The L-curve and its use in the numerical treatment of inverse problems

A study by Per Christian Hansen (2001)

Presentation for the course *Advanced Methods in Applied Statistics*

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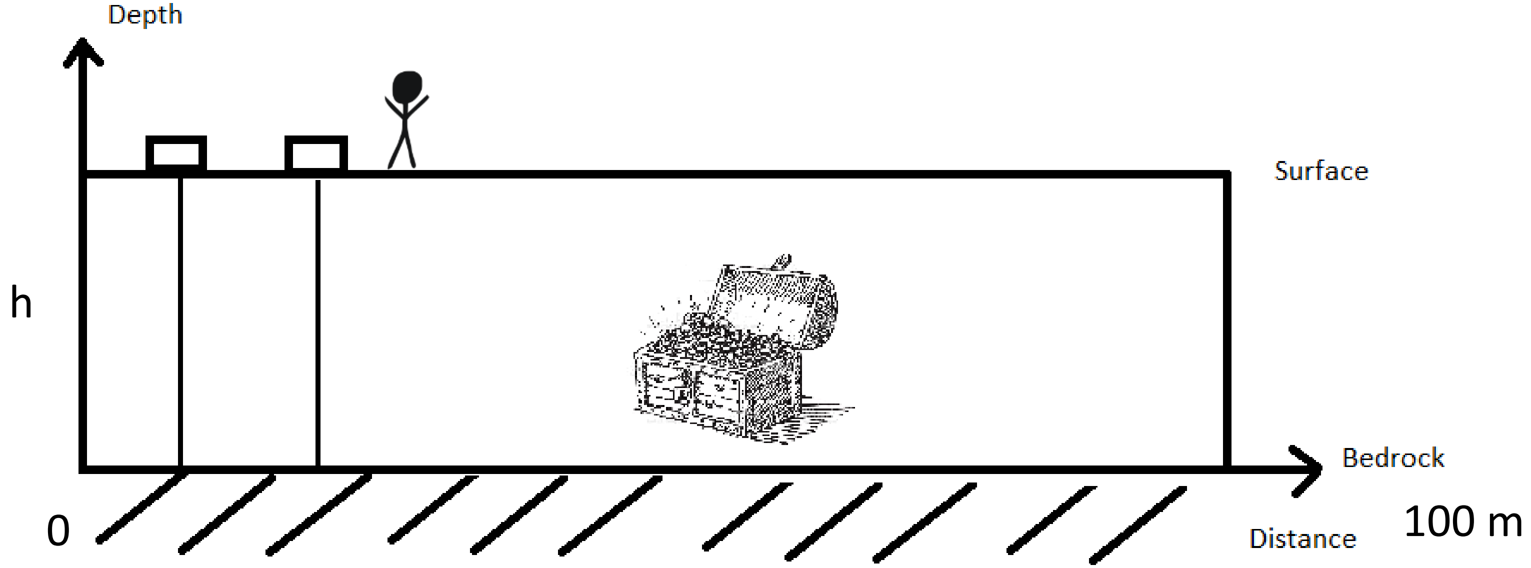
Motivation: Locating a buried treasure

We want to locate a buried treasure

- We know that we can find the treasure by estimating the density deviation $m(x)$
- We are able to measure the gravity anomaly (*d*) 60 times (on locations *s*) on a total distances of 100 meter (*x*)

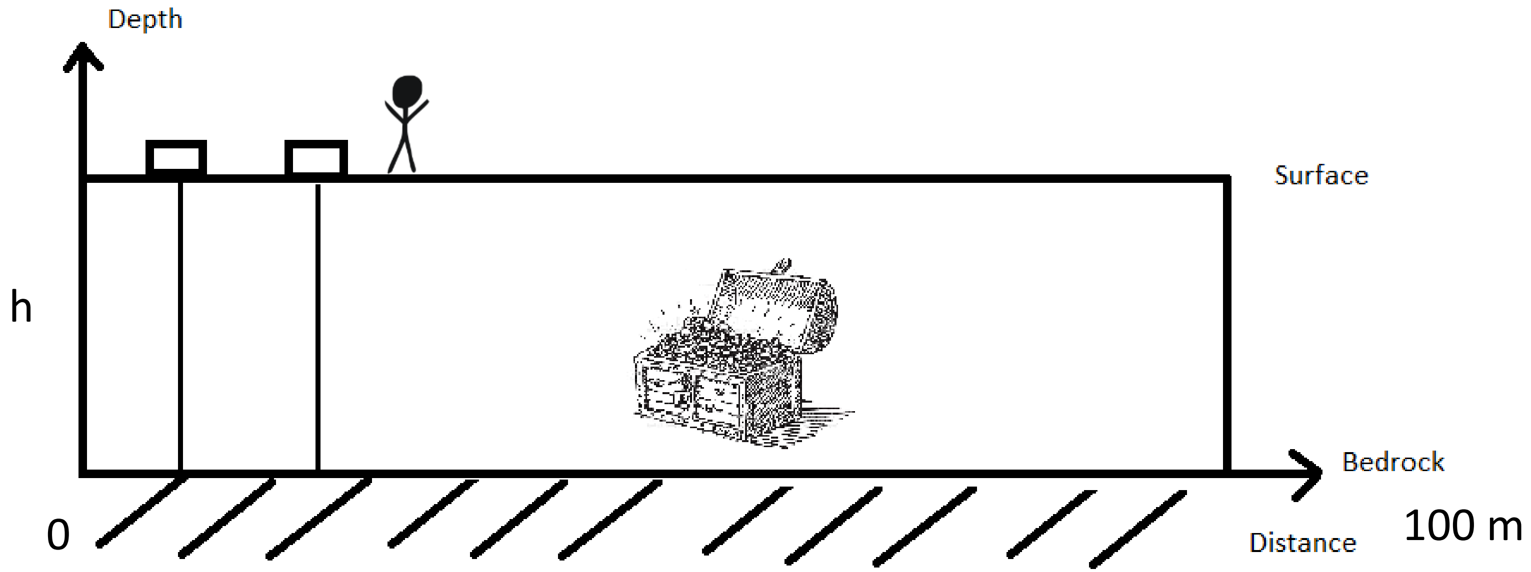


Motivation: Locating a buried treasure

- The measured gravity anomaly can be expressed through a convolution of the density anomaly:

$$d(s) = \int_0^{100} g(x - s)m(x)dx$$

- We can express the convolution kernel *g(x)* as a matrix from the x, h, s values



Motivation: Locating a buried treasure

- We measure the vertical gravity anomaly *d(s),* and we want to estimate the density deviation $m(x)$

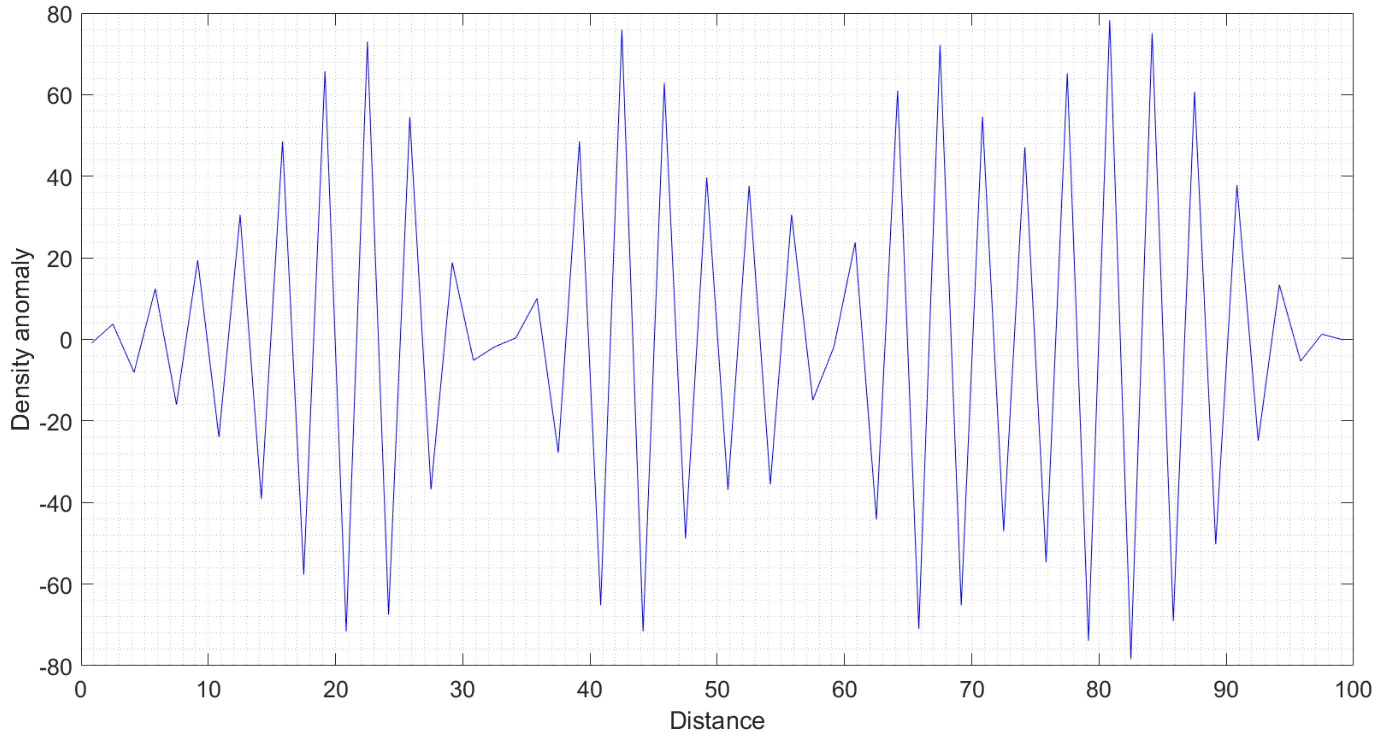
$$d(s) = \int_0^{100} g(x - s)m(x)\mathrm{d}x$$

- This is an inverse problem $\boldsymbol{d} = \boldsymbol{G}\boldsymbol{m}$
  - **d:** The measured data (60x1)
  - **G:** The function (we can express this as a matrix in this case with size (60x60))
  - **m:** The model/parameters we want to estimate (60x1)

- So we know the solution form: $\mathbf{m} = \left(\mathbf{G}^{\mathbf{T}}\mathbf{G}\right)^{-\mathbf{1}}\mathbf{G}^{\mathbf{T}}\mathbf{d}$

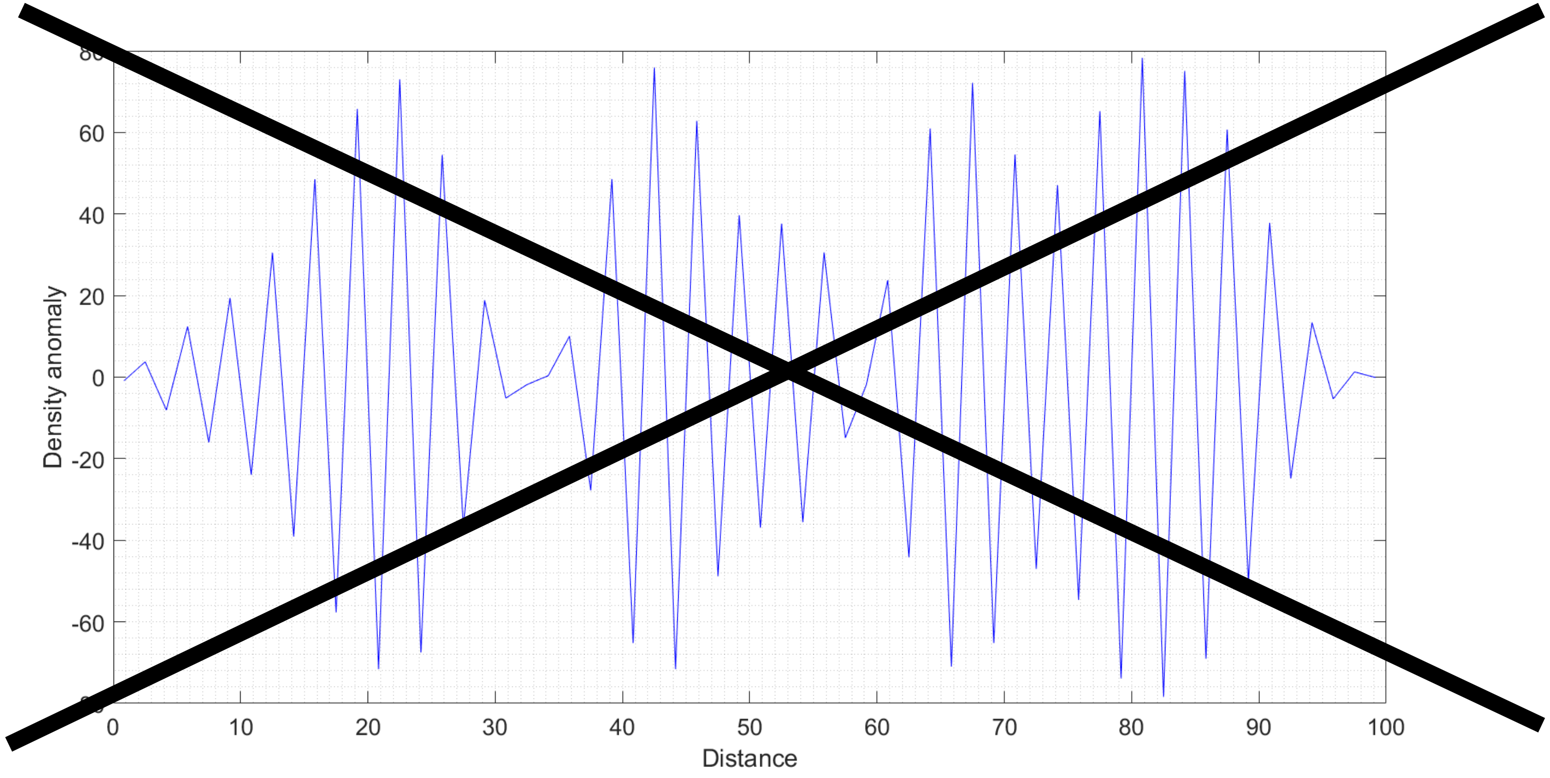
Motivation: Locating a buried treasure

The solution is very noisy and we don't know where the treasure is



Motivation: Locating a buried treasure

- We are unable to locate the treasure as this is an **ill-posed problem** (unstable solution)
  - It happens when the data is dominated by noise



The L-curve and its use in the numerical treatment of inverse problems

- Tikhonov regularization is one way of dealing with an ill-posed inverse problem (also called ridge regression)

- It is a modification to the regular inverse problem $\left(\mathbf{G}^{\mathrm{T}}\mathbf{G}\right)\mathbf{m} = \mathbf{G}^{\mathrm{T}}\mathbf{d}$

- It includes a smoothing term $\alpha^2\ \mathbf{I}$ (The Tikhonov matrix):
  - α being the regularization parameter [0;1]
  - This is a 0th order Tikhonov regularization

$$\left(\mathbf{G}^{\mathrm{T}}\mathbf{G} + \alpha^2\ \mathbf{I}\right)\mathbf{m} = \mathbf{G}^{\mathrm{T}}\mathbf{d}$$

- By adding regularization, we damp the contributions from noise

The L-curve and its use in the numerical treatment of inverse problems

- So we need to estimate the Tikhonov matrix $\alpha^2\,\mathbf{I}$ that provides us with the best solution

- The best solution is found when both the residual $\|\mathbf{Gm}\ -\boldsymbol{d}\|_2$ and solution $\|\boldsymbol{m}\|_2$ are minimal
  - When the residual norm is low we have a good fit between data and model
  - We want the solution norm to be minimal as we don't want our fit to be dominated by data errors
  - 2-norm is similar to least squares:

$$\|\mathbf{Gm}\ -\boldsymbol{d}\|_2 = \sqrt{\sum_{i=1}^{m}\left((\mathrm{G}m)_i - d_i\right)^2} \quad \text{and} \quad \|\mathbf{m}\|_2 = \sqrt{\sum_{i=1}^{m}(m_\mathrm{i})^2}$$

- A 0th order Tikhonov regularization: $\mathbf{m} = \left(\mathbf{G}^\mathrm{T}\mathbf{G} + \alpha^2\,\mathbf{I}\right)^{-\mathbf{1}}\mathbf{G}^\mathrm{T}\mathbf{d}$

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The recipe:

1. Make an array of α values
2. Calculate a series of solutions $\mathbf{m}_{\boldsymbol{i}} = \left(\mathbf{G}^{\mathrm{T}}\mathbf{G} + \alpha_i^2\ \mathbf{I}\right)^{\mathbf{-1}}\mathbf{G}^{\mathrm{T}}\mathbf{d}$ for each value of α
3. Used this $\mathbf{m}_{\boldsymbol{i}}$ to calculate the residual norm $\|\mathbf{G}\mathbf{m}_{\mathbf{i}}\ - \boldsymbol{d}\|_2$ and the solution norm $\|\boldsymbol{m}\|_2$
4. Plot the results on ln-ln plot
5. The best solution is found when the the residual $\|\mathbf{Gm}\ - \boldsymbol{d}\|_2$ and solution $\|\boldsymbol{m}\|_2$ are minimal

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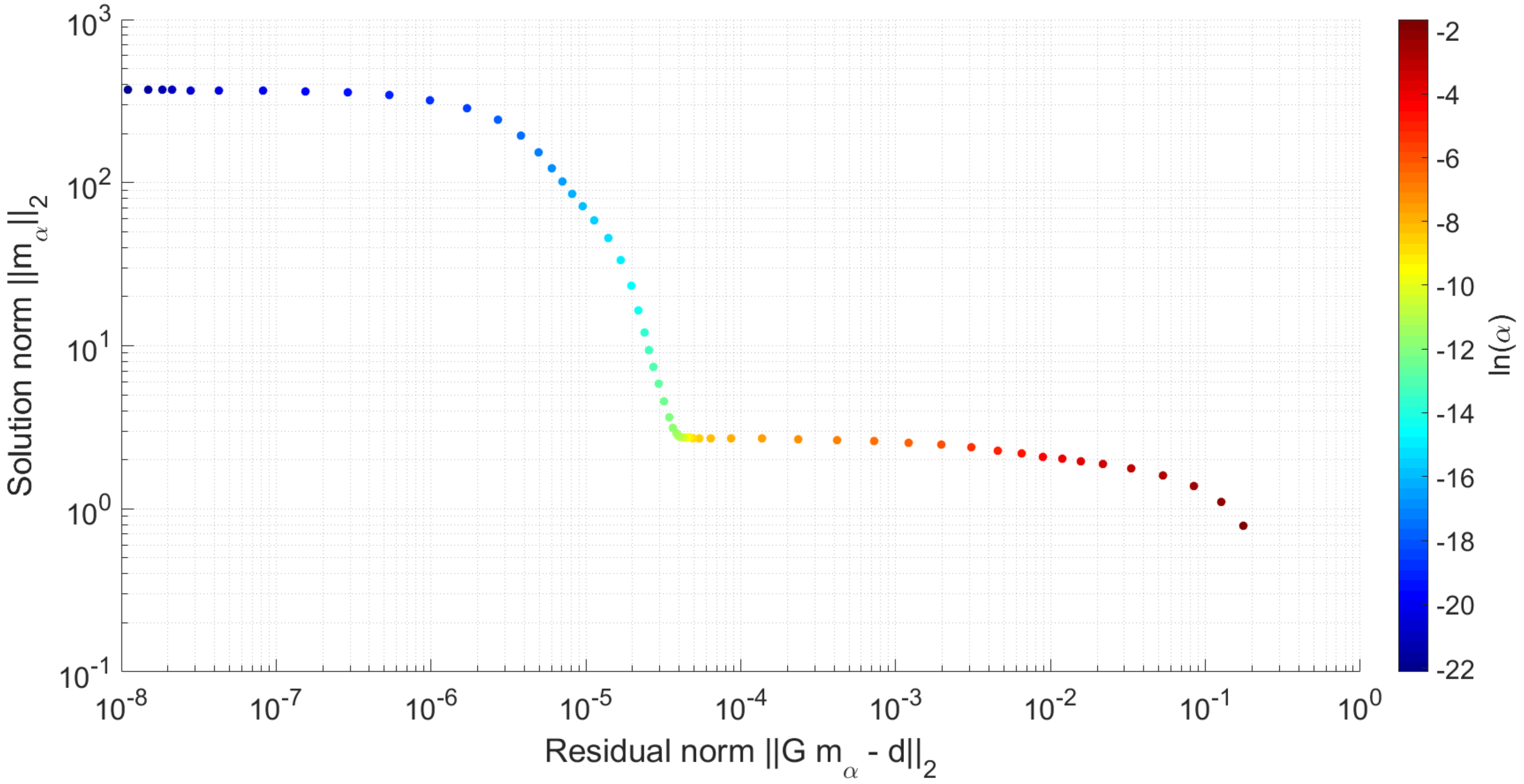
- $||\boldsymbol{m}||_2$ decreases as a function of α and $||\mathbf{Gm}\ - \boldsymbol{d}||_2$ increases as a function of α



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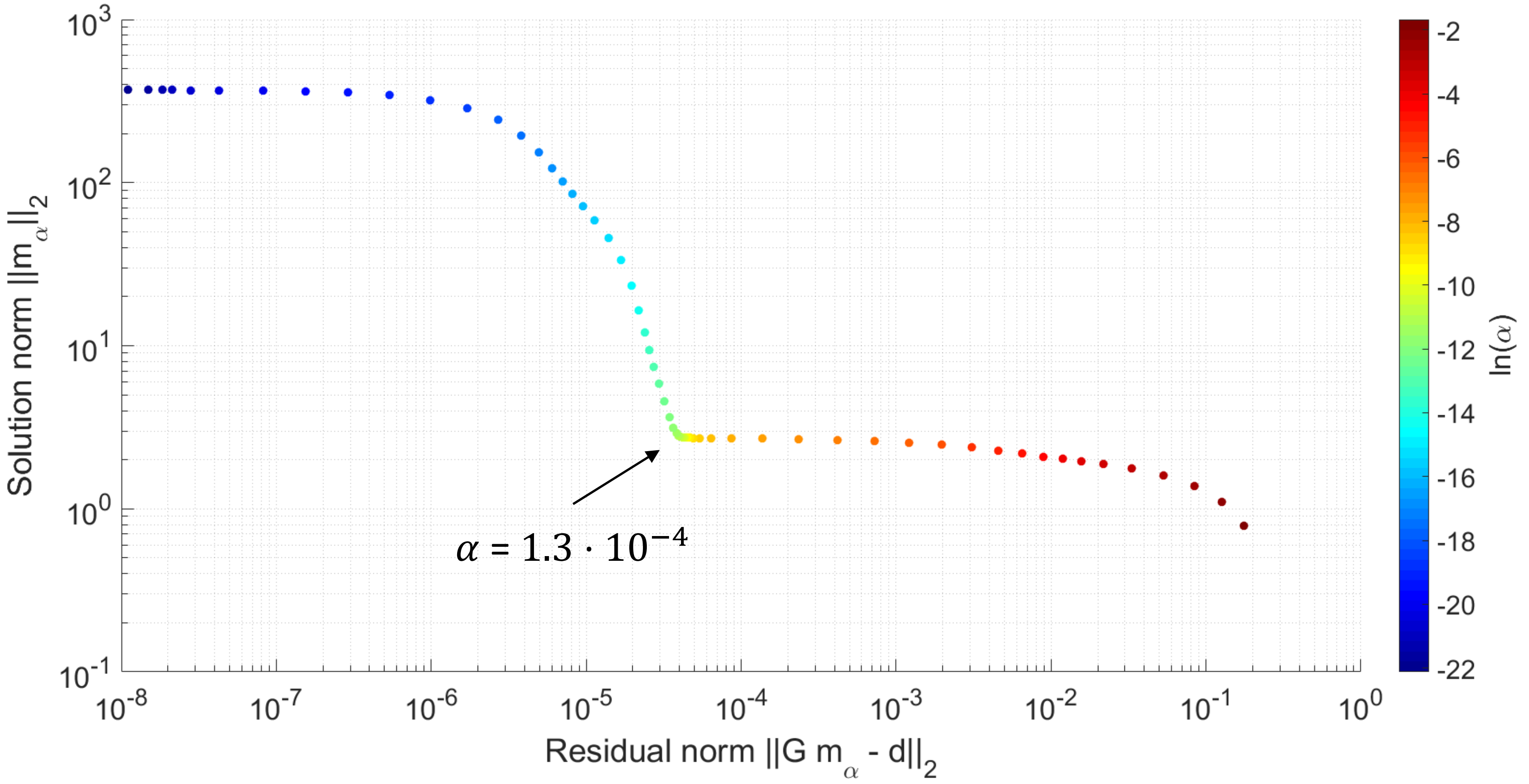
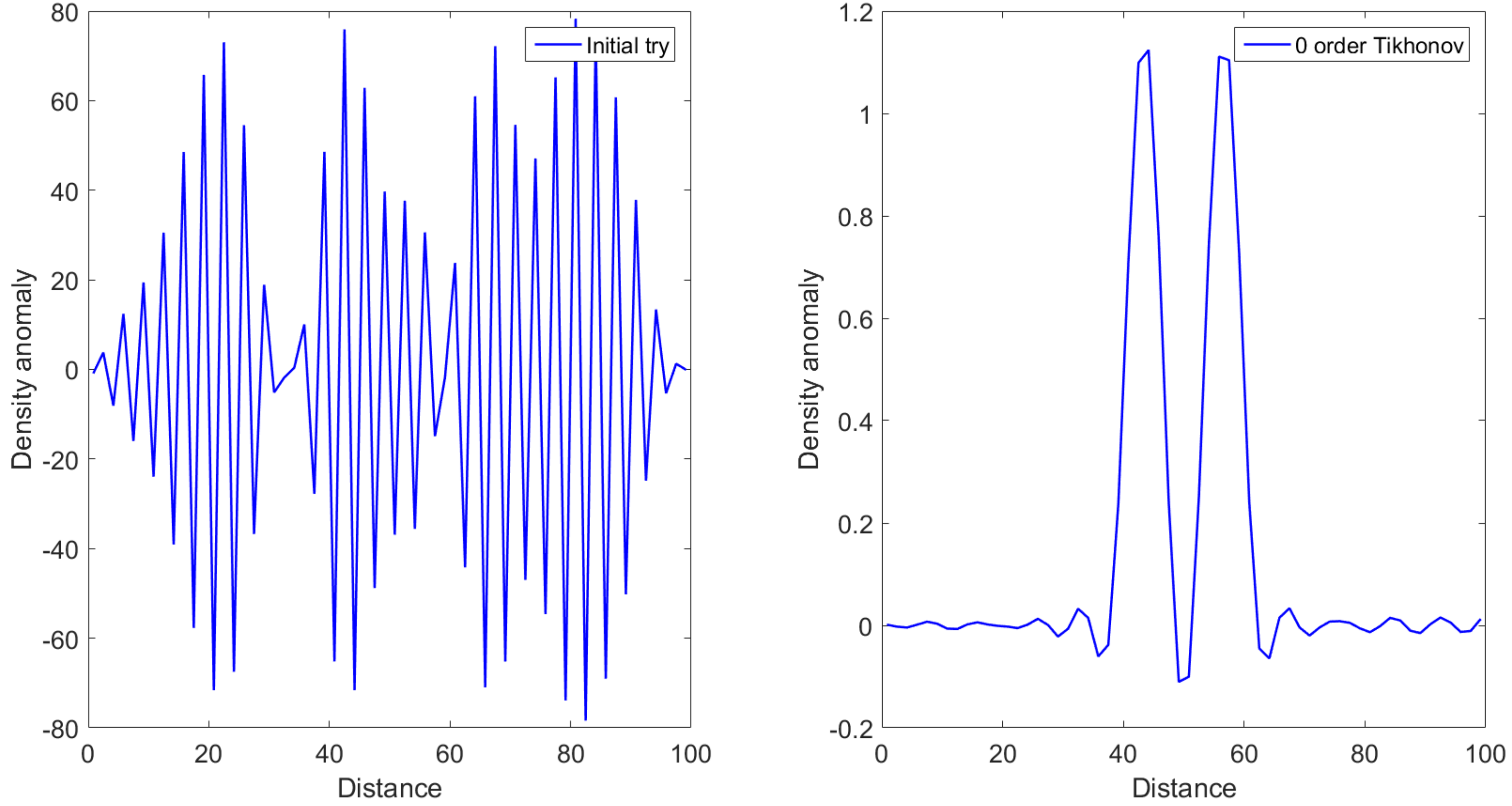
- The best value of α is at the kink/highest curvature



The L-curve and its use in the numerical treatment of inverse problems

- For $\alpha = 1.3 \cdot 10^{-4}$



Conclusion and outlook

- We could locate the treasure by using Tikhonov regularization

- In general, many inverse problems are ill-posed so regularization is needed

- The L-curve can be used to find the best value of the Tikhonov regularization parameter α
  - α can be found at the maximum curvature in the L-curve (the L-curve criterion)

- For more complex functions, a 1 or 2 order Tikhonov regularization can be applied instead

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Questions?