Scaling exponent of the maximum growth probability in diffusion-limited aggregation

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An early (and influential) scaling relation in the multifractal theory of diffusion limited aggregation (DLA) is the Turkevich-Scher conjecture that relates the exponent α_{min} that characterizes the ''hottest'' region of the harmonic measure and the fractal dimension D of the cluster, i.e., $D=1+\alpha_{min}$. Due to lack of accurate direct measurements of both D and α_{min}, this conjecture could never be put to a serious test. Using the method of iterated conformal maps, D was recently determined as $D=1.713\pm0.003$. In this paper, we determine α_{min} accurately with the result $\alpha_{min}=0.665\pm0.004$. We thus conclude that the Turkevich-Scher conjecture is incorrect for DLA.

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Multifractal measures are normalized distributions lying upon fractal sets. As such, they present rich scaling properties that have attracted considerable attention in the last two decades. In this paper, we address the harmonic measure of diffusion limited aggregates (DLA) [1], which is the probability measure for a random walker coming from infinity to hit the boundary of the fractal cluster. This was one of the earliest multifractal measures to be studied in the physics literature [2], but the elucidation of its properties was made difficult by the extreme variation of the probability to hit the tips of a DLA versus hitting the deep fjords. Thus, the understanding of its scaling properties has been a long standing issue. These scaling properties are conveniently studied using the notion of generalized dimensions D_q, and the associated $f(\alpha)$ function [3,4]. The simplest definition of the generalized dimensions is in terms of a uniform covering of the boundary of a DLA cluster with boxes of size ℓ, and measuring the probability for a random walker coming from infinity to hit a piece of boundary that belongs to the ith box. Denoting this probability by $P_i(\ell)$, one considers [3]

$$D_q \equiv \lim_{\ell\to 0}\frac{1}{q-1}\,\frac{\ln\sum_i P_i^q(\ell)}{\ln \ell}, \qquad (1)$$

where the index i runs over all the boxes that contain a piece of the boundary. The limit $D_0\equiv\lim_{q\to 0^+}D_q$ is the fractal or box dimension of the cluster. $D_1\equiv\lim_{q\to 1^+}D_q$ and D_2 are the well known information and correlation dimensions, respectively [5–7]. It is well established by now [4] that the existence of an interesting spectrum of values D_q is related to the probabilities $P_i(\ell)$ having a spectrum of ''singularities'' in the sense that $P_i(\ell)\sim\ell^{\alpha}$ with α taking on values from a range $\alpha_{\min}\leqslant\alpha\leqslant\alpha_{\max}$. The frequency of observation of a particular value of α is determined by the function $f(\alpha)$, where [with $\tau(q)\equiv(q-1)D_q$]

$$f(\alpha)=\alpha q(\alpha)-\tau(q(\alpha)), \quad \frac{\partial\tau(q)}{\partial q}=\alpha(q). \qquad (2)$$

Of particular interest are the values of the minimal and maximal values, α_{min} and α_{max}, relating to the largest and smallest growth probabilities, respectively. The maximal value α_{max} was a subject of a long controversy that was settled only recently (cf. Refs. [8,9], and references therein). The issue of α_{min} appears to be one of the last of the multifractal properties of DLA that has resisted settling. This is the subject of this paper.

Consider DLA clusters containing n particles of radius $\sqrt{\lambda_0}$, and denote the radius of the minimal circle that contains the cluster as R_n. An incoming random walker from infinity has some probability to hit any of the existing particles of the cluster. Denote the maximal of these probabilities as p_{max}. The average of these probabilities over many clusters of n particles appears to scale as

$$\langle p_{max}\rangle \sim \left(\frac{\sqrt{\lambda_0}}{R_n}\right)^{\alpha_{min}} \sim n^{-\alpha_{min}/D}, \qquad (3)$$

where for the last step we have used the obvious scaling law $n\sim(R_n/\sqrt{\lambda_0})^D$. Turkevich and Scher have made the plausible assumption that the position of the cluster particle associated with p_{max} is at the outermost tip of the cluster. Thus, a scaling relation can be derived by stating that upon adding a new particle to the cluster, R_n will grow by one unit $\sqrt{\lambda_0}$ with probability p_{max} or will not grow at all with probability $1-p_{max}$. Then

$$\frac{dR}{dn}\sim\sqrt{\lambda_0}p_{max}\sim n^{1/D-1}, \qquad (4)$$

where again the last step stems from the definition of the fractal dimension. Equating the right-hand side of Eqs. (3) and (4) we get the Turkevich-Scher conjecture [11]

$$D=1+\alpha_{min}. \qquad (5)$$

We will show here that this conjecture is incorrect simply because the position of maximal probability is *not* at the outermost tip of the DLA cluster. In fact, one can introduce

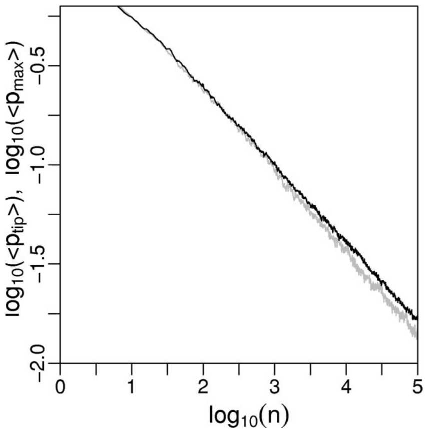

FIG. 1. The average of p_{max} (upper, black line) and p_{tip} (lower, gray line) versus n. The average is over 20 clusters of sizes up to $n=100\,000$. From the values of the slopes we estimate $\alpha_{min}=0.681\pm0.014$ and $\alpha_{tip}=0.713\pm0.012$.

in analogy to Eq. (3) a scaling law for the probability to hit the actual tip of the cluster (the particle which is a furthest away from the origin), i.e.,

$$\langle p_{tip}\rangle \sim \left(\frac{\sqrt{\lambda_0}}{R_n}\right)^{\alpha_{tip}} \sim n^{-\alpha_{tip}/D}. \tag{6}$$

A scaling law

$$D=1+\alpha_{tip} \tag{7}$$

is then a tautology. We will show that for DLA $\alpha_{tip}>\alpha_{min}$.

To achieve accurate estimates of α_{min} (and in passing of α_{tip}), we resort to the method of iterated conformal maps that was shown to be extremely useful for dealing with DLA and related growth processes. The method was amply described before, so we just remind the reader that it is based on compositions of fundamental conformal maps $\phi_{\lambda,\theta}$ which map the exterior of the unit circle to its exterior, except for a little bump at $e^{i\theta}$ of linear size proportional to $\sqrt{\lambda}$. The composition of these mappings is analogous to the aggregation of random walkers in the off-lattice DLA model. We shall here use the mapping introduced in Ref. [10], which produces two square root singularities that we refer to as the branch cuts, and the tip of the bump which we refer to as the microtip. The dynamics is given by

$$\Phi^{(n)}(w)=\Phi^{(n-1)}(\phi_{\lambda_n,\theta_n}(w)), \tag{8}$$

where $\Phi^{(n)}$ maps the exterior of the unit circle to the exterior of the cluster of n bumps. The size of the nth bump is controlled by the parameter λ_n and in order to achieve particles of fixed size we have that, to leading order,

$$\lambda_n=\frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^2}. \tag{9}$$

Using the iterated conformal maps it is very easy to keep track of where the maximum growth probability is located, and where the outermost tip is as more particles are added. Let us assume that at the $(n-1)$th growth step, the site with the largest probability is located at the angle θ_{max} on the unit circle, i.e., for all θ,

$$\frac{1}{|\Phi^{(n-1)\prime}(e^{i\theta_{max}})|}\geqslant\frac{1}{|\Phi^{(n-1)\prime}(e^{i\theta})|}. \tag{10}$$

When we add a new bump in the nth growth step, the position of maximal probability may not change (up to rep-

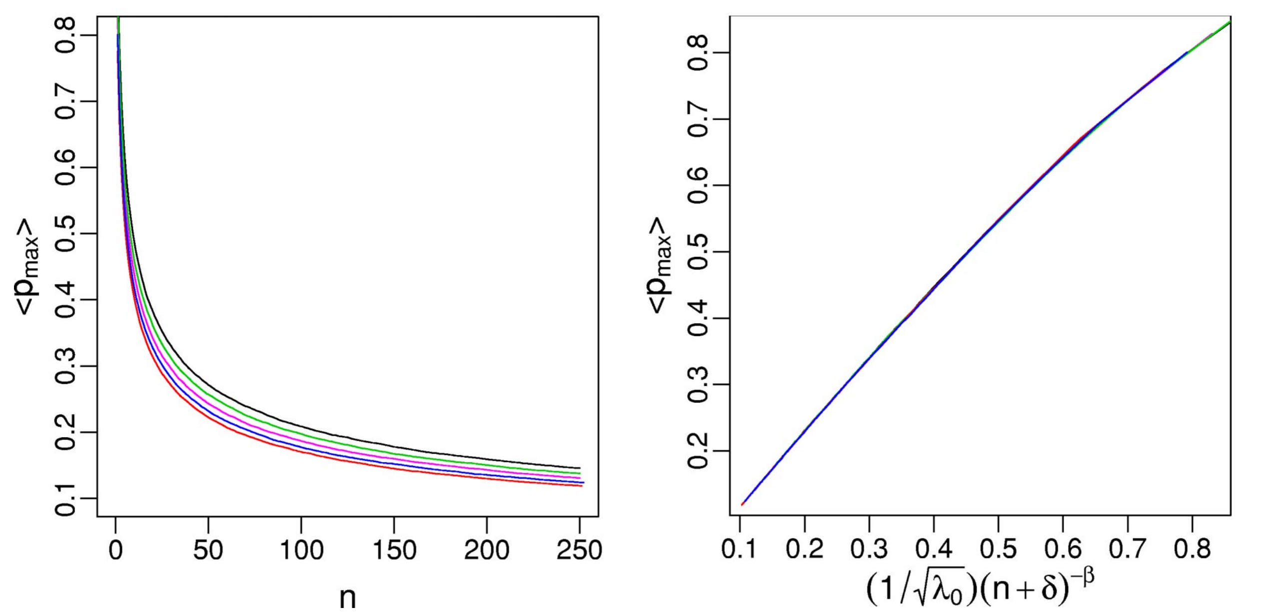

FIG. 2. The upper panel shows data of $\langle p_{max}\rangle$ versus n for $\lambda_0=0.8$, 0.9, 1.0, 1.1, 1.2. The lower panel shows the same data plotted versus the scaling variable defined in Eq. (12). The estimated values of β and δ that lead to the best data collapse, using least squares, are $\beta_{max}=0.389$, $\delta_{max}=0.505$. We therefore have from $D\approx1.713$ that $\alpha_{min}\approx0.666$.

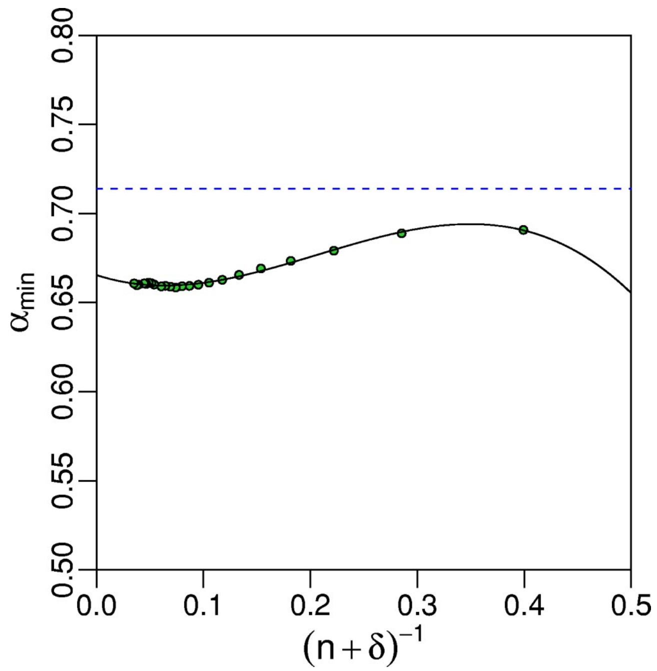

FIG. 3. Data of α_{min} estimated using Eq. (15) with $\tilde{n}=n+1$. The data are fitted with a cubic polynomial. The polynomial intersects the y axis at the value $\alpha_{min}=0.665$. Using upper and lower values of δ, $\delta=0$, and $\delta=1$ we estimate the following bounds on the value of α_{min}, $0.662<\alpha_{min}<0.669$.

arametrization of the angle θ_{max}) or move to the new bump. We can easily find the reparametrized angle and determine the new position from

$$p_{max,n}=\max\left\{\frac{1}{|\Phi^{(n)\prime}(\phi^{-1}_{\lambda_n,\theta_n}(e^{i\theta_{max}}))|},\frac{1}{|\Phi^{(n)\prime}(e^{i\theta_n})|}\right\}. \tag{11}$$

If $p_{max,n}$ is located at θ_n we put $\theta_{max}=\theta_n$ in the $(n+1)$th growth step. Similarly, we track the position $|z|_{max}$ on the cluster by finding the value θ_{tip} that assigns the maximal value of $|\Phi^{(n)}(e^{i\theta})|$. We compute p_{tip} there as $1/|\Phi^{(n)\prime}|$.

A direct measurement of α_{min} and α_{tip} is displayed in Fig. 1. From the direct measurement we find $\alpha_{min}\approx0.681$, while $\alpha_{tip}\approx0.713$. Clearly, the latter is in agreement with Eq. (7), while the former is in disagreement with Eq. (5) (taking as a datum the result of Ref. [12], $D=1.713\pm0.005$).

The direct measurement, while correct, cannot guarantee that very slow convergence of the power laws as a function of n may somehow hide an asymptotic identity of α_{min} and α_{tip}. To remove this problem, we now adopt the scaling function technique of Ref. [12] to achieve an accurate determination of α_{min}. In this approach, one acknowledges that Eq. (3) may be realized only asymptotically for high values of n. For low and medium values of n, $\langle p_{max}\rangle$, which is a function of the discrete n and of λ_0, is in fact a scaling function of a single scaling variable,

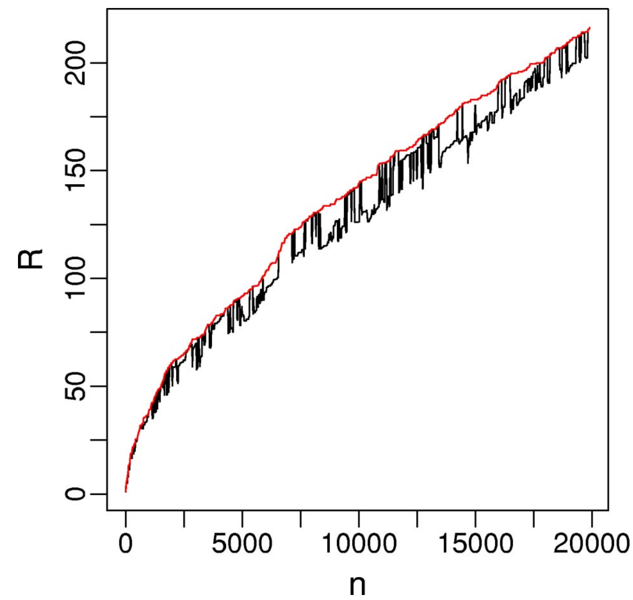

FIG. 4. The value of R_n as a function of n (upper curve) and the position of the bump belonging to α_{min} (lower curve) in a typical DLA.

$$\langle p_{max}\rangle=f_{\delta,\beta}(x),$$
$$x=\frac{1}{\sqrt{\lambda_0}}(n+\delta)^{-\beta}, \tag{12}$$

where we have denoted $\beta=\alpha_{min}/D$. The difference with Eq. (3) is that $f_{\delta,\beta}$ is in general not a linear function of its argument, except at exceedingly small values of x, when n is very large. In Fig. 2, we demonstrate the existence of the scaling function and the excellent data collapse achieved using it. In the upper panel, we plot $f_{\delta,\beta}(\lambda_0,n)$ for five values of λ_0 and $1\le n\le250$. In the lower panel, the same data are collapsed using the single scaling variable. We draw the reader's attention to the following two observations: (i) the data collapse is available immediately, even for the smallest values of n [12], and (ii) the scaling function is not linear throughout the range explored here. Thus, the scaling law (3) is not obeyed yet for values of n of the order of a few hundreds. The set of parameters β and δ which give the best data collapse in the lower panel are $\beta=0.389$ and $\delta=0.505$. These parameters are used in the lower panel and give the estimate $\alpha_{min}=0.666$ when assuming that the fractal dimension is $D=1.713$.

An even more accurate determination of α_{min} is achieved next. Taking the data collapse as an evidence for the existence of a scaling function, we conclude that for any two pairs of numbers (n,λ_0) and $(\overline{n},\overline{\lambda}_0)$ that satisfy the equation

$$\frac{1}{\sqrt{\lambda_0}}(n+\delta)^{-\alpha_{min}/D}=\frac{1}{\sqrt{\tilde{\lambda}_0}}(\tilde{n}+\delta)^{-\alpha_{min}/D}, \tag{13}$$

it follows that

$$f_{\delta,\beta}(\lambda_0,n)=f_{\delta,\beta}(\tilde{\lambda}_0,\tilde{n}). \tag{14}$$

These equations offer a calculational procedure. We find $\langle p_{max}\rangle$ for a given n and λ_0, and then for another value $\overline{n}$ seek the value $\overline{\lambda}_0$ for which $\langle p_{max}\rangle$ is the same. From Eq. (13), we then deduce that

$$\alpha_{min}=\frac{1}{2}D\frac{\ln\lambda_0-\ln\tilde{\lambda}_0}{\ln(n+\delta)-\ln(\tilde{n}+\delta)}. \tag{15}$$

In Fig. 3, we present the results of such a calculation with $\overline{n}=n+1$, and $1\leqslant n\leqslant 250$. Since δ is not known *a priori*, we used the value $\delta=0.505$ that was extracted from the data collapse in Fig. 2. We checked the sensitivity to δ by bracketing the results with $\delta=0$, and $\delta=1$ respectively. The data in Fig. 3 correspond to $\delta=0.505$. Fitting the data with a cubic polynomial and extrapolating to $x\to 0$, we get the value $\alpha_{min}\approx 0.665$. On the other hand, if we repeat the procedure using the values of $0\leqslant\delta\leqslant 1$ we are able to bracket the estimate in the interval

$$0.662<\alpha_{min}<0.669. \tag{16}$$

We thus conclude the analysis with the estimate $\alpha_{min}=0.665\pm 0.004$.

Finally, we explain why the Turkevich-Scher conjecture (5) fails. The reason is that the points corresponding to p_{max} and p_{tip} are not at all the same in typical DLA. In Fig. 4, we present the calculated value of R_n, computed from the position of largest $|z|$ on the cluster, compared with the position corresponding to the maximal harmonic measure. We see that the position of maximal probability fluctuates wildly, and the fluctuations do not appear to go down with the increase in the cluster size. The loss of the conjecture (5) means that there is no clear connection between the spectrum of singularities $f(\alpha)$ and the fractal dimension of DLA. As said above, the relation (7) is a tautology once the existence of the scaling law (6) has been established [13]. Since the value of α_{tip} has nothing to do with the edge of the α spectrum, it appears as hard to determine it from first principles as to determine the dimension D itself.

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