QUANTUM GEOMETRY

A statistical field theory approach
Vol. Ia

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Contents

Preface

The main topic of this book is the work that has been carried out during the last 15 years under the general heading of *random surfaces.* The original motivation for the study of random surfaces came from lattice gauge theory, where one can represent various quantities of interest as weighted sums over surfaces embedded in a hypercubic lattice. A few years later, with the resurrection of string theory, random surfaces were used as regularization of that theory and, most recently, random surface models have been applied to two-dimensional quantum gravity. There is also an impressive body of work on random surfaces that has been carried out by membrane physicists, as well as by condensed matter physicists, so one often finds mathematically identical problems being studied in different branches of physics. Random surfaces are therefore not a physical theory but, rather, a theoretical tool and a methodology that can be applied to various physical problems in the same way as random walks find applications in many branches of science. The formalism that has been developed to deal with random surfaces carries over to the study of higher-dimensional manifolds, which are important for quantizing gravity in higher dimensions.

We address this book primarily to advanced graduate students in theoretical physics but we hope that more experienced researchers in the field, as well as mathematicians, may find it useful. As far as applications are concerned, our point of view is very much biased towards string theory and quantum gravity, where the geometric point of view is most important and powerful. This perhaps justifies the book's title.

The purpose of writing this book was not to provide an encyclopaedic account of all the work on random geometry, rather, we have chosen to focus on mathematically precise results that have wide applications, as well as on key results from numerical simulations which have played an important role in guiding the development of the theory. We do not provide complete proofs of the statements in the text where their content is not illuminating or where they are long and tedious, but try to warn the reader when the discussion becomes conjectural.

At the end of most chapters are notes that are intended to guide the reader to the original literature, as well as review articles and extensions of the discussion in the main text. There is no doubt that we have left out many important papers and we apologize in advance for all those omissions.

Important parts of the work reported in this book have been carried out in collaboration with our friends and colleagues A. Beliakova, D. Boulatov, J. Burda, L. Chekhov, J. Fröhlich, J. Greensite, H. P. Jakobsen, J. Jurkiewicz, V. A. Kazakov, C. F. Kristjansen, Y. M. Makeenko, R. Nest, P. Orland, B. Petersson, G. Thorleifsson, S. Varsted, Y. Watabiki and J. Wheater. We are indebted to J. Jurkiewicz, C. F. Kristjansen and Y. M. Makeenko for reading parts of the first draft of the book and pointing out many things that could be improved. M. Lund and G. Xander gave very valuable assistance with the diagrams, the references and the index.

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Notation

The purpose of this note is to explain a few notational conventions that we have used throughout the text, some of which may not be completely standard.

If f is a function of a positive real variable x, then the equation

$$f(x) = O(x^n)$$

means that there is a constant $C > 0$ such that

$$f(x) \leq Cx^n$$

for all values of x or for x in an asymptotic region, i.e. for either large or small x, depending on the context.

If f is as above, then the equation

$$f(x) = o(x^n)$$

means that

$$\lim_{x \to 0} \frac{f(x)}{x^n} = 0.$$

We use $O(x, y)$ as shorthand notation for $O(x) + O(y)$, and similarly for $o(x, y)$.

Finally, let f and g be functions of a real or complex variable x. We employ the notation

$$f(x) \sim g(x)$$

to mean one of three things: which one we have in mind is either explained explicitly in the text or it is supposed to be clear from the context. The first meaning is that the functions f and g are asymptotic as x approaches some limiting value x_0, i.e.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = C,$$

where C is a non-zero constant. It should always be clear from the context what x_0 is. In particular, x_0 can be infinite. The second meaning is that

$$\lim_{x \to x_0} \frac{\log f(x)}{\log g(x)} = C.$$

This case arises in particular when we wish to say that two functions have the same exponential decay. The third meaning is that f and g have a singularity of the same kind at the limiting point x_0. For example, if f and g are real analytic for $x > x_0$ and the nth derivative of f, $f^{(n)}$, is the lowest derivative of f which does not have a finite limit as $x \downarrow x_0$, then $f(x) \sim g(x)$ means that the limit

$$\lim_{x \to x_0} \frac{f^{(n)}(x)}{g^{(n)}(x)}$$

exists and is non-zero.

1
Introduction

The idea of describing the physical world entirely in terms of geometry has a history dating back to Einstein and Klein in the early decades of the century. This approach to physics had early success in general relativity but the appearance of quantum mechanics guided the development of theoretical physics in a different direction for a long time. During the past quarter of a century the programme of Einstein and Klein has seen a renaissance embodied in gauge theories and, more recently, superstring theory. During this time we have also witnessed the happy marriage of statistical mechanics and quantum field theory in the subject of Euclidean quantum field theory, a development which could hardly have taken place without Feynman's path integral formulation of quantization. In this book we shall work almost exclusively in the Euclidean framework.

The unifying theme of the present work is the study of quantum field theories which have a natural representation as functional integrals or, if one prefers, statistical sums, over geometric objects: paths, surfaces and higher-dimensional manifolds. Our philosophy is to work directly with the geometry as far as possible and avoid parametrization and discretizations that break the natural invariances. In this introductory chapter we give an overview of the subject, put it in perspective and discuss its main ideas briefly.

Lagrangian field theories whose action can be expressed entirely in terms of geometrical quantities such as volume and curvature have a special beauty and simplicity. The simplest example is the theory of a free relativistic particle moving in $\mathbb{R}^d$. If the particle moves from a point x to a point y in $\mathbb{R}^d$ along a path ω, we associate with this movement the action

$$S(\omega) = m \int_\omega dl, \tag{1.1}$$

where dl is the length element along the path and m is the mass of the