THE GEOMETRY OF DYNAMICAL TRIANGULATIONS

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Abstract

The express purpose of these Lecture Notes is to go through some aspects of the simplicial quantum gravity model known as the Dynamical Triangulations approach. Emphasis has been on lying the foundations of the theory and on illustrating its subtle and often unexplored connections with many distinct mathematical fields ranging from global riemannian geometry, moduli theory, number theory, and topology. Our exposition will concentrate on these points so that graduate students may find in these notes a useful exposition of some of the rigorous results one can establish in this field and hopefully a source of inspiration for new exciting problems. We also illustrate the deep and beautiful interplay between the analytical aspects of dynamical triangulations and the results of MonteCarlo simulations. The techniques described here are rather novel and allow us to address successfully many high points of great current interest in the subject of simplicial quantum gravity while requiring very lit-

tle in the way of fancy field theoretical arguments. As a consequence, these notes contains mostly original and yet unpublished material of great potential interest both to the expert practitioner and to graduate students entering in the field. Among the topic addressed here in considerable details there are: *(i)* An analytical discussion of the geometry of dynamical triangulations in dimension $n = 3$ and $n = 4$; *(ii)* A constructive characterization of entropy estimates for dynamical triangulations in dimension $n = 3$, $n = 4$, and a comparision of the analytical results we obtain with tha data coming from Monte Carlo simulations for the 3-sphere $\mathbb{S}^3$ and the 4-sphere $\mathbb{S}^4$; *(iii)* A proof that in the four-dimensional case the analytical data and the numerical data provide the same critical line $k_4(k_2)$ characterizing the infinite-volume limit of simplicial quantum gravity; *(iv)* An analytical characterization of the critical point k_2^{crit} which is in an extremely good agreement with the location of the critical point obtained by MonteCarlo simulations; *(v)* A proof that corresponding to such a critical point, 4-dimensional (simply connected) manifolds undergo a higher order phase transition possibly concealed by a bistable behavior: this settles down in the affirmative a long standing debate as to the nature of the phase transition in 4-dimensional simplicial quantum gravity; *(vi)* A geometrical analysis of the branched polymer phase of $4D$-gravity; *(vii)* We also show that in the three- dimensional case the confrontation between the analytical and numerical data is very satisfactory; *(viii)* A proof that in dimension $n = 3$ there are pronounced hysteresis effects which are possibly responsible for the observed first-order nature of the phase transition at criticality.

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Contents

I. INTRODUCTION

Recent years witnessed a massive introduction of methods of statistical field theory in attempts to quantize gravity along the lines of a conventional field theory [1], [2]. Such an emphasis is the direct outspring of the possibility of discretizing gravity in a way consistent with the underlying reparametrization invariance of general relativity [3], [2]. The basic idea is to use a variant of the standard Regge calculus [4], first suggested by Weingarten [5], in an attempt to make quantum gravity well defined as a lattice statistical field theory. This discretization is known as the theory of Dynamical Triangulations (DT) or simplicial quantum gravity. It consists of a replacement of the (Euclidean) functional integral over all equivalence classes of metrics with a summation over all abstract triangulations of the given manifold. The fixed edge-length of the links of the triangulations plays here the role of *lattice spacing* of the more familiar lattice gauge theories. The basic idea is to search, in the parameter space of these discretized quantum gravity models, for critical points where the *lattice spacing* can be taken to zero and contact can be made to continuum physics. Credit to this picture is lend by noticing that within this framework, Euclidean two-dimensional quantum gravity can be formulated as the scaling limit of a dynamically triangulated collection of surfaces thought of as an ordinary statistical system. Contact with the continuum, as described by Liouville theory [6], can successfully be made, and the robust alliance of the theory of critical phenomena and Monte Carlo simulations even sheds light on many aspects of the theory not liable of an analytical approach.

If we extend the theory to higher dimensions we have to face the fact that we have no continuum theory of Euclidean Quantum Gravity with which we can compare, nonetheless many aspects of the theory can be investigated by computer-assisted simulations with a good degree of precision. These numerical methods allow us to get a rather complete picture of the phase diagram of the discretized theory and they serve as an important inspiration for analytical studies of higher dimensional models of quantum gravity. These lecture notes attempts to partially fill the gap between such numerical studies and a fully- fledged analytical approach to higher dimensional dynamical triangulations. Our emphasis here is on the applications of elementary ideas of Piecewise-Linear (PL) geometry, global riemannian geometry, and number theory in order to understand dynamically triangulated models of quantum gravity in dimension $n = 3, 4$. The occasion for such an analytical approach comes by exploiting the techniques developed for the recent proof [7] of the conjecture concerning the existence of an exponential bound to the number of distinct triangulations on a PL-manifold M^n, $(n \geq 3)$, of given volume and topology. This exponential bound is germane to the definition of dynamical triangulations as a lattice statistical field theory model. Analytical proofs are well known in two dimensions [8], and computer simulations supported very strongly the existence of such bound also in higher dimensions [1]. The actual proof for higher dimensions draws from techniques of controlled topology and geometry initiated by Cheeger, Gromov, Grove, and Petersen [9], [10], and it suggests that such mathematical methods may bear relevance to the whole program of dynamical triangulations. A check with Monte Carlo simulations (in dimension $n = 4$) shows an excellent agreement between the numerical data and the analytical results obtained by pursuing this geometrical approach to higher dimensional dynamically triangulated models. It is not yet possible to reach the level of sophistication of the two-dimensional case, but as we show in these notes, the analytical methods we develop in dimensions $n = 3$ and $n = 4$, provide a good start for an analytical understanding of dynamical triangulations in such dimensions. In particular, we characterize with greater precision the entropy estimates [7], and provide the corresponding generating functions. Among other properties common to both the 3- dimensional and 4-dimensional case, we prove that in 4- dimensions dynamically triangulated models it makes sense to sum over all simply connected manifolds. We also discuss the existence of a polymer phase in these models [11]. We analytically prove that the system admits a critical point $k_2 = k_2^{crit}$ corrisponding to which a dynamically triangulated 4-manifold undergoes a higher order phase transition possibly concealed by a bystable behavior. This is an important result suggesting that dynamical triangulations provide a reliable candidate to a consistent quantum gravity theory, and shows that the importance of the interplay between analytical approaches and Monte Carlo simulations continues unabated.

The arguments presented here should be amenable to considerable improvements, and hopefully these notes can be used as a working tool. For this reason, we have presented a detailed summary of the more important and useful results in section **III**. In section **II** we review the definition of the model. In section **IV** we recall few elementary aspects of the PL-geometry of dynamical triangulations. In section **V** we establish the connection with Gromov's spaces of bounded geometry. This connection provides mathematical foundation to the heuristic arguments which motivate the approximation of distinct riemannian structures with distinct dynamical triangulations. In section **VI** we introduce the *Moduli space* associated with dynamical triangulations. These moduli spaces parametrize the set of inequivalent *deformations* of dynamical triangulations needed to approximate riemannian structures with large symmetries. In sections **VII** and **VIII** we provide all entropy estimates needed for the higher dimensional DT models. Here we exploit elementary number theory in order to provide such estimates and construct the associated generating functions. We discuss also the 2-dimensional case which fully agrees with the known results. The 4-dimensional case, and the 3-dimensional case are discussed in section **IX**. The agreement between the analytical results we obtain and the Monte Carlo simulations is impressive in dimension $n = 4$. We discuss in detail an analytical proof of the existence of the phase transition whose presence is strongly suggested by the computer simulations. As recalled we prove that definitively it is not a first order transition as hinted by some recent simulations.

Finally, we show that also in dimension $n = 3$ the analysis of the data coming from the two approaches is quite satisfactory. In particular, we prove that the correct asymptotics for the canonical partition function has a bistable behavior yielding for the hysteresis effects which are believed to signal a first order nature of the transition point.

II. THE MODEL: SIMPLICIAL QUANTUM GRAVITY

Let M be an n-dimensional, $(n \geq 2)$, manifold of given topology, and with a finite number of fixed $(n-1)$-dimensional boundaries Σ_k, $k = 1, 2, \ldots$. Let $Riem(M)$ and $Diff(M)$ respectively denote the space of riemannian metrics g on M, and the group of diffeomorphisms on M. In the continuum formulation of quantum gravity the task is to perform a path integral over equivalence classes of metrics:

$$Z(\Lambda, G, \Sigma_k, h) = \sum_{Top(M)} \int_{Riem(M)/Diff(M)} \mathcal{D}[g(M)] e^{-S_g[\Lambda, G, \Sigma]} \tag{1}$$

weighted with the Einstein-Hilbert action associated with the riemannian manifold (M, g), *viz.*,

$$S_g[\Lambda, G, \Sigma] = \Lambda \int_M d^n\xi \sqrt{g} - \frac{1}{\sqrt{16\pi G}} \int_M d^n\xi \sqrt{g}\, R + \mathit{boundary\, terms} \tag{2}$$

The boundary terms depend on the metric h and on the extrinsic curvature induced on the boundaries Σ_k, and they are such that for the action obtained by gluing any two manifolds (M_1, g_1) and (M_2, g_2) along a common boundary (Σ, h), we get $S_{g_1+g_2}[\Lambda, G] = S_{g_1}[\Lambda, G, \Sigma] + S_{g_2}[\Lambda, G, \Sigma]$. In this way, the partition function (1) associated with a manifold M with two boundaries (Σ_1, h_1) and (Σ_2, h_2), satisfies the basic composition law

$$Z(\Sigma_1, h_1; \Sigma_2, h_2) = \sum_{\Sigma_i, h_i} Z(\Sigma_1, h_1; \Sigma_i, h_i) Z(\Sigma_i, h_i; \Sigma_2, h_2) \tag{3}$$

which describes how M can interpolate between its fixed boundaries (Σ_1, h_1) and (Σ_2, h_2) by summing over all possible intermediate states (Σ_i, h_i).

Even without pretense of rigor, it is clear that the formal *a priori* measure $\mathcal{D}[g(M)]$ characterizing such sort of path integration should satisfy some basic properties. In particular: *(i)* $\mathcal{D}[g(M)]$ should be defined on the space of riemannian structures $Riem(M)/Diff(M)$ so to avoid counting as distinct any two riemannian metrics $g_1(M)$ and $g_2(M)$ which differ one from the other simply by the action of a diffeomorphism of $\phi: M \to M$, *viz.*, such that $g_2(M) = \phi^* g_1(M)$;

(ii) The measure $\mathcal{D}[g(M)]$ should play a kinematical role and not a dynamical one. More explicitly, for a given $r \in \mathbb{R}^+$, let $[Riem(M)/Diff(M)]_{x_1,x_2;r}$ denote the set of all riemannian structures on a manifold M with two marked points x_1 and x_2 with preassigned distance $d_{g(M)}(x_1, x_2) = r$. When restricted to this set of riemannian structures, $\mathcal{D}[g(M)]$ should factorize: fluctuations in the geometry of M which are localized in widely separated regions, should be statistically independent. This implies that $\mathcal{D}[g(M)]$ should not describe the *a priori* existence of long range correlations on the set of riemannian manifolds considered. Such correlations should be generated by the spectrum of fluctuations of (1) by means of a statistical suppression-enhancement mechanism similar to the *energy* versus *entropy* argument familiar in statistical mechanics. In a field theoretic formalism, this property is translated in the familiar requirement of ultra-locality, namely in the absence of derivatives in the formal L^2 norm on the space of metrics, (the *De Witt supermetric*).

(iii) Finally, $\mathcal{D}[g(M)]$ should be so constructed as to allow for the introduction of a $Diff(M)$-invariant short distance cut-off representing the shortest wavelength allowed in discussing fluctuations of the geometry of M. This cut-off should be removable under appropriate circumstances, in particular when, in a critical phase, long range correlations are generated.

Summing over topologies. Long before addressing the physical and mathematical characterizations of such requirements on $\mathcal{D}[g(M)]$, both in the continuum field-theoretic formalism and in its possible discretized versions, (in this connection see the recent work of Menotti and Peirano [1]), one must also discuss the proper meaning to attribute to the formal sum over topologies appearing in (1): are we summing over homotopy types, homeomorphism types, or over smooth types of manifolds?

The homotopy type of a manifold is a topological notion which (with the exception of dimension two) is too weak to appear of some immediate utility. The homeomorphism type of a manifold M is the notion which is more directly related to a natural summation over topologies, however it is difficult to handle in dimension larger than two, (see below). Finally, the smooth type yields for a summation over the possible distinct differentiable structures that a manifold M can carry. This latter summation appears more relevant to a field- theoretic formalism since in an expression such as (1) the diffeomorphism group of the underlying manifold $Diff(M)$ plays a basic role, but as we shall see momentarily, a summation over the smooth types arises more naturally in simplicial gravity. Thus, this latter interpretation for $\sum_{Top(M)}$ is perhaps the most appealing since it suggests an unexpected bearing of simplicial quantum gravity on the field-theoretic formalism, and in this sense the two approaches are not alternative to each other. Simplicial quantum gravity may be necessary in addressing the important issue of summation over distinct topologies, whereas the field-theoretic formalism is necessary in order to understand the nature of the continuum limit (if any exists) of the discretized models.

In dimension two there is equivalence between smooth structures and the homeomorphism types, and the summation over topologies can be given the unambigous meaning of a summation over the Euler characteristic $\chi(M)$ of the manifolds, since this invariant characterizes surfaces. However, even in such a simple case it is very difficult to obtain clear-cut results for what concerns a reliable procedure for summing over topologies. The use of matrix models of $2D$-gravity and the associated *double scaling* limit have shed some light on this issue, but we are still far from a clear understanding [1].

In dimension three topological manifolds are uniquely smoothable [12], and the question of summing over topologies is again synonimous of summation over smooth structures. Thus, as long as we confine our attention to the topological category or to the smooth category, summation over topologies is reduced to the yet unsolved problem of enumerating the homeomorphism types of three-manifolds. This enumeration cannot be realized as long as the Poincaré conjecture is not proved. For instance, if there were a fake three- sphere then one could prove [13] that there cannot be finitely many homeomorphism types of three-manifolds even under bounds on curvatures of the manifold, (typically, under curvature, volume and diameter bounds one gets topological finiteness theorems for homeomorphism and diffeomorphism types [10] for dimension $n \neq 3, 4$).

In dimension four the situation is even more complex since smoothing theory is not yet completely known. In open contrast to the theory of 3-manifolds, there is no equivalence between topological manifolds and smooth structures, and Donaldson-Freedman's theory shows that there are topological manifolds which are not smoothable as well as manifolds admitting uncountably many inequivalent smooth structures [14]. The situation is further worsened by the fact that the topological classification of all (compact, orientable) 4-manifolds is logically impossible, (again, the fundamental group should be blamed for this). As stressed by Fröhlich [15], this circumstance has even be used to foster the credence that only simply-connected, spinable 4-manifolds should contribute to the gravitational path integral. But clearly this point of view, if not better substained, cannot be advocated. Suppression of a class of manifolds from path integration can be justified only on a dynamical ground, as a form of statistical suppression driven by the spectrum of fluctuations of the theory.

This situation does not improve even if we confine our attention to a manifold M of fixed topology and with a given smooth structure, for, in that case we have the additional problems associated with: *(i)* The unboundedness of the Euclidean action; *(ii)* The mathematical difficulties in defining a proper path integration over the stratified manifold of riemannian structures; *(iii)* The Einstein-Hilbert action is not renormalizable.

Simplicial quantum gravity. The hope behind simplicial quantum gravity is that some of the above problems, concerning both the characterization of the measure $\mathcal{D}[g(M)]$ or the issues related to $\Sigma_{Top(M)}$, can be properly addressed, in a non-perturbative setting, by approximating the path integration over inequivalent riemannian structures with a summation over combinatorially equivalent piecewise linear manifolds. The first attempt of using PL geometry in relativity dates back to the pioneering work of Regge [4]. His proposal was to approximate Riemannian (Lorentzian) structures by PL-manifolds in such a way as to obtain a coordinate-free formulation of general relativity. The basic observation in this approach is that parallel transport and the (integrated) scalar curvature have natural counterparts on PL manifolds once one gives consistently the lengths of the links of the triangulation defining the PL structure. The link length is the dynamical variable in Regge calculus, and classically the PL version of the Einstein field equations is obtained by fixing a suitable triangulation and by varying the length of the links so as to find the extremum of the Regge action. If the original triangulation is sufficiently fine, this procedure consistently provides a good approximation to the smooth spacetime manifold which is the corresponding smooth solution of the Einstein equations. This approach can also be successfully extended so as to provide a quantum analogue to Regge calculus [3], in which we replace the formal path integration over the space of Riemannian structures with an integration over the link variables.

Dynamical triangulations are a variant of Regge calculus in the sense that in this formulation the summation over the length of the links is replaced by a direct summation over abstract triangulations where the length of the links is fixed to a given value a. In this way the elementary simplices of the triangulation provide a *Diff-invariant* cut-off and each triangulation is a representative of a whole equivalence class of metrics. Regge calculus still mantain its validity and it provides both the metric assignement for the PL manifold, obtained by gluing the simplices, and the corresponding action. Since all simplices now are identical, the action will only depend on the numbers, respectively N_n and N_{n-2}, of n- and $(n-2)$-dimensional simplices of the n-dimensional PL manifold. In this way we get that the Einstein- Hilbert action for n-dimensional (Euclidean) gravity formally goes into the combinatorial action

$$S[k_{n-2}, k_n] = k_n N_n(T) - k_{n-2} N_{n-2}(T) + \frac{1}{2} k_{n-2} N_{n-2}(\partial T) \tag{4}$$

where $\frac{1}{2}N_{n-2}(\partial T)$ is the boundary term, and k_n, and k_{n-2} are (bare) coupling constants related to the cosmological constant Λ and to the gravitational coupling G, respectively. In particular, we can view $1/k_{n-2}$ as a bare gravitational coupling constant.

The partition function associated with such discretized action is

$$Z[k_{n-2}, k_n] = \sum_{T \in \mathcal{T}} e^{-k_n N_n + k_{n-2} N_{n-2} - \frac{1}{2} k_{n-2} N_{n-2}(\partial T)} \tag{5}$$

where the summation is over *distinct* triangulations T in a suitable class of triangulations $\mathcal{T}$. Roughly speaking, we consider any two triangulations (say with the some number of vertices) distinct if there is no map between the vertices which is compatible with the assignements of links, triangles, etc., while $\mathcal{T}$ restricts the class of triangulations to those satisfying suitable topological constraints. The proper choice of the class $\mathcal{T}$ is strictly connected to the difficult question of how to sum over topologies in quantum gravity mentioned above.

The interpretation of (5) as providing also a sum over topologies is based on the observation that a PL manifold is uniquely smoothable in low dimension (in particular for $n = 4$, by Cerf theorem, see *e.g.*, [14]), (it is also worth stressing that every smooth manifold admits a natural C^1-smooth PL-structure). In particular, in dimension $n = 4$ there is a bijective correspondence between (isotopy) classes of smooth structures and PL-structures. Thus (5), if no restrictions are imposed on $\mathcal{T}$, subsumes in a rather natural way a summation over topologies if we decide that $\sum_{Top}$ actually means summation over smooth structures. Even if not ambiguous in its definition, (5) blends summation over metric structures and summation over smooth structures only in a formal way. Already in dimension two, the sum (5) is divergent since if the topology is not fixed, the number of distinct triangulations grows factorially with the volume of the manifold (*i.e.*, with the number of simplices). However, if one fixes the topology, the number of distinct triangulations of a 2-dimensional PL manifold is exponentially bounded [8] according to

$$(\Lambda)^{N_2} N_2{}^{(\frac{\chi(\Sigma)}{2})(\gamma_{str}-2)-1} \tag{6}$$

where Λ is a suitable constant and γ_{str}, the *string exponent*, is a topological suscettivity generated by the quantum fluctuations of the metric. In this way, $\sum_{T \in \mathcal{T}} e^{-S(T)}$, where $\mathcal{T}$ is a given PL surface, is well defined. Topology is then allowed to fluctuate (*i.e.*, one attempts to extend the summation to all PL surfaces), by a delicate limiting procedure (*Double Scaling Limit*). In higher dimension is not yet known how to perform the summation (5), but numerical simulation as well as the results of the two-dimensional theory, suggests that (5) makes sense if one fixes the topology. This issue is clearly related to a rigorous characterization of the discretized counterpart of the formal a priori measure $\mathcal{D}[g(M)]$. As a matter of fact, a *necessary condition* for attributing a meaning to (5), *for fixed topology*, is to require that the number of distinct triangulations of a given PL manifolds is exponentially bounded as a function of the number of (top-dimensional) simplices [1]. Recently [7], the existence of the exponential bound in all dimensions and for all (fixed) topologies has been proved. The existence of the exponential bound implies that for a fixed k_{n-2} there is a critical lower value $k_n^{crit}(k_{n-2})$ of k_n such that the partition function is well defined for $k_n > k_n^{crit}(k_{n-2})$ and divergent for $k_n < k_n^{crit}(k_{n-2})$. To be more specific this implies that we can introduce the (canonical) partition function for fixed (lattice) volume: *The effective Entropy*

$$W(N, k_{n-2})_{eff} \doteq \sum_{T \in \mathcal{T}(N_n)} e^{k_{n-2} N_{n-2}(T)} \tag{7}$$

where $\mathcal{T}(N_n)$ denotes the class of distinct triangulations of fixed volume (N_n), fixed topology and boundary conditions. This effective entropy will characterize the *infinite (lattice) volume* limit of the theory, defined by the approach to the critical line $k_n \to k_n^{crit}(k_{n-2})$ in the (k_{n-2}, k_n) coupling constant plane. The existence of the infinite volume limit is a necessary *but not sufficient* condition for the existence of a physically significant *continuum limit* of the theory, and we only expect interesting critical behavior, *i.e.*, the onset of long range correlations, at certain critical values of k_{n-2}. In the rest of these notes we will be mainly interested in the analytical characterization of the canonical partition function (the effective entropy) $W(N, k_{n-2})_{eff}$, and in comparing the obtained results with the existing numerical data coming from Monte Carlo simulations.

It is not difficult to see the origin of the difficulties in dealing with the set of dynamically triangulated manifolds (of given volume and topology) considered as a statistical system described by (7). Roughly speaking we have a collection of identical simplices whose only interaction is basically associated with unpenetrability and gluings according to certain rules.

The energetic term is very simple and controls the average curvature. Thus the free energy of the system is basically characterized by the entropic factor enumerating the number of distinct dynamical triangulations of given topology, volume, and average curvature. Such an estimate is deeply non-local, (*e.g.*, see (6) for $n = 2$), and as such its characterization is conceptually different from the standard approaches used to enumerate distinct configurations, say of spins, on a rigid lattice. This situation is not new in statistical mechanics since it is reminiscent of what happens when dealing with the *hard sphere gas*: a set of identical spherical particles whose only interaction is associated with unpenetrability. Here too, the free energy is of entropic origin, and the characterization of such an entropy is a highly non local problem since it is related to the characterization of the densest packings in spheres: the insertion of a sphere may change the packing up to a distance proportional to the inverse of the average separation between the surfaces of the spheres. For dynamical triangulations, as we shall see, the source of non- locality is subtler since there is a delicate feedback with the topology, (*e.g.*, see (6), where the critical exponent γ_s depends from the Euler characteristic of the surface obtained by gluing the simplices). But these topological difficulties would be present also in the hard sphere gas had we considered the spheres evolving in a topological non-trivial ambient space. Thus, it is perhaps not so surprising to note that the first proof of the existence of an exponential bound for the entropy of dynamical triangulations was carried out [7] by exploiting the geometry of sphere packings in riemannian manifolds. This interplay between the geometry of packings, topology, and the metric properties of riemannian manifolds, (or more general metric spaces), can be traced back to M. Gromov's ideas. These ideas have sprung a renaissance in recent developments in Riemannian geometry, and we will exploit them here by showing that they are a good source of inspiration also in simplicial quantum gravity and in the fascinating field of the statistical mechanics of extended objects.

III. SUMMARY OF RESULTS

For the convenience of the reader we present here a summary of the constructive results that we prove in these notes and that can be useful in applications to higher dimensional dynamical triangulations. If not otherwise stated, we refer to dynamically triangulated manifolds, M, in dimension $n \geq 2$. The dynamical triangulations in question have $N_{n-2}(T) \doteq \lambda + 1$ bones σ^{n-2}, $N_n(T) \doteq N$ top-dimensional simplices σ^n, a given average number, $b(n, n-2) \doteq \frac{1}{2}n(n+1)(N/\lambda + 1)$, of n-simplices incident on a bone, and a minimum number $\hat{q}$ of n-dimensional simplices incident on a bone, (typically $\hat{q} = 3$). Sometimes we need to mark n of these bones and a top-dimensional simplex σ_0^n incident on one of them. We refer collectively to triangulations with such markings as **rooted** triangulations. The topology of the triangulations explicitly enters our results through the *representation variety* $\frac{Hom(\pi_1(M),G)}{G}$: the set of conjugacy classes of representations of the fundamental group of the manifold, $\pi_1(M)$ into a (compact) Lie group G. This representation variety parametrizes the set of inequivalent deformations of dynamical triangulations needed to approximate riemannian structures endowed with G-structures, (for instance, this parameter space tells us how to deform a 3-dimensional dynamical triangulation so as to approximate unambiguosly inequivalent flat 3- tori). Finally, in what follows we also introduce a normalizing factor $0 < c_n \leq 1$, of the form

$$c_n \doteq C_0 N_n^{\tau(n)} \cdot e^{\alpha_n N_n} \tag{8}$$

where the constants $0 < C_0 \leq 1$, $\tau(n) \geq 0$, $\alpha_n \leq 0$, depend only on the dimension n. These normalizing constants are necessary since our enumeration procedure (in establishing entropy estimates) overcounts the number of distinct dynamical triangulations admitted by a manifold of given topology. In dimension $n = 2$, c_n reduces to a constant, whereas in dimension $n = 4$, $\tau(n = 4) = 5/2$ and $\alpha_{n=4} = -\frac{1}{2}\ln(\cos^{-1}(1/4))$ and similarly in dimension $n = 3$. With these notational remarks along the way, we have:

The generating function. *Let $0 \leq t \leq 1$ be a generic indeterminate, and let $p_\lambda(h)$ denote the number of partitions of the generic integer h into (at most) $\lambda + n - 1$ parts, each $\leq (b(n,n-2) - \hat{q})(\lambda+1)$. In a given holonomy represention $\Theta \colon \pi_1(M; \sigma_0^n) \to G$, and for a given value of the parameter $b = b(n,n-2)$, the generating function for the number of distinct rooted dynamical triangulations with $N_{n-2}(T_a^{(i)}) = \lambda + 1$ bones and given number ($\propto h$) of n-dimensional simplices incident on the n marked bones is given by*

$$\mathcal{G}[W(\Theta, \lambda, b; t)] = c_n \cdot \sum_{h \geq 0} p_\lambda(h) t^h = c_n \cdot \begin{bmatrix} (b(n,n-2) - \hat{q})(\lambda+1) + (\lambda+1) - n \\ (\lambda+1) - n \end{bmatrix} \tag{9}$$

where $0 < c_n \leq 1$ is the above-mentioned normalizing factor, and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-t^n)(1-t^{n-1})\dots(1-t^{m+1})}{(1-t^{n-m})(1-t^{n-m-1})\dots(1-t)} \doteq \frac{(t)_n}{(t)_{n-m}(t)_m} \tag{10}$$

is the Gauss polynomial in the variable t.

The Rooted Entropy. *In a given represention $\Theta \colon \pi_1(M; \sigma_0^n) \to G$, and for a given value of the parameter $b = b(n,n-2)$, the number $W(\Theta, \lambda, b)$ of distinct rooted dynamical triangulations with $N_{n-2}(T_a^{(i)}) = \lambda + 1$ bones, is given by*

$$W(\Theta, \lambda, b) = c_n \cdot \begin{pmatrix} (b(n,n-2) - \hat{q})(\lambda+1) + (\lambda+1) - n \\ (\lambda+1) - n \end{pmatrix} \tag{11}$$

Asymptotics for the Entropy. *The number of distinct dynamical triangulations, with $\lambda + 1$ bones, and with an average number, $b \equiv b(n,n-2)$, of n-simplices incident on a bone, on an n-dimensional, ($n \leq 4$), PL- manifold M of given fundamental group $\pi_1(M)$, can be asymptotically estimated according to*

$$W(\lambda, b) \simeq$$

$$W_\pi \cdot \frac{c_n}{\sqrt{2\pi}} \sqrt{\frac{(b-\hat{q}+1)^{1-2n}}{(b-\hat{q})^3}} \left[\frac{(b-\hat{q}+1)^{b-\hat{q}+1}}{(b-\hat{q})^{b-\hat{q}}}\right]^{\lambda+1} \left(\frac{b(n,n-2)}{n(n+1)}\lambda\right)^{D/2} \lambda^{-\frac{2n+3}{2}} \tag{12}$$

where W_π is a topology dependent parameter, and $D \doteq dim[Hom(\pi_1(M), G)]$.

The Canonical Partition Function. *Let us consider the set of all simply-connected n-dimensional, $(n = 3, 4)$, dynamically triangulated manifolds. Let us set*

$$A(k_{n-2}) \doteq$$
$$\left[\frac{27}{2}e^{k_{n-2}} + 1 + \sqrt{(\frac{27}{2}e^{k_{n-2}} + 1)^2 - 1}\right]^{1/3} + \left[\frac{27}{2}e^{k_{n-2}} + 1 - \sqrt{(\frac{27}{2}e^{k_{n-2}} + 1)^2 - 1}\right]^{1/3} - 1, \quad (13)$$

and let k_{n-2}^{crit} denote the unique solution of the equation

$$\eta^*(k_{n-2}) \doteq \frac{1}{3}(1 - \frac{1}{A(k_{n-2})}) = \eta_{max} \quad (14)$$

where $\eta_{max} = 1/4$, (for $n = 4$) and $\eta_{max} = 2/9$, (for $n = 3$). Let $0 < \epsilon < 1$ small enough, then for all values of the inverse gravitational coupling k_{n-2} such that

$$k_{n-2}^{crit} - \epsilon < k_{n-2} < +\infty, \quad (15)$$

the large N-behavior of the canonical partition function (effective entropy) is given by the uniform asymptotics

$$W(N, k_{n-2})_{eff} \doteq \sum_{T \in \mathcal{T}(N_n)} e^{k_{n-2}N_{n-2}(T)} =$$

$$\frac{c_n}{\sqrt{2\pi}}\hat{N}^{-n-1/2}e^{(\hat{N}+1)f(\eta_{max})}\left[\frac{\xi_0}{\sqrt{\hat{N}}}w_0(\psi_{max}(k_{n-2})\sqrt{\hat{N}}) + \frac{\xi_1}{\hat{N}}w_{-1}(\psi_{max}(k_{n-2})\sqrt{\hat{N}})\right] \quad (16)$$

where $w_r(z) \doteq \Gamma(1-r)e^{z^2/4}D_{r-1}(z)$, $(r < 1)$, $D_{r-1}(z)$ and $\Gamma(1-r)$ respectively denote the parabolic cylinder functions and the Gamma function, and where the constants ξ_0 and ξ_1 are given by

$$\xi_0 \doteq -\frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}}\frac{\psi_{max}(k_{n-2})}{f_\eta(\eta_{max})} \quad (17)$$

$$\xi_1 \doteq \frac{\xi_0}{\psi_{max}(k_{n-2})} - \frac{\eta *^{-1/2}(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{max}(k_{n-2})\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \quad (18)$$

Where, for notational convenience we have set

$$f(\eta) \doteq -\eta\log\eta + (1-2\eta)\log(1-2\eta) - (1-3\eta)\log(1-3\eta) + k_{n-2}\eta \tag{19}$$

with $f_\eta \doteq df/d\eta$, and $f_{\eta\eta} \doteq d^2f/d\eta^2$,and

$$\psi_{max}(k_{n-2}) \doteq sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]}. \tag{20}$$

(Note that for the 3-dimensional case the above expression for $W(N, k_{n-2})$ is slightly more general if one uses the variables (N_0, N_3)-see below for details).

The infinite volume limit. *The critical value of the coupling k_n^{crit} corresponding to the infinite-volume limit for simply-connected n-dimensional dynamically triangulated manifolds is given by*

$$k_n^{crit}(k_{n-2}) = \frac{1}{2}n(n+1)\left[\ln\frac{A(k_2)+2}{3} + \lim_{N_n\to\infty}\frac{\ln c_n}{N_n}\right] \tag{21}$$

which, for $0 < \epsilon < 1$ small enough, holds for all $k_{n-2}^{crit} - \epsilon < k_{n-2} < k_{n-2}^{crit}$. Whereas, for $k_{n-2}^{crit} < k_{n-2} < +\infty$ we get

$$k_n^{crit}(k_{n-2}) =$$

$$\frac{1}{2}n(n+1)\left\{\log\left[\frac{(1-2\eta_{max})^{(1-2\eta_{max})}}{\eta_{max}^{\eta_{max}}(1-3\eta_{max})^{(1-3\eta_{max})}}\right] + k_{n-2}\eta_{max} + \lim_{N_n\to\infty}\frac{\ln c_n}{N_n}\right\} \tag{22}$$

The four-dimensional case. *The critical value of the coupling $\tilde{K}_4^{crit}$ corresponding to the infinite-volume limit for a simply-connected 4-manifold, (e.g., the 4-sphere $\mathbb{S}^4$), is given by*

$$\tilde{k}_4^{crit}(k_2) = 10\left\{\log\left[\frac{(1-2\eta_{max})^{(1-2\eta_{max})}}{\eta_{max}^{\eta_{max}}(1-3\eta_{max})^{(1-3\eta_{max})}}\right] + k_2\eta_{max} - \log\left(\frac{e}{2.066}\right)\right\} \tag{23}$$

(with $k_2^{crit} < k_2 < +\infty$), and $\eta_{max} = 1/4$, and

$$k_2^{crit} \simeq 1.387 \tag{24}$$

(note that this is a theoretical upper bound; for instance for $\eta_{max} = 1/4.1$ one would get $k_2^{crit} \simeq 1.2$), whereas one obtains

$$\tilde{k}_4^{crit} = 10 \ln \frac{A(k_2) + 2}{3e/2.066} \tag{25}$$

when $k_2^{crit} - \epsilon < k_2 < k_2^{crit}$. Corresponding to the critical value $k_2 \to k_2^{crit}$ the system undergoes a higher order phase transition.

The three-dimensional case. *Let us consider the set of all simply-connected 3-dimensional dynamically triangulated manifolds described in terms of the variables N_3, N_0. Let k_0^{inf}, and k_0^{crit} respectively denote the unique solutions of the equations*

$$\frac{1}{3}(1 - \frac{1}{A(k_0)}) + \frac{1}{6} = \eta_{min} = \frac{1}{6} \tag{26}$$

$$\frac{1}{3}(1 - \frac{1}{A(k_0)}) + \frac{1}{6} = \eta_{max} = \frac{2}{9} \tag{27}$$

Let $0 < \epsilon < 1$ small enough, then for all values of the inverse gravitational coupling k_0 such that

$$k_0^{inf} + \epsilon < k_0 < k_0^{crit} - \epsilon, \tag{28}$$

the large N-behavior of the canonical partition function $W(N, k_0)_{eff}$ for 3-dimensional simplicial quantum gravity on a simply connected manifold is given by the uniform asymptotics

$$\frac{c_n}{\sqrt{2\pi}} \hat{N}^{-7/2} e^{(\hat{N}+1)f(\eta_{min})} \left[\frac{\alpha_0}{\sqrt{\hat{N}}} w_0(\psi_{min}(k_0)\sqrt{\hat{N}}) + \frac{\alpha_1}{\hat{N}} w_{-1}(\psi_{min}(k_0)\sqrt{\hat{N}}) \right] +$$

$$+ \frac{c_n}{\sqrt{2\pi}} \hat{N}^{-7/2} e^{(\hat{N}+1)f(\eta_{max})} \left[\frac{\xi_0}{\sqrt{\hat{N}}} w_0(\psi_{max}(k_0)\sqrt{\hat{N}}) + \frac{\xi_1}{\hat{N}} w_{-1}(\psi_{max}(k_0)\sqrt{\hat{N}}) \right] \tag{29}$$

where the constants α_0*,* α_1*, and* ξ_0*,* ξ_1 *are given by*

$$\alpha_0 \doteq -\frac{\eta_{min}^{-1/2}(1-2\eta_{min})^{-n}}{\sqrt{(1-3\eta_{min})(1-2\eta_{min})}}\frac{\psi_{min}(k_0)}{f_\eta(\eta_{min})} \tag{30}$$

$$\alpha_1 \doteq \frac{\alpha_0}{\psi_{min}(k_0)} - \frac{\eta *^{-1/2}\,(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{min}(k_0)\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{31}$$

$$\xi_0 \doteq -\frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}}\frac{\psi_{max}(k_0)}{f_\eta(\eta_{max})} \tag{32}$$

$$\xi_1 \doteq \frac{\xi_0}{\psi_{max}(k_0)} - \frac{\eta *^{-1/2}\,(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{max}(k_0)\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{33}$$

The critical value of the coupling k_3^{crit} *corresponding to the infinite-volume limit for the 3-sphere* $\mathbb{S}^3$ *is given by*

$$k_3^{crit}(k_0) = 6\ln\frac{A(0.16k_0)+2}{3[\cos^{-1}(1/3)]^{1/5}} - 0.16k_0 \tag{34}$$

and corresponding to the critical point

$$k_0 \to k_0^{crit} \simeq 3.845 \tag{35}$$

the system described by (29) exhibits a bistable behavior leading to hysteresis effects.

IV. TRIANGULATIONS

In this section we recall few basic notions of Piecewise-Linear (PL) topology in order to provide a self-contained set up for the some of the mathematical problems we wish to discuss in simplicial quantum gravity. Together with standard material, (for which we refer freely to [16]), we also collect few definitions and results which are not readily accessible in textbooks.

A. Preliminaries: simplicial manifolds and pseudo-manifolds

We first list some well known, but necessary, preliminaries.

By an n-simplex $\sigma^n \equiv (x_0, \ldots, x_n)$ with vertices $x_0, \ldots, x_n$ we mean the following subspace of R^d, (with $d > n$),

$$\sigma^n \doteq \sum_{i=0}^{n} \lambda_i x_i \tag{36}$$

where $x_0, \ldots, x_n$ are $n+1$ points in general position in R^d, and $\sum_{i=0}^{n} \lambda_i = 1$ with $\lambda_i \geq 0$.

A face of an n-simplex σ^n is any simplex whose vertices are a subset of those of σ^n, and a simplicial complex K is a finite collection of simplices in R^d such that if $\sigma_1^n \in K$ then so are all of its faces, and if σ_1^n, $\sigma_2^m \in K$ then $\sigma_1^n \cap \sigma_2^m$ is either a face of σ_1^n or is empty. The h-skeleton of K is the subcomplex $K_h \subset K$ consisting of all simplices of K of dimension $\leq h$.

Let K be a (finite) simplicial complex. Consider the set theoretic union $|K| \subset R^d$ of all simplices from K

$$|K| \doteq \cup_{\sigma \in K} \sigma \tag{37}$$

Introduce on the set $|K|$ a topology that is the strongest of all topologies in which the embedding of each simplex into $|K|$ is continuos, (the set $A \subset |K|$ is closed iff $A \cap \sigma^k$ is closed in σ^k for any $\sigma^k \in K$). The space $|K|$ is the underlying *polyhedron*, geometric carrier of the simplicial complex K, it provides the topological space underlying the simplicial complex. The topology of $|K|$ can be more conveniently described in terms of the *star of a simplex* σ, $star(\sigma)$, the union of all simplices of which σ is a face. The open subset of the underlying polyhedron $|K|$ provided by the interior of the carrier of the star of σ is the *open star* of σ. Notice that the open star is a subset of the polyhedron $|K|$, while the star is a sub-collection of simplices in the simplicial complex K. It is immediate to verify that the open stars can be used to define the topology of $|K|$. The polyhedron $|K|$ is said to be triangulated by the simplicial complex K. More generally, a triangulation of a topological space M is a simplicial complex K together with a homeomorphism $|K| \to M$.

Simplicial maps. A simplicial map $f: K \to L$ between two simplicial complexes K and L is a continuous map $f: |K| \to |L|$ between the corresponding underlying polyhedrons which takes n-simplices to n- simplices for all n, (piecewise straight-line segments are mapped to piecewise straight-line segments). The map f is a simplicial isomorphism if $f^{-1}: L \to K$ is also a simplicial map. Such maps preserve the natural combinatorial structure of $\mathbb{R}^n$. Note that a simplicial map is determined by its values on vertices. In other words, if $f: K^0 \to L^0$ carries the vertices of each simplex of K into some simplex of L, then f is the restriction of a unique simplicial map.

Combinatorial properties. Sometimes, one refers to a simplicial complex K as a simplicial division of $|K|$. A subdivision K' of K is a simplicial complex such that $|K'| = |K|$ and each n-simplex of K' is contained in an n- simplex of K, for every n. A property of simplicial complex K which is invariant under subdivision is a combinatorial property or *Piecewise-Linear* (PL) property of K, and a Piecewise-Linear homeomorphism $f: K \to L$ between two simplicial complexes is a map which is a simplicial isomorphism for some subdivisions K' and L' of K and L.

Piecewise-Linear manifolds. A PL manifold of dimension n is a polyhedron $M = |K|$ each point of which has a neighborhood, in M, PL homeomorphic to an open set in R^n.

PL manifolds are realized by simplicial manifolds under the equivalence relation generated by PL homeomorphism. Any piecewise linear manifold can be triangulated, however notice that any particular triangulation of a piecewise linear manifold is not well defined up to piecewise linear homeomorphism, (because such homeomorphisms involve subdivisions of the given triangulations). A triangulated space can be characterized as a PL-manifold according to the [16]

Theorem 1 *A simplicial complex K is a simplicial manifold of dimension n if for all r-simplices $\sigma^r \in K$, the link of σ^r, $link(\sigma^r)$ has the topology of the boundary of the standard $(n-r)$-simplex, viz. if $link(\sigma^r) \simeq \mathbb{S}^{n-r-1}$.*

Recall that the link of a simplex σ in a simplicial complex K is the union of all faces σ_f of all simplices in $star(\sigma)$ satisfying $\sigma_f \cap \sigma = \emptyset$, also recall that the *Cone* on the link $link(\sigma^r)$, $C(link(\sigma^r))$, is the product $link(\sigma^r) \times [0,1]$ with $link(\sigma^r) \times \{1\}$ identified to a point. The above theorem follows [16] by noticing that a point in the interior of an r-simplex σ^r has a neighborhood homeomorphic to $B^r \times C(link(\sigma^r))$, where B^r denotes the ball in $\mathbb{R}^n$. Since $link(\sigma^r) \simeq \mathbb{S}^{n-r-1}$, and $C(\mathbb{S}^{n-r-1}) \simeq B^{n-r}$, we get that $|K|$ is covered by neighborhoods homeomorphic to $B^r \times B^{n-r} \simeq B^n$ and thus it is a manifold. Note that the theorem holds whenever the links of vertices are $(n-1)$-spheres. As long as the dimension $n \leq 4$, the converse of this theorem is also true. But this is not the case in larger dimensions, and there are examples of triangulated manifolds where the link of a simplex is not a sphere. In general, necessary and sufficient conditions for having a manifolds out of a simplicial complex require that the link of each cell has the homology of a sphere, and that the link of every vertex is simply-connected, (see *e.g.*, Thurston's notes [16]). Since we are interested in dimension $2 \leq n \leq 4$, we can simply disregard such subtler characterizations. However (see the next comments on gluings), this issue is actually bypassed by the particular way triangulations are generated in simplicial quantum gravity.

Gluings. In dynamical triangulations theory one is not given from the outset a triangulation of a manifold, rather one generates an n-dimensional PL-manifold by gluing a finite set of

n-simplices $\{\sigma^n\}$. A rather detailed analysis of such gluing procedures is given in Thurston's notes [16], (clearly for different purposes), and here we simply recall the most relevant facts.

Given a finite set of simplices and the associated collection of faces, a *gluing* is a choice of pairs of faces together with simplicial identifications maps between faces such that each face appears in exactly one of the pairs. The identification space, K, resulting from the quotient of the union of the simplices by the equivalence relation generated by the identification maps, is homeomorphic to the polyhedron of a simplicial complex. The gluing maps are linear, and consequently the simplicial complex K obtained by gluing face-by-face n-simplices has the structure of a PL- manifold in the complement of the $(n-2)$-skeleton. Since the link of an $(n-2)$-simplex is a circle, it is not difficult to prove that the PL-structure actually extend to the complement of the $(n-3)$- skeleton, and that the identification space of a gluing among finite n-simplices is a PL-manifold if and only if the link of each cell is PL-homeomorphic to the standard PL-sphere.

Pseudo-manifolds. It is important to stress that in dimension $n > 2$, not every simplicial complex $M = |T|$ obtained by gluing simplices along faces is a simplicial manifold, and in general one speaks of *pseudo-manifolds*.

Definition 1 *$M = |T|$ is an n-dimensional pseudomanifold if: (i) every simplex of T is either an n-simplex or a face of an n- simplex; (ii) each $(n-1)$-simplex is a face of at most two n- simplices; (iii) for any two simplices σ^n, τ^n of T, there exists a finite sequence of n-simplices $\sigma^n = \sigma^n_0, \sigma^n_1, \ldots, \sigma^n_j = \tau^n$ such that σ^n_i and σ^n_{i+1} have an $(n-1)$-face in common, (i.e., there is a simplicial path connecting σ^n and τ^n).*

Recall that a regular point p of a polyhedron $|T|$ is a point having a neighborhood in $|T|$ homeomorphic to an n-dimensional simplex, otherwise p is called a *singular point*. Absence of singular points in a pseudo-manifolds characterizes triangulated manifolds. Moreover, an n-dimensional polyhedron $|T|$ is a pseudo- manifold if and only if the set of regular point in $|T|$ is dense and connected and the set of all singular points is of dimension less than

$n-1$. Thus, in the applications to gravitational physics, the occurrence of pseudo-manifolds is associated with the presence of local irregularities that should not have a sensible effect in the continuum limit of the theory, since the set of regular points has the density properties mentioned above.

Dehn-Sommerville relations. In order to construct a simplicial manifold T by gluing simplices σ^n, through their $n-1$-dimensional faces, the following constraints must be satisfied

$$\sum_{i=0}^{n}(-1)^i N_i(T) = \chi(T) \tag{38}$$

$$\sum_{i=2k-1}^{n}(-1)^i \frac{(i+1)!}{(i-2k+2)!(2k-1)!} N_i(T) = 0 \tag{39}$$

if n is even, and $1 \le k \le n/2$. Whereas if n is odd

$$\sum_{i=2k}^{n}(-1)^i \frac{(i+1)!}{(i-2k+1)!2k!} N_i(T) = 0 \tag{40}$$

with $1 \le k \le (n-1)/2$. These relations are known as the Dehn-Sommerville equations. The first, (38), is just the Euler-Poincaré equation for the triangulation T of which $N_i(T)$ denotes the number of i-dimensional simplices, the *f-vector* of the triangulation T. The conditions (39) or (40), are a consequence of the fact that in a simplicial manifold, constructed by gluings, the link of every $(2k-1)$-simplex (if n is odd) or $2k$-simplex (if n is even), is an odd-dimensional sphere, and hence it has Euler number zero.

Note also that in order to generate an n-dimensional polyhedron there is a **minimum number** $\hat{q}(n)$ of simplices σ^n that must join together at a bone, (for instance, in dimension 2, we have $\hat{q}(2) = 3$, otherwise we have no polyhedral surface). In general, since we will usually consider only the class of regular simplicial manifolds we assume that the number, $q(\sigma^i)$, of n-dimensional simplices σ^n which share the subsimplex σ^i is such that

$$q(\sigma^i) \ge n - i + 1 \tag{41}$$

with $i \leq n-2$.

There is a simple but important relation which must be satisfied by the numbers, $q(\sigma^{n-2})$, of n-dimensional simplices sharing the $(n-2)$-dimensional subsimplices, **the bones** of T, σ^{n-2}, *viz.*,

$$\sum_{\sigma^{n-2}} q(\sigma^{n-2}) = \frac{n(n+1)}{2} N_n \tag{42}$$

which follows by noticing that the number of $(n-2)$-dimensional sub- simplices in an n-dimensional simplex is $\frac{1}{2}n(n+1)$. We shall exploit this relation quite intensively, and for later convenience, let us introduce the average number of simplices σ^n, in T, incident on the $(n-2)$-dimensional subsimplices, σ^{n-2}, namely

$$b(n,n-2) \doteq \frac{1}{N_{n-2}(T)} \sum_{\sigma^{n-2}} q(\sigma^{n-2}) = \frac{1}{2}n(n+1) \left(\frac{N_n(T)}{N_{n-2}(T)} \right) \tag{43}$$

It is easily verified that the relations (38), (39), (40) leave $\frac{1}{2}n-1$, (n even) or $\frac{1}{2}(n-1)$ unknown quantities among the n ratios $N_1/N_0, \ldots, N_n/N_0$, [2]. Thus, in dimension $n = 2, 3, 4$, the datum of $b(n, n-2)$, (trivial for $n = 2$), and of the number of bones N_{n-2}, fixes, through the Dehn-Sommerville relations all the f-vectors $N_i(T)$ of the dynamical triangulation considered.

Together with the Dehn- Sommerville relations, equation (42) implies the following bounds for the average incidence number $b(n, n-2)$, when $N_{n-2} >> 1$:

(i) For $n = 2$:

$$b(2,0) = 6; \tag{44}$$

(ii) for $n = 3$:

$$3 \leq b(3,1) \leq 6; \tag{45}$$

(iii) for $n = 4$:

$$3 \leq b(4,2) \leq 5. \tag{46}$$

Notice however that the lower bounds in (45) and (46) are not optimal. This fact was noticed during computer simulations (see *e.g.* [45]). The actual characterization of the optimal lower bounds for $b(n, n-2)$ in dimension $n = 3$ and $n = 4$ it is rather important since it bears relevance to the location of the critical point in simplicial quantum gravity. It is not trivial to characterize such bounds since they are a consequence of the following result from the combinatorial topology of 3- dimensional PL-manifolds proved by D. Walkup [55]:

Theorem 2 *For any combinatorial* 3*-manifold the inequality*

$$N_1 \geq 4N_0 - 10 \tag{47}$$

holds with equality if and only if it is a stacked sphere

(a stacked sphere is by definition a triangulation of a sphere which can be constructed from the boundary of a simplex by successive adding of pyramids over some facets).

By exploiting this result we can prove the following

Lemma 1 *For any triangulation $T \to M^n$ of a closed n-dimensional PL- manifold M, the actual lower bound for the average incidence number $b(n, n-2)$ is given for $n = 3$ by*

$$b(n, n-2)|_{n=3} = \frac{9}{2} \tag{48}$$

and for $n = 4$ by

$$b(n, n-2)|_{n=4} = 4 \tag{49}$$

Proof. Let us consider first the three-dimensional case. The Dehn-Sommerville relations for 3-manifolds are

$$N_0 - N_1 + N_2 - N_3 = 0 \tag{50}$$

$$N_2 = 2N_3 \tag{51}$$

from which we immediately get $N_3 = N_1 - N_0$. On the other hand, Walkup's theorem implies $N_0 \leq \frac{1}{4}N_1 + \frac{5}{2}$. Thus,

$$N_3 \geq \frac{3}{4}N_1 - \frac{5}{2} \tag{52}$$

which yields the stated result for $n = 3$.

In dimension $n = 4$, the Dehn-Sommerville relations read

$$N_0 - N_1 + N_2 - N_3 + N_4 = \chi \tag{53}$$

$$2N_1 - 3N_2 + 4N_3 - 5N_4 = 0 \tag{54}$$

$$5N_4 = 2N_3 \tag{55}$$

Let us apply Walkup's theorem to to the link of every vertex v in our 4-dimensional triangulation. In this way we get

$$N_1[link(v)] \geq 4_0[link(v)] - 10 \tag{56}$$

and by summing over all vertices one obtains

$$3N_2 \geq 8N_1 - 10N_0 \tag{57}$$

which is the basic inequality that we are going to exploit. From it, (multiplying by 3), we obtain

$$9N_2 \geq 24N_1 - 30N_0 \tag{58}$$

There is another independent relation of this sort which can be obtained if we multiply (53) by 30, and exploit (55), *viz.*,

$$30N_0 - 30N_1 + 30N_2 - 18N_3 = 30\chi \tag{59}$$

From (54) one gets $2N_1 - 3N_2 + 2N_3 = 0$ which, (upon multiplication by 9), yields $18N_3 = 27N_2 - 18N_1$. Inserting this latter expression in (59) we eventually get

$$3N_2 = 30\chi + 12N_1 - 30N_0 \tag{60}$$

which together with (58) provides a direct proof of the bound for $b(n, n-2)|_{n=4}$. Explicitly, let us subtract (60) to (58). We get

$$6N_2 \geq 12N_1 - 30\chi, \tag{61}$$

namely

$$N_2 \geq 2N_1 - 5\chi \tag{62}$$

Since $2N_1 - 3N_2 + 2N_3 = 0$, we get $2N_3 - 2N_2 = N_2 - 2N_1$, thus, according to (62), we have

$$2N_3 - 2N_2 \geq -5\chi \tag{63}$$

And from (55) we obtain

$$5N_4 \geq 2N_2 - 5\chi \tag{64}$$

which, for $N_2 >> \chi$, yields for $b(n, n-2)|_{n=4} \geq 4$. $\Box$

It is also interesting to note that according to relation (42) the incidence numbers $\{q(\sigma^{n-2})\}$ of a simplicial manifold cannot be provided by a monotonically increasing sequence of integers, for otherwise $\Sigma_{\sigma^{n-2}} q(\sigma^{n-2})$ would be $O(N_n^2)$ in plain contrast with (42). The actual distribution of the incidence numbers $q(\sigma^{n-2})$ of a triangulation has typically a peak around a rather small value of $q(\sigma^{n-2})$ (≥ 3) and only a few of the bones σ^{n-2} can support a large number (*i.e.*, $O(N_n)$) of incident n-simplices. In particular, from (42) its is immediate to prove that if we let $q_{max} \doteq \sup_{T(N,B)} \sup_{\sigma^{n-2}} \{q(\sigma^{n-2})\}$ denote the maximum value of $q(\sigma^{n-2})$, as σ^{n-2} varies in the set of triangulations with N simplices σ^n and given value of $b(n, n-2)$, then, for $N_{n-2} >> 1$,

$$q_{max} = \frac{n(n+1)}{2F(n)} \left(\frac{b(n,n-2) - \tilde{q}}{b(n,n-2)} \right) N_n \tag{65}$$

where $F(n)$ is the number of bones over which q_{max} can be attained, and $\tilde{q}$ is the average incidence over the remaining bones, *i.e.*,

$$\tilde{q} \doteq \frac{1}{N_{n-2}} \sum_{F(n)+1}^{N_{n-2}} q(\sigma_\alpha^{n-2}) \tag{66}$$

An elementary example of such a configuration is easily constructed in dimension $n = 2$ by considering two distinct vertices v_1 and v_2 on each one of which $N/2$ triangles are incident. The link of both vertices is an $N/2$-gon, and by gluing the stars of v_1 and v_2 through such links we get a triangulation of the sphere $\mathbb{S}^2$ with N triangles, with $q(v_1) = q(v_2) = \frac{N}{2}$.

Actually, the existence of this sort of triangulation is a rather general fact of some relevance to us. Indeed it provides, for large N_n, a triangulation minimizing the average $\tilde{q}$, ($\tilde{q} = \hat{q} + 1 = 4$ in the above example), with few vertices carrying a large incidence. In general, the existence of such triangulations follows from the construction of a triangulation $T^{n+1} \to B^{n+1}$ of the $(n+1)$-dimensional ball B^{n+1} inducing on $\mathbb{S}^n \simeq \partial B^{n+1}$ a triangulation $T^n \to \mathbb{S}^n$ of the n-dimensional sphere $\mathbb{S}^n$ with N_n simplices and $N_{n-2} = \frac{1}{3!}(n-1)(n+1)N_n + n$ bones. $\frac{N_n}{n}$ of the n-dimensional simplices are incident on n distinguished bones $\{\sigma^{*n-2}_i\}_{i=1,\dots,n}$ spanning an embedded $\mathbb{S}^{n-2}$ in T^n. Thus, for $n = 2$, $\{\sigma^{*n-2}_i\}_{i=1,\dots,n}$ consists of two vertices (*viz.*, $\mathbb{S}^0$ which is indeed topologically identified with a double point). For $n = 3$, $\{\sigma^{*n-2}_i\}_{i=1,\dots,n}$ consists of 3 edges bounding a triangle ($\simeq \mathbb{S}^1$), whereas for $n = 4$ the supporting bones $\{\sigma^{*n-2}_i\}_{i=1,\dots,n}$ yields for 4 triangles bounding a tetrahedron (*viz.*, an $\mathbb{S}^2$).

In order to construct such triangulation of $\mathbb{S}^n$ consider the $(n+1)$-dimensional ball B^{n+1} triangulated by a sequence of N_{n+1}, $(n+1)$-dimensional simplices $\{\sigma^{n+1}\}$ all sharing a common $(n-1)$-dimensional simplex $\sigma_*^{n-1} \doteq \sigma_*$, *i.e.*,

$$\sigma_1^{n+1}(\sigma_*), \sigma_2^{n+1}(\sigma_*), \dots, \sigma_{N+1}^{n+1}(\sigma_*) = \sigma_1^{n+1}(\sigma_*). \tag{67}$$

and such that

$$F[\sigma_j^{n+1}(\sigma_*), \sigma_{j+1}^{n+1}(\sigma_*)] \cap F[\sigma_{j+1}^{n+1}(\sigma_*), \sigma_{j+2}^{n+1}(\sigma_*)] = \sigma_* \tag{68}$$

where $F[\sigma_j^{n+1}(\sigma_*), \sigma_{j+1}^{n+1}(\sigma_*)] \doteq \sigma_j^{n+1}(\sigma_*) \cap \sigma_{j+1}^{n+1}(\sigma_*)$ is the n- dimensional face through which $\sigma_j^{n+1}(\sigma_*)$ and $\sigma_{j+1}^{n+1}(\sigma_*)$ are glued together. The boundary of such $T^{n+1} \to B^{n+1}$ is a triangulation of the sphere $T^n \to \mathbb{S}^n$ with $N_n = nN_{n+1}$ top-dimensional simplices σ^n, and $N_{n-2} = \frac{1}{3!}(n-1)(n+1)N_n + n$ bones. The intersection $\sigma_* \cap T^n$ defines an embedded $\mathbb{S}^{n-2}$ in T^n generated by n bones, $\{\sigma^{*n-2}_i\}_{i=1,\ldots,n}$, on each of which $\frac{N_n}{n}$ simplices σ^n are incident.

These results can be naturally extended to PL-manifolds of arbitrary topology by gluing, the above triangulation of the n-sphere into a *small* triangulation of the given PL-manifod M, (say a triangulation of M with a minimal number of vertices). Explicitly, let $T_{small}(M)$ be a triangulation with few simplices, $N_n(T_{small})$, of an n-dimensional Pl-manifold M of given topology, (for a characterization of such type of triangulations and for examples see the paper by W. Kühnel in [15]). Consider the large N_n triangulation $T(\mathbb{S}^n)$ of the n-sphere $\mathbb{S}^n$ defined above, let $\sigma_1^n(M)$ be a simplex in M and $\sigma_2^n(\mathbb{S}^n)$ a simplex in $\mathbb{S}^n$. Set

$$T(M) \doteq (T_{small}(M) - \hat{\sigma}_1(M)) \cup_f (T(\mathbb{S}^n) - \hat{\sigma}_2^n(\mathbb{S}^n)) \tag{69}$$

where $\hat{\sigma}(\cdot)$ denotes the interior of the given σ's, and f is an homeomorphism from the boundary of $\sigma_1^n(M)$ to the boundary of $\sigma_2^n(\mathbb{S}^n)$. If $N_n(\mathbb{S}^n) >> N_n(T_{small})$, such $T(M)$ provides a large N_n triangulation of M whose distribution of incidence numbers $\{q(\sigma^{n-2})\}$ has the properties of the collection of incidence numbers of $T(\mathbb{S}^n)$.

Anticipating a little bit, when this sort of triangulations entropically dominate in the canonical partition function (7) we should expect that the resulting statistical system as described by (7) exhibits a very large *effective dimensionality* as $N_n \to \infty$. A useful clue to this is to note that an effective dimensionality on a triangulated manifold can be defined by counting the number of vertices visited in a walk of k steps, starting from a typical vertex. If the triangulation is sufficiently regular then, for k large, the number of visited vertices grows as k^n, n being the topological dimension of the underlying PL-manifold. On the other hand, triangulations such as the ones described above are such that all vertices can be visited in a few steps: as N_n increases the system effectively behaves as if we were dealing with a regular *infinite-dimensional* lattice.

Such remarks are elementary, nonetheless they play an important role in what follows since the incidence numbers $\{q(\sigma^{n-2}\}$ are directly related with curvature when discussing dynamical triangulations. Thus there is an intimate relations between the distributions of the incidence numbers $\{q(\sigma^{n-2}\}$, metric properties, (curvature), and effective dimension of the resulting statistical system.

B. Distinct triangulations of the same PL manifold

By the very definition of PL manifolds, it follows that there exist different triangulations, $T^{(i)}$, of the some PL manifold M. For later convenience, it is better to formalize this remark, and recall the following standard characterization by W. Tutte [8]:

Definition 2 *Two triangulations, $T^{(1)}$ and $T^{(2)}$ of the some underlying PL manifold M are identified if there is a one-to-one mapping of vertices, edges, faces, and higher dimensional simplices of $T^{(1)}$ onto vertices, edges, faces, and higher dimensional simplices of $T^{(2)}$ which preserves incidence relations. If no such mapping exists the corresponding triangulations are said to be distinct.*

Notice that sometimes, (*e.g.*, W. Thurston [16],p.105), such triangulations are said to be *combinatorially equivalent.* However, we shall avoid this terminology since, in simplicial quantum gravity, combinatorial equivalence is used as synonimous of PL- equivalence.

Not surprisingly, in connection to the simplicial quantum gravity program natural questions to ask are among the classics of PL- Topology: *(i)* When can a topological manifold be triangulated? *(ii)* Do any two triangulations of a manifold admit isomorphic subdivisions?

The answer to the first question is, for general manifolds, negative [16]: Kirby, Siebenmann and Wall showed that there exist topological manifolds which do not admit combinatorial structures, and that there are non-standard structures for PL-manifolds of dimension at least 5. What Kirby and Siebenmann proved [16] is that for $n \geq 5$ there is a single obstruction $k(M) \in H^4(M, \mathbb{Z}_2)$, (where $H^i(M, \mathbb{Z}_2)$ denotes the i-th cohomology group of M with $\mathbb{Z}_2$ coefficients field), to the existence of a piecewise linear structure on a topological manifold M, and that the vanishing of this obstruction ($k(M) = 0$) allows for the existence of a worth of $|H^3(M, \mathbb{Z}_2)|$ distinct PL structures. Moreover, if $n \leq 7$, every PL manifold admits a compatible differentiable structure which is unique, (up to diffeomorphisms), if $n \leq 6$. The *Hauptvermutung* is the subject of question *(ii)*, and it is false in general. For instance, as hinted before, the 5-sphere $\mathbb{S}^5$ admits PL-inequivalent triangulations. In the dimensions of interest to simplicial quantum gravity, the situation is more encouraging. Indeed, for dimension $n \leq 3$ both questions *(i)*, and *(ii)*, have an affermative answer. However, it is not yet known if every n- dimensional topological manifold is homeomorphic to a polyhedron for $n \geq 4$. Actually, (see the paper by T. Januszkiewicz [50]), there is a closed topological four- manifold M with the following properties: *(i)* M is not homotopy equivalent to a PL manifold; *(ii)* M is not homemorphic to a simplicial complex. The construction of this example is rather involved and exotic (the basic ingredient being the E_8 homology manifold), but shows once more, if necessary, that the interplay between the topology of 4-dimensional manifolds and quantum gravity cannot be easily tackled. For this reason when discussing the 4-dimensional case, we shall not consider explicitly topological 4-manifolds, and work directly in the PL-category. When discussing deformations of dynamical triangulations we shall need to consider smooth structures, but this is simply a technical issue since as already recalled the category of smooth 4-manifolds is equivalent to the category of PL 4-manifolds.

Another natural question to ask is under what conditions a given set of integers N_i is the f-vector of a triangulated polyhedron. This problem is of some relevance to simplicial quantum gravity, since it can be related to the counting of distinct triangulations, (*the*

Entropy problem). Indeed, a possible way of stating the entropy problem is to enumerate all distinct triangulations, on a manifold of given topology, for a given choice of possible f-vectors. However, for general triangulated n-manifolds the characterization of all possible f-vectors is beyond reach. Partial answers can be obtained only under rather severe constraints. Typically, one has to assume that the polyhedron in question is realized as a *convex* polyhedron in Euclidean d-dimensional space, $\mathbb{R}^d$. In this case, the possible f-vectors are characterized by Stanley's theorem [17], according to which the Dehn-Sommerville relations are necessary but not sufficient (they suffice in dimension 2 if we add the obvious condition $N_0 \geq 4$ expressing the fact that to bound a solid takes more than three vertices). In general, for a sequence of integers $N_0, N_1, \ldots, N_{d-1}$, set $N_{-1} = 1$ and $m = [\frac{d}{2}]$. For $0 \leq p \leq d$, define h_p by

$$h_p \doteq \sum_{i=p}^{d} (-1)^{i-p} \begin{pmatrix} i \\ p \end{pmatrix} N_{d-i-1} \tag{70}$$

and for, $1 \leq p \leq m$, set

$$g_p \doteq h_p - h_{p-1} \tag{71}$$

then we have [17]

Theorem 3 *A sequence of integers $N_0, N_1, \ldots, N_{d-1}$ occurs as the number of vertices, edges, ..., $(d-1)$- faces of a d-dimensional convex simplicial polytope if and only if the following conditions hold: $h_p = h_{d-p}$, for $0 \leq p \leq m$, (these are just a rewriting of Dehn-Sommerville relations); $g_p \geq 0$ for $1 \leq p \leq m$; and, if one writes*

$$g_p = \begin{pmatrix} n_p \\ p \end{pmatrix} + \begin{pmatrix} n_{p-1} \\ p-1 \end{pmatrix} + \ldots + \begin{pmatrix} n_r \\ r \end{pmatrix} \tag{72}$$

with $n_p > n_{p-1} > \ldots > n_r \geq r \geq 1$, (there is a unique way of decomposing an integer in this way), then

$$g_{p+1} \leq \begin{pmatrix} n_p + 1 \\ p+1 \end{pmatrix} + \begin{pmatrix} n_{p-1} + 1 \\ p \end{pmatrix} + \ldots + \begin{pmatrix} n_r + 1 \\ r+1 \end{pmatrix} \tag{73}$$

for $1 \leq p \leq m-1$.

This result was conjectured by McMullen [18], sufficiency and later necessity was proven respectively by Billera and Lee [19], and Stanley and McMullen [17], [20]. Finally, it is worth noticing that for the class of all spheres the determination of all f-vectors has been conjectured by Stanley [21]. Owing to the strong limitations imposed by the convexity constraint, it is not very convenient to formulate the entropy problem in terms of the characterization of the possible f-vectors of simplicial manifolds, and in enumerating distinct triangulations we shall follow a different strategy. However, it must be stressed that progress in Stanley's theory, (and its unexpected connections with algebraic geometry) may shed light on this question.

C. Dynamical Triangulations

Let $T \to M = |T|$ be a simplicial manifold, (henceforth, if there is no danger of confusion, we shall use the shorthand notation $M = |T|$ for denoting such a triangulated manifold). A triangulation T can be used to produce a metric on the underlying PL manifold $M = |T|$ by declaring that all simplices in the triangulation are isometric to the standard simplex of the given dimension. This procedure provides a collection of compatible metrics on M, which can be extended to a piecewise-flat metric on M in an obvious way, here we follow the elegant construction provided by Fröhlich [15]. Let us choose an orthonormal frame $\{\vec{e}_a(\sigma^n)\}_{a=1}^n$ in the interior of each simplex $\sigma^n \in T$, (this frame can be conveniently located at the barycentre of σ^n), and introduce another frame by choosing n directed edges $\{\vec{E}_\mu(\sigma^n)\}_{\mu=1}^n$ in the boundary of σ^n. We can write $\vec{E}_\mu(\sigma^n) = \sum_{a=1}^n t_\mu^a(\sigma^n)\vec{e}_a(\sigma^n)$, where the $n \times n$ matrix, the *n-bein*, $(t_\mu^a(\sigma^n))$ is, for each σ^n, a regular matrix, σ^n being a non-degenerate simplex.

Fixed edge-length metric. In terms of the chosen edges $\vec{E}_\mu(\sigma^n)$, we can write the remaining edges in the boundary of σ^n as $\vec{E}_{\mu\nu}(\sigma^n) = \vec{E}_\mu(\sigma^n) - \vec{E}_\nu(\sigma^n)$, with $1 \leq \mu < \nu \leq n$. The metric inside each simplex σ^n is assumed to be the usual Euclidean metric, δ. It can be expressed in terms of the matrix $\delta(\sigma^n) = g_{\mu\nu}(\sigma^n) = \langle \vec{E}_\mu, \vec{E}_\nu \rangle_\delta$. We assume that each edge $\vec{E}_\mu(\sigma^a)$ or $\vec{E}_{\mu\nu}(\sigma^n)$ has a *fixed length* $a = |\vec{E}_\mu(\sigma^n)| = |\vec{E}_{\mu\nu}(\sigma^n)|$, this characterizes fixed edge-length triangulations as compared to the variable edge-length triangulations typical of Regge calculus. In this way we get a metric in each simplex $\sigma^n \in M$ which is extended to M by specifying that the metric is continuous when one crosses the faces of the σ^n's, and that each face of a simplex is a linear flat subspace of the simplex in question. Thus the metric on a fixed edge-length PL manifold is entirely specified by the fixed distance elements a between nearest neighbor vertices of the corresponding simplicial manifold, *viz.*,

Definition 3 *For any $a \in \mathbb{R}^+$, a Dynamical Triangulation T_a of a polyhedron $M = |T|$, with distance cut-off a, and with $N_n(T_a)$ simplices σ^n, is a division of a simplicial manifold $M = |T|$ generated by gluing, along their adjacent faces, $N_n(T_a)$ equilateral simplices $\{\sigma^n\}$ with edge-length $|\vec{E}_\mu(\sigma^n)| = |\vec{E}_{\mu\nu}(\sigma^n)| = a$, $\forall \sigma^n \in T$.*

Note that in order to avoid overcounting dynamical triangulations differing only by a rescaling of the edge-lengths, one always refers to a given value of the cut-off, (typically $a = 1$), when characterizing distinct triangulations.

D. Dynamical triangulations as length-spaces

The metric $g_{\mu\nu}$ generated by extending over $M = |T_a|$ the flat metrics $g_{\mu\nu}(\sigma^n)$ of the simplices $\{\sigma^n\}$, is not the object of primary interest for unravelling the metric geometry of a dynamically triangulated manifold. For one thing, this metric is manifestly singular around the $(n-2)$-dimensional simplices, (which have conical neighborhoods), for the other there is nothing sacred in $g_{\mu\nu}$ and one can capture more explicitly the geometry of M by looking at all curves between given points p_1 and p_2 in M and by setting the distance $d(p_1, p_2)$ equal to the infimum of the length of these curves. Incidentally one can reconstruct g at any (regular) point p from the distance function. Setting $f(\hat{p}) \doteq [d(p, \hat{p})]^2$, it is easily verified that g is provided, to leading order, by the second differential $D^2 f(\hat{p})|_{\hat{p}=p}$.

In order to discuss some aspects of the interior geometry of a dynamical triangulation thought of as a metric space, we introduce few preliminary notions hinting to a characterization of *geodesics* on a dynamical triangulation without using calculus [50].

A *curve* in a dynamically triangulated manifold M is a continuous map $c: I \to M = |T_a|$, where I is an interval. The length $L(c)$ of the curve $c: [a, b] \to M$ is then defined according to

$$L(c) \doteq \sup \sum_{h=1}^{j} d(c(t_{h-1}), c(t_h)) \tag{74}$$

where the supremum is taken over all subdivisions $a = t_0 < t_1 < \ldots < t_j = b$ of $[a, b]$, and $d(,)$ denotes the distance function associated with the given dynamical triangulation $T_a \to M = |T_a|$. We assume that c is rectifiable, (*i.e.*, $L(c) < \infty$), and introduce the *arc length* parametrization as the non-decreasing continuous surjective map

$$s: [a, b] \to [0, L(c)] \tag{75}$$

defined by $s(t) \doteq L(c|[a, t])$. In terms of the arc-length, the curve c is *travelled* at unit speed, *viz.*,

$$\tilde{c}: [0, L(c)] \to M \tag{76}$$

with $\tilde{c}(s(t)) \doteq c(t)$, and

$$L(\tilde{c}|[s_1, s_2]) = |s_1 - s_2| \tag{77}$$

Similarly we can characterize a curve $c: I \to M$ travelled at speed $v \geq 0$ if the curve is such that

$$L(c|[t_1, t_2]) = v|t_1 - t_2| \tag{78}$$

for all $t_1, t_2 \in I$. If a curve $c: I \to M$ has constant speed $v \geq 0$ and if for any $t \in I$ the curve c has a neighborhood U in I such that

$$d(c(t_1), c(t_2)) = v|t_1 - t_2| \tag{79}$$

for all $t_1, t_2 \in U$, then such a c is called a *geodesic* in the dynamically triangulated manifold $M = |T_a|$. Such a geodesic is said to be *minimizing* if (79) holds for all $t_1, t_2 \in I$.

The *interior metric* d_{int} associated to a dynamical triangulation T_a is defined according to [50]

$$d_{int}(x, y) \doteq \inf\{L(c)\} \tag{80}$$

where inf is taken over all curves from x to y. It is easily verified that for a dynamical triangulation such distance function coincides with the original distance function d, $d_{int} = d$, and in such a sense a dynamical triangulation is a *Length space* [9]. It is also a *geodesic space* [50] since for any pair of points x, y in M there is a minimizing geodesic from x to y.

At first sight these facts may seem trivialities, (at least for a dynamically triangulated manifold), however they have important consequences when discussing rigorously the way a dynamically triangulated manifold approximate smooth riemannian manifolds. A first application will be encountered when discussing curvature in the PL-setting.

PL connections and the Incidence Matrix. In general, associated with a PL metric $g_{\mu\nu}(\sigma^n) = \langle \vec{E}_\mu, \vec{E}_\nu \rangle_\delta$, $\sigma^n \in T$, there is also a unique connection (the Levi-Civita connection) which can be characterized by the set of matrices, $\Gamma(\sigma^n_i, \sigma^n_{i+1})$, describing the change of bases $\{\vec{E}_\mu(\sigma^n_i)\}^n_{\mu=1} \to \{\vec{E}_\mu(\sigma^n_{i+1})\}^n_{\mu=1}$, (regarded as different bases of $\mathbb{R}^n$), in passing between two adjacent simplices in M, sharing a common face $F(\sigma^n_i, \sigma^n_{i+1})$. In Regge calculus [15], the Levi-Civita connection so defined is uniquely determined by the length of the edges and by the incidence matrix of T, the $N_n(T) \times N_n(T)$ matrix $I(\sigma^n_i, \sigma^n_j)$, whose entries are 1 if σ^n_i and σ^n_j are glued along a common face, 0 otherwise. It follows that when the metric g is generated by the fixed edge length prescription, the corresponding Levi-Civita connection is uniquely determined by $I(\sigma^n_i, \sigma^n_j)$.

Definition 4 *The family* $\alpha_g \doteq \{\Gamma(\sigma_i^n, \sigma_{i+1}^n)\}_{i=1,\dots}$ *with* σ_i^n *and* σ_{i+1}^n *adjacent in* T_a*, is the Levi-Civita connection associated with the dynamical triangulation* T_a *of the PL manifold* $M = |T|$.

Notice that if in place of the change of bases $\{\vec{E}_\mu(\sigma_i^n)\}_{\mu=1}^n \to \{\vec{E}_\mu(\sigma_{i+1}^n)\}_{\mu=1}^n$ we consider the corresponding change of orthonormal bases $\{\vec{e}_a(\sigma_i^n)\}_{a=1}^n \to \{\vec{e}_a(\sigma_{i+1}^n)\}_{a=1}^n$, in passing from one simplex to another, then the connection α_g can be equivalently expressed in terms of orthogonal matrices $O(\sigma_i^n, \sigma_{i+1}^n)$ given by

$$\Gamma(\sigma_i^n, \sigma_{i+1}^n) = t^{-1}(\sigma_i^n) O(\sigma_i^n, \sigma_{i+1}^n) t(\sigma_{i+1}^n) \tag{81}$$

Simplicial paths and parallel transporters. A trasparent definition of curvature in a simplicial manifold can be provided if together with the standard notion of curve on M, we introduce the combinatorial notion of simplicial path. A *simplicial path* in a simplicial manifold M is defined as a sequence of simplices $\{\sigma_j^n\}_{j=1}^l$, $l = 2, 3, \dots$ with $\sigma_j^n \in M$ and such that σ_j^n and $\sigma_{j+1}^n \neq \sigma_j^n$ share a common face. If $\sigma_1^n = \sigma_l^n$ then we speak of a *simplicial loop* in the manifold M. With any simplicial loop ω, we can associate the parallel transporter

$$R_\omega = \prod_{\sigma_j, \sigma_{j+1} \in \omega} \Gamma(\sigma_j^n, \sigma_{j+1}^n) \tag{82}$$

where the product is path-ordered. Let B denote the generic bone in T_a, and let $\omega(B)$ that unique loop winding around the bone B. If $\sigma_j^n(B)$ denotes the generic simplex containing B, then [15]

$$\omega(B) = \{\sigma_1^n(B), \sigma_2^n(B), \dots, \sigma_m^n(B) = \sigma_1^n(B)\} \tag{83}$$

with $\sigma_1^n(B) \cap \dots \cap \sigma_m^n(B) = B$. The corresponding rotation matrix

$$R_{\omega(B)} = \prod_i R(\sigma_i(B), \sigma_{i+1}(B)) \tag{84}$$

is such that all vectors in the $(n-2)$-dimensional hyperplane spanned by the bone B are eigenvectors with eigenvalue 1, and in the 2-dimensional plane orthogonal to the bone B, $R_{\omega(B)}$ reduces to a rotation, where the rotation angle $\phi(B)$ is given by

$$\phi(B) = \sum_{j=1}^{q(B)} \delta_j(B) \tag{85}$$

$q(B)$ being the number of simplices σ^n incident on the bone B, and $\delta_j(B)$ is the angle between the unique two faces of σ_j^n containing B.

Curvature. In Regge calculus, the quantity $r(B) \equiv 2\pi - \phi(B)$ is the deficit angle at the bone B. Since the angle between two faces sharing a common bone B, in an equilateral simplex σ^n, is given by $\cos^{-1} 1/n$, we can write

$$\phi(B) = q(B) \cos^{-1} \frac{1}{n} \tag{86}$$

where $q(B)$ denotes the number of simplices σ^n incident on the bone B. In defining curvature for dynamically triangulated manifolds one has to change the perspective on Regge calculus somewhat (as advocated by Hamber and Williams [22]) and view the geometry of the triangulation as representing an approximation to some smooth geometry and the local curvature as some average curvature for a small volume, (in the next section we formalize somewhat this point of view by adopting the Gromov-Hausdorff topology on the space of dynamically triangulated manifolds). Thus, to each $(n-2)$-dimensional bone B we assign an n-dimensional volume density given by

$$vol_n(B) \doteq \frac{2}{n(n+1)} \sum_{\sigma_j^n \in \omega(B)} vol(\sigma_j^n) \tag{87}$$

where $\{\sigma_j^n\}$ is the set of n-dimensional simplices in the unique simplicial loop $\omega(B)$ winding around the given bone B. The volume $vol_n(B)$ can be considered as the natural share of the volumes of the n-dimensional simplices to which the bone belongs. It follows that every bone B in a dynamical triangulation T_a carries a curvature given by

$$K(B) = 2\left[2\pi - q(B)\cos^{-1}\frac{1}{n}\right]\frac{vol_{n-2}(B)}{vol_n(B)} \tag{88}$$

where $vol_{n-2}(B)$ is the usual $n-2$-dimensional (Euclidean) measure of the bone B. Recall that the Euclidean volume of a j- dimensional equilateral simplex of edge-length a is

$$vol(\sigma^j) = \frac{a^j\sqrt{(j+1)}}{j!\sqrt{2^j}} \tag{89}$$

thus, we get

$$K(B) = \frac{n^2(n^2-1)}{a^2}\sqrt{\frac{n-1}{n+1}}\left[\frac{2\pi - q(B)\cos^{-1}\frac{1}{n}}{q(B)}\right] \tag{90}$$

For instance, in dimension four we have $vol_{n-2}(B) = \frac{1}{4}\sqrt{3}a^2$, $vol_n(\sigma^n) = \frac{1}{96}\sqrt{5}a^4$, and (88) reduces to

$$K(B) = \frac{240}{a^2}\sqrt{\frac{3}{5}}\left[\frac{2\pi - q(B)\cos^{-1}\frac{1}{n}}{q(B)}\right] \tag{91}$$

Curvature assignements. Up to the choice of the cut- off edge- length, a, and of a numerical factor depending only from the dimension n, curvature in a dynamical triangulation T with $N_{n-2}(T)$ bones $B(\alpha)$, $\alpha = 1, 2, \ldots, N_{n-2}(T)$, is directly provided by the sequence of integers $\{q(\alpha)\}$, where $q(\alpha)$ denotes the number of simplices σ^n incident on the bone $B(\alpha)$. Henceforth, when speaking of *curvature assignements* to the bones $\{B(\alpha)\}$ of a dynamical triangulation T_a, we shall explicitly refer to the sequence of integers $\{q(\alpha)\}$.

The Einstein-Hilbert action for dynamical triangulations. The Regge version of the Einstein-Hilbert action (with cosmological term) for a PL-manifold M, (without boundary), is given by [4], [1], [15]

$$S_{Regge}(\Lambda, G) \doteq \Lambda\sum_{\sigma^n} vol(\sigma^n) - \frac{1}{16\pi G}\sum_B r(B)vol(B) \tag{92}$$

When the PL-manifold M is dynamically triangulated $T_a \to M$, $S_{Regge}(\Lambda, G)$ simplifies considerably reducing to

$$S[k_{n-2}, k_n] \doteq k_n N_n(T_a) - k_{n-2}N_{n-2}(T_a) \tag{93}$$

where we have set according to standard usage

$$k_n \doteq \Lambda vol(\sigma^n) + \frac{1}{2}n(n+1)\frac{\cos^{-1}\frac{1}{n}}{16\pi G}vol(\sigma^{n-2}) \quad (94)$$

$$k_{n-2} \doteq \frac{vol(\sigma^{n-2})}{8G} \quad (95)$$

Thus, for dynamical triangulations the gravitational action reduces to a combinatorial object with two (running) couplings k_{n-2} and k_n. The former proportional to the inverse gravitational coupling, while the latter is a linear combination of $1/16\pi G$ with the cosmological constant Λ. Accordingly, the formal path integration (1) is replaced (see (5)) on a dynamically triangulated Pl manifold M, (of fixed topology), by the (grand-canonical) partition function

$$Z[k_{n-2}, k_n] = \sum_{T\in\mathcal{T}(M)} e^{-k_n N_n + k_{n-2}N_{n-2}} \quad (96)$$

The action $S[k_{n-2}, k_n]$ and the associated partition function are so deceiptively simple that one may wonder under which rug we are dumping all the notorious problems affecting the Einstein-Hilbert action when coming to a path-integral quantization. Obviously most of these problems are now shifted in characterizing the summation over distinct triangulations $\sum_{T\in\mathcal{T}(M)}$ which replaces the formal path integration over distinct riemannian structures, namely in characterizing the dynamical triangulation counterpart of the formal measure $D[g(M)]$. In general, these sort of deals hardly are true bargains. However, as we shall see, this trade off in difficulties will pay off since a rather complex problem (Path Integration over $Riem(M)/Diff(M)$), the mathematical formulation of which is rather ambiguous, has been transformed into a well-defined counting problem: *enumerate all distinct dynamical triangulations $T \in \mathcal{T}(M)$ admitted by a PL manifold of given topology.* However, before addressing this problem, we have to formalize in more precise terms in which sense dynamical triangulations do approximate a riemannian structure on a smooth manifold.

V. APPROXIMATING RIEMANNIAN MANIFOLDS WITH DYNAMICALLY TRIANGULATIONS OF BOUNDED GEOMETRY

The fact that curvature for dynamical triangulations is a combinatorial object has a number of deep consequences, mainly of topological nature, having an important bearing on simplicial quantum gravity. These properties follow from the fact that a dynamically triangulated manifold $M = |T_a|$, with $N_n(T_a)$ n-simplices and with a given average number $b(n, n-2)$ of n- simplices incident on the bones σ^{n-2}, (see (43)), is a piecewise-flat singular metric space of *bounded geometry*, a concept which goes back to A.D. Alexandrov [51]. There is a definite way in which such singular metric spaces approximate (or are approximated by suitably smooth) manifolds, and this will provide the rationale for approximating riemannian manifolds with dynamical triangulations.

A. Dynamical triangulations as singular metric spaces

To start with an intuitive description, let us consider two geodesic rays starting from the some point p in a Riemannian manifold V. Thus, suppose that $l_1\colon [0,1] \to V$ and $l_2\colon [0,1] \to V$ are (minimal) geodesics with $l_1(0) = l_2(0) = p$. Fix $t, s \in (0,1)$ and construct the plane Euclidean triangle with (linear-parametrized) lines $L_1\colon [0,1] \to \mathbb{R}^2$, $L_2\colon [0,1] \to \mathbb{R}^2$ such that $L_1(0) = L_2(0) = (0,0)$, $||L_1(s)|| = d(l_1(0), l_1(s))$, $||L_2(t)|| = d(l_1(0), l_2(t))$, and

$$||L_1(s) - L_2(t)|| = d(l_1(s), l_2(t)). \tag{97}$$

If the sectional curvatures of V are everywhere non-negative, then the geodesics l_1 and l_2 tend to come together compared with the corresponding lines L_1, L_2; *i.e.*,

$$||L_1(1) - L_2(1)|| \geq d(l_1(1), l_2(1)) \tag{98}$$

whereas non-positive sectional curvatures force the goedesics to diverge faster than in the Euclidean situation: $||L_1(1) - L_2(1)|| \leq d(l_1(1), l_2(1))$.

Once one has a notion of geodesics, the above characterization of positive or negative curvature makes sense for any length-spaces, and in that case we speak of a space of positive (or negative) *Alexandrov curvature*. Actually, similar characterizations can be carried out in order to define spaces of Alexandrov curvature bounded above of below by a constant $k \in \mathbb{R}$, by comparing the geodesics l_1, l_2 with the corresponding geodesics on the appropriate simply connected surface of constant Gauss curvature k. These comparision techniques are a standard tool in global riemannian geometry, (see *e.g.*, Chavel's book [50]).

Geodesic triangles. For $k \in \mathbb{R}$, let Σ_k denote the *model surface* of constant Gauss curvature k. Motivated by the above remarks, we characterize dynamical triangulations as singular metric spaces with bounded Alexandrov curvature, by comparing suitable triangles in $M = |T_a|$ with triangles in model surfaces Σ_k. Since in a dynamical triangulation curvature is localized around bones σ^{n-2}, we have to consider geodesic triangles which encircles bones. This can be done according to the following procedure.

Let $\alpha \doteq \sigma^{n-2}$ be the generic bone in a dynamical triangulation M. Consider its associated link, $link(\sigma^{n-2}) \simeq \mathbb{S}^1$, and three distinct vertices, labelled 1, 2, 3 in $link(\sigma^{n-2})$. If $q(\alpha)$ simplices σ^n are incident on the bone α, then $link(\sigma^{n-2})$ has a length

$$||link(\sigma^{n-2})|| = q(\alpha) \cdot a \tag{99}$$

a being the edge-length of the given dynamical triangulation. The vertices 1,2,3 are chosen (up to a rotation) in such a way that their pairwise distance, $d(i,k)$, $i \neq k = 1,2,3,$, measured along the link $link(\sigma^{n-2})$, satisfies

$$d(i,k) \geq [\frac{q(\alpha)}{3}] \tag{100}$$

where the bracket function $[x]$ stands for the greatest integer not exceeding x. Consider now (in the closure of the open star of the bone α) the three geodesic segments pairwise joining the vertices so chosen, *viz.*, $c_1 \doteq l(2,3)$, $c_2 \doteq l(1,3)$, and $c_3 \doteq l(1,2)$, then

Definition 5 *The geodesic triangle $\Delta(\alpha) \doteq (c_1, c_2, c_3)$ consisting of the three geodesic segments c_1, c_2, c_3 in M, (called the sides of the triangle), is the geodesic triangle encircling the bone $\alpha \in M$. A geodesic triangle $\bar{\Delta} \doteq (\bar{c}_1, \bar{c}_2, \bar{c}_3)$ in Σ_k is called an Alexandrov (or comparision) triangle for $\Delta(\alpha)$ if $length(c_i) = length(\bar{c}_i)$, $i = 1, 2, 3$.*

It is not difficult to verify that such a comparision triangle exists, (and it is unique up to a congruence), and we have the following

Lemma 2 *A dynamically triangulated manifold $M = |T_a|$, with $N_n(T_a)$ n-simplices and with a given average number $b(n, n-2)$ of n- simplices incident on the bones σ^{n-2}, is a piecewise-flat singular metric space of bounded Alexandrov curvature k, with $k_{min} \leq k \leq k_{max}$, where k_{min}, k_{max} are characterized by*

$$\frac{1}{2\pi} a \cdot \hat{q} \cos^{-1}(1/n) = \frac{\sin(a\sqrt{k_{max}})}{\sqrt{k_{max}}} \tag{101}$$

and

$$\frac{1}{2\pi} a \cdot q_{max} \cos^{-1}(1/n) = \frac{\sinh(a\sqrt{-k_{min}})}{\sqrt{-k_{min}}} \tag{102}$$

where, (see (65))

$$q_{max} = \frac{n(n+1)}{2F(n)} \left(\frac{b(n, n-2) - \tilde{q}}{b(n, n-2)} \right) N_n. \tag{103}$$

Proof. We start by showing that comparision triangle exists for any bone of the given dynamical triangulation. In order to prove this, let us first remark that the geodesic triangle $\Delta(\alpha)$ has sides satisfying the triangle inequality

$$length(c_i) + length(c_{i+1}) \geq length(c_{i+2}) \tag{104}$$

indices taken modulo 3. However, this is not sufficient to insure the existence of a comparision triangle in Σ_k since, if $k > 0$, the perimeter of $\Delta(\alpha)$ must satisfy the bound

$$length(c_1) + length(c_2) + length(c_3) < \frac{2\pi}{\sqrt{k}} \tag{105}$$

(this bound is trivially extended to the case $k \leq 0$ by setting $\frac{2\pi}{\sqrt{k}} \doteq \infty$ for $k \leq 0$). To prove that (105) holds, let x be any point in the relative interior of the given bone α, and consider the set of all tangent vectors to the geodesic rays (lines) emanating from x, of length a, pointing into σ^n, (the generic simplex incident on α), and orthogonal to α. This set of vectors forms a spherical ($\mathbb{S}^1$)-simplex. Since this set is, up to an isometry, independent of $x \in \alpha$, we shall denote this set as $\mathbb{S}^1 link(\alpha, \sigma^n)$: the *spherical link* of α in σ^n [50]. Geometrically, the spherical link $\mathbb{S}^1 link(\alpha, \sigma^n)$ can be thought of as the arc of circumference *circumscribed* to $link(\alpha, \sigma^n)$, the link of the bone α in σ^n.

If the Regge curvature around the given bone is positive, then

$$q(\alpha) < \frac{2\pi}{\cos^{-1}(1/n)} \tag{106}$$

Since $\cos^{-1}(1/n) \geq \frac{\pi}{3}$ for $n \geq 2$, we get that $q(\alpha) < 6$. The union of the spherical links $\mathbb{S}^1 link(\alpha, \sigma^n)$ over all $q(\alpha)$ simplices σ^n incident on α is then a circumference of length $a \cdot q(\alpha) \cos^{-1}(1/n) < 2\pi a$. Consider now the sphere, $\mathbb{S}^2(r)$, of radius r, such that a great circle of radius a on $\mathbb{S}^2(r)$ has a circumference of length $a \cdot q(\alpha) \cos^{-1}(1/n)$. An elementary computation characterizes the radius r of such $\mathbb{S}^2(r)$ in terms of $q(\alpha)$ as

$$\frac{1}{2\pi} a \cdot q(\alpha) \cos^{-1}(1/n) = r \sin(\frac{a}{r}) \tag{107}$$

or, by introducing the curvature $k \doteq \frac{1}{r^2}$ of $\mathbb{S}^2(r)$

$$\frac{1}{2\pi} a \cdot q(\alpha) \cos^{-1}(1/n) = \frac{\sin(a\sqrt{k})}{\sqrt{k}} \tag{108}$$

Since the geodesic triangle $\Delta(\alpha)$ is inscribed in the circumference of length $a \cdot q(\alpha) \cos^{-1}(1/n)$, we get that (105) holds with k given in terms of $q(\alpha)$ by (108). By developing (108) to leading order in a, we get, not unexpectedly, that k is given by

$$k \simeq \frac{6}{a^2} \left(2\pi - q(\alpha) \cos^{-1}(1/n)\right) \tag{109}$$

namely, up to a numerical factor, by the Regge curvature associated with the bone α. This elementary analysis can be extended to the case of negative Regge curvature obtaining

$$\frac{1}{2\pi}a \cdot q(\alpha)\cos^{-1}(1/n) = \frac{\sinh(a\sqrt{-k})}{\sqrt{-k}} \tag{110}$$

thus proving that a dynamically triangulated manifold $M = |T_a|$, with $N_n(T_a)$ n-simplices and with a given average number $b(n, n-2)$ of n- simplices incident on the bones σ^{n-2}, admits comparision triangles on model surfaces, Σ_k, of constant curvature k, where k is provided by (108) and (110). Since on such set of dynamical triangulations the possible incidence numbers $\{q(\alpha)\}$ are bounded between

$$\hat{q} \leq q(\alpha) \leq q_{max}, \tag{111}$$

(see (65)), the above argument shows that a dynamical triangulation $M = |T_a|$, as a singular metric space, is an Alexandrov space with bounded curvature $k_{min} \leq k \leq k_{max}$ where k_{min} and k_{max} are respectively provided by

$$\frac{1}{2\pi}a \cdot \hat{q}\cos^{-1}(1/n) = \frac{\sin(a\sqrt{k_{max}})}{\sqrt{k_{max}}} \tag{112}$$

and

$$\frac{1}{2\pi}a \cdot q_{max}\cos^{-1}(1/n) = \frac{\sinh(a\sqrt{-k_{min}})}{\sqrt{-k_{min}}}, \tag{113}$$

as stated. □

The notion of bounded geometry associated with the characterization of dynamical triangulations as Alexandrov spaces becomes especially interesting if one considers the set of all dynamical triangulations satisfying the hypotheses of lemma 2. Guided by the intuition of what happens in similar circumstances for ordinary riemannian manifolds [9], [10], we expect that the metric space structures of such dynamical triangulations is controlled by such curvature bounds. To formulate this kind of controlled behavior, it is useful to introduce a topology in the set of dynamical triangulations which arises directly from the classical idea of Hausdorff distance between compact sets in a metric space [9]. The basic rationale is that given a length cut-off ϵ, two metric spaces are to be considered near in this topology, (one is the ϵ-Gromov- Hausdorff approximation of the other), if their metric properties are similar at length scales $L \geq \epsilon$, namely if there is a way of fitting them in a larger matric space so that they are close to each other. To formalize this intuition, we introduce the notion of *rough isometry.*

ϵ-rough isometries. Consider two compact (finite dimensional) metric spaces M_1 and M_2, let $d_{M_1}(\cdot,\cdot)$ and $d_{M_2}(\cdot,\cdot)$ respectively denote the corresponding distance functions, and let $\phi\colon M_1 \to M_2$ be a map between M_1 and M_2, (this map is not required to be continuous). If ϕ is such that: *(i)*, the ϵ-neighborhood of $\phi(M_1)$ in M_2 is equal to M_2, and *(ii)*, for each x, y in M_1 we have

$$|d_{M_1}(x,y) - d_{M_2}(\phi(x),\phi(y))| < \epsilon \tag{114}$$

then ϕ is said to be an *ϵ-Hausdorff approximation.* The *Gromov-Hausdorff distance* between the two metric spaces M_1 and M_2, $d_G(M_1, M_2)$, is then defined according to [9]

Definition 6 *$d_G(M_1, M_2)$ is the lower bound of the positive numbers ϵ such that there exist ϵ-Hausdorff approximations from M_1 to M_2 and from M_2 to M_1.*

Alternatively [23], $d_G(M_1, M_2) \leq \epsilon$ if there is a metric, $d(\cdot,\cdot)$, on the disjoint union $M_1 \coprod M_2$ extendings the metrics on M_1 and M_2 and such that $M_1 \subset \{x \in M_1 \coprod M_2 \colon d(x, M_2) \leq \epsilon\}$, $M_2 \subset \{x \in M_1 \coprod M_2 \colon d(x, M_1) \leq \epsilon\}$. Namely, two compact metric spaces M_1 and M_2 are Gromov-Hausdorff nearby if we can find metrics on the disjoint union of M_1 and M_2 such that M_1 and M_2 look like the same when they are close to each other.

The notion of ϵ-Gromov-Hausdorff approximation is the weakest large-scale equivalence relation between metric spaces of use in geometry, and is manifestly adapted to the needs of simplicial quantum gravity, (think of a manifold and of a simplicial approximation to it).

Notice that d_G is not, properly speaking, a distance since it does not satisfy the triangle inequality, but it rather gives rise to a metrizable uniform structure in which the set of isometry classes of all compact metric spaces, is Hausdorff and complete. This enlarged space does naturally contain riemannian manifolds and metric polyedra. As stressed in [23], the importance of this notion lies not so much in the fact that we have a distance function, but in that we have a way of measuring when metric spaces look alike.

B. Approximating Riemannian manifolds through dynamical triangulations

In this section we prove that, dynamical triangulations T_a, with a given number $N_n(T_a)$ of n-simplices and a given ratio $b(n, n-2)$, are in the Gromov-Hausdorff closure of the ∞-dimensional compact metric space characterized by the following

Definition 7 *For r a real number, D a positive real number, and n a natural number, let us define the associated space of Bounded Geometries, $\mathcal{R}(n, r, D, V)$, as the Gromov-Hausdorff closure of the space of isometry classes of closed connected n-dimensional riemannian manifolds (M, g) with Ricci curvature bounded below by r, viz.,*

$$\inf_{x \in M} \{\inf\{Ric(u, u) \colon u \in T_x M, |u_x| = 1\}\} \geq r$$

and diameter bounded above by D, viz.,

$$diam(M) \equiv \sup_{(p,q) \in M \times M} d_M(p, q) \leq D$$

It is rather immediate to recast this characterization in our current setting so as to prove the following *approximation* property (actually a statement of the Lebesgue covering property) which has far reaching consequences for simplicial gravity:

Lemma 3 *Let $\mathcal{DT}_n(a, b, N)$ denote the set of n-dimensional, $(n = 2, 3, 4)$,dynamically triangulated manifolds $M = |T_a|$, with $N_n(T_a) \equiv N$ top-dimensional simplices and given average incidence $b(n, n-2)$, and let k_{min} be the corresponding lower bound on the Alexandrov* *curvature, (see lemma 2). Correspondingly, let $\mathcal{R}(n,r,D,V)$ denote the space of bouded geometries with*

$$r \geq k_{min} \tag{115}$$

$$V = N \cdot vol(\sigma^n) \tag{116}$$

$$D \leq c_1[N \cdot vol(\sigma^n)]^{1/n} \tag{117}$$

where $vol(\sigma^n)$ denotes the (Euclidean) volume of the equilateral simplex σ^n of edge-length a, and c_1, is a constant (independent from the triangulation considered). Then for any $n = 2,3,4$, any finite $b(n,n-2) \in \mathbb{Q}^+$, and $N \in \mathbb{N}$, there is an $\epsilon = \epsilon(a) > 0$ such that for any riemannian manifold $V \in \mathcal{R}(n,r,D,V)$ we have

$$d_G(V, M = |T_a|) < \epsilon(a) \tag{118}$$

for some dynamical triangulation $M = |T_a| \in \mathcal{DT}_n(a,b,N)$.

In other words, the set of dynamical triangulations $\mathcal{DT}_n(a,b,N)$ uniformly approximates riemannian manifolds of bounded geometry.

Note that we restrict the dimension n to the values $n = 2$, 3, 4, because in such a case, the Dehn-Sommerville relations allow us to control the f-vector of the triangulation in terms of the number of top-dimensional simplices $N_n(T_a) = N$ and of the average incidence number $b(n,n-2)$. By giving the appropriate combinatorial data, the extension to $n \geq 5$ does not present particular difficulties. In order to prove lemma 3, we need to recall some standard results related to the compactness properties of $\mathcal{R}(n,r,D,V)$.

A sequence of compact metric spaces $\{M_i\}$ converges to a compact metric space M provided that $d_G(M_i, M) \to 0$ as $i \to \infty$. Thus we have to prove that there is an $\epsilon = \epsilon(a) > 0$ such that the Gromov- Hausdorff distance between the generic riemannian manifold $M \in \mathcal{R}(n,r,D,V)$ and a suitably chosen dynamical triangulation T_a, is lesser than or equal to

ϵ. Note that the ϵ involved is a function of the cut-off a entering the definition of T_a. The proof of the above lemma is a rather immediate consequence of the following well-known properties [10]:

Theorem 4 *Let $M \in \mathcal{R}(n, r, D, V)$ be the generic riemannian manifold of bounded geometry, and let $d_M(\cdot, p)$ denote the corresponding distance function from a chosen point $p \in M$. Then, for any given $\epsilon > 0$, it is always possible to find an ordered set of points $\{p_1, \ldots, p_m\}$ in M, so that: (i) the open metric balls, (the geodesic balls), $B_M(p_i, \epsilon) = \{x \in M | d(x, p_i) < \epsilon\}$, $i = 1, \ldots, m$, cover M; (ii) the open balls $B_M(p_i, \frac{\epsilon}{2})$, $i = 1, \ldots, m$, are disjoint, i.e., $\{p_1, \ldots, p_m\}$ is a minimal ϵ-net in M.*

We recall that the *filling function* of a minimal ϵ-net is the number m of points $\{p_1, \ldots, p_m\}$, while the first order *intersection pattern* of a minimal ϵ-net in M is the set of pairs $\{(i, j) | i, j = 1, \ldots, m; B(p_i, \epsilon) \cap B(p_j, \epsilon) \neq \emptyset\}$.

It is important to remark that on $\mathcal{R}(n, r, D, V)$ neither the filling function nor the intersection pattern can be arbitrary. The filling function is always bounded above for each given ϵ, and the best filling, with geodesic balls of radius ϵ, of a riemannian manifold of diameter $diam(M)$, and Ricci curvature $Ric(M) \geq (n-1)H$, is controlled by the corresponding filling of the geodesic ball of radius $diam(M)$ on the space form of constant curvature given by H, the bound being of the form [10] $N_\epsilon^{(0)} \leq N(n, H(diam(M))^2, (diam(M))/\epsilon)$.

The multiplicity of the first intersection pattern is similarly controlled through the geometry of the manifold to the effect that its average degree, (*i.e.*, the average number of mutually intersecting balls), is bounded above by a constant as the radius of the balls defining the covering tend to zero, (*i.e.*, as $\epsilon \to 0$). Such constant is independent from ϵ and can be estimated [10] in terms of the parameters n, and $H(diam(M))^2$.

Definition 8 *Two metric spaces (e.g., two riemannian manifolds) M_1 and M_2 in $\mathcal{R}(n, r, D, V)$ are called ϵ-equivalent (or ϵ-rough isometric) if and only if they can be equipped* *with minimal ϵ-nets $\{p_1, \ldots, p_m\}_1$ and $\{p_1, \ldots, p_s\}_2$, respectively, such that $s = m$ and with the same intersection pattern.*

We can introduce a natural minimal geodesic ball covering associated with a dynamical triangulation T_a according to the following

Lemma 4 *Let $(M = |T_a|, T_a)$ denote a dynamically triangulated manifold with fixed edge legth a. With each vertex $p_i \doteq \sigma_i^0$ belonging to the triangulation, we associate the largest open metric ball contained in the open star of p_i. Then the metric ball covering of $M = |T_a|$ generated by such balls $\{B_i\}$ is a minimal geodesic ball covering. It defines the geodesic ball covering associated with the dynamically triangulated manifold (M, T_a).*

It is immediate to see that the set of balls considered defines indeed a minimal geodesic ball covering. The open balls obtained from $\{B_i\}$ by halfing their radius are disjoint being contained in the open stars of $\{p_i\}$ in the baricentric subdivision of the triangulation. The balls with doubled radius cover (M, T), since they are the largest open balls contained in the stars of the vertices $\{p_i\}$ of T_a.

We are now in a position for proving lemma 3.

Proof. Given a riemannian manifold $M \in \mathcal{R}(n, r, D, V)$, endowed with with a minimal ϵ-net $\{q_i\}_{i=1,\ldots,m}$ we can associated with it the dynamical triangulations T_a, with $a = a(\epsilon)$, having the same combinatorial intersection pattern $\{p_i = \sigma_i^0\}_{i=1,\ldots,m}$ and such that $|d(q_i, q_j) - d(p_i, p_j)| < \epsilon$ for all $i, j = 1, \ldots, m$. On the disjoint union $M \coprod T_a$ we can define the metric $d(x, y) = \min\{d(q, q_i) + \epsilon + d(p_i, p) \colon i = 1, \ldots, m\}$ for $q \in M$, $p \in T_a$. More explicitly, this metric is constructed by declaring $d(p_i, q_j) = \epsilon$, and by setting, for $p \in M$ and $q \in T_a$, the distance $d(p, q)$ equal to the infimum of the sums $d(p, x_1) + d(x_1, x_2) + \ldots + d(x_l, q)$ where each consecutive pair (x_i, x_{i+1}), (or (p, x_1), or (x_l, q)) either has both elements in one of M or T_a or has $x_i = p_m$, $x_{i+1} = q_m$. The metric so defined is such that $M \subset \{x \in M \coprod T_a \colon d(x, T_a) \leq \epsilon\}$,

$T_a \subset \{x \in M \coprod T_a : d(x, M) \leq \epsilon\}$, and thus $d_G(M, T_a) \leq 2\epsilon$, which implies the stated approximation result. $\square$

Since a given intersection pattern may be common to distinct dynamical triangulations, there can be many distinct dynamical triangulations that are 2ϵ-rough isometric to a given riemannian manifold. This is not surprising, the equivalence between riemannian manifolds associated with minimal ϵ-nets is very crude, and there are distinct riemannian manifolds sharing the same ϵ- net, (*i.e.*, which are 2ϵ-rough isometric). The point is that all such manifolds as well as all such dynamical triangulations have a Gromov-Hausdorff distance between each other which is smaller than or equal to 2ϵ, and as $\epsilon \to 0$, *viz.*, as the dynamical triangulations become finer and finer such dynamical triangulations converge to a well defined metric space in $\mathcal{R}(n, r, D, V)$.

It is worth noticing that it is possible to obtain a converse of this result, indeed [10], any polyhedral compact subset of a Euclidean space is the Gromov-Hausdorff limit of a sequence of closed Riemannian manifolds.

The density result just obtained provides the mathematical rationale for using dynamical triangulations in simplicial quantum gravity. The original idea was related to the intuition that fixed-edge length triangulations, T_a, (of fixed topology $M = |T_a|$) provide a *grid* of reference manifolds laid upon the ∞-dimensional space of riemannian structures allowed on such M. The compactness results associated with the use of Gromov-Hausdorff convergence support this picture, and the uniform approximation result of lemma 3 makes precise this intuition by proving that dynamically triangulated manifolds do provide a reference grid as long as we look at riemannian manifolds and dynamical triangulations of bounded geometry. At first sight it may appear surprising that no strict topology control is required on such triangulations, but this is actually a bonus coming automatically along with the requirement of bounded geometry, (*i.e.*, fixed N_n (Volume) and fixed average incidence $b(n, n-2)$ (curvature bounds)). As we shall see in the next section these natural bounds severely control topology, by avoiding such patologies as infinite genus surfaces (in dimension $n = 2$), or wild homotopy types, etc. One may question the naturality of such bounds in the sense that dynamical triangulations is an approach to quantizing gravity in the sense of a statistical field theory, and in seeking the associated critical phenomena we eventually need to go to the infinite-volume limit. In this limit the bounds on N_n and $b(n, n-2)$ are removed and one consequently loses control on everything. The point we wish to make is exactly that the above results tell us *how* to carry out such sort of limits. One must rigorously establish estimates for dynamical triangulations, the associated partition functions and m- points correlation functions, in a given space of bounded geometry $\mathcal{D}T_n(a, b, N) \subset \mathcal{R}(n, r, D, V)$, and then consider a nested sequence $\mathcal{D}T_n(a_1, b_1, N_1) \subset \mathcal{D}T_n(a_2, b_2, N_2) \subset \ldots \subset \mathcal{D}T_n(a_i, b_i, N_i) \ldots$ in which the convergence properties of the partition function and the associated infinite-volume limit can be safely discussed. As we shall see, the actual characterization of the infinite volume limit does not call for such a sophisticated machinery. However the discussion of a possible continuum limit may do require the introduction of a suitable projective limit procedure for $\mathcal{D}T_n(a, b, N)$.

C. Topological finiteness theorems for Dynamical Triangulations of bounded geometry

Since dynamical triangulations, (for given N and $b(n, n-2)$), are in a Gromov space of bounded geometries, $\mathcal{R}(n, r, D, V)$, we can control to a considerable extent their topology in terms of the parameters n, r, D, and V, (namely in terms of the number of top-dimensional simplices N and $b(n, n-2)$, in the range of dimension considered). We start with a result expressing finiteness of homotopy types of dynamically triangulated manifolds of bounded geometry [10].

Theorem 5 *For any dimension $2 \leq n \leq 4$, the number of different homotopy-types of dynamically triangulated manifolds realized in $\mathcal{DT}_n(a, b, N) \subset \mathcal{R}(n, r, D, V)$ is finite and is a function of n, $V^{-1}D^n$, and rD^2.*

(Two polyhedra $M_1 = |T_a(1)|$ and $M_2 = |T_a(2)|$ are said to have the same homotopy type if there exists a continuous map ϕ of M_1 into M_2 and f of M_2 into M_1, such that both $f \cdot \phi$ and $\phi \cdot f$ are homotopic to the respective identity mappings, I_{M_1} and I_{M_2}. Obviously, two homeomorphic polyhedra are of the same homotopy type, but the converse is not true).

This homotopy finiteness property of dynamically triangulated manifolds is an obvious adaptation of a more general result proved in a truly remarkable series of papers by P. Petersen, K. Grove, and J. Y. Wu [10]. Actually, for dynamical triangulations, the above theorem can be sharpened so as to provide (rather weak) conditions under which two distinct dynamically triangulated manifolds share a common homotopy type, and we can estimate the number of distinct homotopy types. We have the following

Theorem 6 *For any dimension $2 \leq n \leq 4$, and for a given value a of the cut-off edge-length, the number of different homotopy-types of dynamically triangulated manifolds realized in $\mathcal{DT}_n(a,b,N) \subset \mathcal{R}(n,r,D,V)$ is bounded above by*

$$\sharp Homotopy[\mathcal{DT}_n(a,b,N)] \leq [C(n)^{-1} \cdot N vol(\sigma^n) \cdot a^{-n}]^4 \tag{119}$$

where $C(n)$ is some universal constant depending only on the dimension n, and $vol(\sigma^n)$ is the euclidean volume of the simplex σ^n (of edge-length a).

Proof. Recall [13], [10] that a continuous function $\psi\colon [0,\alpha) \to \mathbf{R}^+$, $\alpha > 0$, with $\psi(0) = 0$, and $\psi(\epsilon) \geq \epsilon$, for all $\epsilon \in [0,\alpha)$, is a local geometric contractibility function for a (finite dimensional) compact metric space M if, for each $x \in M$ and $\epsilon \in (0,\alpha)$, the open ball $B(x,\epsilon)$, of radius ϵ centered at $x \in M$, is contractible in the larger ball $B(x,\psi(\epsilon))$, (which says that in M small metric balls are contractible relative to bigger balls). On a dynamical triangulation $T_a \in \mathcal{DT}_n(a,b,N)$, consider the smallest open ball containing the star of the generic vertex. This is a ball or radius a which is contractible within itself. Thus, for any such T_a we may consider the local geometric contractibility function given by

$$\psi(\epsilon) = \epsilon \tag{120}$$

on $[0, a]$. We can now exploit in our setting a result of P.Petersen [10] to the effect that any two dynamical triangulations T_a^1 and T_a^2 in $\mathcal{DT}_n(a,b,N)$ with Gromov-Hausdorff distance $d_G(T_a^1, T_a^2) < \epsilon^*$ are homotopy equivalent if $\epsilon^* \simeq a/(32n^2)$, (this bound is not optimal, and similarly to what happens for riemannian manifolds with criticality radius bounded below, one should get a sharper bound of the form $a/[25(n+1)]$ [10]). Since $|T_a|$ can be covered by the open metric balls of radius a centered on the vertices of T_a, it also admits a refinenement of this covering generated by open metric balls of radius $\epsilon^*/2$. If $N(\epsilon^*/2)$ denotes the number of such balls, then an argument due to Yamaguchi [10] can be used to show that $\mathcal{DT}_n(a,b,N)$ contains less than $[N(\epsilon^*/2)]^4$ distinct homotopy types. In order to estimate $N(\epsilon^*/2)$, notice that the volume of a metric ball, $B(\epsilon^*/2)$, of radius $\epsilon^*/2$ in $|T_a|$ is bounded below by

$$vol[B(\epsilon^*/2)] \geq \hat{q}(\sigma^0) \cdot vol(\sigma^n)_{\epsilon^*/2} = (n+1) \cdot c(n)(\epsilon^*/2)^n \tag{121}$$

where $\hat{q}(\sigma^0) = (n+1)$ is the minimum number of n- dimensional simplices sharing a vertex, and $vol(\sigma^n)_{\epsilon^*/2} = c(n)(\epsilon^*/2)^n$ is the euclidean volume of a standard euclidean simplex σ^n with edge-length $\epsilon^*/2$, (this characterizes the constant $c(n)$ as $c(n) \doteq \sqrt{n+1}/(n!\sqrt{2^n})$). Since the volume of $|T_a|$ is given by $N \cdot vol(\sigma^n)$, we get

$$N(\epsilon^*/2) \leq \frac{N \cdot vol(\sigma^n)}{(n+1) \cdot c(n)(\epsilon^*/2)^n} \tag{122}$$

Since $\epsilon^* \simeq (a/32n^2)$, by introducing (122) in the Yamaguchi estimate we get the stated result with $C(n) \doteq \frac{c(n)(n+1)}{(64n^2)^n}$. $\Box$

A similar result, obtained by adapting to dynamical triangulations results obtained by Yamaguchi, Grove, Petersen and Wu [10], allows also to control the Betti numbers, (with generic field coeffiecients F), of any $M = |T_a| \in \mathcal{DT}_n(a,b,N)$. Explicitly we have

Theorem 7 *Let $b_i(M,F)$ denote the i-th Betti number with field coefficients F, then*

$$\sum_{i=1}^{n} b_i(M,F) \leq (n+1)^{(n+1)} \left[C(n)^{-1} \cdot N vol(\sigma^n) \cdot a^{-n} \right]^n \tag{123}$$

Proof. The proof is an obvious rewriting of a result of P.Petersen, (see the corollary at p. 393 of P.Petersen's paper [10]).

The above results clearly show that when going to the infinite-volume limit, $\lim_{N\to\infty,a\to 0} < N > vol(\sigma^n) \simeq const.$, (where $< N >$ denotes the statistical average of N with respect to a canonical ensemble of dynamical triangulations), more and more distinct topological types of manifolds come into play. However the resulting *topological complexity* is not too wild since the number of distinct homotopy types (and/or the Betti numbers) grows polynomially with the inverse cut-off a^{-1}.

By considering separately the dimension $n = 2$, $n = 3$, $n = 4$, we can obtain a more specific topological finiteness theorem. This result follows again from the local geometric contractibility which by construction characterizes dynamically triangulated manifolds. Since sufficiently small distance balls on a dynamically triangulated manifold of bounded geometry are always contractible, we get, by specializing a theorem of Grove, Petersen, Wu [10],

Theorem 8 *The set of dynamically triangulated manifolds in $\mathcal{DT}_n(a, b, N) \subset \mathcal{R}(n, r, D, V)$ contains*
(i) finitely many simple homotopy types (all n),
(ii) finitely many homeomorphism types if $n = 4$,
(iii) finitely many diffeomorphism types if $n = 2$ (or $n \geq 5$, in case we remove the constraint $2 \leq n \leq 4$).

Note that quantitative estimates associated with these finiteness results have the same structure of the ones associated with theorems 6 and 7, and can be easily worked out along the same lines. Note also that finiteness of the homeomorphysm types cannot be proved in dimension $n = 3$ as long as the Poincaré conjecture is not proved. If there were a fake three-sphere then one could prove [13] that a statement such as *(ii)* above is false for $n = 3$. Finally, the statement on finiteness of *simple homotopy* types, (which actually holds in any dimension), is particularly important for the applications in quantum gravity we discuss in the sequel. Roughly speaking the notion of simple homotopy is a refinement of the notion of homotopy equivalence, relating (in our case) triangulated manifolds (more in general CW complexes) which can be obtained one from the other by a sequence of expansion or collapses of simplices, (for details see [24], [16]). Roughly speaking, it may be thought of as an intermediate step between homotopy equivalence and homeomorphism.

Topology and enumeration of triangulations. The fact that for a dynamical triangulations with a given number, N, of n-dimensional simplices σ^n, and a given average bone incidence $b(n, n-2)$ we have a good a priori control of topology is the basic reason that allows us to enumerate distinct dynamical triangulations. An important step in this enumeration is to understand how we can reconstruct the triangulation by the knowledge of easy accessible geometrical data. At this stage it is worth recalling that the Levi-Civita connection for a dynamical triangulation is uniquely determined by the incidence matrix $I(\sigma_i^n, \sigma_j^n)$ of the triangulation itself, thus in order to enumerate distinct dynamical triangulations *we can equivalently characterize the set of distinct Levi-Civita connections* $\alpha_g = \{\Gamma(\sigma_i^n, \sigma_{i+1}^n)\}_{i=1,...}$. The problem we have to face is thus reduced to a rather standard question, familiar in gauge theories, where one needs to reconstruct the holonomy of a gauge-connection from data coming from the Wilson loop functionals. Here, we need to reconstruct all discretized Levi-Civita connection from Wilson-loop datas which reduce to the possible set of curvature assignements to the bones, and by taking due care of the non-trivial topological information coming from the moduli of locally homogeneous manifolds, (*e.g.*, constant curvature metrics), which need to be described in terms of deformations of dynamical triangulations. In order to address this problem we exploit the holonomy representation associated with the Levi-Civita connection α_g.

VI. MODULI SPACES FOR DYNAMICALLY TRIANGULATED MANIFOLDS

Let $(T_a, M = |T_a|)$ be a dynamically triangulated manifold. If we denote by $\Omega_\sigma(M)$ the family of all simplicial loops in M, starting at a given simplex σ_0^n, it easily follows that $\Omega_\sigma(M)$, modulo σ_0^n-based homotopic equivalence, is isomorphic to the fundamental group of M, $\pi_1(\sigma_0^n, M)$, based at the given simplex σ_0^n. Moreover, by factoring out the effect of loops homotopic to the trivial σ_0^n-based loop, the mapping

$$R: \omega \in \Omega_\sigma(M) \mapsto R_\omega \tag{124}$$

yields a representation of the fundamental group $\pi_1(\sigma_0^n, M)$. The following definition specilizes to the PL setting some of the well known properties of the holonomy of riemannian manifolds.

Definition 9 *The holonomy group of the (fixed edge length) PL manifold (M, T_a) at the simplex σ_0^n, $Hol(\sigma_0^n)$ is the subgroup of the orthogonal group $O(n)$ generated by the set of all parallel transporters R_w along loops of the simplicial manifold T based at σ_0^n. The subgroup obtained by the loops which are homotopic to the identity, is the restricted holonomy group of (M, T) at σ_0^n, $Hol^0(\sigma_0^n)$.*

If we change the base simplex σ_0^n to the simplex $\bar{\sigma_0^n}$ and fix some path ρ from σ_0^n to $\bar{\sigma_0^n}$, then $Hol(\bar{\sigma_0^n}) = A_\rho Hol(\sigma_0^n) A_\rho^{-1}$, (and similarly for the restricted holonomy group). As a consequence the holonomy groups at the various simplices of T_a are all isomorphic. And we can speak of the holonomy and restricted holonomy group of the (fixed edge length) PL manifold $M = |T_a|$: $Hol(M)$, and $Hol^0(M)$, respectively. Moreover, if ω_0 and ω_1 are simplicial loops based at σ_0^n which represent the same element, $[\omega]$, of $\pi_1(M, \sigma_0^n)$, then R_{ω_0} and R_{ω_1} belong to the same arc component of $Hol(M)$. From this observation and from the definition of restricted holonomy, it naturally follows that there exists a canonical homomorphism

$$\pi_1(M) \to \frac{Hol(M)}{Hol^0(M)} \tag{125}$$

which defines the *Holonomy Representation* of $\pi_1(M)$.

Holonomy representations of this sort are of relevance to dynamical triangulations. This connection is very subtle and it is related to the well-known fact that generally we cannot triangulate locally homogeneous spaces with equilateral flat simplices. For instance, in order to model a space of constant curvature with a dynamical triangulation we have to *slightly* deform the triangulation (following a suggestion anticipated many years ago by Römer and Zähringer [15]). This deformation procedure is apparently trivial in the sense that a small alteration of the chosen edge-length allows one to fit the triangulation on a constant curvature background. However subtle problems arise if there are non- trivial deformations (*i.e.*, moduli) of the underlying constant curvature metric, (*e.g*, distinct flat tori). Formally, these non-trivial deformations are described by the cohomology of the manifold with coefficients in the sheaf of Killing vector fields of the manifold, namely with local coefficients in the (adjoint representation of the) Lie algebra $\mathfrak{g}$ of the group G of isometries considered. This cohomology is isomorphic to the cohomology of the fundamental group $\pi_1(M)$ with value in the holonomy representation of $\pi_1(M)$ in $\mathfrak{g}$, and this is one of the basic mechanisms that calls into play topology in dynamical triangulations. Such issues can be very effectively dealt within the framework of deformation theory in algebraic geometry, (as hinted in the above formal sheaf-theoretic remarks), however, we think appropriate here a more pedagogical approach, also because this issue has not really been considered previously in dynamical triangulation theory.

A. Römer-Zähringer deformations of dynamically triangulated manifolds

As a warm up after the above introductory remarks, let us consider in particular the *flat* Levi-Civita connections. Since the connection is flat, the parallel transport along a closed simplicial loop ω based at σ_0^n depends only on the homotopy class of ω. In this case the parallel transport, through its associated holonomy, gives rise to a representation of the fundamental group

$$\theta: \pi_1(M, \sigma_0^n) \to Hol(M) \subset O(n) \tag{126}$$

the image of which is the holonomy group of the flat Levi-Civita connection considered. Such θ is a homomorphism of $\pi_1(M, \sigma_0^n)$ onto $Hol(M)$, and since $\pi_1(M)$ is countable, $Hol(M)$ is totally disconnected. In this case one speaks of θ as the *holonomy homomorphism.*

It is a well-known fact that connected compact flat n-dimensional riemannian manifolds are covered by a flat n-torus $p: \mathbb{T}^n \to M$, and that the associated group of deck transformations is isomorphic to $Hol(M)$, *viz.*, $M = \mathbb{T}^n / Hol(M)$. Quite surprisingly, any finite group G can be realized as the holonomy group of some flat compact connected riemannian manifold, (this is the Auslander-Kuranishi theorem, see *e.g.* [52]). Accordingly, flat riemannian manifolds are completely determined by homomorphisms of their fundamental group $\pi_1(M)$ onto the (finite) group G, up to conjugation. The kind of difficulties encountered in dynamically triangulating flat riemannian manifolds are particularly clear when discussing the geometry of their covering space.

Flat tori. It is well known that at least in dimension $n \geq 3$, it is not possible to build up a flat space (*e.g.*, a flat three-torus) by gluing flat equilateral simplices $\{\sigma^n\}$, this simply follows from the observation that the angle $\cos^{-1}\frac{1}{n}$, $n \geq 3$, is not an integer fraction of 2π. Thus, for $n \geq 3$, dynamical triangulations cannot directly describe flat tori. Superficially, this does not seem to be a serious problem, since we can deform a few simplices in such a way to match with the 2π- flatness constraint, (see next paragraph for the details). However, this is hardly a sound prescription since, as follows from their definition, there are distinct flat tori. More formally, let us fix a basis $(\vec{e}_1, \ldots, \vec{e}_n)$ of $\mathbb{R}^n$. Then with each n-ple of integers $(a^1, \ldots, a^n) \in \mathbb{Z}^n$ we can associate a transformation, A, of $\mathbb{R}^n$ given by $A(\vec{x}) = \vec{x} + \sum_{j=1}^n a^j \vec{e}_j$. This results in a transformation group isomorphic to $\mathbb{Z}^n$ acting properly discontinuously on $\mathbb{R}^n$. Upon quotienting $\mathbb{R}^n$ by this action we get a manifold diffeomorphic to the n-dimensional torus $\mathbb{T}^n$. Since $\mathbb{R}^n$ is thought of as endowed with its standard flat metric, the resulting $\mathbb{T}^n$

is likewise flat. However, by changing base to $\mathbb{R}^n$ we change the group of transformation associated with A, and the resulting flat torus is not necessarily isometric to the original one. Thus, the naive deformation procedure suggested above is ambiguous as long as it does not specify which particular flat torus one is modelling. In particular, in dimension $n = 2$, let $\vec{a}_1$ and $\vec{a}_2$ be the vectors generating the lattice, and assume that $\vec{a}_1$ is the first basis vector of $\mathbb{R}^2$, $\vec{a}_1 = \vec{e}_1$, and that $\vec{a}_2 = (x, y)$ lies in the first quadrant of $\mathbb{R}^2$. Then the resulting flat tori are parametrized by the points in the plane region

$$\{(x, y)|x^2 + y^2 \geq 1, 0 \leq x \leq \frac{1}{2}, y > 0\} \tag{127}$$

to the effect that flat tori generated by lattices corresponding to two distinct points of this region are distinct. Thus, even in dimension $n = 2$, where the flatness 2π-constraint can be met, it follows that the lattice generated by equilateral triangles, corresponding to a dynamically triangulated torus, can only discretize (up to dilation) the flat torus coming from exagonal lattices, ($x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$): we cannot dynamically triangulate with equilateral triangles all remaining flat tori, (*e.g.*, all the rectangular tori $\vec{a}_2 = (0, y)$), (obviously, this obstruction persists only if we want the triangulation and the lattice to be consistent, otherwise we can dynamically triangulate any 2- torus; in higher dimension we completely loose this freedom).

The difficulties in associating dynamical triangulations with manifolds of large symmetries is not restricted to the case of flat tori just described. Similar problems are encountered in associating dynamically triangulations to any riemannian manifold endowed with a *geometric structure*, namely manifolds locally modelled on homogeneous spaces, for instance manifolds of constant curvature, basically because such manifolds do not generally admit regular tilings by euclidean equilateral simplices.

Deforming a triangulation. Even if there is no regular way to generate locally homogeneous PL manifolds by gluing equilateral simplices, (*e.g.*, flat tori), we can always assume that in trying to model a locally homogeneous riemannian manifold M with a dynamical triangulation T_a, this can be done in such a way that the Gromov- Hausdorff distance between M and T_a is minimal. As a matter of fact, we can, at least in line of principle, construct an approximating sequence $\{T_\mu\}_{\mu=1,2,\ldots}$ of triangulated models of M, interpolating between M and T_a, by using triangulations of M with simplices which are nearly equilateral, (the essence of this remark was very clearly suggested by Römer and Zähringer [15]). With each triangulated manifold T_μ of this sequence we can associate the edge-length fluctuation functional

$$s(T_\mu) = \sum_{\sigma^1} \frac{[|\vec{E}_{ij}(\sigma^1)| - a(T_\mu)]^2}{N_1(T_\mu)} \tag{128}$$

where $a(T_\mu) \equiv \frac{\sum_{\sigma^1} |\vec{E}_{ij}(\sigma^1)|}{N_1}$ is the average edge-length in T_μ. The functional $s(T_\mu)$ has reasonable continuity properties in the Gromov-Hausdorff topology, and since the Gromov space of bounded geometries is compact [9], we can always choose the sequence $\{T_\mu\}_{\mu=1,2,\ldots}$ in such a way that it minimizes $s(T_\mu)$.

However, if there are non-trivial (infinitesimal) deformations of the riemannian structure M, preserving the local homogeneous geometry, (*e.g.*, inequivalent flat tori, or moduli of Riemann surfaces), the Römer-Zähringer procedure may associate distinct moduli to the same approximating dynamical triangulation. Thus, the question remains of how to keep track of which locally homogeneous manifold is Gromov-Hausdorff approximated by the dynamical triangulation considered. This can be done by exploiting the Gromov- Hausdorff continuity of the homotopy type of metric spaces of bounded geometry and by generalizing the previous remark relating the holonomy of flat connections to the representations of the fundamental group, (see eqn. (126)).

B. Dynamical triangulations and locally homogeneous geometries

In order to formalize the above remarks, we need some preparatory material on riemannian metrics on homogeneous spaces and the way a dynamical triangulation can approximate such particular riemannian structures. From the mathematical point of view, most of this material is standard and as a guide, the reader may found profitable to consult the papers by W.Thurston [16] and by W.M.Goldman [26] where the classical geometry of deformations of locally homogeneous riemannian manifolds is beautifully discussed. As far as applications to dynamical triangulation are concerned, the point of view developed here and in the following paragraphs is quite new and full of deep implications for unravelling the subtle connection between topology and simplicial quantum gravity.

Locally homogeneous geometries. Let G be a Lie group, $H \subset G$ a closed (compact) subgroup, we denote by $X = G/H \doteq \{gH : g \in G\}$ the set of cosets. To each $g \in G$ we can associate the left translation $\tau_g : G/H \to G/H$ given by $\tau_g(g'H) = (gg')H$ which makes G act as a transitive Lie transformation group on the homogeneous space X. Such a space X is riemannian homogeneous if G/H is endowed with a riemannian metric relative to which τ_g is an isometry for every $g \in G$. A *locally homogeneous* riemannian manifold M is a riemannian manifold locally modelled on the geometry of a homogeneous space (X, G). Constant curvature metrics are a typical example. They are locally isometric either to euclidean space, the sphere, or hyperbolic space, depending on wether the curvature is zero, positive, or negative respectively. More generally, locally homogeneous riemannian manifolds include all geometric structures used to uniformize surfaces or to discuss Thurston's geometrization program for 3-dimensional manifolds [25].

To be more precise, let X be a real analytic manifold and let G be a finite dimensional Lie group acting analytically and faithfully (*i.e.* $gx = x$ for all $x \in X$ implies $g = id$) on X. We say that a manifold M has a (G, X) structure if we have an open covering $\{U_i\}$ of M, and coordinate charts $\phi_i : U_i \to X$ with transition functions

$$\gamma_{ij} \doteq \phi_i \cdot \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \tag{129}$$

where each γ_{ij} agrees locally with an element of the group G.

Let M be a (X, G) manifold and $\hat{M}$ its universal covering. If we fix a base point x_0 in

M, $\hat{M}$ can be identified with the quotient of the set of pairs (x, β), where $x \in M$ and β is a path from x_0 to x, with respect to the equivalence relation

$$(x, \beta) \simeq (x^*, \beta^*) \tag{130}$$

if and only if $x = x^*$ and $\beta^* \beta^{-1} \simeq 1$ in $\pi_1(M, x_0)$. The projection $p: \hat{M} \to M$ being defined by $p[(x, \beta)] = x$. We wish to recall the characterization of the *Developing map* $D: \hat{M} \to X$ associated with a locally homogeneous manifold M. The basic idea is that given a (X, G) structure on a manifold M it is natural to try to make the charts $\{(U_i, \phi_i)\}$ match up exactly and compatibly with the topology of M so as to tesselate M (or portion of M) for suitably chosen $\{(U_i, \phi_i)\}$, (in our case the $\{(U_i, \phi_i)\}$ will be associated with equilateral simplices or suitable deformations of them).

The developing map is defined through *analytic continuation* along a path α. Fix a basepoint $x_0 \in M$ and a chart (U_0, ϕ_0) the domain of which contains the basepoint x_0. Given a path $\alpha: [0, 1] \to M$ starting at the basepoint subdivide it at points $x_0 = \alpha(t_0)$, $x_1 = \alpha(t_1), \ldots, x_n = \alpha(t_n)$, ($t_0 = 0$ and $t_n = 1$), in such a way that each subpath is contained in the domain of a single coordinate chart (U_i, ϕ_i). Starting with the given ϕ_0 we adjust ϕ_1 at x_1 in such a way that it agrees with the value of ϕ_0 at x_q. Since $\gamma_{ij} \doteq \phi_i \cdot \phi_j^{-1}$ is a locally constant map into the group G, the adjustement of ϕ_1 is easily obtained by replacing $\phi_1(x_1)$ with $\psi_1 \doteq \gamma_{01} \phi_1(x_1)$. By proceeding in this way, the generic k-th adjusted chart is obtained by replacing $\phi_k(x_k)$ with

$$\psi_k \doteq \gamma_{01}(x_1) \gamma_{12}(x_2) \ldots \gamma_{k-1,k}(x_k) \phi_k(x_k). \tag{131}$$

The adjusted charts $\psi_1, \psi_2, \ldots, \psi_n$ form the analytic continuation

$$\phi_0^{[\alpha]} \doteq \psi_n \tag{132}$$

of the chosen ϕ_0 along the path α. It can be verified that this analytic continuation only depends on the homotopy class $[\alpha]$ of the path, in particular it does not depend from the open sets $\{U_i\}$, but *only from the choice of the first open set* (U_0, ϕ_0) containing the basepoint

x_0. We get in this way a well-defined application $D: \hat{M} \to X$. Notice that different choices of (U_0, ϕ_0) characterize applications D differing by a G- translation in X. Formally we have [16]

Definition 10 *For a given choice of the initial chart (U_0, ϕ_0) and of the basepoint $x_0 \in U_0$, the developing map of a locally homogeneous manifold M is the map $D: \hat{M} \to X$ defined by $D \doteq \phi_0^{[\alpha]} \cdot p$ in a neighborhood of $(x, \alpha) \in \hat{M}$.*

Note that in the definition of developing map it is not strictly necessary that the action of G on X is transitive.

Holonomy homomorphism. We can now recall the characterization of the *Holonomy* of a locally homogeneous manifold M, (the holonomy representation (126) is a particular case of such a construction). Let σ be an element of the fundamental group of M, $\pi_1(M, x_0)$. To such a loop there is associated a covering transformation $T_\sigma: \hat{M} \to \hat{M}$ in the universal cover defined by $T_\sigma[(x, \tau)] \doteq [(x, \sigma\tau)]$. Analytic continuation along the loop σ provides $\phi_0^{[\sigma]} = \psi_n$ which is well-defined at the basepoint x_0, (since $x_n = x_0$), and depends only on the homotopy class of the loop σ. We call *Holonomy* of M the map $H: \pi_1(M) \to G$ defined by $\sigma \mapsto h(\sigma)$ where h_σ is that element of G such that

$$\phi_0^{[\sigma]} = h_\sigma \phi_0 \tag{133}$$

Since $D \cdot T_\sigma = h_\sigma \cdot D$, the holonomy map H is a group homomorphism from $\pi_1(M)$ into the structure group of the geometry G, *the holonomy homomorphism.* The holonomy is determined up to conjugacy in G according to the fact that the developing map is determined only up to left composition with elements of G.

Note that if the developing map is a covering map, M is a complete (G, X) manifold and its holonomy group characterizes M, in the sense that M can be reconstructed as the quotient space $X/Hol(M)$, (see Thurston's notes [16] for more details).

More generally [26], the developing map $D: \hat{M} \to X$ can be used to pull back the (X, G) structure from X to $\hat{M}$ which thus inherits a natural structure of locally homogeneous riemannian manifold. Moreover, the holonomy homomorphism $H: \pi_1(M) \to G$ determines the action of $\pi_1(M)$ on the universal cover $\hat{M}$ by (X, G) automorphism. This implies that the pair (D, H) associated with the developing map D and a holonomy representation H in the structure group of the geometry, the so- called *Development pair*, is a useful globalization of a locally homogeneous structure (X, G) defined on a manifold M by local coordinate charts $\{U_i, \phi_i\}$: the pair (D, H) completely determines the locally homogeneous geometry on M.

Moduli of locally homogeneous geometries. For our purposes we need to consider not just a particular locally homogeneous structure (M, g^*) on a compact manifold M, but rather a *space* of such structures. To this end [26], mark a basepoint $x_0 \in M$ and let $\pi_1(M, x_0)$ and $\hat{M} \to M$ be the corresponding fundamental group and universal covering space. We consider the set of homomorphisms of $\pi_1(M, x_0)$ into the structure group of the geometry G, $Hom(\pi_1, G)$ endowed with the pointwise convergence topology, *i.e.*, a sequence $\theta_n \in Hom(\pi_1(M), G)$ is said to converge to a homomorphism θ if $\lim_{n \to \infty} \theta_n(\omega) = \theta(\omega)$ for each $\omega \in \pi_1(M, x_0)$. Let us consider, as before, the coordinate map $\phi_0: U_0 \subset M \to X$, in particular its value at $x_0 \in U_0$. This defines the *germ* at x_0 of the locally homogeneous geometry of M.

The space $\mathcal{I}_{(X,G)}(M)$ of *isotopy classes* of based locally homogeneous structures on (M, x_0) having the same germ $\phi_0: U_0 \subset M \to X$ at x_0 can be defined [26] as the set

$$\mathcal{I}_{(X,G)}(M) = \{(D, \theta) | \theta \in Hom(\pi_1(M), G)\} \tag{134}$$

and where $D: \hat{M} \to X$ is a θ-equivariant nonsingular smooth map, *viz.*, a developing map. If we topologize $\mathcal{I}_{(X,G)}(M)$ using the C^∞ topology on the developing map D, then the map

$$\hat{hol}: \mathcal{I}_{(X,G)}(M) \to Hom(\pi_1(M), G) \tag{135}$$

which associates to the developing pair (D, θ) the corresponding holonomy representation

θ is continuous. Notice that if $Diff_0(M, x_0)$ is the identity component in the group of all diffeomorphisms $M \to M$ fixing x_0, then [26] $Diff_0(M, x_0)$ acts properly and freely on $\mathcal{I}_{(X,G)}(M)$ and the map $\hat{hol}$ is invariant under this action. It is also easily verified that $\hat{hol}$ is G- equivariant also respect to the natural G-actions of G on the developing map and on the representation space $Hom(\pi_1(M), G)$. In the quoted Goldman's paper [26], the intersted reader can find the statement and the proof of an equivariance slice theorem which allows to parametrize the space $\mathcal{I}_{(X,G)}(M)$ of based geometric structures on M in terms of a representation $\theta \in Hom(\pi_1(M), M)$ and of a diffeomorphism $h \in Diff_0(M, x_0)$), (see the deformation theorem at p. 178 in [26]). We state this result in a somehow less technical fashion

Theorem 9 *The map* $\hat{hol}: \mathcal{I}_{(X,G)}(M) \to Hom(\pi_1(M), G)$ *is an open map, and for any given locally homogeneous structure* $u \in \mathcal{I}_{(X,G)}(M)$ *there is a corresponding neighborhood* $W \subset \mathcal{I}_{(X,G)}(M)$ *such that for any two (based) locally inhomogeneous structures* $u_1, u_2 \in W$ *such that* $\hat{hol}(u_1) = \hat{hol}(u_2) = \theta \in Hom(\pi_1(M), G)$ *there is a (based) diffeomorphism* $h \in Diff_0(M, x_0)$ *and* $\gamma \in G$ *with* $(h, \gamma) \cdot u_1 = u_2$.

In other words, in a sufficiently small neighborhood of a given locally homogeneous structure, any two locally homogeneous structures with the same holonomy representations are equivalent, and we can use the holonomy homomorphism as a sort of coordinate map for the space of inequivalent locally homogeneous structures that we can obtain from the given one by local deformation.

It also follows from the above theorem that the map

$$hol: \hat{I}_{(X,G)}(M) \doteq \frac{\mathcal{I}_{(X,G)}(M)}{Diff_0(M, x_0)} \to Hom(\pi_1(M), G) \tag{136}$$

is a local homeomorphism. If we define [26] the *Deformation Space* of locally homogeneous structures on M as the quotient

$$Def_{(X,G)}(M) = \frac{\hat{I}_{(X,G)}(M)}{G} \tag{137}$$

then one would expect that the map hol extends to a local homeomorphism from $Def_{(X,G)}(M)$ and the orbit space $Hom(\pi_1(M), G)/G$ (endowed with the quotient topology). In general this is the case only if we have more information on the structure of the orbit space under the G-action. Typically hol is well-behaved if we restrict our attention to the subspace in $Hom(\pi_1(M), G)/G$ generated by the conjugacy classes of *stable* representations. In any case we shall work mainly with hol: $\hat{I}_{(X,G)}(M) \to Hom(\pi_1(M), G)$, and simply assume, at least at this stage, that the representations considered are such that the local properties of the representation variety $Hom(\pi_1(M), G)/G$ describe, through the holonomy map hol, the local properties of the deformation space $Def_{(X,G)}(M)$ of locally homogeneous structures on M. This is an important point for it will imply that the representations in $\frac{Hom(\pi_1(M),G)}{G}$ provide the missing piece of information needed for describing the inequivalent deformations of dynamical triangulations approximating a given locally homogeneous riemannian manifold.

By exploiting the preparatory material so introduced, we now describe in details the procedure for parametrizing the deformation space of dynamical triangulations.

C. Moduli of dynamical triangulations

Consider on a dynamical triangulation (M, T_a) the set of metric balls of radius a, $\{U_i(M)\}$, centered at the baricenters, p_i, of the $N_{n-2}(M)$ bones $\{\sigma^{n-2}\}$ of M. Each such $U_i(M)$ is homeomorphic to $B^{n-2} \times C(link(\sigma^{n-2}))$, where $C(link(\sigma^{n-2}))$ is the cone on the link of σ^{n-2}, and B^k denotes the topological k-dimensional ball. Since $C(link(\sigma^{n-2})) \simeq B^2$, we get that $U_i(M) \simeq B^2 \times B^{n-2} \simeq B^n$, *viz.* each metric ball $U_i(M)$ is indeed a topological ball. Since their radius is a, the collection of such $U_i(M)$, $i = 1, \ldots, N_{n-2}(M)$, generates an open covering of M.

Assume that the dynamically triangulated manifold (M, T_a), endowed with the covering $\{U_i(M)\}$, is Gromov-Hausdorff near to a locally homogeneous (X, G) riemannian manifold (of bounded geometry) (Ξ, g). This assumption can be translated into the requirement that the riemannian manifold (Ξ, g) admits a geodesic ball covering $\{q_i\}_{i=1,\dots,m} \in \Xi$ generated by metric balls $U_i(\Xi)$, of radius $\epsilon = a$, such that

$$|d_M(p_i, p_j) - d_\Xi(q_i, q_j)| < a. \tag{138}$$

since, by a standard argument already exploited various times, this bound implies that the Gromov-Hausdorff distance between M and Ξ is such that $d_G(M, \Xi) \leq 2a$. By itself, the dynamically triangulated manifold M cannot be endowed, for any fixed value of the cut-off a, with a locally homogeneous metric geometry, (with the exception

of some particular cases in dimension $n = 2$). As already stressed this may be a source of serious ambiguities. In particular, let us assume that the given locally homogeneous riemannian manifold (Ξ, g) admits non-trivial infinitesimal deformations (moduli) $Def_{(X,G)}(\Xi)$, then, up to the cut-off length scale a, the dynamically triangulated manifold (M, T_a) is roughly-isometric to any of the inequivalent locally homogeneous manifolds associated with the moduli of (Ξ, g), *i.e.*, according to (138),

$$d_G(M, \Xi^*) \leq 2a \tag{139}$$

for any $\Xi^* \in Def_{(X,G)}(M)$. Thus, (M, T_a) can be isometric to any locally homogeneous structure in $Def_{(X,G)}(M)$. In order to avoid this ambiguity, we need to *declare* which locally homogeneous geometry (M, T_a) is actually approximating.

To enforce such control, let us consider (M, T_a) endowed with the covering $\{U_i(M)\}$. Choose a basepoint p_0 among the baricenters $\{p_i\}$ of the bones of M, and let $U_0(M)$ denote the corresponding metric ball in the cover $\{U_i(M)\}$. Finally, mark an n-dimensional simplex σ_0^n among those incident on the bone σ_0^{n-2} associated with the given choice of p_0. Notice that with these markings we can associate to $U_0(M)$ a well- defined element, $R_0(\sigma_0^{n-2})$, of the orthogonal group $O(n)$: $R_0(\sigma_0^{n-2})$ is the rotation around the bone σ_0^{n-2} of the vectors defining the marked simplex σ_0^n. Together with these elements, let us also choose a representation

$\theta: \pi_1(M, p_0) \rightarrow G \supseteq O(n)$ of $\pi_1(M, p_0)$ into the structure group $G \supseteq O(n)$ of the locally homogeneous geometry approximated by (M, T_a).

Since (M, T_a) is Gromov-Hausdorff near to a locally homogeneous manifold (Ξ, g), there is by hypothesis an open set $U_0(\Xi)$ on Ξ, corresponding to $U_0(M)$, centered around $q_0 \in \Xi$, and a corresponding map $\phi_0: U_0(\Xi) \rightarrow (X, G)$ associated with the element, $R_0(\sigma_0^{n-2})$, of the orthogonal group describing the rotation around the bone σ_0^{n-2}. If we think of $R_0(\sigma_0^{n-2})$ as an element of $G \supseteq O(n)$ associated with the chosen marked simplex σ_0^n, we can generate ϕ_0 by analytical continuation of $\phi(q_0) \doteq R_0(\sigma_0^{n-2})$ in $U_0(\Xi)$ along the unique simplicial loop winding around σ_0^{n-2}. Thus, associated with the markings of $(U_0(M), \sigma_0^n)$ on M there is a local chart $(U_0(\Xi), \phi_0)$ for the locally homogeneous geometry of Ξ. Since Gromov-Hausdorff nearby metric spaces share a common homotopy type, it also follows that $\pi_1(M, p_0) \simeq \pi_1(\Xi, q_0)$, (provided that a is small enough so that M and Ξ are sufficiently near in the Gromov-Hausdorff distance), and the representation of $\pi_1(M, p_0)$ induces a corresponding representation that we still denote by θ, $\theta: \pi_1(\Xi, q_0) \rightarrow G$.

The relevance of such interplay between the markings of $(U_0(M), \sigma_0^n)$ on M and the corresponding local chart $(U_0(\Xi), \phi_0)$ on Ξ, is twofold and follows from the properties of locally homegeneous geometries recalled in the previous paragraph. First, the local chart $(U_0(\Xi), \phi_0)$ can be used as a basepoint for the development pair $(D, \theta: \pi_1(\Xi, q_0) \rightarrow G)$ defining the locally homogeneous structure (Ξ, g) considered. Moreover, according to the Goldman-Thurston deformation theorem 9, if the geometry (Ξ, g) admits non-trivial deformations $Def_{(X,G)}(\Xi)$, such deformations are locally parametrized by the representation variety $Hom(\pi_1(\Xi), G)/G$. Thus, the data needed in order to specify unambiguosly which locally homogeneous geometry we are approximating with (M, T_a) is basically contained in *the association of a holonomy homomorphism* $\theta: \pi_1(M, p_0) \rightarrow G$ to the dynamical triangulation (M, T_a).

Summing up, we have the following

Theorem 10 *Let $(D, \hat{\theta}: \pi_1(\Xi, q_0) \rightarrow G)$ be the developing pair of a locally homogeneous geometry (Ξ, g) with a non-trivial deformation space $Def_{(X,G)}(\Xi)$, and let $(M, T_a) \in \mathcal{DT}_n(a, b, N)$ denote a dynamical triangulation approximating (Ξ, g). Then the distinct locally homogeneous riemannian manifolds in $Def_{(X,G)}(\Xi)$ are Gromov-Hausdorff approximated by the pairs $[(M, T_a)_{root}, \theta]$ where: (i) $(M, T_a)_{root}$ is the* **rooting** *of the dynamical triangulation (M, T_a) generated by the marking of a bone σ_0^{n-2} of (M, T_a), of a simplicial loop ω_0 winding around σ_0^{n-2}, and of a base simplex $\sigma_0^n \in \omega_0$, while (ii) θ is a holonomy homomorphism*

$$\theta: \pi_1(M, p_0) \rightarrow G \supseteq O(n) \tag{140}$$

varying in a suitable neighborhood of $\hat{\theta}$ in the representation variety $\frac{Hom(\pi_1(M), G}{G}$.

Proof. The markings in *(i)* are simply the markings characterizing the data $(U_0(M), \sigma_0^n)$ on M inducing a local chart $(U_0(\Xi), \phi_0)$ for the locally homogeneous geometry of Ξ, through which Ξ can be *unrolled* by means of the developing map D. These data are necessary for providing a well-defined holonomy representation. Henceforth we shall refer to dynamical triangulations with such marked simplices as **rooted triangulations** for short.

As we have recalled [26], the map

$$\hat{hol}: Def_{(X,G)}(\Xi) \rightarrow Hom(\pi_1(\Xi), G) \tag{141}$$

which associates to the developing pair (D, θ), (*i.e.*, to the locally homogeneous manifold (Ξ, g)), the holonomy representation θ is a continuous and open map. Two locally homogeneous manifolds Ξ_1 and Ξ_2 in $Def_{(X,G)}(\Xi)$ are near in the C^∞ topology (on the respective developing maps D_1 and D_2) and the corresponding geodesic ball coverings $\{U_i(\Xi_1)\}$ and $\{U_i(\Xi_2)\}$ are such that $d_G(\Xi_1, \Xi_2) \leq \epsilon$ where ϵ depends on the C^∞-norm on the respective developing pairs $||(D, \theta_1) - (D, \theta_2)||_\infty$. It follows that (M, T_a) is Gromov- Hausdorff near any locally homogeneous manifolds in $Def_{(X,G)}(\Xi)$.

By choosing, if necessary, a sufficiently small neighborhood of (Ξ, g) in $Def_{(X,G)}(\Xi)$, we can assume that (M, T_a) shares a common homotopy type with all (Ξ^*, g) in $Def_{(X,G)}(\Xi)$.

Thus an immediate application of the Goldman-Thurston deformation theorem 9 implies that we can establish a one-to-one correspondence between each pair $[(M, T_a)_{root}, \theta]$ and a corresponding locally homogeneous manifold in $Def_{(X,G)}(\Xi)$, simply by declaring that a given $[(M, T_a)_{root}, \theta]$ is associated with the locally homogeneous manifold in $Def_{(X,G)}(\Xi)$ defined by the holonomy homomorphism θ, and this correspondence is well-defined for θ varying in a suitable neighborhood of $\hat{\theta}$ in $\frac{Hom(\pi_1(M),G}{G}$. $\square$

Thus, with a slight abuse of language, it seems appropriate to consider the representation variety $\frac{Hom(\pi_1(M),G}{G}$ as the **Moduli Space** for a dynamically triangulated manifold $M \in \mathcal{DT}_n(a, b, N)$.

Since we are interested in closed manifolds M, the study of this moduli space is basically equivalent to the study of discrete cocompact subgroups Γ of the maximal isometry group G that can be ammitted by M, (recall that a discrete subgroup $\Gamma \subset G$ is called cocompact if the quotient space G/Γ is compact). The underlying geometric rationale is that the Lie group in question is the isometry group of the simply connected homogeneous riemannian manifold X which can be approximated by the (universal cover of the) dynamical triangulation M. The discrete subgroup Γ appears (if it has no torsion) as the deck transformation group of the covering $X \to \Xi$ where $\Xi = X/\Gamma$ is the smooth riemannian manifold approximated by M, thus $\Gamma \simeq \pi_1(M)$, (if Γ has torsion we will be dealing with orbifolds). It is intersting to remark that even if not every semisimple Lie group G may be realized as the isometry group of some symmetric space X, every such G is locally isomorphic, (*viz.* the isomorphism is at the level of the universal coverings of the groups in question), to the isometry group of some symmetric space X. The existence of a non-trivial moduli space for dynamical triangulations (of closed manifolds) is thus reduced to the study of the *rigidity of lattices* in such groups G [53].

For the convenience of the reader, we have collected in an Appendix some basic material on the geometry of the representation variety $\frac{Hom(\pi_1(M),G}{G}$. Here we stress that since $\pi_1(M)$ is a finitely generated group, (say with m generators), $Hom(\pi_1(M), G)$ is an analytic subvariety [26] of G^m, (as a variety is defined by the m-tuples $(\Theta(1), \ldots, \Theta(m)) \in G^m$ such that $\rho_i(\Theta(1), \ldots, \Theta(m)) = 1$ for each relation ρ_i associated with the given presentation of the group $\pi_1(M)$). Note that on applying a theorem due to X. Rong [54], one can actually provide a upper bound to the minimal number of generators for $\pi_1(M)$ for $M \in \mathcal{DT}_n(a, b, N)$, (the bound depends on the parameters a,b, and N).

D. Gauge-Fixing of the moduli of a Dynamical Triangulation

Far most of our efforts will be bent toward an enumeration of the distinct dynamical triangulations a PL manifold can carry. Since distinct dynamical triangulations (of bounded geometry) are meant to approximate distinct riemannian structures, we are actually addressing the characterization of a measure on the space of riemannian structures. As we have seen in the previous sections, the correspondence between dynamical triangulations and riemannian structure ceases to be one-to-one nearby locally homogeneous manifolds, no matter how fine is the triangulation. Distinct moduli of locally homogeneous riemannian manifolds may indeed correspond, through a Römer-Zähringer deformation, to a same dynamical triangulation. In order to carry over an enumeration of distinct dynamical triangulations while preserving a one-to-one correspondence between riemannian structures and dynamical triangulations, we have have to impose a suitable *gauge fixing* so as to remove this possible source of ambiguity. The situation is akin to the one familiar in gauge theories where, in order to introduce a suitable path integration on the configuration space of the theory, we have to remove gauge freedom, (by using a slice theorem for the action of the gauge group), to obtain a local (formal) measure on the orbit space labelling the gauge equivalent configurations. Eventually, as in any gauge fixing procedure, in order to obtain the correct count of the number of distinct dynamical triangulations, we have to divide the number of distinct holonomies by the *local volume* of the moduli space parametrizing the conjugacy classes of representation $\Theta\colon \pi_1(M; \sigma_0^n) \to G$ in a neighborhood of the chosen representation. From a combinatorial point of view, the necessity of this factorization comes from the fact that the number of distinct holonomies provides the number of distinct dynamical triangulation *in a given holonomy representation of the fundamental group*, namely it is a labelled or *rooted* enumeration. The unrooted enumeration can be obtained from the rooted one by dividing by all possible labellings of the inequivalent representations.

A heuristic remark. In connection with this counting problem for dynamical triangulation, it is worth remarking a subtle point. Had we addressed the counting of just distinct (in the sense of Tutte) dynamical triangulations on a PL manifold of given volume and topological type, the above gauge fixing problem would have not explicitly appeared. The fact is that in the definition of equivalence according to Tutte, no mention is made of the metric properties of the triangulation considered, and thus in the process of counting, we enumerate also triangulations potentially fitting on locally homogeneous spaces. The requirement of triangulating with equilateral simplices, is a restriction of no consequence as long as we consider generic riemannian manifold, but quite effective when coming to triangulating riemannian manifolds of large symmetry. We enumerate distinct triangulations basically by considering distinct curvature assignements, *viz.*, namely using as labels for distinct triangulations the distinct Levi-Civita connections generated by distinc curvature assignements. This strategy can be considered as the discretized counterpart of the techniques underlying uniformization theory for 2-dimensional surfaces or the geometrization program for 3-dimensional manifolds. The simplest example is provided by 2-dimensional surfaces. In such a case, a given riemannian metric can be conformally deformed to a metric of constant curvature. The conformal factor is related (via the coformal Laplacian) to the gaussian curvature of the surface, and in this sense the datum of the local curvature seems sufficient to reconstruct the original surface. This fails since there are non- equivalent infinitesimal deformations of the base constant curvature metrics (moduli), and in order to reconstruct the metric, we need curvature assignements and the datum of which constant curvature metric, (*e.g.*, which flat metric, if the surface in question is a 2-torus), is used.

Actually, there is a further subtle problem related to the fact that the action of the semi-direct product of the group of diffeomorphisms and of the conformal group makes difficult an actual reconstruction of the metric from the datum of the local curvature and of the moduli. But at this heuristic level, the parallel between the geometric setup of uniformization theory and the reconstruction of dynamical triangulations from the curvature assignements is appropriate.

A measure on the moduli space. According to the above remarks, in order to complete our gauge-fixing procedure, we have to introduce a measure on the moduli space $\mathcal{M}(T_a)$ of a given dynamical triangulation. This procedure is related to the characterization of measures on the representation variety $\frac{Hom(\pi_1(M),G)}{G}$. In dimension two, the study of such measures has been considerably developed by exploiting its connection with the semiclassical limit of Yang-Mills theory on Riemann surfaces [29]. For dynamical triangulations in dimension greater than two, the connection with measure theory on $\frac{Hom(\pi_1(M),G)}{G}$ has been stressed in [7] as the origin of the possible critical behavior in higher dimensional simplicial quantum gravity.

Let us consider the representation, θ, of $\pi_1(M)$ on the Lie algebra $\mathfrak{g}$ generated by composing the given representation $\Theta\colon \pi_1(M) \to G$ with the adjoint action of G on $\mathfrak{g}$, *viz.*, $\theta\colon \pi_1(M) \to_{\Theta} G \to_{Ad} End(\mathfrak{g})$, (henceforth we will always refer to this representation). The tangent space to $\frac{Hom(\pi_1(M),G)}{G}$ corresponding to the conjugacy class of representations $[\theta]$ is provided by $H^1(\pi_1(M),\mathfrak{g})$, the first cohomology group of $\pi_1(M)$ with values in the Lie algebra $\mathfrak{g}$. It is a known fact that this cohomology group is isomorphic to the first cohomology group of the dynamically triangulated manifold $M = |T_a|$, with coefficient in the adjoint bundle $ad(\theta)$ defined by the representation θ, *i.e.*, $H^1(M, ad(\theta))$, (for more details see the Appendix). Notice that $H^1(M, ad(\theta))$ is only a formal tangent space to $\frac{Hom(\pi_1(M),G)}{G}$, since the representation variety has singularities in correspondence to reducible representations and/or linearization unstable representations.

The choice of a well-defined holonomy representation amounts to the choice of a map (basically a section) $S\colon \frac{Hom(\pi_1(M),G)}{G} \to Hom(\pi_1(M),G)$ which associates to a conjugacy class, $[\theta]$, of representations, the holonomy representation $\theta\colon \pi_1(M,p_0) \to \mathfrak{g}$, based at the chosen basepoint.

Let δG denote the generic (infinitesimal) deformations, at θ, of the chosen slice S. Note that $\delta G \in Z^1(\pi_1(M);\mathfrak{g})$, the space of 1-cocycle on $\pi_1(M)$ with coefficients in the Lie algebra $\mathfrak{g}$. This is the (formal) tangent space to $Hom(\pi_1(M),G)$ at the chosen representation θ. A component of the deformation δG may be trivial in the sense that it may lie in the tangent space to the Ad-orbit containing θ. This part of the deformation maps, by Ad-conjugation, $S(\theta)$ to a deformed representation $\tilde{\theta}$ in a nearby slice $S(\tilde{\theta})$ with $[\tilde{\theta}]=[\theta]$. The non-trivial part of the deformation lies in the tangent space to the chosen slice S, *viz.*, $TS[\frac{Hom(\pi_1(M),G)}{G}] \simeq H^1(\pi_1(M),\mathfrak{g})$. This latter part of the deformation maps θ in a neighboring representation which cannot be obtained by Ad-conjugation from θ. As recalled, deformations in $TS[\frac{Hom(\pi_1(M),G)}{G}]$ can be interpreted as inequivalent Römer-Zähringer deformations of the given dynamical triangulation.

A natural [29], [30] volume element on the space of such deformations is the product of a volume form on the Lie algebra $\mathfrak{g}$ and of a volume on $TS[\frac{Hom(\pi_1(M),G)}{G}]$. In order to define the former, consider an Haar measure on the group G, with total volume $Vol(G)$, and let $y_1,y_2,\ldots,y_t$ be Euclidean coordinates on the Lie algebra $\mathfrak{g}$. We choose the y_h in such a way that the measure $\prod_h^t dy_h$ on $\mathfrak{g}$ is the chosen Haar measure at the identity element of G. Analogously, we can introduce a measure, $d\mu$, on the tangent space to the chosen representative of the (smooth component of the) variety $S[\frac{Hom(\pi_1(M),G)}{G}] \subset Hom(\pi_1(M),G)$. Recall that $\frac{Hom(\pi_1(M),G)}{G}$ is an analytic subvariety of G^m, m being the number of generators of $\pi_1(M)$. The fixed Ad-invariant inner product on $\mathfrak{g}$ induces a Riemannian structure on G^m, and hence on $TS[\frac{Hom(\pi_1(M),G)}{G}]$, (at least on its smooth component, to which we are restricting our considerations). The corresponding inner product between tangent vectors in $H^1(M, ad(\theta))$, will be denoted by $<< \cdot, \cdot >>$, and $d\mu$ can be thought of as the associated volume measure. We shall denote by

$$\mathcal{V} \doteq vol[\frac{Hom(\pi_1(M), G)}{G}] \tag{142}$$

the volume of the moduli space $\frac{Hom(\pi_1(M),G)}{G}$ in this measure. We cannot use $d\mu$ as such since we want to localize, as the representation variety becomes smaller and smaller, the choice of the representations θ around the trivial representation. This is dictated by the requirement that as $\mathcal{V} \to 0$, (*e.g.*, for simply connected manifolds), the measures should converge to the atom concentrated on the representation $[\omega, p_0] \mapsto Id_G$, where $[\omega, p_0]$ denotes the trivial loops based at the chosen basepoint p_0.

To this end, we normalize the product measure $d\mu \times \frac{\prod_h^t dy_h}{Vol(G)}$ on $\mathfrak{g} \times TS[\frac{Hom(\pi_1(M),G)}{G}]$, with the heat kernel on the analytic manifold $Hom(\pi_1(M), G) \subset G^m$, and set

$$d\aleph \doteq K\left(N_n vol(\sigma^n)\mathcal{V}, \theta, Id\right) d\mu \times \frac{\prod_h^t dy_h}{Vol(G)} \tag{143}$$

where θ is the given representation in $Hom(\pi_1(M), G)$, Id is the trivial representation, $vol(\sigma^n)$ is the volume of the elementary simplex σ^n, and

$$K(t, x, y)\colon \mathbb{R}^+ \times Hom(\pi_1(M), G) \times Hom(\pi_1(M), G) \to \mathbb{R} \tag{144}$$

is the fundamental solution of the heat equation on $Hom(\pi_1(M), G)$, (since $Hom(\pi_1(M), G)$ is an analytic subvariety of $G \times G \times \ldots \times G$, $K(t, x, y)$ can be constructed with the heat kernel on G). Notice that $N_n vol(\sigma^n)\mathcal{V}$ can be interpreted as the volume of the dynamically triangulated manifold M as **expressed in terms of the volume of the representation variety** $Hom(\pi_1(M), G)/G$.

Thus, the natural normalizing volume element we consider in the procedure of gauge fixing the Römer-Zähringer deformations of a dynamical triangulation is

$$Vol[\mathcal{M}(T_a)] \doteq \int_{\mathfrak{g} \times S[\frac{Hom(\pi_1(M),G)}{G}]} K\left(N_n vol(\sigma^n)\mathcal{V}, \theta, Id\right) d\mu \times \frac{\prod_h^t dy_h}{Vol(G)} \tag{145}$$

Moduli asymptotics. If we exploit the standard properties of the heat kernel, we can get an explicit asymptotic expression for $Vol[\mathcal{M}(T_a)]$ in the limit of small volume for the representation variety.

Theorem 11 *For a given value of N_n and for $\mathcal{V}vol(\sigma^n)$ near zero, the leading contribution to $Vol[\mathcal{M}(T_a)]$ is given by*

$$Vol[\mathcal{M}(T_a)] \rightarrow_{\mathcal{V}vol(\sigma^n)<<1} \rightarrow$$

$$A(\mathcal{V}) \cdot (2\pi N_n \mathcal{V}vol(\sigma^n))^{-D/2}(1 + O(N_n \mathcal{V}vol(\sigma^n))) \tag{146}$$

where $D = \doteq dim[Hom(\pi_1(M), G)]$, and $A(\mathcal{V})$ is a function of $\mathcal{V}$

(The explicit expression of $A(\mathcal{V})$ is provided in the proof).

Proof. We shall consider $Hom(\pi_1(M), G) \subset G^m$ as a compact manifold with a Riemannian structure induced by the Cartan- Killing metric on $\mathfrak{g}$. According to standard estimates [31] on the heat kernel on a compact Riemannian manifold, there is a neighborhood U of the diagonal in $Hom(\pi_1(M), G) \times Hom(\pi_1(M), G)$ such that for any pair of representations $(\theta_1, \theta_2) \in U$ and $Vol[Hom(\pi_1(M), G)/G]$ near 0, we can write

$$K\left(N_n \mathcal{V}vol(\sigma^n), \theta_1, \theta_2\right) =$$

$$(2\pi N_n \mathcal{V}vol(\sigma^n))^{-D/2} \exp\left(-\frac{|\theta_1 - \theta_2|^2}{2N_n \mathcal{V}vol(\sigma^n)}\right)(1 + O(N_n \mathcal{V}vol(\sigma^n))) \tag{147}$$

where $|\theta_1 - \theta_2|$ denotes the riemannian distance, (in the induced Cartan- Killing norm), on $Hom(\pi_1(M), G) \subset G^m$.

Set $t \doteq N_n \mathcal{V}vol(\sigma^n)$, and let us consider [32], [26] the representations θ, for t near 0, as one-parameter family of representations θ^t

$$\theta^t = \exp[tu(a) + O(t^2)]\theta(a) \tag{148}$$

where $a \in \pi_1(M)$, and where $u: \pi_1(M) \to \mathfrak{g}$. Given a and b in $\pi_1(M)$, if we differentiate the homomorphism condition $\theta^t(ab) = \theta^t(a)\theta^t(b)$, we get that the u are one-cocycles of $\pi_1(M)$ with coefficients in the $\pi_1(M)$-module $\mathfrak{g}_\theta$,

$$u(ab) = u(a) + [Ad(\theta(a))]u(b) \tag{149}$$

namely, $u \in Z^1(\pi_1(M), \mathfrak{g})$, the tangent space to $Hom(\pi_1(M), G)$. Among the vectors $u \in Z^1(\pi_1(M), \mathfrak{g})$, those tangent to the Ad-orbit are of the form $u(a) = [Ad(\theta(a))]h - h$, for some $h \in \mathfrak{g}$, these are 1- coboundaries on $\pi_1(M)$ with coefficients in the representation, *i.e.*, elements of $B^1(\pi_1(M), \mathfrak{g})$. We can decompose u in a component $P_H u \in H^1(\pi_1(M), \mathfrak{g}) = Z^1(\pi_1(M), \mathfrak{g})/B^1(\pi_1(M), \mathfrak{g})$ and in a component $P_{\mathfrak{g}} u \in \mathfrak{g}$, P_H and $P_{\mathfrak{g}}$ denoting the respective projection operators. Thus

$$\theta^t - Id = t(P_H u + P_{\mathfrak{g}} u) + O(t^2) \tag{150}$$

Introducing this in (147) we get

$$K\left(N_n \mathcal{V} vol(\sigma^n), \theta, Id\right) =$$

$$(2\pi N_n \mathcal{V} vol(\sigma^n))^{-D_H/2 - D_{\mathfrak{g}/2}} \exp\left(-\frac{t(|P_H u|^2 + |P_{\mathfrak{g}} u|^2)}{2N_n \mathcal{V} vol(\sigma^n)}\right)(1 + O(t)) \tag{151}$$

where D_H is the dimension of the smooth component of $\frac{Hom(\pi_1(M),G)}{G}$ and $D_{\mathfrak{g}} = dimG$. Integrating (151) over $\mathfrak{g} \times S[\frac{Hom(\pi_1(M),G)}{G}]$ and evaluating the gaussian integral over $\mathfrak{g}$ we get to leading order:

$$\int_{\mathfrak{g}\times S} (2\pi N_n \mathcal{V} vol(\sigma^n))^{-D_H/2 - D_{\mathfrak{g}/2}} \exp\left(-\frac{t(|P_H u|^2 + |P_{\mathfrak{g}} u|^2)}{2N_n \mathcal{V} vol(\sigma^n)}\right) =$$

$$(2\pi N_n \mathcal{V} vol(\sigma^n))^{-D_H/2 - D_{\mathfrak{g}}/2} \int_S [det P_{\mathfrak{g}} \cdot det P^*_{\mathfrak{g}}]^{-1/2} \exp\left(-\frac{|P_H u|^2}{2N_n \mathcal{V} vol(\sigma^n)}\right) \tag{152}$$

which, upon setting

$$A(\mathcal{V}) \doteq \int_S [det P_{\mathfrak{g}} \cdot det P^*_{\mathfrak{g}}]^{-1/2} \exp\left(-\frac{|P_H u|^2}{2N_n \mathcal{V} vol(\sigma^n)}\right) \tag{153}$$

yields the stated result. □

Note that the computation of the integral $A(\mathcal{V})$, extended over the moduli space $\frac{Hom(\pi_1(M),G)}{G}$, can be related to the theory of Reidemeister torsion [29], [30].

Moduli asymptotics for DT-surfaces. Both the presence of the volume of the representation variety and of its dimension, are an important aspect of (146). In dimension $n=2$, we can be more explicit, and if we gather information on the dimension of $\frac{Hom(\pi_1(M),G)}{G}$ [26], [33], then in the case of 2-dimensional surfaces theorem 11 specializes to the following

Lemma 5 *For $n=2$ and for $\mathcal{V}vol(\sigma^n)$ near zero, the leading contribution to $Vol[\mathcal{M}(T_a)]$ is given as a function of surface topology by:*

(i) The sphere $\mathbb{S}^2$:

$$Vol[\mathcal{M}(T_a)] \rightarrow_{\mathcal{V}vol(\sigma^n)<<1} \rightarrow 1 \tag{154}$$

(ii) The torus $\mathbb{T}^2$:

$$Vol[\mathcal{M}(T_a)] \rightarrow_{\mathcal{V}vol(\sigma^n)<<1} \rightarrow A(\mathcal{V}) \cdot (2\pi N_n \mathcal{V}vol(\sigma^n))^{-5/2} \tag{155}$$

(iii) Orientable surfaces Σ_h of genus $h>1$, and for representations of $\pi_1(\Sigma_h) \rightarrow G$ whose center has a constant dimension $dimZ$:

$$Vol[\mathcal{M}(T_a)] \rightarrow_{\mathcal{V}vol(\sigma^n)<<1} \rightarrow A(\mathcal{V}) \cdot (2\pi N_n \mathcal{V}vol(\sigma^n))^{-[(2h-1)dimG+dimZ]/2} \tag{156}$$

(If we have a non-trivial centralizer for the representation, then we have to slightly alter the argument in theorem 11).

Proof. For a sphere $dim[H^1(\pi_1(M),\mathfrak{g})]=0$, and trivially $\mathcal{V}=0$. As $\mathcal{V}\to 0$, the measure $d\aleph$ weakly converges to $\delta(Id)$, where $\delta(Id)$ is the Dirac distribution concentrated on the trivial representation. This immediately yields the stated result.

The torus $\mathbb{T}^2$, can be actually considered as a limiting case of (iii). From [26] it follows that if $z(\theta)$ denotes the centralizer of $\theta(\pi_1(\Sigma_h))$ in G, then the dimension of $Hom(\pi_1(\Sigma_h),G)$ is given by

$$(2h-1)dim(G)+dim(z(\theta)) \tag{157}$$

By taking $G=SL(2,\mathbb{R})$, and setting $h=1$ we immediately get the stated result for $\mathbb{T}^2$. The general result for $h>1$ is a trivial consequence of the dimensionality formula (157). $\square$.

We wish to stress that these results (in particular the asymptotics of the volume of moduli space) refer only to the smooth component of the representation space $Hom(\pi_1(M),G)/G$. Singularities are present in correspondence with reducible representations and around representations which are linearization unstable [26]. Also, $Hom(\pi_1(M),G)/G$ may have many distinct connected components. For example, in the relevant case of $G=PSL(2,\mathbb{R})$, $Hom(\pi_1(M),PSL(2,\mathbb{R}))$ has $2^{2h+1}+2h-3$ connected components, [26]. In 2d-Yang- Mills, it is a known fact that, at least in the semiclassical limit, the singular component of the representation variety does not contribute to the volume of the moduli space [30], [29], [34]. Since there are many points of contact between our approach in evaluating the asymptotics of the volume of the moduli space of Römer-Zähringer deformations of a dynamical triangulation, and 2d-Yang-Mills, we may reasonably assume that the singular part of the representation variety is not playing any relevant role in our case, at least in dimension $n=2$. For dimension $n>2$ this is an open issue under current investigation. As we shall see in due course, the asymptotic of the volume of the moduli space of Römer-Zähringer deformations plays a basic role in simplicial quantum gravity being related to the entropy exponent of the theory, thus these matters are quite relevant in addressing on analytical grounds the existence of the continuum limit of the theory.

A compact formula for surfaces. We conclude this section by providing a compact expression for the the asymptotics of the volume of the moduli space of Römer-Zähringer deformations of a dynamical triangulated surface. In the two-dimensional case, according to lemma 5 the small volume behavior of $Vol[\mathcal{M}(T_a)]$ is

$$\tilde{A}_h(\mathcal{V}) \cdot N_n^{\xi(\Sigma_h)} \tag{158}$$

where the constant $\tilde{A}_h(\mathcal{V}) \doteq A(\mathcal{V})(2\pi\mathcal{V}vol(\sigma^n))^{-[(2h-1)dimG+dimZ]/2}$, (for genus $h \geq 1$, whereas $\tilde{A}_h(\mathcal{V}) \doteq A(\mathcal{V})$ for the sphere), and $\xi(\Sigma_h)$ is an exponent depending from the genus, and the group G. This exponent can be given a more compact expression if we introduce an *effective dimension*, D_{eff}, of $Hom(\pi_1(M), G)$ according to

$$D_{eff} \doteq (2h-1)\alpha dim(G) + \alpha dim(z(\theta)) + \beta \tag{159}$$

where the parameters α and β are determined by fitting (159) with the dimensions of the singular case provided by the sphere $\mathbb{S}^2$ and the limiting case of the torus $\mathbb{T}^2$. Explicitly, according to lemma 5, we have

$$\xi(\Sigma_h) = -(2h-1)\frac{\alpha}{2}dim(G) - \frac{\alpha}{2}dim(z(\theta)) - \frac{\beta}{2} + 1 \tag{160}$$

with

$$\begin{aligned} \frac{\alpha}{2}dim(G) - \frac{\alpha}{2}dim(z(\theta)) - \frac{\beta}{2} + 1 &= \xi(\mathbb{S}^2) = 1 \\ -\frac{\alpha}{2}dim(G) - \frac{\alpha}{2}dim(z(\theta)) - \frac{\beta}{2} + 1 &= \xi(\mathbb{T}^2) = 1 - \frac{5}{2} \end{aligned} \tag{161}$$

In this way we get

$$\begin{aligned} \alpha &= \frac{5}{2dimG} \\ \beta &= \frac{5}{2}\left(1 - \frac{dim(z(\theta))}{dimG}\right) \end{aligned} \tag{162}$$

and

$$\xi(\Sigma_h) = \frac{5}{4}\chi(\Sigma_h) - \frac{5}{2} \tag{163}$$

where $\chi(\Sigma_h)\doteq 2-2h$ is the Euler-Poincare characteristic of the surface Σ_h. These observations establish the following

Lemma 6 *The small volume limit asymptotics of the volume of the moduli space of Römer-Zähringer deformations of a dynamical triangulated surface is given by*

$$Vol[\mathcal{M}(T_a)] \rightarrow_{\mathcal{V}vol(\sigma^n)<<1} \rightarrow \tilde{A}_h(\mathcal{V}) \cdot N_n^{\frac{5}{4}\chi(\Sigma_h)-\frac{5}{2}} \tag{164}$$

VII. CURVATURE ASSIGNEMENTS FOR DYNAMICAL TRIANGULATIONS

According to the analysis of the previous sections, the topological information associated with a dynamical triangulation is encoded in the moduli space of its Römer-Zähringer deformations. Here, our purpose is to characterize the metric properties of the triangulations in terms of the (restricted) holonomy of its natural Levi-Civita connections, and reduce the enumeration of distinct dynamical triangulations to the enumeration of distinct curvature assignements.

Simplicial Lassos. We start by introducing the following

Definition 11 *A simplicial loop ν, based at a simplex σ_0^n, is a (small) lasso with nose at the bone B, if it can be decomposed into three simplicial curves $\nu(B) = \tau^{-1} \cdot \omega(B) \cdot \tau$, where τ is a simplicial path from σ_0^n to one of the simplices $\sigma^n(B)$ containing the bone B, τ^{-1} is the same curve going backward, and $\omega(B)$ is the unique loop winding around the bone B.*

Since the open stars of the bones provide an open covering of $M = |T_a|$, any simplicial loop in M, if it is homotopic to zero, is a product of *lassos* whose nose is always contained in the open star of some bone B, (this is the *Lasso lemma* [35]). Thus, in order to evaluate the holonomy around a generic contractible simplicial loop based at a marked simplex σ_0^n we can proceed as follows.

Let us fix a holonomy representation of the fundamental group of M, $\theta\colon \pi_1(M) \to G \to End(\mathfrak{g})$. Recall that the choice of such θ calls for the marking of a bone σ_0^{n-2}, (whose barycenter provides the basepoint p_0), of a simplex $\sigma_0^n \supset \sigma_0^{n-2}$, and of the holonomy around the marked bone. As the notation suggests, we naturally identify the marked simplex σ_0^n with the simplex on which the simplicial lassos are based, also, for notational simplicity, we let $\lambda + 1 = N_{n-2}(T)$ denote the number of bones in the dynamical triangulation T_a. Let us consider the set of simplicial lassos, $\{\nu(\alpha), \sigma_0^n\}$ based at the marked σ_0^n and with their noses corresponding to the $\lambda + 1$ bones $B_0, B(1), \ldots, B(\lambda)$. The effect of parallel transporting, according to the Levi- Civita connection associated with T_a, along any of these lassos is a rotation in a two- dimensional plane, (orthogonal to the bone considered). If ω is the generic (contractible) simplicial loop based at σ_0^n, then the holonomy along the generic loop ω based at σ_0, can be written as

$$R_\omega(\sigma_0) = \prod_{\alpha=0}^{\lambda} \{A(\alpha)^{-1} R(m(\alpha)\phi(\alpha))A(\alpha)\} \tag{165}$$

where $m(\alpha) \in \mathbb{Z}$ are the integers providing the winding numbers of ω around the various bones, $R[m(\alpha)\phi(\alpha)]$ denote the rotations in the planes orthogonal to the bones $B(\alpha)$, and finally, $A(0)$, $A(\alpha)$ denote the orthogonal matrices describing the parallel transport along the ropes, τ_α, of the lassos, ($A(0)$ is associated with the trivial path σ_0^n).

On rather general grounds, it is known [36] that through holonomy we can reconstruct the underlying connection, (up to conjugation). In our case, as already stressed, this reconstruction procedure is particularly relevant since distinct connections are in correspondence with distinct triangulations. It follows that the conjugacy class of the (restricted) holonomy $R_\omega(\sigma_0)$ is G-invariant and fully describes the underlying dynamical triangulation in the given representation $\Theta: \pi_1(M, p_0) \to G$. The natural class functions of the holonomy (and hence of the triangulation) are linear combination of the group characters, namely of the traces of $R_\omega(\sigma_0)$, (these latters being thought of as elements of the fundamental representation of the restricted holonomy group). These traces depend only from the given set of incidence numbers $\{q(\alpha)\}$ of the triangulation, and a question of relevance is to what extent such traces, and hence the curvature assignements, characterize the triangulation itself. In order to answer such a question we exploit the connection between curvature assignements an the theory of partitions of integers.

A. Partitions of integers and curvature assignements

According to (42) the curvature assignements $\{q(\alpha)\}$ of a dynamical triangulation T_a cannot be arbitrary, since they are constrained by the total number $N_n(T^{(i)})$ of n-simplices in T_a. We have

$$q(o) + \sum_{\alpha=1}^{\lambda} q(\alpha) = \frac{1}{2} n(n+1) N_n(T^{(i)}) \tag{166}$$

where the summation extends over the bones of $T_a^{(i)}$, (notice that we have included in the summation the marked bone, by considering also the incidence number $q(0)$). Another natural constraint comes about from the fact that in order to generate an n-dimensional polyhedron there is a minimum number $\hat{q}(n)$ of simplices σ^n that must join together at a bone, (according to the introductory remarks, in dimension $n=2$, $n=3$, $n=4$ we have $\hat{q}(n)=3$, otherwise we have no polyhedral manifold. Rather than use this explicit value, we keep in using the general notation $\hat{q}(n)$). If we define

$$\begin{aligned} Q(0) &\doteq q(0) - \hat{q}(n) \\ Q(\alpha) &\doteq q(\alpha) - \hat{q}(n) \end{aligned} \tag{167}$$

then we can put these two constraints together by writing

$$Q(0) + \sum_{\alpha=1}^{\lambda} Q(\alpha) = \left[\frac{1}{2} n(n+1) \left(\frac{N_n(T^{(i)})}{N_{n-2}(T^{(i)})} \right) - \hat{q}(n) \right] (\lambda + 1) \tag{168}$$

(recall that $N_{n-2}(T^{(i)}) = \lambda + 1$, also recall that $b(n, n-2) \equiv \frac{1}{2} n(n+1)(N_n(T^{(i)})/N_{n-2}(T^{(i)}))$ is the average number of simplices σ^n, in $T^{(i)}$, incident on a bone).

Thus, we have

$$Q(0) + \sum_{\alpha=1}^{\lambda} Q(\alpha) = [b(n, n-2) - \hat{q}(n)] (\lambda + 1) \tag{169}$$

Recall that a *partition* of a positive integer n is a representation of n as the sum $n = 1 \cdot \eta_1 + 2 \cdot \eta_2 + \ldots + n \cdot \eta_n$, where $\eta_1, \eta_2, \ldots, \eta_n$ are non-negative integers, and $\eta_1 + \eta_2 + \ldots + \eta_n$ determines the number of summands of the partition.

This remark suggests that for each given value of $Q(0)$, $b(n, n-2)$, and λ, the set of curvature assignements $q(\alpha) = Q(\alpha) + \hat{q}$ satisfying the constraint (169) may be related to (some of) the possible partitions of the integer $[b(n,n-2)-\hat{q}(n)]\,(\lambda+1)$. Among the attractive features of such an interpretation is the possibility of exploiting the large body of knowledge we have on partition theory, a well-established chapter of combinatorics and analytic number theory. In order to exploit these techniques, we need to modify a little (169) so that it resembles more directly a partition.

First of all note that it is natural to assume that the marked $q(o)$, (or equivalently $Q(0)$), is the largest curvature assignement in the dynamical triangulation considered T_a, *i.e.*, $q(0) \geq q(\alpha)$, (thus $Q(0)$ is a sort of *root blob* of curvature). As we have seen in the introductory chapter, (see (65)), the largest curvature assignements can well be of the order of λ, and the largest curvature assignements which simultaneously minimize the remaining curvature assignements on the other bones are realized on particular triangulations, (see the discussion following (65)). In such triangulations, most of the top-dimensional simplices are incident on n distinguished bones, (bounding an embedded $\mathbb{S}^{n-2}$). Notice that, when generated with equilateral simplices, these particular triangulations have a discrete simmetry associated with the interchange of the n distinguished bones carrying most of the curvature. Since we have to consider (169) as $q(0)$ varies from $\hat{q}$ to q_{max}, we have to factor out these residual symmetries, and we do so by marking not just one bone, (the one providing the basepoint p_0), but rather n bones, $\sigma^{n-2}(0), \sigma^{n-2}(1), \ldots, \sigma^{n-2}(n-1)$, which support, in the extreme case described above, the largest curvatures (with $q(0) = q(1) = \ldots = q(n-1)$) while minimising the incidence on the remaining bones $\{\sigma^{n-2}(\alpha)\}_{\alpha=n,\ldots,\lambda}$. The original marked bone σ_0^{n-2} is one of those n distinguished bones, and can be selected by the required marking of an n-simplex σ_0^n associated with the gauge fixing procedure for the moduli space of dynamical triangulations. Thus, we start by replacing (169) with

$$nQ(0) + \sum_{\alpha=n}^{\lambda} Q(\alpha) = [b(n,n-2)-\hat{q}(n)]\,(\lambda+1) \tag{170}$$

Since $nQ(0) \geq Q(\alpha)$, we can refer the curvature assignements $Q(\alpha)$ to the chosen value for the curvature assignement $nQ(0)$ around the marked bones. To this end, let us add and subtract to (170) the expression $nQ(0)(\lambda+1)$, which allows us to eventually rewrite (169) as

$$\sum_{\alpha=n}^{\lambda}\tilde{Q}(\alpha)=\left[nQ(0)-b(n,n-2)+\hat{q}\right](\lambda+1)-n(n-1)Q(0) \tag{171}$$

where

$$\tilde{Q}(\alpha)\doteq nQ(0)-Q(\alpha) \tag{172}$$

This form (171) of (169) can be interpreted as providing a partition of the integer $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ in at most $\lambda+1-n$ integers $\tilde{Q}(\alpha)$, (some of the $\tilde{Q}(\alpha)$ can be zero), each bounded above by $(b-\hat{q})(\lambda+1)$.

Notice that is not *a priori* true that *each* partition of the integer $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ can be generated by curvature assignements through the $\tilde{Q}(\alpha)$. Thus, we shall denote by

$$p_{\lambda}^{curv}\left([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)\right) \tag{173}$$

the number of partitions of $[nQ(0)-b(n,n-2)+\hat{q}](\lambda+1)-n(n-1)Q(0)$ into at most $\lambda+1-n$ parts, (each $\leq(b(n,n-2)-\hat{q})(\lambda+1)$), arising from actual curvature assignements.

A notational remark. Notice that in comparing with standard formulae for partition theory [37], a more standard notation would have been

$$p_{\lambda+1-n}^{curv}\left([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)\right) \tag{174}$$

however, in order not to burden the notation further, we shall adopt the above simplified version; in some cases, if there is no danger of confusion, we shall simply write p_{λ}^{curv} for the above expression and for other similar expressions involving partitions.

B. Curvature assignements of distinct dynamical triangulations

For small λ's the actual distribution of partitions corresponding to curvature assignements is rather accidental, however for $\lambda>>1$ we have that a large fraction of partitions corresponds to actual curvature assignements. We start by proving that *generically* distinct curvature assignements correspond to distinct dynamical triangulations. We can prove this by the elementary probabilistic methods which are typical in discrete mathematics whenever attention is on asymptotic properties. We also bound the size of the *non-generic* set of distict dynamical triangulations associated to the same set of curvature assignements. The tails associated both to partitions of integers which are not curvature assignements and to distinct dynamical triangulations having the same curvature assignements provide corrections to the naive counting through unrestricted partitions of integers. Such corrections are difficult to control in details, however they have the structure of a rescaling that will allow us to implement a very straightforward counting strategy which will prove very effective.

Let us start by noticing that it is obviously false that every possible set of distinct curvature assignement corresponds to triangulations having distinct incidence matrices. One may easily construct particular examples violating such statement. Consider for instance two dynamically triangulated (exagonal) flat tori. By flipping links, we may generate two curvature bumps on each torus, (on each torus, the bumps may correspond to distinct curvature assignements) . By inserting in each torus a copy of these curvature bumps in such a way that their distance is different in the two tori, we get distinct triangulations with the same curvature assignements. As another more general example, (again in dimension 2), consider a two-dimensional triangulation with a large number of vertices, let abc, acd, and efg, egh four triangles pairwise sharing a common edge (ac for the former pair, and eg for the latter), but otherwise largely separated, (so that their incidence numbers on the respective vertices are uncorrelated). Let us assume that the corresponding incidence number are given by: $q(a) = \alpha$, $q(b) = \gamma$, $q(c) = \beta$, $q(d) = \delta$, for the first pair of triangles, while for the second we set: $q(e) = \gamma + 1$, $q(f) = \alpha - 1$, $q(g) = \delta + 1$, $q(h) = \beta - 1$. It is immediate to check that a flip move will interchange the curvature assignements of the first pair of adjacent triangles with those of the second pair, thus changing the incidence relations in the triangulation. Nonetheless, the sequence of curvature assignements remains unchanged, and again we have two distinct triangulations with the same sequence of curvature assignements.

Clearly, these counterexamples work either because we carefully chose a particularly symmetric triangulation or because we adjusted the curvature assignements to the particular action of the flip move. However, when moving from a dynamical triangulation to another, with a set of ergodic moves (Pachner ([38])), the curvature assignements are not fixed, and in this sense the above counterexamples are not generic, and if we denote by $\{T_a^{(i)}\}_{curv}$ the set of distinct triangulations sharing a common set of curvature assignements, we can prove that as $\lambda >> 1$, $\{T_a^{(i)}\}_{curv}$ is in a suitable sense a small subset in $\{T_a^{(i)}\}$. We can formalize this remark according to the

Lemma 7 *For a PL-manifold M, let $\Theta\colon \pi_1(M;p_0) \to G$, be a given holonomy representation, and let $\{T_a^{(i)}\}_\lambda$ denote the set of distinct dynamical triangulations of $M = |T_a^{(i)}|$ with $N_{n-2}(T_a^{(i)}) = \lambda + 1$ bones, and let $\{T_a^{(i)}\}_{\lambda \in \mathbb{Z}}$ be the associated inductive limit space generated as λ grows. Let $\mathcal{P}_T \doteq (\{T_a^{(i)}\}_{\lambda\in\mathbb{Z}}, \mathcal{B}, m_T)$ be the probability space endowed with the normalized counting measure m_T on every non-empty open set $\mathcal{B} \subset \{T_a^{(i)}\}_{\lambda\in\mathbb{Z}}$. For any $T_a \in \{T_a^{(i)}\}_\lambda$, let $\{q(\alpha)\}$ describe the sequence of integers providing the incidence of the simplices $\{\sigma_i^n\}_{i=1}^{N_n(T_a)}$ over the $\lambda + 1$ bones $\{\sigma_k^{n-2}\}_{k=1}^{N_{n-2}(T_a)}$. Then, as $\lambda \to \infty$, the curvature assignements $\{q(\alpha)\}$ characterize with probability one the dynamical triangulation $T_a[q(\alpha)]$.*

Proof. We start by creating a probability space $\mathcal{P}_T$ whose elements are the distinct triangulations in $\{T_a^{(i)}\}_{\lambda\in\mathbb{Z}}$, and whose Borel probability measure, m_T, is the normalized counting measure on every non-empty open set $\mathcal{B} \subset \{T_a^{(i)}\}_{\lambda\in\mathbb{Z}}$. Note that we have to consider triangulation with variable λ, since the known set of ergodic moves [38] mapping a triangulation $T_a^{(i)}(1)_\lambda$ into a distinct triangulation $T_a^{(i)}(2)_\lambda$, (with the same λ), go through intermediate steps which do not preserve volume (hence the number of bones λ at a given $b(n, n-2)$). If, for a given value of λ, $\{T_a^{(i)}\}_{curv}$ is the subset of distinct triangulations with a same set of curvature assignements, we can write (λ fixed)

$$m_T[\{T_a^{(i)}\}_{curv}] = \frac{Card\{T_a^{(i)}\}_{curv}}{Card\{T_a^{(i)}\}} \tag{175}$$

Recall [38] that on $(\mathcal{P}_T, \mathcal{B}, m_T)$, the Pachner moves characterize a continuous and ergodic dynamical system with respect to m_T. If

$$\lim_{\lambda\to\infty} \frac{Card\{T_a^{(i)}\}_{curv}}{Card\{T_a^{(i)}\}} > 0 \tag{176}$$

then by Birkhoff ergodic theorem [39] we would get

$$m_T[\{T_a^{(i)}\}_{curv}] > 0 \tag{177}$$

and the Pachner moves would fix m_T-almost everywhere the set $\{T_a^{(i)}\}_{curv}$ thus contradicting their ergodicity. Hence we must have

$$\lim_{\lambda\to\infty} \frac{Card\{T_a^{(i)}\}_{curv}}{Card\{T_a^{(i)}\}} = 0 \tag{178}$$

which reflects the fact that Pachner moves can fix the curvature assignements only under special circumstances. $\square$

Even if this formal result may at first sight appear very reassuring we still have to face the fact that distinct dynamical triangulations associated to a same set of curvature assignements may give rise to significant subleading corrections to the counting of distinct triangulations of a given PL-manifold. Thus we need a way for controlling how large the set $\{T_a^{(i)}\}_{curv}$ can be. This control is provided by the following result

Lemma 8 *There are constants, $\mu(n) > 0$, and $\tau(n) \geq 0$, depending only on the dimension n, such that*

$$Card\{T_a^{(i)}\}_{curv} \leq \mu(n) \cdot \lambda^{\tau(n)} \tag{179}$$

Proof. For a given set of curvature assignements over the $\lambda+1$ bones of a dynamically triangulated PL-manifold M, let $\{T_a^{(i)}\}_{curv}$ denote the set of distinct dynamical triangulations (with $\lambda+1$ bones), having the same curvature assignements. Consider the corresponding dual complexes, *viz.*, for a given $T_a \in \{T_a^{(i)}\}_{curv}$, consider its barycentric subdivision, then the closed stars (in the subdivision) of the vertices of T_a form the collection of n-cells of the complex dual to the given T_a. Distinct triangulations in $\{T_a^{(i)}\}_{curv}$ correspond to distinct dual complexes. The use of the dual complex rather than of the triangulation itself is related to the fact that the incidence matrix of the dual complex is directly related to the curvature assignements of the original triangulation. This is most easily seen in dimension $n=2$, whereas in dimension $n>2$ the curvature assignements of a dynamical triangulation T_a, over the respective bones, can be read from the incidence matrix $A \doteq \{a_{ij}\}_{0\leq i,j\leq\lambda}$ associated with the 2- skeleton of the dual complex, (*e.g.*, in dimension $n=3$ think of a froth pattern in space). Explicitly, one has that the line sums of the 2- skeleton incidence matrix A are such that

$$\sum_{j=0}^{\lambda} a_{kj} = q(k) \tag{180}$$

where $q(k)$ is the number of simplices in $\{T_a^{(i)}\}_{curv}$ which are incident on the bone $\sigma^{n-2}(k)$. A similar relation holds for the column sums. In order to estimate $Card\{T_a^{(i)}\}_{curv}$ for a given set of curvature assignements $\{q(k)\}_{0\leq k\leq\lambda}$ we have to enumerate the set of such distinct incidence matrices. The generic matrix A of such a kind, is a symmetric $(0,1)$ matrix with vanishing diagonal entries satisfying, for each line, at least the relation (180), (other constraints, associated with the Dehn-Sommerville relations, have to be satisfied). In order to enumerate such matrices we can proceed as follows. We have

$$\begin{pmatrix} \lambda-1 \\ q(1) \end{pmatrix} \tag{181}$$

choices for the first row $\{a_{1j}\}$ of A, and in general, since A is a symmetric traceless matrix, once chosen the $k-1$-th row, we have

$$\begin{pmatrix} \lambda - k \\ q(k) - (\sum_{j=1}^{k-1} a_{kj}) \end{pmatrix} \tag{182}$$

choices for the k-th row.

$Card\{T_a^{(i)}\}_{curv}$ is bounded above by the number of distinct $(0,1)$ matrices satisfying the constraint (180), thus we get

$$Card\{T_a^{(i)}\}_{curv} \leq \prod_{k=0}^{\lambda} \begin{pmatrix} \lambda - k \\ q(k) - (\sum_{j=1}^{k-1} a_{kj}) \end{pmatrix} \tag{183}$$

By taking logarithms and on applying Jensen's inequality, $(\ln \sum_k h_k \geq \sum_k \ln h_k)$, we get

$$\ln[Card\{T_a^{(i)}\}_{curv}] \leq \ln \sum_{k=0}^{\lambda} \begin{pmatrix} \lambda - k \\ q(k) - (\sum_{j=1}^{k-1} a_{kj}) \end{pmatrix} \tag{184}$$

We set

$$\tau(n) \doteq \max_k [q(k) - \sum_{j=1}^{k-1} a_{kj}] \tag{185}$$

and bound the binomial coefficient in (184) according to

$$\begin{pmatrix} \lambda - k \\ q(k) - (\sum_{j=1}^{k-1} a_{kj}) \end{pmatrix} \leq \frac{(\lambda - k)!}{(\lambda - k - \tau(n))!} \tag{186}$$

On applying Stirling's formula $\lambda! = (\lambda/e)^{\lambda}\sqrt{2\pi\lambda}e^{\alpha_\lambda}$, where $1/(12\lambda+1) < \alpha_\lambda < 1/(12\lambda)$, we easily get from (184) that for $\lambda >> 1$

$$Card\{T_a^{(i)}\}_{curv} \leq \mu(n) \cdot \lambda^{\tau(n)}, \tag{187}$$

as stated. $\square$

The above bound on $Card\{T_a^{(i)}\}_{curv}$ is quite rough but sufficient for our purposes, since it implies that $Card\{T_a^{(i)}\}_{curv}$ can, at worst, give rise to a subleading polynomial correction to the counting associated with the curvature assignements.

This remark implies that the knowledge of the asymptotics of the number of partitions p_λ^{curv} of the integer $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ which are curvature assignements provides a way of counting distinct triangulations of a given PL-manifold, at least when λ is large.

Trivially $p_\lambda^{curv} < p_\lambda$, and a first estimate of the asymptotic behaviour of p_λ^{curv} can be obtained by the asymptotics of p_λ:

Lemma 9 *Let $\mathcal{D}T_n(a,b,\lambda)$ denote the set of distinct dynamical triangulations, of an n-dimensional PL manifold M, with $N_{n-2}(T_a)=\lambda+1$ bones, and average incidence $b(n,n-2)=b$. Then, for large λ the number of distinct partitions of the integer $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ (into $\lambda+1-n$ parts) coming from curvature assignements of a dynamical triangulation $T_a\in\mathcal{D}T_n(a,b,\lambda)$ is exponentially bounded above according to*

$$p_\lambda^{curv}\leq e^{[2+\ln(b-\hat{q})]\lambda}\lambda^{-2} \tag{188}$$

Proof. The asymptotic behavior of $p_\lambda([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0))$ follows by noticing that (see *e.g* [40], pp. 32) $p_\lambda([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0))$ depends only on the value of $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ modulo $(\lambda+1-n)!$, and that it is a polynomial of degree $\lambda-n$ whose leading term is

$$\frac{\{[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)\}^{\lambda-n}}{(\lambda+1-n)!(\lambda-n)!} \tag{189}$$

On applying Stirling's formula, we get for $\lambda >> n$

$$p_\lambda([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0))\simeq\frac{e^{2\lambda}}{2\pi}\frac{\{[nQ(0)-b+\hat{q}]\lambda\}^\lambda}{\lambda^{2\lambda}[nQ(0)-b(n,n-2)+\hat{q}]\lambda} \tag{190}$$

As $Q(0)$ varies, $p_\lambda([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0))$ is bounded above by the right side of the above expression evaluated for $\max[nQ(0)]=(b-\hat{q})\lambda$, which yields $p_\lambda\simeq e^{[2+\ln(b-\hat{q})]\lambda}\lambda^{-2}$. Since $p_\lambda^{curv}<p_\lambda$, the result follows. $\Box$

The bound (188) may be consistent as well with a subexponential growth for p_λ^{curv}, a subexponential growth that, if present, may make the actual control of p_λ^{curv} quite unassailable. However, more can be said on the actual asymptotics of the curvature assignements in $\mathcal{DT}_n(a,b,\lambda)$. Since some of the $\tilde{Q}(\alpha)$ may be zero, the number of distinct partitions of the integer $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ into $\lambda+1-n$ parts coming from curvature assignements is approximated with greater and greater accuracy, for λ large, by the unrestricted partition of $[nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$ into curvature assignements, (*viz*, we no longer require to have a partition into $\lambda+1-n$ parts). This latter partition function is in general much more difficult to estimate than the original one, however its asymptotics can be qualitatively controlled through known techniques of analytic number theory, and together with the elementary bound (188) we can readily establish an exact exponential asymptotics for p_λ^{curv}.

For a given value of b, and $\lambda >> 1$, let $\mathcal{C}(b,\lambda,T_a)$, ($\mathcal{C}$ for short), be the set of positive integers occurring as curvature assignements of some rooted triangulation T_a of bounded geometry in $\mathcal{DT}_n(a,b,\lambda)$. Set $k \doteq [nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)$, and for t denoting a generic indeterminate, let

$$f_{curv}(t) \doteq \sum_{k>0} p^{curv}(k)t^k \tag{191}$$

be the generating function for the set of unrestricted partitions, $p^{curv}(k)$, of the integer k into curvature assignements of some $T_a \in \mathcal{DT}_n(a,b,\lambda)$. Then, by exploiting the formula for the sum of a finite geometric series, one can easily show that for $|t|<1$

$$f_{curv}(t) = \prod_{k\in\mathcal{C}} (1-t^k)^{-1} \tag{192}$$

where $k\in\mathcal{C}$ is a shorthand notation for denoting a partition of $k=(k_1,k_2,\ldots)$ whose parts belong to $\mathcal{C}$, (recall that for large λ distinct partitions generically correspond to distinct triangulations), and where

$$\prod_{k\in\mathcal{C}} (1-t^k)^{-1} \doteq (1-t^{k_1})^{-1}\times(1-t^{k_2})^{-1}\times(1-t^{k_3})^{-1}\times\ldots \tag{193}$$

If we denote the partition $k \in \mathcal{C}$ in the form

$$k = (1^{a_1} 2^{a_2} 3^{a_3} \ldots) \tag{194}$$

where exactly $a_i$of the $\{k_j\}$ are equal to i, we can eventually rewrite $f_{curv}(t)$ as the unconstrained product

$$f_{curv}(t) = \prod_{j=0}^{\infty} (1 - t^j)^{-a_j}. \tag{195}$$

At this stage, we can exploit a general result due to G. Meinardus [37] providing the structural asymptotics of partition functions which are associated with infinite product generating functions. Such a technique is neatly described in Andrews' monography, and we shall follow closely his exposition, (see [37], Chap.VI).

Let us consider the Dirichlet series associated with the set of multiplicities $\{a_j\}$ associated with a curvature assignement partition $k \in \mathcal{C}$, *viz.*,

$$D(s) = \sum_{j=1}^{\infty} \frac{a_j}{j^s} \tag{196}$$

with $s \doteq u + iv$, and let $\alpha \in \mathbb{R}^+$ be its abscissa of convergence, (*i.e.*, (196) is assumed to converge for $s > \alpha$). Let us further assume that $D(s)$ admits an analytic continuation in the strip $u \geq -C_0$ with $0 < C_0 < 1$, and that in this region $D(s)$ is analytic with the exception of a pole of order one located at $s = \alpha$ with residue $Res[D(s)]_{s=\alpha} = Res(\alpha)$. Under a number of further technical assumptions which are not relevant for our purposes, we get the following result which is a straightforward application of Meinardus' theorem:

Lemma 10 *As $k \to \infty$, the leading asymptotics of $p^{curv}(k)$ is given by*

$$p^{curv}(k) \simeq C k^{\eta} \exp\left\{ k^{\alpha/(\alpha+1)} (1 + \frac{1}{\alpha}) [Res(\alpha) \Gamma(\alpha+1) \zeta(\alpha+1)]^{1/(\alpha+1)} \right\} (1 + \ldots) \tag{197}$$

where C is a suitable constant, (see e.g., Theorem 6.2 in [37]), $(1 + \ldots)$ stands for higher order terms, $\Gamma(\alpha+1)$, $\zeta(\alpha+1)$ respectively denote the Euler gamma function and the Riemann zeta function, and finally

$$\eta \doteq \frac{D(0) - 1 - \frac{1}{2}\alpha}{1+\alpha} \tag{198}$$

In our case, k stands for $[nQ(0) - b + \hat{q}](\lambda + 1) - n(n-1)Q(0)$ and for $nQ(0) = (b - \hat{q})\lambda$, we get from (197)

$$p^{curv} \simeq C[(b-\hat{q})\lambda^2]^{\eta} \exp\left\{[(b-\hat{q})\lambda^2]^{\alpha/(\alpha+1)}(1+\frac{1}{\alpha})[Res(\alpha)\Gamma(\alpha+1)\zeta(\alpha+1)]^{1/(\alpha+1)}\right\}(1+\ldots) \tag{199}$$

This asymptotics is consistent with the bound (188) only if the abscissa of convergence of the series $D(s)$ is $\alpha = 1$, and if its residue $Res(\alpha)$ and the value $D(0)$ are bounded according to

$$Res(\alpha) \leq \sqrt{\frac{6}{b-\hat{q}}}\frac{2+\ln(b-\hat{q})}{2\pi} \tag{200}$$

$$D(0) \leq -\frac{1}{2} \tag{201}$$

C. The counting principle

The bound (188) and the asymptotics of lemma 10 establish that p_{λ}^{curv} has an exponential growth with λ, and that the coefficients a_n characterizing the Dirichlet series $D(s)$, (and the partitions in curvature assignements), have to satisfy some growth condition (related to the above bounds). Clearly, more geometric information would be necessary for getting a complete control of p_{λ}^{curv}, however what really matters for our purposes is the observation stemming from the above analysis that the ratio

$$\frac{p_{\lambda}^{curv}}{p_{\lambda}} \doteq c_n \tag{202}$$

where $0 \leq c_n \leq 1$, either is a constant or an exponentially decreasing function of λ, and $b(n, n-2)$ *viz.*, $c_n = \exp[(\alpha_1 - \alpha_2 b)\lambda + \alpha_0]$ for some suitable constants α_0, α_1, α_2. This functional dependence of c_n on λ can be conveniently recast in the form

$$\frac{p_\lambda^{curv}}{p_\lambda} \doteq c_n = C_0 e^{\alpha_n N_n(T_a)} \tag{203}$$

where the constants $0 < C_0 \leq 1$, and $\alpha_n \leq 0$ depend only on the dimension n.

Since c_n has exactly the structure of the exponential of the *volume part* of the dynamical triangulation action, and since p_λ^{curv} generically enumerates distinct dynamical triangulations, we can as well as use for such a purpose the much more manageable counting function p_λ provided that in the final results we allow for a *renormalization* of the coupling k_n, according to

$$k_n \rightarrow k_n + \delta_n \tag{204}$$

with $\delta_n \leq 0$, a constant depending only on the dimension n.

In other words, by using p_λ as an enumerator, we are using a discrete measure on a space (the space of distinct partitions of the integer $[nQ(0) - b + \hat{q}](\lambda + 1) - n(n-1)Q(0))$ which contains the relevant space of curvature assignements, but which is definitely larger. The discrete measure associated with p_λ induces a corresponding measure, p_λ^{curv} on the space of distinct dynamical triangulations, and the two measure scales according to (203), (which appears as a the discrete Radon-Nykodim derivative of the measure p_λ^{curv} with respect p_λ). The nature of this scaling allows us to use directly the *overcounting* measure p_λ since in seeking the infinite volume limit and a possible scaling limit for the partition function (96) we have to vary both couplings k_n, k_{n-2}. Roughly speaking, in using directly p_λ in place of p_λ^{curv} it is more or less like including tadpoles and self-energies in the triangulation counting. The shift (203) and (204) allows us to remove, in the final results, the effect of such inclusions. We formalize the results of the lemmas 8, 9, 10 and of the associated remarks, in the following lemma providing the rationale of our *counting strategy*,

Lemma 11 *There exists constants, $0 < C_0 \leq 1$, $\alpha_n \leq 0$, and $\tau(n) \geq 0$ depending only on* *the dimension n, such that the number of distinct curvature assignements of rooted dynamical triangulations $T_a \in \mathcal{D}T_n(a,b,\lambda)$ is enumerated, up to the scaling*

$$\frac{Card\{T_a^{(i)}\}}{p_\lambda} \doteq c_n = C_0 N_n^{\tau(n)} e^{\alpha_n N_n(T_a)}, \tag{205}$$

by the set of distinct partitions p_λ of the integer $[nQ(0) - b + \hat{q}](\lambda+1) - n(n-1)Q(0)$ into at most $\lambda + 1 - n$ parts, (each $\leq (b-\hat{q})(\lambda+1)$).

D. A geometrical remark

According to the above results, curvature assignements *generically* characterize dynamical triangulations. In the two-dimensional case this can be related to the fact that curvature assignements provide the rotation matrices defining the holonomy, and thus the triangulation. Let us verify that geometrically also for $n = 3$, and $n = 4$ curvature assignements provide most of the information characterizing the holonomy for dynamical triangulations. We work out in details the three-dimensional case by using an $SU(2)$ connection. The case for dimension $n = 4$ can be easily discussed along the same lines by exploiting the isomorphism $SO(4) \approx \frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$. This analysis is due to J. Lewandowsky, to whom we wish to express special gratitude.

Consider the marked simplex σ_0^n, (in this case a tetrahedron), and fix an orthonormal basis $\{\vec{e}_{x,y,z}\}$ in σ_0^n. Among the six bones (edges) in the boundary of σ_0^n, we can choose $\frac{1}{2}n(n-1) = 3$ bones, τ_i, $i = 0,1,2$, one of which is the marked one, τ_0, and all sharing a common vertex, (recall the need of marking n bones in interpreting (169) in terms of partition of integers!). These bones are defined by the vectors $\vec{E}(\tau_i) \in \mathbb{R}^3$, (with components referred to the fixed orthonormal basis $\{\vec{e}_{x,y,z}\}$). Set

$$\vec{\tau}_i = \sigma_x \frac{E_x(\tau_i)}{|\vec{E}(\tau_i)|} + \sigma_y \frac{E_y(\tau_i)}{|\vec{E}(\tau_i)|} + \sigma_z \frac{E_z(\tau_i)}{|\vec{E}(\tau_i)|} \tag{206}$$

with σ_x, σ_y, σ_z denoting the Pauli matrices. The holonomy matrix describing the 2-dimensional plane rotations generated by winding around the bones τ_i can be written as the $SU(2)$ matrix

$$U(\tau_i) = I\cos(\phi(i)/2) + \vec{\tau}_i \sin(\phi(i)/2) \tag{207}$$

where I is the identity operator, and $\phi(i) = q(i)\cos^{-1}\frac{1}{3}$ denotes the rotation angle around the bone τ_i. In general, we can reconstruct the holonomy matrix $U(\tau_i)$ for every distinguished bone τ_i in the marked base simplex σ_0^n, by giving the traces of $U(\tau_i)$, and the traces of $U(\tau_i)U(\tau_j)$, $i \neq j$. The traces $tr[U(\tau_i)]$ are proportional to the incidence numbers $q(i)$, $i = 0, 1, 2$, while $tr[U(\tau_i)U(\tau_j)]$ are proportional to the scalar product in $SU(2)$ between the spinors $\vec{\tau}_i$ and $\vec{\tau}_j$, respectively associated to the bones τ_i, τ_j. These latter data are actually already known, since the base simplex is equilateral. Thus the $SU(2)$-holonomies around the base simplex σ_0^n are trivially determined once we give the corresponding curvature assignements. The reconstruction of the $SU(2)$- holonomies around the remaining $\lambda + 1 - \frac{1}{2}n(n+1) = \lambda - 5$ bones $\tau_\alpha = \{\sigma_\alpha^{n-2}, 0 < \alpha < \lambda, \sigma_\alpha^{n-2} \notin \sigma_0^n\}$, is similar. Formally, for such a reconstruction, we would need the assignement of the traces $\chi(\alpha) = tr[U(\tau_\alpha)]$, and of $\chi(\alpha, j) = tr[U(\tau_\alpha)U(\tau_j)]$, with $j = 0, 1, 2$. Indeed, with the characters $\chi(\alpha) = tr[U(\tau_\alpha)] = 2\cos\frac{\phi(\alpha)}{2}$ we can reconstruct the rotation angles $\phi(\alpha)$, while with the characters $\chi(\alpha, j) = tr[U(\tau_\alpha)U(\tau_j)]$, *viz.*,

$$\chi(\alpha, j) = 2\left(\cos\frac{\phi(\alpha)}{2}\cos\frac{\phi(j)}{2} - < \vec{\tau}_\alpha, \vec{\tau}_j > \sin\frac{\phi(\alpha)}{2}\sin\frac{\phi(j)}{2}\right) \tag{208}$$

we can reconstruct the $SU(2)$ inner products

$$< \vec{\tau}_\alpha, \vec{\tau}_j > \doteq -\frac{1}{2}Tr[\vec{\tau}_\alpha \cdot \vec{\tau}_j] \tag{209}$$

and hence obtain the spinors $\vec{\tau}_\alpha$ through which the $SU(2)$-holonomies $U(\tau_\alpha) = I\cos(\phi(\alpha)/2) + \vec{\tau}_\alpha \sin(\phi(\alpha)/2)$ are defined.

The characters $\chi(a) = tr[U(\tau_a)]$ with $a = 0, 1, \ldots, \lambda$ are given (up to a constant factor) by the incidence numbers $\{q(a)\}_{a=0}^{\lambda}$, and thus are known. Since the simplices in the triangulations are all equilateral, the bones $\vec{\tau}_j$ are not arbitrarily distributed with respect to the reference bones $\vec{\tau}_\alpha$, but are in correspondance with the 12 points on the unit sphere

$\mathbb{S}^2$ marked by the vertices of the inscribed Icosahedron, (arrange the icosahedron in space so that the uppermost vertex coincides with $(0,0,1)$). Hence, the $SU(2)$ inner products $< \vec{\tau}_\alpha, \vec{\tau}_j >$ are known, and the characters $\chi(\alpha, j) = tr[U(\tau_\alpha)U(\tau_j)]$ are given in terms of the $\{q(a)\}_{a=0}^{\lambda}$. It follows that the curvature assignements $\{q(a)\}_{a=0}^{\lambda}$ provide *grosso modo* the $SU(2)$ holonomy matrices, (up to conjugation), what is missing at this detailed level, (which is independent from any asymptotics), is the combinatorial data of how to put together the local holonomies so as to reconstruct the underlying dynamical triangulation T_a.

VIII. ENTROPY ESTIMATES

According to the results of the previous section, dynamical triangulations are *generically* characterized by the corresponding curvature assignements $\{q(\alpha)\}$, and the simple counting of the possible curvature assignements provides a very accurate estimate of the number of distinct dynamical triangulations a PL manifolds (with given fundamental group) can support. The previous results have already evidentiated the close connection between the theory of partitions of integers and the curvature assignements, our strategy is to further exploit this connection in order to estimate the number of distinct dynamical triangulations.

A. The generating functions for the enumeration of dynamical triangulation

Since the partitions of $[nQ(0)-b(n,n-2)+\hat{q}](\lambda+1)-n(n-1)Q(0)$ into at most $\lambda+1-n$ parts, (each $\leq (b(n,n-2)-\hat{q})(\lambda+1)$), can be thought of as enumerating, in the large λ limit, distinct curvature assignements, theorem 7 implies that they also enumerate distinct rooted dynamical triangulations. This observation establishes the following

Theorem 12 *Let $\Theta\colon \pi_1(M;p_0) \rightarrow G$, be a given holonomy representation of an n-dimensional PL-manifold M, $n \geq 2$. The number $W(\Theta,\lambda,b,Q(0))$, of distinct rooted dynamical triangulations on M, with $\lambda+1$ bones and with a given average number, $b \doteq b(n,n-2)$, of n-simplices incident on a bone, is given, in the large λ limit, by*

$$W(\Theta,\lambda,b,Q(0)) = c_n \cdot p_\lambda([nQ(0)-b+\hat{q}](\lambda+1)-n(n-1)Q(0)) \tag{210}$$

where c_n is the rescaling factor of lemma 11.

Remark. Note that selecting the value $Q(0)$ of the curvature at the n marked bones is equivalent to considering dynamical triangulations with a boundary, (the links of the marked bone), of variable volume, (the length being proportional to the chosen $Q(0)$). Thus we have the

Lemma 12 *The function* $W(\Theta, \lambda, b, Q(0)) = c_n \cdot p_\lambda([nQ(0) - b + \hat{q}](\lambda + 1) - n(n-1)Q(0))$ *provides the number of distinct dynamical triangulations with boundary* ∂T_a *consisting of the disjoint union of* $n =$ *spheres of dimension* $n-1$, $\mathbb{S}^{n-1}$, *each one of volume* $Vol[\partial T_a] = q(0)vol[\sigma^{n-1}]$.

Proof. This trivially follows from the above theorem by noticing that by removing from T_a the open star of the marked bones, (on each of which $q(o)$ simplices $\{\sigma^n\}$ are incident), we get dynamical triangulations with a boundary $\partial T_a \simeq \coprod \mathbb{S}^{n-1}$ of volume $Vol[\partial T_a] = nq(0)vol[\sigma^{n-1}]$. $\Box$

B. Gauss polynomials and dynamical triangulations

By exploiting the properties of the partitions $p_\lambda([nQ(0) - b + \hat{q}](\lambda+1) - n(n-1)Q(0))$ we can actually characterize the generating function for the number of distinct rooted dynamical triangulations.

Theorem 13 *Let* $0 \leq t \leq 1$ *be a generic indeterminate, and let* $p_\lambda(h)$ *denote the number of partitions of the generic integer* h *into (at most)* $\lambda + 1 - n$ *parts, each* $\leq (b(n, n-2) - \hat{q})(\lambda + 1)$. *In a given holonomy represention* $\Theta: \pi_1(M; \sigma_0^n) \to G$, *and for a given value of the parameter* $b = b(n, n-2)$, *the generating function for the number of distinct rooted dynamical triangulations with* $N_{n-2}(T_a^{(i)}) = \lambda + 1$ *bones and given number*

$$q(0) = \frac{h + (b - \hat{q})(\lambda + 1)}{n(\lambda + 2 - n)} + \hat{q} \tag{211}$$

of n*-dimensional simplices incident on the* n *marked bones is given by*

$$\mathcal{G}[W(\Theta, \lambda, b; t)] = c_n \cdot \sum_{h \geq 0} p_\lambda(h) t^h = c_n \cdot \begin{bmatrix} (b(n, n-2) - \hat{q})(\lambda + 1) + (\lambda + 1) - n \\ (\lambda + 1) - n \end{bmatrix} \tag{212}$$

where $0 < c_n \leq 1$ *is an exponential rescaling factor, (see lemma 11), and*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1 - t^n)(1 - t^{n-1}) \ldots (1 - t^{m+1})}{(1 - t^{n-m})(1 - t^{n-m-1}) \ldots (1 - t)} \doteq \frac{(t)_n}{(t)_{n-m}(t)_m} \tag{213}$$

is the Gauss polynomial in the variable t.

Note that the Gauss polynomial $\frac{(t)_n}{(t)_{n-m}(t)_m}$ is a polynomial in t of degree $(n-m)m$, and that

$$\lim_{t\to 1}\begin{bmatrix} n \\ \\ m \end{bmatrix} = \frac{n!}{m!(n-m)!} = \begin{pmatrix} n \\ \\ m \end{pmatrix} \tag{214}$$

(see [37] for an extensive review of the properties of Gaussian polynomials).

Proof. The proof is a straightforward consequence of the fact [37] that the Gauss polynomials are the generating function of the partitions $p_\lambda(h)$. $\square$.

From these observations it follows that the generating function for dynamical triangulations with (spherical) boundaries is provided, up to an exponential rescaling, (see lemma 11), by a Gauss polynomial of degree $(b-\hat{q})(\lambda+1)(\lambda+1-n)$, and we expect that their properties bear relevance to simplicial quantum gravity. It is interesting to remark that these polynomials already play a basic role in the celebrated solution of the *Hard Hexagon Model* by R.J. Baxter [41]. A first important property, related to the theory of Gaussian polynomials, that we can exploit for understanding the geometry of dynamical triangulations is related to the *unimodality* of the Gaussian polynomials. Recall that a polynomial $P(t) = a_0 + a_1 t + \ldots + a_n t^n$ is said to be unimodal [37] if there exists m such that $a_0 \le a_1 \le a_2 \le \ldots \le a_m \ge a_{m+1} \ge a_{m+2} \ge \ldots \ge a_n$. A simple applications of the unimodal properties to our case yields the following

Theorem 14 *For all $(b(n,n-2)-\hat{q})(\lambda+1)$, λ, $h \ge 0$*

$$W(\Theta,\lambda,b,h) = W(\Theta,(b-\hat{q})(\lambda+1),b,h) \tag{215}$$

where $W(\Theta,(b-\hat{q})(\lambda+1),b,h) = c_n \cdot p_{(b-\hat{q})(\lambda+1)}(h)$, with $p_{(b-\hat{q})(\lambda+1)}(h)$ denoting the number of partitions of the integer h into at most $(b-\hat{q})(\lambda+1)$ parts, each $\le \lambda+1-n$. Moreover

$$W(\Theta,\lambda,b,h) = W(\Theta,\lambda,b,(b-\hat{q})(\lambda+1)(\lambda+1-n)-h) \tag{216}$$

and, for $0 < h \le \frac{(b-\hat{q})(\lambda+1)(\lambda+1-n)}{2}$,

$$W(\Theta,\lambda,b,h)-W(\Theta,\lambda,b,h-1)\geq 0 \tag{217}$$

Proof. The proof is obtained by the known proof [37] of unimodality for $p_\lambda(h)$, by replacing $p_\lambda(h)$ with $W(\Theta,\lambda,b,h)=c\cdot p_\lambda(h)$. $\Box$

Since $W(\Theta,\lambda,b,h)$ enumerates the distinct rooted dynamical triangulations with a marked curvature assignement related to the value of the parameter h by

$$h=[nQ(0)-(b-\hat{q})](\lambda+1)-n(n-1)Q(0) \tag{218}$$

the above unimodality properties tells us that *grosso modo* the largest entropic contribution to the dynamical triangulation generating function comes from triangulations such that

$$q(0)\simeq\frac{(b-\hat{q})(\lambda+1)}{2n} \tag{219}$$

(this bound being attained on the n marked bones).

C. Asymptotics and Entropy estimates

If in the generating function $\mathcal{G}[W(\Theta,\lambda,b;t)]$ we let $t\to 1$ we get the enumerator of the distinct dynamical triangulations, for a given representation of the fundamental group. Roughly speaking, for $t\to 1$, $\mathcal{G}[W(\Theta,\lambda,b;t)]$ reduces to the sum over all possible curvature assignements $q(0)$ in the set of distinct dynamical triangulations in the given holonomy representation. We have the following

Theorem 15 *In a given represention $\Theta\colon\pi_1(M;\sigma_0^n)\to G$, and for a given value of the parameter $b=b(n,n-2)$, the number $W(\Theta,\lambda,b)$ of distinct dynamical triangulations with $N_{n-2}(T_a^{(i)})=\lambda+1$ bones, n of which are marked, is given by*

$$W(\Theta,\lambda,b)=c_n\cdot\begin{pmatrix}(b(n,n-2)-\hat{q})(\lambda+1)+(\lambda+1)-n\\(\lambda+1)-n\end{pmatrix} \tag{220}$$

Proof. The sum $\lim_{t\to 1}\mathcal{G}[W(\Theta,\lambda,b;t)]$ involving the partions $p_\lambda(h)$ can be evaluated by exploiting the property (214) of the Gaussian polynomials, *viz.*,

$$\lim_{t\to 1}\mathcal{G}[W(\Theta,\lambda,b;t)] = c_n\cdot\sum_{h\geq 0}^{(\lambda+1-n)(\lambda+1)(b-\hat{q})} p_\lambda(h)$$

$$= c_n\cdot\begin{pmatrix}(b(n,n-2)-\hat{q})(\lambda+1)+(\lambda+1)-n\\(\lambda+1)-n\end{pmatrix} \tag{221}$$

establishing the desired result. $\Box$.

On applying Stirling's formula, we easily get that the above result yields, for large λ, the asymptotics

$$W(\Theta;\lambda;b)\simeq\frac{c_n}{\sqrt{2\pi}}\sqrt{\frac{(b-\hat{q}+1)^{1-2n}}{b-\hat{q}}}\left[\frac{(b-\hat{q}+1)^{b-\hat{q}+1}}{(b-\hat{q})^{b-\hat{q}}}\right]^{\lambda+1}\lambda^{-\frac{1}{2}}\left(1+O(\lambda^{-\frac{3}{2}})\right) \tag{222}$$

In order to obtain from (222)the asymptotics of the number of distinct dynamical triangulations T_a, with $\lambda+1$ bones, we have to factor out the particular rooting we exploited for our enumerative purposes. We have first of all to factor out the marking of the n bones $\sigma^{n-2}(0),\sigma^{n-2}(1),\ldots,\sigma^{n-2}(n-1)$, and the marking of the base simplex σ_0^n. To factor out the marking of the n bones, we have to divide (222) by λ^n. There can be at most $[\frac{(b(n,n-2)-\hat{q})}{n}](\lambda+1)$ simplices σ^n sharing one of the marked bones $\{\sigma^{n-2}(i)\}_{i=0,\ldots,n-1}$, (actually the actual count is $[\frac{(b(n,n-2)-\tilde{q})}{n}](\lambda+1)$ where $\tilde{q}$ can be larger than $\hat{q}$; we use $\hat{q}$ for simplicity). Since the marked simplex σ_0^n can be incident on any of the n marked bones, we still have to divide by n. For $\lambda>>1$, we get in this way a normalization factor $[b(n,n-2)-\hat{q}]\lambda^{n+1}$.

We have also marked an orthogonal representation Θ_0 of $\pi_1(M;p_0)$ corresponding to the marked base simplex σ_0^n, and as discussed in section IV, besides the above purely combinatorial factors, we have to divide (222) also by the volume of the moduli space, $Vol[\mathcal{M}(T_a)]$, associated with the possible non-trivial deformations of the class of dynamical triangulations considered. According to theorem 11 this volume is provided by

$$Vol[\mathcal{M}(T_a)]\simeq A(\mathcal{V})\left(4\pi\mathcal{V}vol(\sigma^n)\frac{b}{n(n+1)}\lambda\right)^{-D/2} \tag{223}$$

where $\mathcal{V}\doteq Vol[\frac{Hom(\pi_1(M),G)}{G}]$ and $D\doteq dim[Hom(\pi_1(M),G)]$. Hence, in order to get an asymptotic estimate for the number of distinct dynamical triangulations, we have to divide (222) by

$$A(\mathcal{V})[b(n,n-2)-\hat{q}]\lambda^{n+1}\left(4\pi\mathcal{V}vol(\sigma^n)\frac{b}{n(n+1)}\lambda\right)^{-D/2} \tag{224}$$

The Entropy Function. These observations establish the following

Theorem 16 *The number of distinct dynamical triangulations, with $\lambda+1$ bones, and with an average number, $b\equiv b(n,n-2)$, of n-simplices incident on a bone, on an n-dimensional, $(n\leq 4)$, PL- manifold M of given fundamental group $\pi_1(M)$, can be asymptotically estimated according to*

$$W(\lambda,b)\simeq$$

$$W_{\pi}\cdot\frac{c_n}{\sqrt{2\pi}}\sqrt{\frac{(b-\hat{q}+1)^{1-2n}}{(b-\hat{q})^3}}\left[\frac{(b-\hat{q}+1)^{b-\hat{q}+1}}{(b-\hat{q})^{b-\hat{q}}}\right]^{\lambda+1}\left(\frac{b(n,n-2)}{n(n+1)}\lambda\right)^{D/2}\lambda^{-\frac{2n+3}{2}} \tag{225}$$

where $0<c_n\leq 1$ is the exponential rescaling factor of lemma 11, and where we have introduced the topology dependent parameter

$$W_{\pi}\doteq A^{-1}(\mathcal{V})[4\pi\mathcal{V}vol(\sigma^n)]^{D/2} \tag{226}$$

$\square$.

As we shall see momentarily, the estimate (225) fits extremely well with the data coming from numerical simulations in dimension $n\geq 3$, and in dimension $n=2$, (where $b(n,n-2)=6$), it is in remarkable agreement with the known analytical estimates [8]. Notice in particular that for a two-dimensional sphere the integration over the representation variety drops, and (225) yields

$$W(\lambda;S^2)\simeq\frac{c_2}{24\sqrt{6\pi}}\left[\frac{4^4}{3^3}\right]^{v}v^{-\frac{7}{2}} \tag{227}$$

where we have denoted by $v = \lambda + 1$ the number of vertices of the triangulation. This estimate should be compared with the known result provided long ago by Tutte [8], *viz.*,

$$\frac{1}{64\sqrt{6\pi}}\left[\frac{4^4}{3^3}\right]^v v^{-\frac{7}{2}} \tag{228}$$

Thus, if the exponential rescaling $0 < c_2 \leq 1$, characterized by lemma 11, just reduces to a constant rescaling ($c_2 \simeq 24/64$), the agreement between the two asymptotic analysis would be excellent.

D. The 2-dimensional case

If we exploit the compact formula (164) of lemma 6, providing the asymptotics of the volume of Römer-Zähringer deformations for surfaces, then the agreement between our analysis and the known results in dimension $n = 2$ is complete. For a surface Σ_h of genus h, and for representations of $\pi_1(\Sigma_h)$ in a general semi-simple Lie group G, the properties of the representation variety [26] exploited in lemma 6 imply that (225) takes the form

$$W(\lambda)|_{n=2} \simeq W_\pi \cdot \frac{c_2}{24\sqrt{6\pi}}\left[\frac{4^4}{3^3}\right]^{\lambda+1} \lambda^{-1-\frac{5}{4}\chi(\Sigma_h)}(1+\ldots) \tag{229}$$

where $\chi(\Sigma_h)$ is the Euler characteristic of the surface Σ_h, and ... stands for terms of higher order.

If we rewrite (229) as

$$W_\pi \cdot (\frac{4^4}{3^3})^\lambda \lambda^{\gamma_s - 3}(1+\ldots) \tag{230}$$

where γ_s is the *critical exponent*, then we get

$$\gamma_s(M) = 2 - \frac{5}{4}\chi(M) \tag{231}$$

The exponent $\gamma_s(M)$ defines the universal critical exponent of pure $2D$-gravity. The expression (231) coincides with the well-known expression obtained by the use of Liouville theory.

E. The $n \geq 3$-dimensional case

In order to check our analytical results against the MonteCarlo simulations which up to now are the only available results in dimension $n \geq 3$, let us rewrite the partition function for dynamical triangulations (5) explicitly in terms of the entropy function $W(\lambda, b)$, as

$$Z[k_{n-2}, k_n] = \sum_N \left(\sum_{\lambda_{min}}^{\lambda_{max}} W[\lambda, N = \frac{2b(\lambda+1]}{n(n+1)}) e^{k_{n-2}\lambda} \right) e^{-k_n N_n} \tag{232}$$

where a sum over all possible λ appears. This sum is needed since triangulations with distinct values of λ contribute to the number of distinct triangulations with a given volume N. Notice that in dimension $n = 2$ this further summation is absent since, in that case, the average value of 2-simplices incident on a bone (vertex) is always $b = 6$.

The presence of this summation over λ shows that in dimension greater than 2, what entropically characterizes the infinite volume limit of a dynamically triangulated model as a statistical system is not just the entropy function $W(\lambda, b)$, (which can be identified with the entropy in the micro-canonical ensamble) but rather the **effective entropy** provided by the canonical partition function

$$W(N, k_{n-2})_{eff} \equiv \sum_{\lambda_{min}}^{\lambda_{max}} W(\lambda, N) e^{k_{n-2}\lambda} \tag{233}$$

In order to evaluate the canonical partition function and compare the results with the know numerical simulations we can limit ourselves to considering dynamical triangulations of simply connected manifolds, since numerical data are available just for the 3-sphere $\mathbb{S}^3$ and the 4-sphere $\mathbb{S}^4$. Already in this case computations are far from being trivial, and much care is needed in handling the asymptotic estimation of the sum (233).

According to proposition 15, the entropy for simply-connected PL-manifolds is

$$W(\lambda, b) = \frac{c_n}{[(b-\hat{q})(\lambda+1)^{n+1}]} \begin{pmatrix} (b(n, n-2) - \hat{q})(\lambda+1) + (\lambda+1) - n \\ (\lambda+1) - n \end{pmatrix} \tag{234}$$

where the factor $[(b-\hat{q})(\lambda+1)^{n+1}]$ comes from removing the labellings associated with the rootings, (see the discussion following theorem 15). For discussing the asymptotics of (234), we need Stirling's formula in the form $k! \simeq \sqrt{2\pi}(k+1)^{k+\frac{1}{2}}e^{-k-1}$, which is accurate also for small k.

With these caveats along the way, and setting explicitly $\hat{q}=3$, a long but elementary computation provides the asymptotics

$$W(\lambda, N) \simeq \frac{c_n}{\sqrt{2\pi}} \frac{\lambda^{-1/2}(\hat{N}-2\lambda+1)^{-n}}{\hat{N}\sqrt{(\hat{N}-3\lambda+1)(\hat{N}-2\lambda+1)}} e^{h(\hat{N},\lambda)} \tag{235}$$

where $\hat{N} \doteq \frac{1}{2}n(n+1)N$, and where we have set

$$h(\hat{N},\lambda) \doteq -\lambda\log\lambda + \log\frac{[\hat{N}-2\lambda+1]^{(\hat{N}-2\lambda+1)}}{[\hat{N}-3\lambda+1]^{(\hat{N}-3\lambda+1)}} \tag{236}$$

Thus, (233) reduces to

$$W(N, k_{n-2})_{eff} = \frac{c_n}{\sqrt{2\pi}} \sum_{\lambda_{min}}^{\lambda_{max}} \frac{\lambda^{-1/2}(\hat{N}-2\lambda+1)^{-n}}{\hat{N}\sqrt{(\hat{N}-3\lambda+1)(\hat{N}-2\lambda+1)}} e^{h(\hat{N},\lambda)+k_{n-2}\lambda} \tag{237}$$

where for $n=3$

$$\frac{\hat{N}}{6} = \lambda_{min} \le \lambda \le \lambda_{max} = \frac{2}{9}\hat{N} \tag{238}$$

and for $n=4$

$$\frac{\hat{N}}{5} = \lambda_{min} \le \lambda \le \lambda_{max} = \frac{\hat{N}}{4} \tag{239}$$

We can estimate the sum (237) by noticing that the function appearing in the exponent, *viz.*,

$$\begin{aligned} &f(\hat{N},\lambda) \doteq h(\hat{N},\lambda) + k_{n-2}\lambda = \\ &-\lambda\log\lambda + (\hat{N}-2\lambda+1)\log(\hat{N}-2\lambda+1) - (\hat{N}-3\lambda+1)\log(\hat{N}-3\lambda+1) + k_{n-2}\lambda \end{aligned} \tag{240}$$

has a sharp maximum in correspondence of the solution, $\lambda = \lambda^*(k_{n-2})$, of the equation

$$\log \frac{[\hat{N}-3\lambda+1]^3}{\lambda[\hat{N}-2\lambda+1]^2} = -k_{n-2} \tag{241}$$

A straightforward computation provides

$$\lambda^* = \frac{\hat{N}+1}{3}(1-\frac{1}{A(k_{n-2})}) \tag{242}$$

where for notational convenience we have set

$$A(k_{n-2}) \doteq \left[\frac{27}{2}e^{k_{n-2}}+1+\sqrt{(\frac{27}{2}e^{k_{n-2}}+1)^2-1}\right]^{1/3} + \left[\frac{27}{2}e^{k_{n-2}}+1-\sqrt{(\frac{27}{2}e^{k_{n-2}}+1)^2-1}\right]^{1/3} - 1 \tag{243}$$

The structure of (242) suggests the change of variable

$$\lambda = (\hat{N}+1)\eta \tag{244}$$

(we wish to thank G. Gionti for this remark), and by replacing the sum (237) with an integration, we get

$$W(N,k_{n-2})_{eff} = \frac{c_n}{\sqrt{2\pi}}\hat{N}^{-n-1/2}\int_{\eta_{min}}^{\eta_{max}} d\eta \frac{\eta^{-1/2}(1-2\eta)^{-n}}{\sqrt{(1-3\eta)(1-2\eta)}}e^{(\hat{N}+1)f(\eta)} \tag{245}$$

where

$$f(\eta) \doteq -\eta\log\eta + (1-2\eta)\log(1-2\eta) - (1-3\eta)\log(1-3\eta) + k_{n-2}\eta \tag{246}$$

and $\eta_{min} \doteq \eta(\lambda_{min})$, $\eta_{max} \doteq \eta(\lambda_{max})$. For $n=4$, we get $\eta_{min} \simeq 1/5$, $\eta_{max} \simeq 1/4$, whereas for $n=3$, $\eta_{min} \simeq 1/6$ and $\eta_{max} \simeq 2/9$.

The obvious strategy is to estimate (245) with Laplace method, however attention must be paid to the possibility that, as k_{n-2} varies, the maximum

$$\eta^*(k_{n-2}) = \frac{1}{3}(1-\frac{1}{A(k_{n-2})}) \tag{247}$$

crosses the integration limits η_{min} and η_{max}. It will become clear in a moment that the quantities telling us when we are *nearby* these particular regions are

$$\psi_{min}(k_{n-2}) \doteq -sgn(\eta^* - \eta_{min})\sqrt{2[f(\eta^*) - f(\eta_{min})]} \tag{248}$$

and

$$\psi_{max}(k_{n-2}) \doteq sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]} \tag{249}$$

This suggests that in order to control the large N behavior of the effective entropy as k_{n-2} varies, we have to use uniform Laplace estimation in terms of ψ. The following general theorem provides such uniform asymptotics. We state it first for the case involving the crossing of the lower integration limit η_{min}. A completely analogous result holds, *mutatis mutandis*, for the upper crossing η_{max}, and we state it as an obvious corollary .

Theorem 17 *Let us consider the set of all simply-connected n-dimensional, $(n = 3, 4)$, dynamically triangulated manifolds. Let k_{n-2}^{inf} denote the unique solution of the equation*

$$\frac{1}{3}(1 - \frac{1}{A(k_{n-2})}) = \eta_{min} \tag{250}$$

Let $0 < \epsilon < 1$ small enough, then for all values of the inverse gravitational coupling k_{n-2} such that

$$k_{n-2}^{inf} - \epsilon < k_{n-2} < k_{n-2}^{inf} + \epsilon, \tag{251}$$

the large N-behavior of the canonical partition function $W(N, k_{n-2})_{eff}$ is given by the uniform asymptotics

$$\frac{c_n}{\sqrt{2\pi}}\hat{N}^{-n-1/2}e^{(\hat{N}+1)f(\eta_{min})}\left[\frac{\alpha_0}{\sqrt{\hat{N}}}w_0(\psi_{min}(k_{n-2})\sqrt{\hat{N}}) + \frac{\alpha_1}{\hat{N}}w_{-1}(\psi_{min}(k_{n-2})\sqrt{\hat{N}})\right] \tag{252}$$

where $w_r(z) \doteq \Gamma(1-r)e^{z^2/4}D_{r-1}(z)$, $(r < 1)$, $D_{r-1}(z)$ and $\Gamma(1-r)$ respectively denote the parabolic cylinder functions and the Gamma function, and where the constants α_0 and α_1 are given by

$$\alpha_0 \doteq -\frac{\eta_{min}^{-1/2}(1-2\eta_{min})^{-n}}{\sqrt{(1-3\eta_{min})(1-2\eta_{min})}}\frac{\psi_{min}(k_{n-2})}{f_\eta(\eta_{min})} \tag{253}$$

$$\alpha_1 \doteq \frac{\alpha_0}{\psi_{min}(k_{n-2})} - \frac{\eta *^{-1/2}(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{min}(k_{n-2})\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{254}$$

Notice that the above expression, (in which $f_\eta \doteq df/d\eta$, and $f_{\eta\eta} \doteq d^2f/d\eta^2$), provides the leading asymptotics. The full asymptotics is discussed during the proof of the theorem. Notice also that in some circumstances, (notably in the 3-dimensional case when employing as variables N_0, and N_3), there can be an equally important constribution to $W(N, k_{n-2})_{eff}$ due to the upper crossing η_{max}. In that case $W(N, k_{n-2})_{eff}$ is, at leading order, characterized by the sum of the uniform asymptotics around both $f(\eta_{min})$ and $f(\eta_{max})$.

Proof. We have to provide a uniform asymptotic estimation for large $\hat{N}$ of the integral appearing in (245), *viz.*,

$$I \doteq \int_{\eta_{min}}^{\eta_{max}} d\eta \frac{\eta^{-1/2}(1-2\eta)^{-n}}{\sqrt{(1-3\eta)(1-2\eta)}} e^{(\hat{N}+1)f(\eta)} \tag{255}$$

The uniformity requirement stated here refers to the existence of a $\delta \in \mathbb{R}^+$ such that for all $|\psi(k_{n-2})| \leq \delta$ the error after a finite number of terms in the asymptotic expansion should be smaller in N than the last term which is kept [48]. As discussed above, the integral I has various distinct asymptotic regimes, according to the relative location of the maximum η^* with respect to the lower integration limit η_{min} or to the upper limit η_{max}. We explicitly discuss for simplicity the case where η^* may approach η_{min}. The case in which $\eta^* \to \eta_{max}$, can be dealt with similarly, and we shall indicate the necessary modifications in the proof and state the final result.

In order to carry out the required asymptotic estimation, we first transform the exponential in the integral I by introducing the variable $p = p(\eta)$ according to

$$f(\eta) - f(\eta_{min}) = -\left(\frac{p^2}{2} + \psi_{min}(k_{n-2})p\right) \tag{256}$$

where for $\eta = \eta_{min}$ we assume $p = 0$ and $(d\eta/dp)|_{p=0} > 0$, so that the orientation of the path of integration remains unchanged. Differentiating this expression we get

$$\frac{df}{d\eta} = -(p + \psi_{min})\frac{dp}{d\eta} \tag{257}$$

In order to have $d\eta(p)/dp$ finite and non-zero everywhere we require that $p \to -\psi_{min}$ as $\eta \to \eta^*$. Thus, by L'Hôpital rule

$$\lim_{p \to -\psi} \frac{d\eta}{dp} = \sqrt{-\frac{1}{f_{\eta\eta}(\eta^*)}} \tag{258}$$

To determine the expression of the parameter $\psi_{min}(k_{n-2})$, (see (248)), controlling the uniform expansion, we evaluate $f(\eta) - f(\eta_{min})$ at $\eta = \eta^*$, (*viz.*, for $p = -\psi_{min}$):

$$f(\eta^*) - f(\eta_{min}) = \frac{1}{2}\psi_{min}^2 \tag{259}$$

whence

$$\psi_{min} = \pm\sqrt{2[f(\eta^*) - f(\eta_{min})]} \tag{260}$$

The branch of the square root is selected by examining $\frac{d}{dp}[f(\eta) - f(\eta_{min})]$ at $\eta = \eta_{min}$, (*i.e.*, for $p = 0$). We get

$$\frac{df}{dp}|_{p=0} = \frac{df}{d\eta}|_{p=0} = -\frac{d}{dp}(\frac{p^2}{2} + \psi_{min}p)|_{p=0} = -\psi_{min} \tag{261}$$

Since $(d\eta/dp)|_{p=0} > 0$ by hypothesis, and $(df/d\eta)|_{\eta_{min}} > 0$, for $\eta^* > \eta_{min}$, $((df/d\eta)|_{\eta_{min}} < 0$, for $\eta^* < \eta_{min})$, we get that

$$sgn\psi_{min} = -sgn(\eta^* - \eta_{min}). \tag{262}$$

Thus

$$\psi_{min}(k_{n-2}) = -sgn(\eta^* - \eta_{min})\sqrt{2[f(\eta^*) - f(\eta_{min})]} \tag{263}$$

(note that $\psi_{min}(k_{n-2})$ is a monotonically decreasing function of k_{n-2}).

With these remarks out the way, we can write

$$I = e^{(\hat{N}+1)f(\eta_{min})} \int_0^\infty G(p) e^{-(\hat{N}+1)(\frac{1}{2}p^2+\psi_{min}p)} dp \tag{264}$$

where we have extended the upper limit of integration to ∞, (on introducing a Heaviside function), and where we have set

$$G(p) \doteq \left[\frac{\eta^{-1/2}(1-2\eta)^{-n}}{\sqrt{(1-3\eta)(1-2\eta)}} \right] \frac{d\eta}{dp} \tag{265}$$

If $p = -\psi_{min} \in [0, +\infty)$, then we can expand the function $G(p)$ in a neighborhood of $p = -\psi_{min}$, (corresponding to $\eta = \eta^*$). Otherwise, if $p = -\psi_{min} \notin [0, +\infty)$ we need to expand around $p = 0$, (corresponding to $\eta = \eta_{min}$). According to a standard procedure [48], we can take care of both cases by setting

$$G(p) \doteq \alpha_0 + \alpha_1 p + p(p+\psi) G_1(p) \tag{266}$$

The integral I now becomes

$$\begin{aligned} I = I_0 + I_1 = {} & e^{(\hat{N}+1)f(\eta_{min})} \int_0^\infty (\alpha_0 + \alpha_1 p) e^{-(\hat{N}+1)(\frac{1}{2}p^2+\psi_{min}p)} dp \\ & + e^{(\hat{N}+1)f(\eta_{min})} \int_0^\infty p(p+\psi_{min}) G_1(p) e^{-(\hat{N}+1)(\frac{1}{2}p^2+\psi_{min}p)} dp \end{aligned} \tag{267}$$

Recall that for $r < 1$

$$w_r(z) \doteq \int_0^\infty p^{-r} e^{-(\frac{1}{2}p^2+zp)} dp = \Gamma(1-r) e^{z^2/4} D_{r-1}(z) \tag{268}$$

where $\Gamma(1-r)$ is the Euler Gamma function and $D_{r-1}(z)$ is the parabolic cylinder function [49]. Thus we get

$$I_0 = e^{(\hat{N}+1)f(\eta_{min})} \left[\frac{\alpha_0}{\sqrt{\hat{N}}} w_0(\psi_{min}(k_{n-2})\sqrt{\hat{N}}) + \frac{\alpha_1}{\hat{N}} w_{-1}(\psi_{min}(k_{n-2})\sqrt{\hat{N}}) \right] \tag{269}$$

where the constants α_0 and α_1 are determined by

$$\alpha_0 = G(0) \tag{270}$$

and

$$\alpha_1 = \frac{G(0) - G(-\psi_{min})}{\psi_{min}} \tag{271}$$

Namely:

$$\alpha_0 = -\frac{\eta_{min}^{-1/2}(1-2\eta_{min})^{-n}}{\sqrt{(1-3\eta_{min})(1-2\eta_{min})}} \frac{\psi_{min}(k_{n-2})}{f_\eta(\eta_{min})} \tag{272}$$

$$\alpha_1 \doteq \frac{\alpha_0}{\psi_{min}(k_{n-2})} - \frac{\eta *^{-1/2}\,(1-2\eta)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}} \left[\frac{1}{\psi_{min}(k_{n-2})\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{273}$$

The remaining integral I_1 can be reduced to the same structure of I and an inductive argument [48] shows that the asymptotics of I is provided by

$$I = e^{(\hat{N}+1)f(\eta_{min})} \left\{ \frac{w_0(\psi_{min}(k_{n-2})\sqrt{\hat{N}+1})}{\sqrt{\hat{N}+1}} \left[\sum_{i=0}^{m} \frac{\alpha_{2i}}{(\hat{N}+1)^i} + O\left(\frac{1}{\hat{N}^{m+1}}\right)\right]\right\}$$

$$+e^{(\hat{N}+1)f(\eta_{min})} \left\{ \frac{w_{-1}(\psi_{min}(k_{n-2})\sqrt{\hat{N}+1})}{\hat{N}+1} \left[\sum_{i=0}^{m} \frac{\alpha_{2i+1}}{(\hat{N}+1)^i} + O\left(\frac{1}{\hat{N}^{m+1}}\right)\right]\right\} \tag{274}$$

where the constants α_{2i} and α_{2i+1} are determined recursively from α_0 and α_1 according to

$$\alpha_{2i} = G_i(0) \tag{275}$$

$$\alpha_{2i+1} = \frac{G_i(0) - G_i(-\psi_{min})}{\psi_{min}} + \frac{dG_i}{dp}|_{p=-\psi} \tag{276}$$

and where the functions $G_i(p)$ satisfy the equation

$$G_i(p) + p\frac{dG_i(p)}{dp} = \alpha_{2i} + p\alpha_{2i+1} + p(p+\psi_{min})G_{i+1}(p). \tag{277}$$

Thus, the leading uniform asymptotics is given by

$$I = e^{(\hat{N}+1)f(\eta_{min})}\left\{\frac{w_0(\psi_{min}(k_{n-2})\sqrt{\hat{N}+1})}{\sqrt{\hat{N}+1}}\left[\alpha_0 + O\left(\frac{1}{\hat{N}}\right)\right]\right\}$$

$$+e^{(\hat{N}+1)f(\eta_{min})}\left\{\frac{w_{-1}(\psi_{min}(k_{n-2})\sqrt{\hat{N}+1})}{\hat{N}+1}\left[\alpha_1 + O\left(\frac{1}{\hat{N}}\right)\right]\right\} \tag{278}$$

which provides the stated result.

A similar analysis can be carried out when η^* approaches $\eta_{max} = \frac{1}{4}$. In this case, it is convenient to make the preliminary change of variable $y = 1 - 3\eta$ thus reducing the integral I to

$$I_\infty = 3^n \int_{1-3\eta_{max}}^{1-3\eta_{min}} dy \frac{(1-y)^{-1/2}(1+2y)^{-n}}{\sqrt{y(1+2y)}} e^{(\hat{N}+1)f[\eta(y)]} \tag{279}$$

If we set

$$\psi_{max}(k_{n-2}) = -sgn(y(\eta^*))\sqrt{2[f(\eta^*) - f(\eta_{max})]} \tag{280}$$

and introduce the variable $p = p(y)$ according to

$$f[\eta(y)] - f[\eta(0)] = -\left(\frac{p^2}{2} + \psi_{max}(k_{n-2})p\right) \tag{281}$$

we get again an integral whose asymptotic estimation can be reduced to the integral representation of the parabolic cylinder function and to a set of parameters β_{2i} and β_{2i+1} recursively determined from the function $H(p)$ defined by

$$H(p)p^{-1/2} \doteq \left[\frac{(1-y(p))^{-3/2}}{\sqrt{y(p)(1+2y(p))}}\right]\frac{dy(p)}{dp} \tag{282}$$

Explicitly we obtain

$$I_\infty = e^{(\hat{N}+1)f(\eta_{max})}\left\{\frac{w_0(\psi_{max}(k_{n-2})\sqrt{\hat{N}+1})}{(\hat{N}+1)^{1/2}}\left[\sum_{i=0}^{m}\frac{\xi_{2i}}{(\hat{N}+1)^i} + O\left(\frac{1}{\hat{N}^{m+1}}\right)\right]\right\}$$

$$+e^{(\hat{N}+1)f(\eta_{max})}\left\{\frac{w_{-1}(\psi_{max}(k_{n-2})\sqrt{\hat{N}+1})}{(\hat{N}+1)}\left[\sum_{i=0}^{m}\frac{\xi_{2i+1}}{(\hat{N}+1)^i}+O\left(\frac{1}{\hat{N}^{m+1}}\right)\right]\right\} \tag{283}$$

where

$$\xi_0 = H(0) \tag{284}$$

and

$$\xi_1 = \frac{H(0)-H(-\psi_{max})}{\psi_{max}} \tag{285}$$

The constants ξ_{2i} and ξ_{2i+1} are determined recursively from ξ_0 and ξ_1 according to

$$\xi_{2i} = H_i(0) \tag{286}$$

$$\xi_{2i+1} = \frac{H_i(0)-H_i(-\psi_{max})}{\psi_{max}} + \frac{dH_i}{dp}|_{p=-\psi} \tag{287}$$

and the functions $H_i(p)$ satisfy the equation

$$H_i(p) + p\frac{dH_i(p)}{dp} = \xi_{2i} + p\xi_{2i+1} + p(p+\psi_{max})H_{i+1}(p) \tag{288}$$

From these expressions it is straightforward to obtain the leading $\hat{N}$-asymptotics for I as k_{n-2} is such that η^* crosses η_{max}. $\Box$.

Summing up we have the following

Lemma 13 *Let us consider the set of all simply-connected n-dimensional, ($n = 3,4$), dynamically triangulated manifolds. Let k_{n-2}^{crit} denote the unique solution of the equation*

$$\frac{1}{3}(1-\frac{1}{A(k_{n-2})}) = \eta_{max} \tag{289}$$

Let $0<\epsilon<1$ small enough, then for all values of the inverse gravitational coupling k_{n-2} such that

$$k_{n-2}^{crit}-\epsilon < k_{n-2} < +\infty, \tag{290}$$

the large N-behavior of the canonical partition function $W(N,k_{n-2})_{eff}$ is given by the uniform asymptotics

$$\frac{c_n}{\sqrt{2\pi}}\hat{N}^{-n-1/2}e^{(\hat{N}+1)f(\eta_{max})}\left[\frac{\xi_0}{\sqrt{\hat{N}}}w_0(\psi_{max}(k_{n-2})\sqrt{\hat{N}})+\frac{\xi_1}{\hat{N}}w_{-1}(\psi_{max}(k_{n-2})\sqrt{\hat{N}})\right] \qquad (291)$$

The notation k_{n-2}^{crit} characterizing the value of k_{n-2} for which we have the upper limit crossing, (as compared with the notation k_{n-2}^{inf} appearing in theorem 17) is justified by the observation, discussed in the next paragraph, that corresponding to this values of k_{n-2} we have the phase transitions observed in numerical simulations of simplicial quantum gravity.

F. Distinct asymptotic regimes

As k_{n-2} varies around k_{n-2}^{inf} and k_{n-2}^{crit}, the two parameters $\psi_{min}(k_{n-2})$ and $\psi_{max}(k_{n-2})$ vary monotonically from positive values to negative values. This monotonicity allows us to describe in details the asymptotic behavior of I in terms of k_{n-2}. Strictly speaking we should consider separately both a neighborhood of k_{n-2}^{inf} as well as a neighborhood of k_{n-2}^{crit}. However, corresponding to the former, available numerical simulations are so polluted by finite size effects that the final confrontation between the analytical results and the numerical results would be quite unreliable. Thus we confine ourself to a detailed analysis of what happens nearby k_{n-2}^{crit}, this corresponds to the value of the inverse gravitational couplings probed, reliably, by the MonteCarlo simulations. In any case, it is trivial to extend the conclusions we reach to k_{n-2}^{inf}. Thus, limiting ourself to a neighborhood of k_{n-2}^{crit} we can distinguish the following cases:

(i) **Strong coupling**: $k_{n-2} < k_{n-2}^{crit} - \epsilon$, where ϵ is implicitly characterized by the condition $2\psi_{max}^2(k_{n-2}^{crit})\hat{N} \simeq 1$.
In dimension $n = 4$, this implies

$$k_2 < 1.387 - \epsilon \qquad (292)$$

(notice, that 1.387, is a theoretical bound-see the discussion in the next paragraph). Since k_{n-2} is proportional to the inverse gravitational coupling, this asymptotic behavior describes a strong coupling regime for simplicial quantum gravity. For such a range of values of k_{n-2}, $\eta^* < \eta_{max}$ and we get

$$\psi_{max}(k_{n-2}) = sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]} = \sqrt{2[f(\eta^*) - f(\eta_{max})]} < 0 \tag{293}$$

In such a case the appropriate asymptotic expansion for the parabolic cylinder function is

$$D_r(z) \simeq e^{-z^2/4} z^r \left(1 - \frac{r(r-1)}{2z^2} + \frac{r(r-1)(r-2)(r-3)}{2 \cdot 4z^4} - \dots\right) - \\ - \frac{\sqrt{2\pi}}{\Gamma(-r)} e^{r\pi i} e^{z^2/4} z^{-r-1} \left(1 + \frac{(r+1)(r+2)}{2z^2} + \frac{(r+1)(r+2)(r+3)(r+4)}{2 \cdot 4z^4} + \dots\right) \tag{294}$$

(which holds for all complex z such that $|z| >> 1$, $|z| >> r$, and $\frac{\pi}{4} < arg(z) < \frac{5}{4}\pi$). Setting $z = \psi_{max}\sqrt{\hat{N}+1}$ and $r = -1, -2$ in (278) we get that the leading asymptotics for I is

$$I \simeq \frac{\eta *^{-1/2} (1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}} \sqrt{-\frac{2\pi}{f_{\eta\eta}(\eta^*)}} (\hat{N}+1)^{-1/2} e^{(\hat{N}+1)f(\eta^*)} \tag{295}$$

(this, as expected, is twice the expression obtained in the limiting case $k_{n-2} = k_{n-2}^{crit}$-see below, *(ii)*, for the discussion of this limiting case). The corresponding expression for the effective entropy in this range of k_{n-2} is given by

$$W(N, k_{n-2})_{eff} = c_n \left(\frac{(A(k_{n-2})+2)}{3A(k_{n-2})}\right)^{-n} N^{-n-1} \,.$$

$$\cdot \exp\left[[\frac{1}{2}n(n+1)\ln\frac{A(k_{n-2})+2}{3}]N\right](1 + O(N^{-3/2})) \tag{296}$$

We have a $\hat{N}^{-n-1}$ subleading behavior which changes as $k_{n-2} \to k_{n-2}^{crit}$.

(ii) **Critical coupling**: $k_{n-2}^{crit} - \epsilon < k_{n-2} < k_{n-2}^{crit} + \epsilon$.

The above asymptotic is no longer appropriate when, for a given (large) value of N, ψ is so small that

$$2\psi^2(k_{n-2}^{crit} \pm \epsilon)\hat{N} << 1 \tag{297}$$

In such a regime, the parabolic cylinder function is more correctly described by

$$D_r(z) \simeq \frac{\sqrt{\pi}}{2^{-r/2}\Gamma((1-r)/2)} \exp[-\sqrt{-r-1/2}z + v_1] \tag{298}$$

where

$$v_1 \simeq -sgnz \frac{\frac{2}{3}(\frac{1}{2}z)^3}{2\sqrt{-r-1/2}} - \frac{(\frac{1}{2}z)^2}{(2\sqrt{-r-1/2})^2} - \ldots \tag{299}$$

(this holds for $-r - \frac{1}{2} >> z^2$, see [49] pp.689). Setting $r = -1$ and $z = \psi_{max}\sqrt{\hat{N}+1}$, we get for the canonical partition function the expression

$$W(N, k_{n-2})_{eff} = \frac{c_n}{\sqrt{2\pi}}(\psi_{max}\sqrt{\hat{N}+1}) \left[\ln \frac{\eta_{max}(1-2\eta_{max})^2}{(1-3\eta_{max})^3} - k_{n-2}\right]^{-1} \cdot$$

$$\cdot \frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}}(\hat{N}+1)^{-n-3/2} e^{(\hat{N}+1)f(\eta_{max}) - \psi_{max}\sqrt{\frac{\hat{N}+1}{2}} - \ldots} \tag{300}$$

where ... stands for higher order correction terms (which can be obtained by the appropriate substitutions from (299)).

Notice that in this latter regime, the subleading asymptotics of (300) is no longer polynomial, but rather exponential in $\psi_{max}\sqrt{N}$. This transient in the subleading asymptotics is necessary for taking care of the fact that as $k_{n-2} \to k_{n-2}^{crit}$ the term $(-1/f_\eta)$ tends to diverge. This divergence is compensated for by the vanishing of $\psi_{max}\sqrt{\hat{N}+1}$, and indeed, the singularity does not manifest itself in the exponential leading behavior of the effective entropy $W(N, k_{n-2})_{eff}$ which goes sufficiently smootly from $e^{(\hat{N}+1)f(\eta_{max})}$ into $e^{(\hat{N}+1)f(\eta^*)}$, but rather in a change of the polynomial part $\hat{N}^{-n-3/2}$ of the subleading asymptotics, which changes by a factor $\hat{N}^{1/2}$. This transient exponential subleading asymptotics is of relevance to the MonteCarlo simulations since it appears consistent with the numerical data for k_{n-2} nearby k_{n-2}^{crit}.

Notice that the range of k_{n-2}, for which the exponential subleading asymptotics is active depends on the actual value of N, and as $N \to \infty$, the range $k_{n-2}^{crit} - \epsilon < k_{n-2} < k_{n-2}^{crit} + \epsilon$ becomes smaller and smaller since $\epsilon = \epsilon(N)$ is characterized by the condition $2\psi^2(k_{n-2}^{max} \pm \epsilon)\hat{N} \simeq 1$ which implies that $\lim_{N\to\infty} \epsilon(N) = 0$. In this case we have

$$\psi_{max}(k_{n-2}) = sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{min})]} = 0 \tag{301}$$

and by setting $\psi_{max} = 0$ in (278) we get that the leading asymptotics for I is

$$I \simeq \frac{\eta *^{-1/2} (1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}} \sqrt{-\frac{\pi}{-2f_{\eta\eta}(\eta^*)}} (\hat{N}+1)^{-1/2} e^{(\hat{N}+1)f(\eta^*)} \tag{302}$$

The corresponding expression for the canonical partition function is easily seen to be

$$W(N, k_{n-2})_{eff} = \frac{c_n}{2} \left(\frac{(A(k_{n-2})+2)}{3A(k_{n-2})} \right)^{-n} N^{-n-1} \, .$$

$$\cdot \exp \left[[\frac{1}{2} n(n+1) \ln \frac{A(k_{n-2})+2}{3}] N \right] (1 + O(N^{-3/2})) \tag{303}$$

(iii) **Weak coupling**: $k_{n-2}^{crit} + \epsilon < k_{n-2} < +\infty$.

This is a weak coupling regime in a rather loose sense, since the asymptotics we obtain holds both for k_{n-2}'s in a small right- neighborhood of k_{n-2}^{crit} and in a large left-neighborhood of $k_{n-2} = +\infty$.

For such range of values of k_{n-2}, $\eta^* > \eta_{max}$ and

$$\psi_{max}(k_{n-2}) = sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]} = \sqrt{2[f(\eta^*) - f(\eta_{max})]} > 0 \tag{304}$$

In such a case, and as long as $\psi^2 \hat{N} >> 1$, the appropriate asymptotic expansion for the parabolic cylinder function is

$$D_r(z) \simeq e^{-z^2/4} z^r \left(1 - \frac{r(r-1)}{2z^2} + \frac{r(r-1)(r-2)(r-3)}{2 \cdot 4z^4} - \ldots \right) \tag{305}$$

(which holds for all complex z such that $|z| >> 1$, $|z| >> r$, and $|arg(z)| < \frac{3}{4}\pi$). Setting $z = \psi_{max}\sqrt{\hat{N}+1}$ and $r = -1, -2$ in (278) we get that the leading asymptotics for I is

$$I \simeq \frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}} \left(-\frac{1}{f_\eta(\eta_{max})}\right) (\hat{N}+1)^{-1} e^{(\hat{N}+1)f(\eta_{max})} \tag{306}$$

which provides the following expression for the canonical partition function in this regime

$$W(N, k_{n-2})_{eff} = \frac{c_n}{\sqrt{2\pi}} \left[\ln \frac{\eta_{max}(1-2\eta_{max})^2}{(1-3\eta_{max})^3} - k_{n-2}\right]^{-1} \cdot$$

$$\cdot \frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}} (\hat{N}+1)^{-n-3/2} e^{(\hat{N}+1)f(\eta_{max})} \tag{307}$$

which clearly exhibits a different subleading asymptotic behavior, $\hat{N}^{-n-3/2}$, with respect to the previous one discussed in the strong coupling case *(i)*.

Obviously, these results could have been obtained by standard Laplace approximation working in the appropriate range of values of the inverse gravitational coupling, but the expression (291), (and its related counterpart (252)), has the advantage of describing in a uniform way all possible regimes for k_{n-2} in an open neighborhood of k_{n-2}^{crit}. Roughly speaking, the asymptotics (296) corresponds to the location of maximum of $f(\eta)$ *well* inside the integration interval $\eta_{min} < \eta < \eta_{max}$, whereas the asymptotics (307) occurs when η^* is larger than η_{max}, (but sufficiently near to it). Notice also that $A(k_{n-2}) > 1$, $\forall k_{n-2} \in \mathbb{R}$.

We wish to stress the fact that the transition between the two regimes (296) towards (307) manifests itself in a change of the subleading asymptotics: the growth N^{-n-1} which characterizes the expression (296) for $W(N, k_{n-2})_{eff}$ changes into the $N^{-n-3/2}$ subleading behavior characterizing (307), and this change is *mediated* by an exponential subleading asymptotics in the range $k_{n-2}^{crit} - \epsilon \leq k_{n-2} \leq k_{n-2}^{crit} + \epsilon$.

The infinite volume limit. From the above asymptotics we can also read the critical value of the coupling k_n^{crit} corresponding to which the *infinite-volume* limit may be taken. In general, *i.e.*, for all values of the inverse gravitational coupling k_{n-2}, this critical value is provided by

$$k_n^{crit}(k_{n-2}) = \lim_{N\to\infty} \frac{\log W(N, k_{n-2})_{eff}}{N} \tag{308}$$

where $W(N, k_{n-2})_{eff}$ is given by (291). By considering separately the various ranges of k_{n-2} we explicitly get

$$k_n^{crit}(k_{n-2}) = \frac{1}{2}n(n+1)\left[\ln\frac{A(k_2)+2}{3} + \lim_{N\to\infty}\frac{\ln c_n}{N}\right] \tag{309}$$

which, for $0 < \epsilon(N) < 1$ characterized by the condition $2\psi^2(k_{n-2}^{crit} \pm \epsilon)\hat{N} \simeq 1$, holds for all $k_{n-2} < k_{n-2}^{crit} - \epsilon$. Whereas, for $k_{n-2}^{crit} + \epsilon \leq k_{n-2} < +\infty$ we get

$$k_n^{crit}(k_{n-2}) = \frac{1}{2}n(n+1)\left\{\ln\left[\frac{(1-2\eta_{max})^{(1-2\eta_{max})}}{\eta_{max}^{\eta_{max}}(1-3\eta_{max})^{(1-3\eta_{max})}}\right] + k_{n-2}\eta_{max} + \lim_{N\to\infty}\frac{\ln c_n}{N}\right\} \tag{310}$$

Notice that if one works at *finite* N, (as is the case in MonteCarlo simulations), care must be taken in the transition region $k_{n-2}^{crit} - \epsilon \leq k_{n-2} \leq k_{n-2}^{crit} + \epsilon$ since in that case a non-trivial exponential subleading asymptotics is present, (see *(ii)*).

With these elements in our hands, we can compare our analytical estimates with some of the known numerical data coming from MonteCarlo simulations. We start with the 4-dimensional case.

IX. ANALYTICAL VS. NUMERICAL DATA

The interplay between the asymptotic estimates arising from (296) and the results coming from Montecarlo simulations requires some care, since there are manifolds which are *algorithmically unrecognizable*. Denote such a manifold by M_0, assume that $n = 4$, and let M_0 be finitely described by a dynamical triangulation $T(M_0)$. Monte Carlo simulations make use of two kinds of moves altering the initial triangulation $T(M_0)$: a finite set of *local moves* [38], and global *baby universe surgery* moves. The local moves are ergodic, since given any two distinct triangulations of the manifold it is possible in a finite number of moves, carried out successively, to change on triangulation into the other. The algorithmic unrecognizability of M_0 means that there exists no algorithm which allows us to decide wether another manifold M, again finitely described by a triangulation $T(M)$, is combinatorially equivalent (in the PL-sense), to M_0. It has been proved [42] that the number of ergodic moves [43] needed to connect two triangulations of M_0, T and T^*, with $N_4(T) = N_4(T^*)$, cannot be bounded by any recursive function of N_4. This result implies that there can be very large barriers between some classes of triangulations of M_0 and that there would be triangulations which can never be reached in any reasonable number of steps, to the effect that the local moves, although ergodic, can be computationally non-ergodic. The manifold used in actual Monte Carlo simulations of four-dimensional simplicial gravity is the four-sphere $\mathbb{S}^4$, which is not known to be algorithmically recognizable or not. Up to now, there is no numerical evidence settling this issue, even using the non-local moves associated with baby-universes surgery [44], [45]. This implies that either the class of configurations separated by high barriers contains only very special configurations which are of measure zero and not important for numerical simulations, or the barriers expected are so high that they have not be encountered yet in the actual simulations. It is important to have these remarks clearly in mind when discussing the comparision with the analytical data.

A. The four-dimensional case

In order to discuss how the asymptotics associated with (296) compares with MonteCarlo simulations, let us rewrite $W(N,k_2)_{eff}$ as

$$W(N,k_2)_{eff} = 10c_n \cdot \frac{e}{\sqrt{2\pi}} e^{k_4^{crit}(k_2)N} N^{\gamma_s(k_2)-3}(1+\ldots) \tag{311}$$

where according to theorem 17 and lemma 13, the critical k_4^{crit} is provided by:

$$k_4^{crit}(k_{n-2}) = 10 \ln \frac{A(k_2)+2}{3} + \lim_{N\to\infty} 10 \frac{\ln c_n}{N} \tag{312}$$

for $k_2 \leq k_2^{crit} - \epsilon$, and

$$k_4^{crit}(k_2) = 10 \left\{ \log \left[\frac{(1-2\eta_{max})^{(1-2\eta_{max})}}{\eta_{max}^{\eta_{max}}(1-3\eta_{max})^{(1-3\eta_{max})}} \right] + k_2\eta_{max} + \lim_{N\to\infty} \frac{\ln c_n}{N} \right\} \tag{313}$$

for $k_2 > k_2^{crit} + \epsilon$ and $\eta_{max} = \frac{1}{4}$, with

$$A(k_2) \equiv \left[\frac{27}{2}e^{k_2} + 1 + \sqrt{(\frac{27}{2}e^{k_2}+1)^2 - 1} \right]^{1/3} + \left[\frac{27}{2}e^{k_2} + 1 - \sqrt{(\frac{27}{2}e^{k_2}+1)^2 - 1} \right]^{1/3} - 1 \tag{314}$$

and where the factor c_n is the normalizing factor of lemma 11.

The critical exponent $\gamma_s(k_2)$ is also provided by lemma 291, (see the discussion on the distinct asymptotic regimes), according to

$$\gamma_s(k_2) = 2 - n \tag{315}$$

for $k_2 < k_2^{inf} - \epsilon$,

$$\gamma_s(k_2) = -\infty \tag{316}$$

for $k_2^{inf} - \epsilon < k_2 < k_2^{inf} + \epsilon$, (recall that in this range, for large but finite N, (300) provides a non-polynomial subleading asymptotics which formally corresponds to such a critical exponent), and finally

$$\gamma_s(k_2) = \frac{3}{2} - n \tag{317}$$

for $k_2^{inf} + \epsilon < k_2$.

Notice that these critical exponents can get an additive normalization by the term $\tau(n)$, (depending only on the dimension n), associated with the rescaling factor c_n, (see (205)). Thus, the actual critical exponents are provided by

$$\tilde{\gamma}_s(k_2) = \gamma_s(k_2) + \tau(n). \tag{318}$$

This remark reminds us that we still have to fix the scale factor c_n, affecting the canonical partition function $W(M; N, k_2)_{eff}$. According to (204) and lemma 11 this factor normalizes the counting of the partitions p_λ to the partitions p_λ^{curv} coming from actual curvature assignements. In dimension $n = 4$, a simple exponential (down)shift times a constant polynomial subleading correction is all is needed, (later on we see that this latter correction to the subleading asymptotics is related to a polymerization mechanism), and if we set

$$c_n = (\frac{e}{2.066})^{-10N} N^{5/2} \tag{319}$$

then we get an extremely good agreement with MonteCarlo simulations. It is interesting to note that the exponential scaling factor $(\frac{e}{2.066})^{-10N}$ is a very close approximation to

$$[\cos^{-1}(\frac{1}{n})|_{n=4}]^{-10N} \tag{320}$$

suggesting that the ratio $p_\lambda^{curv}/p_\lambda$ is, as expected, of geometrical origin. We wish to stress that we do not have yet a convincing geometrical proof that the scale factor c_n is actually given by (320), and leave this as a conjecture to be proven by the interested reader.

The above choice (319) for the rescaling factor c_n provides

$$\tilde{K}_4^{crit} = 10 \ln \frac{A(k_2) + 2}{3e/2.066} \tag{321}$$

with $k_2 < k_2^{inf} - \epsilon$, and

$$\tilde{k}_4^{crit}(k_2) = 10\left\{\log\left[\frac{(1-2\eta_{max})^{(1-2\eta_{max})}}{\eta_{max}^{\eta_{max}}(1-3\eta_{max})^{(1-3\eta_{max})}}\right] + k_2\eta_{max} - \log\left(\frac{e}{2.066}\right)\right\} \tag{322}$$

(with $k_2 > k_2^{inf} + \epsilon$).

(Note that the subleading correction $N^{5/2}$ is irrelevant for the characterization of (321) and (322)).

The rescaled critical value of k_4 so obtained gives the critical k_4 for any simply-connected closed 4-manifold, in particular for the 4-sphere $\mathbb{S}^4$. Indeed, it provides an extremely good fit with the numerical data relative to the characterization of the ∞-volume line in the (k_4, k_2)- couplings space. Explicitly, if we denote by $K_4^{crit}(Mont.Car)$ the critical value of k_4 obtained from Montecarlo simulations [46] for dynamically triangulated four-spheres, then the following table demonstrates how well this agreement works out

$$k_2 = 0.0 \ldots K_4^{crit}(Mont.Car) = 0.98 \ldots \tilde{K}_4^{crit} = 1.07$$

$$k_2 = 0.2 \ldots K_4^{crit}(Mont.Car) = 1.45 \ldots \tilde{K}_4^{crit} = 1.47$$

$$k_2 = 0.3 \ldots K_4^{crit}(Mont.Car) = 1.65 \ldots \tilde{K}_4^{crit} = 1.67$$

$$k_2 = 0.4 \ldots K_4^{crit}(Mont.Car) = 1.89 \ldots \tilde{K}_4^{crit} = 1.87$$

$$k_2 = 0.5 \ldots K_4^{crit}(Mont.Car) = 2.07 \ldots \tilde{K}_4^{crit} = 2.07$$

$$k_2 = 0.6 \ldots K_4^{crit}(Mont.Car) = 2.30 \ldots \tilde{K}_4^{crit} = 2.33$$

$$k_2 = 0.7 \ldots K_4^{crit}(Mont.Car) = 2.54 \ldots \tilde{K}_4^{crit} = 2.55$$

$$k_2 = 0.8 \ldots K_4^{crit}(Mont.Car) = 2.78 \ldots \tilde{K}_4^{crit} = 2.78$$

$$k_2 = 0.9 \ldots K_4^{crit}(Mont.Car) = 3.02 \ldots \tilde{K}_4^{crit} = 3.01$$

$$k_2 = 1.0 \ldots K_4^{crit}(Mont.Car) = 3.25 \ldots \tilde{K}_4^{crit} = 3.25$$

$$k_2 = 1.1 \ldots K_4^{crit}(Mont.Car) = 3.49 \ldots \tilde{K}_4^{crit} = 3.49$$

$$k_2 = 1.3 \ldots K_4^{crit}(Mont.Car) = 3.97 \ldots \tilde{K}_4^{crit} = 3.97$$

$$k_2 = 1.5 \ldots K_4^{crit}(Mont.Car) = 4.47 \ldots \tilde{K}_4^{crit} = 4.46$$
$$k_2 = 1.8 \ldots K_4^{crit}(Mont.Car) = 5.20 \ldots \tilde{K}_4^{crit} = 5.19 \tag{323}$$

(note that the numerical data are affected by an error of the order ± 0.02, and the analitical data are rounded to the largest second significant figure). According to the distinct asymptotic regimes provided by theorem 17 and lemma 291, we have used (322), (evaluated for $\eta_{min} = 1/5$), in the range for $0 \leq k_2 \leq 0.5$, (however we stress that in such region, and in particular near $k_2 \simeq 0$, the MonteCarlo data are rather unreliable being strongly polluted by finite size effects). For $0.6 \leq k_2 \leq k_2^{crit} \simeq 1.38$ we have used (321). Incidentally, it is not difficult to check that the use of (321) works quite well also for $0 \leq k_2 \leq 0.5$, with just a sligth discrepancy between the MonteCarlo simulations and the analytical data. Finally, for $k_2 > k_2^{crit}$ we have used (322). The numerical data in (323) come from simulations [46] which are already a few years old, however we wish to stress that the agreement with the analytical estimates holds beautifully also when using more recent and accurate MonteCarlo data, a fact that confirms that our simple counting strategy based on the use of the generating function of partitions of integers p_λ has a sound foundation in the geometry of dynamical triangulations.

B. Polymerization

A well known consequence of the numerical simulations in 4- dimensional simplicial quantum gravity is the onset of a branched polymer phase [11] in the weak coupling regime, *i.e.*, for $k_2 > k_2^{crit}$. This polymerization is very effectively described by the asymptotics of theorem 17 and lemma 291, and it is generated again by a mechanism of entropic origin.

Let us start by noticing that if in the expression yielding for the canonical partition function (245) we replace the upper integration limit $\eta_{max} = 1/4$ with the limit $\tilde{\eta} = 1/3$, associated with the algebraic singularity of the integrand at $\eta = 1/3$ (and corresponding to the unphysical average incidence $b(n, n-2) = 3$), then we would get a canonical partition function $W(N, k_{n-2})_{poly}$ whose large N limit is provided by the asymptotics (296), *viz.*,

$$W(N, k_{n-2})_{poly} = c_n \left(\frac{(A(k_{n-2}) + 2)}{3A(k_{n-2})} \right)^{-n} N^{-n-1} \cdot$$

$$\cdot \exp \left[[\frac{1}{2} n(n+1) \ln \frac{A(k_{n-2}) + 2}{3}] N \right] (1 + O(N^{-3/2})) \qquad (324)$$

for all $k_2^{inf} + \delta \leq k_2 < +\infty$, where k_2^{inf} corresponds to the lower crossing $\eta^* \to \eta_{min} = 1/5$, and $\delta \in \mathbb{R}^+$ is a suitably small number.

In the range $k_2^{inf} + \delta < k_2 < k_2^{crit} - \epsilon$ this is exactly the expression for the asymptotics of the canonical partition function $W(N, k_{n-2})_{eff}$, but this identification breaks down as soon as k_2 approaches k_2^{crit}. The geometrical reason for such a behavior is very simple: as $k_2 \to k_2^{crit}$, the possible maximum value of the (positive) average curvature for a given volume, ($vol(M) \propto N$), is saturated: no more positive average curvature is allowed for that volume, (think of the average curvature on a sphere of given volume: it is maximized for the round metric corresponding to that volume). On the other hand, (324) as it stands, would allow, as k_2 grows, for more and more (positive) average curvature at fixed volume, and in order to comply with the constraint relating volume to positive curvature we have to turn to the correct weak coupling description of the partition function, *viz.*, (307). However, (324) can still be interpreted as the partition function of a large positive curvature PL manifold of given volume, but now the manifold in question is *disconnected.* Such a disconnected manifold is generated by means of a *polymerization mechanism*, namely it results as the (Gromov-Hausdorff) limit of a network of many smaller manifolds carrying large curvature. Such curvature *blobs* are connected to each other by thin tubes of negligible volume so that the sum of the volumes of the costituents manifolds adds up to the given total volume, (the standard example is afforded by a collection of highly curved small spheres connected by tubules of negligible area, *vs.* the standard round sphere of the some total area). The basic observation is that for $k_2^{crit}+\epsilon<k_2<+\infty$, the effective entropy (324) is *larger* than (307): thus in the weak coupling region, for a dynamical triangulation is energetically favourable to polymerize rather than to comply to the curvature-volume constraint with a *large* connected manifold.

The above remarks can be easily formalized. According to (88), the average curvature (around the collection of bones $\{B\}$) on a dynamically triangulated manifold is proportional to

$$< K(B) >\doteq \frac{1}{\hat{N}}\sum_{B} K(B)q(B) \propto \frac{1}{\hat{N}}\sum_{B}^{N_{n-2}}\left[2\pi - q(B)\cos^{-1}\frac{1}{n}\right] \tag{325}$$

Namely

$$< K(B) >\propto \frac{2\pi - b(n,n-2)\cos^{-1}(\frac{1}{n})}{b}. \tag{326}$$

The canonical average, $<< K(B) >>$, of $< K(B) >$ over the ensemble of dynamical triangulation considered is, up to an inessential positive constant, provided by (see (245)),

$$<< K(B) >>= \int_{\eta_{min}}^{\eta_{max}} d\eta \frac{\eta^{-1/2}(1-2\eta)^{-n}}{\sqrt{(1-3\eta)(1-2\eta)}}[2\pi\eta^{-1} - \cos^{-1}(\frac{1}{n})|_{n=4}]e^{(\hat{N}+1)f(\eta)} \times$$

$$\times \left[\int_{\eta_{min}}^{\eta_{max}} d\eta \frac{\eta^{-1/2}(1-2\eta)^{-n}}{\sqrt{(1-3\eta)(1-2\eta)}} e^{(\hat{N}+1)f(\eta)}\right]^{-1} \tag{327}$$

For $k_2=k_2^{crit}-\epsilon$, with $\epsilon\in\mathbb{R}^+$ sufficiently small, the maximum for $f(\eta)$, (see (245)), is attained for an η^* which uniformly approaches $\eta=\frac{1}{4}$ as $\epsilon\to 0$. Correspondingly, the integrands in (327) are peaked around $\eta\simeq 1/4$ and $<< K(B) >>$ is positive, (explicit expression can be easily obtained from the uniform asymptotics of the previous paragraphs). Let us now extend (327) to $\eta_{max}=\frac{1}{3}$, namely consider the ensemble average, $<< K(B) >>_{poly}$ of $< K(B) >$ over $W(N,k_{n-2})_{poly}$. It is trivially checked that $<< K(B) >>_{poly}$ formally remains positive for all $k_2^{crit}<k_2<+\infty$, (corresponding to the maximum η^* ranging from

$\frac{1}{4} \leq \eta^* \leq \frac{1}{3}$). But according to lemma 1, there cannot be connected four-manifolds (regardless of topology) with $b < 4$, (*i.e.*, $\eta > \frac{1}{4}$), because for any dynamically triangulated 4-manifold

$$5N_4 \geq 2N_2 - 5\chi \tag{328}$$

An inequality which implies that as $\eta \to \frac{1}{4}$, the average curvature $< K(B) >$ attains its maximum $\propto [2\pi - 4\cos^{-1}(\frac{1}{4})]/4$ compatibly with the given volume, ($\propto N$).

There is a natural way of circumventing the implications of the constraint (328): we have to remove the connectivity requirement for M.

Let us assume that we have a collection $\{M_\alpha\}$ of dynamically triangulated manifolds of fixed volume $\propto N_4(M_\alpha) = N$, and where each M_α is generated by a large number m of dynamically triangulated 4-manifolds $\{\Xi(i)\}_{i=1,\ldots,m}(\alpha) \subset M_\alpha$, connected to each other by tubes of negligible volume. Such a sequence of manifolds naturally converges, in the Gromov-Hausdorff topology, to the disjoint union of a set of m 4-manifolds, *viz.*,

$$\{M_\alpha\} \to M \doteq \coprod_i^m \Xi(i) \tag{329}$$

such that $N_4(M) = N_4(M_\alpha) = N$. The constraint (328), which holds for each connected manifold M_α of the sequence, carries over to the disconnected limit space M and implies that

$$5\sum_i^m N_4(\Xi(i)) \geq 2\sum_i^m N_2(\Xi(i)) - 5\sum_i^m \chi(\Xi(i)) \tag{330}$$

namely,

$$b_M \doteq 10\frac{\sum_i^m N_4(\Xi(i))}{\sum_i^m N_2(\Xi(i))} \geq 4 - 10\frac{\sum_i^m \chi(\Xi(i))}{\sum_i^m N_2(\Xi(i))} \tag{331}$$

Thus, if each $\Xi(i)$ is topologically a 4-sphere, $\mathbb{S}^4$, and $N_2(\Xi(i))$ is $O(1)$, we can make b_M take any value in the range $3 \leq b \leq 4$. In particular, if

$$\tilde{N}_2(\Xi(i)) \doteq \frac{1}{m}\sum_{i}^{m} N_2(\Xi(i)) = 20 \tag{332}$$

and

$$\tilde{N}_4(\Xi(i)) \doteq \frac{1}{m}\sum_{i}^{m} N_4(\Xi(i)) = 6 \tag{333}$$

then for such $M = \coprod \Xi(i)$, we get

$$b_M \to 3 \tag{334}$$

Such an M is the disjoint union of $m \simeq \frac{N_4(M)}{6}$ *small* 4-dimensional spheres, (recall that the standard triangulation of a 4-sphere obtained as the boundary of the standard 5-simplex σ^5 is characterized by the f-vector $(N_0, N_1, N_2, N_3, N_4) = (6, 15, 20, 15, 6)$).

It is trivially checked that for $k_2^{crit} + \epsilon < k_2 < +\infty$, the effective entropy (324) is *larger* than (307). Thus in the weak coupling region, it is energetically favourable to generate dynamical triangulations associated with a disconnected manifold $M = \coprod \Xi(i)$, rather than to comply with the curvature-volume constraint on a *large* connected manifold. From the asymptotics (307) and (296) it follows that the critical exponent associated with such a polymer phase, γ_{poly}, is related to $\gamma_s(k_2) = 2 - n$, (see (315)), by the relation

$$\gamma_{poly} = \gamma_s(k_2) + \frac{1}{2} \tag{335}$$

This jump of $\frac{1}{2}$ has been observed in the numerical simulation [11] exactly around $k_2^{crit} \simeq 1.3$. In such simulations there is also some indication that $\gamma_s(k_2 \simeq k_2^{crit}) \simeq 0$, so that γ_{poly} is exactly the critical exponent of branched polymers, but this evidence is not yet conclusive due to critical slowing down near the transition point.

According to our counting strategy, we still have the freedom to add to the critical exponents the term $\tau(n)$ associated with the count of distinct dynamical triangulations with the same curvature assignements. The correct critical exponents to be confronted with the MonteCarlo data are then provided by

$$\tilde{\gamma}_s(k_2) = \gamma_s(k_2) + \tau(n) \tag{336}$$

and

$$\tilde{\gamma}_{poly} = \gamma_{poly} + \tau(n) \tag{337}$$

Since the critical exponent of a branched polymer is exactly $\gamma_{poly} = \frac{1}{2}$, and as we have shown (324) seems to geometrically describe a polymer phase, we can use this result to fix the value of $\tau(n)$. Indeed, if at least provisionally, we accept this heuristic argument, from (315) we find that the critical exponent scaling factor $\tau(n)$ must satisfy the relation

$$2 - n + \tau(n)|_{n=4} = \frac{1}{2} \tag{338}$$

namely, $\tau(n)|_{n=4} = \frac{5}{2}$, a result already anticipated in the expression (319) of the normalizing factor c_n. Not surprisingly, this position provides also the correct critical exponent near criticality, *viz.*,

$$\gamma_s(k_2) = 0 \tag{339}$$

These latter results, although not fully rigorous, seem to be conforted by the numerical simulations.

We wish to stress that the *polimerization phase* is not properly speaking described by the dominance of a disconnected phase (*i.e.*, by a *gas* of weakly interacting small 4-spheres), but rather by the dominance, in the statistical sum, of the dynamically triangulated 4-manifolds M_α generated by a large number of curvature blobs $\{\Xi(i)\}_{i=1,\ldots,m}(\alpha)$, connected to each other by tubes of negligible volume.

Few remarks are also in order for what concerns the transition between the various phases. The transition threshold towards a polymer phase occurs, in our modelling, somewhere around the value of k_2 such that $\eta^* = \eta_{max}$. Explicitly, for a $k_2 \simeq k_2^{crit}$ such that

$$\frac{A(k_2)-1}{A(k_2)}=\frac{3}{4} \tag{340}$$

This condition provides $k_2^{crit} \simeq 1.387$ which appears quite in a good agreement with the value indicated by the numerical simulations. For instance in [11], the critical point is located around $k_2^{crit}(Mont.Carl) \simeq 1.336(\pm 0.006)$. More recent simulations [45] tend to move this point towards $k_2^{crit} \simeq 1.22$. The explanation of this discrepancy is related to the observation that $\eta^* = \eta_{max}$ is a theoretical lower bound, and that k_2^{crit} changes very rapidly with small variations of the ratio $\frac{A(k_2)-1}{A(k_2)}$. Thus for instance, if we set $\eta_{max} = 1/4.1$ we would get $k_2^{crit} \simeq 1.2$. As is clearly indicated by our analysis, this is simply a question related to the fact that it is very delicate to exactly locate numerically the true critical point.

C. Summing over simply-connected 4-dimensional manifolds

It is important to stress that the asymptotic estimates of theorem 17 refers to **all simply-connected 4-manifolds**, not just to the 4-sphere case, the only for which numerical data are currently available.

Besides $\mathbb{S}^4$, well-known example of closed simply-connected topological 4-manifolds are provided by $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, the Kummer surface $K_3 \doteq \{[Z_0, Z_1, Z_2, Z_3] \in CP^3 | Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 = 0\}$, and more in general by all nonsingular algebraic surfaces of degree n in $\mathbb{C}P^3$. The basic topological invariant for closed simply-connected 4- manifolds is the intersection form [14]. This is a natural symmetric bilinear form, $\omega_M(,)$, on the second homology group $H_2(M,\mathbb{Z})$ of the manifold: for α, $\beta \in H_2(M,\mathbb{Z})$, let α^* be the Poincaré dual of $\alpha \in H_2(M,\mathbb{Z})$, then $\omega_M(\alpha,\beta) \doteq <\alpha^*, \beta>_K$, where $<,>_K$ is the Kronecker pairing between the cohomology group and the homology group. The case of 2-dimensional (connected) surfaces Σ provides a direct illustration of what is meant by this construction. Here the relevant homology group is $H_1(\Sigma,\mathbb{Z}_2)$, every homology class in $H_1(\Sigma,\mathbb{Z}_2)$ can be represented by a simple closed curve $C \subset \Sigma$, and if C_1, C_2 are any two such curves, their intersection number $C_1 \cdot C_2 = C_2 \cdot C_1 \in \mathbb{Z}_2$ is defined. Note that the self-intersection number $C \cdot C$ is zero as soon as the curve C_1 has a neighborhood which is orientable, (by deforming the curve $C \to \hat{C}$, while remaining in the same homology class, we can make $\hat{C}$ disjoint from C), while if such a neighborhood is non-orientable, the self-intersection number will be odd, (think of a deformation of the curve on a Möbius band). Actually, by Poincaré duality we can make $H_1(\Sigma, \mathbb{Z}_2)$ an inner product space over $\mathbb{Z}_2$ if we use the intersection number $C_1 \cdot C_2$ as inner product. It can be proven that any two closed connected surfaces Σ_1 and Σ_2 are homeomorphic if and only if the associated inner product spaces $H_1(\Sigma_1, \mathbb{Z}_2)$ and $H_1(\Sigma_2, \mathbb{Z}_2)$ are isomorphic (see Milnor-Husemoller's book [14]), and that every inner product space over $\mathbb{Z}_2$ is isomorphic to $H_1(\Sigma, \mathbb{Z}_2)$ for some closed connected surface Σ.

In a similar fashion, the intersection form $\omega_M(\alpha, \beta)$ on $H_2(M, \mathbb{Z})$ may be thought of as representing the transversal intersection of two oriented surfaces α and β in the compact, simply-connected 4-manifold M, α and β being cycles representing elements in $H_2(M, \mathbb{Z})$. Again, we can think of $H_2(M, \mathbb{Z})$ as an inner product space over $\mathbb{Z}$ if we use the intersection number $\omega_M(\alpha, \beta)$ as inner product. Moreover, it can be proven that any two closed oriented simply- connected 4-dimensional manifolds M_1 and M_2 have the same homotopy type, (*i.e.*, there exists an orientation preserving homotopy equivalence $M_1 \to M_2$), if and only if the associated symmetric inner product spaces $H_2(M_1, \mathbb{Z})$ and $H_2(M_2, \mathbb{Z})$ are isomorphic, (Whitehead's theorem [14]). Actually, as in the case of surfaces more can be stated in terms of this isomorphism, since the set of simply-connected oriented closed 4-manifolds is in correspondence [14] with the set of all unimodular, (any matrix representing $\omega(,)$ has determinant ± 1), bilinear symmetric form, (the manifold is unique when the form is even, and there are exactly two distinct manifolds when the form is odd), (Freedman's theorem [14]). However, if the 4-manifolds in question carry differentiable structures, the correspondence is with positive-definite intersection forms, (Donaldson's theorem [14]).

Let us denote by $W(M; N, k_2)_{eff}$ the canonical partition function enumerating the distinct triangulations on a 4-dimensional simply-connected manifold M of given topology and given volume (N). The set of such M is formally labelled by the set of all positive-definite, (we are dealing with PL 4-manifods), unimodular bilinear symmetric forms ω_M. According to theorem 17, the asymptotics (252) refers to all simply-connected 4-dimensional PL-manifolds of fixed volume, thus we can write

$$\begin{aligned}W(N,k_2)_{eff} &= \sum_{\omega_M} W(M;N,k_2)_{eff} \\ &= W(\mathbb{S}^4;N,k_2)_{eff} + W(\mathbb{S}^2\times\mathbb{S}^2;N,k_2)_{eff} + W(K^3;N,k_2)_{eff} + \ldots \end{aligned} \tag{341}$$

For N large but fixed, the dynamical triangulations entering (341) are of bounded geometry, thus according to theorem 5, there are only a finite number of distinct homotopy types of 4-manifolds contributing to the above sum, and Whitehead's theorem mentioned above implies that likewise finite is the number of distinct types of (isomorphism classes of) intersection forms ω entering (341).

Thus from (341) we get

$$\begin{aligned}&10c_n \cdot \frac{e}{\sqrt{2\pi}} e^{k_4^{crit}(k_2)N} N^{\tilde{\gamma}_s(k_2)-3} \\ &= W(\mathbb{S}^4;N,k_2)_{eff} + W(\mathbb{S}^2\times\mathbb{S}^2;N,k_2)_{eff} + W(K^3;N,k_2)_{eff} + \ldots \end{aligned} \tag{342}$$

where $\tilde{\gamma}_s(k_2)$ is the scaled critical exponent (337). The identity (342) makes clear that the expression (291) for the canonical partition function being common to all dynamically triangulated 4-manifold (closed) with trivial fundamental group, tends to *overestimate* the canonical partition function of any 4-manifold M in (342). However, guided by the analogy with the 2-dimensional case, and by the agreement between our analytical results and the MonteCarlo simulations for the 4-sphere, we may tentatively assume that all simply-connected DT manifolds contribute in (342) to the total entropy $W(N,k_2)_{eff}$ with the some exponential behavior and that the topological differences come about only in the subleading asymptotics. Since (341) is a finite sum, (342) implies that the subleading asymptotics in each entropy function $W(M;N,k_2)_{eff}$ is of polynomial type, (at least in the range of k_2 considered). Thus, we are led to the following *ansatz*:

$$W(M;N,k_2)_{eff} = 10c_n \cdot \frac{e}{\sqrt{2\pi}} e^{\tilde{k}_4^{crit}(k_2)N} N^{\tilde{\gamma}_s(\omega,k_2)-3}(1+\ldots) \tag{343}$$

where $\tilde{\gamma}_s(\omega, k_2)$ is the critical exponent for the topological 4-manifold characterized by the intersection form ω. Since by hypothesis the exponential leading term does not depend on topology, the above ansatz allows us to rewrite (342) as

$$\sum_{\omega} N^{\tilde{\gamma}_s(\omega,k_2)-3} = N^{\tilde{\gamma}_s(k_2)-3} \tag{344}$$

The above hypothesis is fully consistent with the finiteness result for the topology of dynamical triangulations of bounded geometry discussed in Section V, (theorem 6), according to which the number of distinct homotopy types for such triangulations grows at most polynomially with N.

In order to be more specific, let us recall that the *rank* $r(\omega)$ of a unimodular symmetric bilinear form is the dimension of the space on which ω is defined, namely $r(\omega) = dim[H_2(M, \mathbb{Z})]$. The *signature* $s(\omega)$ of ω is defined by choosing an orthogonal basis $b_1, \ldots, b_n$ for the (rational) inner product space $\mathbb{Q} \otimes H_2(M, \mathbb{Z})$, then [14] $s(\omega) = s(\mathbb{Q} \otimes H_2(M, \mathbb{Z})) \in \mathbb{Z}$ is the difference between the number of basis elements b_i with $\omega(b_i, b_i) > 0$ and the number of basis elements b_j with $\omega(b_j, b_j) < 0$. Roughly speaking, $s(\omega)$ is the number of positive eigenvalues minus the number of negative eigenvalues of ω (thought of as a real form). Finally, the *type* of ω is defined by stating that ω is *even* (or of type II) if $\omega(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in H_2(M, \mathbb{Z})$, (*i.e.*, the diagonal entries of ω are even), otherwise ω is *odd*, (or of type I [14]). The relevance of such definitions lies in the fact that any two (indefinite) inner product spaces $(H_2(M_1, \mathbb{Z}), \omega_1)$ and $(H_2(M_2, \mathbb{Z}), \omega_2)$ are isomorphic if ω_1 and ω_2 have the same rank, type, and signature [14]. According to this observation, we can replace the sum over the possible intersection forms in (344) with a sum over the possible ranks, signatures and types. To this end, let us introduce the functions $A_{\pm}(r, s)$ enumerating the distinct (isomorphism classes of) intersection forms ω of rank $r(\omega) = r$, signature $s(\omega) = s$ of even type, $(A_+(r, s))$, and of odd type, $(A_-(r, s))$ on a 4-manifold of bounded geometry. Then, we can write (344) as

$$\sum_{r=0}^{\hat{r}}\sum_{s=0}^{\hat{s}}[A_+(r,s)+A_-(r,s)]N^{\tilde{\gamma}_s(\omega,k_2)}=N^{\tilde{\gamma}_s(k_2)} \tag{345}$$

where $\hat{r}<+\infty$ and $\hat{s}<+\infty$ are the largest value for the rank and signature allowed on a 4-manifold of bounded geometry. In analogy with the 2-dimensional case, (where the critical exponent depend linearly on topology), let us make the further natural ansatz

$$\tilde{\gamma}_s(\omega,k_2)=\alpha r+\beta s+\delta \tag{346}$$

where α, β, δ are constants possibly depending on the inverse gravitational coupling k_2. Notice that since $H_2(\mathbb{S}^4,\mathbb{Z})=0$, δ is the critical exponent for the 4-sphere.

The ansatz concerning the structure of the effective entropy of each simply- connected 4-manifold M, implies that

$$\sum_{r=0}^{\hat{r}}\sum_{s=0}^{\hat{s}}[A_+(r,s)+A_-(r,s)]N^{\alpha r+\beta s+\delta}=N^{\tilde{\gamma}_s(k_2)} \tag{347}$$

which is a very strong bound on the set of possible intersection form for 4-manifolds of bounded geometry. Indeed, without the control on geometry, the number of inequivalent intersection forms, say of given rank r, grows with r at an extremely fast rate. To make a rather well-known example, (see Milnor-Husemoller book quoted in [14]), consider the subcase of positive-definite intersection forms of odd type. The number of such forms is known to grow with the rank r at least as

$$c\left(\frac{r}{2\pi e\sqrt{e}}\right)^{r^2/4}\left(\frac{8\pi e}{r}\right)^{r/4}r^{-1/24} \tag{348}$$

Thus, the interplay between our analytical approach and MonteCarlo simulations seems to indicate results of great potential interests in the borderline between $4D$-simplicial quantum gravity and the topology of 4-manifolds.

D. Higher order phase transitions for simply connected 4- manifolds

To conclude, we will directly prove that for a dynamically triangulated 4- dimensional simply-connected manifold M, (*e.g.*, the sphere $\mathbb{S}^4$), the jump in the critical exponents $\tilde{\gamma}(M, k_2)$, as the inverse bare gravitational coupling k_2 approaches the value $k_2 = k_2^{crit}$, corresponds to a higher order phase transition of our statistical system. This result settles down in the affermative a long standing debate about the real nature of the critical point k_2^{crit} characterized by the MonteCarlo simulations. As we shall see, some of the contraddictory results of the numerical simulations find their rationale in our analysis. We wish to stress that most of the material in this section is based on joint work with D. Gabrielli and G. Gionti, and we wish to thank them for granting us permission to present part of their results here.

The optimal way for discussing the onset of a phase transition for simplicial quantum gravity on a 4-manifold M, would require the characterization of the two-point function associated with the set of all dynamical triangulations of M. The evaluation of such two-point function requires the enumeration of all distinct dynamical triangulations on M with two marked simplex whose distance is declared to be some running parameter r. As already remarked, this counting problem is, at the moment of writing these notes, not yet solved. However, a reliable indication that our system exhibits a phase transition may came also from exhamining the simplicial version of the *curvature susceptibility*. This is defined by

$$\chi_R \doteq \frac{1}{\hat{N}} \frac{\partial^2 \ln W(\mathbb{S}^4; N, k_2)_{eff}}{\partial k_2^2} = \frac{1}{\hat{N}} \left[< \lambda^2 > - < \lambda >^2 \right] \tag{349}$$

where $\hat{N} \doteq \frac{1}{2} n(n+1) N|_{n=4} = 10N$. Since we work at fixed $N \doteq N_4$, (*i.e.*, at fixed volume), such a χ_R can be equivalently written as

$$\chi_R = \langle (\frac{1}{b})^2 \rangle - \langle \frac{1}{b} \rangle^2 \tag{350}$$

where as usual $b(n, n-2) \doteq \frac{1}{2} n(n+1) \frac{N}{\lambda+1}$. Thus χ_R characterizes the fluctuations in the average incidence b on the set of all possible dynamical triangulations (at fixed volume) on $\mathbb{S}^4$. For this reason, it is not difficult to show on heuristic grounds that such a χ_R can be actually considered as the dynamical triangulation counterpart of a connected curvature-curvature correlation function:

$$\langle \int_{S^4} d^4\xi_1 d^4\xi_2 \sqrt{g} R(\xi_1) \sqrt{g} R(\xi_2) \rangle_{connected} \tag{351}$$

It is worth noticing that such a χ_R has been considered for describing the onset of phase transitions also in computer-assisted simulations. However, its direct use in such a setting is hampered by the fact that in such computer simulations one is forced to allow for volume fluctuations. Moreover, it has been observed that the peak of such χ_R is rather asymmetric and difficult to locate. Thus, in MonteCarlo simulations it is difficult to provide a clear cut interpretation of the behavior of χ_R.

Such difficulties in using the curvature susceptibility have a clear origin in the analytical computation of χ_R. Indeed, contrary to what one would like to think, at the critical point χ_R *does not* diverge with N, it rather exhibits a finite jump. According to standard statistical mechanics, such a jump may or may not be associated with the onset of long range order, and not surprisingly, for characterizing the order of the transition at k_2^{crit}, we are naturally led to a finite scaling argument *a' la Fisher* not much dissimilar from the kind of argument adopted in the actual MonteCarlo simulations.

Adopting the notation iof theorem 17 and lemma 13, the possible existence of long range order at the critical point $k_2 = k_2^{crit}$ rests on the following

Theorem 18 *As* $|k_2 - k_2^{crit}| \to 0$ *and* N *varies there exist pseudo critical points* $k_2 = k_2^{crit}(N)$ *characterized by the condition*

$$\psi_{max}(k_2^{crit}(N))\sqrt{\hat{N}} = \delta \tag{352}$$

where $0 < \delta \leq 1$, is a fixed (suitably small) constant (indipendent from N-see (297)), such that

$$|k_2^{crit}(N) - k_2^{crit}| \simeq_{N>>1} \delta \frac{\sqrt{-f_{\eta\eta}(\eta^*)}}{\sqrt{\hat{N}}} \tag{353}$$

and

$$\lim_{N\to\infty} \chi_R(k_2 = k_2^{crit}) = \frac{1}{2}\left[-\frac{1}{f_{\eta\eta}(\eta^*)}\right]. \tag{354}$$

This indicates that at the critical point $k_2 = k_2^{crit} \simeq 1.387$ the dynamically triangulated 4-manifold M undergoes a higher order phase transition possibly associated with the onset of long range order.

Proof. According to the uniform asymptotic estimation of the canonical partition function $W(N, k_2)_{eff}$ provided by theorem 17 and its corollaries, the approach to the critical point $k_2^{crit} \simeq 1.387$ is characterized by the parameter (see (249)),

$$\psi_{max}(k_2) \doteq sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]} \tag{355}$$

through the condition

$$\psi_{max}(k_2)\sqrt{\hat{N}} \to 0. \tag{356}$$

Whereas, the separation between the weak (strong) coupling regions and the critical region corresponds to values of the inverse gravitational coupling k_2 such that

$$\psi_{max}(k_2)\sqrt{\hat{N}} \simeq 1. \tag{357}$$

It follows that if we choose a δ, with $0 < \delta \leq 1$, sufficiently small, then through the condition (352), we select the *rounding interval* $k_2^{crit} - k_2^{crit}(N) \leq k_2 \leq k_2^{crit} + k_2^{crit}(N)$

corresponding to which the critical behavior occurs. As the notation suggests, this condition characterizes a value of the coupling $k_2 \doteq k_2^{crit}(N)$ according to

$$\psi_{max}(k_2^{sup}(N))\sqrt{\hat{N}} = \delta \tag{358}$$

which localizes the maximum η^* of the function $f(\eta, k_2)$ to an exponentialy small neighborhood of the transition point $\eta = \eta^{max}$. For a given N, this $k_2^{crit}(N)$ maximizes the corresponding value of the curvature susceptibility $\chi_R(N)$, and as such it defines a pseudo critical point. Explicitly, from the definition of χ_R and from the proof of theorem 17, we compute

$$\chi_R = \frac{1}{\hat{N}}\frac{\partial^2}{\partial {k_2}^2}\ln\int_0^{\infty} G(p)e^{-(\hat{N}+1)(\frac{1}{2}p^2+\psi_{max}p)}dp \tag{359}$$

By exploiting the uniform asymptotics estimation of the integral appearing in the right member of (359), we can write, up to order $O(N^{-m-1})$,

$$\chi_R = \frac{1}{\hat{N}}\frac{\partial^2}{\partial {k_2}^2}\ln\left\{\frac{w_0(z)}{(\hat{N})^{1/2}}\sum_{i=0}^{m}\frac{\xi_{2i}}{(\hat{N})^i} + \frac{w_{-1}(z)}{(\hat{N})}\sum_{i=0}^{m}\frac{\xi_{2i+1}}{(\hat{N})^i}\right\} \tag{360}$$

where $z \doteq \psi_{max}(k_{n-2})\sqrt{\hat{N}}$, (henceforth we will omit the annoying 1 whenever $\hat{N}+1$ appears). If we introduce explicitly the expression of the Whittaker functions $w_0(z)$ and $w_{-1}(z)$ in terms of the parabolic cylinder function $D_r(z)$, *viz.*, $w_r(z) = \Gamma(r-1)e^{z^2/4}D_{r-1}(z)$, then a long but straightforward computation provides

$$\chi_R = \frac{1}{2}\left(\frac{d}{dk_2}\psi_{max}(k_2)\right)^2 + \frac{1}{2}\psi_{max}\frac{d^2\psi_{max}(k_2)}{dk_2^2} +$$

$$\frac{1}{\hat{N}}\frac{\partial^2}{\partial {k_2}^2}\ln\left\{\frac{D_{-1}(z)}{(\hat{N})^{1/2}}\sum_{i=0}^{m}\frac{\xi_{2i}}{(\hat{N})^i} + \frac{D_{-2}(z)}{(\hat{N})}\sum_{i=0}^{m}\frac{\xi_{2i+1}}{(\hat{N})^i}\right\} \tag{361}$$

In the critical region characterized by $\psi_{max}(k_2)\sqrt{\hat{N}} << 1$, the asymptotics (298) of the parabolic cylinder functions shows that, to leading order in N, we have

$$\chi_R \simeq \frac{1}{2}\left(\frac{d}{dk_2}\psi_{max}(k_2)\right)^2 + \frac{1}{2}\psi_{max}\frac{d^2\psi_{max}(k_2)}{dk_2^2} \tag{362}$$

In order to compute the above derivatives, let us consider the following limit

$$\lim_{k_2\to k_2^{crit}} \frac{|\psi_{max}(k_2)|}{|k_2 - k_2^{crit}|} = \lim_{k_2\to k_2^{crit}} \sqrt{\frac{2[f(\eta^*) - f(\eta^{max})]}{(k_2 - k_2^{crit})^2}} \tag{363}$$

By l'Hôpital rule one easily obtains

$$\lim_{k_2\to k_2^{crit}} \frac{|\psi_{max}(k_2)|}{|k_2 - k_2^{crit}|} = \left(\frac{d\eta^*}{dk_2}|_{k_2=k_2^{crit}}\right) = \sqrt{\eta^*(1-3\eta^*)(1-2\eta^*)}. \tag{364}$$

But this latter is exactly the expression of $-f_{\eta\eta}^{-1}(\eta^*)$, thus one computes

$$\lim_{k_2\to k_2^{crit}} \frac{|\psi_{max}(k_2)|}{|k_2 - k_2^{crit}|} = \sqrt{-\frac{1}{f_{\eta\eta}(\eta^*)}} \tag{365}$$

This latter result implies that for $|k_2 - k_2^{crit}|$ small enough we may write

$$|\psi_{max}(k_2)| \simeq \sqrt{-\frac{1}{f_{\eta\eta}(\eta^*)}}|k_2 - k_2^{crit}| \tag{366}$$

and thus, not unexpectedly, we obtain

$$\chi_R = \frac{1}{2}\left(\sqrt{-\frac{1}{f_{\eta\eta}(\eta^*)}}\right) \tag{367}$$

which proves that at criticality the susceptibility does not diverge with N, but rather it has a finite jump.

We can rewrite (367) as

$$\chi_R = \frac{\psi_{max}^2}{2(k_2 - k_2^{crit})^2} \tag{368}$$

Thus, if $k_2^{max}(N)$ is defined as the value of k_2 for which

$$\psi_{max}(k_2 = k_2(N))\sqrt{\hat{N}} = \delta \tag{369}$$

we obtain

$$\chi_R(\delta, N) = \left(\frac{\delta}{\hat{N}}\right) \frac{1}{(k_2^{crit}(N) - k_2^{crit})^2} \tag{370}$$

which shows that *at fixed* N, and for a given δ, (fixing the rounding interval), χ_R attains its maximum at $k_2^{crit}(N)$. This indicates that $k_2^{crit}(N)$ is a pseudo critical point for the given rounding interval and (366) provides the way such pseudo critical points approach, as N increases, the true critical point k_2^{crit}. Indeed, from (366) and (369) we obtain

$$|k_2^{crit}(N) - k_2^{crit}| \simeq_{N>>1} \delta \frac{\sqrt{-f_{\eta\eta}(\eta^*)}}{\sqrt{\hat{N}}} \tag{371}$$

as stated.

As is well known, the scale fixing the rounding of a first- order transition takes place on a scale proportional to the inverse of the total volume of the system. In our case, this rounding takes place on a scale proportional to the square root of the volume of the manifold. Thus the phase transition at k_2^{crit} is **not** a first order transition and it is possibly associated with the onset of long range order in our statistical system. □

This result strongly suggests that the ensemble of 4-manifolds will develop a genuine extension when $k_2 \to {k_2}^{crit}$, and make this transition point a potential candidate for a non-perturbative theory of quantum gravity. As already emphasized, in order to carry out this program (and eventually remove all the *possibly*'s in the above statements concerning the existence of long range order), we have to compute the two point function and characterize its scaling properties at criticality. The almost spectacular agreement between the analytical approach and the MonteCarlo data indicates that the picture of 4-dimensional simplicial quantum gravity which would emerge from this program should be very similar to that one already elaborated on the basis of computer simulations. However, a delicate point emerges here for what concerns the assessment of the nature of the transition at k_2^{crit}. Some recent MonteCarlo simulations have cast doubts to the standard wisdom concerning its second order nature. It has been observed [45] a double peak in the distribution of order of vertices, and this has been interpreted of some indication of a first order nature of the transition. On the other hand, finite size scaling results, as above, point to a higher order transition. From the numerical simulations point of view the situation is not yet settled. In fact the situation might be quite similar to the one we encounter for compact QED in four dimensions: we have a strong coupling phase were the string tension is different from zero (to be compared with the crumpled phase in 4-dimensional gravity). The string tension scales to zero at the critical point (in 4D-gravity this would correspond to the mass parameter scaling to zero at the critical point). At this critical point it is unclear if there is a two-phase signal, and if the transition is first or higher order. A two-phase signal has first been observed, then it has disappeared in a sort of see-saw game, and in a way akin to the situation we are experiencing in 4D-gravity. This similarity persists in the weak coupling phase: for small coupling constant we have the coulomb phase in QED, *viz.*, for all coupling constants smaller than the critical one we have a continuum massless phase, and indeed in 4D-gravity for all (bare) gravitational coupling constants smaller than the critical one we have branched polymers, which indeed admit a continuum limit. Since, as we have shown, there is more than a superficial agreement between MonteCarlo simulations and our analytical approach, one wonders if there is a way of understanding the origin of a possible two-phase signal and how this may be compatible with the results above concerning the higher order nature of the transition.

The explanation is that the uniform asymptotics controlling the leading behavior of the canonical partition function may generate a bistable behaviour: we have discussed the approach to criticality as if the only leading contribution to $W(k_2, N)_{eff}$ comes from developing in a neighborhood of $\psi_{max}(k_2)\sqrt{N} \to 0$. However, as it is easily checked, $f(\eta_{min})$ is not much different from $f(\eta_{max})$, and $W(k_2, N)_{eff}$ may receive an important contribution from developing around $\psi_{min}(k_2)$, too. It is this further contribution that may generate a double peak and similar hysteresis effects, in a way completely analogous to what happens in the three-dimensional case and which we discuss in details in the next section. These effects tend to obliterate the nature of the transition. They are more manifest and more important in the three-dimensional case essentially because the difference $|f(\eta_{max}) - f(\eta_{min})|$ is smaller than in the four-dimensional case.

E. Hysteresis effects in the three-dimensional case

The confrontation between MonteCarlo simulations and the analytical estimates is quite satisfactory also in the three-dimensional case.

Numerical data in dimension $n = 3$ often employ [43] the pair of variables N_3 and N_0 rather than N_3 and N_1, thus we have to adapt our asymptotic estimates to this new pair of variables before comparing with the simulations. This can be trivially done by noticing that the Dehn-Sommerville relations $N_0 - N_1 + N_2 - N_3 = 0$ and $N_2 = 2N_3$ imply $N_1 = N_3 + N_0$, so that the partition function can be written as

$$\begin{aligned} Z[k_1(k_0), k_3] &= \sum_N \left(\sum_{\lambda=N+N_0} W(\lambda = N + N_0, N) e^{k_1(\lambda=N+N_0)} \right) e^{-k_3 N} \\ &= \sum_N \left(\sum_{\lambda=N+N_0} W(\lambda = N + N_0, N) e^{k_1 N_0} \right) e^{(k_1-k_3)N} \end{aligned} \tag{372}$$

where $N \doteq N_3$, and $\lambda = N_1$.

The delicate point to stress is that in dimension $n = 3$, we cannot any longer use the results of theorem 17 and lemma 13 as they stand. The basic observation is that the function $f(\eta)$, (now expressed in terms of the variable N_0), is, in the interval $\eta_{min} \leq \eta \leq \eta_{max}$, a concave function symmetric with respect to the point η^* where it attains its maximum, *viz.*,

$$f(\eta_{min}) \simeq f(\eta_{max}) \tag{373}$$

Since the parameters driving the uniform asymptotics of the saddle-point evaluation of the canonical partition function are, (see (248) and (249)),

$$\psi_{min}(k_{n-2}) \doteq -sgn(\eta^* - \eta_{min})\sqrt{2[f(\eta^*) - f(\eta_{min})]} \tag{374}$$

and

$$\psi_{max}(k_{n-2}) \doteq sgn(\eta^* - \eta_{max})\sqrt{2[f(\eta^*) - f(\eta_{max})]} \tag{375}$$

then (373) implies that the correct asymptotics of the canonical partition function, in the range $\eta_{min} \leq \eta \leq \eta_{max}$ is provided by a linear combination of the two uniform asymptotics obtained by developing the integral (255) in terms of both $\psi_{min}(k_1)$ and $\psi_{max}(k_1)$, *viz.*,

Theorem 19 *Let us consider the set of all simply-connected 3-dimensional dynamically triangulated manifolds. Let k_0^{inf}, and k_0^{crit} respectively denote the unique solutions of the equations*

$$\frac{1}{3}(1 - \frac{1}{A(k_0)}) + \frac{1}{6} = \eta_{min} = \frac{1}{6} \tag{376}$$

$$\frac{1}{3}(1 - \frac{1}{A(k_0)}) + \frac{1}{6} = \eta_{max} = \frac{2}{9} \tag{377}$$

Let $0 < \epsilon < 1$ small enough, then for all values of the inverse gravitational coupling k_0 such that

$$k_0^{inf} + \epsilon < k_0 < k_0^{crit} - \epsilon, \tag{378}$$

the large N-behavior of the canonical partition function $W(N, k_0)_{eff}$ for 3-dimensional simplicial quantum gravity on a simply connected manifold is given by the uniform asymptotics

$$\frac{c_n}{\sqrt{2\pi}}\hat{N}^{-7/2}e^{(\hat{N}+1)f(\eta_{min})}\left[\frac{\alpha_0}{\sqrt{\hat{N}}}w_0(\psi_{min}(k_0)\sqrt{\hat{N}}) + \frac{\alpha_1}{\hat{N}}w_{-1}(\psi_{min}(k_0)\sqrt{\hat{N}})\right] +$$

$$+\frac{c_n}{\sqrt{2\pi}}\hat{N}^{-7/2}e^{(\hat{N}+1)f(\eta_{max})}\left[\frac{\xi_0}{\sqrt{\hat{N}}}w_0(\psi_{max}(k_0)\sqrt{\hat{N}}) + \frac{\xi_1}{\hat{N}}w_{-1}(\psi_{max}(k_0)\sqrt{\hat{N}})\right] \tag{379}$$

where the constants α_0, α_1, and ξ_0, ξ_1 are given by

$$\alpha_0 \doteq -\frac{\eta_{min}^{-1/2}(1-2\eta_{min})^{-n}}{\sqrt{(1-3\eta_{min})(1-2\eta_{min})}}\frac{\psi_{min}(k_0)}{f_\eta(\eta_{min})} \tag{380}$$

$$\alpha_1 \doteq \frac{\alpha_0}{\psi_{min}(k_0)} - \frac{\eta*^{-1/2}(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{min}(k_0)\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{381}$$

$$\xi_0 \doteq -\frac{\eta_{max}^{-1/2}(1-2\eta_{max})^{-n}}{\sqrt{(1-3\eta_{max})(1-2\eta_{max})}}\frac{\psi_{max}(k_0)}{f_\eta(\eta_{max})} \tag{382}$$

$$\xi_1 \doteq \frac{\xi_0}{\psi_{max}(k_0)} - \frac{\eta*^{-1/2}(1-2\eta^*)^{-n}}{\sqrt{(1-3\eta^*)(1-2\eta^*)}}\left[\frac{1}{\psi_{max}(k_0)\sqrt{-f_{\eta\eta}(\eta^*)}}\right] \tag{383}$$

From this result it follows that $k_3^{crit}(k_0)$ can be formally obtained from an expression similar to (309) by the substitution $k_1 \to \alpha k_0$ and $k_3^{crit}(k_1) \to k_3^{crit}(k_1 \to \alpha k_0) - \alpha k_0$, where $0 < \alpha \leq 1$ is a suitable constant, *viz.*,

$$k_3^{crit}(k_0) = 6\ln\frac{A(\alpha k_0)+2}{3} - \alpha k_0 + 6\ln_{N\to\infty}\frac{\ln c_n}{N} \tag{384}$$

where

$$A(\alpha k_0) \equiv \left[\frac{27}{2}e^{\alpha k_0} + 1 + \sqrt{(\frac{27}{2}e^{\alpha k_0}+1)^2 - 1}\right]^{1/3} +$$

$$+\left[\frac{27}{2}e^{\alpha k_0} + 1 - \sqrt{(\frac{27}{2}e^{\alpha k_0}+1)^2 - 1}\right]^{1/3} - 1 \tag{385}$$

If we fix the rescaling factor c_n according to

$$c_n = [\cos^{-1}(\frac{1}{3})]^{-(6/5)N} \tag{386}$$

and set the constant $\alpha \simeq 0.16$, then it is not difficult to check that the rescaled $k_3^{crit}(k_0)$ has a distribution of values, as k_0 increases, which is in very good agreement with the corresponding values determined by MonteCarlo simulations. Explicitly we get

$$
\begin{aligned}
k_0 = 0 \ldots K_3^{crit}(Mont.Car) = 2.050 \ldots K_3^{crit} = 2.04 \\
k_0 = 1 \ldots K_3^{crit}(Mont.Car) = 2.055 \ldots K_3^{crit} = 2.07 \\
k_0 = 2 \ldots K_3^{crit}(Mont.Car) = 2.070 \ldots K_3^{crit} = 2.11 \\
k_0 = 3 \ldots K_3^{crit}(Mont.Car) = 2.100 \ldots K_3^{crit} = 2.15 \\
k_0 = 4 \ldots K_3^{crit}(Mont.Car) = 2.220 \ldots K_3^{crit} = 2.20
\end{aligned}
\tag{387}
$$

The critical point (corresponding to the crossing of the upper limit $\eta_{max} = \frac{2}{9}$) occurs for $k_0^{crit} \simeq 3.845$ which agrees well with the location of the critical point $k_0^{crit}(Mont.Car) \simeq 3.9$ obtained by the MonteCarlo simulations.

Some aspects of the nature of the transition at the above critical point k_0^{crit} are also rather easily characterized. It is standard wisdom among groups performing MonteCarlo simulations that this transition should be a first order transition, a credence fostered, among other things, also by the fact that in three-dimensional gravity we expect no dynamical graviton to be present, (but this is a very misleading concept in a non-perturbative approach to quantum gravity!). More seriously, the assumed first order nature of the transition is associated to quite pronounced hysteresis effects, the same hysteresis effects that (even if at a much less pronounced level) seem to be manifest also in some recent simulation of 4-dimensional simplicial quantum gravity.

From theorem 19 is clear that as $k_0 \to {k_0}^{crit}$ the system shows a cross-over behavior with the phase connected with small values of k_0, since as k_0 approaches k_0^{crit} both ψ_{max} and ψ_{min} tend to go to 0 and the system shows a bistable behavior yielding for an hysteresis associated with the distinct phases described by ψ_{max} and ψ_{min}. This bistability, if strong enough, can be consistent with a first order nature of the transition at k_0^{crit}.

It is very important to develop the above analysis in order to study in detail the neighborhood of the transition point either in 3-dimensions as well as in 4-dimensions. The analytical approach developed here already provides useful indications, but it can be certainly improved, hopefully to bring to full power dynamical triangulations as a most effective approach to deal with non- perturbative issues in euclidean quantum gravity.

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X. APPENDIX: THE TANGENT SPACE TO THE REPRESENTATION VARIETY

Given the representation $\Theta\colon \pi_1(M) \to G$ let us consider the associated representation θ on the Lie algebra $\mathfrak{g}$ generated by composing Θ with the adjoint action of G on $\mathfrak{g}$, *viz.*, $\theta\colon \pi_1(M) \to_{\Theta} G \to_{Ad} End(\mathfrak{g})$, (henceforth we will always refer to this representation). The representation θ defines a flat bundle, $\mathfrak{g}_\theta$, over the underlying space M of the dynamical triangulation considered. This bundle is costructed by exploiting the adjoint representation of G on its Lie algebra $\mathfrak{g}$, *i.e.*, $Ad\colon G \to End(\mathfrak{g})$, and by considering the action of $\pi_1(M)$ on $\mathfrak{g}$ generated by composing the adjoint action and the representation Θ:

$$\mathfrak{g}_\theta = \hat{M} \times \mathfrak{g}/\pi_1 \otimes [Ad(\Theta(\cdot))]^{-1} \tag{388}$$

where $\pi_1 \otimes [Ad(\Theta(\cdot))]^{-1}$ acts, through $\pi_1(M)$, by deck transformations on $\hat{M}$ and by $[Ad(\Theta(\cdot))]^{-1}$ on the Lie algebra $\mathfrak{g}$. More explicitly, if $\hat{\sigma}_1$, $\hat{\sigma}_2$ are simplices in $\hat{M}$, and g_1, g_2 are elements of $\mathfrak{g}$, then $\hat{\sigma}_1, g_1) \sim (\hat{\sigma}_2, g_2)$ if and only if $\hat{\sigma}_2 = \hat{\sigma}_1 a$, and $g_2 = [Ad(\Theta(a))]^{-1} g_1$, for some $a \in \pi_1(M)$.

Given a dynamical triangulation T_a and a flat bundle $\mathfrak{g}_\theta$ over $M = |T_a|$, we can define homology and cohomology groups with value in $\mathfrak{g}_\theta$, (namely homology and cohomology with values in the sheave of Killing vectors of M_{riem}). On T_a we define a p- *chain* with values in $\mathfrak{g}_\theta$ to be a formal sum $\sum_{i=1}^m g_i \sigma_i^p$ where $\sigma_i^p \in T_a$ is an ordered p-simplex and g_i is an element of the fiber of $\mathfrak{g}_\theta$ over the first vertex of σ_i^p. Namely, first we consider chains with coefficients in the Lie algebra $\mathfrak{g}$, *viz.*, $\sum_j g_j \hat{\sigma}_j^i$ with $g_j \in \mathfrak{g}$, and then quotient the resulting chain complex $C(\hat{M}) \otimes \mathfrak{g}$ by the action of $\pi_1 \otimes [Ad(\theta(\cdot))]^{-1}$. We denote the group of such chains by $C_p(M, \mathfrak{g}_\theta)$. There is an action of $\pi_1(M)$ on the above chains expressed by

$$a(\sum_j g_j \hat{\sigma}_j^i) \to \sum_j ([Ad(\theta(a))]^{-1} g_j) a(\hat{\sigma}_j^i) \tag{389}$$

for any $a \in \pi_1(M)$, (*i.e.*, we are considering $\mathfrak{g}$ as a $\pi_1(M)$-module). This action commutes with the boundary operator, and we may define homology groups $H_*(M, \mathfrak{g}_\theta)$ with local coefficients in the flat bundle $\mathfrak{g}_\theta$. By dualizing one defines the cohomology $H^*(M, \mathfrak{g}_\theta)$, which enjoys the usual properties of a cohomology theory.

These homology groups are naturally related to the geometry of the infinitesimal deformations of representations of the fundamental group of M. Given the representation $\theta\colon \pi_1(M) \to G$ let us consider the associated representation on the Lie algebra $\mathfrak{g}$ generated by composing θ with the adjoint action of G on $\mathfrak{g}$, *viz.*, $\theta\colon \pi_1(M) \to G \to End(\mathfrak{g})$, (henceforth we will always refer to this representation). Then a 1-*cocycle* on $\pi_1(M)$ with coefficients in θ is a map $c\colon \pi_1(M) \to \mathfrak{g}$ such that for γ, $\delta \in \pi_1(M)$ we have

$$c(\gamma\delta) = c(\gamma) + \theta(\gamma) \cdot c(\delta) \tag{390}$$

The space of such cocycles (often called crossed homomorphisms) is denoted by $Z^1(\pi_1(M), \mathfrak{g})$. We denote by $B^1(\pi_1(M), \mathfrak{g}) \subset Z^1(\pi_1(M), \mathfrak{g})$, the subspace generated by the cocycles which are 1-coboundaries, namely the set of $c \in Z^1(\pi_1(M), \mathfrak{g})$ for which there exists a $g \in \mathfrak{g}$ such that

$$c(\gamma) = \theta(\gamma)g - g \tag{391}$$

for all $\gamma \in \pi_1(M)$. The *first cohomology group* of $\pi_1(M)$ with values in $\mathfrak{g}$ is then defined by

$$H^1(\pi_1(M), \mathfrak{g}) = \frac{Z^1(\pi_1(M), \mathfrak{g})}{B^1(\pi_1(M), \mathfrak{g})} \tag{392}$$

The definition of the higher cohomology groups is somewhat more complicated, and the interested reader may consult [47]. In order to relate $H^1(\pi_1(M), \mathfrak{g})$ to the infinitesimal deformations of the representations of $\pi_1(M)$, let $\frac{Hom(\pi_1(M),G)}{G}$ denote the set of all conjugacy classes of representations of the fundamental group $\pi_1(M)$ into the Lie group G. Notice that if θ and $F\theta F^{-1}$ are two conjugate representations of $\pi_1(M)$ in G, then through the map $Ad(F)\colon \mathfrak{g} \to \mathfrak{g}$ we get a natural isomorphism between the groups $H_i(\pi_1(M), \mathfrak{g})$ and $H_i(\pi_1(M), F\mathfrak{g}F^{-1})$.

We are interested in understanding the structure of $\frac{Hom(\pi_1(M),G)}{G}$ when deforming a particular representation $\theta\colon \pi_1(M) \to G$ through a differentiable one-parameter family of representations θ_t with $\theta_0 = \theta$ which are not tangent to the G-orbit of $\theta \in Hom(\pi_1(M), G)$.

To this end, let us rewrite, for t near 0, the given one-parameter family of representations θ_t as [32], [26]

$$\theta_t = \exp[tu(a) + O(t^2)]\theta(a) \tag{393}$$

where $a \in \pi_1(M)$, and where $u\colon \pi_1(M) \to \mathfrak{g}$. In particular, given a and b in $\pi_1(M)$, if we differentiate the homomorphism condition $\theta_t(ab) = \theta_t(a)\theta_t(b)$, we get that u actually is a one-cocycle of $\pi_1(M)$ with coefficients in the $\pi_1(M)$-module $\mathfrak{g}_\theta$, *viz.*,

$$u(ab) = u(a) + [Ad(\theta(a))]u(b) \tag{394}$$

Moreover, any u verifying the above cocycle condition leads to a map $\theta_t\colon \pi_1(M) \to G$ which, to first order in t, satisfies the homomorphism condition. This remark implies that the (Zariski) tangent space to $Hom(\pi_1(M), G)$ at θ, can be identified with $Z^1(\pi_1(M), \mathfrak{g})$. For this reason, the space $Z^1(\pi_1(M), \mathfrak{g})$ is called the space of infinitesimal deformation of the representation θ.

In a similar way, it can be shown that the tangent space to the Ad-orbit through θ is $B^1(\pi_1(M), \mathfrak{g})$. Thus, the (Zariski) tangent space to $\frac{Hom(\pi_1(M),G)}{G}$ corresponding to the conjugacy class of representations $[\theta]$ is $H^1(\pi_1(M), \mathfrak{g})$ which, therefore, can be identified with the formal tangent space to the representation space.

It is not difficult to establish an explicit isomorphism between the tangent space $H^1(\pi_1(M), \mathfrak{g})$ and the first cohomology group with coefficient in the flat bundle $\mathfrak{g}_\theta$, $H^1(M, \mathfrak{g}_\theta)$, associated with the underlying space M of the dynamical triangulation T_a. We choose a base $x_0 \in M$, (say a vertex of the base simplex σ_0^n), and a corresponding basepoint $\tilde{x}_0 \in \tilde{M}$, the universal cover of M, such that $\pi(\tilde{x}_0) = x_0$, $\pi\colon \tilde{M} \to M$ being the covering map. If g_0 is the fiber of $\mathfrak{g}_\theta$ over x_0, then g_0 is also the fiber of $\pi^*\mathfrak{g}_\theta$ over $\tilde{x}_0$. Since parallel transport (for the flat connection) is independent from the path on $\tilde{M}$, we can parallel transport the fibers of $\pi^*\mathfrak{g}_\theta$ to g_0, and this generates a map $\epsilon\colon C^0(\tilde{M}, \pi^*\mathfrak{g}_\theta) \to g_0$, where $C^0(\tilde{M}, \pi^*\mathfrak{g}_\theta)$ is the space of functions assigning to each vertex of $\tilde{M}$ an element of the fiber of $\pi^*\mathfrak{g}_\theta$ over the vertex considered. By exploiting this map, we can define, for every 1-cocycle α on M with values in $\mathfrak{g}_\theta$, a function $\phi_\alpha \colon \pi_1(M, x_0) \to g_0$ by setting

$$\phi_\alpha(\gamma) = \epsilon(\pi^*\alpha(\tilde{\gamma})) \tag{395}$$

where $\gamma \in \pi_1(M, x_0)$, and $\tilde{\gamma}$ is a simplicial lift of γ to $\tilde{M}$ (starting at x_0). Such ϕ_α is a 1-cocycle on $\pi_1(M, x_0)$ with values in $\mathfrak{g}$, and the mapping $\alpha \to \phi_\alpha$ establishes the seeked isomorphism from the cohomology group $H^1(M, \mathfrak{g}_\theta)$ to the cohomology group $H^1(\pi_1(M), \mathfrak{g})$.

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