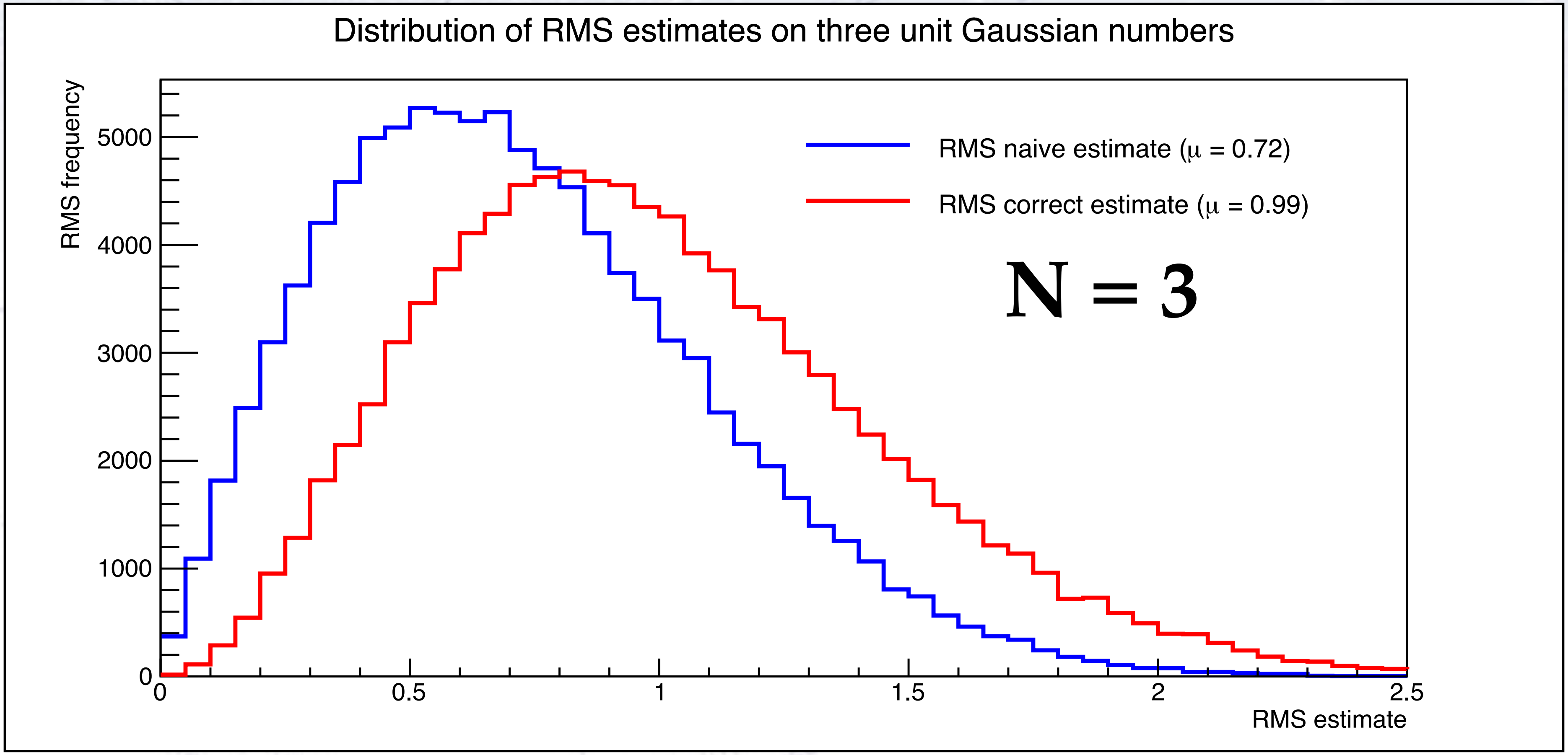


However, now the mean is off (not terribly so) and the SD way off (terribly so!). If we had used the true mean in the formula, it would not have been a problem.

How incorrect is the naive SD?

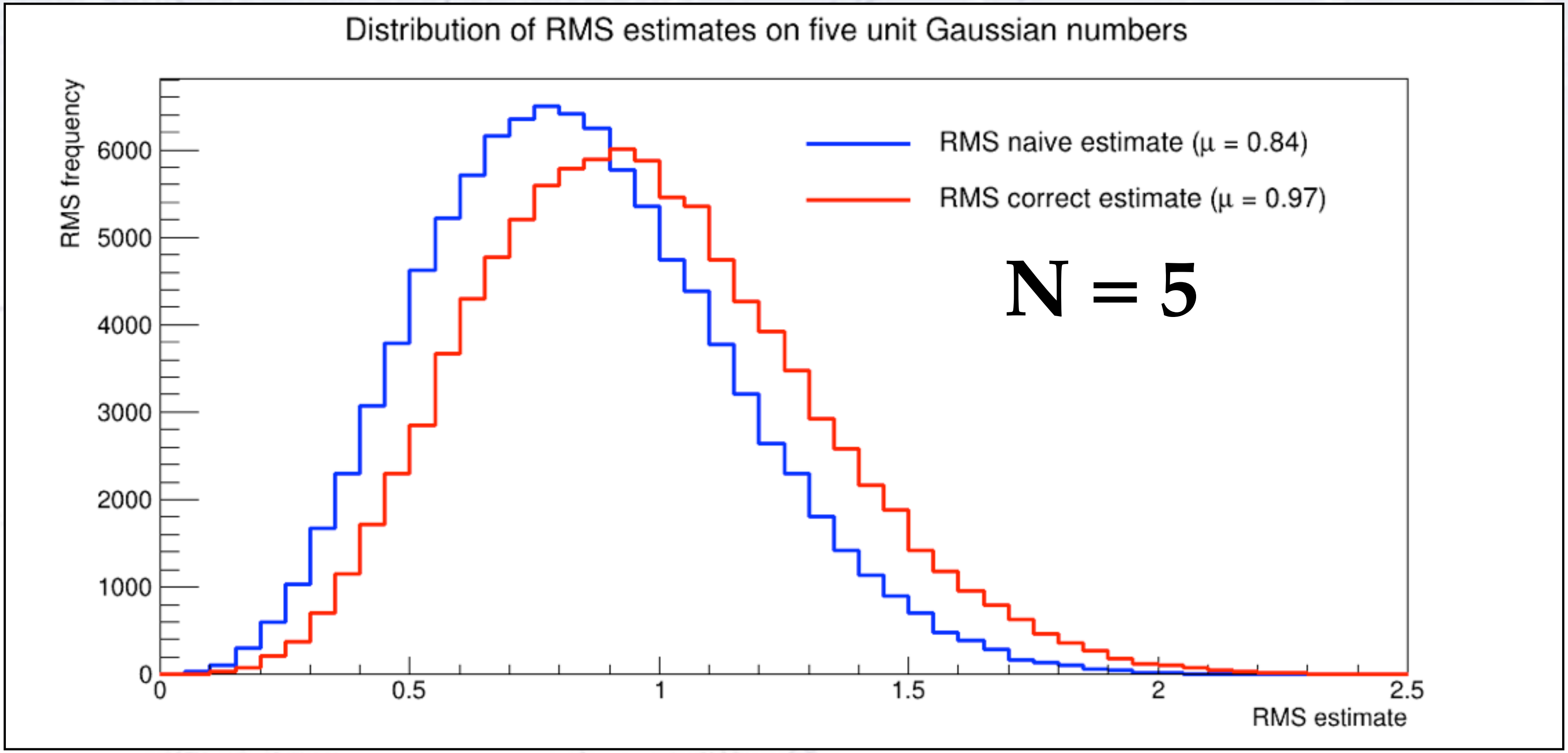
Such questions can most easily be answered by a small simulation…
Produce N=3 numbers from a unit Gaussian, and calculate the SD estimate:



So, the "naive" SD underestimates the uncertainty significantly…

How incorrect is the naive SD?

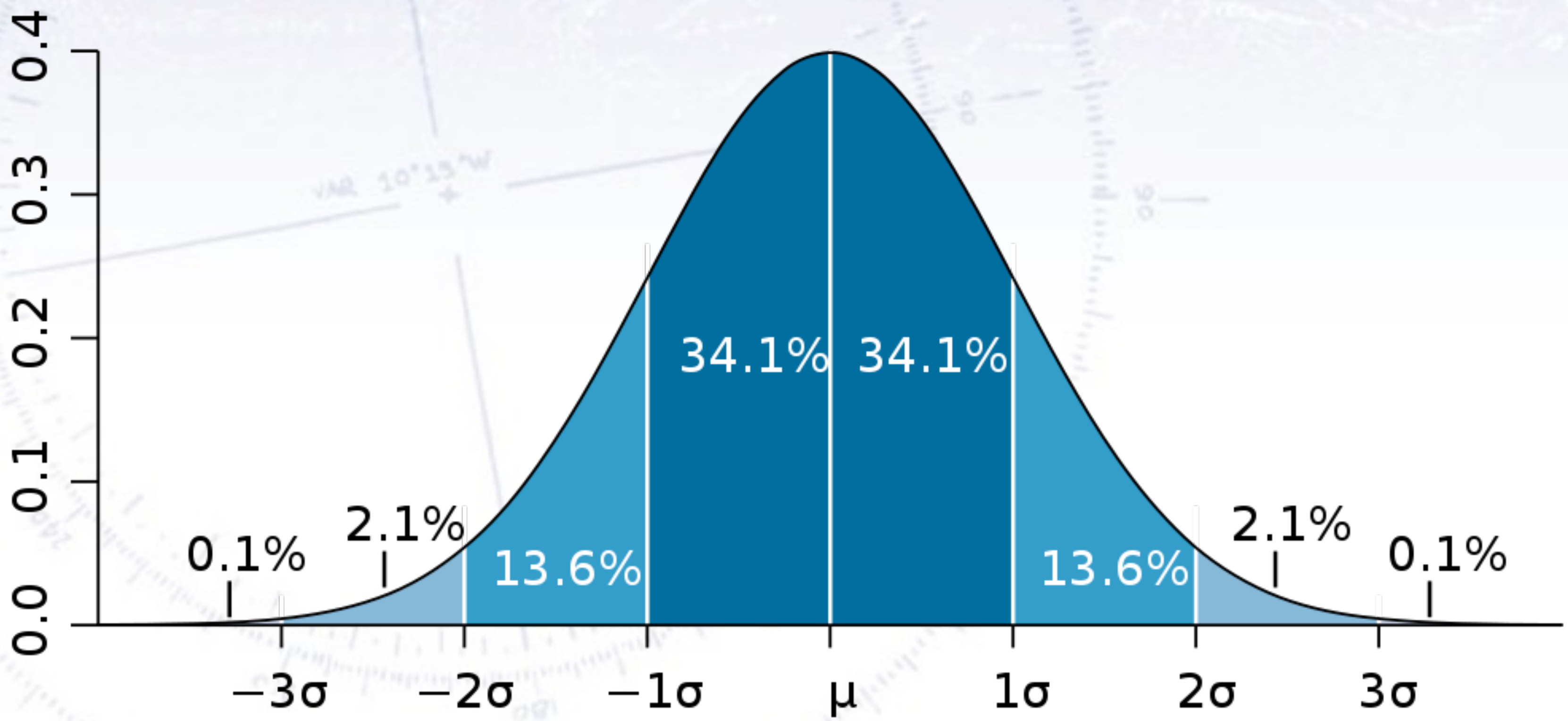
Such questions can most easily be answered by a small simulation…
Produce N=5 numbers from a unit Gaussian, and calculate the SD estimate:



Here, the "naive" SD underestimates the uncertainty a bit...

SD and Gaussian σ relation

When a distribution is Gaussian, **the Std. corresponds to the Gaussian width σ**:



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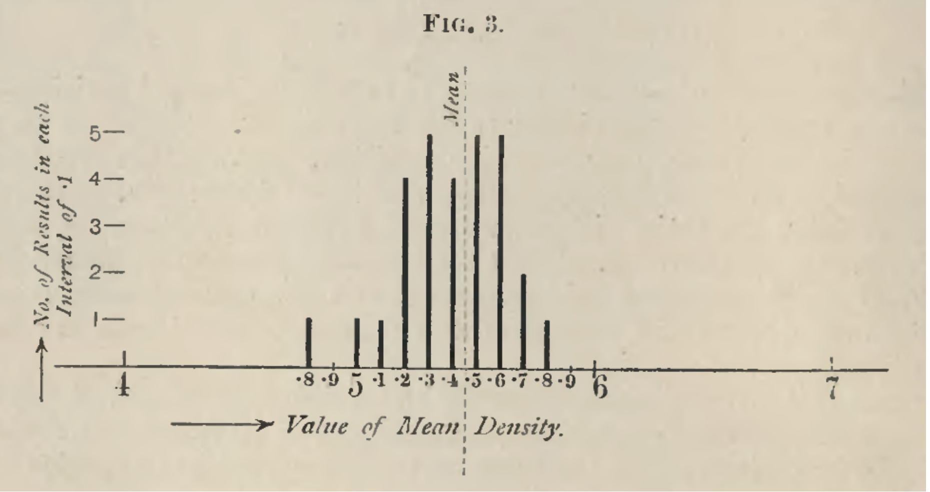
<u>Example:</u>
Cavendish Experiment
(measurement of Earth's density)
N = 29
mu = 5.42
sigma = 0.333
sigma(mu) = 0.06
Earth density = 5.42 ± 0.06



Mean and Width

What is the **uncertainty on the mean?** And how quickly does it improve with more data?

$$\hat{\sigma}_{\mu} = \hat{\sigma} / \sqrt{N}$$

Please, commit to memory now!

<u>Example:</u>
Cavendish Experiment
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N = 29
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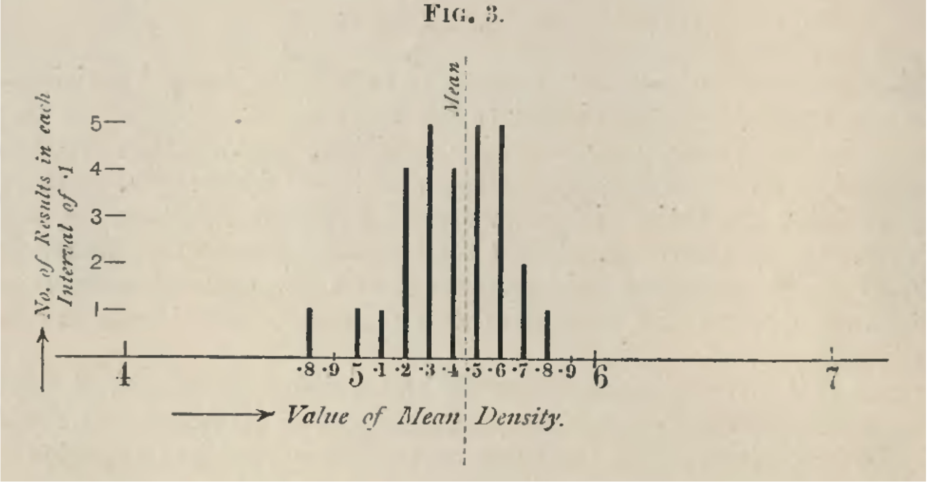
FIG. 3.



Weighted Mean

What if we are given data, which has different uncertainties?
How to average these, and what is the uncertainty on the average?

$$\hat{\mu} = \frac{\sum x_i/\sigma_i^2}{\sum 1/\sigma_i^2}$$

For measurements with varying uncertainty, there is no meaningful SD!
The uncertainty on the mean is:

$$\hat{\sigma}_\mu = \sqrt{\frac{1}{\sum 1/\sigma_i^2}}$$

Can be understood intuitively, if two persons combine 1 vs. 4 measurements

Weighted Mean

What if we are given data, which has different uncertainties?
How to average these, and what is the uncertainty on the average?

$$\sum x_i/\sigma_i^2$$

Note that when doing a weighted mean, one should check if the measurements agree with each other!
This can be done with a ChiSquare test.

For measur… SD!
The uncerta…

$$\hat{\sigma}_\mu = \sqrt{\frac{1}{\sum 1/\sigma_i^2}}$$

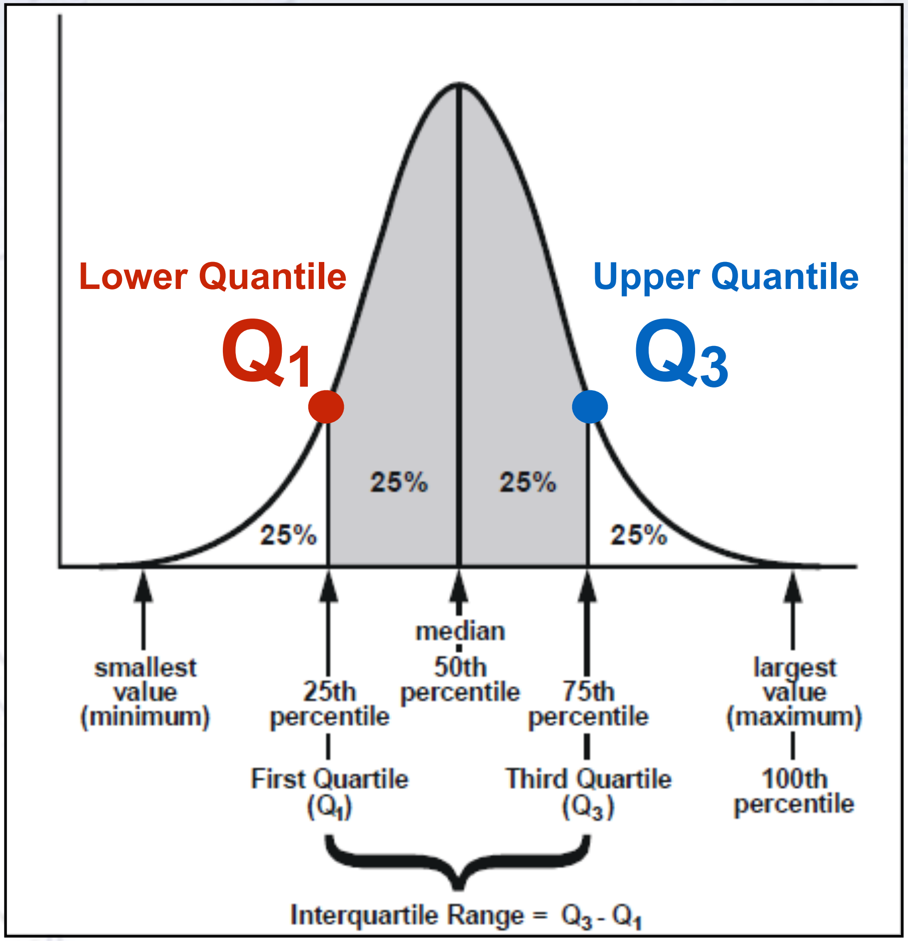
Can be understood intuitively, if two persons combine 1 vs. 4 measurements

Resolution using InterQuantile Range

A useful measure of resolution is the InterQuantile Range (IQR), as this is not affected by long tails.

IQR measures **statistical dispersion**, calculated as the difference

$$\mathbf{IQR} = Q_3 - Q_1$$



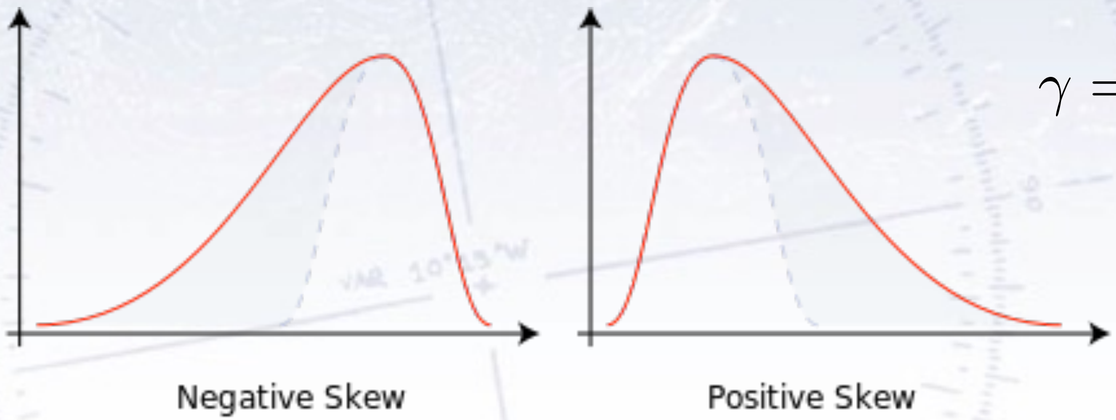
The InterQuantile Efficiency (IQE) is defined as:

$$\mathbf{IQE} = \mathbf{IQR} / 1.349$$

The factor $1.349 = 2\ \Phi^{-1}(0.75)$ ensures that IQR = 1 for a unit Gaussian.

Skewness and Kurtosis

Higher moments reveal something about a distributions asymmetry and tails:



$$\gamma = \frac{\frac{1}{N}\sum_i (x_i - \bar{x})^3}{(\frac{1}{N}\sum_i (x_i - \bar{x})^2)^3/2}$$

$$\kappa = \frac{\frac{1}{N}\sum_i (x_i - \bar{x})^4}{(\frac{1}{N}\sum_i (x_i - \bar{x})^2)^2} - 3$$