Accelerators
Transverse motion

Peter H. Hansen

University of Copenhagen

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Preface

- The homepage for these 2×2 hour lectures is `http://www.nbi.dk/~phansen/acc`
- The contents stem mostly from:

  CERN school of accelerator physics, CERN Yellow Reports, 1995

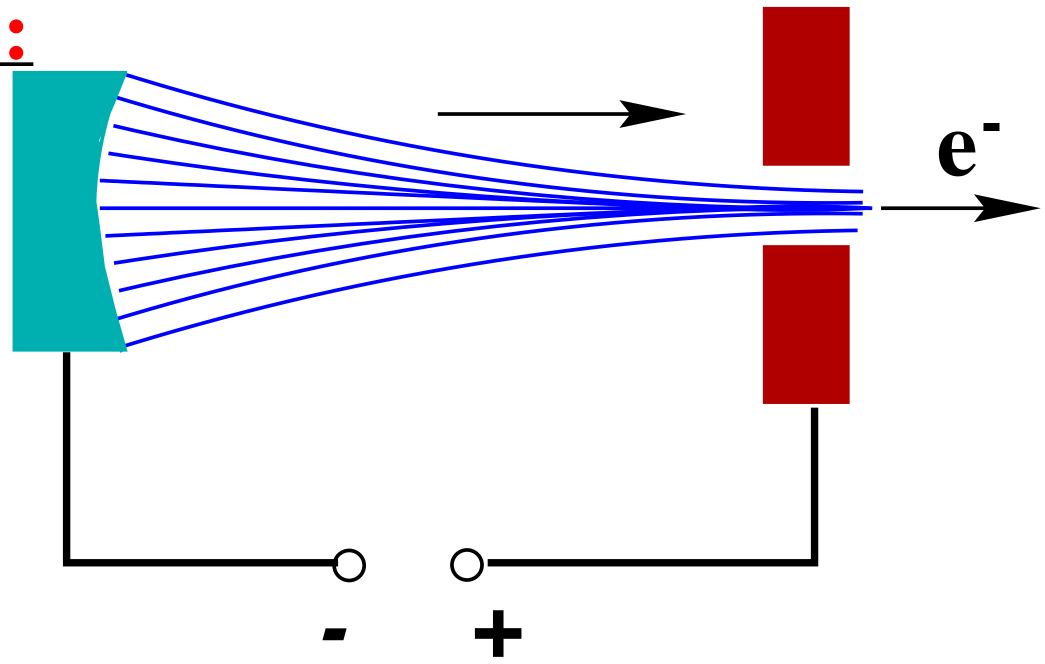
  Here you can find the missing proofs, a large bibliography and many more subjects. All is on the net. Extracts will be distributed.
- For a real text-book, see An introduction to particle accelerators, Wilson, Oxford University Press, 1995.
- or An introduction to the physics of particles accelerators, Mario Conte and William W. MacKay, World Scientific, 1991.

History - the first accelerators

- The first accelerator was the Lenard Cathode Ray Tube (1895), used by Thomson to discover the electron. Both the electron and the CRT have found many practical applications later!
- The Cockcroft-Walton circuit (1932) could rectify and multiply an alternating input voltage. This made possible the first split atomic nucleus. Nuclear fission had many practical applications afterwards, and so had the Cockcroft-Walton.
- The Van de Graff generator is a workhorse of nuclear physics. It brought you nuclear structure and is today used for analysis of trace elements in many sciences – and for hair rising experiments in school.

Particle Sources:

e^-:



Cathode Rays

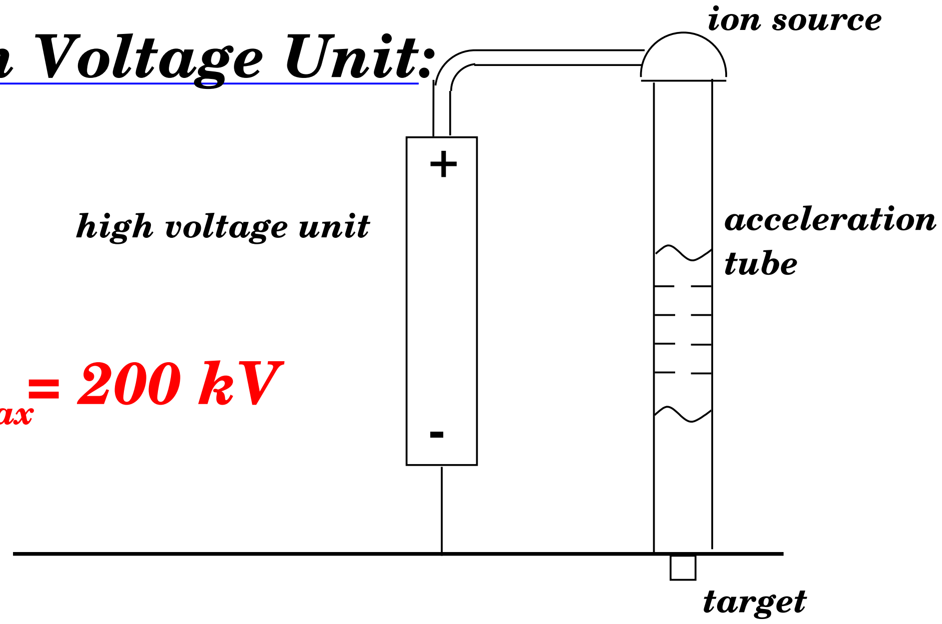
p^+: ***Cathode Tube with H***

$$H_2 + e^- \rightarrow H_2^+ + 2e^-$$
$$H_2^+ + e^- \rightarrow H^+ + H + e^-$$
$$H + e^- \rightarrow H^+ + 2e^-$$

Antimatter: ***Pair Production***

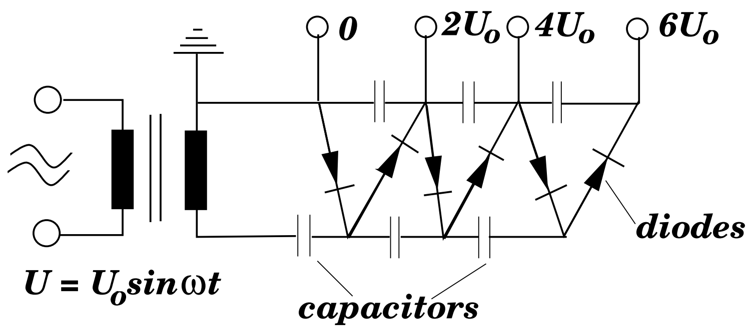
Electrostatic Fields

High Voltage Unit:



$V_{max} = 200\ kV$

Cascade Generator:

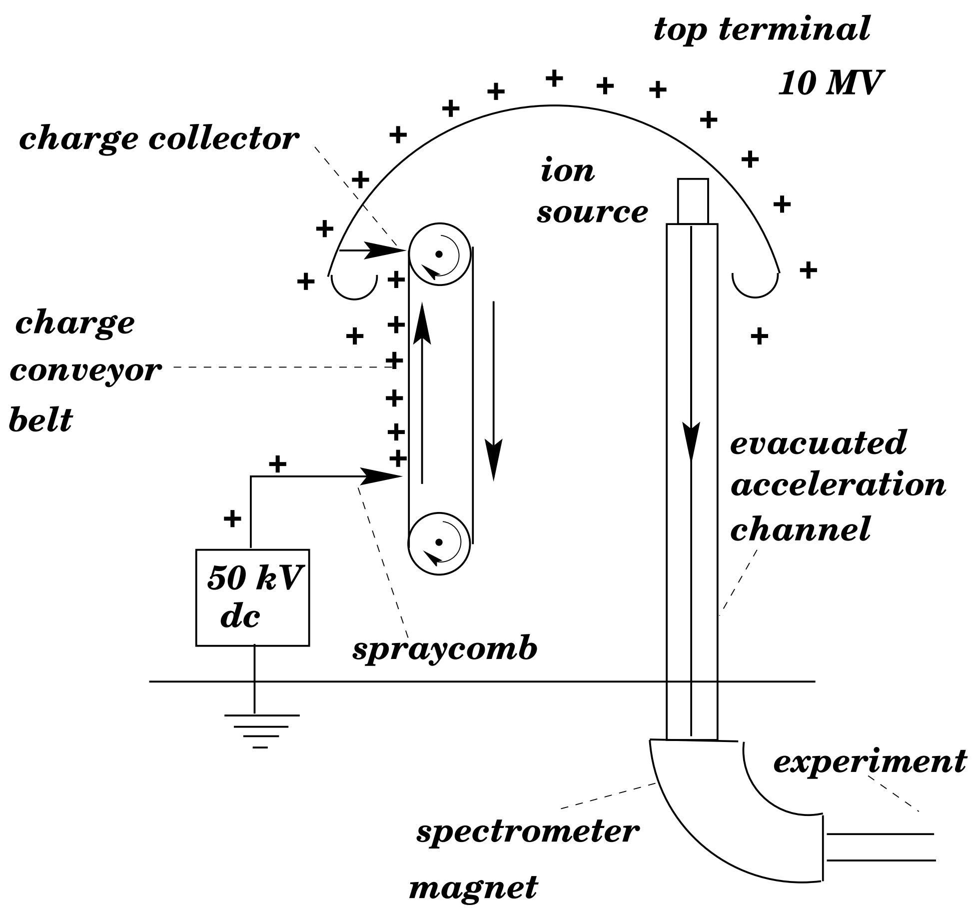


- **1928:** Cockroft + Walton 800kV
- **1932:** $p + Li \rightarrow 2\ He$ 700kV (p)

  (Nobel Prize 1951)

Van de Graaf Generator

Single Unit:



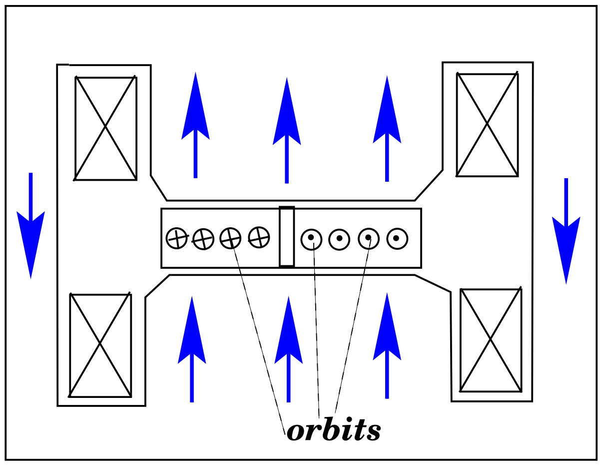
$\longrightarrow$ V_{max} = 10 MVolt

Cyclotrons

- The cyclotron (Lawrence 1929) is like a shoe-polish box, split along its diagonal, with an alternating electric field applied across the gap and a constant perpendicular magnetic field all over the box.
  The particle velocity is increased each time the gap is crossed, and the resulting larger orbit ensures that it will hit the alternating field at the same phase the next time around.
  This device reaches higher energy than the Van de Graff, but does not work at highly relativistic energies.

Circular Accelerators I

- **1929:** Lawrence $\longrightarrow$ Cyclotron



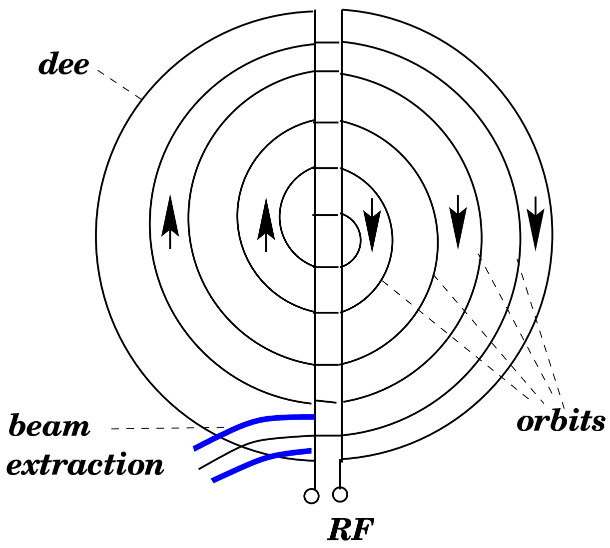
$$\omega = \frac{Q}{m} \cdot B$$

$$r = \frac{m}{Q} \cdot \frac{v}{B}$$

$m = const$

$\longrightarrow$ $f_{RF} = const$

$B = const$



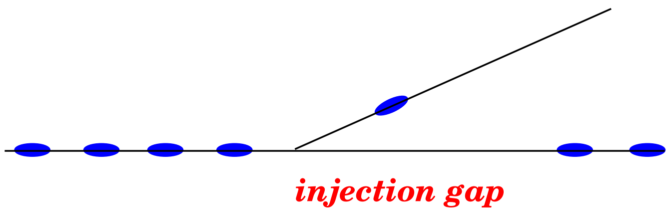
- **1931:** Livingston $\longrightarrow$ H^- to 80 keV
- **1932:** Lawrence $\longrightarrow$ p to 1.2 MeV (NP 1939)

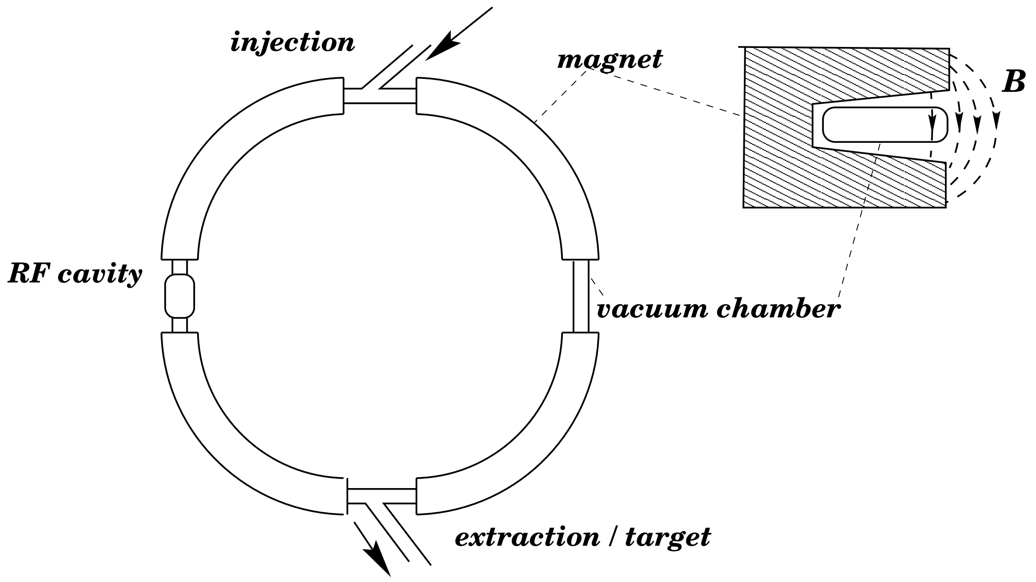
Synchrotrons

- The synchrotron (McMillan, Veksler 1944), replaces the split shoe-polish box by a circular vacuum chamber immersed in a vertical magnetic field. As the energy of the beam particle increases by repeated passages of accelerating cavities, the frequency of the cavities is changed and the magnetic field is ramped up to maintain the radius of the orbit.
- The The first synchrotrons were weak focusing machines. At high energy, this leads to very bulky and expensive dipole magnets. The concept of strong focusing (Livingstone, Courant, Snyder, Cristofilos 1952) allowed to keep the dimensions of the vacuum chamber small. All high energy synchrotrons are strong focusing. (More about that later).

Synchrotron Inventory

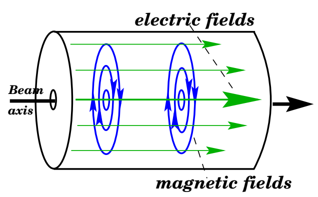
- ***Injection:***





- ***Ejection***

- ***RF Cavity***



- ***Bending Magnets***

Storage rings

- The Storage rings builds up high intensity particle bunches and have them rotating for many hours. An example is ASTRID in Aarhus where ions are stored and cooled by laser techniques.
- If two stored beams are brought to collision, we have a colliding beam machine. The first proton-collider was the ISR, CERN 1972-1983, colliding up to 30 GeV proton-beams. At the collision points each bunch meets much fewer particles than the N_A atoms encountered in a fixed target, but they meet the other bunches again and again.
- More importantly, the center-of-mass energy, available for production of new particles, grows linearly with the beam energy and not like the square-root as in a fixed target experiment.

Luminosity

- In Storage rings the name of the game is to maximize the luminosity:

$$N_{\mathrm{events}}/\mathrm{s} = L\sigma \ ([L] = \mathrm{cm}^{-2}/\mathrm{s})$$

- The luminosity is given by

$$L = \frac{n_{\mathrm{bunches}} N_1 N_2 f_{\mathrm{rev}}}{A}$$

- The limitations are mainly due to collective beam-beam interactions, blowing up the beam area, and magnet hardware.

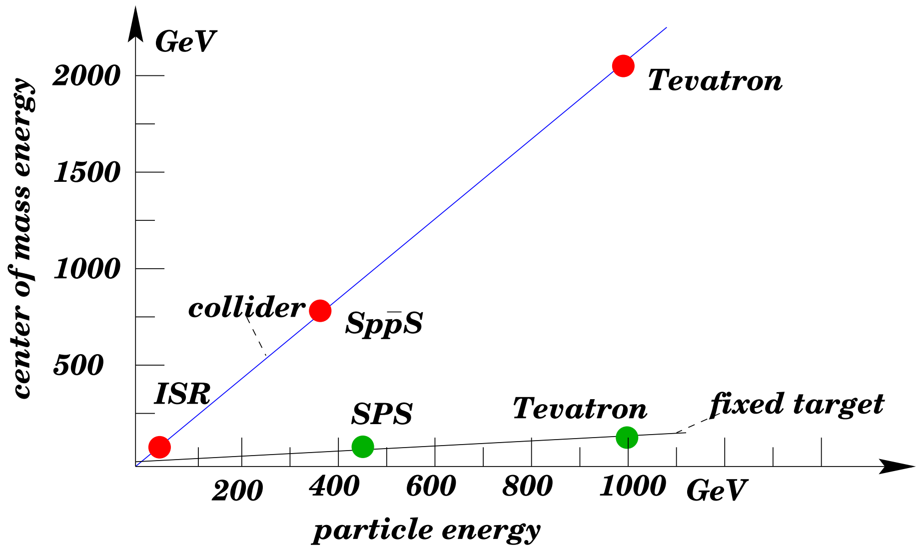
Collider Rings

1960: fixed target physics (bubble chamber)

But:

$$E_{cm} = 2 \cdot m_0 \cdot c^2 \left(\sqrt{1 + \frac{E}{2 \cdot m_0 \cdot c^2}} - 1 \right)$$

Collider: $E_{CM} = 2 \cdot E_p$



1960 ↗ : e^+ / e^- collider

1970 ↗ : p^+ / p^- collider

Cooling of positrons

- In a storage ring, the same vacuum chamber and magnet system can be used to have anti-particle bunches rotating the other way around. However, these particles are created with strongly diverging momenta and need to be “cooled” to fit into the beam-pipe.
- For electrons and positrons, the cooling takes place automatically from their emission of synchrotron radiation, carrying away the “heat”.
- Two very successful examples of e^+e^- storage rings were SPEAR, where the τ and the J/Ψ were discovered, and LEP where the Standard Model stood its toughest test.

Cooling of anti-protons

- The Stochastic cooling (Van der Meer, 1972) enabled storing of anti-protons. This led to the SPS collider (CERN 1981-1988), where the W and Z were discovered, and the Fermilab Tevatron, where the top-quark was discovered and where the highest energy collisions today of 1×1 TeV are available.
- A technique called electron cooling works better at lower energies. Here the “heat” of a hot anti-proton beam is transferred to a cold electron beam.

Linear accelerators

- The first Linear accelerator structures were built by Widerø (1928). The most famous realization is the Stanford Linear Accelerator, where the quark structure of the proton was demonstrated in 1969.
- They are today used as injectors to circular accelerators and for many practical purposes (e.g. sterilizing hospital equipment).
- In the future, linear e^+e- colliders will provide precision measurements at the energy frontier and provide X-rays of laser intensity.

Applications

- -Particle physics is always at the Energy Frontier and drives the development of still more powerful and compact accelerator structures.
- -Nuclear physics has at least three main lines of research: exotic nuclei using high intensity ion cyclotrons or synchrotrons, quark structure using high intensity electron synchrotrons and quark-gluon plasma using high energy colliding heavy ions.
- -Atomic physics has about 10^4 machines around the world to create ion beams from zero to about 10MeV. The important thing is to create very cold beams.
- -Astrophysics is interested in any of the above.

Applications

- In Condensed matter physics, chemistry and material science increasing use is made of X-rays from electron synchrotron radiation. The need is for ever more intense and higher energy X-ray sources.
- Complementary to the X-rays, neutron spallation sources provide knowledge about spin-structure and the hydrogen component in materials. Such a source requires an extremely intense proton accelerator.
- In Biology and medicine the same tools can determine the structure of proteins in crystalline form.

Applications

In the future there are plans of huge advances in these fields. Most spectacular is DESY's TESLA proposal which includes a Free Electron Laser increasing the X-ray power by a staggering factor of 10^{10} over existing machines. The proposal for a new European Neutron Spallation Source will also significantly increase performance over existing neutron sources.

Applications

- -Cancer treatment is sometimes impossible with standard techniques without fatally damaging other essential tissue. A solution may be provided by ion beam therapy, typically using a few hundred MeV light ion beams, which deliveres its effect at a well defined depth of tissue.
- -Geology, Archeology and Environmental Sciences uses smaller accelerators for isotope-analysis of specimens for dating and trace-element finding.
- -Metal and plastic manufactoring industry uses “pocket-size” accelerators for ion-implantation of tools and products. Suitable ion-implantation can increase hardness and resistance to corrosion by factors of 100 or more.

Applications

- -In the electronics industry, ion-implantation is a standard technique to change the transport properties of some substrate, and large scale studies are made to investigate the replacement of the standard “Fabs” by synchrotron X-ray sources etching smaller structures into silicon.
- -In Energy Production there are some brilliant ideas of making a beam catalyzed fission reactor using natural Uranium or Thorium. This would be completely safe, use an unlimited natural supply and could at the same time be used to get rid of longlived radioactive waste.

Equations of motion

- Consider a accelerator where the design particle orbits in a circle with radius ρ and momentum $p_0 = qB_0\rho$.
- Define a local co-moving system with x radially outwards and y upwards perpendicular to the accelerator plane. Define $R = \rho + x$.
- Since the variation of energy with time is negligible, we have
  the equation of horizontal motion for any particle:

$$\frac{d}{dt}(\gamma m \dot{x}) \approx \gamma m(\ddot{x} - \frac{v^2}{R}) = \gamma m \left(\frac{v^2}{R^2}\frac{d^2x}{d\theta^2} - \frac{v^2}{R}\right) = -qvB_y$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{qB_y}{\gamma m v}R - 1\right)R = 0$$

Equations of motion

- Expanding in $x << \rho$, we get

$$\frac{d^2x}{d\theta^2} + \left[\left(1 + \frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right)\left(1 + \frac{x}{\rho}\right) - 1\right]\rho\left(1 + \frac{x}{\rho}\right) \approx 0$$

$$\frac{d^2x}{d\theta^2} + \left(\frac{x}{\rho} + \frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right)\rho \approx 0$$

Equations of motion

- Defining now the Field Index:

$$n = -\frac{\rho}{B_0}\left(\frac{\partial B_y}{\partial x}\right)_{x=0}$$

we get for the horizontal motion:

$$\frac{d^2x}{d\theta^2} + (1-n)x = 0$$

Equations of motion

- Clearly, simultaneous focusing in both planes can not occur unless

$$0 < n < 1$$

- Under this condition the particles undergo socalled betatron oscillations in the transverse plane.

Weak focusing synchrotron

Consider, for example, a ring of identical radially opening dipoles:

$$B_y = B_0 - gx\ ,\ \ B_x = -gy$$

Such an accelerator is called a Weak Focusing Synchrotron. Here we have

$$n = \frac{g}{B_0}\rho$$

Weak focusing synchrotron

Changing back to the time variable using $\theta = \omega_0 t$ where ω_0 is the cyclotron frequency $\frac{eB_0}{m}$, we get

$$\ddot{x} + \omega_0^2(1-n)x = 0$$

$$\ddot{y} + \omega_0^2 n y = 0$$

So the frequency of the betatron oscillations is smaller than the revolution frequency. The amplitude therefore scales with radius and therefore poses increasing demands on the size and cost of the vacuum chamber and the magnets with increasing momentum.

Solutions

- If $0 < n < 1$ solutions to the horizontal equations of motion are:

$$x(\theta) = A\cos\sqrt{1-n}\theta + B\sin\sqrt{1-n}\theta$$

$$\frac{dx}{d\theta} = \sqrt{1-n}(-A\sin\sqrt{1-n}\theta + B\cos\sqrt{1-n}\theta)$$

- Defining $x_0 = x(0)$ and $x' = dx/ds = (1/\rho)(dx/d\theta)$ we get:

$$A = x_0 \ , \ B = \frac{\rho}{\sqrt{1-n}}x_0'$$

Solutions

- Summarizing in matrix notation:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{1-n}\theta & \frac{\rho}{\sqrt{1-n}}\sin\sqrt{1-n}\theta \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{1-n}\theta & \cos\sqrt{1-n}\theta \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- and similarly for the vertical motion:

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{n}\theta & \frac{\rho}{\sqrt{n}}\sin\sqrt{n}\theta \\ -\frac{\sqrt{n}}{\rho}\sin\sqrt{n}\theta & \cos\sqrt{n}\theta \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

These are the Transfer Matrices, $M_H(\theta)$ and $M_V(\theta)$.

Momentum dispersion

Allowing now for deviations from a monochromatic beam, the horizontal equation of motion becomes (to first order):

$$\frac{d^2x}{d\theta^2} + \left[\left(1 - \frac{\delta p}{p}\right) \left(1 + \frac{1}{B_0}\frac{\partial B_y}{\partial x}x\right) \left(1 + \frac{x}{\rho}\right) - 1 \right] \rho \left(1 + \frac{x}{\rho}\right) \approx 0$$

$$\frac{d^2x}{d\theta^2} + (1-n)x \approx \rho \frac{\delta p}{p}$$

Momentum dispersion

Solutions are a particular solution plus any solution to the homogeneous equation. The one fullfilling the boundary conditions is (without proof):

$$\begin{pmatrix} x \\ x' \\ \frac{\delta p}{p} \end{pmatrix} = \begin{pmatrix} \cos\sqrt{1-n}\theta & \frac{\rho}{\sqrt{1-n}}\sin\sqrt{1-n}\theta & \frac{\rho}{1-n}(1-\cos\sqrt{1-n}\theta \\ -\frac{\sqrt{1-n}}{\rho}\sin\sqrt{1-n}\theta & \cos\sqrt{1-n}\theta & -\frac{1}{\sqrt{1-n}}\sin\sqrt{1-n}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x'_0 \\ \left(\frac{\delta p}{p}\right)_0 \end{pmatrix}$$

For $n = 0$, we have a dipole sector magnet. If furthermore $\theta = \pi$, we have a mass spectrometer, where $\pm\frac{\delta p}{p}$ will cause $\delta x = \pm 2\rho\frac{\delta p}{p}$.

Free drift space

- In a region of length l without magnetic fields the transport matrix in both directions is:

$$M = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multipole expansion of B-fields

- From Maxwells equations with $\vec{j} = \vec{0}$:

$$\mathrm{div}\vec{B} = 0 \;\Rightarrow\; \vec{B} = \mathrm{rot}\vec{A} \;\Rightarrow\; \vec{A} = (0, 0, A_s)$$

$$\mathrm{rot}\vec{B} = \vec{0} \;\Rightarrow\; \vec{B} = -\nabla\vec{V}$$

Multipole expansion of B-fields

- The analycity of the complex function $\tilde{A} = A_s + iV$ ensures that it can be written as a polynomium:

$$\tilde{A} = \sum k_n z^n \;, \quad z = x + iy = r(\cos\phi + i\sin\phi)$$

$$B_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = B_0 \sum (b_n \cos n\phi + a_n \sin n\phi) \left(\frac{r}{r_0}\right)^{n-1}$$

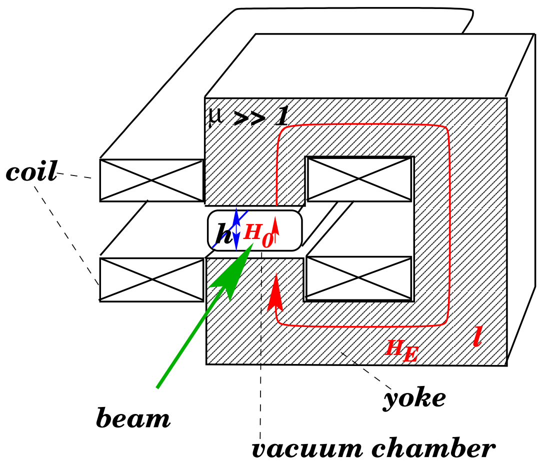
$$B_r = -\frac{\partial V}{\partial r} = B_0 \sum (-a_n \cos n\phi + b_n \sin n\phi) \left(\frac{r}{r_0}\right)^{n-1}$$

- An ideal $2n$-pole has $b_n = 1$. To obtain this, the iron pole shoe is formed as an equipotential plane:

$$(r/r_0)^{n-1} \sin n\phi = 1$$

Bending Magnet

- $\oint H = I \cdot N$ $\qquad$ $B = \mu_0 \cdot \mu \cdot H$



$\mu < 1$: *Dia*

$\mu > 1$: *Para*

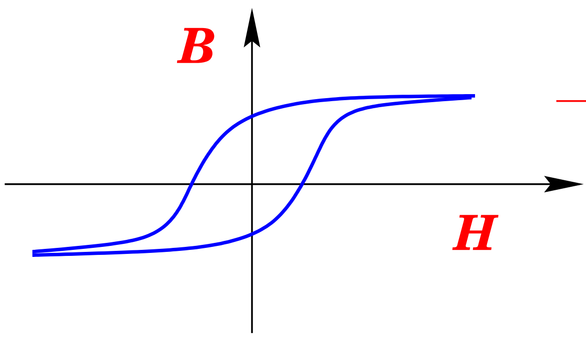
$\mu >> 1$: *Ferro*

- ***Maxwell Equations:***

$$B_{0\perp} = B_{E\perp}$$

$$\oint H = h \cdot H_0 + l \cdot H_E$$

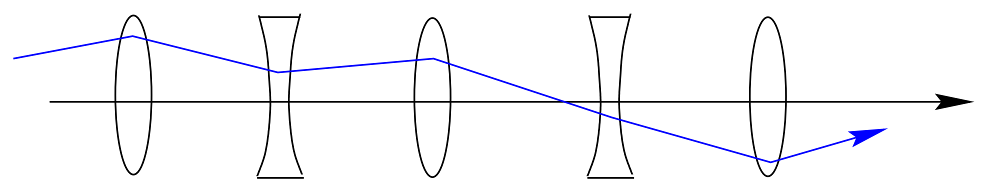
$$H_0 = \mu \cdot H_E$$



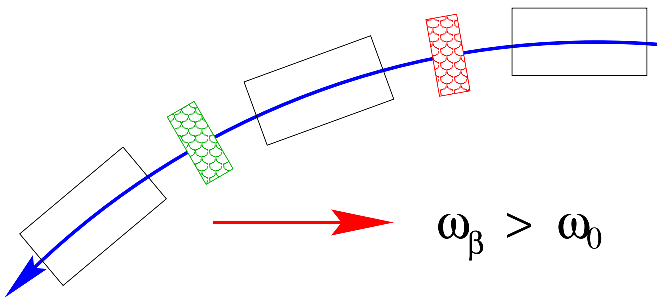
$$B_0 = \mu_0 \frac{N\,I}{h}$$

$$\frac{1}{\rho}\,[m^{-1}] = \frac{e \cdot B}{p} = 0.3 \cdot \frac{B\,[T]}{p\,[GeV]}$$

Alternate Gradient Focusing



Idea: ***cut the arc sections in focusing and defocusing elements***



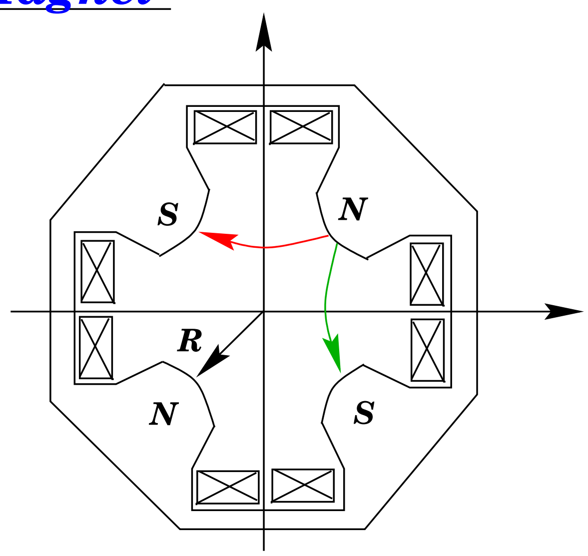
Quadrupole Magnet

$$B_x = -g \cdot y$$

$$B_y = -g \cdot x$$

$$F_x = -q \cdot v \cdot B_y$$

$$F_y = q \cdot v \cdot B_x$$



Quadrupole

- Let the quadrupole field be $B(r) = gr$, and define its focusing strength: $k = eg/p = 0.3g/p$ (with g in T/m and p in GeV/c). The focal length is $f = 1/kl$.
- The vertical equation of motion is for positive k:

$$y'' + ky = 0$$

where the differentiation is w.r.t. path length, s.

Quadrupole

- Define furthermore a phase $\phi = l\sqrt{|k|}$.
- The transport matrix is focusing in the vertical and defucusing in the horizontal direction:

$$M_y = \begin{pmatrix} \cos\phi & \frac{1}{\sqrt{|k|}}\sin\phi & 0 \\ -\sqrt{|k|}\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_x = \begin{pmatrix} \cosh\phi & \frac{1}{\sqrt{|k|}}\sinh\phi & 0 \\ \sqrt{|k|}\sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- If $k < 0$ it is opposite (focusing in the horizontal and defocusing in the vertical plane).

Thin lens

- Let the length l of a quadrupole go to zero while keeping kl constant, so that

$$f = \frac{1}{kl} >> l$$

- The transport matrix now simplifies to:

$$M_{x(y)} = \begin{pmatrix} 1 & 0 & 0 \\ \pm\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the Thin Lens Approximation which holds in larger accelerators.

Strong focusing unit cell

- Consider now a quadrupole doublet, i.e. a defocusing lense and a focusing lense separated by a drift-space l.
- In the horizontal plane, we get (in the vertical plane, f is replaced by $-f$):

$$
\begin{aligned}
M_{\mathrm{Doublet},x} &= M(\frac{1}{f})\ M_{\mathrm{Drift}}(l)\ M(-\frac{1}{f}) \\
&= \begin{pmatrix} 1-\frac{l}{f} & l & 0 \\ -\frac{l}{f^2} & 1+\frac{l}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

- It is seen from the $-l/f^2$ term (and known from geometrical optics) that such a doublet has a net focusing effect in both directions.
- This is the basis of the Strong Focusing Synchrotron, where periodic focusing and defocusing lenses keep the transverse dimensions of the beam small.
- A typical unit cell is the FODO cell (Focus-nofield-Defocus-nofield). This is repeated all around the accelerator with dipole magnet bending-arcs put in between.

Strong focusing unit cell

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\end{aligned}
$$

Hill's equation

- Consider a periodic lattice of quadrupoles, dipoles and drift-spaces. Ignoring, for now, the momentum dispersion, the equations of motion takes for both of the two coordinates the form of the Hill's equation:

$$\begin{aligned} y'' + K(s)y &= 0 \\ K(s+L) &= K(s) \end{aligned}$$

Solutions to Hill's equation

- The solution, in terms of the transport matrix $M(s|s_0)$ for the periodic cell, is

  stable if $M^n(s+L,s)$ is finite.

- To find stable solutions, we look for eigenvalues of M. Using $\det M = 1$, the eigenvalue equation reads:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda I\right) = 0$$

$$\lambda^2 - \lambda \mathrm{Tr} M + 1 = 0$$

$$\lambda_1 = e^{i\mu}\ , \quad \lambda_2 = e^{-i\mu}$$

where

$$\cos\mu = \frac{1}{2}\mathrm{Tr} M \neq \pm 1$$

Twiss parameters

- Such a matrix can be written in Twiss form:

$$M = I\cos\mu + J\sin\mu\ ,\ \text{where } J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\det M = 1 \ \Rightarrow\ \beta\gamma - \alpha^2 = 1$$

so that

$$J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

(just like the complex i).

Twiss parameters

- For n cells we get:

$$M^n = I \cos n\mu + J \sin n\mu$$

  So the solution is stable if $|\mathrm{Tr} M| < 2$, or in other words, if μ is real.

- Floquets Theorem states that the two linearly independent solutions to the periodic lattice can be written as

$$y = e^{\pm i\mu s/L}\, p(s)$$

  where μ is constant and $p(s)$ is composed of periodic functions α, β and γ.

The β-function

One of the Twiss parameters is easily eliminated:

$$\det M = 1 \;\Rightarrow\; \gamma(s) = \frac{1 - \alpha^2}{\beta}$$

The β-function

The next is not so easy. For the eigenfunctions we have:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_{s+L} = M \begin{pmatrix} y \\ y' \end{pmatrix}_s = e^{\pm i\mu} \begin{pmatrix} y \\ y' \end{pmatrix}_s$$

$$\Rightarrow \; y\alpha + y'\beta = \pm iy \; \Rightarrow \; \frac{y'}{y} = \frac{\pm i - \alpha}{\beta}$$

$$\Rightarrow \; \alpha = -\frac{1}{2}\beta'(s)$$

(where many steps have been omitted). Anyway, we got rid of everything, except the β-function, $\beta(s)$.

The orbit solution

Substituting for α, we get the differential equation:

$$\frac{y'}{y} = \frac{\pm i + \frac{1}{2}\beta}{\beta}$$

$$\Rightarrow \; y = a\sqrt{\beta(s)}\, e^{\pm i\Phi(s)}, \text{ where } \Phi' = \frac{1}{\beta(s)}$$

The orbit solution

The parameters of the solutions are:

- local amplitude $a\sqrt{\beta(s)}$
- local wavelength $2\pi\beta(s)$
- phase advance per cell $\mu = \int_s^{s+L} \frac{dt}{\beta(t)}$
- oscillations per turn $Q = \frac{N\mu}{2\pi}$

Emmittance

- The real part of the solution:

$$y(s) = a\sqrt{\beta(s)}\cos[\Phi(s) - \delta]\ ;\ \Phi(s) = \int \frac{ds}{\beta(s)}$$

$$y'(s) = -\frac{a}{\beta(s)}\left(\sin[\Phi(s) - \delta] + \alpha\cos[\Phi(s) - \delta]\right)$$

parametrizes an ellipse in the y, y' plane, centered at zero, and reaching

$y'_{\max} = \sqrt{\epsilon\gamma}$ and $y_{\max} = \sqrt{\epsilon\beta}$.

Emmittance

- The area of the largest possible ellipse is $\pi\epsilon = \pi a_{\rm max}^2$, where ϵ is called the emmittance.
- The ellipse equation can be written:

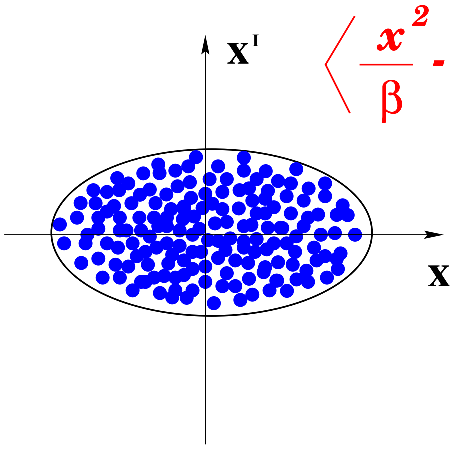
$$\gamma y^2 + 2\alpha y y' + \beta' y'^2 = \epsilon$$

  which is the Courant-Snyder equation.
- Typicallly, we want $\epsilon \approx 10\text{mm mrad}$ in order to have a beam-pipe of few cm.

- β ***and*** ϕ ***are determined by the arrangement of the magnets in the tunnel***

- ***individual trajectories are determined by*** A ***and*** ϕ_0

- ***beam ensemble:***



$$\left\langle \frac{x^2}{\beta} - \beta' \cdot x \cdot x' + \beta \cdot x'^2 \right\rangle = \varepsilon$$

$$\varepsilon = \frac{area}{\pi}$$

→ ε ***describes the beam quality***

→ $\sigma = \sqrt{\varepsilon \cdot \beta}$ ***describes the beam size***

Dispersion

- Consider a deviation from the design momentum by $\frac{\Delta p}{p_0}$. This gives rise to a deviation of the horizontal closed orbit, as described by the revolution matrix:

$$M_x = \begin{pmatrix} C & S & D \\ \pm C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion

- This matrix has a third eigenvalue $\lambda_3 = 1$ (since $\mathrm{Tr}M = \sum \lambda$). We parametrize the eigenvector as:

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} \delta$$

  where $\eta(s) = dx/d\delta$ is called the dispersion funcion.

- Obviously, we want to avoid integer Q. In fact, any "simple" rational number for either Q_H or Q_V is bad and may lead to resonances where the beam blows up. This is because an off-momentum particle gets the same kick in the same phase every few turns.

Dispersion

- Solving the eigenvector equation, we get

$$\begin{aligned} \eta &= \frac{(1-S')D+SD'}{2-C-S'} = \frac{(1-S')D+SD'}{2(1-\frac{1}{2}\mathrm{Tr}M)} \\ &= \frac{(1-S')D+SD'}{2(1-\cos 2\pi Q)} = \frac{(1-S')D+SD'}{4\sin^2 \pi Q} \end{aligned}$$

- Obviously, we want to avoid integer Q. In fact, any "simple" rational number for either Q_H or Q_V is bad and may lead to resonances where the beam blows up. This is because an off-momentum particle gets the same kick in the same phase every few turns.

Dispersion

- The total deviation from the design orbit has now two components:

$$x(s) = x_\beta(s) + x_p(s)$$

with a spread of:

$$\sigma_{\text{tot}} = \sqrt{\sigma_\beta^2 + \left(\eta \frac{\sigma_p}{p}\right)^2}$$

Summary Focusing

- ***beam divergence***

- ***geometrical focusing***

  $\longrightarrow$ ***horizontal stability***

- ***strong focusing***

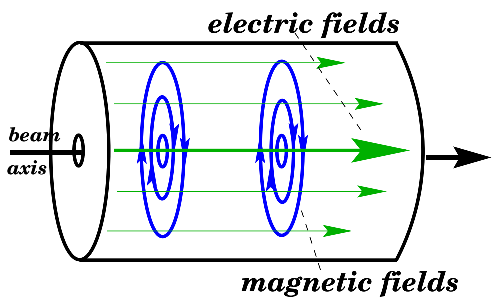
  $\longrightarrow$ ***horizontal and vertical stability***

- ***optic functions:*** β, ϕ

- ***beam size:*** $\sigma = \sqrt{\beta \cdot \varepsilon}$

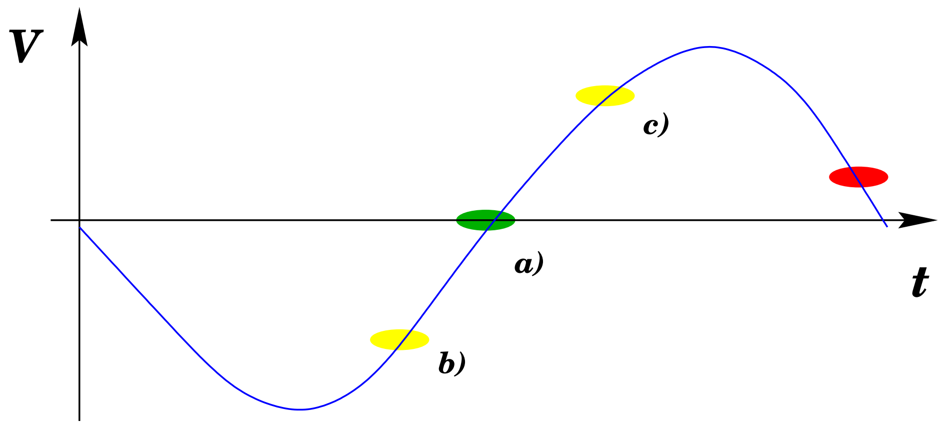
Longitudinal Stability

- ***RF Cavity***



- ***assume:*** $p = p_0 + \Delta p \longrightarrow \omega = \omega_0 + \Delta\omega$

- ***voltage in cavity:***



$\longrightarrow$ ***longitudinal stability***