two have a common error (variance a^2), but this time the values are anticorrelated. This might happen, for example, if the sum of the two variables is a constant. Then

$$S = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} + \begin{pmatrix} s^2 & s^2 & 0 \\ s^2 & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} . \quad (28.22)$$

If unequal amounts of the common baseline were subtracted from variables 1, 2, and 3—*e.g.*, fractions f_1, f_2, and f_3, then we would have

$$S = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} + \begin{pmatrix} f_1^2 s^2 & f_1 f_2 s^2 & f_1 f_3 s^2 \\ f_1 f_2 s^2 & f_2^2 s^2 & f_2 f_3 s^2 \\ f_1 f_3 s^2 & f_2 f_3 s^2 & f_3^2 s^2 \end{pmatrix} . \quad (28.23)$$

While in general this "two-vector" representation is not possible, it underscores the procedure: Add zero-determinant correlation matrices to the matrix expressing the independent variation.

Care must be taken when fitting to correlated data, since off-diagonal contributions to χ^2 are not necessarily positive. It is even possible for all of the residuals to have the same sign.

Example: straight-line fit

For the case of a straight-line fit, $y(x) = \alpha_1 + \alpha_2 x$, one obtains, for independent measurements y_i, the following estimates of α_1 and α_2,

$$\widehat{\alpha}_1 = (g_1 \Lambda_{22} - g_2 \Lambda_{12})/D , \quad (28.24)$$

$$\widehat{\alpha}_2 = (g_2 \Lambda_{11} - g_1 \Lambda_{12})/D , \quad (28.25)$$

where

$$(\Lambda_{11}, \Lambda_{12}, \Lambda_{22}) = \sum (1, x_i, x_i^2)/\sigma_i^2 , \quad (28.26a)$$

$$(g_1, g_2) = \sum (1, x_i) y_i / \sigma_i^2 . \quad (28.26b)$$

respectively, and

$$D = \Lambda_{11} \Lambda_{22} - (\Lambda_{12})^2 . \quad (28.27)$$

The covariance matrix of the fitted parameters is:

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \Lambda_{22} & -\Lambda_{12} \\ -\Lambda_{12} & \Lambda_{11} \end{pmatrix} . \quad (28.28)$$

The estimated variance of an interpolated or extrapolated value of y at point x is:

$$(\widehat{y} - y_{\text{true}})^2 \Big|_{\text{est}} = \frac{1}{\Lambda_{11}} + \frac{\Lambda_{11}}{D} \left(x - \frac{\Lambda_{12}}{\Lambda_{11}} \right)^2 . \quad (28.29)$$