SOLUTION BY SEPARATION OF VARIABLES OF THE SPHERICAL LAPLACE EQUATION

The Laplace equation is a second order partial differential equation in a scalar field u. We often write the Laplace equation as the divergence of the gradient of u, i.e.

$$\nabla \cdot \nabla u = 0 \tag{1}$$

or simply as

$$\nabla^2 u = 0 \tag{2}$$

A basic equation in physics, which involves the Laplace operator, is the Helmholtz equation

$$\nabla^2 u + \kappa^2 u = 0 \tag{3}$$

In a three dimensional Cartesian coordinate system, the Helmholtz equation has the following form

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} + \kappa^2 u(x,y,z) = 0, \tag{4}$$

whereas in spherical coordinates, with

$$x = r \sin\theta \cos\varphi \tag{5}$$

$$y = r \sin\theta \sin\varphi \tag{6}$$

$$z = r \cos\varphi, \tag{7}$$

the Helmholtz equation of $u(r,\theta,\varphi)$ can be written on the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} + \kappa^2 u = 0 \tag{8}$$

We consider solutions to the Helmholtz equation where the scalar field is written as a product of three functions in the variables r,θ and φ, i.e.

$$u(r,\theta,\varphi) = R(r)Y(\theta)P(\varphi) \tag{9}$$

A solution on this form is called a solution by separation of variables and an equation for which such a solution exists is sometimes called separable. While we can always try and see if such a solution exist, most equations will in general not have separable solutions or even if they had, the solutions cannot be guaranteed to match the boundary conditions.

If we insert Eq. (9) in Eq. (8), we get

$$0 = R''YP + \frac{2}{r}R'YP + \frac{1}{r^2}RY''P + \frac{\cot\theta}{r^2}RY'P + \frac{1}{r^2\sin^2\theta}RYP'' + \kappa^2 RYP, \tag{10}$$

where $R' = \partial_r R$, $Y' = \partial_\theta Y$ and $P' = \partial_\varphi P$. If we multiply by r^2 on both sides and divide through by RYP, we get

$$0 = \frac{r^2R'' + 2rR' + \kappa^2 r^2 R}{R} + \frac{Y'' + \cot\theta Y'}{Y} + \frac{1}{\sin^2\theta}\frac{P''}{P} \tag{11}$$

or similarly, we have

$$\frac{r^2R'' + 2rR' + \kappa^2 r^2 R}{R} = -\frac{Y'' + \cot\theta Y'}{Y} - \frac{1}{\sin^2\theta}\frac{P''}{P}, \tag{12}$$

where the left-hand side only depends on the variable r and the right-hand side only depends on the angular variables. This is only possible if both the left and right-hand sides are equal to a constant.

The Legendre Differential Equation

Let us for a moment consider the case where the solution does not depend on the variable φ, i.e. we consider a problem that is rotation-symmetric around the z-axis. In this case we end up with an equation

$$-\frac{Y'' + \cot\theta Y'}{Y} = k \tag{13}$$

or

$$Y'' + \cot\theta Y' + kY = 0 \tag{14}$$

The $\cot\theta$ is rather disturbing and we would therefore prefer if the equation could be expressed in a different form. A simple change of variable $x = \cos\theta$ does the trick. We then get that

$$Y'(\theta) = \frac{\partial Y}{\partial \theta} = \frac{\partial Y}{\partial x}\frac{\partial x}{\partial \theta} = -\sin\theta\frac{\partial Y}{\partial x} \tag{15}$$

and

$$Y''(\theta) = \frac{\partial Y'}{\partial \theta} = -\cos\theta\frac{\partial Y}{\partial x} + \sin^2\theta\frac{\partial^2 Y}{\partial x^2} = -x\frac{\partial Y}{\partial x} + (1-x^2)\frac{\partial^2 Y}{\partial x^2} \tag{16}$$

By setting $k = \ell(\ell+1)$ and by using the new variable x, Eq. (14) becomes Legendre's differential equation

$$(1-x^2)\frac{\partial^2 Y}{\partial x^2} - 2x\frac{\partial Y}{\partial x} + \ell(\ell+1)Y = 0 \tag{17}$$

The Associated Legendre Differential Equation

We now consider solutions that depends on the azimuthal angle φ. If we assume that the function $P(\varphi)$ can be decomposed in Fourier modes $e^{im\varphi}$, we achieve from Eq. (12),

$$\ell(\ell+1) = -\frac{Y'' + \cot\theta Y'}{Y} + \frac{m^2}{\sin^2\theta}. \tag{18}$$

Going through the same substitution calculation as above, we end up with the associated Legendre differential equation

$$(1-x^2)\frac{\partial^2 Y}{\partial x^2} - 2x\frac{\partial Y}{\partial x} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] Y = 0 \tag{19}$$

The Bessel Equation

With the separation constant $\ell(\ell+1)$, the radial part of Eq. (12) turns into the equation

$$r^2 R'' + 2rR' + \left[\kappa^2 r^2 - \ell(\ell+1)\right] R = 0. \tag{20}$$

The Bessel equation typically follows from cylindrical coordinates and not spherical coordinates as we have considered here. However, we bring Eq. (20) to same form as follows from consideration in cylindrical coordinates by performing the substitution,

$$R(r) = r^{-\frac{1}{2}} y(r) \tag{21}$$

where

$$R' = -\frac{1}{2} r^{-\frac{3}{2}} y + r^{-\frac{1}{2}} y' \quad \text{and} \quad R'' = r^{-\frac{1}{2}} y'' - r^{-\frac{3}{2}} y' + \frac{3}{4} r^{-\frac{5}{2}} y \tag{22}$$

By inserting these expressions in Eq. (20) and multiplying with $\sqrt{r}$, we end with the Bessel equation

$$r^2 y'' + ry' + \left[\kappa^2 r^2 - \left(\ell + \frac{1}{2}\right)^2\right] y = 0 \tag{23}$$

We then perform the final substitutions $x = \kappa r$ and $\nu = \ell + 1/2$, and achieve

$$x^2 y'' + xy' + \left[x^2 - \nu^2\right] y = 0 \tag{24}$$