COURSE 1

QUANTIZATION OF GEOMETRY

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J. Dalibard, J.M. Raimond and J. Zinn-Justin, eds.
Les Houches, Session LIII, 1990
Systèmes Fondamentaux en Optique Quantique
/Fundamental Systems in Quantum Optics

photograph of lecturer

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1. Introduction

The unification of the theory of critical phenomena and Euclidean quantum field theory has been immensely yielding for both areas. The theory of critical phenomena improved in a very important way our understanding of renormalization. The concepts of universality classes, critical points, marginal operators etc. have made many aspects of the renormalization procedure and the renormalization group equations much more intuitive for the field theorists, while the full machinery of Green functions and Feynman integrals have been a most important technical ingredient, which has influenced the theory of critical phenomena.

There is a special area of field theory which is most intimately linked to statistical mechanics, but where it nevertheless was felt that one could not use the methods from statistical mechanics: the quantization of geometrical objects, i.e. the first quantization of the free relativistic particle, the relativistic string and the quantization of gravity. In all cases the classical action is defined entirely in terms of simple geometrical expressions which, however, when written in terms of an explicit parametrization, become ugly and difficult to treat by analytical methods: the length of a curve will involve the square root of the coordinates, just to take an example. Having chosen a parametrization one gets the additional problem that the final result should not depend on this parametrization. Rather absurdly, this is often viewed as a virtue: the theory has a large invariance, namely invariance under diffeomorphism. In reality it reflects our inability to quantize the physical, geometric degrees of freedom. Much is lost compared to the beauty present if we simply view the system as a statistical ensemble of geometrical objects, the partition function being defined as the integral over all such objects, the weight being the exponential of the classical action which itself is defined entirely in terms of the geometry.

When the link is made between statistical systems and Euclidean quantum field theory, the discretization of space is usually a key ingredient. By discretizing ordinary space and restricting the volume to be finite we approximate the field theoretical problem by a finite dimensional problem, which can often be viewed as a generalized lattice spin problem. In the infinite volume limit we can look for phase transitions of the spin system, and if these are characterized by a divergent correlation length, it is possible to forget the underlying lattice structure: we can take the lattice spacing to zero compared with the correlation length, and in this limit we might recover a continuum field theory. Usually masses and coupling constants are defined not *at* the critical point but by *the approach to* the critical point. In this process space is just playing a spectator role. We do not demand any *local* invariance maintained and this is why we can discretized space without any problems. Lattice operators which break explicitly Euclidean invariance will be suppressed in the scaling limit. Local invariance usually mixes high an low frequencies and is much more difficult to discretize in a simple way. This is often used as an argument against attempts to discretize geometrical theories and it is presumably correct that it is rather fruitless to attempt a discretization of a given parametrization of the geometry. If we, on the contrary, choose to discretize the geometry directly there will be no problems at all. The main theme of these lectures will be that a discretization of geometry is natural, it fits perfectly with a statistical mechanics interpretation of the theories and the whole machinery of critical phenomena and scaling limits can be applied in a very powerful way to geometrical theories. In addition we will get a pleasant surprise: some of the theories are easier to solve directly at the discretized level than in the continuum, and this fact allows us to study the scaling limit in considerable detail.

What is meant by "geometrical theories"? These are theories which describe the propagation of (d-1)-dimensional manifolds by summing over an appropriate class of d-dimensional manifolds of which they are boundaries. The action might depend only on the intrinsic geometry of the d-dimensional manifolds. If the manifolds are embedded in R^D the action might in addition depend on the extrinsic geometry.

The simplest of such theories describes the propagation of point particles, i.e. we consider the theory of paths $P(x,y)$ between two space points x and y which belong to R^D. The simplest action for the paths will be

$$S[P(x,y)] = m \int_{P(x,y)} \mathrm{d}l + \lambda \int_{P(x,y)} \mathrm{d}l \; |k| + \cdots . \tag{1.1}$$

The first term only refers to an "intrinsic" property of the path, its length, which can be defined without reference to the target space where it is embedded. The second term, a curvature term, refers explicitly to the embedding. In principle we can add higher powers of the curvature and also torsion terms to the action.

If we move up one dimension we get a theory which describes the propagation of one dimensional objects (strings), i.e. we consider the theory of surfaces $S(l_i)$ spanned between boundary strings (or loops) l_i, $i = 1, .., n$. The simplest action for such surfaces will be:

$$S[M(l_i)] = \mu \int_{M(l_i)} \mathrm{d}A(M) + \lambda \int_{M(l_i)} \mathrm{d}A(M) H^2 \cdot \tag{1.2}$$

where $M(l_i)$ denotes a manifold with boundaries l_i and $\mathrm{d}A(M)$ the area element of the manifold. Here a number of interpretations are possible. We can choose to view the area as the induced area of a surface embedded in, say, R^D. This is analogous to the interpretation given in (1.1). Then it is possible to add further terms depending on the extrinsic geometry as indicated in eq. (1.2) where H denotes the extrinsic curvature term. In two dimensions we have in addition the possibility to formulate the theory without reference to a target space (we have the same possibility in one dimension, but the theory will be trivial unless matter fields are added, as will be discussed later). By viewing the variables as intrinsic we consider two-dimensional quantum gravity and the action can be written as

$$S[M(l_i)] = \int_{M(l_i)} \mathrm{d}^2\xi \sqrt{g} \left(\mu - \lambda_1 R + \lambda_2 R^2 \cdots\right). \tag{1.3}$$

In this formula g_{ab} denotes an internal metric. The first term is still the area term now written in terms of the internal metric. R denotes the intrinsic or Gaussian curvature of the manifold.

When we finally move to higher dimensions we usually have no interest in referring to some embedding space and we will discuss the propagation of $(d-1)$-dimensional manifolds b_i via d-dimensional manifolds which have the b_i's as their boundary entirely in terms of intrinsic variables. The action will be

$$S[M(b_i)] = \int_{M(b_i)} \mathrm{d}^d\xi \sqrt{g} \left(\mu - \lambda_1 R + \lambda_2 R^2 \cdots\right) + \text{b.t.}, \tag{1.4}$$

where *b.t.* denotes boundary terms. In a formula like (1.4) the natural variables to consider are equivalence classes $[g_{ab}]$ of metrics. In a continuum formulation it is quite difficult to work directly with such variables. One almost inevitably ends up with g_{ab} themselves. Somewhat surprisingly, the regularized (discretized) quantum theory offers the possibility to work directly with equivalence classes.

The natural way to define the quantum theory corresponding to the above classical actions is to use the path integral. (First) quantization tells us that the propagator $G(b_1, ..., b_n)$ between $d-1$ dimensional boundaries will be

$$G(b_i) = \int \mathrm{d}M(b_i) \ \mathrm{e}^{-S[M(b_i)]}, \tag{1.5}$$

where the summation is over d-dimensional geometries. From this point of view eq. (1.5) defines the theory of fluctuating geometries. We have already defined the action. One has to contribute a meaning to the integration over geometries. A key ingredient in doing so will be to approximate in a natural way the smooth structures by piecewise linear structures. In this way (1.5) will be an ordinary statistical system. It is possible to discuss the critical properties of this system and they agree with the results obtained by continuum methods whenever they are known. In addition (1.5) will provide us with a nonperturbative definition of the theory in cases were continuum methods seem powerless.

In these lectures the intrinsic properties of our geometric objects will be described by the metric. However, as will be clear, this description can be replaced by any other, using more appropriate variables, if needed. The only requirement seems to be that these variables have a natural description on piecewise linear structures.

In the following I will try systematically to develop the quantum theory of geometric objects starting from the simplest one-dimensional objects in (1.1) and ending with the four-dimensional objects in eq. (1.5) relevant to quantum gravity.

2. Bosonic propagators and random paths

2.1. Quantization

The classical action of the free relativistic particle in R^D moving from x to y is, as already mentioned, expressed by:

$$S[P(x,y)] = m_0 \int_{P(x,y)} \mathrm{d}l, \tag{2.1}$$

The classical equations of motion are derived by choosing a parametrization of eq. (2.1) :

$$x(\xi) : [0,1] \to R^D, \qquad x(0) = x, \quad x(1) = y\,. \tag{2.2}$$

With this parametrization one gets:

$$S = m_0 \int_0^1 \mathrm{d}\xi \, \sqrt{(\dot{x}^\mu)^2}\,, \tag{2.3}$$

$$\frac{\delta S}{\delta x^\mu(\xi)} = \frac{\mathrm{d}}{\mathrm{d}\xi}\,(\dot{x}^\mu/|\dot{x}|) = 0, \tag{2.4}$$

where $\dot{x} \equiv \mathrm{d}x/\mathrm{d}\xi$. The obvious solution to the classical equation of motion is $\dot{x} = const.$, i.e. the straight line between x and y, but any reparametrization of the straight line:

$$x^\mu(\xi) \to x^\mu(f(\xi)), \qquad f(0) = 0, \;\; f(1) = 1, \quad \dot{f} > 0, \tag{2.5}$$

is a solution too. This is a reflection of the reparametrization invariance of the geometrical action (2.1)

First quantization of the system is implemented via the path integral. We get the propagator $G(x,y)$ by summing over *all* paths connecting x and y and weighted by $\mathrm{e}^{-S[P]}$. The quantum aspect comes precisely from the fact that not only the path which solves the classical equations of motion contributes to the propagation:

$$G(x,y;m_0) = \int \mathcal{D}P(x,y)\;\, \mathrm{e}^{-S[P(x,y)]}\,. \tag{2.6}$$

Each path should be counted only once in eq. (2.6). Reparametrizations like (2.5) should not be counted as different paths.

In order to contribute a meaning to $\int \mathcal{D}P$ we have to introduce a cut-off. Note that while we usually think about smooth paths, the action is in fact defined on a larger class of paths, the ones which are only piecewise smooth. Let us introduce a cut-off by considering only piecewise linear paths where each step on the path is of length a. This implies that we refrain from discussing structures smaller than a. Note that the *cut-off introduced in this way by definition is reparametrization invariant*, since it refers directly to a length in R^D. The possible length of the paths will now be a multiple of a : $l = na$ and the action of such a path will be $S = m_0 l$. For each piecewise linear path between x and y consisting of n pieces of length a we have to integrate over the possible positions of the $n-1$ interior points compatible with the length assignment a. If we denote the vectors of the n linear parts by $a\hat{e}_i$, $\hat{e}_i$ being a unit vector in R^D, we have:

$$\int \mathcal{D}P(x,y) \to \sum_{n=1}^{\infty} \int \prod_{i=0}^{n} \mathrm{d}\hat{e}_i \; \delta(a \sum \hat{e}_i - (y-x)), \tag{2.7}$$

$$G_a(x,y;m_0) = \sum_{n=1}^{\infty} \mathrm{e}^{-m_0 a n} \int \prod_{i=0}^{n} \mathrm{d}\hat{e}_i \; \delta(a \sum \hat{e}_i - (y-x)). \tag{2.8}$$

We can calculate the propagator at the discretized level by getting rid of the δ-function by a Fourier transformation:

$$G_a(p;m_0) = \int \mathrm{d}x \; \mathrm{e}^{-ip(x-y)} \, G(x,y;m_0) = \sum_{n=0}^{\infty} \mathrm{e}^{-m_0 a \cdot n} \int \prod_{i=1}^{n} \mathrm{d}\hat{e}_i \; \mathrm{e}^{-iap\cdot\hat{e}_i} \, . \tag{2.9}$$

We have

$$\int \mathrm{d}\hat{e} \; \mathrm{e}^{-ip\cdot\hat{e}} = 2\pi^{D/2} \left[\frac{\mathrm{J}_{\frac{D-1}{2}}(pa)}{(pa/2))^{\frac{D-1}{2}}} \right] \equiv f(pa),$$

where

$$f(0) = \mathrm{Vol}(S^{D-1}), \qquad f(pa) \approx f(0)(1 - c^2(pa)^2 + \cdots). \tag{2.10}$$

The final expression for $G_a(p;m_0)$ is

$$G_a(p;m_0) = \sum_{n=0}^{\infty} \left(\mathrm{e}^{-m_0 a} \, f(pa)\right)^n = \frac{1}{1 - \mathrm{e}^{-m_0 a} \, f(pa)}. \tag{2.11}$$

This is an exact expression for the regularized propagator. We have to take $a \to 0$ to get the continuum limit and it is seen that we get the free relativistic propagator if we at the same time *renormalize the bare mass* m_0 such that

$$\mathrm{e}^{-m_0 a} \, f(pa) \to 1 - c^2 \, m_{\mathrm{ph}}^2 a^2, \quad \text{i.e.} \quad m_0 = \frac{\log f(0)}{a} + c^2 \, m_{\mathrm{ph}}^2 a \tag{2.12}$$

From eqs. (2.10) and (2.11) it follows that:

$$G_a(p; m_0) \to \frac{1}{c^2\, a^2} \frac{1}{p^2 + m_{\rm ph}^2} = \frac{1}{c^2\, a^2} G_c(p; m_{\rm ph}), \tag{2.13}$$

where $G_c(p; m_{\rm ph})$ denotes the continuum propagator. The divergent factor which relates G_a and G_c is a kind of wave function renormalization, but it has a physical meaning since the power of a which appears reflects directly the short distance behavior of the propagator, as we shall discuss in detail later.

It is worth rephrasing the above results in terms of dimensionless quantities, and in this way make the statistical mechanics aspect more visible. Introduce $\mu = m_0 a$ and $q = pa$ and consider coordinates in R^D as dimensionless. The steps in the random walk will be of length one and (2.11) reads

$$G_\mu(q) = \sum_{n=0}^{\infty} \mathrm{e}^{-\mu n}\, f(q)^n = \frac{1}{1 - \mathrm{e}^{-\mu}\, f(q)}. \tag{2.14}$$

μ acts like a chemical potential for inserting additional sections in the piecewise linear walk. We have a *critical value* $\mu_c = \log f(0)$ *of the chemical potential* μ. For $\mu > \mu_c$ the sum is convergent for all q and the average number of steps in the random walk is finite. For $\mu < \mu_c$ the sum is divergent for some range of q and the average length of the paths not defined. For $\mu \to \mu_c$ very long paths will dominate the sum and this is where we can take a continuum limit. We can write

$$f(q) \sim \mathrm{e}^{\mu_c}(1 - c^2 q^2), \qquad G_\mu(q) \approx \frac{1}{\mu - \mu_c + c^2 q^2}, \tag{2.15}$$

and at this point we can introduce the physical length scale a, and the physical momentum $p_{\rm ph}$ and the physical mass $m_{\rm ph}$, both intended to be kept fixed for $a \to 0$, by

$$\mu - \mu_c = m_{\rm ph}^2 a^2, \qquad q = c p_{\rm ph} a. \tag{2.16}$$

This defines a as a function of μ by

$$a(\mu) = \frac{1}{m_{\rm ph}} \sqrt{\mu - \mu_c}. \tag{2.17}$$

Later we will later discuss relations like (2.16) in great detail.

It should be emphasized that the critical behavior we have found this way is universal. Any "reasonable" class of random paths should result in the same scaling limit. The piecewise linear paths are convenient because the results, even at the discretized level, are Euclidean invariant. If we choose to regularize the summation over all paths by considering the subclass of paths which can be formed by links on a hyper-cubic lattice in R^D, the regularized propagator will be:

$$G_a = \sum_n \mathrm{e}^{-m_0 a n} \sum_{P_n} \delta(a \sum \hat{e}_i - (y-x)).$$

Again the regulatized propagator can be computed by (lattice) Fourier transformation and in the scaling limit one gets the same results as for the piecewise linear random walks.

2.2. One-dimensional gravity

Let us now turn to a somewhat different quantization of the free propagator. The action (2.1) has a beautiful geometrical interpretation and from the discretized point of view there was no problems associated with the quantization, as explained above. However, the square root, which appears in eq. (2.3) after a choice of parametrization, makes it very difficult to use the action in formal continuum manipulations. For this reason it might be preferable to use a different action which at the classical level is equivalent with the one given in eq. (2.1):

$$S[x,g]_{(x,y)} = \frac{1}{\alpha'} \int_0^1 \mathrm{d}\xi \sqrt{g(\xi)} \left[g^{ab}(\xi) \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\mu}{\partial \xi^b} + \mu \right]. \tag{2.18}$$

In eq. (2.18) $x^\mu(\xi)$ denotes a path in R^D such that

$$x^\mu(0) = x^\mu, \qquad x^\mu(1) = y^\mu,$$

and g_{ab} is an *internal* metric on the one-dimensional manifold given by the parametrization ξ. The indices a, b can only take the value 1, but we have written the action in a general covariant way. Like (2.1) the action (2.18) is invariant under the reparametrization $\xi \to f(\xi)$ defined in eq. (2.5) provided g satisfies

$$g^{(f)}(f(\xi)) = \frac{1}{\dot{f}^2} g(\xi). \tag{2.19}$$

This transformation rule ensures the invariance of internal distance:

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b = g^{(f)}_{ab}\,\mathrm{d}f^a\,\mathrm{d}f^b; \qquad a,b=1. \tag{2.20}$$

The classical equations of motion, obtained by considering g_{ab} and x^μ as independent variables, agree with the classical equations obtained from eq. (2.1):

$$\frac{\delta S}{\delta\sqrt{g}} = -\frac{1}{\sqrt{g}}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\right)^2 + \mu = 0$$
$$\frac{\delta S}{\delta x^\mu} = -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\sqrt{g}}\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\right) = 0.$$

However, it is by no means obvious that the quantum systems defined from eqs. (2.1) and (2.18) agree. The quantum equivalent of eq. (2.6) is:

$$G(x,y) = \int \frac{\mathcal{D}g_{ab}}{\mathrm{Vol(diff)}} \int \mathcal{D}_g x \;\; \mathrm{e}^{-S[x,g]_{(x,y)}}\,, \tag{2.21}$$

where $x(0) = x$ and $x(1) = y$. The integration variable is equivalence classes of metrics, i.e. metrics which are related by reparametrization. This is indicated by the symbolic division by the "volume" of the diffeomorphism group.

In order to define the functional integral in eq. (2.21) we first introduce *a reparametrization invariant cut-off "a"*, i.e. we consider only paths $x(\xi)$ which have no structure below the length scale $\mathrm{d}s = a$, $\mathrm{d}s$ given by eq. (2.20). This can be achieved by restricting ourselves to piecewise linear paths where the *internal length* of the individual pieces is a. For a given smooth metric $g_{ab}(\xi)$ and a given smooth path we can, if we want, approximate the action by a corresponding action of a piecewise linear path: first we discretize the manifold $[0,1]$ parametrized by ξ according the above prescription. Here it is important to note that the only reparametrization invariant quantity which characterizes the manifold is the total length:

$$l = \int_0^1 \mathrm{d}\xi\,\sqrt{g(\xi)} \tag{2.22}$$

The length l is clearly invariant, and for any metric g_{ab} satisfying (2.22) we can transform it to the constant metric $g^{(c)}_{ab} = l^2\delta_{ab}$ by choosing the function f in (2.19) as

$$f(\xi) = \frac{1}{l} \int_0^{\xi} \mathrm{d}\xi' \sqrt{g(\xi')}. \tag{2.23}$$

A given discretized path will always have the length na. For a given l we simply take $n = [l/a]$, and given g_{ab} we can calculate the points ξ_i on the manifold parametrized by ξ where the internal length from the boundary is ia:

$$ia = \int_0^{\xi_i} \mathrm{d}\xi \sqrt{g(\xi)}. \tag{2.24}$$

For a given metric we have discretized the manifold. The continuum action can now be approximated on the discretized manifold in a natural way by using that $\mathrm{d}\xi(\xi_i) \approx a/\sqrt{g(\xi_i)}$ and $\sqrt{g}g^{ab} = 1/\sqrt{g}$ in one dimension:

$$\int_0^1 \mathrm{d}\xi \left[\frac{1}{\sqrt{g}} \left(\frac{\mathrm{d}x_\mu}{\mathrm{d}\xi}\right)^2 + \mu\sqrt{g}\right] \approx \sum_{i=1}^{n-1} a \left[\frac{(x^\mu(\xi_i) - x^\mu(\xi_{i-1}))^2}{a^2} + \mu\right]. \tag{2.25}$$

Note that the discretized action contains no explicit reference to the metric g_{ab}. The ξ_i's appear as one would have expected it from a constant metric. But this is consistent with the observation that any metric is equivalent to a constant metric and the fact that the action reparametrization invariant.

We can now combine the information given above: the integration of equivalence classes of metrics reduces in one dimension to an integration over the length l and in the discretized approach this integration is replaced by a summation over $n = l/a$. The weight of (discretized) configurations will be determined by eq. (2.25). Let us at this stage change to dimensionless quantities, as advocated above. By redefining x and μ we can dispose of α' and choose $a = 1$, and the discretized version of eq. (2.21) will be

$$G_\mu(x,y) = \sum_{n=1}^{\infty} \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n} \prod_{\mu=1}^{D} \mathrm{d}x_i^\mu \; \mathrm{e}^{-\sum_{i=1}^{n+1}(x_i^\mu - x_{i-1}^\mu)^2} \tag{2.26}$$

A few remarks about the formula: we have replaced the index ξ_i with i since it is just a dummy label of an integration variable. We have also defined $x_0 = x$ and $x_{n+1} = y$. Note finally that the index g in the measure $\mathcal{D}_g\phi$ in formula (2.21) is explicit present in the measure in eq. (2.26): it is n.

Since the integrals in eq. (2.26) are Gaussian they can be performed explicitly:

$$\int \prod_{i=1}^{n} \prod_{\mu=1}^{D} \mathrm{d}x_i^\mu \; \mathrm{e}^{-\sum_{i=1}^{n+1}(x_i^\mu - x_{i-1}^\mu)^2} = \pi^{nD/2} \, \frac{\mathrm{e}^{-(x-y)^2/4n}}{n^{D/2}}, \tag{2.27}$$

and we can write

$$G_\mu(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^{D/2}} \; \mathrm{e}^{(\mu_0-\mu)n-(x-y)^2/4n^2}, \qquad \mu_0 = D \log \sqrt{\pi}. \tag{2.28}$$

Again we see that there is a critical point μ_0 above which $G_\mu(x,y)$ is convergent and below which it is divergent.

If we take the Fourier transformation of $G_\mu(x,y)$ we immediately get (2.15) and we conclude that the theories defined by eqs. (2.1) and (2.18) equivalent both at the classical and the quantum level.

The reader is invited to compare the above calculation of $G(x,y)$ defined by eq. (2.21) with a calculation performed entirely in the continuum, where one first has to introduce parametrization and ghosts and in the end a cut-off in order to calculate ill-defined determinants. The advantage of being able to work directly with equivalence classes of metrics should be obvious.

2.3. Scaling relations

The free relativistic quantum particle is described as the scaling limit of the random walk. In the following we will consider models which cannot be solved as completely as was the case above. In general we will not be interested in the complete solution at the discretized level, but only in the scaling limit where we approach the critical point. Let us therefore discuss scaling relations using the random walk as an example. The relations to be derived will be valid in a much broader context and we will use them many times in the rest of these lectures.

Consider a model for random walks from point 0 to x in R^D. Step number n will be characterized by an initial position x_{n-1}, and an (unnormalized) probability distribution $P(v)$ for a step to $x_n = x_{n-1} + v$. Let us assume that P is only a function of $|v|$. These assumptions can (and will be) considerable relaxed in the following, but they are convenient for a first discussion. If we use the notation $x_0 = 0$ and $x_{n+1} = x$ for a random walk of n steps from 0 to x, our generalized propagator can be written as:

$$G_\mu(x) = \sum_n \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n} \mathrm{d}x_i \prod_{i=1}^{n+1} P(|x_i - x_{i-1}|), \qquad \int \mathrm{d}v\, P(v) = \mathrm{e}^{\mu_c}\,. \tag{2.29}$$

Note that $\int \mathrm{d}x\, G_\mu(x)$ can be performed since the integration over probability distributions reduces to a product of single integrals:

$$\chi(\mu) \equiv \int \mathrm{d}x\; G_\mu(x) = \frac{1}{1-\mathrm{e}^{-(\mu-\mu_c)}}. \tag{2.30}$$

$\chi(\mu)$ is called the *susceptibility*[1]. It is now clear that the critical point is μ_c. For $\mu > \mu_c$ (2.29) is convergent for all x. For $\mu < \mu_c$ (2.29) is divergent for all x. One can solve the model given by eq. (2.29) in the scaling limit since it follows from the central limit theorem that the convolution of P many times reduces to the normal distribution, i.e. we get precisely the Gaussian model considered in the last subsection. Let us, however, discuss some general properties of eq. (2.29) which will be of use for the more general models to be considered in the following.

Theorem: $G_\mu(x)$ falls off exponentially for $\mu > \mu_c$.

$p(x) = G_\mu(x)/\chi(\mu)$ is the probability density for a random walk from 0 to x, i.e. $\Delta p(x) = p(x)\Delta x$ is the probability for a random walk from 0 to x and we have the the inequality $\Delta p(x+y) \geqslant \Delta p(x)\Delta p(y)$, simply because the random walks from 0 to $x+y$ which pass through x are a subset of all random walks from 0 to $x+y$. This means that $-\log \Delta p(x)$ is a *sub-additive* function of $|x|$:

$$-\log \Delta p(x) \leqslant -\log \Delta p(\alpha x) - \log \Delta p((1-\alpha)x), \qquad 0<\alpha<1. \tag{2.31}$$

From the sub-additivity alone it follows that the following limit exists

$$\lim_{|x|\to\infty} \frac{-\log G_\mu(x)}{|x|} = m(\mu), \qquad m(\mu) \geqslant 0. \tag{2.32}$$

[1] This notation is borrowed from spin systems, where the spin susceptibility is the second derivative of the free energy with respect to the magnetic field, but also has the interpretation as the integral of the spin-spin correlation function over space.

Since $G_\mu(x)$ by the definition (2.29) is a decreasing function of μ it follows that $m(\mu)$ must decrease as μ decreases towards μ_c. Since $\int \mathrm{d}x\, G_\mu(x)$ exists for $\mu > \mu_c$ we conclude that $m(\mu) \geqslant 0$. It is now clear that $m^{-1}(\mu)$ will serve as the correlation length for the random walk and *we can only obtain a correlation length which is large compared to the individual step length provided* $m(\mu) \to 0$ *for* $\mu \to \mu_c$. We will assume this is the case and introduce the following critical exponents:

$$m(\mu) \sim (\mu - \mu_c)^\nu \qquad \text{for} \quad \mu \to \mu_c. \tag{2.33}$$

$$G_\mu(x) \sim \frac{1}{|x|^{D-2+\eta}} \qquad \text{for} \quad 1 \ll |x| < \frac{1}{m(\mu)}. \tag{2.34}$$

$$\int \mathrm{d}x\, G_\mu(x) \sim \frac{1}{(\mu - \mu_c)^\gamma} \qquad \text{for} \quad \mu \to \mu_c. \tag{2.35}$$

The mass exponent ν, the anomalous scaling dimension η and the susceptibility exponent γ are not independent. They satisfy *Fischer's scaling relation*:

$$\gamma = \nu(2 - \eta). \tag{2.36}$$

The proof of this relation is simple. From the behavior assumed for $G_\mu(x)$ in eqs. (2.33)-(2.35) we can cut off the integration over x in the susceptibility at $1/m(\mu)$, i.e. eq. (2.35) can be estimated as follows:

$$\chi(\mu) \sim \int_{|x|<1/m(\mu)} \mathrm{d}x\, \frac{1}{x^{D-2+\eta}} \sim \frac{1}{m(\mu)^{2-\eta}}.$$

Eq. (2.36) follows from the definition of γ and ν.

For our simple random walk case it follows from eq. (2.30) that $\gamma = 1$ and it follows from the results in the last subsection that $\nu = 1/2$. This implies that $\eta = 0$ and this is why it is called the anomalous scaling dimension: If η is different from zero it is anomalous with respect to the free particle.

The exponent ν has a direct geometric meaning:

$$d_H = 1/\nu, \tag{2.37}$$

where d_H denotes the *Hausdorff dimension* of the random walk. It can be defined in the following way: Let $\langle L \rangle_x$ denote the average length of a

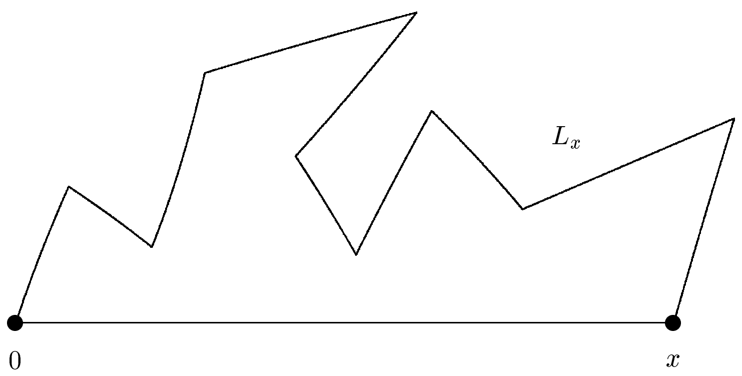

Fig. 1. A typical piecewise linear path between 0 and x.

path from 0 to x in the ensemble of paths defined by eq. (2.29). This is illustrated in fig. 1. Let us define

$$\langle L\rangle_x \sim x^{d_H} \quad \text{for} \quad |x| \to \infty, \quad m(\mu)|x| = \text{const.} \tag{2.38}$$

Note that for any power k we can write for large n:

$$\frac{\int \prod_{i=1}^{n} \mathrm{d}x_i \left(\sum_{i=1}^{n+1} |x_i - x_{i-1}|^k\right) \prod_{i=1}^{n+1} P(x_i - x_{i-1})}{\int \prod_{i=1}^{n} \mathrm{d}x_i \prod_{i=1}^{n+1} P(x_i - x_{i-1})} \sim c_k n.$$

Let us use this for $k = 1$:

$$\langle L\rangle_x \equiv \frac{\sum_n \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n} \mathrm{d}x_i \left(\sum_{i=1}^{n+1} |x_i - x_{i-1}|\right) \prod_{i=1}^{n+1} P(x_i - x_{i-1})}{\sum_n \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n+1} \mathrm{d}x_i \prod_{i=1}^{n+1} P(x_i - x_{i-1})}, \tag{2.39}$$

i.e. we can write:

$$\langle L\rangle_x \sim \langle n\rangle_x = -\frac{\frac{\partial}{\partial\mu} G_\mu(x)}{G_\mu(x)} \sim m'(\mu)|x|. \tag{2.40}$$

For $\mu > \mu_c$ fixed and $|x| \to \infty$ we just get $\langle L\rangle \sim |x|$ and the reason is clear: $\langle n\rangle_x$ is finite for fixed $|x|$ and just goes to infinity proportional to $|x|$. This implies that only paths which are close to the straight line from 0 to x contribute for $\mu > \mu_c$. However, the coefficient $m'(\mu)$ diverges as $\mu \to \mu_c$

and we are interested in the limit where $m(\mu)|x|$ is constant. This is the limit where we can introduce a physical "lattice" length $a(\mu)$, a physical length $x_{\rm ph}$ and a physical mass $m_{\rm ph}$ by

$$m(\mu) = m_{\rm ph} a(\mu), \quad x_{\rm ph} = x\, a(\mu), \quad \text{i.e. } a(\mu) \sim (\mu - \mu_c)^{\nu}, \tag{2.41}$$

such that $x_{\rm ph}$ and $m_{\rm ph}$ are kept fixed for $\mu \to \mu_c$, where $a(\mu) \to 0$. In this limit we have $(\mu - \mu_c)^{\nu} \sim |x|^{-1}$ and

$$\langle L \rangle_x \sim \frac{m'(\mu)}{m(\mu)} \sim \frac{1}{\mu - \mu_c} \sim x^{1/\nu}. \tag{2.42}$$

Since the random walk described by (2.29) has $\nu = 1/2$ we arrive at the well known result that $d_H = 2$, but (2.37) will be valid if $\nu \neq 1/2$. We will meet such situations for the smooth random walks considered in the next subsection.

It should also be emphasized that the "physical" length of $\langle L \rangle_x$ will diverge in the scaling limit if we define $L_{\rm ph} = La(\mu)$. It has to be this way for dimensional reasons:

$$\langle L_{\rm ph} \rangle \sim \frac{x_{\rm ph}^{1/\nu}}{a(\mu)^{(1-\nu)/\nu}}. \tag{2.43}$$

The propagator itself will be singular in the scaling limit. As already mentioned it can be viewed as a kind of wave function renormalization. From the assumed short distance behavior (2.34) we get from (2.41)

$$G_\mu(x) \sim a^{D-2+\eta} G_c(x_{\rm ph}, m_{\rm ph}) \quad \text{for} \quad a \to 0, \tag{2.44}$$

or, for the Fourier transformed where $q = p_{\rm ph} a$:

$$G_\mu(q) \sim a^{\eta-2} G_c(p_{\rm ph}, m_{\rm ph}). \tag{2.45}$$

2.4. Smooth random walks

In the last subsection we saw that ordinary random walks have Hausdorff dimension $d_H = 2$. This result is quite universal as follows from the general expression (2.29) for a random walk. In the scaling limit the different random walk representations all agreed with the direct discretization of the action (2.1). Let us consider the first non-trivial generalization of purely

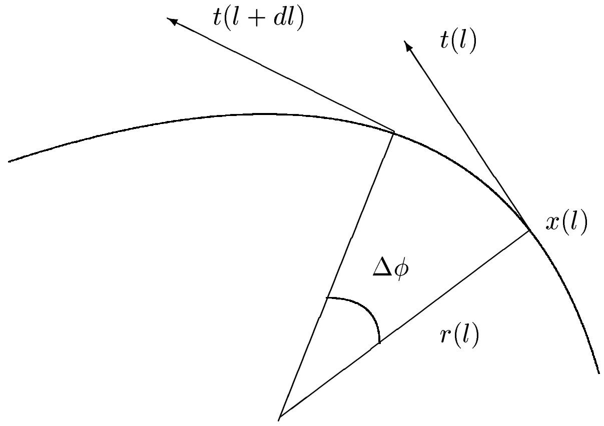

Fig. 2. The geometry related to the calculation of curvature. $r(l) = 1/\kappa(l)$.

geometrical nature:

$$S[P(x,y)] = \mu \int_{P(x,y)} \mathrm{d}l + \lambda \int_{P(x,y)} \mathrm{d}l\ |\kappa(l)|, \tag{2.46}$$

where $k(l)$ denotes the curvature of the path P.

Recall how the curvature of a curve in R^D is defined. Let $x^\mu(\xi)$ be a parametrization of the curve. Use as ξ the length l of the curve. Let $t^\mu(l)$ denote the normalized tangent of the curve:

$$t^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}l} \equiv \dot{x}^\mu, \qquad t^\mu t^\mu = 1, \quad t^\mu \dot{t}^\mu = 0. \tag{2.47}$$

It follows that $\dot{t}^\mu$ is orthogonal to t^μ and we write

$$\dot{t}^\mu = \kappa n^\mu, \tag{2.48}$$

where κ *is the curvature and* n^μ *the principal normal.* It lies in the osculating plane of the curve. The geometry is shown in fig. 2. It is seen that

$$|\Delta t| = 2 \sin \frac{\Delta\phi}{2} \approx \Delta\phi, \tag{2.49}$$

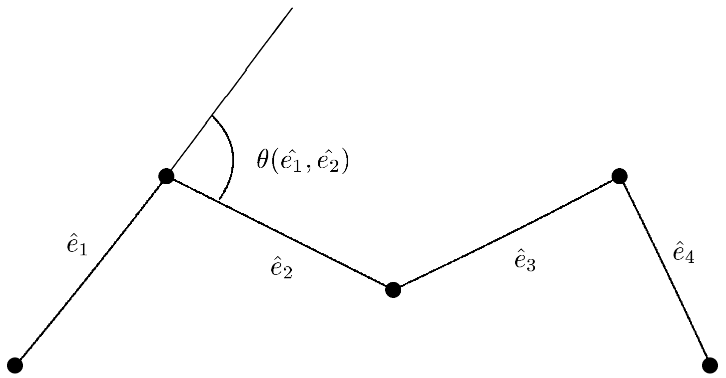

Fig. 3. The angle $\theta(\hat{e}_1,\hat{e}_2)$ between successive tangent vectors in the piecewise linear random walk.

$\Delta\phi$ is called the angle of contingency and we have

$$\kappa(l) = \frac{\mathrm{d}\phi}{\mathrm{d}l}, \qquad \kappa(l) = \frac{1}{r(l)}, \tag{2.50}$$

where $r(l)$ is the radius of curvature. It is the radius of the osculating circle which is defined by the quadratic approximation to the curve and has its center on the principal normal at the distance $r(l)$ from $x^\mu(l)$.

To quantize the theory defined by the action (2.46) we have to perform the path integral[1]. Again it is useful to regularize the sum over all paths by restricting the sum to be over all piecewise linear paths, the length of the individual paths being a (which we choose equal 1) [2]. If the path consists of n linear pieces, the i's piece will be characterized by the a unit vector $\hat{e}_i$. Although there is no universal definition of curvature for such paths, eq. (2.49) is a useful guide, where we simply take $\Delta\phi_i$ to be difference $\theta(\hat{e}_{i+1},\hat{e}_i)$ in angles between $\hat{e}_i$ and $\hat{e}_{i+1}$. This is shown in fig. 3. The discretized version of the action (2.46) reads:

$$S[P_n,\mu,\lambda] = \mu\sum_{i=1}^{n}|\hat{e}_i| + \lambda\sum_{i=1}^{n-1} 2\sin\frac{\theta(\hat{e}_{i+1},\hat{e}_i)}{2}$$

[1] There is an extensive literature on canonical quantization, saddle-point calculations and large d expansions of the theory. Ref. [1] is a very incomplete list.

$$= \mu n + \lambda \sum_{i=1}^{n-1} f(\theta(\hat{e}_{i+1}, \hat{e}_i)), \tag{2.51}$$

where we have already generalized the curvature term to a function $f(\theta)$ satisfying

$$f(0) = 0, \qquad f'(\theta) > 0 \quad \text{for} \quad 0 \leqslant \theta \leqslant \pi. \tag{2.52}$$

The path integral for the propagator from 0 to x can now be written as:

$$G(x; \mu, \lambda) = \int \mathcal{D}P(x) \; \mathrm{e}^{-S[P(x);\mu,l]} \tag{2.53}$$

$$\sim \sum_n \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n} \mathrm{d}\hat{e}_i \; \prod_{i=1}^{n-1} \mathrm{e}^{-\lambda f(\theta(\hat{e}_{i+1}, \hat{e}_i))} \; \delta(\sum \hat{e}_i - x).$$

In order to solve this model let us introduce the probability $K_\lambda(\hat{e}_2, \hat{e}_1)$ for a step $\hat{e}_2$ provided the former step was $\hat{e}_1$:

$$K_\lambda(\hat{e}_2, \hat{e}_1) = \frac{1}{N_\lambda} \mathrm{e}^{-\lambda f(\theta(\hat{e}_2, \hat{e}_1))}, \quad N_\lambda = \int \mathrm{d}\hat{e}_1 \; \mathrm{e}^{-\lambda f(\theta(\hat{e}_2, \hat{e}_1))}. \tag{2.54}$$

With this notation we can write:

$$G(x; \mu, \lambda) \sim \sum_n \left(\mathrm{e}^{-\mu} N_\lambda\right)^n \int \prod_{i=1}^{n} \mathrm{d}\hat{e}_i \prod_{i=1}^{n-1} K_\lambda(\hat{e}_{i+1}, \hat{e}_i) \; \delta(\sum \hat{e}_i - x). \tag{2.55}$$

We can calculate the susceptibility since the additional integration over x just removes of the δ-function in eq. (2.55) and the integrations of $\hat{e}_i$ become trivial ($\int \mathrm{d}\hat{e}_1 K_\lambda(\hat{e}_2, \hat{e}_1) = 1$):

$$\chi(\mu, \lambda) = \int \mathrm{d}x \; G(x; \mu, \lambda) \sim \frac{1}{1 - \mathrm{e}^{-\mu} N_\lambda}. \tag{2.56}$$

For a fixed λ it follows from eq. (2.56) that the critical point $\mu_c(\lambda)$ is determined by:

$$\mathrm{e}^{\mu_c(\lambda)} = N_\lambda, \qquad \text{i.e.} \quad \chi(\mu, \lambda) \sim \frac{1}{\mu - \mu_c(\lambda)}. \tag{2.57}$$

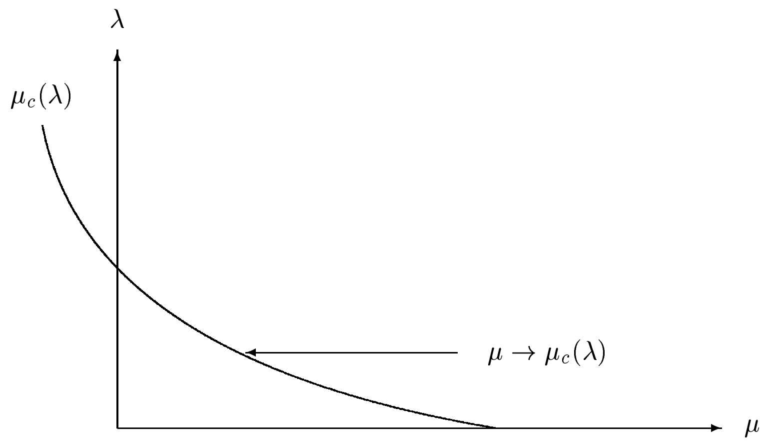

Fig. 4. The phase diagram in the (μ, λ)-plane. The theory is defined to the right of the critical line $\mu_c(\lambda)$.

In the (μ, λ)-coupling constant plane we have a critical line $\mu_c(\lambda)$, as shown in fig. 4. The model (2.55) is defined to the right of the critical line and the scaling limit for a fixed λ leads to the susceptibility (2.57), *i.e. the susceptibility exponent* $\gamma = 1$. From the general arguments presented in the last subsection it follows that $G(x;\mu,\lambda)$ falls of exponentially for large $|x|$ for any (μ, λ) to the right of the critical line:

$$\lim_{|x|\to\infty} \frac{-\log G(x;\mu,\lambda)}{|x|} = m(\mu,\lambda). \tag{2.58}$$

However, in this model we have in addition an independent correlation between tangents:

$$\langle \hat{e}_n \cdot \hat{e}_1 \rangle \equiv \int \prod_{i=1}^{n} \mathrm{d}\hat{e}_i \; [\hat{e}_n \cdot K(\hat{e}_n, \hat{e}_{n-1}) \cdots K(\hat{e}_2, \hat{e}_1) \hat{e}_1]. \tag{2.59}$$

For a fixed λ the tangent-tangent correlation function falls of exponentially with the number of steps:

$$\langle \hat{e}_n \cdot \hat{e}_1 \rangle \sim \mathrm{e}^{-n m_t(\lambda)}, \qquad m(\lambda) > 0. \tag{2.60}$$

This implies that we for a fixed λ just have an ordinary random walk in the scaling limit. After $n_0 \sim 1/m_t(\lambda)$ steps the initial orientation of the a tangent will be lost. If we group together n_0 steps we will have an ordinary random walk where there is no correlation between successive groups of steps. Can we ever get any non-trivial behavior? The proof of (2.60) will show the way to a non-trivial random walk behavior.

In order to prove eq. (2.60) let us view $K_\lambda(\hat{e}_2, \hat{e}_1)$ as a kernel for an operator $\hat{K}_\lambda$ acting on $L^2(S^{D-1})$:

$$(\hat{K}_\lambda \phi)(\hat{e}_2) = \int \mathrm{d}\hat{e}_1 \; K_\lambda(\hat{e}_2, \hat{e}_1)\phi(\hat{e}_1), \qquad \phi \in L^2(S^{D-1}). \tag{2.61}$$

The kernel is symmetric and $K_\lambda(\hat{e}_2, \hat{e}_1) > 0$ and $\hat{K}_\lambda$ is compact. The Perron-Fröbenius theorem tells us that the largest eigenvalue is non-degenerate and that the corresponding eigenfunction is the only one which can be chosen positive. Since the constant function clearly is an eigenfunction of $\hat{K}_\lambda$ with eigenvalue 1 we conclude that 1 is the largest eigenvalue. It is easy to show that -1 is not an eigenvalue. Finally $\phi_a(\hat{e}) = \hat{a} \cdot \hat{e}$ is an eigenfunction since

$$(\hat{K}_\lambda \phi_a)(\hat{e}) \equiv \int \mathrm{d}\hat{e}_1 \; K_\lambda(\hat{e}, \hat{e}_1) \; \hat{a} \cdot \hat{e}_1 = \alpha(\lambda) \; \hat{a} \cdot \hat{e}\,, \tag{2.62}$$

The equality follows since the integral is linear in $\hat{a}$ and invariant under simultaneous rotation of $\hat{a}$ and $\hat{e}$, i.e. proportional to $\hat{a} \cdot \hat{e}$. From Perron-Fröbenius it follows that $\alpha(\lambda) < 1$, and direct calculation shows:

$$0 \leqslant \alpha(\lambda) \to 1 \qquad \text{for} \quad \lambda \to \infty. \tag{2.63}$$

We can directly apply (2.62) in eq. (2.59) to get

$$\langle \hat{e}_n \cdot \hat{e}_1 \rangle = \alpha^n(\lambda) = \mathrm{e}^{-n m_t(\lambda)}, \qquad m_t(\lambda) = \log \frac{1}{\alpha(\lambda)}. \tag{2.64}$$

The only possibility to get a non-trivial scaling limit is to take $\lambda \to \infty$ simultaneously with $\mu \to \mu_c(\lambda)$ since the tangent-tangent correlator can only approach macroscopic distances for $\lambda \to \infty$. At the same time it is necessary that $\mu \to \mu_c(\lambda)$ since this is the only possibility for the two point function to be non-trivial according to (2.58). *Such a scaling limit exists.* In order to construct the limit it is useful to introduce the propagator $G(\hat{e}_f, \hat{e}_i; x)$ which depends explicitly on the first and last step in the random walk: it is defined by (2.55) except that there is no integration over $\hat{e}_1 = \hat{e}_i$ and $\hat{e}_n = \hat{e}_f$. $G(\hat{e}_f, \hat{e}_i; x)$ can be viewed as a kernel for an operator $\hat{G}$ on

$L^2(S^{D-1})$ precisely as $K_\lambda(\hat{e}_2, \hat{e}_1)$ is the kernel of $\hat{K}_\lambda$. Let $|1\rangle$ denote the constant function: $1 = 1(\hat{e})$. We have by definition:

$$G(x;\mu,\lambda) = \langle 1|\, \hat{G}\, |1\rangle \equiv \int \mathrm{d}\hat{e}_f \, \mathrm{d}\hat{e}_i \; G(\hat{e}_f, \hat{e}_i; x). \tag{2.65}$$

By Fourier transformation we can get rid of the δ-function in (2.55). If we introduce the notation $\Delta\mu = \mu - \mu_c(\lambda)$ and denote the Fourier transformation of $G(\hat{e}_f, \hat{e}_i; x)$, $\hat{G}(x)$ and $G(x;\mu,\lambda)$ by $G(\hat{e}_f, \hat{e}_i; q)$, $\hat{G}(q)$ and $G(q;\mu,\lambda)$, we can write:

$$\hat{G}(q) = \sum_n \mathrm{e}^{-\Delta\mu n} \left[\mathrm{e}^{-iq\cdot\hat{e}}\, \hat{K}_\lambda \right]^n = \frac{1}{1 - \mathrm{e}^{-\Delta\mu - iq\cdot\hat{e}}\, \hat{K}_\lambda} \tag{2.66}$$

In this formula $q\cdot\hat{e}$ is viewed as a multiplication operator, i.e.:

$$(f(q\cdot\hat{e})\phi)(\hat{e}) = f(q\cdot\hat{e})\phi(\hat{e}). \tag{2.67}$$

We have in analogy with (2.65):

$$G(q;\mu,\lambda) = \langle 1|\hat{G}(q)|1\rangle \equiv \int \mathrm{d}\hat{e}_f \, \mathrm{d}\hat{e}_i \, G(\hat{e}_f, \hat{e}_i; q). \tag{2.68}$$

While these expressions look somewhat formal, they allow a rather transparent discussion of the scaling limit. Let us consider a fixed λ and take $\Delta\mu$ and q to zero. Consider the matrix element:

$$\langle 1|1 - \mathrm{e}^{-\Delta\mu - iq\cdot\hat{e}}\, \hat{K}_\lambda|1\rangle \approx \langle 1|\Delta\mu + iq\cdot\hat{e} + \frac{1}{2}(q\cdot\hat{e})^2|1\rangle = \Delta\mu + cq^2. \tag{2.69}$$

Let us now introduce the ordinary scaling for the random walk:

$$\Delta\mu \sim m_{\mathrm{ph}}^2 a^2, \qquad q \sim p_{\mathrm{ph}} a, \qquad a(\mu,\lambda) \sim \sqrt{\mu - \mu_c(\lambda)}. \tag{2.70}$$

For fixed λ there is a finite mass gap from 1 to the next lowest eigenvalue. This implies that the only matrix element which contributes to (2.68) to leading order in a is (2.69):

$$\begin{aligned} G(q;\mu,\lambda) &= \langle 1|\frac{1}{1 - \mathrm{e}^{-\Delta\mu - iq\cdot\hat{e}}\, \hat{K}_\lambda}|1\rangle \\ &\sim \frac{1}{a^2}\left[\frac{1}{p_{\mathrm{ph}}^2 + m_{\mathrm{ph}}^2} + O(\sqrt{a}) \right], \end{aligned} \tag{2.71}$$

i.e. just the ordinary propagator. However, if $\lambda \to \infty$ the mass gap goes to zero and more matrix elements will contribute to $\langle 1|\hat{G}(q)|1\rangle$ and the expansion in (2.71) is no longer valid. By Taylor expanding $\phi(\hat{e})$ around a fixed vector $\hat{e}_0$ and using that $e^{-\lambda f(\theta(\hat{e}_0,\hat{e}))}$ will be peaked around $\hat{e}_0$ one can show:

$$\hat{K}_\lambda \to \mathrm{e}^{-c(\lambda)L^2}, \quad c(\lambda) \sim \lambda^{-2} \;\text{ for }\; \lambda \to \infty. \tag{2.72}$$

where L^2 denotes the Beltrami-Laplace operator on S^{D-1}. Let us now introduce a lattice length scale by

$$c(\lambda) = \lambda_{\mathrm{ph}} a(\lambda), \qquad m(\mu,\lambda) \equiv m_{\mathrm{ph}} a(\lambda) = \Delta\mu. \tag{2.73}$$

The last equation fixes μ as a function of λ and defines the approach to $\mu_c(\lambda)$ for $\lambda \to \infty$. From (2.71) we get

$$\begin{aligned} G(q;\mu,\lambda) &= \langle 1| \frac{1}{1-\mathrm{e}^{-\Delta\mu - iq\cdot\hat{e}}\,\hat{K}_\lambda}|1\rangle \\ &\sim \frac{1}{a(\lambda)}\,\langle 1|\frac{1}{\lambda_{\mathrm{ph}}L^2 + m_{\mathrm{ph}} + ip_{\mathrm{ph}}\cdot\hat{e}}|1\rangle. \end{aligned} \tag{2.74}$$

This is our final expression. The matrix element on the rhs is expressed in terms of continuum variables, and the scaling factor in front tells us that $\eta = 1$ if we compare with eq. (2.45). Likewise eq. (2.41) and $m(\mu,\lambda) = \Delta\mu$ shows that $\nu = 1$, i.e. $d_H = 1$. *We have a new class of smooth random walks ($d_H = 1$).* We have already shown that $\gamma = 1$ and the exponents γ, ν and η is our first example of a set of non-trivial exponents. They satisfy Fischer's scaling relation.

The ordinary random walk has a stochastic interpretation as a Brownian motion of a particle, i.e. each step is performed according to some probability distribution P, but is independent of the former steps. With the extrinsic curvature term added the step also depends on the direction of the former step. For a finite coupling constant this does not change the universality class of the random walk, but as we take $\lambda \to \infty$ we enter a new class of random processes characterized by different critical exponents. The interpretation of these is that the *velocity, rather than the position of the particle itself, is changed stochastically according to some probability distribution P*. The path in an ordinary random walk will be continuous, but, with probability one, nowhere differentiable. If the velocity is stochastic the typical path will be differentiable and the first derivative continuous but nowhere differentiable. There is a number of stochastic processes which have this feature. The well known Ornstein-Uhlenbeck process [3] is one of them. It can be shown that the propagation of particles in such processes are described by propagators of the type (2.74), but we have to refer to the original articles for details [2].

The progagator (2.74) is also related to the propagation of a spinning particle with infinitely many components. This will be discussed at the end of the next subsection.

2.5. Fermionic random walks

For the ordinary bosonic propagator we found the following representation:

$$\langle 1|\hat{G}(q)|1\rangle \sim \frac{1}{\Delta\mu + q^2} = \frac{1}{a^2}\frac{1}{m_{\rm ph}^2 + p_{\rm ph}^2}. \tag{2.75}$$

The fermionic propagator in D dimensions is obtained by the change

$$\frac{1}{p^2+m^2} \to \frac{1}{ip^\mu\gamma^\mu + m}, \tag{2.76}$$

where the D γ matrices satisfy:

$$\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}, \qquad (\gamma^\mu)^\dagger = \gamma^\mu \tag{2.77}$$

The lowest dimensional representation of this so-called Clifford algebra is by matrices of dimension $n = [D/2]$. In two dimensions one can take the Pauli matrices σ_1 and σ_2. In three dimensions one can use all three Pauli matrices. The γ-matrices can be used to construct a representation of $\mathrm{SO}(D)$: The matrices

$$s^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] \tag{2.78}$$

satisfy the commutation relations for the generators of $\mathrm{SO}(D)$ and rotations will be generated by

$$K(\omega) = \mathrm{e}^{-is^{\mu\nu}\omega^{\mu\nu}/2}. \tag{2.79}$$

These rotations act on *spinors*, i.e. vectors in C^n, $n = [D/2]$. If

$$\omega^{\mu\nu} = \theta n^{\mu\nu}(\hat{e}_1,\hat{e}_2), \qquad n^{\mu\nu} = \frac{\hat{e}_1^\mu\hat{e}_2^\nu - \hat{e}_1^\nu\hat{e}_2^\mu}{2\sin\theta}, \tag{2.80}$$

where θ is the angle between $\hat{e}_1$ and $\hat{e}_2$, the rotation (2.79) is a rotation with angle θ in the plane spanned by $\hat{e}_1$ and $\hat{e}_2$. An important aspect of the spinor representation of the rotation group is that a 2π-rotation gives -1. This well known fact will be of outmost importance for the fermionic random walk.

Let $x^\mu(\xi)$ be a curve in R^D and let $t^\mu(\xi)$ be the normalized tangent vector, defined by (2.47)-(2.48). t^μ and $\dot{t}^\mu$ span the osculating plane of the curve and we can write:

$$\omega^{\mu\nu} \equiv \frac{1}{2}(t^\mu \dot{t}^\nu - t^\nu \dot{t}^\mu) = \kappa n^{\mu\nu}. \tag{2.81}$$

Recall that the curvature κ is related to the angle of contingency by (2.50). This implies that we can write

$$\omega^{\mu\nu}\,\mathrm{d}l = \mathrm{d}\theta n^{\mu\nu} \tag{2.82}$$

where $n^{\mu\nu}$ is the antisymmetric tensor which defines the osculating plane and $d\theta$ is the angle between tangent vectors $t(l)$ and $t(l + \mathrm{d}l)$. If we consider a discretized random walk and as usual denote the two successive unit vectors as $\hat{e}_i$ and $\hat{e}_{i+1}$ and the angle between them as $\theta(\hat{e}_{i+1}, \hat{e}_i)$ the discrete analogy of (2.82) is

$$\omega^{\mu\nu}(\hat{e}_{i+1}, \hat{e}_i) = \theta(\hat{e}_{i+1}, \hat{e}_i) n^{\mu\nu}(\hat{e}_{i+1}, \hat{e}_i) \tag{2.83}$$

Let $\psi \in C^n$ be a spinor. Let us imagine it "propagates" along the given path in such a way that it is always rotated according to the orientation of the curve, i.e. for each discrete step we perform a rotation given by the rotation matrix

$$K(\hat{e}_{i+1}, \hat{e}_i) = \mathrm{e}^{-is^{\mu\nu}\omega^{\mu\nu}(\hat{e}_{i+1}, \hat{e}_i)}\,. \tag{2.84}$$

The total rotation during a travel along the path P_n will be given by

$$K(P_n) = K(\hat{e}_n, \hat{e}_{n-1}) \cdots K(\hat{e}_2, \hat{e}_1). \tag{2.85}$$

A formal continuum version of $K(P_n)$ for a smooth path is

$$K(P) = \mathcal{P}\,\mathrm{e}^{-\frac{i}{2}\int_P \mathrm{d}l\; s^{\mu\nu}\omega^{\mu\nu}(l)} \tag{2.86}$$

where $\mathcal{P}$ denotes the path ordered integral. This factor appears in the famous *Strominger-Polyakov representation of the fermionic propagator*[4]:

$$G(x,y) = \int \mathcal{D}P(x,y) \;\; e^{-m\int_{P(x,y)} \mathrm{d}l} K(P(x,y)). \tag{2.87}$$

The definition is identical to the one for the bosonic particle except for the matrix K, which rotates a spinor "along the curve". The factor $K(P)$ is rather ill defined and it has been difficult to use this expression. However, from the above definitions it is clear how to write down a well defined discretized version of (2.87) [5]:

$$G_\mu(\hat{e}_f, \hat{e}_i, x) = \sum_n \mathrm{e}^{-\mu n} \int \prod_{i=1}^{n} \mathrm{d}\hat{e}_i \; \prod_{i=0}^{n} K(\hat{e}_{i+1}, \hat{e}_i) \; \delta(\sum \hat{e}_i - x), \tag{2.88}$$

where $\hat{e}_0 = \hat{e}_i$ and $\hat{e}_f = \hat{e}_{n+1}$. This expression is very similar to (2.55) for the bosonic particle and we can directly use the machinery developed in the last subsection. The matrix $K(\hat{e}_2, \hat{e}_1)$ can be viewed as the kernel for an operator $\hat{K}$ which acts on wave functions belonging to $\mathcal{H} = L(S^{D-1}) \times C^n$, i.e. spinors on S^{D-1}. If we normalize K we have:

$$(\hat{K}\psi)(\hat{e}) \equiv \int \mathrm{d}\hat{e}_1 \; K(\hat{e}, \hat{e}_1) \; \psi(\hat{e}_1), \qquad \int \mathrm{d}\hat{e}_1 \; K(\hat{e}, \hat{e}_1) = \hat{1}. \tag{2.89}$$

In the same way we can view $G_\mu(\hat{e}_f, \hat{e}_i, x)$ as the kernel of an operator $\hat{G}(x)$ on $\mathcal{H}$. We get rid of the δ-function in eq. (2.88) by Fourier transformation and express the Fourier transformed $\hat{G}(q)$ as

$$\hat{G}_\mu(q) = \frac{1}{1 - \mathrm{e}^{-\mu - iq\cdot\hat{e}} \, \hat{K}} \tag{2.90}$$

in the same way as for the scalar particle (see eq. (2.66)).

Let us now define the following scaling limit:

$$\mu = m_{\mathrm{ph}} a, \quad q = p_{\mathrm{ph}} a, \tag{2.91}$$

i.e. the propagator becomes:

$$\hat{G}_\mu(q) = \frac{1}{a} \left[\frac{1}{\frac{1-\hat{K}}{a} + m_{\mathrm{ph}} + ip_{\mathrm{ph}} \cdot \hat{e} + O(a)} \right]. \tag{2.92}$$

As usual $p \cdot \hat{e}$ should be viewed as an multiplication operator. As for the scalar particle it can be shown that 1 is the largest eigenvalue and that there is a gap to the next lowest eigenvalue (there is no extrinsic curvature which allows us to tune the gap to zero). It follows that only eigenvectors corresponding to the eigenvalue 1 will propagate for long paths and contribute to matrix elements of $\hat{G}_\mu(q)$ for $a \to 0$. For the scalar particle we only got a non-trivial result from the scaling ansatz (2.91) in the limit where the extrinsic curvature term reduced the mass gap to zero. *Here it is different because the eigenvalue 1 is degenerate.* It is $2n$-times degenerate and the operator $\hat{G}_\mu(q)$ acting on the Hilbert space $\mathcal{H}$ reduces in the scaling limit to $2n \times 2n$ matrix acting on the eigenspace V corresponding to the eigenvalue 1.

Theorem: The constant vectors and the column vectors of the matrix $\gamma^\mu \hat{e}^\mu$ span the $2n$ dimensional eigenspace V of $\hat{K}$ corresponding to eigenvalue 1.

That constant vectors on C^n are eigenfunctions of $\hat{K}$ of eigenvalue 1 is clear. By definition of the rotation we have:

$$K(\hat{e}_2, \hat{e}_1)\gamma \cdot \hat{e}_1 K^{-1}(\hat{e}_2, he_1) = \gamma \cdot \hat{e}_2. \tag{2.93}$$

This shows that the columns of the matrix $\gamma \cdot \hat{e}$ are eigenfunctions of eigenvalue 1 since:

$$\hat{K}\gamma \cdot \hat{e} = \int \mathrm{d}\hat{e}_1 \; K(\hat{e}, \hat{e}_1) \; \gamma \cdot \hat{e}_1 = \gamma \cdot \hat{e} \int \mathrm{d}\hat{e}_1 K(\hat{e}, \hat{e}_1) = \gamma \cdot \hat{e}.$$

We leave it as an interesting exercise to show that there are no other eigenvectors corresponding to the eigenvalue 1 (see [5]).

The projection operators $P^\pm = \frac{1}{\sqrt{2}}(1 \pm \gamma \cdot \hat{e})$ commutes with $\hat{K}$ and split $\mathcal{H}$ into "±-chirality spaces":

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad V = V_+ \oplus V_-.$$

Theorem: On $V_\pm$ the multiplication operators $\hat{e}^\mu$ can be replaced by $\pm\gamma^\mu$-matrices.

The proof is simply by calculation: Let $\psi(\hat{e})$ and $\phi(\hat{e})$ be two vectors in $\mathcal{H}$. The scalar product on $\mathcal{H}$ (and V) is

$$\langle\phi|\psi\rangle = \int \mathrm{d}\hat{e}\; \phi^*(\hat{e})_\alpha \psi(\hat{e})_\alpha$$

where α is a spinor index. Let us now apply this to n vectors $v_\alpha^{(\beta)}(\hat{e}) = P^+_{\alpha\beta}(\hat{e})$. These are orthonormal vectors on V_+. It follows that

$$\langle v^{(\beta_1)}|\hat{e}^\mu|v^{\beta_2}\rangle = \int \mathrm{d}\hat{e}\; v_\alpha^{*(\beta_1)}(\hat{e})\hat{e}^\mu v_\alpha^{(\beta_2)}(\hat{e}) = \frac{1}{D}\gamma^\mu_{\beta_1\beta_2}.$$

We can finally state that the scaling limit (2.92) is given by

$$\hat{G}_\mu(q) \to \frac{1}{a}\begin{pmatrix} \frac{1}{m_{\mathrm{ph}}-ip_{\mathrm{ph}}\cdot\gamma} & 0 \\ 0 & \frac{1}{m_{\mathrm{ph}}+ip_{\mathrm{ph}}\cdot\gamma} \end{pmatrix}. \tag{2.94}$$

We recognize two copies of the Dirac operator corresponding to $V_\pm$. This doubling is needed by the Nielsen-Ninomiya theorem.

On the way we have determined the critical exponents for the fermionic random walk. From eq. (2.92) and $\hat{G}_\mu(q=0) = \int \mathrm{d}x\, \hat{G}(x)$ we get $\gamma = 1$. From $m(\mu) = \mu$ we get $\nu = 1$ and from $G_\mu(q) \sim 1/a$ we get $\eta = 1$. *The fermionic random walk has the same critical exponents as the smooth random walk.* This implies that $d_H = 1$ and effectively we have smooth paths. However, we have no extrinsic curvature term to produce the smoothness. Rather, it comes about because of cancellations between a large number of bosonic paths. To show this let us consider two dimensions (but the mechanism is the same in higher dimensions). From (2.82) it follows that we can write:

$$\omega^{\mu\nu}s^{\mu\nu} = \frac{\mathrm{d}\theta}{\mathrm{d}l}\sigma_3 \tag{2.95}$$

and the eigenvalues of σ_3 is precisely the split of $\mathcal{H}$ in $\mathcal{H}_\pm$. On one of these spaces $K(P)$ (2.86) becomes a phase factor and (2.87) allows a scalar interpretation:

$$G(x) = \int \mathcal{D}P(x)\; \mathrm{e}^{-m\int \mathrm{d}l + \frac{i}{2}\int \mathrm{d}\theta}, \tag{2.96}$$

where $\theta(l)$ is the angle of the tangent relative to a fixed direction in the plane. *It is now essential that we have spinors since the* -1 *resulting from a*

$\Delta\theta = 0$ $\Delta\theta = 2\pi$

Fig. 5. The cancellation between two paths. For the first we get a phase factor $e^{i\Delta\theta/2} = 1$, while the second gives a phase factor $e^{i\Delta\theta/2} = -1$.

2π rotation leads to a cancellation between intersecting and non-intersecting paths as shown in fig. 5. Only smooth paths survive this cancellation.

Even if the mechanism for producing the smooth paths seems very different for the bosonic and the fermionic particles there is a connection. As is rather clear from the derivation of the bosonic propagator (2.74) the tangent vectors are effectively performing a random walk on S^{D-1}. This is just what one would expect from a classical spin and this is made clear from the expression for the operator $\hat{K}_\lambda$ in the scaling limit:

$$\hat{K}_\lambda \sim \mathrm{e}^{-\lambda_{\mathrm{ph}} L^2 a}\,. \tag{2.97}$$

Since $\hat{K}_\lambda$ is the *transfer matrix* for proper time evolution along the path and since the Hausdorff dimension is one, L^2 can be viewed as the continuum Hamiltonian, and we know that it is precisely the Hamiltonian for a classical spinning particle. This interpretation is substantiated by canonical quantization of the particle with extrinsic curvature. The Lagrangian contains higher derivatives and this implies that $\dot{x}/|\dot{x}|$ will be a new additional canonical coordinate which will serve as a the spin. We have to refer to [6] for further discussion. Here it is worth emphasizing that this result is quite natural in the discretized approach. In addition it is seen that the wave function of such a classical spinning particle is an infinite component object as is clear from eq. (2.74). It is closely related to the infinite component spinor introduced by Majorana in an attempt to avoid the negative eigenstates of the Dirac operator. From this point of view it is possible to consider the Dirac propagator in a more a more general context. If the matrix $K(\hat{e}_1, \hat{e}_2)$ had been generalized to contain an extrinsic curvature term, one would have obtained a scaling limit similar to (2.74), the only difference being that

$$\lambda_{\mathrm{ph}} L^2 \to \lambda_{\mathrm{ph}}(J^2 - s^2), \tag{2.98}$$

where s^2 is the spin operator while J^2 refers to the total angular momentum operator. For $\lambda_{\mathrm{ph}} \to \infty$ it will project to the finite dimensional subspace which we obtained above for the ordinary Dirac operator but in a limiting process where the Hausdorff dimension always is one.

It is natural to ask the following question: Although we have found a nice discretized version of (2.87) and have shown that it leads to the Dirac propagator in the scaling limit, (2.87) itself is not a path integral of an action. Is there a path integral which leads to (2.87)? The answer is yes! The supersymmetric generalization of the bosonic action (2.18). The supersymmetric partner of the field x^μ is a Grassmann variable ψ^μ while the supersymmetric partner of $e(\xi) \equiv \sqrt{g(\xi)}$ is a "gravitino" field $\chi(\xi)$ and we have

$$S_B = \int \mathrm{d}\xi \; \left[\frac{1}{e}\dot{x}^2 + e \right] \to$$

$$S_F = \int \mathrm{d}\xi \left[\frac{1}{e}\dot{x}^2 - \psi^\mu \dot{\psi}^\mu + \frac{1}{e}\chi\psi^\mu\dot{x}^\mu + e - \frac{1}{2}\chi \left(\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{-1} \chi \right]. \qquad (2.99)$$

It is possible to show that the path integral over the Grassmann variable χ and ψ results in the factor $K(P)$ in (2.86). Details can be found in [4].

3. Random surfaces and strings

3.1. Definition of the model

The theory of random paths described the relativistic particle. We expect the theory of random surfaces to describe the relativistic string. The strings sweep out a surface while they propagate. The path integral is a sum over all such surfaces with a weight given by the classical action. It is our goal to define this sum and analyze it in detail.

As for the relativistic particle we have two actions which are equivalent at the classical level. The first action is geometrical, only determined by the area spanned between the starting position and the end position of the closed string[2]. Let $M(l_i)$ denote the manifold with boundaries l_i, $i = 1, ..., n$, $F(l_i)$ a corresponding surface in R^D and $x^\mu(\xi)$ the coordinates of F in R^D. The action is

$$S[F(l_i)] = \mu \int_{F(l_i)} \mathrm{d}A(F) \qquad (3.1)$$

[2] For simplicity we consider here only the theory of closed strings.

$$= \mu \int_{M(l_i)} \mathrm{d}^2\xi \sqrt{\left(\frac{\partial x^\mu}{\partial \xi^1}\right)^2 \left(\frac{\partial x^\mu}{\partial \xi^2}\right)^2 - \left(\frac{\partial x^\mu}{\partial \xi^1}\frac{\partial x^\mu}{\partial \xi^2}\right)^2},$$

An alternative description is obtained by introducing an *internal metric* g_{ab}, $a, b = 1, 2$ on $M(l_i)$ and use the following action:

$$S[g,x] = \frac{1}{\alpha'} \int_{M(l_i)} \mathrm{d}^2\xi \sqrt{g} \left[g^{ab} \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial x^\mu}{\partial \xi^b} + \mu \right]. \tag{3.2}$$

The classical equations for the actions (3.1) and (3.2) agree, but it is not at all obvious that the quantum theories are identical. In the case of strings we have many "natural" objects, in contrast to the situation for the free particle where one only has the two-point function. A surface can join an arbitrary number of strings and one is led to consider the n-loop amplitude $G(l_1, ..., l_n)$ between the n loops or strings. The formal path integral expression for the n-loop is written as:

$$G(l_1, \ldots, l_n) = \int \mathcal{D}F(l_i)\ \mathrm{e}^{-S[F(l_i)]}, \tag{3.3}$$

where the integration is over *physical distinct surfaces F in R^D*, or

$$G(l_1, \ldots, l_n) = \int_{M(l_i)} \frac{\mathcal{D}g_{ab}}{\mathrm{Vol(diff)}} \int \mathcal{D}_g x\ \mathrm{e}^{-S[g,x]}. \tag{3.4}$$

where the integration is over all equivalence classes of metrics on $M(l_i)$ and all embeddings $x^\mu(\xi)$.

In order to define (3.3) and (3.4) we have to introduce a *reparametization invariant cut-off.* We follow the procedure outlined for the relativistic particle. In the case (3.3) it amounts to use as a building block a smallest triangle in R^D and glue together these triangles in all possible ways to surfaces with the given boundary conditions. Alternatively one could consider the hyper-cubic lattice surfaces where the surfaces are made of plaquettes [10, 11]. Many of the results we will obtain in the following are valid (and easier to prove) for these models than for the model defined by eq. (3.4). However, eq. (3.4) relates closer to quantum gravity, since it is just two-dimensional gravity coupled to D free scalar fields x^μ, and for this reason it is convenient to use here eq. (3.4) rather than eq. (3.3) as we are going to consider quantum gravity in some detail [7–9] (again there is a large number of articles describing this approach [13, 12, 14, 15], just to mention some of the articles which concentrated on computer simulations of the model).

In the case of the random walk the reparametrization invariant cut-off introduced was related to the shortest distance. In the two dimensional case it is natural to combine shortest distance and smallest area in a single cut-off. The fundamental building block will in this way be an equilateral triangle with edge length a. At this point we encounter a new problem compared with the one-dimensional situation: The gluing is in no way unique. In the process of gluing together triangles to form a two-dimensional manifold the order of a given vertex (i.e. the number of triangles to which the vertex belongs) is almost arbitrary. In the case where we use eq. (3.3) and glue together triangles directly in R^D the answer is clear: We should include all *distinct different*[3] ways of gluing compatible with the boundary conditions since we will get different surfaces in R^D. But also for the model (3.4) where the triangles are defined with respect to the internal metric the freedom of gluing will go hand in hand with the fact that a closed surface, apart from the total area (the equivalence of the total length of the path), also has a new *local* invariant: $R(\xi)$: the Gaussian curvature. $R(\xi)$ cannot be changed by a reparametrization of the surface. In the following it will be argued that the sum over triangulations in a precise way captures this new degree of freedom.

Let us make a short digression and discuss curvature. The curvature (or Riemann) tensor R^d_{abc} can be defined in terms of the metric. It describes the deviation from flat space. This is manifest in the formula for parallel transport of a vector S^a around an infinitesimal closed curve:

$$\Delta S_a = \frac{1}{2} R^d{}_{abc} S_d \oint x^c \, \mathrm{d}x^b. \tag{3.5}$$

The once contracted tensor R_{ab} is called the Ricci tensor and the scalar obtained by contracting the Ricci tensor is called the scalar curvature R:

$$R_{ab} = R^c{}_{acb}, \qquad R = R^a{}_a.$$

In two dimensions there is only one independent component:

$$R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R, \quad R_{ab} = \frac{1}{2} g_{ab} R, \quad R = 2K. \tag{3.6}$$

[3] By different ways of gluing we have in mind that the resulting (abstract) triangulations are different in the way defined below.

where K denotes the Gaussian curvature on the surface. K has the simple geometrical interpretation as being the product of the principal curvatures associated the normal planes intersecting the surface which is assumed to be embedded in R^D in order for this interpretation to make sense. But R itself is of course independent of this embedding. Eq. (3.5) simplifies:

$$\Delta S_a = \frac{1}{2} R S^b \,\mathrm{d}A_{ab}, \qquad d\theta = \frac{1}{2} R \,\mathrm{d}A = K \,\mathrm{d}A, \tag{3.7}$$

where $\mathrm{d}A_{ab} = \frac{1}{2}\oint (x_a \,\mathrm{d}x_b - x_b \,\mathrm{d}x_a)$ is the area tensor and $d\theta$ the infinitesimal angle by which S^a has been rotated during the parallel transport. There is a nice integrated version of this relation, known as (one of the versions of) the Gauss-Bonnet theorem: Let T be a geodesic triangle on the surface, i.e. a triangle where the sides are geodesic curves, with angles β_1, β_2 and β_3. The sum of the angles is no longer π but the deviation from the Euclidean value is given by the integral of the Gaussian curvature over the interior of the triangle:

$$\int_T K \,\mathrm{d}A = \beta_1 + \beta_2 + \beta_3 - \pi \equiv \varepsilon_T, \tag{3.8}$$

where ε_T is called the excess angle of the triangle. The situation is illustrated in fig. 6. Parallel transport of a vector along the boundary of the triangle will rotate the vector by the angle ε_T.

Due to Regge [16] we can define curvature and parallel transport in a natural way on piecewise linear surfaces. The curvature cannot be located in the interior of the triangles since we view the interior as flat. Since the curvature is defined as an intrinsic geometric quantity it is clearly *bending invariant*[4]. Since we can bend the surface around an edge without changing anything we cannot use the edges either. In this way we are lead to locate the curvature of the piecewise linear surfaces at the vertices. To each vertex v we associate a deficit angle ε_v by:

$$\varepsilon_v = 2\pi - \sum_{t \ni v} \alpha_v(t), \tag{3.9}$$

where the summation is over the v-angles of the triangles to which v belongs. This is illustrated in fig. 7 where a geodesic triangle with the vertex v

[4] Historically the Gaussian curvature was defined in terms of the principal curvatures κ_1 and κ_2, which explicitly depended on the embedding. It came as a surprise that $K = \kappa_1\kappa_2$ is independent of the embedding (bending invariant). This is a "Theorema egregium", a "most excellent theorem", wrote Gauss.

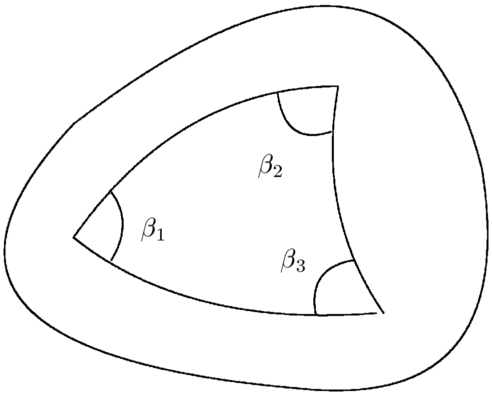

Fig. 6. A geodesic triangle on a smooth surface.

of the piecewise linear manifold is shown. It follows from fig. 7 that $\varepsilon_T = \varepsilon_v$ on the piecewise linear surface, i.e. we can write:

$$\int_T K \;\mathrm{d}A \sim \varepsilon_v, \qquad \int K \;\mathrm{d}A \sim \sum_v \varepsilon_v. \tag{3.10}$$

The lhs of the equations are intended to be valid for smooth surfaces, the rhs for piecewise linear surfaces. In the last equation we have generalized the formula to any region where the boundary is a piecewise geodesic curve and the summation is intended to be over interior vertices in the triangulation. In particular the formula will be valid for closed surfaces.

Let us split ε_v in a curvature term and an area term by assigning the following area to the vertex v:

$$A_v = \frac{1}{3}\sum_{t\ni v} A_t, \qquad K_v = \frac{\varepsilon_v}{A_v}, \tag{3.11}$$

i.e. the area of each triangle is distributed equally among its three vertices. With these definitions we have finally:

$$\int \mathrm{d}A \sim \sum_v A_v, \qquad \int K \;\mathrm{d}A \sim \sum_v K_v A_v. \tag{3.12}$$

Regge originally intended to use these formulas by constructing a sensible sequence of piecewise linear approximations to a *given smooth surface* such

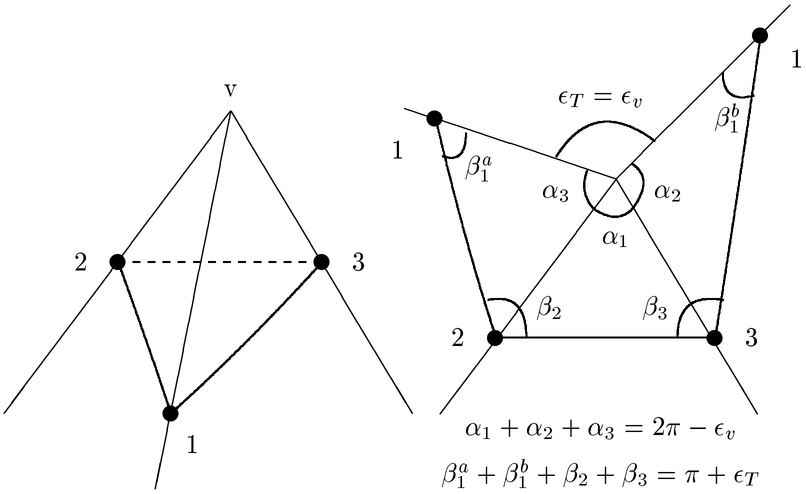

Fig. 7. A geodesic triangle on a piecewise linear surface. The vertex v is an interior point of the triangle. (fig. 7a). Fig. 7b shows the excess angle after the piecewise linear neighborhood has been cut open along a link and unfolded in the plane.

that

$$\sum_v A_v f_v \to \int \mathrm{d}A(\xi) f(\xi), \tag{3.13}$$

$$\sum_v f_v K_v \, A \to \int f(\xi)\, K(\xi)\; \mathrm{d}A(\xi). \tag{3.14}$$

On the same grid one can discretize the covariant action of the free scalar fields in a natural way. For a given triangulation we denote the vertices by indices $i, j, k, \ldots$. The triangulation is characterized by its coincidence matrix, which specifies the neighbor vertices to a given vertex (see below for definition), and by the length l_{ij} of the links between vertices i and j. A natural coordinate system is introduced on the piecewise linear surface by assigning coordinates y_i to the vertices i such that $l_{ij} = |y_i - y_j|$. The y_i's live in some ambient space R^n. The interior of the the triangle (ijk) is parametrized by barycentric coordinates and for a field ϕ defined at the vertices by ϕ_i we use in the same way the linear extension to the interior of the triangle:

$$y = \xi^1 y_i + \xi^2 y_j + (1 - \xi^1 - \xi^2) y_k \tag{3.15}$$

$$\phi(y) = \xi^1 \phi_i + \xi^2 \phi_j + (1 - \xi^1 - \xi^2) \phi_k \tag{3.16}$$

The metric will be defined by

$$g_{ab} = \frac{\partial y^\alpha}{\partial \xi^a} \frac{\partial y^\alpha}{\partial \xi^b}, \quad a, b = 1, 2 \tag{3.17}$$

and it is straightforward to calculate g_{ab} and $g^{ab}\partial_a\phi\partial_b\phi$

$$g_{ab} = \begin{pmatrix} l_{ik}^2 & \frac{1}{2}(l_{ik}^2 + l_{jk}^2 - l_{ij}^2) \\ \frac{1}{2}(l_{ik}^2 + l_{jk}^2 - l_{ij}^2) & l_{jk}^2 \end{pmatrix}, \tag{3.18}$$

$$\begin{aligned} g^{ab}\partial_a\phi\partial_b\phi = \frac{1}{g} \big[& l_{ij}^2(\phi_i - \phi_k)(\phi_j - \phi_k) + \\ & l_{jk}^2(\phi_j - \phi_i)(\phi_k - \phi_i) + l_{ik}^2(\phi_i - \phi_j)(\phi_k - \phi_j) \big], \end{aligned} \tag{3.19}$$

where

$$g = (2\Delta_{ijk})^2 = \frac{1}{2}\left(l_{ik}^2 l_{jk}^2 + l_{ij}^2 l_{kj}^2 + l_{ik}^2 l_{ij}^2 - \frac{1}{2}(l_{ij}^4 + l_{ik}^4 + l_{jk}^4) \right). \tag{3.20}$$

After this digression let us return to the problem of regularizing the integration of Riemannian structures of two dimensional manifolds. The reparametrization invariant regularization suggested above consists of constructing all piecewise linear manifolds obtainable by gluing together equilateral triangles of side-length a and assigning to these the metric structure offered to us by Regge calculus. In the case of equilateral triangles the formulas above simplify a lot: Let n_v denote the order of vertex v in a given triangulation, i.e. the number of triangles which contain vertex v. From eq. (3.11) we get:

$$A_v = \frac{1}{3} n_v \frac{\sqrt{3}}{4} a^2, \qquad R_v A_v = \frac{2\pi}{3}(6 - n_v). \tag{3.21}$$

Let us for simplicity of notation absorb $\sqrt{3}/4$ in a^2 and put the resulting $a = 1$ in the following. The formulas for curvature and for the Gaussian action then read:

$$A_v = \frac{1}{3} n_v, \qquad R_v = 2\pi \frac{6 - n_v}{n_v}, \tag{3.22}$$

$$\sum_v A_v = N_T, \quad \sum_v A_v R_v = 4\pi\chi, \quad \chi = N_T - N_L + N_V. \tag{3.23}$$

Here N_T, N_L and N_V denote the number of triangles, links and vertices in the triangulation, and χ denotes the Euler characteristic of the manifold. (2.33) is the discretized version of the Gauss-Bonnet theorem mentioned above[5]. Finally the action of the free Gaussian field given by eq. (3.19) in the case of equilateral triangles becomes

$$\int \mathrm{d}^2\xi \sqrt{g} g^{ab} \partial_a \phi \partial_b \phi \to \sum_{(ij)} (\phi_i - \phi_j)^2 \tag{3.24}$$

From (3.22) one observes that different triangulations, i.e. triangulations which cannot be mapped onto each other by a simple relabeling of the vertices, lead to different local curvature assignments and consequently inequivalent metric structures. In two dimensions a closed manifold is characterized entirely by its Euler number. Given a manifold we want to integrate over equivalence classes of metrics. Since all the different triangulations we can construct by gluing the equilateral triangles together correspond to inequivalent metrics it is clear that one should sum over all such triangulations. By this prescription one approximates a continuous integration over metrics by the summation over a grid of points in the space of inequivalent metrics. *The conjecture is that this grid becomes uniformly dense when the number of triangles N_T of the triangulations goes to infinity.* We shall later on verify this conjecture.

Although we have used Regge's prescription for assigning curvature, the philosophy outlined is very different from the one which motivated Regge. In the classical Regge calculus the objective was to approximate a given smooth surface by a piecewise linear manifold. A fixed triangulation was chosen and the link length treated as the dynamical variable which should

[5] It is easy to prove eq. (2.33) from Euler's formula which states that $\chi = N_P - N_L + N_V$ for any polygon net covering a surface of topology characterized by χ. If one use Euler's formula on a triangulation ($2N_T = 3N_L$) and combine it with (3.22) one arrive at the formula in (3.23).

be adjusted to get the best approximation to the given manifold[6]. In particular, different link assignment will not necessarily result in a different metric assignment, as is clear by considering triangulations of the plane. Clearly there is a lot of room for moving the vertices around (and thereby changing the link length) without changing the metric at all. Integration over link length is not an integration over equivalence classes of metrics, but involves a highly non-trivial Jacobian. Here we are not interested in approximating specific manifolds, but in using different triangulations to label different equivalence classes of metrics.

The regularized definition of the multi-loop Green functions (3.4) reads:

$$G_\mu(l_1,\ldots,l_n) = \sum_{T\in\mathcal{T}(l_1,\ldots,l_n)} \mathrm{e}^{-\mu N_T} \int \prod_{i\in T/\{l_1,\ldots,l_n\}} \mathrm{d}x_i \;\mathrm{e}^{-\sum_{(ij)}(x_i-x_j)^2}. \tag{3.25}$$

In this formula T denotes an abstract triangulation, defined by its vertices i and a table which tell us the neighbor vertices to a given i. This information is encoded in the *coincidence matrix* $\tilde{C}_T$ which is an $N_V \times N_V$ matrix where the (ij) entry is -1 if i and j are neighbors, 0 if they are not neighbors and n_v, the order of the vertex, on the diagonal. The links are defined to be pairs (ij) of neighbor vertices and the triangles are defined as triples of neighbors (ijk) such that each link (ij), which is not a boundary link, belongs to precisely two triangles (ijk_1) and (ijk_2). The summation is over *different* triangulations. Two triangulations are considered as identical[7] if there is a map between the vertices compatible with the assignment of links and triangles. l_i has a two-fold meaning as a fixed polygon loop in target space R^D and an abstract boundary in the triangulation T and $\mathcal{T}(l_1,...,l_n)$ denotes a suitable class of triangulations with the given boundaries. Usually we have in mind all triangulations of a given topology χ. However, occasionally it is convenient to enlarge the class of simplexes considered, such that they strictly speaking do not form a combinatorial manifold. Local "irregularities" of this kind should not be important, since they are related to short distance effects which should not play any role

[6] The given smooth manifold in the Regge approach was the one given by solution to Einsteins equations with suitable boundary conditions and it would be an extremum of the Einstein-Hilbert action. An important feature in this context is the convergence (3.14) (in higher dimensions where the Einstein-Hilbert action is not a topological invariance), which ensured a good approximation to the action if the triangulations were chosen well in accordance with the geometry of the problem.

[7] Often the word *equivalent* is used for such two triangulations. However, we are going to reserve this notation to triangulations which have a common subdivision.

in the continuum. More serious is the restriction on topology. Formula (3.25) is very tantalizing in the sense that it has no reference to topology. Is it possible that eq. (3.25) provides a non-perturbative definition of the summation over topologies? It has always been an annoying aspect of the continuum formula (3.4) that we only know how to interpret it for a given manifold, i.e. a given topology in the two-dimensional case. A summation over different topologies has to be performed by hand. It turns out that *eq. (3.25) can be used to study the summation over topologies, but not directly as it stands.* A special limit, *the double scaling limit* has to be taken. This will be discussed later. At the moment we will always restrict the class of triangulations $\mathcal{T}$ to mean triangulations with a fixed topology, usually the simplest, the spherical topology. In the following we will also use the notation spherical topology for surfaces with boundaries where we recover the sphere after closing the boundary.

It is often useful to consider a number of special cases of (3.25). If there is no loops at all we talk about *the partition function for closed surfaces*:

$$Z(\mu) = \sum_{T \in \mathcal{T}} \frac{1}{S_T} \, \mathrm{e}^{-\mu N_T} \int \prod_{i \in T / \{i_0\}} \mathrm{d}x_i \; \mathrm{e}^{-\sum_{(ij)} (x_i - x_j)^2} . \tag{3.26}$$

In this formula is included an additional symmetry factor S_T for the triangulation. It is similar to the additional factor which appears in vacuum Feynman diagrams, and it reflects the additional symmetry which can be present for surfaces without a marked boundary: A permutation of the vertices might leave unchanged the links and triangles and in this way not change the surface. S_T is equal to the order of the automorphism group of the graph T. A vertex i_0 is excluded from the integration in order to kill the mode associated with translational invariance. The Gaussian integration is independent of this choice and alternatively one could have chosen to fix the center of mass.

Another limiting case arises if we contract the loops to points, i.e. marked vertices. Strictly speaking this cannot be done in a continuous way on the triangulations. We denote the n-point function $G(x_1, \ldots, x_n)$. It will be given by

$$G_\mu(x_1, \ldots, x_n) = \sum_{T \in \mathcal{T}(i_1, \ldots, i_n)} \frac{\mathrm{e}^{-\mu N_T}}{S_T} \int \prod_{i \in T / \{i_1, \ldots, i_n\}} \mathrm{d}x_i \; \mathrm{e}^{-\sum_{(ij)} (x_i - x_j)^2} \tag{3.27}$$

where the symmetry factor can be different from zero for the 1- and 2-point

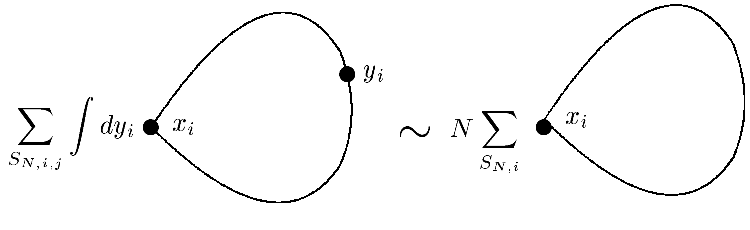

Fig. 8. To each surface $S_{N,i}$ with one marked point x_i corresponds $N-1$ surfaces $S_{N,i,j}$ with two marked points since we can put y_i at the N_1 other vertices. This is true up to symmetry factors.

function. Note that the 1-point function (which by translational invariance is independent of the target space point x_1) is (essentially) equal to (minus) the derivative of the partition function with respect to μ, while the integral of the 2-point function is (essentially) equal to the double derivative of the partition function:

$$G_\mu(x_1) \sim -Z'(\mu), \qquad \int \mathrm{d}x \; G_\mu(x,y) \sim Z''(\mu). \tag{3.28}$$

The equations follow from the observation that differentiation of $Z(\mu)$ multiplies each triangulation with a factor N_T coming from $\mathrm{e}^{-\mu N_T}$. The 1-point function is the summation over marked triangulations, but there are N_V of these for each triangulation without a marked vertex, up to symmetry factors which play no role for the generic large triangulation. Relations like (3.28) will be valid in the limit where triangulations with large N_T dominate. This is illustrated in fig. 8. As we shall see shortly the model will have *a critical point* μ_c. For μ above the critical point all G_μ will be analytic functions of μ, but at the critical point they will contain non-analytic parts. These are the universal parts which have our interest. They are determined by the large N_T part of the triangulations since finite N_T's only produce analytic contributions. The word "essentially" above and the symbol "$\sim$" in (3.28) refers to this non-analytic part determined by the large N_T's. The line of argument can be extended to the n-point function. Let the generalized susceptibility $\chi^{(n)}(\mu)$ be defined as the integral over $n-1$ of the arguments. $\chi^{(n)}(\mu)$ is "essentially" equal to the derivative of $\chi^{(n-1)}(\mu)$ since the derivative brings down a factor N_T in the definition (3.27) while $\chi^{(n)}(\mu)$ contains an additional marked point which produces a factor N_V ($\approx N_T/2$) in the counting of surfaces. It is summarized in the formulas:

$$\chi^{(n)}(\mu) \equiv \int \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}\; G_\mu(x_1,\ldots,x_n) \sim (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\mu^n} Z(\mu). \tag{3.29}$$

3.2. Physical observable

Eqs. (3.25) -(3.28) defined the regularized loop-functions. The following theorem ensures the existence of a critical point μ_c like for the random walk:

Theorem: For surfaces of spherical topology exist a critical point μ_c such that $G_\mu(\{l_i\})$ is convergent for $\mu > \mu_c$ and divergent for $\mu < \mu_c$. μ_c is independent of the boundary loops $\{l_i\}$.

Conjecture: The theorem is true for surfaces of any fixed topology χ and μ_c is the same for all χ.

The theorem will not be proven here, but it is not difficult to show that $Z(\mu)$ is well defined for μ sufficiently large. Two steps are needed. First we have to bound the Gaussian integral for a given triangulation. In fact it is not difficult to show that there exists constants c_l and c_u such that

$$\mathrm{e}^{c_l N_T} \leqslant \int \prod_{i\in T/\{i_0\}} \mathrm{d}x_i \;\mathrm{e}^{-\sum_{(ij)}(x_i-x_j)^2} \leqslant \mathrm{e}^{c_u N_T}\,. \tag{3.30}$$

Secondly we have to use that the number of triangulations of a fixed topology is exponentially bounded. Let us denote the number of triangulations with topology χ, which can be constructed from N triangles as $\mathcal{N}(N,\chi)$: Constants d_l and d_u exist such that

$$\mathrm{e}^{d_l N_T} \leqslant \mathcal{N}(N_T,\chi) \leqslant \mathrm{e}^{d_u N_T}\,. \tag{3.31}$$

Combining eqs. (3.31) and (3.30) we conclude that for any χ:

$$c_l + d_l \leqslant \mu_c \leqslant c_u + d_u. \tag{3.32}$$

For the rest of the theorem we refer to the original article [7]. The conjecture that μ_c is independent of χ is almost certainly true. It has been proven for sums over triangulations coupled to matter with central charge $c \leqslant 1$.

It is worth to notice that the exponential bound on the number of triangulations plays an important role for the existence of the critical point. If we try to define the summation over all topologies directly from eq. (3.25) we will fail due to the entropy of triangulations. The number of triangulations of N triangles $\mathcal{N}(N)$, with no restriction on topology grows faster than factorially and from eq. (3.30) it follows that the sum (3.25) is ill defined for any choice of μ! Later we will discuss some attempts to make sense of the sum (3.25) after all.

After the existence of a critical point is established it is of interest to study the critical behavior of the n-loop and n-point functions when we approach the critical point.

The most important quantity in this context is the mass gap since it determines the possible scaling. The mass gap can be defined by the exponential decay of the two-loop function as the distance d between the loops goes to infinity:

$$m(\mu) = -\lim_{d\to\infty} \frac{G_\mu(l, l_d)}{d}, \tag{3.33}$$

where l_d denotes the loop l displaced a distance d.

Theorem: The two-loop function falls off exponentially with the distance between the two loops.

The argument is the same as for the random walk: since $G_\mu(l, l_{d_1+d_2})$ is the unnormalized probability density for a propagation of a string from l to $l_{d_1+d_2}$ it will, multiplied by an appropriate normalization factor which converts it into a probability, be larger than the corresponding product of $G_\mu(l, l_{d_1})$ and $G_\mu(l_{d_1}, l_{d+d_2})$, since this product (correctly normalized) imposes the constrain that the surfaces from l to $l_{d_1+d_2}$ should pass through l_{d_1}. This is illustrated in fig. 9. I.e. *the (correctly normalized) two-loop function is sub-additive* and since the normalization factor is independent of d the limit (3.33) exists. It follows from the definition that $m(\mu)$ is an increasing function of μ. Unfortunately there exists no proof that $m(\mu)$ goes to zero for $\mu \to \mu_c$. *We will assume this is the case.* We can then define the same critical exponents and fractal dimensions as for the random walk, but let us first introduce a new critical exponent ρ corresponding to the string tension. We define the string tension $\sigma(\mu)$ by the exponential decay of the 1-loop function $G_\mu(l_A)$ for a large planar loop l_A which encloses

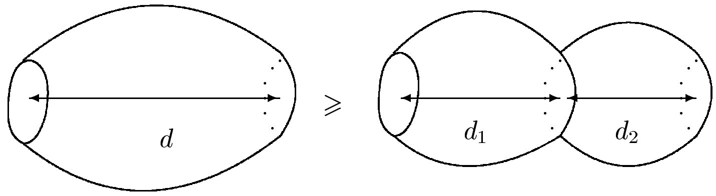

Fig. 9. The sum over surfaces from l to l_d is larger than the restricted sum there is a "bottle neck" at distance d_1.

an area $A \sim l^2$:

$$G_\mu(l_A) \sim A^\beta \, \mathrm{e}^{-\sigma(\mu)A} \,. \tag{3.34}$$

The proof that the $G_\mu(l_A)$ falls off exponentially is again based on sub-additivity: Let A denote both the planar region enclosed by l_A and its area. The number of surfaces which has l_A as boundary is larger than the sub-ensemble where we divide A in two pieces with area A_1 and A_2 by introducing a new boundary in the middle of A which we force the surfaces to respect. We conclude that $G_\mu(l_A) \geqslant G_\mu(l_{A_1})G_\mu(l_{A_2})$ and this implies (3.34)

Why is $\sigma(\mu)$ called the string tension? We can view $G_\mu(l_A)$ as the partition function $Z(A)$ for an ensemble of surfaces which are allowed to fluctuate, but where the boundary, i.e. the frame, is fixed. The Gibbs free energy of the system is $F(A) = -\log Z(A)$. The string tension is defined by change in free energy per unit area if we change the area A

$$\Delta F = \sigma \Delta A. \tag{3.35}$$

From (3.34) it will precisely be our $\sigma(\mu)$ for large A.

Let us now introduce the scaling parameters:
(1): The critical mass exponent ν is defined by the assumed scaling of $m(\mu)$ to zero. This allows us to introduce the physical mass and a length scale which goes to zero at the critical point.

$$m(\mu) \sim (\mu - \mu_c)^\nu, \qquad m(\mu) = m_{\mathrm{ph}} a(\mu). \tag{3.36}$$

(2): The short distance behavior of the 2-point function is characterized by the anomalous scaling exponent η:

$$G_\mu(x,y) \sim |x-y|^{2-d-\eta}, \qquad 1 \ll |x-y| \ll 1/m(\mu), \tag{3.37}$$

(3) The susceptibility $\chi(\mu)$ and the susceptibility exponent γ_s is defined by:

$$\chi(\mu) = \int \mathrm{d}x \; G_\mu(x,y) \sim \frac{1}{(\mu-\mu_c)^{\gamma_s}}. \tag{3.38}$$

(4) The *string tension* $\sigma(\mu)$ is defined by the exponential decay of the 1-loop function and we assume

$$\sigma(\mu) \sim (\mu-\mu_c)^{\rho}. \tag{3.39}$$

From dimensional arguments we expect $\rho = 2\nu$. However, we will prove that $\sigma(\mu)$ does not scale to zero for the simplest model.
(5) The extrinsic Hausdorff dimension d_H of the ensemble of surfaces is defined by

$$\langle Area \rangle_r \sim \langle N_T \rangle_r \sim r^{d_H}, \tag{3.40}$$

where the average area is over an ensemble of surfaces with two marked points a distance r apart in target space R^D.

Let the topology of the surface be spherical except for possible boundaries. If the mass scales to zero it follows, by arguments identical to the ones presented for the random walk, that

$$\gamma_s = \nu(2-\eta), \qquad d_H = 1/\nu, \tag{3.41}$$

i.e. Fischer's scaling relation and the relation between the mass exponent and the Hausdorff dimension. For most systems one has that $0 \leqslant \eta \leqslant 2$. In ordinary statistical systems $\eta = 2$ at the infinite temperature limit, while $\eta = 0$ is the Gaussian approximation. For such systems it is clear that $\nu > 0$ implies $\gamma > 0$. In the following we will often encounter systems where $\gamma_s < 0$. It is important to realize that such systems exist. As a simple model one can take the closed random walk with two marked points x_i and y_j kept fixed in R^D. It has *some* analogy with the 2-point function for random surfaces[8]. Since it is clearly the product of two ordinary random

[8] It does *not* describe the propagation of the desired geometrical object, the particle, between x and y, while the two-point function for the closed string indeed describes the propagation of the string between x and y.

walks between x and y the mass, which is determined by the exponential decay at large distances, will be twice that of the ordinary random walk, i.e. it has the same critical exponent ν. The short distance behavior will be

$$\tilde{G}_\mu(x) \sim \frac{1}{|x|^{2(d-2)}}, \quad \text{i.e.} \quad \eta = d-2. \tag{3.42}$$

For $d > 4$ it follows that $\eta > 2$. It is easy to show directly that $\gamma_s = 2 - d/2$, i.e. Fischer's scaling relation is valid even if $\gamma < 0$.

There exists a theorem which indicates that it might be difficult to obtain a positive γ_s for random surfaces.

Theorem: $\gamma_s \leqslant 1/2$ for the random surfaces model (3.25) with spherical topology.

Conjecture: $\gamma_s > 0$ implies $\gamma_s = 1/2$ for the random surface model (3.25) with spherical topology.

The theorem *and the conjecture* are known to be true for the hyper-cubic random surface model [11]. Rather than giving the rigorous arguments let me present the underlying geometrical reason for the theorem and the conjecture. But first a remark about the technical point which should be dealt with if the arguments should be made exact. In order to apply the cutting and sewing arguments below one would have to introduce somewhat more complicated objects than the n-point functions we have been considering until now: the correct objects to consider are n-loop functions where each boundary loop consists of, say, three links. Fix the center of mass of the boundary vertices of loop i to be x_i, the corresponding point in the n-point function, but integrate over the positions of the boundary vertices compatible with these constraints. The corresponding n-loop function will be a function of x_i, precisely as the n-point function and we expect that they coincide in the scaling limit where the "bare" distances x_i all scale to infinity while the "physical" distances $x_{\text{ph}} = x\, a(\mu)$ stay fixed as $\mu \to \mu_c$, i.e. $a(\mu) \to 0$. The reason is that any contribution where the distances between the boundary vertices are larger than 1 will be exponentially suppressed by the Gaussian action and in the scaling limit distances of order 1 mean physical distances of order $a(\mu)$, i.e. of the order of the "lattice spacing". Had we considered surfaces made of plaquettes living on the hyper-cubic lattice or surfaces in R^D build directly from the gluing of equilateral triangles these problems would be absent and the arguments to be presented would

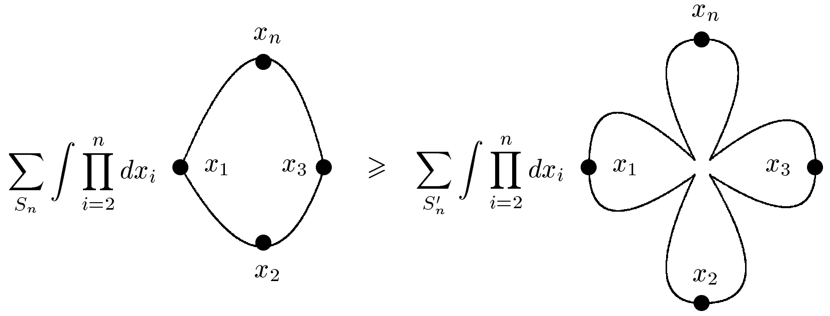

Fig. 10. The sum $\sum_{S_n} \int \cdots$ over all surfaces with n punctures is larger than the restricted sum $\sum_{S'_n} \int \cdots$ over over surfaces S'_n with n punctures where the surfaces S'_n are characterized by a "joint" where the n "bubbles" connected to the x_i's get together. Since we integrate over the position of the joint this effectively implies a factorization in integrated 2-point functions from x_i to the joint.

be exact in the sense that one would not have to integrate over positions of the boundary vertices. In the following we will ignore these complications for the sake of argument.

Consider the generalized susceptibility $\chi^{(n)}(\mu)$ as defined by eq. (3.29). For $n > 2$ we have the obvious geometrical inequality (see fig. 10)

$$\chi^{(n)}(\mu) \geqslant \left(\chi^{(2)}(\mu)\right)^n. \tag{3.43}$$

Assume $\gamma_s > 0$. Eq. (3.38) implies that the 2-point function diverges as $\mu \to \mu_c$, i.e. the non-analytic part dominates and for $\mu \to \mu_c$ it is legal to use eqs. (3.29) and (3.38) in (3.43):

$$\frac{c_n}{(\mu - \mu_c)^{\gamma_s + n - 2}} \geqslant \frac{c_2}{(\mu - \mu_c)^{n\gamma_s}} \quad \text{for} \quad \mu \to \mu_c. \tag{3.44}$$

We conclude that

$$\gamma_s \leqslant 1 - \frac{1}{n-1}, \qquad n > 2. \tag{3.45}$$

It is tempting to apply the formula for $n = 2$, in which case we get $\gamma_s \leqslant 0$. However, (3.43) is not valid for $n = 2$ since there is not a unique decomposition in "joints", as illustrated in fig. 11 which gives the correct decomposition for $n = 2$.

Let us consider the following two random surface models: In one model we allow the gluing of triangles such that a minimal loop-length on the surface can be two. This can only happen if the surface is pinched in a bottle neck consisting of these two loops. Such bottle necks are precisely what we have in mind in fig. 11. We call this class of triangulations $\mathcal{T}_2$. The other class of triangulations differs from $\mathcal{T}_2$ only by not allowing such two-loops[9]. In this class, which is denoted $\mathcal{T}_3$ we can still have bottle necks, the only difference is that the length of the bottle neck loop will be three. In the scaling limit we clearly expect no difference between the random surface models constructed from $\mathcal{T}_2$ and $\mathcal{T}_3$ since the bottle neck loops anyway will be of the order of the cut-off. Denote the n-loop and n-point functions of the two models by G_μ and $\bar{G}_{\bar{\mu}}$. Up to the technical complications mentioned above (but rigorous for the other classes of random surface models mentioned) we have the following identity for the 1-point function, (or more precisely, for the one-loop function where the boundary consists of two links in the way described above).

$$G_\mu = \sum_{\bar{T}\in\mathcal{T}_3} \mathrm{e}^{-\mu N_{\bar{T}}} \left(1+G_\mu)\right)^{N_{\bar{L}}} = \sum_{\bar{T}\in\mathcal{T}_3} \mathrm{e}^{-\bar{\mu}N_{\bar{T}}} = \bar{G}_\mu.$$

$$\bar{\mu} = \mu - \frac{3}{2}\log(1+G_\mu) \tag{3.46}$$

The interpretation of the following: In the class of triangulations $\mathcal{T}_2$ each link serves as a potential source of a bottle neck from which a new *baby universe* can grow. For a given triangulation $T \in \mathcal{T}_2$ we can cut away the maximal size baby universes and close the corresponding two-link boundary. This will leave us with a triangulation which belongs to $\mathcal{T}_2$. In this way we get all triangulations of $\mathcal{T}_2$ with one boundary by summing over $\mathcal{T}_3$ and for each link either do nothing or add a whole one-loop universe, i.e. G_μ itself. Finally we note that $N_{\bar{L}} = 3N_{\bar{T}}/2$.

Eqs. (3.46) and (3.46) define the relation between the two models. Let us use that $\gamma_s < 1$, i.e. that $G_\mu = \text{const.} + (\mu-\mu_c)^{1-\gamma_s}$ is finite at the critical point. Differentiating eqs. (3.46) and (3.46) after $\bar{\mu}$ and μ we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}\bar{\mu}} = 1 - \frac{3}{2}\frac{\bar{\chi}(\bar{\mu})}{1+\bar{G}_{\bar{\mu}}} \tag{3.47}$$

$$\chi(\mu) = \frac{\bar{\chi}(\bar{\mu})}{1-\frac{3}{2}\frac{\bar{\chi}(\bar{\mu})}{1+\bar{G}_{\bar{\mu}}}}, \tag{3.48}$$

[9] In terms of the dual ϕ^3 graphs the difference is that self-energy diagrams are excluded. In both cases ϕ^3 tadpole diagrams are excluded. They correspond to one-loop diagrams.

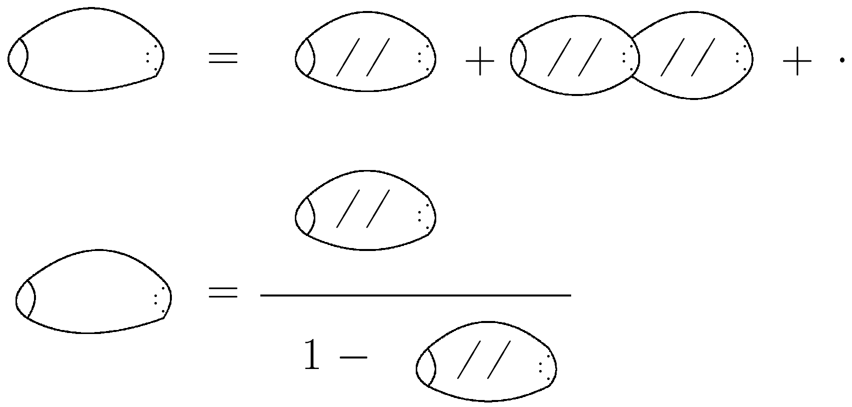

Fig. 11. The decomposition of the 2-loop function for the class of triangulations in parts which cannot be cut in two part along a 2-loop bottle neck such that the two boundary loops are separated. It is possible to perform the sum over the "irreducible" components as shown in the lower part of the figure.

i.e. the algebraic version of the fig. 11. The factor $3/2(1+G_\mu)$ multiplying each bottle neck is a combinatorial factor associated with the outgrow of the baby universes *at* the bottle neck.

By universality the two models based on $\mathcal{T}_2$ and $\mathcal{T}_3$ have the same critical exponents. Let us assume that $\gamma_s > 0$. It implies that $\chi(\mu) \to \infty$ for $\mu \to \mu_c$. The same is true for $\bar{\chi}(\bar{\mu})$ for $\bar{\mu} \to \bar{\mu}_c$, the critical point of the $\mathcal{T}_3$ model. We conclude from (3.48) that $\bar{\mu}(\mu_c) > \bar{\mu}_c$ since $\bar{\chi}(\bar{\mu}(\mu_c)) = 3(1+G_{\mu_c})/2 < \infty$ and we can Taylor expand the rhs of (3.48) around $\bar{\mu}(\mu_c)$:

$$\chi(\mu) \sim \frac{1}{\bar{\mu} - \bar{\mu}(\mu_c)} \sim \frac{1}{\sqrt{\mu - \mu_c}}. \tag{3.49}$$

The last equation follows from (3.47) *which shows that the transformation from $\bar{\mu}$ to μ is non-analytic in μ_c*: At the critical point we have $d\mu/d\bar{\mu} = 0$, and since $\bar{\chi}(\bar{\mu})/(1+\bar{G}_{\bar{\mu}})$ is monotonic decreasing we have $d^2\mu/d\bar{\mu}^2 > 0$:

$$\mu - \mu_c = \text{const.}(\bar{\mu} - \bar{\mu}(\mu_c)^2, \qquad \bar{\mu} - \bar{\mu}(\mu_c) = \text{const.}\sqrt{\mu - \mu_c}. \tag{3.50}$$

The above line of arguments is rigorous for the hyper-cubic random surface model [11]. For the Gaussian model considered it is plausible but not completely proven due to the technical assumptions mentioned. In a later section we shall see that in more elaborate theories with more than one coupling constant it is possible to find a loop-hole in the argument, and indeed a different critical behavior.

The arguments presented above can easily be generalized. The important relation (3.48) is valid not only for $\chi(\mu)$ but for the Fourier transformed $G_\mu(p)$ of the two-point function $G_\mu(x,y)$. Recall that

$$\chi(\mu) = \int \mathrm{d}x \; G_\mu(x,y) = G_\mu(p=0).$$

The extension of (3.48) to $p \neq 0$ has the simple graphical interpretation that a momentum p is flowing through the bubbles in fig. 11. We have

$$G_\mu(p) = \frac{\bar{G}_{\bar{\mu}}(p)}{1 - \frac{3}{2}\frac{\bar{G}_{\bar{\mu}}(p)}{1+\bar{G}_{\bar{\mu}}}}, \tag{3.51}$$

Since $\bar{\mu}$ did not approach $\bar{\mu}_c$ for $\mu \to \mu_c$ we can in this region expand $\bar{G}_{\bar{\mu}}(p)$ as

$$\bar{G}_{\bar{\mu}}(p^2) = \bar{\chi}(\bar{\mu}) - c\,p^2 + \cdots \tag{3.52}$$

where c is constant. The rhs of (3.51) can be expanded around $p=0$ and after the use of (3.48) and little algebra this leads to

$$G_\mu(p) \sim \frac{1}{\chi(\mu)^{-1} + cp^2 + \cdots}. \tag{3.53}$$

This relation shows that *if* $\gamma_s > 0$, i.e. $\chi(\mu)$ diverges for $\mu \to \mu_c$, we have:

$$m(\mu) \sim \chi(\mu)^{-\frac{1}{2}}, \qquad \text{i.e.}\;\; \nu = \frac{1}{2}\gamma_s \quad \text{and} \quad \eta = 0, \tag{3.54}$$

where the last relation follows from Fischer's scaling relation $\gamma_s = \nu(2-\eta)$. In fact (3.54) shows that *for strings embedded in R^D the mass $m(\mu)$ scales to zero if and only if $\chi(\mu)$ is divergent at the critical point.*

A as final application of technique leading to (3.48) let us consider the string tension. We consider a large loop l_A where the boundary length $|l_A| \sim \sqrt{A}$ and the sum of all random surfaces with this loop as boundary. Again we can cut away two-loops and we get directly the analogue of (3.46)[10]

$$G_\mu(l_A) = \bar{G}_{\bar{\mu}}(l_A), \qquad \text{i.e.} \;\; \sigma(\mu) = \bar{\sigma}(\bar{\mu}). \tag{3.55}$$

This relation tells us that the string tension does not scale to zero if $\gamma_s > 0$ since $\bar{\mu}$ does not go to $\bar{\mu}_c$ for $\mu \to \mu_c$ in that case. Let us instead write:

$$\sigma(\mu) = \sigma_0 + c(\mu - \mu_c)^\rho. \tag{3.56}$$

This still defines ρ as a critical exponent and differentiating (3.55) after μ and using (3.47) :

$$\frac{d\sigma}{d\mu} \sim \chi(\mu), \qquad \text{i.e.} \;\; \rho = 1 - \gamma_s. \tag{3.57}$$

If we combine this with $\gamma_s = 1/2$ we get $\rho = 2\nu$ as one would expect from dimensional analysis if the string tension was scaling. It seems still to be satisfied with the definition (3.56). We will return to this definition when we consider strings with extrinsic curvature.

3.3. Non-scaling of the string tension

We have defined the string tension $\sigma(\mu)$ as the exponential decay of the one-loop Green function for large loops (see (3.34)). Above we presented arguments in favor of a non-scaling string tension. The arguments did to constitute a proof since we had to make certain technical assumptions. It is therefore important that there exists a simple rigorous proof of the non-scaling of the string tension [17].

Theorem: $\sigma(\mu) > 0$ for all $\mu \geqslant \mu_c$.

Let us for simplicity assume that we have a large square loop l_{L^2} of area L^2 :

$$G_\mu(l_{L^2}) \sim \mathrm{e}^{-\sigma(\mu) L^2} \quad \text{for} \quad L \to \infty. \tag{3.58}$$

It is possible to bound the Green function $G_\mu(l_{L^2})$ in the following way: The points at the boundary are kept fixed and are not integrated over, as

[10] It is assumed that one cannot have two-loops directly at the boundary. If this assumption is dropped there will be a perimeter term $(1+G_\mu)^{|l_A|}$ on the rhs of (3.55). This term plays no role for the string tension argument.

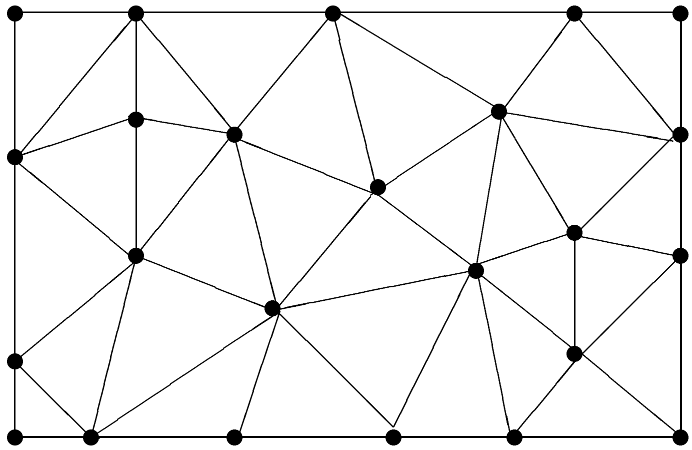

Fig. 12. A typical triangulation of a square loop. When mapped into R^D we integrate over the interior vertices, while the boundary is kept fixed.

it is also assumed in the general notation for the loop Green functions. It is natural to imagine that the density of boundary points is proportional to the length of the perimeter but it is not essential for the following. Let T be one of the triangulations (see fig. 12) in the sum (3.25). The action

$$S[x,T] = \sum_{(i,j)} (x_i - x_j)^2 \tag{3.59}$$

can be bounded because of the following decomposition :

$$S[x,T] = S_{min}(T, l_{L^2}) + S[x', T'] \tag{3.60}$$

For the given (abstract) triangulation T we let $S_{min}(T, l_{L^2})$ denote the minimum of $S[x,T]$ as a function of the coordinates $x^\mu(i)$ of the vertices $i \in T/\partial T$. T' denotes the triangulation where all boundary points are identified. For the surface embedded in R^D we can view it as a contraction of the boundary loop l_{L^2} to a single point 0_L of order $|\partial T| \propto L$. This is illustrated in fig. 13. The decomposition (3.60) follows from the quadratic nature (3.59) of $S[x,T]$.

The loop Green function $G_\mu(l_{L^2})$ for the large square loop l_{L^2} can now

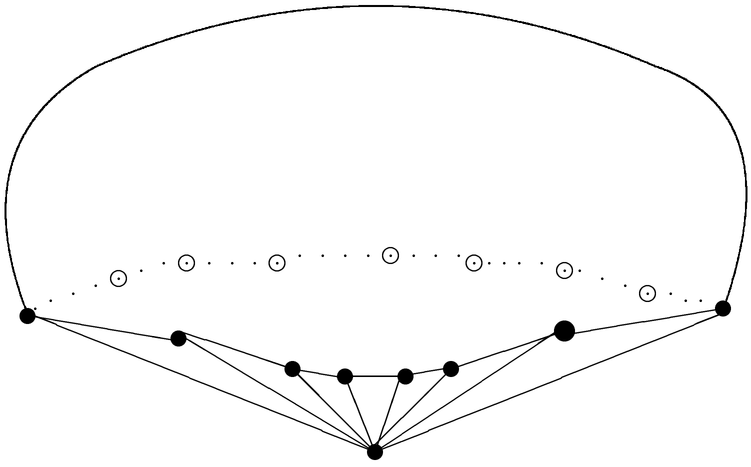

Fig. 13. The surface obtained by contraction the boundary loop of a graph like the one in fig. 12 to one point.

be written as

$$G_\mu(l_{L^2}) = \sum_{T \in \mathcal{T}(l_{L^2})} \mathrm{e}^{-S_{min}(T, l_{L^2}) - \mu N_T} \int \prod_{i \in T/\partial T} \mathrm{d}x(i) \ \mathrm{e}^{-S[x,T']} . \tag{3.61}$$

Next we note that the sum of squares of the length of any two sides of a triangle is $\geqslant 2$ times its area. It follows that

$$S_{min}(T, l_{L^2}) \geqslant 2L^2 \tag{3.62}$$

and from (3.61) we can write

$$G_\mu(l_{L^2}) \leqslant \mathrm{e}^{-2L^2} \ G_\mu(0_L) \tag{3.63}$$

In eq. (3.63) $G_\mu(0_L)$ denotes the loop Green function where the loop l_{L^2} is contracted to one point of order $|l_{L^2}| \propto L$. Since μ_c is independent of boundaries this Green function has the same critical point as ordinary Green functions like $G_\mu(l_{L^2})$ and it can be bounded by

$$G_\mu(0_L) \leqslant \mathrm{e}^{c(\mu)L} \tag{3.64}$$

where $c(\mu)$ is finite for $\mu > \mu_c$. This is a consequence of $\gamma_s \leqslant 1/2$, which implies that the one-point function is finite at the critical point.

From the definition of the string tension it finally follows that

$$\sigma(\mu) \geqslant 2. \tag{3.65}$$

People performing strong coupling expansions will recognize estimates like (3.63) as typical strong coupling estimates. What usually happens is that the function $G_\mu(0_L)$, which is based on a strong coupling approximation, becomes dominant before one reaches μ_c. However, in this case we can control it all the way down to μ_c since $\gamma_s \leqslant 1/2$.

What are the consequences of this non-scaling of the string tension? We have to assume that $m(\mu)$ scales in order to take the continuum limit. As discussed above this uniquely fixes how the lattice spacing scales to zero as a function of μ: $a(\mu) \sim m(\mu)$. It follows that the physical string tension scales to infinity since we have (for dimensional reasons) :

$$\sigma_{\text{ph}} a^2(\mu) = \sigma(\mu) \geqslant 2. \tag{3.66}$$

Since the physical string tension scales to infinity fluctuations including any surfaces having an area different from the minimal area for the given boundary l_{L^2} will be strongly suppressed. When we approach the critical point μ_c we will be left with a class of surfaces consisting of a minimal surface, depending on the Green loop function in question, and singular, spiky, branched polymers growing out everywhere on this surface. Such polymers are essentially one-dimensional objects with no or very little area.

For large dimensions D there is little doubt that this picture is correct. It might still be that it is too coarse an approximation to consider the surfaces strictly as polymers for lower dimensions. According to the theory of such polymers the generic values of γ_s and ν for polymers are $\gamma_s = 1/2$ and $\nu = 1/4$ as we will now explain.

3.4. Branched polymers

Let us return to eq. (3.48) as shown in fig. 11. If $\gamma_s = 1/2$ we concluded that the modified theory based on the class $\mathcal{T}_3$ of triangulations is not critical for $\mu \to \mu_c$ in the model based on $\mathcal{T}_2$. This implies that the individual bubbles in fig. 11 are not critical, i.e. they are of lattice size. The only way the number of triangles can grow to infinity is by the successive gluing of bubbles. All dynamics lie in this gluing and it seems that we get a perfect model of this dynamics by consider a model of *branched polymers*:

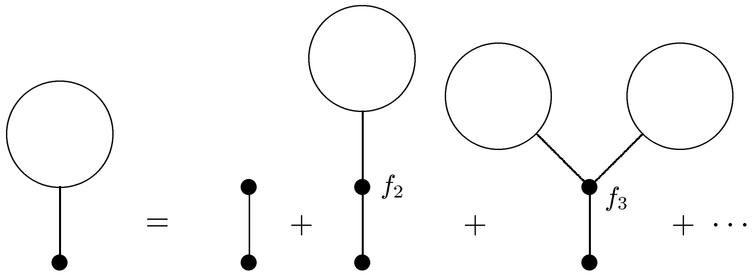

Fig. 14. The equation for rooted branched polymers.

Each individual bubble is represented as a link with an associated chemical potential which we denote μ as in the original model and with a weight factor f_n associated with the joining of n bubbles at a vertex v_n. In the original surface theory we have the possibility of gluing n bubbles to n links which share two vertices. This is the motivation for introducing the factor f_n associated with a branched polymer vertex v_n of order n. If we consider only spherical surfaces (as we will do in the following) *the branched polymer graphs will be tree graphs.* There can be no loops. The partition function for the branched polymers can be written as

$$Z(\mu) = \sum_{BP} \frac{1}{S_{BP}} \mathrm{e}^{-\mu N_{BP}} \prod_v f_{n(v)} \tag{3.67}$$

where the summation is over all branched polymers, i.e all tree graphs, and the product is over all vertices in the tree graphs. It is convenient to consider *rooted trees*, i.e. with one marked link, since the symmetry factor S_{BP} in this case drops out. It corresponds to the one-loop function G_μ considered for the full surface theory, and we will also denote the corresponding BP function G_μ. It satisfies the self-consistent equation [7]:

$$G_\mu = \mathrm{e}^{-\mu}(1 + f_2 G_\mu + f_3 G_\mu^2 \cdots), \tag{3.68}$$

which is shown in fig. 14. We can solve for μ as a function of G_μ:

$$\mathrm{e}^{\mu} = \frac{1 + f_2 G_\mu + f_3 G_\mu^2 \cdots}{G_\mu} \equiv F(G_\mu). \tag{3.69}$$

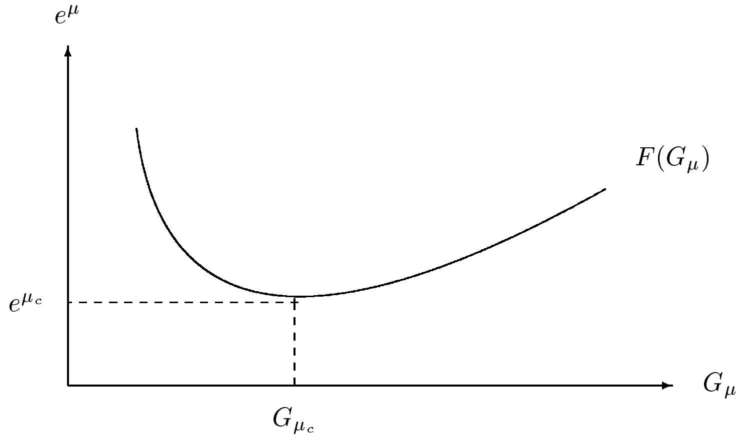

Fig. 15. The graphic solution of the branched polymer model.

This relation is shown in fig. 15 and for the weights f_n positive[11] we conclude that the lowest value μ_c of μ for which eq. (3.69) has a solution is the minimum of $F(G_\mu)$:

$$\mu - \mu_c \sim c(G_{\mu_c} - G_\mu)^2, \qquad \text{i.e.} \;\; G_\mu = G_{\mu_c} - \tilde{c}\sqrt{\mu - \mu_c}. \tag{3.70}$$

Since G_μ is the one-point function it is expected to have the critical behavior $G_\mu \sim (\mu - \mu_c)^{1-\gamma_s}$ and from (3.70) we conclude that *the generic value of γ_s for branched polymers is 1/2.*

One can consider branched polymer models directly in R^D, for instance with Gaussian interactions between the vertices. As long as the branched polymers are tree graphs it is possible to perform the Gaussian integration explicitly, as for the random walk, and for the calculation of γ_s one immediately gets the above considered model. Finally $\nu = 1/4$ and $\eta = 0$. This has the following interpretation: Consider a branched polymer propagating from x to y. For a given branched polymer there is a unique path of minimal length along the links going from x to y. This path can be viewed as a random walk path and the summation over branched polymers is a summation over all random walks from x to y where each vertex can be the source of an outgrowth of a rooted branched polymer. This observation allows us to solve the problem as a random walk problem, only is the

[11] There is a loop-hole in this argument if we allow infinite branching and the weights f_n in addition satisfy certain convergence relations which move the critical point μ_c out to the radius of convergence of the rhs of eq. (3.69). I refer to the original articles for discussion [7].

chemical potential $e^{-\mu}$ renormalized by the factor $(1+G_\mu)$, i.e. we get a random walk with $\bar{\mu}$ given by:

$$\mathrm{e}^{-\bar{\mu}} = \mathrm{e}^{-\mu}(1+G_\mu), \tag{3.71}$$

i.e. expanding around μ_c:

$$\bar{\mu}-\bar{\mu}_c \sim \left(1-\frac{1}{1+G_\mu}\,\frac{\mathrm{d}G_\mu}{\mathrm{d}\mu}\right)(\mu-\mu_c) \sim \sqrt{\mu-\mu_c}. \tag{3.72}$$

The exponential decay of the random walk is given by

$$m(\bar{\mu}) \sim (\bar{\mu}-\bar{\mu}_c)^{\frac{1}{2}} \sim (\mu-\mu_c)^{\frac{1}{4}}. \tag{3.73}$$

The exponent $\nu = 1/4$ is due to the non-analyticity of the coupling constant transformation (3.71)-(3.72), a phenomena we have encountered a number of times by now. Since we still have $\nu = 1/d_H$ we conclude that the Hausdorff dimension of the ensemble of branched polymers is 4, the "product" of two random walks.

3.5. Extrinsic curvature terms (I)

It is natural to ask if there are more elaborate random surface models where both the mass and the string tension scales. One can view the situation much as in the random walk case. The generic random walk had $\nu = 1/2$, but by tuning an extrinsic curvature term to infinity it was possible to enter a new universality class of smooth random walks. We saw that this universality class was related to spinning particles in the sense that one got the same critical exponents as for the Dirac particle and that the random walk with extrinsic curvature term could be viewed as a classical spinning particle, the tangent vector of the path playing the role of the classical spin. For the Dirac particle one could further relate it to an underlying world line supersymmetry.

Could a similar scenario be present for random surfaces? The theory of random surfaces is not so well understood yet and we cannot present analytic arguments in the same detail as was the case for the random walk. However, in many ways the situation is identical to that of the random walk [18, 19]. First, it is possible to add extrinsic curvature terms to the action in a natural way. In the context of statistical mechanics of membranes these terms are well known (see the lectures of Peliti and Nelson)[12]. The terms

[12] Also from the point of view of the continuum string theory there is a vast literature

suppress the branched polymer outgrows and their presence could result in a phase transition where the new phase is characterized by smoother surfaces. In the case of the random walks we had to take the bare coupling constant of the extrinsic curvature term to infinity in order to reach a new phase. For the random surfaces there are strong indications that the transition takes place for a finite value of the coupling constant. In addition numerical simulations indicate that *both the mass and the string tension scales to zero at the transition point.* Therefore this point is of interest if we want to discuss continuum limits of the random surfaces theory. One could further ask if there is any hint of extrinsic curvature terms coming from a fermionic surface theory. The answer is yes. An old result of Wiegmann [21] shows that the integration over fermionic variables will produce such terms. To be more precise the results are the following: If we consider a fermionic string theory, i.e. a string theory where we have local worldsheet supersymmetry, i.e. our bosonic variables $x^\mu(\xi)$ have supersymmtric partners $\psi^\mu(\xi)$, one can explicitly integrate out the fermions and arrive at the effective bosonic string action:

$$S_{eff} = S_{bos} + \tau \int \mathrm{d}^2\xi \sqrt{h} \left[(D_a n_i^\mu)^2 + (e_a^\mu \partial_b e_c^\mu)^2 \right] + \frac{ik}{8} \mathrm{Tr} \left(\frac{1}{2} \int \mathrm{d}^2\xi \; \mathrm{d}A \wedge A + \frac{1}{3\pi} \int_D d^3\xi A \wedge A \wedge A \right). \tag{3.74}$$

The terms in eq. (3.74) refers explicitly to the *extrinsic* geometry of the surface defined by the bosonic variables $x^\mu(\xi)$. τ depends on the fermionic representation, k is an integer, e_a^μ, $a = 1,2$ are unit tangent vectors of the surface, n_i^μ are $D-2$ normal vectors and D_a the covariant derivative with respect to the generic *normal bundle* associated with the surface:

$$A_{ij}^a = n_i^\mu \partial^a n_j^\mu, \qquad A^a = A_{ij}^a M_{ij}, \quad M_{ij} \in \mathrm{so}(D-2). \tag{3.75}$$

The Lie algebra elements A^a will be generators of parallel transport in the (D-2)-dimensional vector space orthogonal to the tangent space. The last term in eq. (3.74) is a WZW-like term. $\int \mathrm{d}^3\xi$ is the integration over a three-dimensional manifold which has the two-dimensional manifold as its boundary. This term, which in Euclidean space is a pure phase factor, is analogous to the term $\mathcal{P} \exp i \int \omega^{\mu\nu} s^{\mu\nu}/2$ for the Dirac particle. It is a very interesting question whether it serves to cancel the contributions between

and many different motivations for including extrinsic curvature terms. In [20] a very incomplete list of references is provided.

various rough surfaces in the path integral, leaving only smoother ones, as was the case for the fermionic particle. *The term $(D_a n_i^\mu)^2$ definitely acts in favor of smoother surfaces.*

To see this recall the following facts from the classical theory of surfaces embedded in D dimensions. Let h_{ab} denote the induced metric and $\Gamma_{ab}{}^c$ the corresponding connection:

$$h_{ab} \equiv \partial_a x^\mu \partial_b x^\mu, \qquad \Gamma_{ab;c} = \frac{1}{2}\left(\partial_a h_{bc} + \partial_b h_{ac} - \partial_c h_{ab}\right) = \partial_a \partial_b x^\mu \partial_c x^\mu.$$

where the indices $a, b, c...$ are lowered and raised with h_{ab} and the inverse h^{ab}. The second fundamental form is given by

$$K_{i;ab} = -\partial_a n_i^\mu \partial_b x^\mu = n_i^\mu \partial_a \partial_b x^\mu.$$

The basic equations which are satisfied *if* $x^\mu(\xi)$ represents a surface embedded in R^D are the Gauss-Weingarten equations:

$$\begin{aligned} \partial_a \partial_b x^\mu &= \Gamma_{ab}{}^c \partial_c x^\mu + K_{i;ab} n_i^\mu \\ \partial_a n_i^\mu &= -n_i^\nu \partial_a n_j^\nu - K_{i;ab} h^{bc} \partial_c x^\mu \end{aligned}$$

Using the definition (3.75) the last equation reads:

$$D_{a;ij} n_j^\mu = -K_{i;ab} h^{bc} \partial_c x^\mu, \qquad D_{a;ij} \equiv \partial_a \delta_{ij} + A_{a;ij}.$$

Let $\tilde{D}_a$ denote the ordinary covariant derivative with respect to the connection $\Gamma_{ab}{}^c$ and $\Box \equiv \tilde{D}_a \tilde{D}^a$. Using the covariant constance of h_{ab}: $\tilde{D}^a h_{ab} = 0$ one can check that:

$$\Box x^\mu = -\frac{1}{\sqrt{h}} \partial_a \sqrt{h} h^{ab} \partial_b x^\mu = h^{ab}(\partial_a \partial_b - \Gamma_{ab}{}^c \partial_c) x^\mu = K_{i;a}^a n_i^\mu.$$

The final result is that

$$(\Box x^\mu)^2 = (h^{ab} K_{i;ab})^2 = (h^{ab} D_a n_i^\mu D_b n_i^\mu)^2. \tag{3.76}$$

The principal curvatures κ_1 and κ_2 of the embedded surface is determined by the second fundamental form: If we define the *mean curvature* $H = (\kappa_1 + \kappa_2)/2$ and the Gauss curvature (as already mentioned) $K = R/2 = \kappa_1 \kappa_2$, we have

$$(h^{ab} K_{i;ab})^2 = 4H^2, \qquad (h^{ab} K_{i;ab})^2 - h^{ad} h^{cb} K_{i;ab} K_{i;cd} = R/2 \tag{3.77}$$

To summarize the situation we seemingly get a number of terms which can act to produce smoother bosonic surfaces if we integrate out the fermionic degrees of freedom for a string with world sheet supersymmetry. Here we will consider only the simplest of these terms: the extrinsic curvature term.

We have two different versions of the action available: one which refers exclusively to the geometry of the surface F embedded in R^D and one which is a hybrid between terms referring to extrinsic and intrinsic geometry:

$$S[F;\mu,\lambda] = \int_F \mathrm{d}A(\mu + \lambda H^2), \tag{3.78}$$

$$S[g,x;\mu,\lambda] = \int \mathrm{d}^2\xi\sqrt{g}\left[\mu + g^{ab}\partial_a x^\mu \partial_b x^\mu + \lambda g^{ab} D_a n_i^\mu D_b n_i^\mu\right]. \tag{3.79}$$

In the last equation the induced metric enters via the equation of Weingarten. Let us for simplicity discuss the situation in $D = 3$ (the results are easily generalized to any dimensions) and let us choose the action given by eq. (3.79) since it is easier to use in numerical simulations. In $D = 3$ there is only one normal n^μ to the surface and $D_a n_i$ reduces to $\partial_a n$. If we want to discretize the system there is no unique way to include such higher derivative terms but as in the case of the random walk it can be done in a natural way. Let T be an abstract triangulation and let i be a vertex and (ijk) a triangle. x_i is the coordinate in R^3 and $(x_i x_j x_k)$ will define a triangle $\triangle$ in R^3 and a normal $n_\triangle$. For a given triangulation T, i.e. a given choice of equivalence class of internal metrics g, the discretized version of eq. (3.79) is

$$S[T,x;\mu,\lambda] = \mu N_T + \sum_{(ij)} (x_i - x_j)^2 + \lambda \sum_{(\triangle_i,\triangle_j)} (n_{\triangle_i} - n_{\triangle_j})^2, \tag{3.80}$$

and the for the path integral we can write:

$$\begin{aligned} Z[\mu,\lambda] &= \int \frac{\mathcal{D}g_{ab}}{\mathrm{Vol(diff)}} \int \mathcal{D}x \;\, \mathrm{e}^{-S[g,x;\mu,\lambda]} \\ &\to \sum_{T\in\mathcal{T}} \frac{1}{S_T} \int \prod_{i\in T/\{i_0\}} \mathrm{d}x \;\, \mathrm{e}^{-S[T,x;\mu,\lambda]} \,. \end{aligned} \tag{3.81}$$

This is a theory with two coupling constants, as for the random walk with extrinsic curvature. Qualitatively the phase diagram looks identical

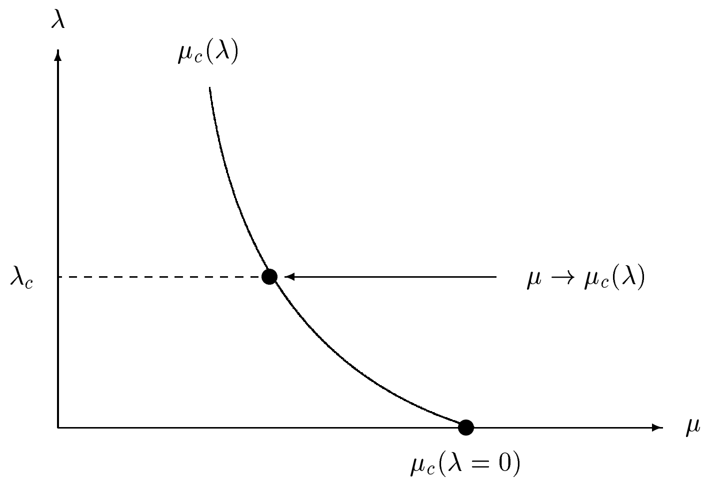

Fig. 16. The phase diagram for the model (3.81)

to the diagram for the random walk except that in the two-dimensional system we have *the possibility for a phase transition for a finite value of* λ as shown in fig. 16. Will there be a transition for a finite value λ_c of λ? It is not known rigorously, but extensive numerical simulations seem to support the idea that we have such a transition and that surfaces for $\lambda > \lambda_c$ are flat [22]. Could this be the transition we asked for where the string tension scales to zero together with the mass. Again numerical simulations seem to support this idea [23, 24]!

Let us briefly discuss the possible scaling at the critical point. For $\lambda = 0$ we have seen that the string tension $\sigma(\mu, \lambda = 0)$ does not scale to zero. Let $\mu_c(\lambda)$ denote the critical line in the (μ, λ)-plane (see fig. 16) and let $\Delta\mu = \mu - \mu_c(\lambda)$ parametrize the approach to $\mu_c(\lambda)$. Let us assume the string tension has the form:

$$\sigma(\mu, \lambda) = \sigma_0(\lambda) + c(\lambda)\Delta\mu^{2\nu(\lambda)} \qquad \text{for} \quad \Delta\mu \to 0. \tag{3.82}$$

Since $\sigma_0(0) > 0$ we need to have $\sigma_0(\lambda) \to 0$ for $\lambda \to \lambda_c$ in order to have a scaling of the string tension. Let us finally assume that

$$\sigma_0(\lambda) \sim (\lambda_c - \lambda)^{\alpha} \qquad \text{for} \quad \lambda \to \lambda_c. \tag{3.83}$$

As discussed above a number of times the continuum limit is dictated by the exponential decay of our correlators. Here we consider two: the one-loop function $G_{\mu,\lambda}(l_A)$ which falls off like $e^{-\sigma(\mu,\lambda)A}$ and the two-point function

$G_{\mu,\lambda}(x,y)$ which falls off like $e^{-m(\mu,\lambda)|x-y|}$. There is no reason not to expect the mass $m(\mu,\lambda)$ to scale to zero for $\Delta\mu \to 0$ for all $\lambda \leqslant \lambda_c$ and we can write at λ_c:

$$\begin{aligned} \sigma(\mu,\lambda_c)A &= c(\lambda_c)\Delta\mu^{2\nu(\lambda_c)}A = \sigma_{\mathrm{ph}}A_{\mathrm{ph}} \\ m(\mu,\lambda_c)x &= \tilde{c}(\lambda_c)\Delta\mu^{\tilde{\nu}(\lambda_c)}x = m_{\mathrm{ph}}x_{\mathrm{ph}}. \end{aligned} \tag{3.84}$$

The scaling limit is one where $(\cdot)_{\mathrm{ph}}$ are kept fixed while A and x goes to infinity as $\Delta\mu \to 0$. We can introduce two different exponents $\nu(\lambda)$ and $\tilde{\nu}(\lambda)$. But as we have already seen for the string without extrinsic curvature they have to agree if the string tension scales: $\nu(\lambda_c) = \tilde{\nu}(\lambda_c)$ and they will be related to the Hausdorff dimension: $\nu = 1/d_H$.

From (3.83) and (3.84) it is thus possible to define a consistent scaling limit for $\lambda \to \lambda_c$ and $\Delta\mu \to 0$ such that $m^2(\mu,\lambda) \to 0$ and $\sigma(\mu,\lambda) \to 0$ while the ratio stays constant. The only requirement is that

$$\Delta\lambda^{\alpha} \sim \Delta\mu^{2\nu}. \tag{3.85}$$

It should be emphasized that it is indeed possible to measure the exponents $\nu(\lambda)$, $\tilde{\nu}(\lambda)$ and α by Monte Carlo simulations of the statistical system (3.81) [24]. In such simulations it is convenient to transform from the "grand canonical" ensemble (3.81) to a "canonical" ensemble where the number N of triangles is fixed. This is a Legendre transformation in μ and N. There is no space to discuss the details, but it should be mentioned that a very nice feature of this transformation is that the string tension $\sigma(\mu,\lambda)$ just becomes the expectation value of the simplest local observable, the Gaussian action itself:

$$\sigma(\mu,\lambda) = \lim_{N\to\infty} \frac{\langle S_G\rangle_N - \frac{D}{2}(N-1)}{A},$$

where the average is taken in the canonical ensemble and the formula is true apart from subleading corrections in $1/A$.

The numerical simulations lead to a value of ν close to 0.25 i.e. $d_H \approx 4$ (more precisely $3.5 < d_H < 4.5$). It is also possible to measure the string susceptibility at the critical point[25]: $\gamma_s(\lambda_c) \approx 1/4$. The same values were obtained some years ago in a related hyper-cubic model [26] and suggests universality between the two models. We will return to the interpretation of $\gamma_s(\lambda_c) \approx 1/4$ in the context of conformal field theories with central charge $c > 1$ coupled to two-dimensional quantum gravity.

3.6. Supersymmetric random surfaces

Part of the motivation to consider the surfaces with extrinsic curvature came from the study of surfaces with worldsheet supersymmetry. Is it possible to implement the local supersymmetry at a discretized level? The answer is no. Local supersymmetry cannot be put on a lattice since the generators relate to translations and rotations which are explicitly broken by the presence of the lattice. However, if we consider the Green-Schwarz formulation of the superstring the situation is somewhat different. In this formulation there is no local worldsheet supersymmetry, but space-time supersymmetry. This is no problem in the discretized approach which is Euclidean invariant. In fact one can directly write down an action which is supersymmetric [27, 28].

Recall the continuum formulation of the Green-Schwarz superstring: The simplest supersymmetric action is obtained by the replacement:

$$\partial_a x^\mu \to \Pi^\mu = \partial_a x^\mu - i\bar{\theta}\gamma^\mu \partial_a \theta, \tag{3.86}$$

in the bosonic string action:

$$\int \mathrm{d}^2\xi \sqrt{g} g^{ab} \partial_a x^\mu \partial_a x^\mu \to \int \mathrm{d}^2\xi \sqrt{g} g^{ab} \partial_a \Pi^\mu \partial_a \Pi^\mu. \tag{3.87}$$

This action possesses an obvious supersymmetry if $\theta(\xi)$ like x^μ is a worldsheet scalar, but an anticommuting space-time spinor. The global supersymmetry is generated by the infinitesimal transformations:

$$x^\mu \to x^\mu - \frac{i}{2}(\bar{\theta}\gamma^\mu \varepsilon - \varepsilon^* \gamma^\mu \theta), \qquad \theta \to \theta + \varepsilon. \tag{3.88}$$

At the discretized level we have assigned a bosonic variable x_i to each vertex in a given triangulation T. We now assign an additional fermionic variable θ_i to the vertex and the discretized supersymmetric action is:

$$S_T[x,\theta] = (x_i^\mu - x_j^\mu - \Omega_{ij}^\mu)^2 \tag{3.89}$$

$$\Omega_{ij}^\mu \equiv \frac{i}{2}(\bar{\theta}_i \gamma^\mu \theta_j - \bar{\theta}_j \gamma^\mu \theta_i) \tag{3.90}$$

The path integral would now be written as a summation over all triangulations and for a given triangulation the integration is over all x_i^μ's and

θ_i's:

$$Z(\mu) = \sum_{T \in \mathcal{T}} \frac{1}{S_T} \int \prod_{i \in T/\{i_0\}} \mathrm{d}x_i \, \mathrm{d}\theta_i \, \mathrm{d}\bar{\theta}_i \, \mathrm{e}^{-S_T[x,\theta]} . \tag{3.91}$$

The problem with the above action at the continuum level is that it is not understood if it leads to the theory we want. Even at the classical level the continuum system seems quite impenetrable. The Dirac bracket prescription leads to complicated expressions which seem impossible to disentangle. A naive counting of degrees of freedom shows that in spite of the global supersymmetry there is not a perfect match between the fermionic and bosonic degrees of freedom. In dimensions $D = 3, 4, 6$ and 10 it is possible to add an additional term to the action which simplifies the situation drastically:

$$\frac{i}{2} \int \mathrm{d}^2\xi \; \varepsilon^{ab} \partial_a x^\mu (\bar{\theta}\gamma^\mu \partial_b \theta - \partial_b \bar{\theta} \gamma^\mu \theta). \tag{3.92}$$

This term has the correct symmetry properties in the above mentioned dimensions if the fermions are chosen as Majorana spinors in $D = 3$ and 6, Majorana or Weyl spinors in $D = 4$ and Majorana-Weyl spinors in $D = 10$. In all of these cases the resulting fermionic degrees of freedom will be $2(D-2)$. If we naively state that the bosonic degrees of freedom will be $D-2$, since two degrees will be absorbed in reparametrizations of the surface, we still have an incorrect number of degrees of freedom. It can be shown that eqs. (3.87) and (3.92) together lead to an additional *local* fermionic symmetry on the worldsheet, the so-called κ-symmetry, which effectively allows a decoupling of $D-2$ fermionic degrees of freedom. Unfortunately the local nature of the κ symmetry makes it difficult to enforce at the discretized level. A suggestion for a discretized version of (3.92) for a given metric, i.e. a given triangulation, has been [27]:

$$i \sum_{\triangle(ijk) \in T} \varepsilon_{ijk} x_i^\mu \Omega_{jk}^\mu . \tag{3.93}$$

However, we have no exact κ-symmetry even with this term, so from this point of view it is not clear that we need to add the term.

To summarize, it is nice that one can write down a regularized version of a superstring partition function, which even at the discretized level has space-time supersymmetry. However, it is not based on the action used in the continuum. Nevertheless one should be aware that the continuum action (3.87) was discarded not because it was wrong, but because a simpler alternative arose after adding the term given by eq. (3.92) to the action (3.87). The action (3.87) might still serve well in a non-perturbative framework, like the one presented here.

4. Matrix models and two-dimensional quantum gravity

4.1. Matrix models

For the bosonic string without extrinsic curvature term the results of the discretization was somewhat disappointing in the sense that we did not find a string theory. However, it is worth to recall that it is possible to calculate γ_s using continuum methods [29]:

$$\gamma_s = \frac{c-1-\sqrt{(c-1)(c-25)}}{12}. \tag{4.1}$$

The formula is not valid for $c > 1$. The bosonic string has $c = D$ and we are probing a most difficult region.

For $c < 1$ some results are known from Liouville theory [30], e.g. (4.1). The concept of discretization works very well for general covariant theories in the same region. A number of aspects of the theories can be solved both by continuum methods and directly at the discretized level. Historically many of the results were obtain first using the discretized approach. The exact solution allows one to study in detail the scaling limit. In this section I will outline how to solve some of the theories with $c \leqslant 0$ coupled to quantum gravity, using very elementary tools.

The continuum action of two-dimensional quantum gravity coupled to matter is given by:

$$S_{\rm eh}[g;\mu,G] + S_m[\phi,g;\lambda] = \int \mathrm{d}^2\xi\sqrt{g}\left[\mu - \frac{1}{4\pi G}R(\xi) + \mathcal{L}_m(\phi,g;\lambda)\right]. \tag{4.2}$$

Here $S_{\rm eh}$ denotes the Einstein-Hilbert action, $\mathcal{L}_m$ an invariant Lagrangian density, $\phi(\xi)$ a matter field and λ a coupling constant. μ and G are the cosmological and the gravitational coupling constants, respectivly. Since the integration over the curvature term is a topological invariant for closed two-dimensional surfaces, only the cosmological term will play a role, except if we sum over topologies. If we consider a specific manifold, i.e. if the topology is fixed and characterized by its Euler characteristic χ, and if A

denotes the area of the manifold for an equivalent class of metrics, the Einstein-Hilbert action is

$$S_{\rm eh}[g;\mu,G] = \mu A - \frac{\chi}{G}. \tag{4.3}$$

The quantum partition function can be written as

$$Z[\mu,G,\lambda] = \int \frac{\mathcal{D}g_{ab}(\xi)}{\mathrm{Vol(diff)}}\ \mathrm{e}^{-S_{\rm eh}[g;\mu,G]} \int \mathcal{D}_g\phi(\xi)\ \mathrm{e}^{-S_m[\phi,g;\lambda]}. \tag{4.4}$$

We have already considered the discretization of (4.4) in the case of D Gaussian fields ϕ^μ. Let us consider pure two-dimensional quantum gravity, i.e. eq. (4.4) without any matter fields ϕ. Recall the discretized translation of the gravity part:

$$\int \frac{\mathcal{D}g_{ab}(\xi)}{\mathrm{Vol(diff)}} \to \sum_{T\in\mathcal{T}} \tag{4.5}$$

$$S_{\rm eh}[g;\mu,G] \to \mu N_T - \frac{\chi(T)}{G}, \tag{4.6}$$

and the partition function (4.4) can be written as:

$$Z[\mu,G] = \sum_{T\in\mathcal{T}} \mathrm{e}^{-\mu N_T + \chi(T)/G} \tag{4.7}$$

As usual the summation is over a suitable class of (abstract) triangulations. As noted in the last section eq. (4.7) is not well defined since we can split it as follows

$$Z[\mu,G] = \sum_K \mathrm{e}^{-\mu K} \sum_{T\in\mathcal{T}_K} \mathrm{e}^{\chi(T)/G}, \tag{4.8}$$

where $\mathcal{T}_K$ denotes the abstract triangulations constructed from K triangles. The number of inequivalent triangulations which can be build from K triangle grows faster than $K!$. For a given K the Euler characteristic $\chi(T) \geqslant -K/2$ and it is clear that the sum (4.8) is divergent for all μ. In the continuum it is not known how to define the path integral (4.4) except for a fixed topology. Eq. (4.8) shows that even in the discretized approach where we have introduced a cut-off we can still not define such a sum in a straightforward manner. We can split (4.8) in a sum over triangulations of fixed topology:

$$Z[\mu, G] = \sum_{\chi} \mathrm{e}^{\chi/G} \sum_{K}^{\infty} \mathrm{e}^{-\mu K} \sum_{T \in \mathcal{T}_K(\chi)} . \tag{4.9}$$

The number of triangulations of fixed topology has an exponential bound:

$$\sum_{T \in \mathcal{T}_K(\chi)} \equiv \mathcal{N}(K, \chi) = \mathrm{e}^{\mu_c K} \, K^{\gamma_\chi - 3}(1 + O(1/K)), \tag{4.10}$$

contrary to the factorial bound valid when all topologies are included. As a consequence we can make sense of eq. (4.9) if we restrict the topology:

$$Z_\chi(\mu) = \sum_{K} \mathrm{e}^{-\mu K} \, \mathcal{N}(K, \chi). \tag{4.11}$$

For a given χ the critical cosmological term is the minimal value of μ for which the sum (4.11) is convergent. This value is precisely the value μ_c in (4.10) and it can be shown to be independent of χ. For a fixed topology we can try to define a continuum limit by approaching μ_c from above. By combining eqs. (4.10) and (4.11) we see that sufficiently high derivatives of $Z_\chi(\mu)$ will diverge for $\mu \to \mu_c$. If we define

$$\langle K^n \rangle_\chi = \frac{\sum_K K^n \ \mathrm{e}^{-\mu K} \sum_{T \in \mathcal{T}_K(\chi)}}{Z_\chi(\mu)} = (-1)^n \frac{1}{Z_\chi(\mu)} \frac{\mathrm{d}^n Z_\chi}{\mathrm{d}\mu^n} \tag{4.12}$$

we find for $\mu \to \mu_c$ and $n > 2 - \gamma_\chi$:

$$\langle K^n \rangle_\chi \sim \frac{1}{(\mu - \mu_c)^{\gamma_\chi - 2 + n}}. \tag{4.13}$$

This is an indication that large K will dominate for $\mu \to \mu_c$ and that it makes sense to introduce a scaling parameter a such that $A = Ka^2$ is viewed as the *physical area* of our two-dimensional world.

While these results have been obtained by the mathematicians [31] by explicit counting the number of ways to glue together triangles to form closed combinatorial manifolds, it is convenient from the point of view of physics to make this counting "automatic". This is done by representing the triangles by means of Hermitian matrices [32, 33]: Label the vertices of the i^{th} triangle by abstract indices $\alpha_i, \beta_i, \gamma_i$ and attach an Hermitian

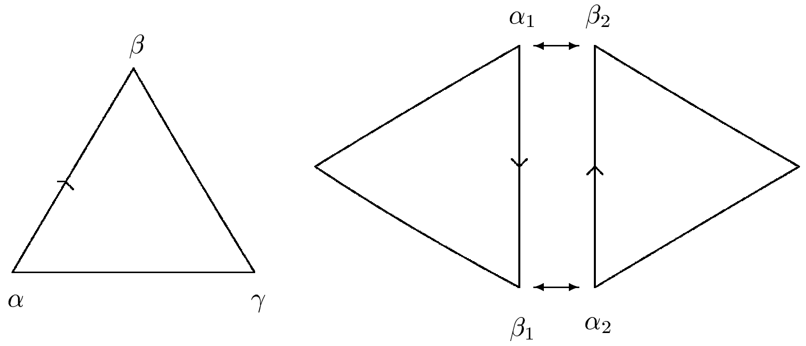

Fig. 17. The matrix representation of triangles which converts the gluing along links to a Wick contraction.

matrix $\phi_{\alpha_i\beta_i}$ to the oriented link from α_i to β_i. In this way we can attach the scalar quantity

$$\phi_{\alpha_i\beta_i}\phi_{\beta_i\gamma_i}\phi_{\gamma_i\alpha_i} = \mathrm{Tr}\ \phi^3 \tag{4.14}$$

to each of the K triangles. The Gaussian integral

$$\int \mathrm{d}\phi\ \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\ \phi^2}\ \frac{1}{K!}\left(\frac{1}{3}\mathrm{Tr}\ \phi^3\right)^K \tag{4.15}$$

where

$$\mathrm{d}\phi \equiv \prod_{\alpha\leqslant\beta} \mathrm{dRe}\phi_{\alpha\beta} \prod_{\alpha<\beta} \mathrm{dIm}\phi_{\alpha\beta} \tag{4.16}$$

can be performed by doing all possible Wick contractions of ϕ-fields. This corresponds to performing all possible gluings of surfaces of K triangles. The reason being that each Wick contraction in the Gaussian integral glues together two links:

$$\langle\phi_{\alpha_i\beta_i}\phi_{\alpha_j\beta_j}\rangle \equiv \int \mathrm{d}\phi\ \mathrm{e}^{-\frac{1}{2}|\phi_{\alpha\beta}|^2}\ \phi_{\alpha_i\beta_i}\phi_{\alpha_j\beta_j} = \delta_{\alpha_i\beta_j}\delta_{\beta_i\alpha_j}\,. \tag{4.17}$$

This is illustrated in fig. 17. After all Wick contractions are performed on the lhs of eq. (4.15) the K triangles have been glued together in all possible ways. The surfaces created in this way will consist of disconnected parts, but we get the connected graphs[13] by taking the logarithm of all graphs. Furthermore we can calculate the contribution from a particular graph constituting a closed surface: in the process of successive gluing we pick up a factor N, N being the number of indices, whenever a vertex becomes an internal vertex in the process of gluing together links by Wick contractions[14]. This means that we get a total factor N^V, where V is the number of vertices. If we make the substitution

$$\mathrm{Tr}\ \phi^3 \to \frac{g}{\sqrt{N}} \mathrm{Tr}\ \phi^3 \tag{4.18}$$

it is seen that the factors of N for K triangles combine to $N^{V-K/2} = N^{\chi}$, since the Euler characteristic for a triangulation of K triangles, L links and V vertices is

$$\chi = V - L + K = V - K/2. \tag{4.19}$$

By multiplying each triangle $\mathrm{Tr}\ \phi^3$ by $1/\sqrt{N}$ the weight of a given triangulation only depends on its topology. In addition the sum over all triangulations exponentiates. Collecting this information we can write:

$$Z(\mu, G) = \log \frac{Z(g,N)}{Z(0,N)} \tag{4.20}$$

where $Z(\mu, G)$ is defined by (4.9) and

$$Z(g,N) = \int \mathrm{d}\phi\ \exp\left(-\frac{1}{2}\mathrm{Tr}\ \phi^2 + \frac{g}{3\sqrt{N}}\mathrm{Tr}\ \phi^3\right) \tag{4.21}$$

---

[13] Among the connected graphs created in such an unrestricted gluing will be graphs with one-loops and two-loops. These are strictly speaking not combinatorial manifolds but they have a clear identification as "surfaces" with a specific χ. As already mentioned we do not expect such short distance phenomena to play any role in the scaling limit and this is substantiated by the known fact that although the value of μ_c in eq. (4.10) will depend on the particular class of graphs we consider (one-loops excluded, two-loops excluded etc), the exponent γ_s will not. This is in accordance with the general behavior in the theory of critical phenomena: the positions of the critical points are not universal, only the critical exponents.

[14] It is important to stress that N is a formal expansion parameter which should always be taken to ∞ at the end of a calculation since the analogy with surfaces is based on the fact that the indices of the different vertices are independent.

provided we make the identification:

$$\frac{1}{G} = \log N, \qquad \mu = -\log g. \tag{4.22}$$

Below it is shown that eq. (4.21) allows a $1/N^2$ expansion. This expansion is therefore identical to the topological expansion (4.9) of the random surfaces and eq. (4.21) is an attempt to perform a summation of this expansion. A glance at eq. (4.13) gives an idea of the physics involved in this summation: Since $\gamma_s \leqslant 1/2$ the partition function $Z(\mu)$ is finite at the critical point μ_c and for $Z(\mu)$ itself we get:

$$Z(\mu, G) = \sum_{\chi} \frac{e^{\chi/G}}{(\mu - \mu_c)^{\gamma_s(\chi)-2}} + \text{less singular terms.} \tag{4.23}$$

The only way we can imagine at all a summation over χ for $\mu \to \mu_c$ is to conjecture that $\gamma_s(\chi) - 2 = \tilde{c}\chi$ and that the *bare* gravitational coupling constant G is renormalized:

$$\frac{1}{G_R} = \frac{1}{G} - \log \frac{1}{(\mu - \mu_c)^{\tilde{c}}}. \tag{4.24}$$

Although the Einstein-Hilbert action is topological in two dimensions it will play a non-trivial role if we attempt to perform the summation over topologies. The conjecture $\gamma_s(\chi) - 2 = \tilde{c}\chi$ turns out to be true, and $\tilde{c} > 0$. This implies that the continuum limit $\mu \to \mu_c$ forces $G \to 0$ if we want to perform the summmation (4.23). This limit is called *the double scaling limit.* [34]. If we introduce the renormalized cosmological constant Λ by

$$\mu - \mu_c = \Lambda a^2 \tag{4.25}$$

this a is precisely the lattice a which goes into the discretizations of triangles by $A_{ph} = N_T a^2$, where A is the continuum area of the surface. Equation (4.25) tells us that that cosmological term has an additive renormalization. In terms of N the double scaling limit reads:

$$N a^{2\tilde{c}} = e^{\frac{1}{G_R}} \qquad \text{for} \quad a \to 0, \quad N \to \infty \tag{4.26}$$

We will later verify the conjecture

$$\gamma_s(\chi) - 2 = \tilde{c}\chi, \qquad (\text{in fact } \tilde{c} = 5/4.) \tag{4.27}$$

A few comments are necessary at this point. Clearly a formula like (4.21) makes no sense as it stands. The matrix integral is not convergent and we have only defined and identified it with a sum over triangulations in a perturbative sense, i.e. by expanding $\exp(g\mathrm{Tr}\ \phi^3/(3\sqrt{N})$ in a power series, and performing the resulting Gaussian integral, i.e. gluing the triangles together via the Wick contractions. It is of course a very interesting question if it is possible to make sense of the integral (4.21) in a non-perturbative way. The original continuum functional integral which led to (4.9) is vague when it comes to the question of summing over different topologies. In the case of string theory, and in the case of random surfaces which are intended to be a regularized version of string theory, *we have to sum over all topologies* in the way indicated in (4.9) by unitarity: A closed string can split in two, which can later join again, in this way changing the topology of the surface from that of a sphere (with two boundaries) to that of a torus (with two boundaries). In the case of gravity it is not clear that such a change is required if we disregard any connection between gravity and string theory. It is nevertheless tempting to assume that the summation should be performed. In the general case of random surfaces we have already mentioned that our regularized (discretized) approach has little to say about the sum over topologies: It is a non-perturbative regularization of the string path integral for a fixed topology, but perturbative in topology. In the special case of strings in $d = 0$, i.e. pure two-dimensional gravity, *a closed formula like (4.21) seems to offer some possibility for a non-perturbative definition of the sum over all topologies.* An obvious first suggestion is to define the functional integral by analytic continuation. By the rotation $\phi \to e^{i\pi/6}\phi$ we can define a matrix integral which has the *same* perturbative expansion as (4.21)

$$\tilde{Z}(g,N) = \mathrm{e}^{-i\pi N^2/6} \int \mathrm{d}\phi\ \exp\left(-\,\mathrm{e}^{i\pi/3}\,\mathrm{Tr}\ \phi^2 + i\frac{g}{3\sqrt{N}}\mathrm{Tr}\ \phi\right). \quad (4.28)$$

Contrary to (4.21) this integral is well defined. If we expand the interaction term in powers of g and perform the Gaussian integrals we get identical results to the "Wick-gluing" underlying the formal expression (4.21). The problem with an expression like (4.28) is that we cannot be sure it is real. In fact it is not: it can be shown that it contains a complex part which is *non-perturbative* in the coupling constant g: it has the form

$$\mathrm{Im}\tilde{Z}(g,N) \sim e^{-\mathrm{const.}/g} \quad (4.29)$$

i.e. it will never show up in the perturbative expansion in g. It seems from these considerations that we have not yet succeeded in a satisfatory definition of a non-perturbative summation over topology, but it shows the potential power of the discretized approach that one is able to discuss these questions at all .

A second, and much more simple minded remark is that the problem with the definition of the matrix integral (4.21) is not due to the unboundedness of the Tr ϕ^3 term. Although from a geometrical point of view it is natural to use triangles as building blocks, in the context of two-dimensional quantum gravity one could use squares, pentagons etc.. Had we chosen to glue together squares, we would have a term $g\mathrm{Tr}\ \phi^4/N$ in the action instead of the cubic term, but it would still appear with the wrong sign, i.e. the action would be unbounded from below, since we want all "surfaces" build from squares to appear with a positive weight in our functional integral. In general the gluing of n-gons will be generated by the matrix integral

$$Z(g_n, N) = \int \mathrm{d}\phi \exp\left(-\frac{1}{2}\mathrm{Tr}\ \phi^2 + \frac{g_n}{nN^{n/2}}\mathrm{Tr}\ \phi^n\right) \tag{4.30}$$

For the purpose of a general (perturbative) analysis of the matrix integral (4.21) it is convenient to consider the generalization of (4.30) to an arbitrary set of coupling constants $\{g_i\}$:

$$Z(g_1, g_2, \ldots) = \int \mathrm{d}\phi\ \mathrm{e}^{-N\mathrm{Tr}\ V(\phi)} \tag{4.31}$$

where

$$V(\{g_i\}) = \sum_{n=1}^{\infty} \frac{g_n}{n}\phi^n. \tag{4.32}$$

In eq. (4.32) we have of convenience scaled $\phi \to \sqrt{N}\phi$. In this way the topological nature of the expansion is still preserved: All two-dimensional complexes of Euler characteristic χ will have a factor N^χ associated with them. The interpretation of (4.31) is intended to be as before: we have in mind a Gaussian integral around which we expand, i.e. $g_2 > 0$ and $g_n \leqslant 0$ with the sign convention used in (4.32). The convenience of considering an arbitrary potential is that the general coupling constants g_n act as sources for terms like Tr ϕ^n, and by differentiating Z with respect to g_n we can calculate expectation values of "observables" like Tr ϕ^n. $\langle \mathrm{Tr}\ \phi^n/N\rangle$ has the following obvious interpretation: It represent the summation over all "surfaces" which have a n-sided polygon as boundary. This follows from the gluing procedure realized by Wick contractions of the Gaussian integrals. In a similar way

$$\frac{1}{N^2}\langle \mathrm{Tr}\ \phi^n \mathrm{Tr}\ \phi^m \rangle - \frac{1}{N^2}\langle \mathrm{Tr}\ \phi^n \rangle \langle \mathrm{Tr}\ \phi^m \rangle \tag{4.33}$$

will represent the sum over all connected two-dimensional complexes which connect one boundary consisting of n links with another boundary consisting of m links. Since two-dimensional quantum gravity describes the amplitude between one-dimensional geometries such expectation values are precisely what we are looking for (in the end we will of course have to take some kind of scaling limit in order to make contact with continuum physics). Let us define the generating functional for connected loop correlators. The expectation value of an arbitrary observable is defined by

$$\langle Q(\phi)\rangle \equiv \frac{1}{Z}\int \mathrm{d}\phi\ \mathrm{e}^{-N \mathrm{Tr}\ V(\phi)}\ Q(\phi). \tag{4.34}$$

The generating function for s-loop correlators, which we will also, somewhat inaccurate, denote the s-loop correlator, is defined by

$$W(z_1,\ldots,z_s) \equiv N^{s-2} \sum_{k_1,\ldots,k_s}^{\infty} \frac{\left\langle \mathrm{Tr}\ \phi^{k_1}\cdots \mathrm{Tr}\ \phi^{k_s}\right\rangle_{conn}}{z^{k_1+1}\ldots z^{k_s+1}} \tag{4.35}$$

where *conn* refers to the connected part as defined by (4.33), or its generalization to more correlators. One can rewrite (4.35) as

$$W(z_1,\ldots,z_s) = N^{s-2}\left\langle \mathrm{Tr}\ \frac{1}{z_1-\phi}\ \cdots\ \mathrm{Tr}\ \frac{1}{z_n-\phi}\right\rangle_{conn}. \tag{4.36}$$

In particular, we can drop the index *conn* for the 1-loop correlator:

$$W(z) = \frac{1}{N}\left\langle \mathrm{Tr}\ \frac{1}{z-\phi}\right\rangle = \frac{1}{N}\sum_{k=1}^{\infty}\frac{\left\langle \mathrm{Tr}\ \phi^k\right\rangle}{z^{k+1}} = \frac{\mathrm{d}F}{\mathrm{d}V(z)} \tag{4.37}$$

where we have introduced *the free energy* F by

$$Z(\{g_i\}) = \mathrm{e}^{N^2 F(\{g_i\})} \tag{4.38}$$

and the so-called *loop insertion operator* by

$$\frac{\mathrm{d}}{\mathrm{d}V(z)} \equiv -\sum_{k=1}^{\infty} \frac{k}{z^{k+1}} \frac{\mathrm{d}}{\mathrm{d}g_k}. \tag{4.39}$$

The name "loop insertion operator" is natural since it follows from the definition (4.35) that

$$W(z_1,\ldots,z_s) = \frac{\mathrm{d}}{\mathrm{d}V(z_s)} \frac{\mathrm{d}}{\mathrm{d}V(z_{s-1})} \cdots \frac{\mathrm{d}}{\mathrm{d}V(z_2)} W(z_1) \tag{4.40}$$

and this equation shows that if the 1-loop operator is known for an arbitrary potential, all multi-loop correlators can be calculated.

The 1-loop correlator is related to the density $\rho(\lambda)$ of eigenvalues defined by the matrix integral as follows :

$$\rho(\lambda) = \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle \tag{4.41}$$

where λ_i, $i = 1,\ldots,N$ denote the N eigenvalues of the matrix ϕ. With this definition we have

$$\frac{1}{N} \langle \mathrm{Tr}\ \phi^n \rangle = \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \rho(\lambda)\lambda^n, \quad \forall n \geqslant 0. \tag{4.42}$$

and therefore

$$W(z) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \frac{\rho(\lambda)}{z-\lambda} \tag{4.43}$$

For $N \to \infty$ exist, as we shall see, consistent solutions where the support of ρ is confined to a finite interval $[y, x]$ on the real axis. In this case $W(z)$ will be an analytic function in the complex plane, except for a cut at the support of ρ and it follows from Schwartz's reflection principle that

$$2\pi i\rho(\lambda) = \lim_{\varepsilon\to 0} W(\lambda + i\varepsilon) - W(\lambda - i\varepsilon) \tag{4.44}$$

4.2. The loop equations

Amazingly few assumptions enter in the derivation of a set of equations which allow us to solve, as an expansion in large N, the above defined matrix model [35–38]. Let us explore the invariance of the matrix integral (4.31) under field redefinitions of the type:

$$\phi \to \phi + \varepsilon \phi^n. \tag{4.45}$$

We consider ε as an infinitesimal parameter and one can prove that to first order in ε the measure $\mathrm{d}\phi$ defined by eq. (4.16) will transform like

$$\mathrm{d}\phi \to \mathrm{d}\phi (1 + \varepsilon \sum_{k=0}^{n} \mathrm{Tr}\, \phi^k \ \mathrm{Tr}\, \phi^{n-k}). \tag{4.46}$$

The action will change as

$$\mathrm{Tr}\, V(\phi) \to \mathrm{Tr}\, V(\phi) + \varepsilon \mathrm{Tr}\, \phi^n V'(\phi). \tag{4.47}$$

We can use these formulas to study the transformation of the measure under more general field redefinitions of the form

$$\phi \to \phi + \varepsilon \sum_{k=0}^{\infty} \frac{\phi^k}{p^{k+1}} = \phi + \varepsilon \frac{1}{p - \phi}. \tag{4.48}$$

This kind of field redefinitions only make sense if p is chosen on the real axis outside the support of the eigenvalues of ϕ. As mentioned above we will verify that this scenario is realized for $N \to \infty$. Under the transformation (4.48) the measure and the action change as

$$\mathrm{d}\phi \to \mathrm{d}\phi \left(1 + \varepsilon \mathrm{Tr}\, \frac{1}{p - \phi} \ \mathrm{Tr}\, \frac{1}{p - \phi} \right) \tag{4.49}$$

$$\mathrm{Tr}\, V(\phi) \to \mathrm{Tr}\, V(\phi) + \varepsilon \mathrm{Tr}\, \left(\frac{1}{p - \phi} V'(\phi) \right). \tag{4.50}$$

The integral (4.31) will be invariant under a redefinition of the integration variables by eq. (4.48) and the change of measure and action has to cancel to first order in ε. By use of eqs. (4.49) and (4.50) this leads to the following equation:

$$\int \mathrm{d}\phi \left\{ \left(\mathrm{Tr}\, \frac{1}{p - \phi} \right)^2 - N \mathrm{Tr}\, \left(\frac{1}{p - \phi} V'(\phi) \right) \right\} \mathrm{e}^{-N \mathrm{Tr}\, V(\phi)} = 0. \tag{4.51}$$

The first term in this equation is by definition

$$N^2 W(p) W(p) + W(p,p). \tag{4.52}$$

The second term in eq. (4.51) can be written as an integral over the the 1-loop correlator by means of the density of eigenvalues:

$$\frac{1}{N}\left\langle \mathrm{Tr}\, \frac{V'(\phi)}{p-\phi} \right\rangle = \int \mathrm{d}\lambda\, \rho(\lambda)\, \frac{V'(\lambda)}{p-\lambda} = \oint_C \frac{\mathrm{d}\omega}{2\pi i}\, \frac{V'(\omega)}{p-\omega} W(\omega). \tag{4.53}$$

The second equality is obtained by the rewriting

$$\int \mathrm{d}\lambda\, \rho(\lambda) \left[\oint_C \frac{\mathrm{d}\omega}{2\pi i}\, \frac{1}{\lambda-\omega}\, \frac{V'(\omega)}{p-\omega} \right] = \oint_C \frac{\mathrm{d}\omega}{2\pi i}\, \frac{V'(\omega)}{p-\omega} \left[\int \mathrm{d}\lambda\, \rho(\lambda)\, \frac{1}{\lambda-\omega} \right]$$

where the curve C should enclose λ but not p. However, for the change of the order of integration it is essential that we can choose C such that it encloses *all* eigenvalues and not p. This is possible if the density ρ has compact support on the real axis. With this assumption we finally can write (4.51) in the standard form, known as the loop equation:

$$\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} W(\omega) = W(z)^2 + \frac{1}{N^2} W(z,z) \tag{4.54}$$

where z is outside the interval $[y,x]$ on the real axis. In addition, since we have seen that $W(z)$ is analytic outside the support of ρ, eq. (4.54) will be valid in the rest of the complex plane by analytic continuation, again provided C does not enclose z (see fig. 18).

We can solve the loop equation by standard contour integration if we ignore the last term on the rhs of eq. (4.54). This is in accordance with a perturbative expansion in $1/N$, since the correlators are normalized to order $O(1)$ in N. The approximation where we ignore the last term on the rhs of (4.54) is called the large N approximation. Translating back to the language of surfaces the $1/N^2$ expansion is the expansion in topology of the manifolds, and the large N approximation implies a restriction to spherical topology.

Let us for simplicity assume that the potential is quadratic:

$$V(\phi) = \frac{1}{2}\phi^2. \tag{4.55}$$

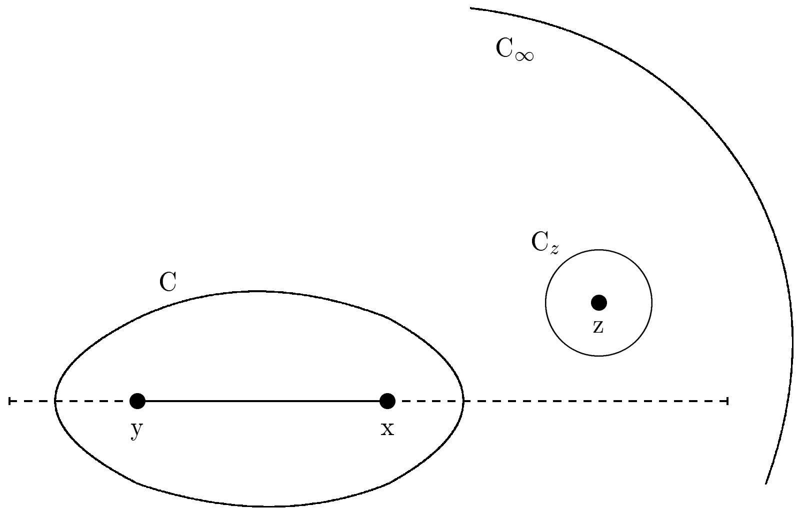

Fig. 18. The integration contour C and the cut from y to x. When deforming the contour to infinity we get two contributions: one from the circle C_∞ and one from the circle C_z around the pole z.

From the definition (4.37) $W(z)$ falls off like

$$W_0(z) = \frac{1}{z} + O(z^{-2}) \tag{4.56}$$

and C can be deformed to ∞ only picking up a simple pole $(zW_0(z))$ at z, (see fig. 18). The contour at infinity contributes with

$$\oint_{C_\infty} \frac{\mathrm{d}\omega}{2\pi i} \frac{1}{z-\omega} = -1 \tag{4.57}$$

and eq. (4.54) reads

$$zW_0(z) - 1 = W_0(z)^2, \quad W_0(z) = \frac{1}{2}\left(z - \sqrt{(z-2)(z+2)}\right) \tag{4.58}$$

$W_0(z)$ has a square root cut at the real axis ($[y,x] = [-2,2]$) and according to (4.44) the eigenvalue density is related to $W_0(\lambda + i\varepsilon) - W_0(\lambda - i\varepsilon)$ and we have that

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{(2-\lambda)(2+\lambda)} \tag{4.59}$$

which is Wiener's famous semicircle law.

For a general potential we can find a solution which has essentially the same structure as for the quadratic potential. By deforming the contour C out to infinity we pick up a pole term $V'(z)W_0(z)$ at z, and at C_∞ we get the following integral

$$\oint_{C_\infty} \frac{\mathrm{d}\omega}{2\pi i}\,\frac{V'(\omega)}{z-\omega}\,W_0(\omega), \tag{4.60}$$

which is a polynomium $Q(z)$ in z, as is seen by expanding the integrand in powers of $1/\omega$:

$$\frac{1}{z-\omega} = -\sum_{j=1}^{\infty}\frac{z^{j-1}}{\omega^j}, \quad W_0(\omega) = \sum_{k=1}^{\infty}\frac{W_k}{\omega^{k+1}}, \tag{4.61}$$

Only the term with total power -1 will contribute, i.e. if V is a polynomium of power n then $Q(z)$ will be of order $n-2$:

$$Q(z;g_i;W_k) = -\sum_{i=2}^{n} g_i \sum_{\substack{j+k=n-1,\\ j>0,k>0}} z^j W_k. \tag{4.62}$$

It follows that the loop equation does not determine uniquely the $n-2$ "moments" $W_1,\ldots,W_{n-2}$, and the polynomium $Q(z)$ is consequently an arbitrary polynomium of order $n-2$ which enters into the resulting second order equation for $W_0(z)$

$$\begin{aligned} W_0(z)^2 &= V'(z)W_0(z) - Q(W_k,z,g_i), \\ W_0(z) &= \frac{1}{2}(V'(z) - \sqrt{(V'(z))^2 - 4Q(z)} \end{aligned} \tag{4.63}$$

Not every choice of $Q(z)$ is allowed since $W_0(z)$ cannot have a cut away from the real axis by construction. Let us here assume that the eigenvalue density is as close in structure as possible to (4.59):

$$\sqrt{(V'(z))^2 - 4Q(z)} = M(z)\sqrt{(z-x)(z-y)} \tag{4.64}$$

where $M(z)$ is a polynomium of degree $n-2$. The eigenvalue density is

$$\rho(\lambda) \sim M(\lambda)\sqrt{(y-\lambda)(x-\lambda)}, \tag{4.65}$$

and in this sense it is close to Wiener eigenvalue distribution (4.59). Nevertheless an interesting critical behavior appears when zeros of $M(\lambda)$ accumulate near x (or y) [39]. We can write:

$$W_0(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z-x)(z-y)}\right) \tag{4.66}$$

and the requirement that $W_0(z) = O(1/z)$ uniquely determines $M(z)$ since (4.66) allows us to write

$$M(z) = \oint_{C_\infty} \frac{\mathrm{d}\omega}{2\pi i}\frac{M(\omega)}{z-\omega} = \oint_{C_\infty} \frac{\mathrm{d}\omega}{2\pi i}\frac{V'(\omega)}{z-\omega}\frac{1}{\sqrt{(\omega-x)(\omega-y)}}, \tag{4.67}$$

as the part of the integral which involves $W_0(z)$ vanishes. By expanding the last integrand in powers of $1/(\omega - x)$ we can write:

$$M(z) = \sum_{k=1}^{\infty} M_k (z-x)^{k-1},$$

$$M_k[x,y;g_i] = \oint_C \frac{\mathrm{d}\omega}{2\pi i}\frac{V'(\omega)}{(\omega-x)^{k+\frac{1}{2}}(\omega-y)^{\frac{1}{2}}}. \tag{4.68}$$

$W_0(z)$ can now be expressed in terms of the "moments" M_k:

$$W_0(z) = \frac{1}{2}\left(V'(z) - \sqrt{(z-x)(z-y)}\sum_{k=1}^{\infty} M_k[x,y;g_i](z-x)^{k-1}\right). \tag{4.69}$$

The endpoints x and y of the cut is determined selfconsistently by the following boundary conditions

$$M_{-1}[x,y,g_i] = 2, \qquad M_0[x,y;g_i] = 0, \tag{4.70}$$

These two equations follows by contracting the contour in the last integral in (4.67) to C. In this way we pick up a pole term at z which precisely cancels $V'(z)$ in (4.66) and we get yet another expression for $W_0(z)$:

$$W_0(z) = \sqrt{(z-x)(z-y)} \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{z-\omega} \frac{1}{\sqrt{(\omega-x)(\omega-y)}}. \tag{4.71}$$

By expanding in $1/z$ and using that $W_0(z)$ contains no constant term and $W_1 = 1$ we get (4.70).

4.3. Complete solution to leading order in $1/N^2$

In principle the complete solution at spherical level is given by (4.69)-(4.70). These equations define $W_0(z)$ and we can apply the loop inserting operator to obtain any multi-loop correlator. Quite surprisingly one can obtain an explicit formula if we change the matrix model slightly. Instead of Hermitian matrices we use general complex matrices and consider a potential:

$$V(\phi^\dagger \phi) = \sum_{n=1}^{\infty} \frac{g_n}{n} \text{Tr } (\phi^\dagger \phi)^n. \tag{4.72}$$

The one-loop correlator is given by:

$$W_0(z) \equiv \frac{1}{N} \sum_n \frac{\left\langle \text{Tr } (\phi^\dagger \phi)^n \right\rangle}{z^{2k+1}}. \tag{4.73}$$

This model has a surface representation [40]. Tr $(\phi^\dagger\phi)^n$ represents a $2n$-gon where the boundary links have alternating black and white colors, corresponding the ϕ and $\phi^\dagger$. The effect of Gaussian integration with respect to the complex matrices is to glue together such "checker-board" polygons just as Hermitean matrices glued together ordinary polygons. The only additional rule is that only "white" and "black" links can be glued together since $\langle \phi^\dagger \phi \rangle \neq 0$ while $\langle \phi^2 \rangle = 0$ and $\langle \phi^\dagger \phi^\dagger \rangle = 0$. Such short distance differences in gluing should be unimportant in the continuum limit.

We can write down the loop equations for this model. Since the potential is symmetric with respect to $\phi \to -\phi$ we have that $y = -x$ and we can write:

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{z^2 - \omega^2} W(\omega) = W^2(z) + \frac{W(z,z)}{N^2}, \qquad \oint_C \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{\sqrt{\omega^2 - x^2}} = 2, \tag{4.74}$$

where the last equation expresses that $W_0(z) \sim 1/z$ for $|z| \to \infty$.

The solution can be written as

$$W_0(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{z^2 - x^2}\right),$$

$$M(z) = \oint_{C_\infty} \frac{\mathrm{d}\omega}{4\pi i} \frac{\omega V'(\omega)}{(\omega^2 - z^2)\sqrt{\omega^2 - x^2}} \tag{4.75}$$

If we expand the denominator of $M(z)$ in powers of z^2/ω^2 we can write:

$$M(z) = \sum_{k=1}^{\infty} M_k[x, g_i](z^2 - x^2)^{k-1},$$

$$M_k[x, g_i] = \oint_C \frac{\mathrm{d}\omega}{4\pi i} \frac{\omega V'(\omega)}{(\omega^2 - x^2)^{k+1/2}}, \tag{4.76}$$

and the position x of the cut is determined by the last equation in (4.74):

$$M_0[x, g_i] = 2. \tag{4.77}$$

While these formulas look rather complicated it is a pleasant surprise that things simplify a lot for the higher loop correlators. In order to apply the loop insertion operator we note that

$$\frac{\mathrm{d}}{\mathrm{d}V(z)} \equiv -\sum_k \frac{k}{z^{2k+1}} \frac{\mathrm{d}}{\mathrm{d}g_k} = \frac{\partial}{\partial V(z)} - \frac{x^2}{M_1(z^2 - x^2)^{3/2}} \frac{\mathrm{d}}{\mathrm{d}x^2}. \tag{4.78}$$

where

$$\frac{\partial}{\partial V(z)} \equiv -\sum_k \frac{k}{z^{2k+1}} \frac{\partial}{\partial g_k}, \qquad \frac{\partial V'(\omega)}{\partial V(z)} = \frac{2\omega z}{z^2 - \omega^2}. \tag{4.79}$$

It is now straightforward, though tedious, to apply the loop insertion operator. We find [38, 41]:

$$W_0(z_1, z_2) = \frac{1}{4(z_1^2 - z_2^2)^2}\left[z_2^2\sqrt{\frac{z_1^2 - x^2}{z_2^2 - x^2}} + z_1^2\sqrt{\frac{z_2^2 - x^2}{z_1^2 - x^2}} - 2z_1 z_2\right]. \tag{4.80}$$

$$W_0(z_1,z_2,z_3) = \frac{x^4}{16M_1}\,\frac{1}{\sqrt{(z_1^2-x^2)(z_2^2-x^2)(z_3^2-x^2)}}. \tag{4.81}$$

$$W_0(z_1,\ldots,z_s) = \left(\frac{1}{M_1}\frac{d}{dx^2}\right)^{s-3}\frac{1}{2x^2M_1}\prod_{k=1}^{s}\frac{x^2}{2(z_k^2-x^2)^{3/2}}. \tag{4.82}$$

Notice the following [41, 42] (see also [43–45]):
(1): The above formulas are valid for *any* potential V. All dependence on the coupling constants are hidden in M_1 and x.
(2): From (4.76) it follows that $dM_k/dx^2 = (k+1/2)M_{k+1}$. For an arbitrary potential V and $s>1$ the s-loop correlator $W_0(z_1,\ldots,z_s)$ is a simple algebraic function of z_k and only depends on x and $M_1,\ldots,M_{s-2}$.
(3): The same statements are true for the Hermitean matrix model, except that we havetwo independent end points x and y and therefore two independent set of "moment" M_k and J_k defined by:

$$M_k[x,y,g_i] = \oint_C \frac{\mathrm{d}\omega}{2\pi i}\frac{V'(\omega)}{(\omega-x)^{k+1/2}\sqrt{\omega-y}}, \qquad J_k[x,y,g_i] = M_k[y,x,g_i]. \tag{4.83}$$

(4): If we iterate the loop equation after $1/N^2$ this simplicity continue to hold. For genus g surfaces the Hermitean s-loop correlator $W_g(z_1,...,z_s)$ is a simple rational function of (z_k-y) and (z_k-x) and depends apart from x and y on at most $2(3g-2+s)$ moment M_k, J_k, $k\leqslant 3g-2+s$. This is even true for $s=0$ if $g>0$. As an example it can be shown that the free energy F_g for genus $g=1$ is

$$F_{g=1} = -\frac{1}{24}\log M_1J_1(x-y)^4. \tag{4.84}$$

This formula is valid for all potentials $V(\phi)$!

The proofs of the above statements are all trivial (but sometimes tedious) and involve nothing beyond elementary linear algebra.

4.4. The scaling limit

In the case of the simplest potential (Tr ϕ^3 for the Hermitean matrix model, Tr $(\phi^\dagger\phi)^2$ for the complex matrix model), we have one independent coupling constant g if we fix the coupling constant in front of the Gaussian term. Eq. (4.22) gives the relation between the bare cosmological coupling constant μ and the coupling constant g of the matrix models. We have seen that there is a critical μ_c such that the continuum limit should be taken for $\mu \to \mu_c$. Corresponding to μ_c there will be a g_c. If we now introduce the lattice spacing a the area of our universe for a given triangulation is given by $N_T a^2$, N_T being the number of triangles in T. It is natural to introduce the *renormalized* cosmological constant Λ by:

$$\mu - \mu_c = \Lambda a^2, \qquad \text{i.e.} \quad g_c - g \sim \Lambda a^2. \tag{4.85}$$

In this way the renormalization follows the renormalization of one-dimensional gravity and is in accordance with the general additive renormalization of dimensionful coupling constants. In case we extend our model and consider the gluing of arbitrary polygons (but with positive weight) we expect nothing new, except that the critical point g_c will now be a (n-1) dimensional hyper-surface if we have n coupling constants. The identification of this hyper-surface is easy in the present formalism. Let us for simplicity consider the complex matrix model. The density of eigenvalues $\rho(\lambda)$ is given by (4.65) and (4.76):

$$\rho(\lambda) \sim M(\lambda)\sqrt{x^2 - \lambda^2}, \qquad M(\lambda) = \sum_k M_k (x^2 - \lambda^2)^{k-1}. \tag{4.86}$$

The only non-analytic behaviour is associated with the endpoints of the distribution $|\lambda| \to x$. If $M_1[g_i]$ is positive the behavior will be identical to that of the Gaussian model. M_1 involves a relation between the positive coupling constant in front of the Gaussian term and the negative coupling constants of the polygons. It can be fine tuned to zero on a $n-1$ dimensional hyper-surface. All higher M_k will be negative if the coupling constants g_k are negative. We conclude that the critical hyper-surface is determined by:

$$M_1[x(g_i), g_i] = 0. \tag{4.87}$$

A glance on $W_0(z_1, ..., z_s)$ corroborates this observation since it will be singular precisely when $M_1 = 0$. Let us denote a critical point on the hyper-surface by g_i^c and the corresponding endpoint of the eigenvalue distribution by x_c. In statistical mechanics the masses and running coupling constants are defined *by the approach to the critical point.* We have already seen examples of this in the case of random walks and non-critical strings. Let us therefore move slightly away from the critical hyper-surface by scaling the coupling constants according to (4.85):

$$g_i = g_i^c(1 - \Lambda a^2) = g_i^2 + \delta g_i. \tag{4.88}$$

Corresponding to this change there will be a change $x_c^2 \to x_c^2 - \delta x^2$. It can be calculated directly from (4.77) using $dM_k/dx^2 = (k+1/2)M_{k+1}$ and $M_1[x_c, g_i^c] = 0$. Let us introduce the notation:

$$M_i^c = M_i[x_c, g_i^c], \qquad \text{i.e } \; M_0^c = 2, \; M_1^c = 0, \; M_c^2 \neq 0.$$

Expanding $M_0[x, g_i]$ leads to:

$$2 = M_0[x, g_i] = M_0^c(1 - \Lambda a^2) + \frac{3}{8} M_2^c \, (\delta x^2)^2 + O(a^3), \tag{4.89}$$

or

$$(\delta x^2) = \frac{16}{3M_2^c} \, \Lambda a^2, \qquad \text{i.e.} \quad (\delta x^2)^2 \sim \delta g_i. \tag{4.90}$$

In addition $M_1[x(g_i), g_i]$ will now be different from zero:

$$M_1[x, g_i] = \frac{3}{2M_2^c} \, \delta x^2 + O(a^2). \tag{4.91}$$

By a trivial rescaling of the cosmological constant we can there *define* it by

$$x^2 = x_c^2 - a\sqrt{\Lambda}. \tag{4.92}$$

Let us now calculate γ_s. As discussed in detail in the last section γ_s will be related to the s-point function by[15]:

$$W_0(i_1, \ldots, i_s) \sim \frac{1}{(\mu - \mu_c)^{\gamma_2 + s - 2}} \sim \frac{1}{(\Lambda a^2)^{\gamma_s + s - 2}}, \tag{4.93}$$

where (4.85) is used. We get the contribution to the s-point function in the scaling limit from (4.82) by first performing the contour integral with some finite powers of z_i. This produce a non-singular function of x. Using (4.90) and (4.91) we can take $a \to 0$ and we get:

$$W_0(i_1, \ldots, i_s) \sim (\Lambda a^2)^{5/2 - s}. \tag{4.94}$$

We conclude that $\gamma_{str} = -1/2$

[15] In the string case we had to integrate the s-point function over all space, but here is no target space and the integration simply drops out.

Let us now turn to objects which are not readily calculated in Liouville theory, *correlation functions between genuine macroscopic loops* [41]. These are object of fundamental interest in a quantum gravity theory.

From the explicit formulas for the multi-loop correlators it follows that the complex variable z appears in the combination $z^2 - x^2$ and it is natural to introduce a scaling of z^2:

$$z^2 = x_c^2 + a\pi. \tag{4.95}$$

With this notation we can write in the scaling limit :

$$M(z) = M_1 + M_2(z^2 - x^2) + \cdots = aM_2^c(\pi - \frac{1}{2}\sqrt{\Lambda}) + O(a^2) \tag{4.96}$$

π serves the same role in the scaling limit as z at the discretized level. Knowing $W_0(z_1, ...z_s)$ allow us to reconstruct the multi-loop correlators consisting of discretized boundary loops of length $n_1, ..., n_s$ by multiple contour integration. In the scaling limit the physical length of these loops will be $l_i = n_i a$, i.e. they will go to zero. If we want *genuine macroscopic loops* we have to scale n_i to ∞ at the same time as $a \to 0$ such that l_i is constant. By substituting (4.92) and (4.95) in the expressions for $W_0(z_1, ...z_s)$ we get an expression $W_0(\pi_1, ...\pi_s; \Lambda)$ and we can reconstruct the corresponding multi-loop amplitude $W_0(l_1, ..., l_s; \Lambda)$ by an inverse Laplace transformation. The reason why the contour integration is changed into an inverse Laplace transformation is that the cut $[-x, x]$ (or $[y, x]$ in the Hermitean matrix model) by the substitution (4.92) and (4.95) is changed into a cut $]-\infty, \sqrt{\Lambda}]$. The contour integration around the cut can now be deformed:

$$\oint_{C_i} \prod_{i=1}^{s} \mathrm{d}z_i\, z^{2n_i}\; W_0(z_1, \ldots, z_s) \to \tag{4.97}$$

$$x_c^{2(n_1+\cdots+n_s)} \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{s} \mathrm{d}\pi_i\; \mathrm{e}^{l_i \pi_i}\; W_0(\pi_1, \ldots, \pi_s; \Lambda)\,, \qquad c > \sqrt{\Lambda},$$

since we have

$$z^{2n} = x_c^{2n}(1 + a\pi)^{l/a} \sim x_c^{2n}\, \mathrm{e}^{\pi l}, \quad l \equiv n/ax_c^2. \tag{4.98}$$

The highly divergent factor $x_c^{2(l_1+\cdots+l_s)/ax_c^2}$ is a wave function renormalization of the macroscopic boundaries. It is to be expected since it is possible in the continuum to add a term

$$S_{\partial\mathcal{M}} = \lambda \int_{\partial\mathcal{M}} \mathrm{d}l = \lambda \int \mathrm{d}\xi \; g^{\frac{1}{4}}(\xi) \tag{4.99}$$

to the action. This is just the induced one-dimension gravity on the boundary of the manifold. Since λ has the dimension of mass we expect in the discretized version that the bare coupling constant will undergo an additive renormalization (like the cosmological constant itself), i.e. we will have to cancel a term like x_c^{2n}.

From eq. (4.82) we immediately get the generating functional for macroscopic multi-loop amplitudes:

$$W_0(z_1,\ldots,z_s) \sim a^{5-7s/2} w_0(\pi_1,\ldots,\pi_s;\Lambda), \tag{4.100}$$

$$w_0(\pi_1,\ldots,\pi_s;\Lambda) = \frac{\mathrm{d}^{s-3}}{\mathrm{d}\Lambda^{s-3}} \frac{1}{\sqrt{\Lambda}} \prod_{k=1}^{s} \frac{1}{(\pi_k+\sqrt{\Lambda})^{3/2}}, \qquad s \geqslant 3. \tag{4.101}$$

The expressions for $w_0(\pi,\Lambda)$ and $w_0(\pi_1,\pi_2;\Lambda)$ are slightly more complicated since $W_0(z)$ contains a non-universal part $(V'(z)/2)$. and we will not give them here.

We can calculate the inverse Laplace transform (4.97)

$$\begin{aligned} w_0(l_1,\ldots,l_s;\Lambda) &= \int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{s} \mathrm{d}\pi_i \; \mathrm{e}^{\pi_i l_i} \; w_0(\pi_1,\ldots,\pi_s;\Lambda) \\ &= \frac{\mathrm{d}^{s-3}}{\mathrm{d}\Lambda^{s-3}} \frac{\sqrt{l_1\cdots l_s}}{\sqrt{\Lambda}} \; \mathrm{e}^{-\sqrt{\Lambda}(l_1+\cdots+l_s)}\,. \end{aligned} \tag{4.102}$$

4.5. Generalizations

In the last subsection we considered the spherical limit of ordinary discretized gravity. All graphs appeared with positive weight and the scaling limit was independent of the precise class of graphs used: Any polynomial

$$V(\phi^\dagger\phi) = \sum_{n=1}^{n_0} \frac{g_n}{n}(\phi^\dagger\phi)^n, \quad g_1 > 0, \quad g_n < 0, \; n > 2 \tag{4.103}$$

leads to the same scaling limit.

Let us lift the constraint in eq. (4.103). In this way we clearly move away from pure gravity since some of the polygons are glued together with negative weight. However, as already mentioned, formulas like (4.80)-(4.82) are valid for *any* potential where $g_1 > 0$. Without the constraint it is possible to get a new critical behavior in the scaling limit [39] since one can fine tune

$$M_k[x(g_i), x_i] = 0, \quad k < m, \qquad M_m[x(g_i), g_i] \neq 0. \tag{4.104}$$

The hyper-surface in coupling constant space satisfying (4.104) is called *the* m^{th} *multicritical hyper-surface* and pure gravity corresponds to $m = 2$. On this surface the eigenvalue density $\rho(\lambda) \sim M(\lambda)\sqrt{x^2 - \lambda^2}$ changes from the generic Wiener form to :

$$\rho(\lambda) \sim (x^2 - \lambda^2)^{m+1/2}. \tag{4.105}$$

Let us move away from the critical surface in a way similar to (4.88):

$$g_i = g_i^c(1 - (\Lambda a^2)^{m/2}) = g_i^c + \delta g_i. \tag{4.106}$$

We want to maintain (4.92) and (4.95) which allow us to identify a with the link-length. This being the case, (4.104) forces us to replace (4.90) and (4.91) with

$$(\delta x^2)^m \sim \delta g_i, \qquad M_1[x_i(g_i), g_i] \sim (\delta x^2)^{m-1}, \tag{4.107}$$

and one is finally led to the power $a^m \Lambda^{m/2}$ used in (4.106). We can repeat the arguments used for pure gravity and find corresponding to (4.94):

$$W_0(i_1, \ldots, i_s) \sim (\Lambda a^2)^{m+\frac{1}{2}} \left[(\Lambda a^2)^{m/2}\right]^{-s} \tag{4.108}$$

If we compare this formula for $s = 0$ to the free energy $F \sim (\Lambda a^2)^{2-\gamma_s}$ we conclude:

$$\gamma_s = -m + \frac{3}{2}. \tag{4.109}$$

Let us tentatively compare this to a (p, q) minimal conformal field theory coupled to gravity. The central charge is $c = 1 - 6(p-q)^2/pq$ and from (4.1) we have:

$$\gamma_s = 1 - \frac{q}{p}, \quad q > p \;\; \text{and co} - \text{primes}. \tag{4.110}$$

(4.109) leads to *the identification of the m^{th} multicritical model with a (2,2m-1) minimal conformal theory coupled to quantum gravity.* It might be confusing that the s-point function $W_0(i_1, ..., i_s)$ does not behave like (4.93). But the explanation is perfectly in accordance with the identification suggested. (p,q)-models have operators of negative scaling dimension, the most negative being $\Delta_0 = (1-(p-q)^2)/4pq$. After coupling to gravity they might produce a more singular behavior than the cosmological term. The change in potential (4.106) excite all operators, including possible negative dimensional ones. These will dominate over the cosmological term for $m > 2$ and the scaling dimension of the most negative one integrated over the whole surface produces the observed behavior $(\Lambda a^2)^{m/2}$. This implies that each time one inserts "an arbitrary puncture" on the surface, i.e an excitation which involves the negative dimensional operator, it will multiply the partition function by $\tilde{\Lambda}^{-1}$, where $\tilde{\Lambda} = \Lambda^{m/2}$. From this point of view $\tilde{\Lambda}$ acts like the cosmological term in the unitary theories and we can rewrite eq. (4.40) as:

$$W_0(i_1, \dots, i_s) \sim (\tilde{\Lambda} a^m)^{2+\frac{1}{m}} \left[(\tilde{\Lambda} a^m) \right]^{-s}$$

from which one would be tempted to define a

$$\tilde{\gamma}_s = -\frac{1}{m}.$$

It is of course a matter of definition whether one uses (4.109) or $\tilde{\gamma}_s$.

The second point concerns the expansion beyond the spherical limit. Using the loop equation it can be performed without any problem. Let us here just mention the result in the double scaling limit: Approaching the m^{th} multicritical hyper-surface (4.104) according to $\delta g_i \sim O(a^m)$ leads to the following behavior of the moments M_k:

$$M_k \sim \mu_k a^{m-k}, \quad 1 \leqslant k \leqslant m, \qquad M_k = O(1), \quad k \geqslant m. \tag{4.111}$$

The free energy has a genus expansion [46, 42]:

$$F = \sum_{g=0}^{\infty} \frac{1}{N^2 a^{2m+1}} F_g, \qquad F_g = \sum_{\alpha_i > 1} \langle \alpha_1 \cdots \alpha_s \rangle_g \frac{\mu_{\alpha_1} \cdots \mu_{\alpha_s}}{\mu_1^{2g-2+s} d^{g-1}} \tag{4.112}$$

where $\sum_j \alpha_j = 3g - 3 + s$ and $d = x - y$ is the length of the cut. The second equation is valid only for $g > 0$. For $g = 0$ there are additional non-universal parts. The numbers $\langle \alpha_1 \cdots \alpha_s \rangle_g$ are independent of the multicritical point and can be identified with so-called *intersection indices* in the moduli space $\mathcal{M}_{g,s}$ of Riemann surfaces of genus g with s punctures.

This expansion points towards two apects of the theory which we have no space to discuss further
(1): *The double scaling limit* where $N^2 a^{2m+1}$ is kept fixed should be identified with the renormalized gravitational constant via (4.26). In fact we get $\tilde{c} = (2m+1)/4$.
(2): The strong indication of topological nature of the theory due to the surprising and beautiful appearance of the intersection indices of moduli spaces of Riemann surfaces. (see [46] for a review, and [45] for the connection to the so-called Kontsevich model.)

What we would rather like to stress here is the perfect analogy to standard statistical mechanics. The "masses" of the theory (μ_k) are fixed by the specific approach to the critical surfaces. These are hyper-surfaces in an (in principle) infinite dimensional coupling constant space and they are all of finite co-dimension, precisely as we expect in the general analysis of critical phenomena.

5. The mystery of $c > 1$

5.1. The Ising model

The coupling of any matter fields to two-dimensional quantum gravity at the discretized level is in principle simple. First consider the theory in ordinary two-dimensional space. Most two-dimensional field theories can be "latticized", i.e. they can be formulated as statistical models on a regular two-dimensional lattice in such a way that the continuum limit is recovered as the scaling limit where the "bare" coupling constants are scaled to a critical point. At this critical point a correlation length is divergent and one can forget the underlying lattice. In order to couple the theory to gravity we formulate the model on random lattices and take the *annealed average over a suitable ensemble of random lattices* which can be identified with surfaces of a certain topology. Usually we have in mind triangulations, but as we saw in the last section, it is possible to glue together a large variety of polygons without changing the critical behavior of pure gravity. To the extend that it is possible to formulate the matter theory on such polygons, we expect the same universality after coupling to matter. After taking the annealed average there might still be a phase transition and at the transition point it might be possible to define a continuum limit of the theory. This continuum limit will then qualify as an explicit realization of the original theory coupled to quantum gravity. In general the critical properties of the theory defined by taking the annealed average over the class of random lattices will differ from the corresponding critical properties of the theory defined on a regular lattice. This change is interpreted as the influence of quantum gravity on the matter fields. However, the weight attributed to the random surfaces is influenced by the presence of the matter fields and *at* the critical point the non-analytic part of the partition function can change. Critical exponents like γ_s change too, and this has the interpretation as a *back reaction of the matter on gravity.*

In sec. 4 this program was realized, although the identification with the $(2, 2m-1)$ minimal conformal field theories were made in a rather indirect way for $m > 2$. They all corresponded to $c < 0$ for $m > 2$, i.e. to non-unitary field theories. Conceptually it is more interesting to consider unitary theories, i.e. theories with $c > 0$. The Ising model was the first model with $c > 0$ where it was possible to take the annealed average and in this way calculate the critical exponents of the theory after coupling to gravity [47, 85]. Later the results were confirmed by the use of continuum methods.

Following the strategy just outlined we define the model on a regular lattice. This is just the ordinary Ising model[16]. Next we define the Ising model on a random triangulation as follows:

$$Z(\mu,\beta) = \sum_{T\in\mathcal{T}} \frac{1}{S_T}\, \mathrm{e}^{-\mu N_T} \sum_{\{\sigma_i\}} \exp\left(\frac{\beta}{2}\sum_{(ij)}(\sigma_i\sigma_j - 1)\right) \tag{5.1}$$

where i, j refers to the triangles in T and $\sum_{(ij)}$ is over pairs of neighboring triangles. We assume that the triangulation has spherical topology. The explicit solution of this model is made possible by mapping it on a two-matrix model. To each triangle i is associated a spin σ_i variable which can take two values (± 1) and the model (5.1) has the following representation in terms of triangulations: we have to glue together two kind of triangles (with labels $\pm$) in all possible ways compatible a given topology (here

[16] On a regular lattice the spins are usually placed at the sites of the lattice. Alternatively one could place the spins in the centers of the simplexes or hyper-cubes etc. depending on the structure of the lattice. By connecting the centers we can view the latter assignment as a spin system on the dual lattice. When it comes to critical properties of the system one would not expect any difference since we will usually be interested in long distance behavior where we *want* to forget about the underlying lattice. Historically the spins on dynamical triangulations were put at the centers of the triangles and we will follow this convention.

taken to be spherical). The weight of the gluing along links will be 1 if the triangles are identical and $e^{-\beta}$ if they are different. In addition we have the usual weight given by the total number of triangles. This generalized gluing process can be realized by a two matrix model where the two kind of triangles are represented by two different matrix potentials ϕ^3 and ψ^3, ϕ and ψ being Hermitean matrices as in the one-matrix model. In order to perform the gluing we use a Gaussian part:

$$\frac{1}{2(1-\mathrm{e}^{-2\beta})}\mathrm{Tr}\,\left(\phi^2+\psi^2-2\,\mathrm{e}^{-\beta}\,\phi\psi\right). \tag{5.2}$$

The inverse of this quadratic form will be given by

$$\begin{pmatrix} 1 & \mathrm{e}^{-\beta} \\ \mathrm{e}^{-\beta} & 1 \end{pmatrix}\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2,\beta_2} \tag{5.3}$$

where α, β refers to the matrix indices, while the 2×2 matrix refers the indices $\pm$. It glues triangles with different spin with weight $e^{-\beta}$ as desired.

This two-matrix model can be solved explicitly, as was the case for the one-matrix model. We will not discuss the solution here, only mention the result. For large β (low temperature) the system is magnetized. The spins are aligned. All triangulations have the same magnetic energy for this spin configuration and this means that gravity can fluctuate as if there where no spins at all: The fractal geometry will be as in the pure gravity case and $\gamma_s = -1/2$. For very low β (high temperature) the spins will fluctuate wildly and not care about any underlying lattice structure either. The magnetization will be zero, the geometry will still be independent of the spin system and $\gamma_s = -1/2$. As β is increased from zero the spin system will start to interact with geometry. This is possible since the magnetic energy is proportional to the length of boundaries between spin clusters. On a regular lattice a large spin cluster of area A will have a length $L \geqslant \sqrt{A}$. This is not necessarily so on a dynamical lattice. A glance at fig. 19 shows that we can have very large spin clusters separated by boundaries of only a few links. In this way it is clear that the matter system will have a tendency to deform the triangulations towards geometry with small "bottle necks" like the ones shown in fig. 19. *This gives us a direct visualization of the back-reaction of matter on geometry.* For the Ising model this back-reaction is not sufficiently strong to the change γ_s before we reach a β so high that it is favorable for the spins to align: we have a transition. At the transition γ_s changes to $-1/3$. Above the transition it jumps back to $-1/2$. Not only does the spin system affect the fractal structure of the geometry. The fluctuating geometry will change the critical properties of the spin system. Intuitively one would expect it to soften the transition and this is what happens. The second order transition for the Ising model on a regular lattice is changed to a third order transition and the specific heat exponent α is changed from 0 to -1. The critical exponents β for the magnetization and the spin susceptibility exponent γ (not to be confused with γ_s) are also changed. Needless to say, the exponents calculated this way agree with the KPZ exponents of a $c = 1/2$ conformal field theory coupled to two-dimensional quantum gravity.

5.2. Multiple Ising spins

It is now trivial to couple many Ising spins to quantum gravity. We simply put several independent species of Ising spins on each triangle. On a regular lattice this does not lead to any new physics since they are non-interacting. The only change is that the central charge will be $c = n/2$, n being the number of Ising copies. On dynamical triangulations they will interact in a non-trivial way via the back-reaction on geometry as described above. It is clear how one can formulate the models as *multi-matrix models*. However, we cannot solve these. If the number of copies n of Ising models is larger than 2 we enter into the region of strong interaction between geometry and matter in the sense that the KPZ formula (4.1) breaks down. In this region it is still not fully understood what happens. This is the same region where the non-critical string is defined. However, for $n \to \infty$ it is possible to analyze what happens using mean field theory[17].

As a starting point we use some very important inspiration from the extensive numerical simulations of multiple Ising models on dynamical triangulations [49–52]. The simulations indicate that:

1) There is still a critical point β_c below which there is no magnetization and above which the system is magnetized.

2) Above β_c the geometry seems to be that of pure gravity. Below β_c there is a region where the situation is not clear, and where the fractal structure of the surface is very pronounced. For sufficiently small β the geometry again is that of pure $2d$ gravity.

3) For a large number of spins it seems as if the system increasingly fast will be totally magnetized when $\beta > \beta_c$. I.e. for $\beta > \beta_c$, but quite close to β_c, the system will essentially be identical to the one at $\beta = \infty$.

4) For a sufficiently large number of Ising spins $\gamma_s(\beta_c)$ will be positive.

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[17] The discussion here follows [53], since the interpretation in terms of random surfaces is very simple. However, the same mean field arguments have appeared in slightly different contexts in a number of papers [54–58].

In view of point 2) and 3) it seems reasonable to attempt a description of the model (for a large number of Ising copies) in a region around β_c and for large β in terms of an effective model which has only spin excitations with minimal boundaries separating $\pm$ regions since these excitations have a minimum energy. Let us for convenience of the following arguments specify the class of dynamical triangulations $\mathcal{T}_2$ used in the annealed average to include all gluings of triangles such that the corresponding surfaces are spherical and the minimal length of a closed loop of links is two. Such minimal loops are shown in fig. 19. For this class of triangulations a minimal boundary between spin clusters will have length two and we can view the surfaces as glued together of $\pm$ decorated baby universes, i.e. parts of the surface connected to the rest by a small loop of length two. In other words we consider the summation over all triangulations in the class $\mathcal{T}_2$ and on a given triangulation we consider self-consistent iterations of the minimal spin-energy excitation. This is obviously identical to the first term in a low temperature expansion on a regular lattice. Here it will interact non-trivially with the geometry.

Let us in this mean field model consider the one-point (or more precisely one-loop) function $G(\mu,\beta)$, where the boundary just consist of two (marked) links. The boundary eliminates the symmetry factor $1/S_T$ in eq. (5.1) and we have

$$G(\mu,\beta) = \sum_{T\in\mathcal{T}_2'} \mathrm{e}^{-\mu N_T} {\sum_{\{\sigma_i\}}}' \exp\left(\frac{\beta}{2}\sum_{(ij)}(\sigma_i\sigma_j - 1)\right) \tag{5.4}$$

where $\mathcal{T}_2'$ denotes the class of triangulations where the boundary is a loop consisting of two (marked) links and ${\sum_{\{\sigma_i\}}}'$ denotes the summation over the restricted class of spin configurations. Similarly, one defines n-point functions that are essentially derivatives w.r.t. μ of the one-point function. The susceptibility $\chi(\mu,\beta)$ is defined as

$$\chi(\mu,\beta) = -\frac{\partial G}{\partial\mu} \tag{5.5}$$

and the string susceptibility exponent $\gamma_s(\beta)$ is given by

$$\chi(\mu,\beta) = \frac{c(\beta)}{(\mu-\mu_c(\beta))^{\gamma_s(\beta)}} + \text{less singular terms}, \tag{5.6}$$

for $\mu \to \mu_c(\beta)$, where $\mu_c(\beta)$ denotes the critical cosmological constant as a function of β.

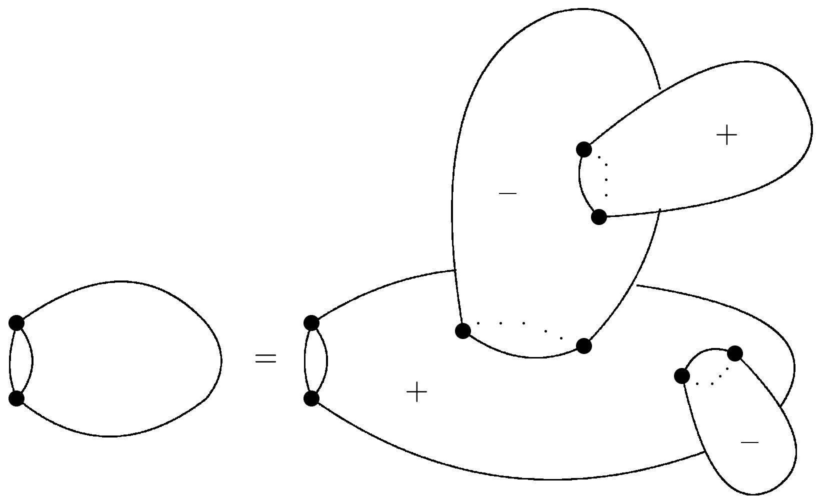

Fig. 19. A graphical representation of eq. (5.7). The complete one-loop function allows a recursive decomposition into baby universes with definite spin assignment.

The corresponding quantities in pure 2d gravity will be denoted by $\chi_0(\mu)$, $\gamma_s^{(0)}$ and μ_0.

Recall that the reason why the Ising model on the dynamical triangulations could be solved was the ability to map it to a random surface model (gluing two kind of triangles). For our mean field model we have this mapping explicitly given to us by the recursive decomposition in baby universes of alternating spin orientation as illustrated in fig. 19.

Using this decomposition procedure and summing first over baby universes with different spin configurations we obtain

$$G(\mu,\beta) = \sum_{\bar{T}\in\mathcal{T}_2'} \mathrm{e}^{-\mu N_{\bar{T}}} \left(1+\mathrm{e}^{-2\beta}\, G(\mu,\beta)\right)^{L_{\bar{T}}} \tag{5.7}$$

where $L_{\bar{T}}$ denotes the number of links in $\bar{T}$, i.e. $L_{\bar{T}} = 1+\frac{3}{2}N_{\bar{T}}$. In eq. (5.7) the factor $\mathrm{e}^{-2\beta}$ represents the coupling of a baby universe across the phase boundary to the rest of the surface and the factor 1 in the parentheses originates from the empty baby universe. We can now write

$$G(\mu,\beta) = \sum_{T\in\mathcal{T}_2'} \mathrm{e}^{-\bar{\mu}N_T} = G_0(\bar{\mu}), \qquad \bar{\mu} = \mu - \frac{3}{2}\log\left(1+\mathrm{e}^{-2\beta}\, G(\mu,\beta)\right). \tag{5.8}$$

Note that the last equation can be written as

$$\mu = \bar{\mu} + \frac{3}{2} \log \left(1 + \mathrm{e}^{-2\beta} G_0(\bar{\mu})\right) \tag{5.9}$$

which expresses μ in terms of known functions of $\bar{\mu}$ and β since pure gravity can be solved.

From eqs. (5.8) and (5.9) we get

$$\chi(\mu, \beta) = \chi_0(\bar{\mu}) \frac{\partial \bar{\mu}}{\partial \mu}, \tag{5.10}$$

$$\frac{\partial \bar{\mu}}{\partial \mu} = \frac{\mathrm{e}^{2\beta} + G_0(\bar{\mu})}{\mathrm{e}^{2\beta} - (\frac{3}{2}\chi_0(\bar{\mu}) - G_0(\bar{\mu}))}. \tag{5.11}$$

It is clear that the derivation and the equations are very similar to the ones used to prove $\gamma_s = 1/2$ for the non-critical strings. Indeed, we will find the same results here, but the additional coupling constant β allows a new critical behavior as will be clear.

Since the string susceptibility exponent $\gamma_s^{(0)} = -1/2 < 0$ in the case of pure gravity both $G_0(\mu_0)$ and $\chi_0(\mu_0)$ are finite. This implies that a β_c exists such that the denominator in (5.11) is different from zero for all $\mu \geqslant \mu_c(\beta)$ provided $\beta > \beta_c$. β_c will be the phase transition point. Fig. 20 shows the phase diagram in the (μ, β)-plane.

Let us first consider region I in fig. 20. The only chance for a singular behavior is that $\bar{\mu}(\mu_c(\beta) = \mu_0$ and in this case the leading singularity has to come from $\chi_0(\bar{\mu})$. We conclude:

$$\bar{\mu}(\mu_c(\beta)) = \mu_0 \quad \text{and} \quad \gamma_s(\beta) = \gamma_s^{(0)} \quad \text{for} \quad \beta > \beta_c. \tag{5.12}$$

This is the phase where the model is magnetized, where the spin fluctuations are small and where the geometry of the surfaces is not affected by the spins.

The number β_c is characterized by being the largest β for which $\partial\mu/\partial\bar{\mu}$ equals zero for some $\bar{\mu}$, i.e. (by (5.11)) the largest β for which the equation

$$\mathrm{e}^{2\beta} = \frac{3}{2}\chi_0(\bar{\mu}) - G_0(\bar{\mu}) \ \ (= \sum_{T \in \mathcal{T}_2'} (3N_T/2 - 1)\, \mathrm{e}^{-\bar{\mu} N_T}) \tag{5.13}$$

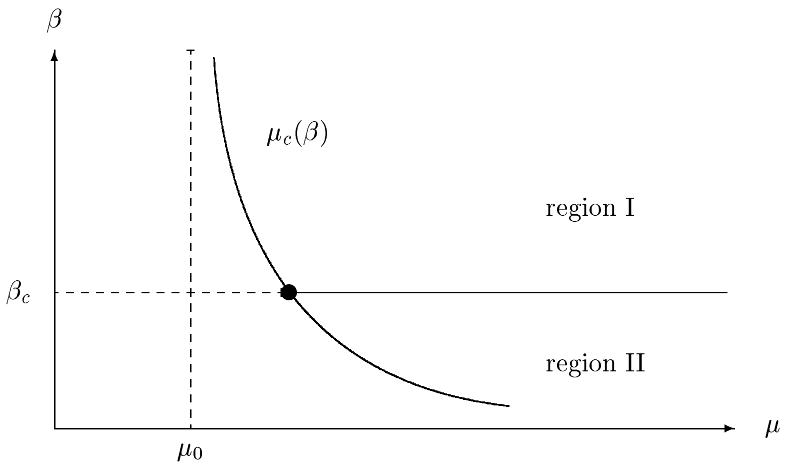

Fig. 20. The phase diagram. The partition function is defined (convergent) to the right of the critical line $\mu_c(\beta)$. The infinite volume limit is obtained by keeping β fixed and approaching $\mu(\beta)$ from the right. For $\beta \to \infty$ $\mu(\beta)$ approaches μ_0 for pure gravity. β_c denotes the phase transition point.

has a solution. If we define $\bar{\mu}_c(\beta)$ by

$$\left.\frac{\partial \mu}{\partial \bar{\mu}}\right|_{\bar{\mu}_c(\beta)} = 0 \quad \text{for} \quad \beta \leqslant \beta_c, \tag{5.14}$$

then $\bar{\mu}_c(\beta)$ obviously solves eq. (5.13) and we have

$$\bar{\mu}_c(\beta_c) = \mu_0, \qquad \bar{\mu}_c(\beta) > \mu_0 \quad \text{for} \quad \beta < \beta_c. \tag{5.15}$$

Let us now consider region II in fig. 20 where $\beta < \beta_c$. If we use eq. (5.10) and (5.11) this implies that the lhs will be singular for $\bar{\mu} \to \bar{\mu}_c(\beta)$. In fact, since $\bar{\mu}_c(\beta) < \mu_0$ both $\chi_0(\bar{\mu})$ and $G_0(\bar{\mu})$ will be regular around $\bar{\mu}_c(\beta)$ and we can Taylor expand the lhs of eq. (5.10):

$$\chi(\mu, \beta) \sim \frac{c}{\bar{\mu} - \bar{\mu}_c(\beta)} \sim \frac{\tilde{c}}{\sqrt{\mu - \mu_c(\beta)}} \tag{5.16}$$

To derive the last equation we have used (5.11) and (5.14) which tell us that

$$\mu = \mu_c(\beta) + \text{const.}(\bar{\mu} - \bar{\mu}_c(\beta))^2 + \cdots. \tag{5.17}$$

We conclude that $\gamma_s(\beta) = 1/2$ for $\beta < \beta_c$. In this phase baby universes are dominant. Effectively we have branched polymers and the total magnetization of the system is zero.

Let us finally consider the system at the critical point β_c. This point is characterized by the fact that $\bar{\mu}_c(\beta_c)$ coincides with μ_0. Although the singularity of $\chi(\mu, \beta_c)$ for $\mu \to \mu_c(\beta_c)$ is still dominated by the zero of $\partial\mu/\partial\bar{\mu}$ we can no longer Taylor expand $\mu(\bar{\mu}, \beta_c)$ around $\bar{\mu}_c(\beta_c)$ $(= \mu_0)$ since the functions in (5.11) are singular in μ_0. On the other hand we can use the known singular behavior of the pure gravity functions G_0 and χ_0 at μ_0 to deduce from (5.11), remembering that $\gamma_s^{(0)} < 0$,

$$\frac{\partial\mu}{\partial\bar{\mu}} \sim (\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_s^{(0)}}, \qquad \text{i.e.} \quad \mu - \mu_c(\beta_c) \sim (\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_s^{(0)}+1}. \tag{5.18}$$

From (5.10) we finally get, using (5.18)

$$\chi(\mu, \beta_c) \sim \frac{c}{(\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_s^{(0)}}} \sim \frac{\tilde{c}}{(\mu - \mu_c(\beta_c))^{-\gamma_s^{(0)}/(-\gamma_s^{(0)}+1)}}, \tag{5.19}$$

and we have derived the remarkable relation:

$$\gamma_s(\beta_c) = \frac{-\gamma_s^{(0)}}{-\gamma_s^{(0)} + 1} = \frac{1}{3}. \tag{5.20}$$

The model can easily be extended to include the coupling to a magnetic field and one can explicitly verify that the system is magnetized for $\beta > \beta_c$ and has zero total magnetization for $\beta < \beta_c$. Further the transition is a third order transition (like in the full Ising model on dynamical triangulations). The model captures the essential features observed numerically for a large number of Ising models on dynamical triangulations. It also strongly suggests the existence of a region below β_c where the surfaces consist of numerous totally magnetized baby universes with different spin orientations such that the total magnetization is zero. The spin transition observed is just the final alignment of the spins of the baby universes.

The following should be noticed:

(1): For a finite number of Ising models the mean field approximation above clearly fails for $\beta \to 0$. In the infinite temperature limit γ_s has to return to the value $-1/2$. In the mean field approximation γ_s stays equal $1/2$. For high temperatures there will be important spin configurations which are different from the ones considered in the mean field approximation. If we for sufficiently high central charge c *assume* the existence of an interval below β_c where $\beta_s = 1/2$, it is an interesting and unsolved question how the transition to $\gamma_s = -1/2$ takes place. Is there a single transition, a continuous change or a cascade of transitions, and how are these transitions characterized?

(2): The transition at β_c had $0 < \gamma_s < 1/2$ and it was related to $\gamma_s^{(0)}$ for pure gravity by eq. (5.20). The formula reflects that the individual baby universes are totally magnetized, i.e. have the fractal structure of pure gravity. This can be generalized [59]:

Theorem: given a multiple spin model with $0 < \gamma_s < 1/2$ it follows that

$$\gamma_s = \frac{-\gamma_n}{-\gamma_n + 1} = \frac{1}{n+1}, \qquad \gamma_n = -\frac{1}{n}. \tag{5.21}$$

The interpretation of eq. (5.21) is that the individual baby universes have a fractal structure corresponding to a unitary conformal field theories *with* $c < 1$ *coupled gravity.* Such theories are characterized by a $c_n = 1-6/n(n+1)$ and a $\gamma_s = -1/n$. Formula (5.20) represented the simplest example, $c = 0$, but it can be shown, using renormalization group arguments, that (5.21) is the only possible critical behavior if $\gamma_s > 0$. It rules out the possibility that γ_s can change continuously while larger than zero. It also tells us the possible solution to the $c > 1$ disaster in the continuum approach: *The interaction between matter and geometry becomes so strong that the surfaces disintegrate dynamically into baby universes which individually have* $c < 1$. What is missing in our understanding is how to predict a specific $c_n \leqslant 0$ from $c \geqslant 1$. The only thing we know is that for $c \to \infty$ it follows that $c_n = 0$.

5.3. Random surfaces with extrinsic curvature (II)

Let us return to the random surface theory with extrinsic curvature. Recall that numerical simulations indicated that $\gamma_s \approx 1/4$. Can we understand the theory in the general setting outlined in the beginning of this section? To investigate this possibility consider the model *on a fixed regular trian-*

gulation T:

$$Z_T(\lambda) = \int \prod_{i \in T/\{i_0\}} \mathrm{d}x_i \ \mathrm{e}^{-\sum_{(ij)}(x_i - x_j)^2 - \lambda \sum_{\triangle_i,\triangle_j}(n_{\triangle_i} - n_{\triangle_j})^2}. \tag{5.22}$$

In (5.22) i denotes a vertex in T which is mapped into $x_i \in R^3$ (but it is easy to formulate the model in R^D) and $n_\triangle$ is a normal to the oriented triangle $(x_i x_j x_k)$ where $\triangle = (ijk) \in T$.

Eq. (5.22) defines the model of so-called *crystalline surfaces* [60] (see the lectures of Peliti and Nelson for details). Since T is regular we can view it as a lattice theory, either in terms of the variables x_i, in which case it is non-polynomial, or in terms of the normals $n_\triangle$ after integrating out some of the degrees of freedom of the x_i's. If we go to the limit $\lambda = 0$ it is a purely Gaussian model and it can be solved trivially. Viewed as a lattice model in terms of the normals it is a weird theory [61]. If we use a continuum notation and denote the lattice coordinates by ξ the normal-normal correlator is:

$$\langle \boldsymbol{n}(0)\boldsymbol{n}(\xi)\rangle \sim \frac{\delta_\Lambda^{(2)}(\xi)}{\xi^2} - \frac{1 - \mathrm{e}^{-\Lambda^2\xi^2}}{2\pi\xi^4} \sim \frac{-1}{2\pi\xi^4} \tag{5.23}$$

for $\xi \neq 0$. In this formula Λ is just a convenient continuum proper-time cut-off instead of the lattice spacing[18]. Except at coinciding points the normals are anti-correlated. This truly remarkable situation can only appear because the surface is totally crumpled in R^3. It is easy to calculate the average radius of the surface in R^3:

$$\langle r\rangle_{N_T} \sim \log N_T, \tag{5.24}$$

i.e. the Hausdorff dimension $d_H = \infty$.

With increasing λ the short distance correlation between the normals becomes positive, but the long distance tail remains negative, which implies

[18] To be more precise the continuum Gaussian action is used and, if Λ is an ultraviolet cut-off and L an infrared cut-off, the propagator is

$$\langle x^\mu(0)x^\nu(\xi)\rangle = \delta_{\mu\nu} \int_{L^{-2}}^{\Lambda^2} d\alpha e^{-\alpha\xi^2}/4\pi\alpha.$$

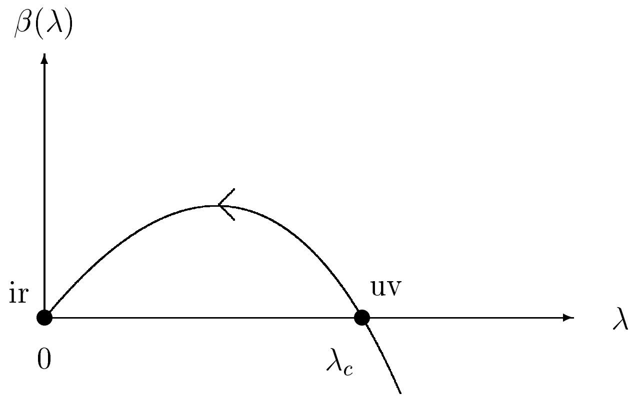

Fig. 21. The β-function for the crystalline surface model (5.22).

that $d_H = \infty$. However, for a finite value $\lambda = \lambda_c$ the system undergoes a phase transition, the so-called *crumpling transition*, after which the normal-normal correlation becomes positive everywhere and the Hausdorff dimension finite. *The transition seems to be second order.* Large scale computer simulations have verified the existence of this transition and strongly suggest that it is second order [62–64]. Calculations of the β-function $\beta(\lambda)$ for continuum toy models related to the model defined by eq. (5.22) lead to [65]

$$\beta(\lambda) = c_1\lambda - c_2\lambda^2, \qquad c_i > 0, \tag{5.25}$$

i.e. an ultraviolet fixed point and applying the renormalization group we would have the flow shown in fig. 21. There should be a central charge c associated with this transition. Its value is not known, but it would be most interesting if $c > 1$. If $\lambda = 0$ we have 3 Gaussian fields and $c = 3$. Since $\lambda = 0$ is an infrared fixed point (see fig. 21) one would naively expect from the c-theorem that $c_{crump} \geqslant 3$.

The summarize: Although weird, the model fulfills the requirement for a spin model we can couple to quantum gravity. It is done by taking the anneal average of (5.22) over random triangulations (of the sphere), i.e. we arrive precisely at the random surface model studied earlier. Numerical simulations showed that this model had a phase transition to smooth surfaces. *It is natural to conjecture that it is the quantum gravity version of the crumpling transition.* If $c_{crump} > 1$ we will be in same situation as for the multiple Ising models. The numerical measurement of γ_s gives some support to this idea. It would be most interesting to have a better analytic understanding of the transition since it is relevant both from the viewpoint of string theory, as described above, and from the viewpoint of membrane physics: It is a model of a fluid membrane, not entirely physical since it is self-intersecting, but still of considerable interest.

6. Euclidean quantum gravity in $d > 2$

6.1. Basic questions in Euclidean quantum gravity

Until now we have considered various aspects of two-dimensional quantum gravity. As long as we restrict topology we deal with a well defined theory. There might even be some hope that one could define the summation over all two-dimensional topologies although this question is not yet settled. At the moment we address Euclidean quantum gravity in dimensions higher than two a frightening number of basic questions appear. Let us just list some of them:
(1): How do we cure the unboundedness of the Einstein-Hilbert action in $d > 2$?
(2): Does the non-renormalizability of the gravitational coupling constant not spoil any hope of making sense of the theory?
(3): What is the relation between Euclidean and Lorentzian signature in the absence of any Osterwalder-Schrader axioms to ensure that we can rotate from Euclidean space to Lorentzian space-time?
(4): What is the role of topology, keeping in mind that four-dimensional topologies cannot be classified?

It is possible to take these questions and our inability to answer them in a fully satisfactory way as an indication that there exists no theory of Euclidean quantum gravity for $d = 4$. Let us briefly discuss the problems. While (1) and (2) in principle are unrelated the solution of one of the problems often solves the other problem too. The unboundedness of the Euclidean Einstein action is due to the conformal mode. An obvious way to cure this problem is to assume that the Einstein-Hilbert action is only an infrared approximation to the complete theory. A simple minded way to make the action bounded is to add higher derivative terms like R^2 or $R_{\mu\nu}R^{\mu\nu}$. Such terms also tend to cure the problem of renormalizability since the propagators will contain higher inverse powers of the momentum. In fact adding suitable combinations of R^2 terms one can get a theory which is renormalizable and where the Euclidean action is bounded from below [66]. It is not known whether the theory qualifies as a unitary field theory if we apply the standard rotation back to Lorentzian signature. A more general discussion along the lines of viewing the Einstein-Hilbert term as the first (infrared leading term) term of the full action goes back to Weinberg who introduced to concept of asymptotic safety [67]. The idea is that we should be able, by means of the renormalization group, to work our way back from the long distance region where the Einstein-Hilbert action is a good effective description to some non-trivial ultraviolet fixed point. In addition the associated critical surface is assumed to be of finite co-dimension, which means that only a finite number of parameters need to be fine tuned to reach the critical surface and from this point of view the theory will not differ in spirit from ordinary renormalizable theories. The effective Lagrangian description of the field theory by means of fields suitable for the infrared fixed point might then be an infinite series

$$\mathcal{L} = \sqrt{g}\left[\Lambda - \frac{1}{16\pi G}R + f_2 R^2 + f_2' R_{\mu\nu}R^{\mu\nu} + \cdots\right] \tag{6.1}$$

which might even be non-polynomial, but which *might* make sense both with Riemannian and Lorentzian signatures.

A weakness in such a scenario is that the existence of the ultraviolet fixed point is hypothetical up to now and we have not exactly been flooded with examples of non-trivial fixed points in four-dimensional field theory. However, the interesting results obtained by use of the expansion in $2+\varepsilon$ dimensions give some support to this idea [68].

Finally one could hope that once the correct non-perturbative formulation of the quantum theory has been found these problems will resolve themselves and it will become clear why one should not necessarily think in terms of perturbative expansions around fixed background metrics.

Not much is known about the question of summing over topologies. As long as we think in terms of continuum physics and write down the path integral it offers the possibility of summing over different manifold structures as well as integrating over inequivalent Riemannian structures of a given manifold:

$$Z = \sum_{\mathcal{M}\in \mathrm{Top}} \int_{\mathcal{M}} \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol(diff)}}\, \mathrm{e}^{-S[g]}\,. \tag{6.2}$$

In two and three dimensions we do not have to worry about the meaning of *Top*, since there is equivalence between smooth manifolds and topological manifolds. In two dimensions the manifolds are uniquely characterized by their Euler characteristic χ and the summation over *Top* is simply a summation over χ or the genus $g = 1 - \chi/2$ of the surfaces. In spite of this simple prescription surprisingly little progress has been made in defining the sum in (6.2) using continuum methods. Matrix models gave non-perturbative definitions, as described in sec. 4. However, so far the results have been ambiguous. If we move to three dimensions we encounter a slight classification problem in the sense that no simple parametrization like χ of the various topologies exists. However, the problem seems to get completely out of control when we move to four dimensions. For four-dimensional manifolds there is not equivalence between smooth and topological structures. Topological manifolds exist which do not admit smooth structures and some topological manifolds admit infinitely many inequivalent smooth structures. If we insist on summing over all smooth structures $\sum_{Top}$ will be rather unwieldy. To complicate the operational meaning of $\sum_{Top}$ it should be added that four-dimensional manifolds are not algorithmic classifiable, i.e. no finite algorithm *in the sense of Turing* exists which allows us to decide if two arbitrary four dimensional manifolds are equivalent. On the other hand arguments (not known to me) might exist which dictate a restriction of the allowed class of manifolds. Since there seems to be fermions in the world one could argue that the manifold should be a spin manifold. If one makes the additional (rather arbitrary) restriction that it should be simply connected it is possible to show that such a (smooth) manifold is characterized by its Euler number and its signature where these in four dimensions are given by:

$$\chi(M) = \frac{1}{128\pi^2} \int_M \mathrm{d}^4\xi \sqrt{g}\; R_{abcd} R_{a'b'c'd'} \varepsilon^{aba'b'} \varepsilon^{cdc'd'} \tag{6.3}$$

$$\tau(M) = \frac{1}{96\pi^2} \int_M \mathrm{d}^4\xi \sqrt{g}\; R_{abcd} R^{ab}{}_{c'd'} \varepsilon^{cdc'd'} \tag{6.4}$$

For simply-connected spin manifolds the signature τ is a multiple of 16, while χ is an integer $\geqslant 2$. Thus eq. (6.4) seems a minor extension compared to two dimensions where the summation is over χ, but the restriction to simply connected spin manifolds is not natural at this stage of a quantum theory of space-time.

While all these problems seem to discourage any attempt to make sense of the path integral of Euclidean quantum gravity, it is still our obligation to try to investigate if it is possible. Below I will argue that the use of dynamical triangulations, which works so well in two dimensions, allows us to discuss several of the above issues in some detail and might be a candidate for non-perturbative definition of quantum gravity even in higher dimensions.

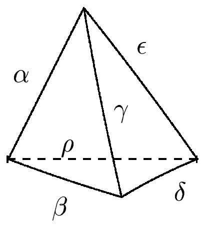

Fig. 22. Labels of a tetrahedron

6.2. Definition of simplicial quantum gravity for $d > 2$

Let us define simplicial quantum gravity as a generalization of the construction in sec. 3 and sec. 4 for two dimensions: In d dimensions (where $d = 3$ or $d = 4$) we construct all closed (abstract) simplicial manifolds from K d-dimensional simplexes. As in two dimensions we imagine that the length of the links in the simplexes are a (which we take as 1 unless explicitly stated). For such a combinatorial or, equivalently, piecewise linear manifold we can apply Regge calculus and in this way assign a Riemannian metric to the manifold. By such an assignment we see that the discretization is able, for finite K, to assign a meaning to the sum and integral in (6.2)[69, 70]

$$\sum_{\mathcal{M}\in \mathrm{Top}} \int_{\mathcal{M}} \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol(diff)}} \to \sum_{T}. \tag{6.5}$$

The discretization has the same virtue as in two dimensions: In principle it allows a unified treatment of the summation over topologies and Riemannian structures.

A few remarks should be said about the formula (6.5). First one could try as in two dimensions to make the gluing automatic, and in this way arrive at a generalized matrix model [70–72]. In d dimensions the task is to glue together (d-1)-dimensional sub-simplexes. Let us discuss the situation in $d = 3$ (the generalization to higher dimensions is trivial). In fig. 27 we have shown a building block for three-dimensional simplicial complexes: a tetrahedron (the generalization of the equilateral triangle used up to now) where the edges has labels $\alpha, \beta, \gamma, \delta, \varepsilon, \rho$, which takes values $1, ..., N$, where N is an abstract label which we in the end want to take to infinity like in the two-dimensional case. To each oriented face of the tetrahedron, α, β, γ, say, we assign a complex variable $\phi_{\alpha\beta\gamma}$ whose value is invariant under even permutations of the indices but conjugated under odd permutations (which correspond to a reversal of orientation). To the tetrahedron we associate the generalization of Tr ϕ^3 for the triangles:

$$A(\phi) = \sum_{\alpha\beta\gamma\delta\varepsilon\rho} \phi_{\alpha\beta\gamma}\phi_{\gamma\delta\varepsilon}\phi_{\varepsilon\rho\alpha}\phi_{\beta\rho\delta} \tag{6.6}$$

Consider the following integral:

$$Z(g) = \int \mathrm{d}\phi \; \exp\left(-\frac{1}{6}\sum_{\alpha\beta\gamma} |\phi_{\alpha\beta\gamma}|^2 + gA(\phi)\right). \tag{6.7}$$

If the exponent is expanded in a power series in g, and all possible Wick contractions are performed on the powers of $A(\phi)$ it corresponds to all possible gluings of the tetrahedra along triangles. The expansion leads to

$$\log Z(g) \sim \sum_T \frac{1}{S_T} N^{N_1(T)} g^{N_3(T)}, \tag{6.8}$$

where the sum is over all possible gluings T. $N_1(T)$ denotes the number of links in T and $N_3(T)$ the number of tetrahedra. Let us postpone the discussion of the terms $N^{N_1(T)}g^{N_3(T)}$ and concentrate on $\sum_T$. This kind of "blind" gluing of d-dimensional simplexes along their $d-1$ dimensional sub-simplexes will not in general create combinatorial manifolds, but only so-called *combinatorial pseudo-manifolds*: i.e simplicial complexes where $d-1$-dimensional sub-simplexes which are not boundary simplexes are contained in precisely two d-simplexes and where it is possible to connect any two d-simplexes by a sequence of d-simplexes, each intersecting along some $d-1$ simplex. For such pseudo-manifolds the neighborhood of a vertex will not necessarily be equivalent with a d-dimensional ball. Whether or not this is important for quantum gravity is not clear. In the two-dimensional case it did not course any problems. One could create one-loops and two-loops which cannot be present on a combinatorial manifold, but the pseudo-manifold still had a transparent surface interpretation and the Euler number was perfectly well defined. In the three dimensional case the modification is more severe in the sense that the Euler characteristic can be different from zero, while $\chi = 0$ for any odd dimensional *manifold*. In the following we will (to be conservative) restrict ourself to so-called combinatorial manifolds, i.e. combinatorial pseudo manifolds where the neighborhood of any sub-simplex is homeomorphic to a d-simplex. For closed manifolds this enforces certain relations between the number of i-simplexes

N_i in the triangulation, known as the Dehn-Sommerville relations:

$$N_i = \sum_{k=i}^{d} (-1)^{k+d} \binom{k+1}{i+1} N_k. \tag{6.9}$$

If we define $N_{-1} = \chi$, the Euler characteristic of the manifold, (6.9) is valid for $i = -1, ..., d-1$.

Secondly, in light of the complicated relation between topology and diffeomorphism for four-dimensional manifolds one could be worried that similar problems arise when we compare combinatorial, i.e. piecewise linear, manifolds and smooth manifolds. However, for dimensions $d < 7$ we have equivalence between piecewise linear and smooth structures. Whatever subtleties might be involved in defining the sum on the lhs of eq. (6.5) it should be captured at the rhs of eq. (6.5). Of course eq. (6.5) itself is rather formal as it stands. Even in the two-dimensional case the precise meaning of the lhs was not clear from a mathematical point of view and in sec. 4 we tried to use the rhs to *define* what is meant by the summation over all two-dimensional manifolds. The problem was non-trivial in the sense that the number of triangulations grows as fast as $K!$, K being the number of triangles. No reasonable discretized action could kill this entropy factor [69, 70] and

$$Z = \sum_T \mathrm{e}^{-S_T} \tag{6.10}$$

will be divergent. This problem will of course be present also when we move to higher dimensions. The naive expression (6.10) makes no sense. In two dimensions the way out was to fix topology. This bounds the number of triangulations exponentially and eq. (6.10) will be well defined. Only afterwards the topology is allowed to fluctuate and the *double scaling limit taken*. In higher dimensions it is sensible, as a minimum, to try to define eq. (6.10) for a fixed topology. We need the following conjecture :

Conjecture: The number of combinatorial equivalent d-dimensional manifolds is an exponentially bounded function of the number of d dimensional simplexes.

We call two simplicial complexes combinatorial equivalent if they have a common subdivision and when we talk about equivalence classes of piecewise linear manifolds we have in mind combinatorial equivalence. While the conjecture is true in two dimensions, there is no general proof in higher dimensions[19]. For $d \geqslant 4$ manifolds exist which are not algorithmic recognizable, the reason being that any finitely presented group can appear as the fundamental group for some four-dimensional manifold. However, finitely presented groups cannot be algorithmically classified due to the famous "word problem". As a curiosity we can mention that for such manifolds their number as a function of N_4, the number of 4-simplexes, is not algorithmic calculable. This does not mean that the number cannot be exponentially bounded, only that there is no finite algorithm which allows us to calculate the exact number. In fact numerical simulations seem to work well for the simplest topology (S^4) and the give support to the conjecture for S^4 [75, 77, 78][20].

After these general remarks it is natural as a first explorative step to fix the topology of our four-dimensional manifold to be the simplest possible, that of S^4. The Einstein-Hilbert action for a simplicial manifold can be calculated by Regge calculus. However, we do not need the full machinery in our case where the simplexes are identical and all link length equal. The Regge version of the Einstein action is the sum over deficit angles of the $d-2$ dimensional sub-simplexes times their $d-2$ dimensional volume. In our case the deficit angle associated with a $d-2$ dimensional sub-simplex n_{d-2} is $2\pi - \mathrm{const.}o(n_{d-2})$, where $o(n_{d-2})$ is the *order* of n_{d-2}, i.e. the number of d dimensional simplexes of which n_{d-2} is a sub-simplex. It follows that

$$\left\{ \int \mathrm{d}^d \xi \sqrt{g} R \right\}_{Regge} \propto \sum_{n_{d-2}} (2\pi - \mathrm{const.}o(n_{d-2})). \tag{6.11}$$

If we note that the number of $d-2$-dimensional sub-simplexes in a d-dimensional simplex is $d(d+1)/2$ we can write:

$$S[g;\lambda,G] = \int \mathrm{d}^d \xi \sqrt{g} \left(\lambda - \frac{1}{16\pi G} R \right) \to$$

$$S_T[k_d, k_{d-2}] = k_d N_d(T) - k_{d-2} N_{d-2}(T), \tag{6.12}$$

where the piecewise linear manifold is given defined by the triangulation T and $N_d(T)$ and $N_{d-2}(T)$ denote the number of d- and $d-2$-dimensional simplexes in the triangulation T. We can view $1/k_{d-2}$ as a bare gravitational coupling constant. At first sight the action might seem much too

[19] Very recently there has appeared a proof [73].

[20] Recently some controversy arose [79], but my opinion the question is now settled.

simple to have anything to do with gravity. Our point of view will be the opposite: *The fact that the action is so simple reflect the beauty and simplicity of quantum gravity, and hopefully this simplicity will be reflected in the solution of the theory.*

Our final prescription will be:

$$
\begin{aligned}
Z[\lambda, G] &= \int_{S^d} \frac{\mathcal{D}g_{\mu\nu}}{\mathrm{Vol(diff)}}\, \mathrm{e}^{-S[g;\lambda,G]} \to \\
Z[k_{d-2}, k_d] &= \sum_{T\in S^d} \mathrm{e}^{-S_T[k_{d-2},k_d]}\,. \qquad (6.13)
\end{aligned}
$$

We will be interested in $d = 4$ and sometimes in $d = 3$, in which cases we get

$$Z[k_1, k_3] = \sum_{T\in S^3} \mathrm{e}^{-k_3 N_3(T)+k_1 N_1}\,. \qquad (6.14)$$

$$Z[k_2, k_4] = \sum_{T\in S^4} \mathrm{e}^{-k_4 N_4(T)+k_2 N_2}\,. \qquad (6.15)$$

For $d = 3, 4$ there are only two independent coupling constants as long as we only want an action which depends on global quantities like N_i. It follows from the Dehn-Sommerville relations. If we include higher derivative terms in the action we will certainly loose the simplicity of eqs. (6.14) and (6.15). The higher derivative terms will contain explicit the order of the sub-simplexes which carry the curvature. We will not discuss the lattice implementation of these.

Let us now discuss the phase diagram. Assume $d = 4$. Since it is easy to prove that $N_2(T) \leqslant \mathrm{const.}N_4(T)$, the conjecture above implies that for each k_2, a $k_4^c(k_2)$ exists such that the lhs of (6.13) for a given k_2 is well-defined for $k_4 > k_4^c(k_2)$ and divergent for $k_4 < k_4^c(k_2)$. A potential continuum limit should be taken as $k_4 \to k_4^c$ from above. The corresponding phase diagram is shown in fig. 23. If we for a fixed k_2 approach k_4^c we will be probing the infinite volume limit of the discretized system. It does not imply that there necessarily will be a continuum limit. Rather we should view the system as a lattice system where the infinite volume limit is taken. For some specific values of the bare couplings critical points might exist where a correlation length diverges and where a continuum limit exists. Such a point is tentatively indicated at the figure. Approaching this point in a specific way will then define the renormalized cosmological constant and the

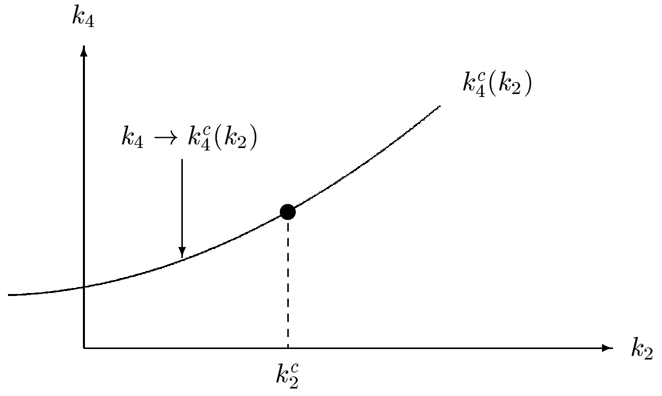

Fig. 23. A hypothetical phase diagram for four-dimensional gravity.

renormalized gravitational constant. Since we are in unchartered territory one should be open-minded for other possibilities, e.g. the possibility that a whole range of k_2's might correspond to a topological gravity where the metric, and correspondingly concepts like divergent distances, play no role.

Let us for a moment return to the three-dimensional partition function given by eq. (6.15). Comparing with eq. (6.8) derived from the tensor model one has the identification:

$$k_3 = -\log g, \qquad k_1 = \log N. \tag{6.16}$$

This relation is the same as we encountered in two dimensions and the relation will be valid for the matrix model in any dimension:

$$k_d = -\log g, \qquad k_{d-2} = \log N. \tag{6.17}$$

From a formal point of view the $N \to \infty$ limit corresponds to taking the "bare" gravitational coupling constant $G \to 0$. In two dimensions this limit played a very important role: since the Einstein-Hilbert action in two dimensions is topological, an expansion in $1/G$ automatically becomes an expansion in topology. However, in three and higher dimensions we have not reason to expect the large N limit to classify topologies for us, and it does not!. It has not yet been possible to construct anything like the double scaling limit in higher dimensions than two.

6.3. Observables

It is possible to define the same critical exponents for higher dimensional simplicial quantum gravity as we have already defined for the various two-dimensional theories we have considered.

First we can define an entropy or susceptibility exponent γ_s. In the following it will always be assumed that the topology is spherical, i.e. the combinatorial manifolds are combinatorial equivalent to the boundary of a 5-simplex. According to the conjecture the number $\mathcal{N}(N_4)$ of triangulations which can be constructed from N_4 4-simplexes is exponentially bounded. Let us now fix k_2. According to the remarks above there will be a critical point $k_4^c(k_2)$. This implies that

$$\mathcal{N}(k_2, N_4) \equiv \sum_{T \in S^4(N_4)} \mathrm{e}^{k_2 N_2} \sim \mathrm{e}^{k_4^c(k_2) N_4} f_{k_2}(N_4) \tag{6.18}$$

where $f_{k_2}(N_4)$ stands for subleading corrections. *If* the subleading correction is power-like we define $\gamma_s(k_2)$ by

$$\mathcal{N}(N_4) \sim N_4^{\gamma_s(k_2)-3}\, \mathrm{e}^{\kappa_4^c N_4}(1 + O(1/N)). \tag{6.19}$$

However, it is not clear that we have such a behavior. It is possible to imagine an asymptotic behavior like:

$$\mathcal{N}(N_4) \sim \mathrm{e}^{\kappa_4^c N_4 - c(k_2) N_4^{\alpha}(k_2) - \cdots}(1 + O(1/N)) \tag{6.20}$$

where $0 < \alpha < 1$. In this case the exponential correction given by α will always dominate over the power-like correction determined by $\gamma_s(k_2)$. There are strong indications that there are several regions with different asymptotic behavior, depending on the value of k_2. This will be discussed below.

Apart from the entropy- or susceptibility exponent γ_s we can introduce the critical exponents ν and η already discussed numerous times in the context of strings and random walks where they were determined from the properties of the two-point function in target space. However, here these quantities will refer to *intrinsic* geometry. Let us define the two-point function as follows: Consider the ensemble of combinatorial manifolds with two marked vertices i_1 and i_2. For each manifold, build of regular 4-simplexes, we have by Regge's prescription a metric. This means that we can define the geodesic distance between the vertices i_1 and i_2. As a rough definition one can consider paths along the links in the triangulation and define the geodesic distance as the shortest link-path[21] between i_1 and i_2. Let us consider the sub-ensemble of manifolds where the geodesic distance between i_1 and i_2 is fixed to be r. If we use the link-paths in the definition of geodesic length r will be an integer. Let us denote this ensemble of manifolds by $\mathcal{T}(2,r)$. We define the two-point function by:

$$G_{k_4}^{(k_2)}(r) = \sum_{T \in \mathcal{T}(2,r)} e^{-S_T[k_4,k_2]} \tag{6.21}$$

Provided that we have the asymptotic behavior (6.19) we expect a generic behavior of this spherical two-point function (we suppress the explicit dependence on k_2):

$$G_{k_4}(r) \sim r^{-\alpha} e^{-m(k_4)r}, \qquad r \gg 1/m(k_4); \tag{6.22}$$

$$G_{k_4}(r) \sim r^{1-\eta}, \qquad 1 \ll r \ll 1/m(k_4); \tag{6.23}$$

$$\chi(k_4) \equiv \sum_r G_{k_4}(r) \sim \frac{1}{(k_4 - k_4^c)^{\gamma_s}}. \tag{6.24}$$

The short distance behavior of $G_{k_4}(r)$ arises from the generic behavior since there is an angular average involved in the definition.

If $m(k_4)$ *scales to zero at the critical point* k_4^c:

$$m(k_4) \sim (k_4 - k_4^c)^{\nu} \tag{6.25}$$

we can define in a simple way the continuum limit: Introduce the continuum parameters $R = r \cdot a$ with the dimension of length and $m_{\rm ph} = m(k_4)/a$ and keep them fixed as $k_4 \to k_4^c$. This fixes $a(k_4) \sim (k_4 - k_4^c)^{\nu}$. In addition one can readily prove Fischer's scaling relation: $\gamma_s = \nu(2-\eta)$.

As usual one can prove that the exponent ν is related to the Hausdorff dimension of the ensemble of manifolds. However, in this case we have in mind an *intrinsic Hausdorff dimension* d_h. For a manifold with N_4 4-simplexes the volume $\sim N_4$ as long as we take $a = 1$. The average volume

[21] Alternatively one could talk about the distance between the 4-simplexes, defined as by moving between successive 4-simplexes via their common 3-simplex boundary. Again this is not a true geodesic distance, but should not differ drastically from the genuine geodesic distance. For a given combinatorial manifold there might be a significant deviation between link distances and 4-simplex distances, but we expect concepts like Hausdorff dimensions etc. to be the same.

is therefore $\sim \langle N_4 \rangle_r$, where $\langle \cdot \rangle_r$ refers to the ensemble $\mathcal{T}(2,r)$. We can now define the internal Hausdorff dimension d_h by

$$\langle N \rangle_r \sim r^{d_h} \quad \text{for} \;\; r \to \infty, \;\; m(k_4) r = \text{const.} \tag{6.26}$$

From (6.21) we have

$$\langle N \rangle_r = -\frac{\mathrm{d} \log G_{k_4}(r)}{\mathrm{d} k_4} = \frac{\mathrm{d} m(k_4)}{\mathrm{d} k_4}\, r, \tag{6.27}$$

and using (6.25) we get

$$\langle N \rangle_r \sim r^{1/\nu}, \quad \text{i.e.} \;\; d_h = 1/\nu. \tag{6.28}$$

As is seen from the above consideration one can apply without any problems general scaling considerations known from critical phenomena provided that we have the asymptotic behavior (6.19). It can be formulated even stronger: It is hard to imagine that any continuum limit of a conventional kind can be taken unless that mass $m(k_4) \to 0$ for $k_4 \to k_4^c(k_2)$.

A good test case could be to return to $2d$ gravity, but surprisingly it turns out that $m(\mu)$ ($\mu \equiv k_4$ in $d = 2$) and the associated critical exponents have only recently been calculated even if they are the most fundamental scaling variables in two dimensional quantum gravity. Unfortunately there is no space for a detailed discussion of the calculation, which uses the so-called transform matrix formulation [80]. Details can be found in [81] and the surprising result is that the two-point function $G_\mu(r)$ can be calculated even at the discretized level:

$$G_\mu(r) = \text{const.}\, (\triangle\mu)^{3/4} \frac{\cosh\left[(\triangle\mu)^{\frac{1}{4}}\beta r\right]}{\sinh^3\left[(\triangle\mu)^{\frac{1}{4}}\beta r\right]}, \tag{6.29}$$

where *const.* and β are positive constants of $O(1)$ and $\triangle\mu = \mu - \mu_c$.

We conclude the following:

(i) $G_\mu(r)$ falls of like $\mathrm{e}^{-2(\triangle\mu)^{\frac{1}{4}}\beta r}$ for $r \to \infty$, i.e. the critical exponent $\nu = \frac{1}{4}$ and the Hausdorff dimension $d_H = 4$.

(ii) $G_\mu(r)$ behaves like r^{-3} for $1 \ll r \ll \triangle\mu^{-\frac{1}{4}}$, i.e. the scaling exponent $\eta = 4$.

(iii) The pre-exponential factor to $\mathrm{e}^{-2(\triangle\mu)^{\frac{1}{4}}\beta r}$ is $m(\mu)^{\eta-1}$, and this factor is precisely the one needed if we should be able to define a continuum two-point function up to an allover scaling factor by the assignment (6.26), which states that $r\, m(\mu)$ is constant when $r \to \infty$ and $\mu \to \mu_c$, since the short distance behavior is given by $r^{1-\eta}$.

(iv) The exponent α giving the power correction to the exponential decay is zero. This can be understood intuitively from from the fact universes for $r \gg 1/m(\mu)$ have to be thin tubes, and it is not difficult to show that these have no correction to the exponential decay.

(v) From Fisher's scaling relation we get $\gamma_s = \nu(2-\eta) = -1/2$. This well known result can of course also be derived directly from

$$\chi(\mu) = \sum_{r=1}^{\infty} G_\mu(r) = \text{const.} - c^2 (\triangle\mu)^{\frac{1}{2}} + \cdots \tag{6.30}$$

by use of (6.29), but it should be clear that the explicit calculation in (6.30) is nothing but a specific example of the general calculation used in proving Fisher's scaling relation. What is somewhat unusual compared to ordinary statistical systems is that the anomalous scaling dimension $\eta > 2$. $\eta = 0$ is the ordinary free field result, while $\eta = 2$ is the infinite temperature limit, and for statistical systems we expect $\eta < 2$.

(vi) The continuum two-point function can now be defined as

$$G(R;M) = \lim_{a\to 0} (\sqrt{a})^{\eta-1} G_\mu(r) \sim M^3 \frac{\cosh[M\,R]}{\sinh^3[M\,R]}. \tag{6.31}$$

where R is a continuum distance and M a continuum mass proportional to $\Lambda^{1/4}$, the unusual power of the "mass" being due to $d_h = 4$. The factor in front of $G_\mu(r)$ is the usual "wave function renormalization" present in the path integral representation of the propagator.

It is interesting to give a direct physical interpretation of the short distance behavior of $G_{k_4}(r)$ as defined by (6.21)[22] (in the following we suppress the index k_2). In order to do so let us change from the grand canonical ensemble given by (6.21) to the canonical ensemble defined by $\mathcal{T}(2,r,N_4)$, the class of triangulations with N_4 4-simplexes where two of the 4-simplexes are marked and separated by a distance r. On this ensemble we can define the discretized analogue of $G_{k_4}(r)$:

$$G(r,N_4) = \sum_{T\in\mathcal{T}(2,r,N_4)} 1, \tag{6.32}$$

For $r = 0$ we have the following N_4 dependence (for the r dependence see

[22] The remarks to follows are of course equally valid for $G_\mu(r)$ in two-dimensional gravity.

(6.37) below)

$$G(0, N_4) \sim N_4^{\gamma_s - 2}\ \mathrm{e}^{\mu_c N_4}, \tag{6.33}$$

assuming that γ_s exists. This remark is important since it is by no means obvious that that γ_s exists in four-dimensional gravity, as discussed above. The partition function$Z(N_4) \sim N_4^{\gamma_s-3}\ \mathrm{e}^{\kappa_4^c N_4}$ and the one-point function for large N_4 is proportional to $N_4 Z(N_4)$ since it counts the triangulations with one marked simplex For $r = 0$ (or just small) there is essentially no difference between the one-point function and $G(0, N_4)$.

$G(r, N_4)$ is related to $G_{k_4}(r)$ by a (discrete) Laplace transform:

$$G_{k_4}(r) = \sum_{N_4} G(r, N_4)\ \mathrm{e}^{-k_4^c N_4}\,. \tag{6.34}$$

The *long distance behavior* of $G(r, N_4)$ is determined by the long distance behavior of $G_{k_4}(r)$. Close to the scaling limit it follows by direct calculation (e.g. a saddle point calculation[23]) that

$$\begin{aligned} G_{k_4}(r) &\sim \mathrm{e}^{-r\,(k_4 - k_4^c)^{1/d_h}} \quad \Rightarrow \\ G(r, N_4) &\sim \mathrm{e}^{-c\left(r^{d_h}/N_4\right)^{\frac{1}{d_h - 1}}}\ \mathrm{e}^{k_4^c N_4} \quad \text{for} \quad r^{d_h} > N_4, \end{aligned} \tag{6.35}$$

where $c = (d_h - 1)/d_h^{d_h/(d_h-1)}$.

On the other hand the *short distance behavior* of $G_{k_4}(r)$ is determined by the short distance behavior of $G(r, N_4)$ which is simple. Eqs. (6.26)-(6.28) define the concept of Hausdorff dimension in the grand canonical ensemble. A definition in the canonical ensemble would be: Take $N_4^{1/d_h} \gg r$ and simply count the volume (here number of 4-simplexes) of a "spherical shell" of thickness 1 and radius r from a marked simplex, sum over all triangulations T_{N_4} with one marked 4-simplex and divide by the total number of such triangulations. Call this number $\langle n(r) \rangle_{N_4}$. The Hausdorff dimension is then defined by

$$\langle n(r) \rangle_{N_4} \sim r^{d_h - 1} \quad \text{for} \quad 1 \ll r \ll N_4^{1/d_h}. \tag{6.36}$$

[23] In addition to the exponentially decaying part of $G(r, N_4)$ there is also a power correction coming from the quadratic integration in the saddle point approximation. We shall not consider the explicit form of the power correction here.

It follows from the definitions that we can write

$$\langle n(r)\rangle_{N_4} \sim \frac{G(r,N_4)}{G(0,N_4)}, \quad \text{i.e}$$
$$G(r,N_4) \sim r^{d_h-1} N_4^{\gamma-2}\, \mathrm{e}^{k_4^c N_4} \quad \text{for} \quad 1 \ll r \ll N_4^{1/d_h}. \tag{6.37}$$

We can finally calculate the short distance behavior of $G_\mu(r)$ from eq. (6.34). For $k_4 \to k_4^c$, we get:

$$G_{k_4}(r) \sim r^{d_h-1} \sum_{N_4} N_4^{\gamma-2}\ \mathrm{e}^{-c\left(r^{d_h}/N_4\right)^{\frac{1}{d_h-1}}} \sim r^{\gamma_s d_h - 1} \tag{6.38}$$

This shows that

$$\eta = 2 - \gamma_s d_h, \quad \text{i.e.} \quad \gamma_s = \nu(2-\eta), \tag{6.39}$$

which is Fishers scaling relation. The above arguments highlight that the anomalous scaling dimension η is a function of the two kinds of fractal structures we can define on the ensemble of piecewise linear manifolds: the Hausdorff dimension and the baby universe proliferation probability. In addition the arguments show that the canonical and grand canonical definitions of Hausdorff dimension agree.

6.4. Branched polymers

The model of branched polymers (BP) provides us with a simple, but non-trivial example of the above scenario [61] and will play an important role in the following. In a certain way it can be viewed as the lowest dimensional fractal structure and it will appear as the limiting case of higher dimensional gravity theories.

Let us define branched polymers as the sum over all tree graphs (no loops in the graphs) with certain weights given to the graphs according to the following definition of the partition function:

$$Z(\mu) = \sum_{BP} \frac{1}{C_{BP}}\, \rho(BP)\ \mathrm{e}^{-\mu|BP|}, \tag{6.40}$$

where $|BP|$ is the number of links in a BP and μ is a chemical potential for the number of links, while

$$\rho(BP) = \prod_{i\in BP} f(n_i), \tag{6.41}$$

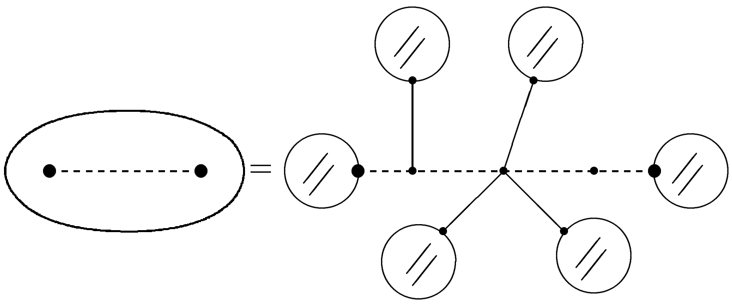

Fig. 24. The graphical representation of the two-point function for branched polymers. The dashed line represents the unique shortest path between the two marked vertices. The "blobs" represent the contribution from all rooted polymers branching out from a vertex.

where i denotes a vertex, n_i the number of links joining at vertex i and $f(n_i)$ is non-negative. $f(n_i)$ can be viewed as the unnormalized branching weight for one link branching into $n_i - 1$ links at vertex i. Finally C_{BP} is a symmetry factor such that rooted branched polymers, i.e. polymers with the first link marked, is counted only once.

This model can be solved [7]. It has a critical point μ_c (depending on f) and close to the critical point we have:

$$Z''(\mu) \sim (\mu - \mu_c)^{-1/2}, \tag{6.42}$$

i.e. $\gamma_s = 1/2$ for branched polymers. On the branched polymers we define the "geodesic distance" between two vertices as the shortest link path, which is unique since we consider tree-graphs. The graphic representation of the two-point function is shown in fig. 24. Had it not been for the ability to branch, the two-point function would simply be

$$G_\mu(r) = \mathrm{e}^{-\mu r}\,. \tag{6.43}$$

However, the insertion of one-point functions at any vertex leads to a non-analytic coupling constant renormalization and the result is changed to [61]

$$G_\mu(r) = \text{const. } \mathrm{e}^{-\kappa\, r\sqrt{\mu-\mu_c}} \quad \text{for} \quad \mu - \mu_c \to 0, \tag{6.44}$$

where κ is a positive constant depending on f. We can now find $G(r, N)$

by an inverse Laplace transform:

$$G(r, N) = \text{const.}\, N^{-3/2} r \, \mathrm{e}^{-\kappa^2 r^2/4N}\,. \tag{6.45}$$

We confirm from this explicit expression that the (internal) Hausdorff dimension of branched polymers is 2 (like a smooth surface !) and that $\gamma_s = 1/2$ since the prefactor of $G(r, N)$ for small r should be $N^{\gamma_s - 2} r^{d_H - 1}$.

6.5. Numerical simulations

Presently it has not been possible to make much progress in analyzing (6.13) by analytical methods except for $d = 2$. However, the action is well suited for the use of Monte Carlo simulations and the results to be discussed later for $d = 3$ and $d = 4$ have been obtained by such simulations. Let me shortly discuss the principles involved in such simulations since there are a number of interesting aspects involved compared to ordinary lattice simulations.

Monte Carlo simulations usually operate by means of a stochastic process, usually taken to be a Markov chain. The chain is selected such that it has a stationary distribution equal to the desired probability distribution, in this case the probability distribution given by (6.13). The Markov chain is a prescription for moving around in the configuration space, each step being independent of the former steps and chosen with a certain probability among a set of possible steps. Two conditions exist which, if fulfilled, are sufficient to ensure that the chain converges to the desired probability distribution. The first condition is *ergodicity*, i.e. by applying the available steps successively one should be able to move between any two configurations. The other condition is *detailed balance*. Let the steps be defined by a probability $P(A \to B)$ for going from state A to state B and let the desired probability distribution be given by $\mathrm{e}^{-S(A)}$. The equation of detailed balance state:

$$\mathrm{e}^{-S(A)}\, P(A \to B) = \mathrm{e}^{-S(B)}\, P(B \to A). \tag{6.46}$$

Let us first discuss the question of ergodicity. Given the class combinatorial triangulations of a manifold we want to find a sequence of so-called moves which are ergodic. Such moves have been known for a long time and they are known as the Alexander moves [82]. They can be simplified somewhat and in d dimensions there are $d + 1$ of them [83]. They can be described as follows: Given a sub-simplex i of order $d{+}1{-}i$ the $d{+}1{-}i$ d-simplexes which share i form a d-ball $B(i)$. Remove i and all the higher dimensional simplexes to which i belongs. Instead insert an "orthogonal"

$d-i$ dimensional sub-simplex, i.e. the sub-simplex constructed from the $d-i$ vertices of $\partial B(i)$ which did not belong to i, together with all the higher dimensional sub-simplexes. In this way $B(i)$ has been exchanged with a new d-ball $B'(d-i)$ with the same boundary: $\partial B(i) = \partial B'(d-i)$. This is illustrated for $d=2$ and $d=3$ in fig. 25.

This set of moves is ergodic for a given combinatorial manifold. In two dimensions it is seen that one of the moves preserves N_2, the number of triangles in the triangulation. In fact it can be shown that this move alone is ergodic on the set of triangulations which a fixed topology *and* and fixed volume (i.e. N_2). From a computational point of view it is very convenient to be able to keep the volume fixed. Can it be done in higher dimensions with a suitable set of moves of the same local nature as the ones mentioned above? The answer is no, at least in four- and higher dimensions, the reason being that it is known that in these dimensions manifolds exist which are not algorithmic recognizable. More precisely this means that manifolds $\mathcal{M}_0$ exist such that no finite algorithm in the sense of *Turing* allows us to decide if an arbitrarily chosen manifold is combinatorial equivalent to the given $\mathcal{M}_0$. *If* there were ergodic moves which kept the volume fixed one can argue that it takes only a finite number of operations, computable as a function of the volume, to list all combinatorial manifolds equivalent with $\mathcal{M}_0$, and for a given combinatorial manifold one can now by inspection check if it is in the list. Again it can be argued that this only takes a finite number of operations, computable as a function of the volume [74].

It is seen that there are deep reasons which makes it impossible to have a local set of moves which preserves the volume and are ergodic for all manifolds. The situation is even more drastic: Let N be the volume and let us ask the question: what is the smallest number of moves needed in order to connect any two configuration with volume N. The answer is that for manifolds like $\mathcal{M}_0$ this number cannot be bounded by any recursively definable function. This result is only possible if we on the way from one configuration to another with volume N has to make large detours to configurations with a volume $N' >> N$ [74].

Let us finally discuss the implementation of detailed balance: The transition from a configuration A to a neighboring one B can be realized in two steps. First we pick an i-simplex shared by $d+1-i$ d-simplexes. The probability of doing this is $1/n_i(A)$, $n_i(A)$ being the number of i-simplexes in A. Afterwards we exchange i with an $d-i$ simplex with some probability

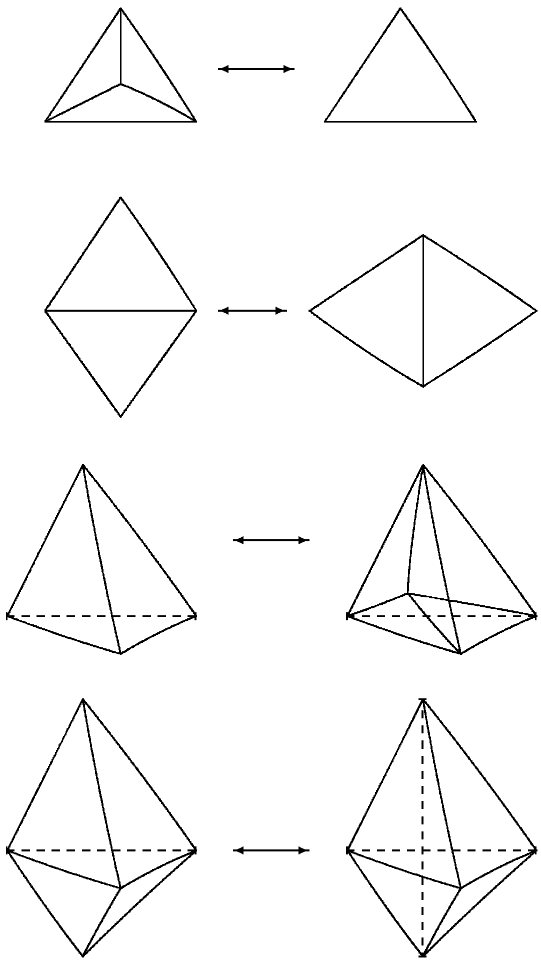

Fig. 25. The moves in $d = 2$ and $d = 3$.

$P_i(A \to B)$. The equation for detailed balance reads:

$$\mathrm{e}^{-S(A)} \frac{1}{n_i(A)} P_i(A \to B) = \mathrm{e}^{-S(B)} \frac{1}{n_{d-i}(B)} P_{d-4}(B \to A), \tag{6.47}$$

where $i = 0, 1, ..., d$. One can now choose P_i's according to some standard Metropolis algorithm, but it is worth to emphasize the appearance of the combinatorial prefactors $1/n_i$ which are usually absent in ordinary lattice simulations in the sense that they cancel out. They appear here because we have a dynamical lattice which change during the updating.

6.6. Results

Let me now discuss the results of the numerical simulations. They can be summarized as follows [90, 91, 94, 96, 95, 92, 93]
(1): There seem to be two different regions, as a function of the bare *inverse* gravitational coupling constant k_2: For small or negative values of k_2 the typical quantum universe will be very crumpled, with almost no extension and a very large, if not infinite Hausdorff dimension, while the universes for large values of k_2 will be elongated with a Hausdorff dimension as small as two. In fig. 26 we have shown the average radius for universes of size 9000, 16000, 32000 and 64000 4-simplexes as a function of k_2. The two phases are separated by a phase transition which is of order two or higher. *At* the transition point, k_2^c, the Hausdorff dimension might be finite (the precise value is not well determined, but it could be close to four).
(2): The same results are valid for three dimensional simplicial quantum gravity except that the phase transition seems to be of *first order*, rather than of higher order [84–87].

From fig. 26 it is seen that the change between the elongated region and the crumpled region becomes increasing visible as the size of the system increases. In addition the critical point seems to move to higher values of k_2. We observe a so called pseudo critical point $k_2^c(N_4)$. The indication of convergence to a limiting value k_2^c as the volume $N_4 \to \infty$ is shown in fig. 27 and we conclude that we have a genuine phase transition. From the general theory of finite size scaling the pseudo critical point $k_2^c(N_4)$ of a first order phase transition scales to the limiting value k_2^c as $1/N_4$. Since we observe a scaling like $1/\sqrt{N_4}$ we conclude that the transition most likely is a higher order transition. In three dimensional simplicial quantum gravity it is difficult to perform this kind of measurement since we observe pronounce hysteresis around the transition point. This is a typical sign of a *first order* transition.

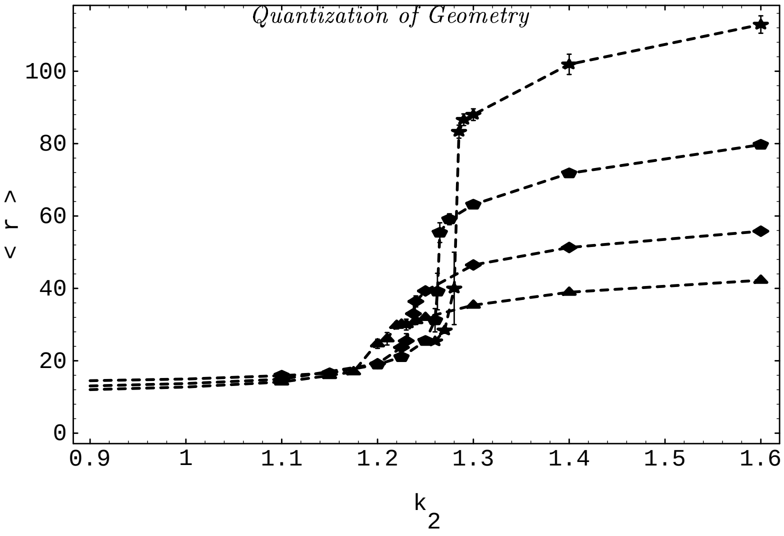

Fig. 26. The average radius of the universes of sizes 9000 (triangles), 16000 (squares), 32000 (pentagons) and 64.000 (stars) versus k_2.

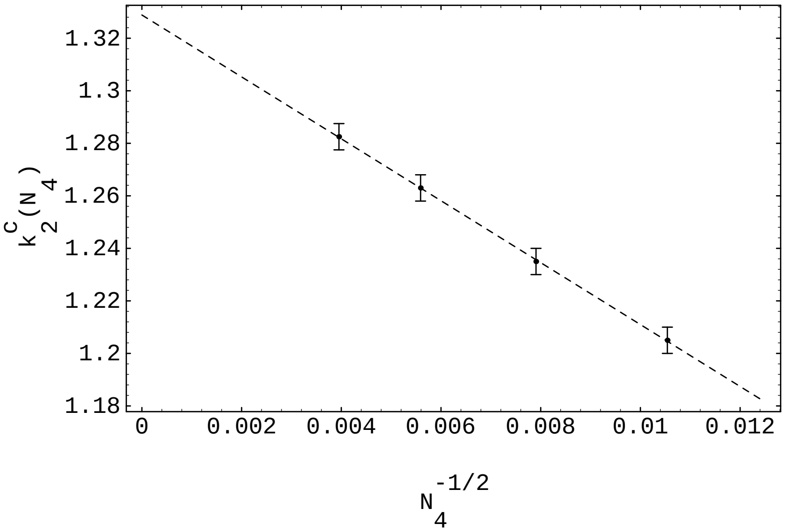

Fig. 27. The pseudo critical point $k_2(N_4)$.

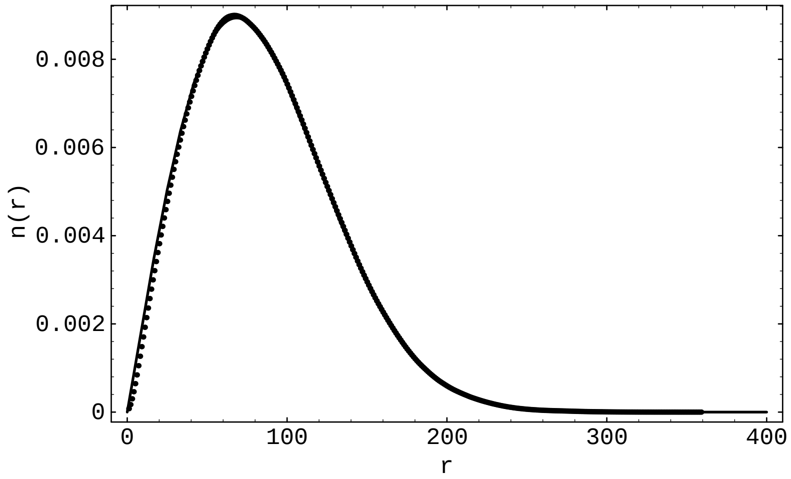

Fig. 28. The measured distribution (dots, the error bars smaller than the dots), and the fit (the curve) using the functional form $r\exp(-r^2/N_4)$.

Let us first discuss the measurements in the elongated phase, i.e. for $k_2 > k_2^c$. An ideal quantity to measure in the computer simulations is the number of four-simplexes $\langle n(r)\rangle_{N_4}$ within a spherical shell of thickness 1, a geodesic distance r from a marked four simplex, the average taken over all spherical triangulations with one marked four-simplex. It will depend on the coupling constant k_2 and it is related to the two-point function precisely as discussed above:

$$G(r,N_4) \sim N_4^{\gamma(k_2)-2}\,\mathrm{e}^{k_4^c(k_2)N_4}\,\langle n(r)\rangle_{N_4}, \tag{6.48}$$

provided the subleading corrections are power-like. In fig. 28 we show the measured distribution $\langle\, n(r)/N_4\rangle_{N_4}$ just above the transition to the elongated phase and a corresponding fit to $r\exp(-r^2/N_4)$. We see a very good agreement and it becomes even better if we move further into the elongated phase. Now recall the general scaling relations

$$\langle n(r)\rangle_{N_4} \sim r^{d_h-1} \quad \text{for} \quad 1 \ll r \ll N_4^{1/d_h}, \tag{6.49}$$

$$\langle n(r)\rangle_{N_4} \sim r^{\alpha}\,\mathrm{e}^{-c(r^{d_h}/N_4)^{1/(d_h-1)}} \quad \text{for} \quad N_4^{1/d_h} < r < N_4, \tag{6.50}$$

we conclude that $d_h = 2$ (and $\alpha = 1$). In addition we can measure the critical exponent γ_s very conveniently by baby universe counting. Again the arguments are identical to ones presented in the two-dimensional case. The result is shown in fig. 29 and it is natural from the figure to conjecture

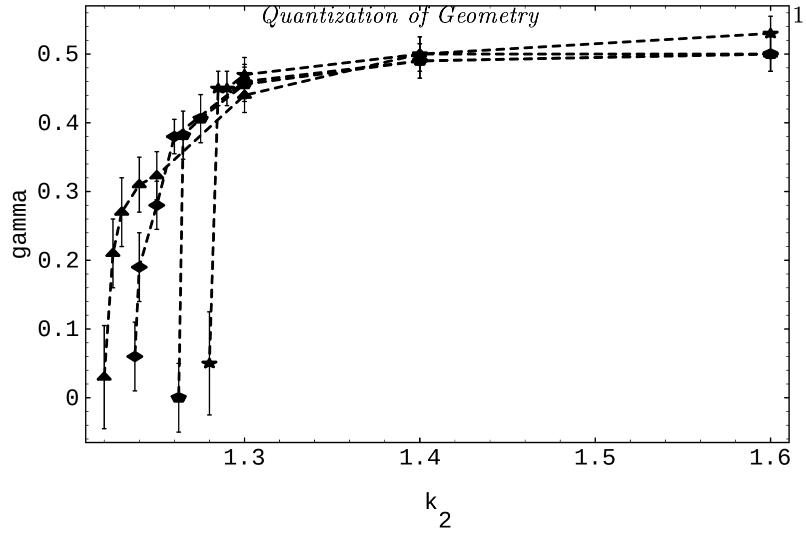

Fig. 29. The measure γ in the elongated phase for various size lattices (N_4 = 9000, 16000, 32000 and 64000).

that $\gamma = 1/2$ as long as we are in the elongated phase. From (6.48) and $\langle n(r)\rangle_{N_4}$ we know $G(r, N_4)$ and we can construct the two-point function $G_{k_4}(r; k_2))$ by Laplace transform (as in the two-dimensional case):

$$G_{k_4}(r; k_2) = \sum_{N_4=1}^{\infty} N^{-3/2} e^{-\Delta(k_4)N_4}\; r\, \mathrm{e}^{-cr^2/N_4} \sim \mathrm{e}^{-\tilde{c}\, r\sqrt{\Delta k_4}}, \qquad (6.51)$$

where $\Delta(k_4) = k_4 - k_4^c(k_2)$ is assumed to be small. This function is precisely the two-point function of the so called branched polymers, which are known to have internal Hausdorff dimension $d_h = 2$. We conclude that the numerical simulations provide convincing evidence that the elongated phase of simplicial quantum gravity corresponds to a well known statistical theory, the one of branched polymers. *The tendency to create baby universes is so pronounced in this phase that the geometry degenerates to the generic lowest dimensional fractal structure possible, i.e. that of branched polymers.*

When we lower the value of k_2 and move below the critical point k_2^c the fractal structure of our ensemble of piecewise linear manifolds changes drastically. A glance on fig. 26 shows that the average radius hardly changes with the volume. This is an indication that the Hausdorff dimension is large or maybe infinite. If we move deep into this phase the average curvature is negative and in addition there are only few baby universes and they are small. This could lead to the idea that we entered a phase with "smooth" manifolds of negative curvature. For such manifolds one would expect that the volume of geodesic balls of radius r would grow exponentially with the radius, which is what we observe. Clearly this is a "fake" infinite Hausdorff dimension and indeed we should observe the dimension $d_h = 4$ in the sense that $\langle n(r)\rangle_{N_4} \sim r^3$ for small geodesic distances. A closer look at "typical" members of the computer generated manifolds indicates that they cannot be considered as "smooth". Rather they have a few vertices of very high orders which connect to almost any other vertex in the manifold and in such a situation it is not surprising that the linear extension will be small. In addition we have not been able to fit to any sensible power dependence r^{d_H-1} for small r. A plot of $\log\langle n(r)\rangle_{N_4}$ shows indeed a linearly growing function of r up to some $r_0(N_4)$ which is not much different from the average value $\langle r\rangle_{N_4}$. A reasonable fit to $\langle r\rangle_{N_4}$ is

$$\langle r\rangle_{N_4} = a(k_2) + b(k_2)\log N_4. \tag{6.52}$$

This again gives some support to the idea that the Hausdorff dimension is infinite in this phase since it appears as a limit of $\langle r\rangle_{N_4} \sim N^{1/d_H}$ for $d_H \to \infty$. Finally the extrapolation of (6.50) to infinite Hausdorff dimension indicates that in such a case one should observe

$$\langle n(r)\rangle_{N_4} \sim c_1\, \mathrm{e}^{-m_1(k_2)\,r} \tag{6.53}$$

down to quite small distances $r \gg N_4^{1/d_h}$, $d_h \to \infty$, which naturally is replaced by $r > a_1 \log N_4 + a_2$. This is indeed what we measure.

The observation that $\langle n(r)\rangle_{N_4}$ grows exponentially from $r \approx 6$ out to $r \approx r_0$ and then falls off exponentially indicates that we deal with an infinite Hausdorff dimension at all distances and it is easy to get a quite good "phenomenological" fit to $\langle n(r)\rangle_{N_4}$ which incorporates both these features by choosing e.g.:

$$\langle n(r)\rangle_{N_4} \sim \exp\left(-m_1(k_2)r - c_2\,\mathrm{e}^{-m_2(k_2)r}\right). \tag{6.54}$$

It will grow like $\mathrm{e}^{(c_2 m_2 - m_1)r}$ for small distances and fall off like

$$\langle n(r)\rangle_{N_4} \sim c_1\, \mathrm{e}^{-m_1(k_2)\,r} - c_2\, \mathrm{e}^{-(m_1(k_2)+m_2(k_2)\,r} + \cdots \tag{6.55}$$

for large distances, while a N_4 dependence in the coefficient c_2 would explain the observed N_4 dependence of r_0. The data and a fit of the form

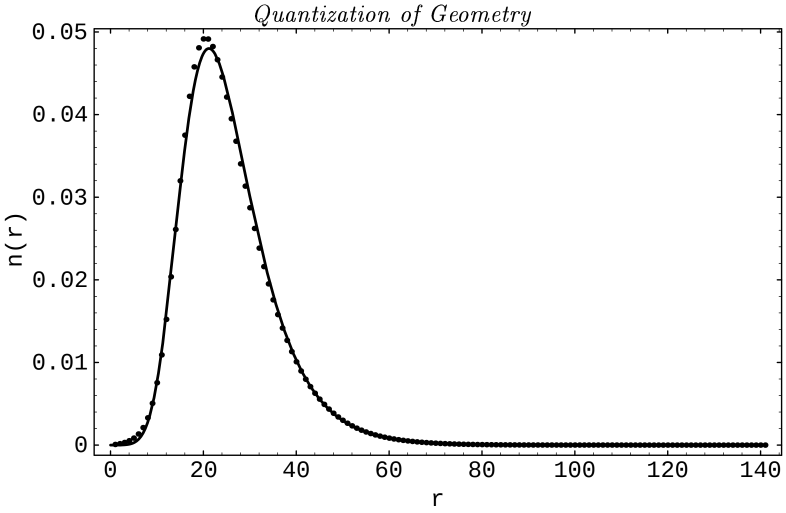

Fig. 30. exponential fit (curve) of the form (6.54) to the measured $n(r; N_4)$ (dots, error bars less than dot-size) in the crumpled phase ($N_4 = 64000$, $k_2 = 1.26$).

(6.54) are shown in fig. 30 for $k_2 = 1.26$ and $N_4 = 64000$, i.e. right below the transition to the crumpled phase, where the fit is worst. But even so close to critical point (6.54) works quite well over the whole range of $r > 6$. It should be mentioned that the coefficient in front of the second exponential in eq. (6.55) is negative. This implies that the term cannot be given the interpretation as an additional heavier mass excitation. However, just looking at the long distance tail the distribution $\langle n(r)\rangle_{N_4}$ allows us to determine m_1, m_2 and c_2 from (6.55). On the other hand we can determine $c_2 m_2 - m_1$ from the short distance exponential growth alone and find good agreement. This indicates that long and short distance behavior are intertwined in the case of infinite Hausdorff dimension, as they are in the case of finite Hausdorff dimension, where d_H appears both in the short distance and long distance expression for $\langle n(r)\rangle_{N_4}$ (see e.g. (6.49)-(6.50)).

We conjecture that the internal Hausdorff dimension is infinite for $k_2 < k_2^c$.

In accordance which this conjecture we have in the elongated phase no "mass" term $m(k_4) = (k_4 - k_4^c(k_2))^\nu$ which scales to zero as we approach $k_4^c(k_2)$. The exponential coefficient $m_1(k_2)$ is finite in the infinite volume limit $k_4 \to k_4^c(k_2)$. However, it is most interesting that $m_1(k_2)$ scales to zero as $k_2 \to k_2^c$, the phase transition point between the crumpled and the elongated phase. *This gives a strong indication that the system at the transition might have a finite Hausdorff dimension, which could very well*

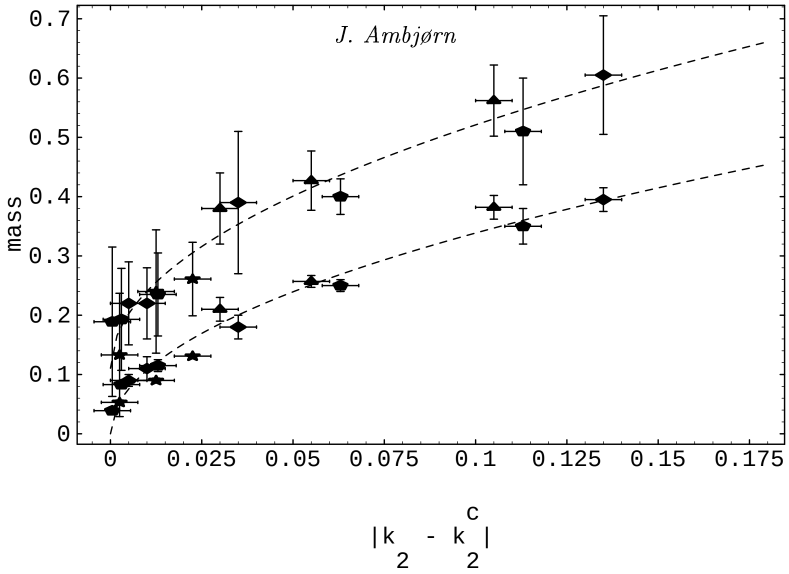

Fig. 31. The behavior of the masses $m_1(k_2)$ and $m_1(k_2) + m_2(k_2)$ from (6.54) in the crumpled phase as a function of $k_2^c - k_2$.

be larger that the generic $d_h = 2$ found in the elongated phase. In fig. 31 we have shown the scaling of the mass $m_1(k_2)$ as a function of k_2.

The above mentioned numerical "experiments" suggest the following scenario: The typical quantum universe, determined without any Einstein action (i.e. $k_2 = 0$) has (almost) no extension. Its Hausdorff dimension might be infinite and internal distances between points are always "at the Planck scale". By that we simply mean that no consistent scaling can be found which will be compatible with finite continuum volume and finite Hausdorff dimension. For a finite value of the bare gravitational coupling constant there is a phase transition (second or higher order) to a phase with a completely different geometry with pronounced fractal structures. It is tempting to view the transition between the two kind of geometries as a transition where excitations related to the conformal mode are liberated, since large k_2 is a region which formally corresponds to small values of the gravitational coupling constant. Right at the transition it seems as if we have the chance to encounter genuine extended structures with a finite Hausdorff dimension. Maybe the fact that the transition between the two types of geometry is of second (or higher) order can be used as the starting point for a non-perturbative definition of quantum gravity.

Of course it is crucial to be able to perform high statistics simulations *at* the critical point in order to investigate this possibility in greater detail. It would be even better if we could perform some analytical calculations. The branched polymer picture indicates that mean field methods should be available in some regions of the coupling constant space.

7. Discussion

We have seen in some detail that it is possible to discretize reparametrization invariant theories and apply with success the methods known from the theory of critical phenomena. In this way we deal with the theories of fluctuating geometries. However, we did not really answer the most interesting question: How wildly should we allow the geometries to fluctuate? For a fixed topology it was possible to formulate a regularized Euclidean quantum theory. In two dimensions one can take the scaling limit of the regularized theory and the corresponding quantum theory correctly describes the interaction between matter and gravity. In higher dimensions it is not yet known if it leads to any interesting theory, but at least there is a well defined non-perturbative procedure for how to investigate this question. In addition the discretized action in this approach is remarkable simple and one could hope that it will be possible to solve the theory in the same detail as in two dimensions. At the moment we let loose topology we have also lost control of the theory. In two dimensions the double scaling limit gave us a hint that it might be possible to perform the summation over topologies. At least there are some prescriptions for how it should be done. They are not yet unambiguous and this reflects our lack of understanding of the physics which goes beyond a simple perturbative expansion in topology. Nevertheless I view it as very encouraging that one at least seems to have by now the tools which allow such questions to be asked. These tools have their origin in the discretized approach and even in three and four dimensions the discretized approach has the advantage that it in a natural way is able to combine the summation over Riemannian structures and topology with the volume of the universe acting as the cut-off. It remains to be seen if the approach will result in an interesting theory of gravity, but it offers at least a playground for a fascinating interplay between pure mathematics, theoretical physics and computational physics.

Acknowledgment It is a pleasure to thank the organizers and the participants of the school for creating a most simulating environment during the

lectures. The presentation has be heavily flavored by discussions and collaborations with Bergfinnur Durhuus, Thordur Jonsson, Jerzy Jurkiewicz, Charlotte Kristjansen, Yuri Makeenko and Gudmar Thorleifsson. They are responsible for most of the interesting observations presented, while any mistakes can only be blamed on me.

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