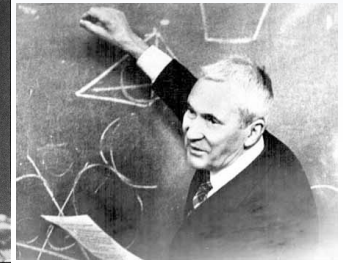
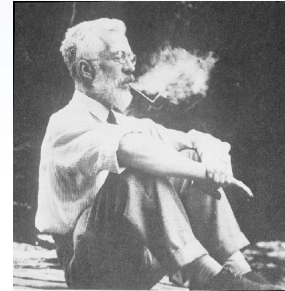
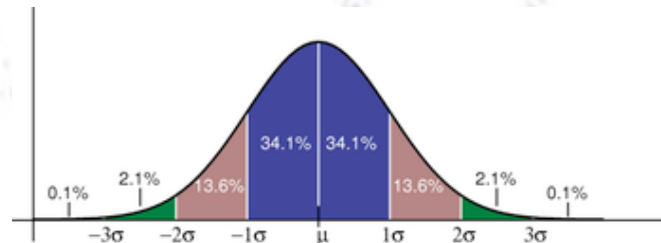


# Applied Statistics

## Error propagation



Troels C. Petersen (NBI)



*"Statistics is merely a quantisation of common sense"*

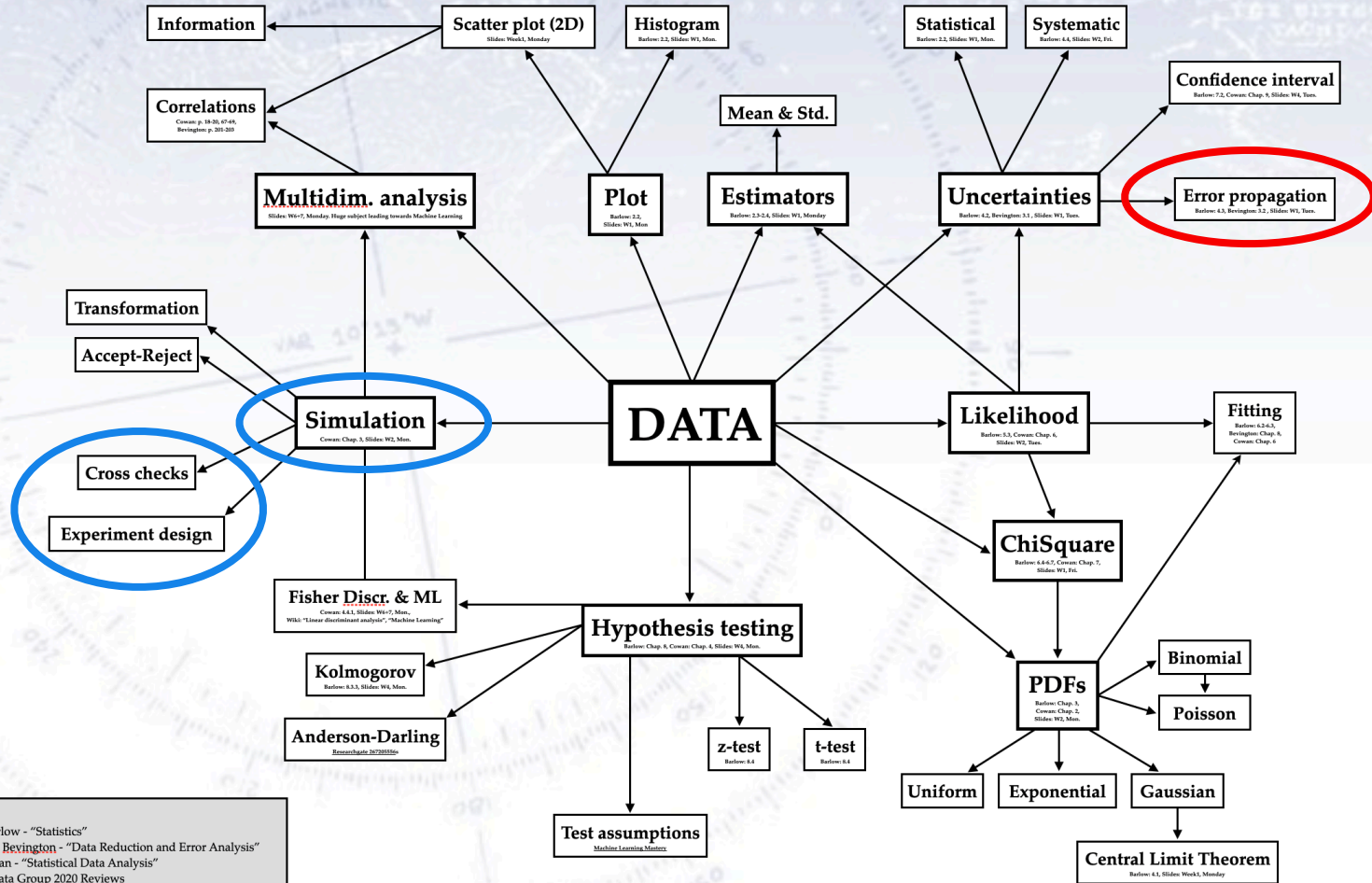
# Error propagation

Applied Statistics

Describe data (Quantify & Visualise)

Overview of subjects

Version 1.2, 6. Nov. 2020



Simulate data (Design & Cross Check)

Model data (Predict & Understand)

References:  
 Barlow: R. J. Barlow - "Statistics"  
 Bevington: P. H. Bevington - "Data Reduction and Error Analysis"  
 Cowan: G. Cowan - "Statistical Data Analysis"  
 PDG: Particle Data Group 2020 Reviews  
 Slides: T. C. Petersen - "Applied Statistics 2020" course (W = Week)  
 Wiki: Good reference for ALL subjects (only specified when essential)  
 SciPy: SciPy Statistical Functions and (very brief) documentation

Test hypotheses on data (Decide)

# Error propagation

Imagine that  $y$  is a function of  $x_i$

$$y(x_i)$$

and that we wish to find the error on  $y$  from the errors on  $x_i$ .

$$\sigma(x_i) = 0.8$$

$$\sigma(y(x_i)) = ?$$



# Error propagation

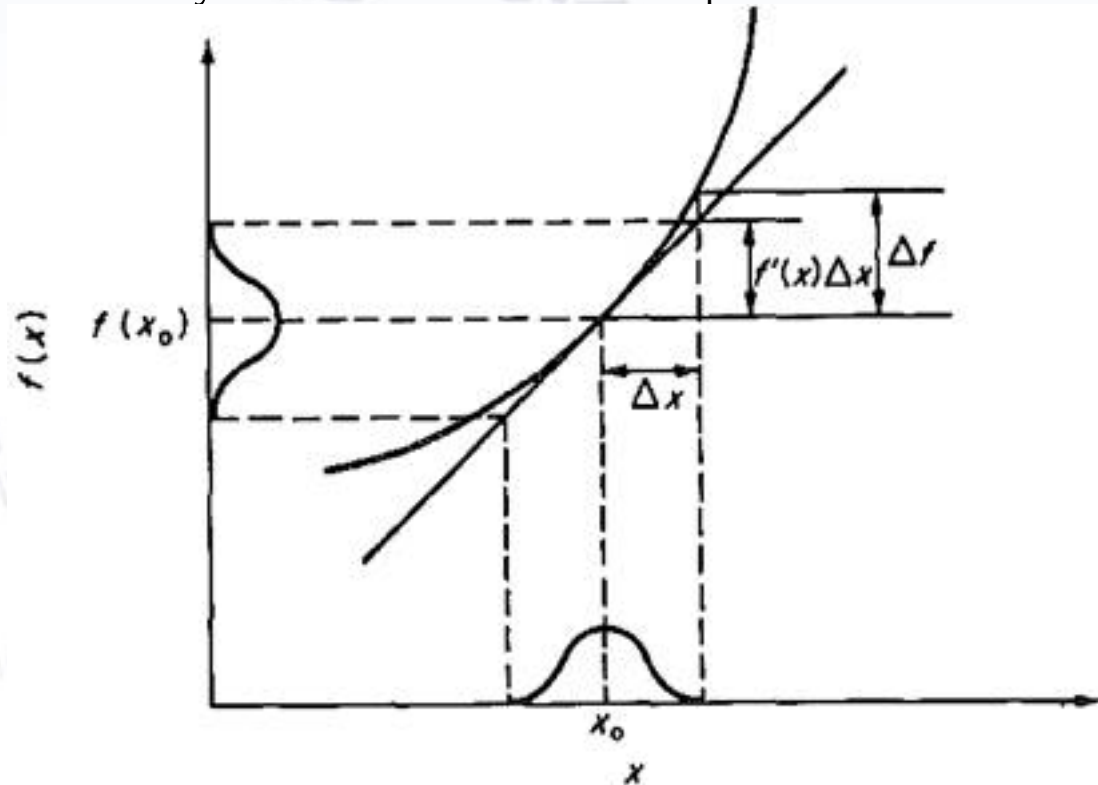
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# Error propagation

Imagine that  $y$  is a function of  $x_i$

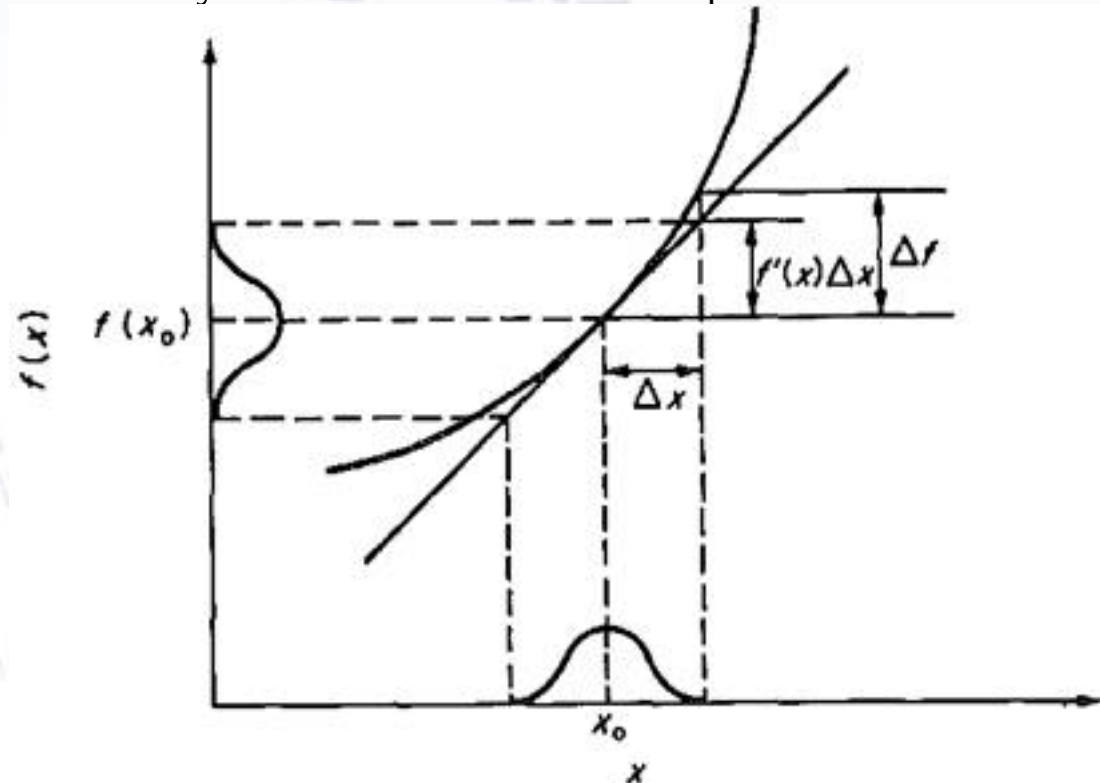
$$y(x_i)$$

and that we wish to find the error on  $y$  from the errors on  $x_i$ .

$$\sigma(x_i) = 0.8$$

$$\sigma(y(x_i)) =$$

$$\frac{\partial y}{\partial x_i} \times 0.8$$



# Error propagation

Note, the approximation here:

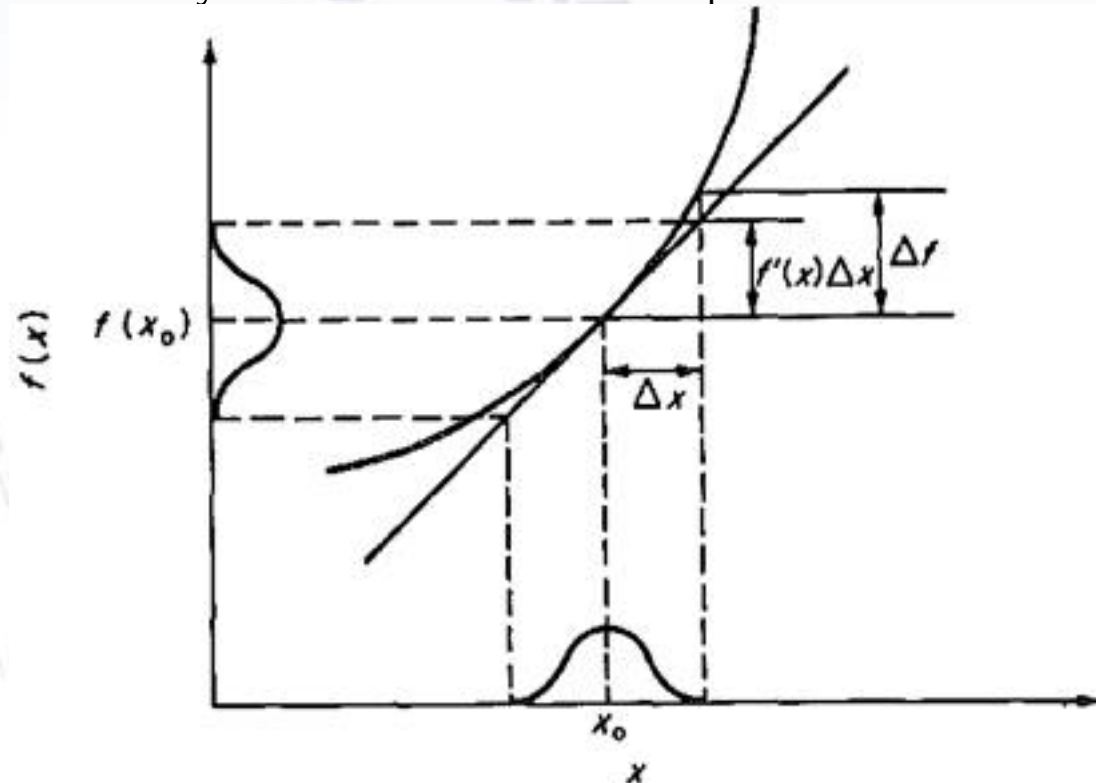
The derivative of  $y - dy / dx_i$  - should be relatively constant.  
If not, the error propagation formula breaks down.

and that we wish to find the error on  $y$  from the errors on  $x_i$ .

$$\sigma(x_i) = 0.8$$

$$\sigma(y(x_i)) =$$

$$\frac{\partial y}{\partial x_i} \times 0.8$$





# General formula

(i.e. can always be used!)

# Error propagation

Imagine that  $y$  is a function of  $x_i$ , and that we wish to find the error on  $y$  from the errors on  $x_i$ . Making a Taylor expansion of the function  $y$  gives:

$$y(\bar{x}) \simeq y(\bar{\mu}) + \sum_i^n \frac{\partial y}{\partial x_i} (x_i - \mu_i)$$

In order to get the uncertainty of  $y$  as a function of the variables  $x_i$  we calculate:

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = E[x^2] - E^2[x]$$

$$E[y(\bar{x})] \simeq y(\bar{\mu})$$

$$E[y^2(\bar{x})] \simeq y^2(\bar{\mu}) + \sum_{i,j}^n \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right] V_{ij}$$



# Error propagation formula

Subtracting the two formulae, we obtain:

$$\sigma_y^2 = \sum_i^n \sum_j^n \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right] V_{ij}$$

# Error propagation formula

Subtracting the two formulae, we obtain:

$$\sigma_y^2 = \sum_i^n \sum_j^n \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right] V_{ij}$$

If there are no correlations, only the diagonal (individual errors) enter:

$$\sigma_y^2 = \sum_i^n \left[ \frac{\partial y}{\partial x_i} \right]_{\bar{x}}^2 \sigma_i^2$$

# Error propagation formula

The “simple” (i.e. uncorrelated) formula is thus:

$$\sigma_y = \sqrt{\left[ \frac{\partial y}{\partial x_1} \right]_{\bar{x}_1}^2 \sigma_1^2 + \left[ \frac{\partial y}{\partial x_2} \right]_{\bar{x}_1}^2 \sigma_2^2 + \dots}$$

# Error propagation formula

The “simple” (i.e. uncorrelated) formula is thus:

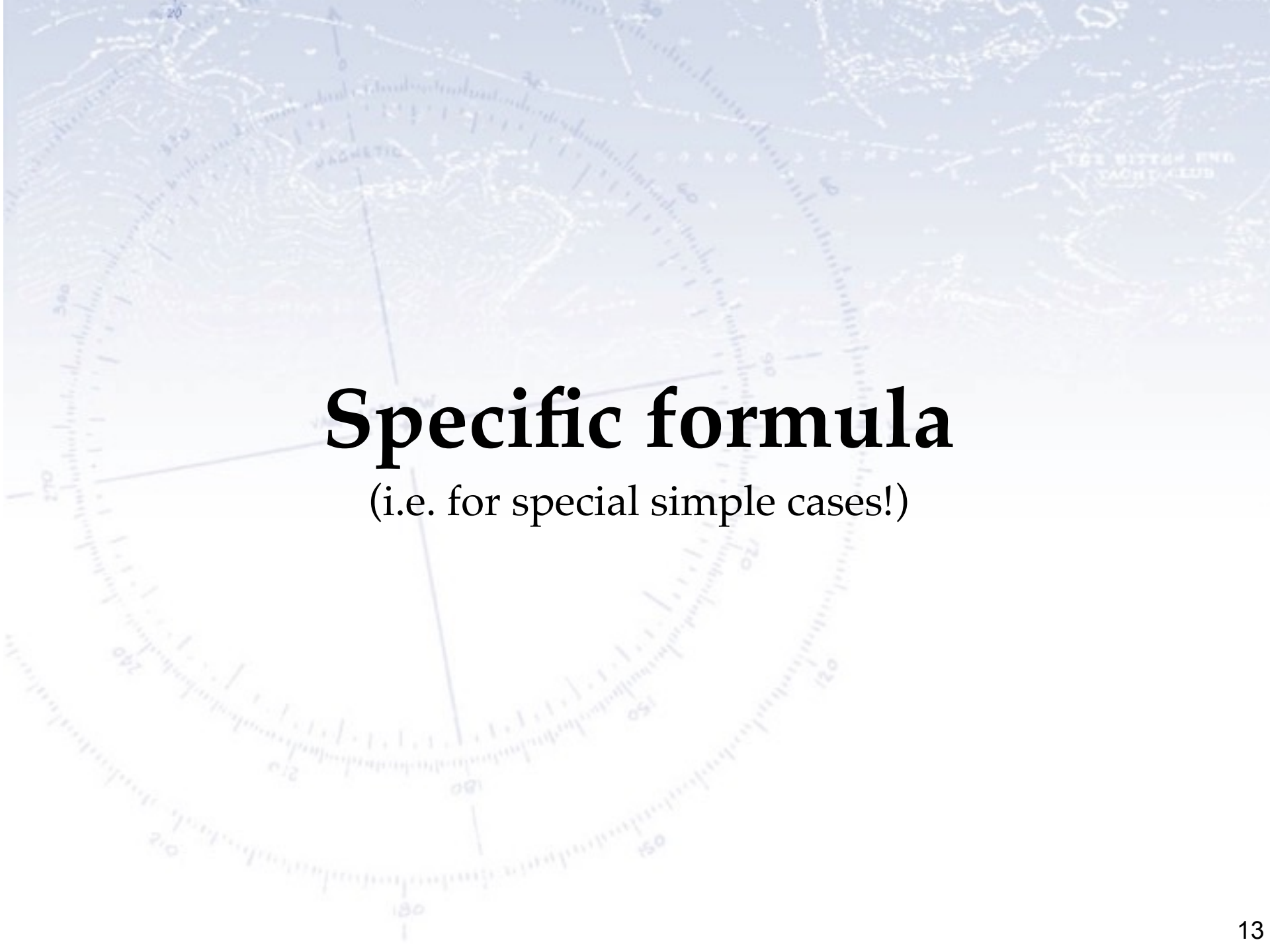
$$\sigma_y = \sqrt{\left[ \frac{\partial y}{\partial x_1} \right]_{\bar{x}_1}^2 \sigma_1^2 + \left[ \frac{\partial y}{\partial x_2} \right]_{\bar{x}_1}^2 \sigma_2^2 + \dots}$$

Note, that **each term** represents the **individual contributions** of  $x_i$  to the uncertainty on  $y$ .

Thus, the uncertainty on  $y$  from e.g.  $x_1$  is:

$$\sigma_y = \sqrt{\left[ \frac{\partial y}{\partial x_1} \right]_{\bar{x}_1}^2 \sigma_1^2} = \frac{\partial y}{\partial x_1} \sigma_1$$





# Specific formula

(i.e. for special simple cases!)

# Specific error propagation formula

## Addition

$$y = x_1 + x_2$$

$$\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2V_{x_1, x_2}$$

$$y = ax_1 + bx_2$$

$$\sigma_y^2 = a^2\sigma_{x_1}^2 + b^2\sigma_{x_2}^2 + 2abV_{x_1, x_2}$$

*“When adding numbers, their errors add in quadrature”*

# Specific error propagation formula

## Multiplication

$$y = x_1 x_2$$

$$\sigma_y^2 = (x_2 \sigma_{x_1})^2 + (x_1 \sigma_{x_2})^2 + 2x_1 x_2 V_{x_1, x_2}$$

Dividing by  $x^2$  to get relative terms, we obtain:

$$\frac{\sigma_y^2}{y^2} = \frac{\sigma_{x_1}^2}{x_1^2} + \frac{\sigma_{x_2}^2}{x_2^2} + 2 \frac{V_{x_1, x_2}}{x_1 x_2}$$

*“When multiplying numbers, their RELATIVE errors add in quadrature”*



# Error propagation at work...

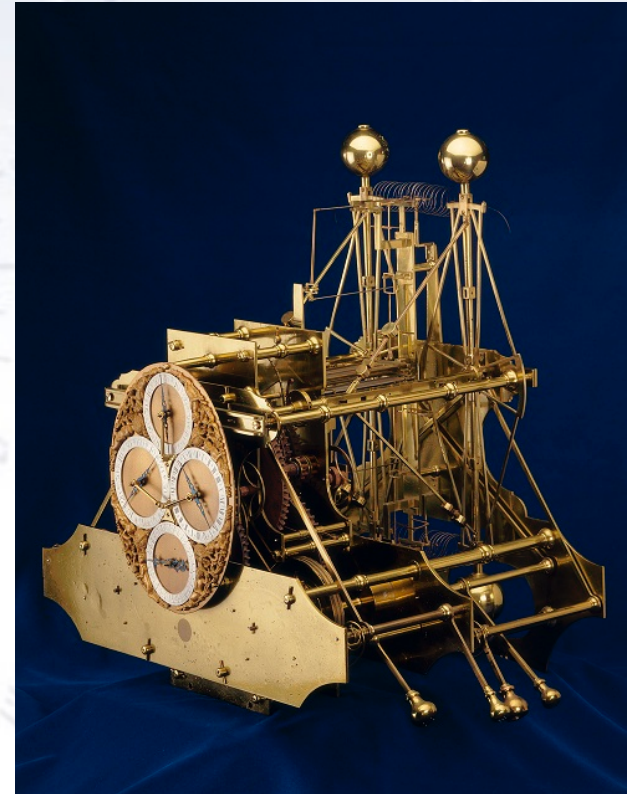


**John Harrison** (24 March 1693 – 24 March 1776)

British clockmaker extraordinaire

“Won” the Longitude Act prize (3 sec/day).

Harrison's first sea clock (H1)



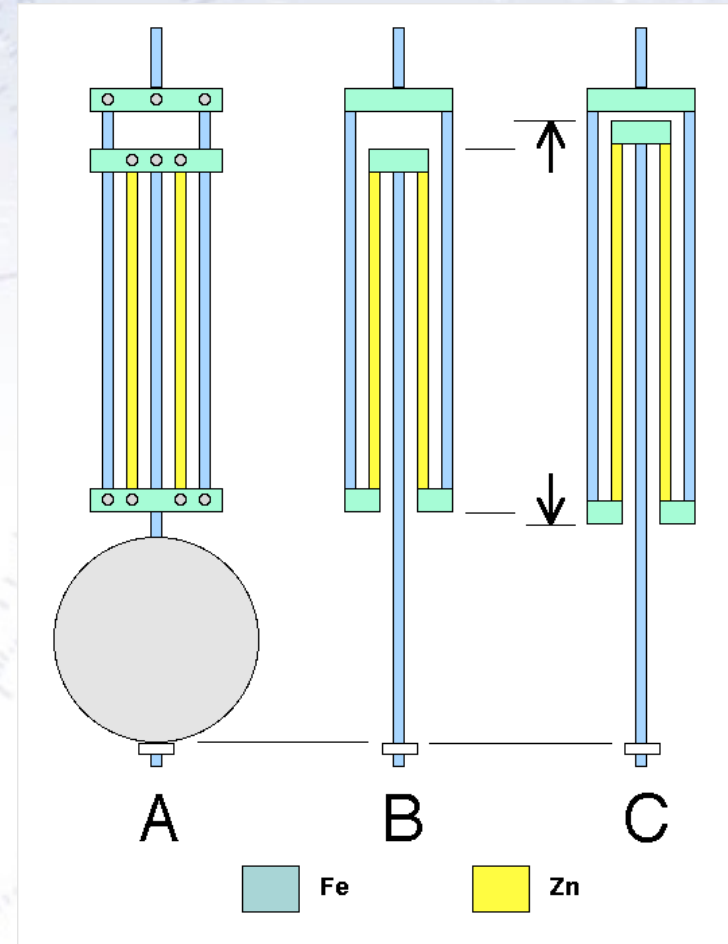
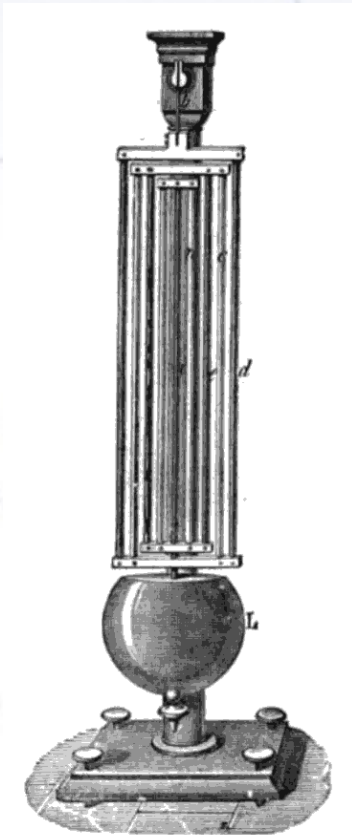
**Harrison build H1-H5.**

K1 (Copy of H4) was used by James Cook.



# Error propagation at work...

Harrison's Gridiron pendulum is designed to cancel the change in length (in fact moment of inertia) with temperature.



Coefficient of thermal expansion:  
Iron =  $11.8 \times 10^{-6} / C^{\circ}$  Zinc =  $30.2 \times 10^{-6} / C^{\circ}$

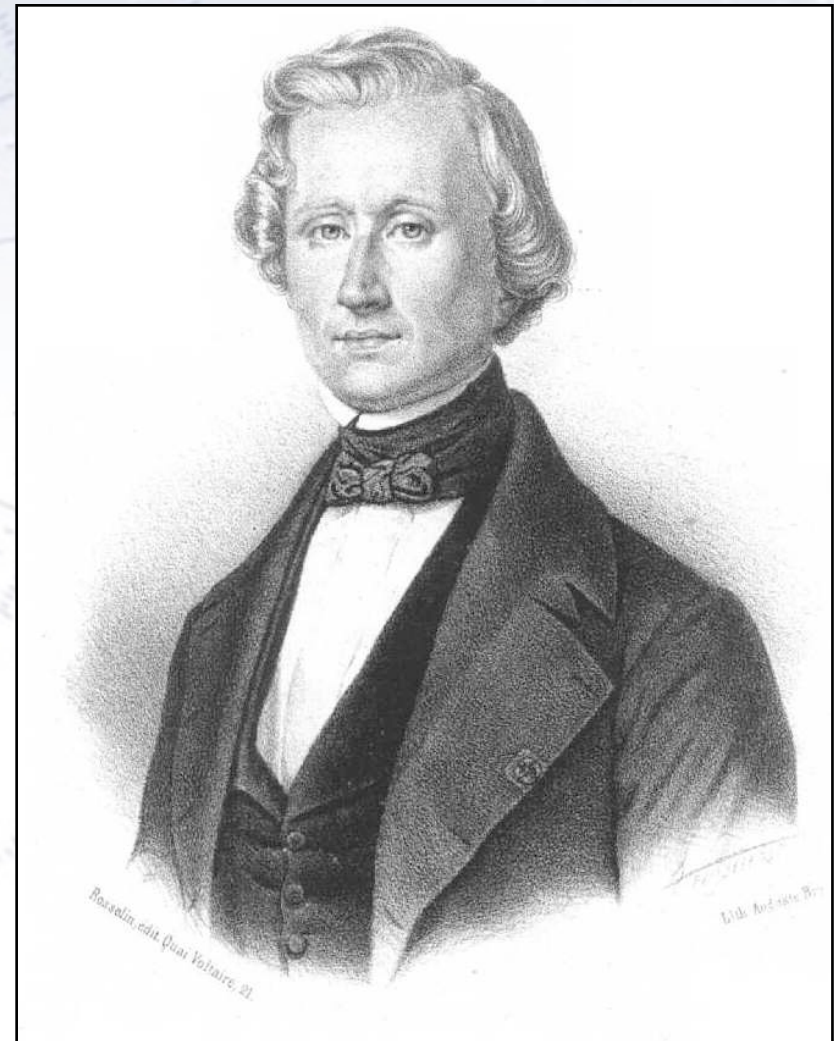
# Error propagation at more work...

Analysis of tiny differences in Uranus' orbit from Newtonian prediction led to the prediction and discovery of Neptune!

Continuing with Mercury...

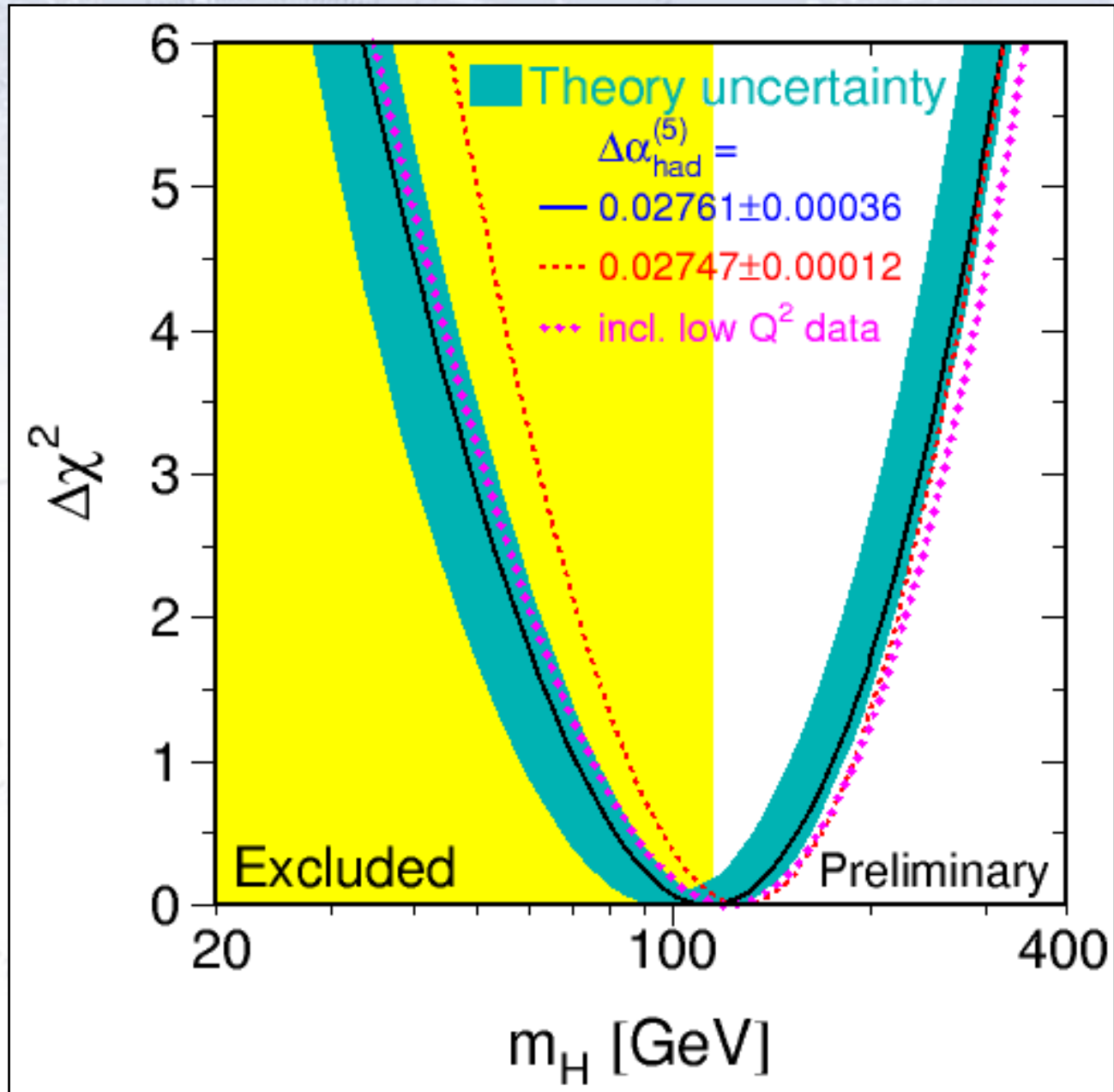
TABLE II. Contributions to the motion of the perihelia of Mercury and the earth.

Cause	$m^{-1}$		Motion of perihelion	
			Mercury	Earth
Mercury	6 000 000	$\pm 1\ 000\ 000$	$0''.025 \pm 0''.00$	$-13''.75 \pm 2''.3$
Venus	408 000	$\pm 1\ 000$	$277.856 \pm 0.68$	$345.49 \pm 0.8$
Earth	329 390	$\pm 300$	$90.038 \pm 0.08$	
Mars	3 088 000	$\pm 3\ 000$	$2.536 \pm 0.00$	$97.69 \pm 0.1$
Jupiter	$1\ 047.39 \pm 0.03$		$153.584 \pm 0.00$	$696.85 \pm 0.0$
Saturn	3 499	$\pm 4$	$7.302 \pm 0.01$	$18.74 \pm 0.0$
Uranus	22 800	$\pm 300$	$0.141 \pm 0.00$	$0.57 \pm 0.0$
Neptune	19 500	$\pm 300$	$0.042 \pm 0.00$	$0.18 \pm 0.0$
Solar oblateness			$0.010 \pm 0.02$	$0.00 \pm 0.0$
Moon				$7.68 \pm 0.0$
General precession (Julian century, 1850)			$5025.645 \pm 0.50$	$5025.65 \pm 0.5$
Sum			$5557.18 \pm 0.85$	$6179.1 \pm 2.5$
Observed motion			$5599.74 \pm 0.41$	$6183.7 \pm 1.1$
Difference			$42.56 \pm 0.94$	$4.6 \pm 2.7$
Relativity effect			$43.03 \pm 0.03$	$3.8 \pm 0.0$



Urbain Le Verrier (1811-1877)

Advanced example of error propagation (Higgs particle mass):



# Reporting uncertainties

The systematic uncertainties of a measurement should be reported in a table, and if measurements are combined, the correlation needs consideration.

Selection criteria	Systematic uncertainties (%)
$K/\pi$ PID	1.0
$\mu$ PID	0.6
Muon selection	0.6
Trigger	1.0
Yields of reference channel	0.4
Efficiency modeling	5.3



# Reporting uncertainties

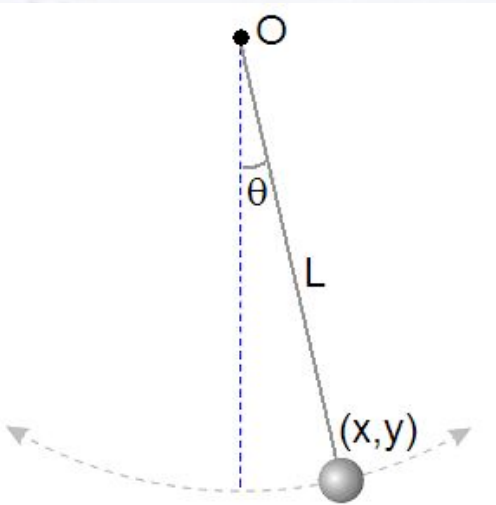
The systematic uncertainties of a measurement should be reported in a table, and if measurements are combined, the correlation needs consideration.

CDF II preliminary		L = 200 pb <sup>-1</sup>		
m <sub>τ</sub> Uncertainty [MeV]	Electrons	Muons	Common	
Lepton Scale	30	17	17	
Lepton Resolution	9	3	0	
Recoil Scale	9	9	9	
Recoil Resolution	7	7	7	
u <sub>  </sub> Efficiency	3	1	0	
Lepton Removal	8	5	5	
Backgrounds	8	9	0	
p <sub>T</sub> (W)	3	3	3	
PDF	11	11	11	
QED	11	12	11	
Total Systematic	39	27	26	
Statistical	48	54	0	
Total	62	60	26	

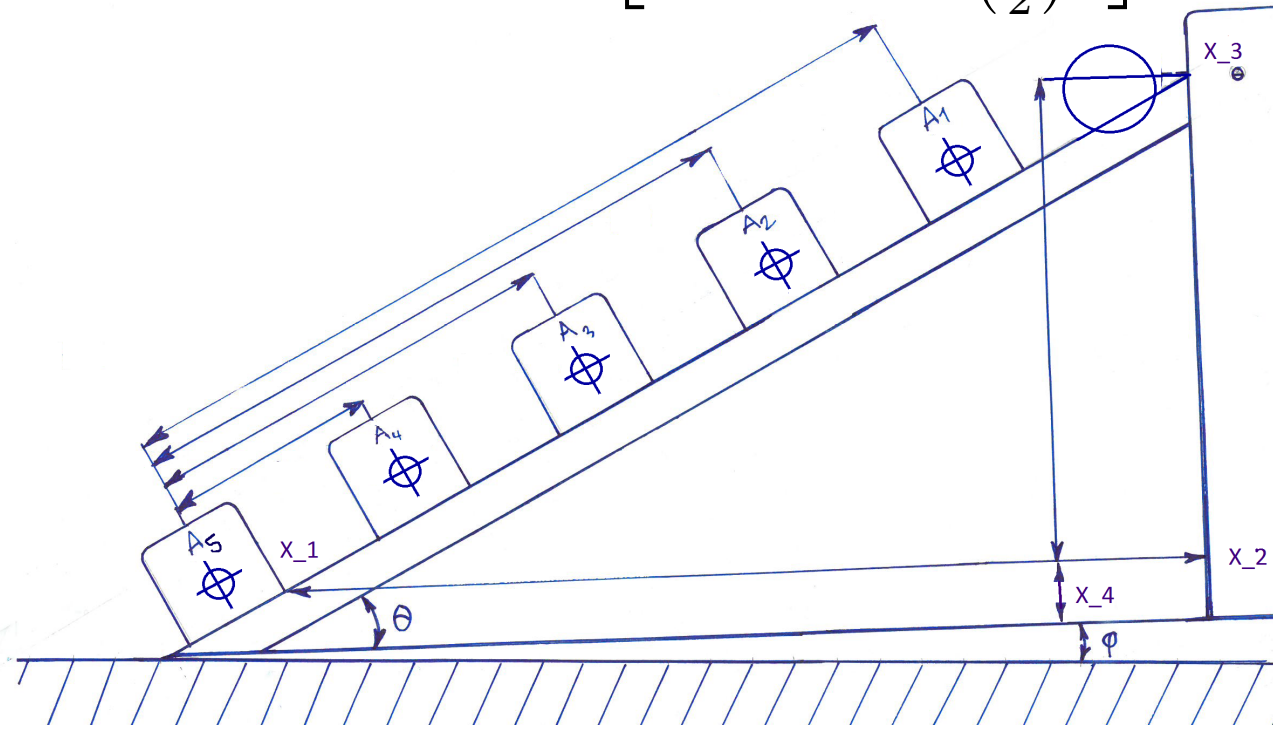
# Applying error propagation

For the project, we'll be working with two measurements, which result from two formulae. Work out the error propagation formula for these two cases, and use these, when we discuss typical size of uncertainties.

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2} \right]$$



$$g = L \left( \frac{2\pi}{T} \right)^2$$



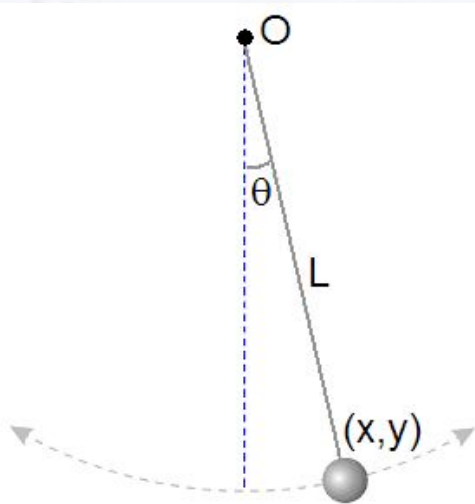
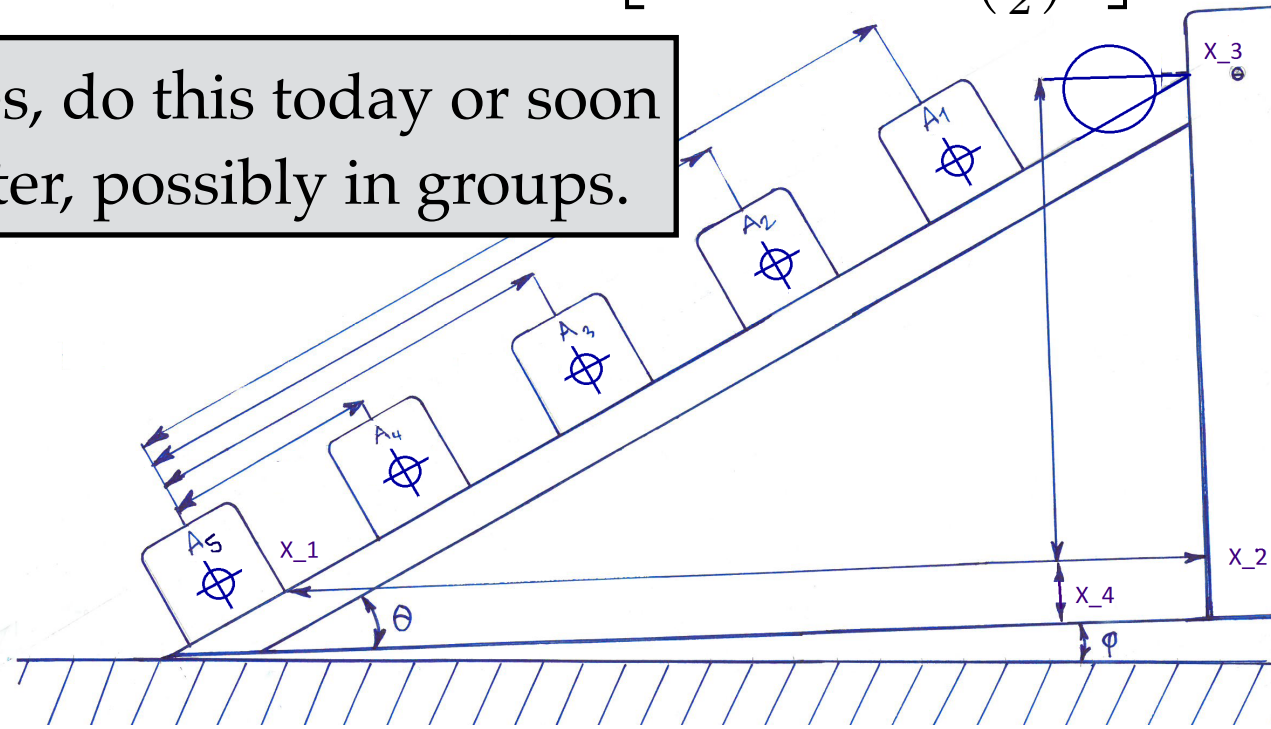
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$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2} \right]$$

Yes, do this today or soon after, possibly in groups.

$$g = L \left( \frac{2\pi}{T} \right)^2$$





# Simulating error propagation

Imagine that  $y$  is a very complicated function of  $x_i$ , perhaps not even parametric (i.e. not a function, but rather a model).

A simple method is to use simulation:

- Choose random values of  $x_i$ , corresponding to mean and Std of each  $x$ .
- Calculate  $y(x_i)$  and record the resulting values.
- The standard deviation (and distribution) of  $y$  reflects the impact of  $x_i$ .

Note that the distribution of  $y$  may NOT be Gaussian, if the error propagation formula breaks down. It is then important to make this clear to the reader.

However, simulation exactly allows one to see to what degree the resulting distribution in  $y$  is Gaussian.



# Errors on errors

The “uncertainty on the uncertainty” follows the approximate rule:

$$\sigma_{\sigma} = \frac{1}{\sqrt{2N - 2}}$$

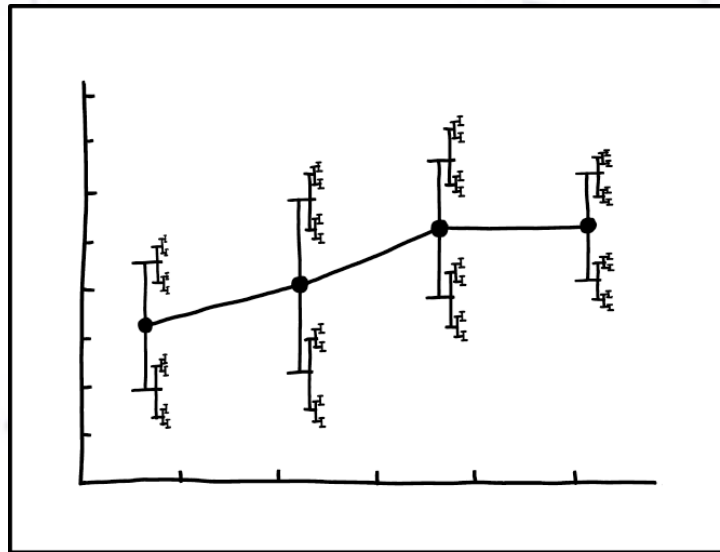
As we don't want to pursue an infinite line of uncertainties, we simply state the uncertainty, and only include one or two significant digits.

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I DON'T KNOW HOW TO PROPAGATE  
ERROR CORRECTLY, SO I JUST PUT  
ERROR BARS ON ALL MY ERROR BARS.