Communication Dynamics in Finite Capacity Social Networks

Jan O. Haerter,[1] Bjørn Jamtveit,[2] and Joachim Mathiesen[1,2]

[1]*Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*
[2]*Physics of Geological Processes, University of Oslo, Oslo, Norway*

In communication networks, structure and dynamics are tightly coupled. The structure controls the flow of information and is itself shaped by the dynamical process of information exchanged between nodes. In order to reconcile structure and dynamics, a generic model, based on the local interaction between nodes, is considered for the communication in large social networks. In agreement with data from a large human organization, we show that the flow is non-Markovian and controlled by the temporal limitations of individuals. We confirm the versatility of our model by predicting simultaneously the degree-dependent node activity, the balance between information input and output of nodes, and the degree distribution. Finally, we quantify the limitations to network analysis when it is based on data sampled over a finite period of time.

Limitations on the processing capacities of nodes and links have a profound impact on the flow of information in online communication networks [1,2], the spreading of diseases in human encounter networks [3], and in social networks [4–7], where links between interacting individuals can be highly volatile [8]. It is often assumed that communication takes place in an unrestrained way on a set of established connections, thereby neglecting, that structure and dynamics are interdependent. Here we consider the evolution of a network where links form as a result of non-Markovian interaction between nodes. In a time-limited environment, communication demands prioritization which is evident from the analysis of correspondence patterns [7,9]. Hence, information flow on a network is a result of individuals' choices which are influenced by the state of surrounding nodes. In natural [10] and online [11–15] social networks, the nodes' activity is a nontrivial function of their degree. The activity level can be quantified by the number of social relationships simultaneously maintained by an individual. This number has been suggested to reflect basic cognitive capabilities of primates [10] and humans [11,14,15]. Here we model a network of individuals acting under time constraints and compare it with a complete data set of email communication in a large organization. The model is discussed in the context of other communication networks. We predict the information processing capacity of individuals as well as the structure of the network that they form.

We use representative communication data from a large social organization, the University of Oslo. The data comprise a complete time-ordered list of 2.3×10^7 emails between 5600 employees, 30 000 students and approximately 10^6 people outside the organization over a period of three months (Sep.–Nov., 2010). The email content was not recorded and identities of individuals were encrypted. We limit the influence of unsolicited bulk emails by disregarding those simultaneously sent to more than five recipients. However, the results are not sensitive to the filtering of bulk emails [16]. Previous work on email data has considered static network structures [17–22].

Results.—We show that the communication is non-Markovian by comparing random and directed information flow. (i) Random flow is given by random walks on the network. The walker follows an empirical time-independent jump probability $p_{ij} = N_{ij}/\sum_k N_{ik}$ from node i to node j. The sum is taken over all nodes and N_{ij} is the number of emails sent from i to j during the time span of the data. (ii) Directed flow is given by the chronological email exchange. Starting from a random node i, we wait for i to send an email, say to j. We then jump to j and wait for the next message j sends either back to i or to a new node k. Repeating this, we obtain a finite trajectory within the time span of the data. The numbers of unique nodes visited by the directed and random flow as function of the number of jumps are compared by averaging over trajectories originating from all nodes (Fig. 1). On average, directed flow visits relatively fewer nodes than random flow, indicating a significant correlation between sent and received messages.

Our model requires nodes to perform a trade-off between replying to others and initiating new conversations. Specifically, we consider $\mathcal{N}$ nodes, each initially connected to one other node. The nodes have a limited capacity and can send a maximum of N_{max} messages in a time step $\Delta t = 1$ day. The dynamics follows from three possible actions for a node i of out-degree k_i.

(a) i processes received emails and if i has sent less than N_{max} messages, any received email is replied to with a probability proportional to the sender's degree. Emails not replied to within Δt are subsequently deleted. In total, δ_2 replies are sent by this action.

(b) If less than N_{max} emails have been sent in (a), the remaining capacity $N_{max} - \delta_2$ is available for sending
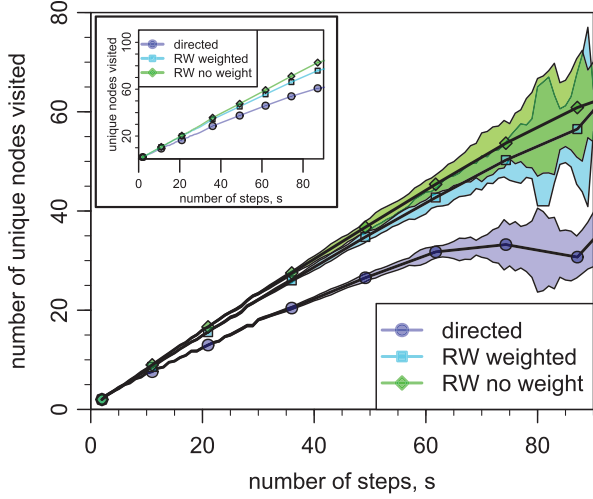
FIG. 1 (color). Weighted random, unweighted random, and directed information flow. The error bars are estimated by bootstrapping. Inset: Similar plot using model data. The quantitative discrepancy between model and data results from the relative dominance of degree-one nodes in the empirical data.

messages, called δ_1, to previously established contacts. The probability of sending a message to a contact is given by a constant r_{ini}. Hence, granted sufficient capacity on average $r_{\mathrm{ini}}k_i$ messages are initiated by i. Nodes with low k_i will generally not reach their full capacity.

(c) Nodes establish new contacts by sending requests with a probability r_{req}. The probability that a request is sent to a node j is proportional to the degree of j, k_j. A link is established between i and j, if j in the next time step according to (a) replies to i. In reality, contacts might as well be established by face-to-face encounters, i.e., via channels not recorded explicitly in our data.

The total number of messages δ sent by a node in Δt is the sum $\delta \equiv \delta_0 + \delta_1 + \delta_2$. Analogously, messages received by a node in the same time step are termed $\alpha \equiv \alpha_0 + \alpha_1 + \alpha_2$. Nodes have an average lifetime τ and are therefore removed from the network with a probability $\Delta t/\tau$. For every node removed, a new node with a single random connection to an existing node is introduced. τ is estimated to be 5.8 years from the known mean email user turnover time in the organization. The parameters r_{ini}, r_{req} and $N_{\max}$ are determined below.

According to (c), a link is established between i and j if one of the nodes sends a message to the other and receives a reply. The probability, P_{ij}, that a message is sent from i to j in Δt is proportional to k_j,

$$P_{ij} = \frac{r_{\mathrm{req}}k_j}{\sum_{\ell \neq i}k_\ell} \approx \frac{r_{\mathrm{req}}k_j}{\mathcal{N}\langle k\rangle}, \qquad (1)$$

where we in the approximation assume that $k_i \ll \sum_\ell k_\ell$. According to (a), the mean number of requests that j receives during a time step is proportional to r_{req} and k_j. The probability for j to reply to a request from nodes of

degree k is proportional to $\beta k n(k)$, where β is a constant and $n(k)$ is the number of nodes with degree k. The number of replies written by j is the product of Eq. (1) and the integral over nodes

$$\frac{r_{\mathrm{req}}k_j}{\mathcal{N}\langle k\rangle} \int \beta k n(k)dk = \beta r_{\mathrm{req}}k_j. \qquad (2)$$

Since nodes reply to requests and therefore establish new links with a probability proportional to the sender degree, $kn(k)$, the mean degree k_c of a node's contacts is $k_c \equiv \int k^2 n(k)dk / \int kn(k)dk = \langle k^2\rangle/\langle k\rangle$, a number generally larger than the mean degree $\langle k\rangle$ (Fig. 3).

Consequently the average degree increase of nodes of degree k per time step becomes $r(k)\Delta t \equiv 2\beta r_{\mathrm{req}}k\Delta t$. The factor of 2 reflects the symmetry of sending and replying. The rate of losing links is inversely proportional to τ, $d \equiv k/\tau$. Hence, the net degree-growth rate becomes $\Delta k/\Delta t = kr_0$, where $r_0 \equiv (2\beta r_{\mathrm{req}} - \tau^{-1})$. As long as a node has sufficient capacity to reply to all requests its degree increases approximately exponentially, $k(t) \sim \exp(r_0 t)$.

The degree distribution follows from the consideration that during Δt, a fraction of nodes $n(k)$ of degree k change their degree, $r_0[(k-1)n(k-1) - kn(k)]$, and a fraction $1/\tau$ is removed. A continuum-limit approximation yields

$$\frac{\partial n(k)}{\partial t} = -r_0\left[k\frac{\partial n(k)}{\partial k} + n(k)\right] - \frac{n(k)}{\tau}. \qquad (3)$$

The steady-state solution has the form $n(k) = n(1)k^{-\gamma}$, where $\gamma \equiv (1 - 1/2\beta r_{\mathrm{req}}\tau)^{-1}$. The constant $n(1)$ is fixed by integrating Eq. (3) over k and by demanding that the total number of nodes $\mathcal{N} = \int dk n(k)$ be constant. This yields $n(1) = \mathcal{N}(\gamma - 1)$. The condition $0 < n(1) < \mathcal{N}$ bounds the power-law exponent: $1 < \gamma < 2$. The data yield $\gamma \simeq 1.85$ (Fig. 2 inset).

So far we have assumed that nodes have infinite capacity. As a node's degree increases, it receives more messages and this assumption becomes invalid. Consider the number of messages received by i per time step. Contact requests from other nodes amount to $\alpha_0 \equiv r_{\mathrm{req}}k_i/\langle k\rangle$ messages. The senders of these messages are drawn from a distribution $n(k)/\mathcal{N}$. The probability for i to receive a message from its contacts is proportional to r_{ini} and k_i, hence $\alpha_1 \equiv r_{\mathrm{ini}} \cdot k_i$. Analogously, as defined in (a), i issues $\delta_0 \equiv r_{\mathrm{req}}$ requests to recipients distributed according to $\rho_1(k)$ [where $\rho_\ell(k) \equiv k^\ell n(k)/\int k'^\ell n(k')dk'$] due to the weighting of probabilities by the recipient degree. In the same time step i sends $\delta_1 = \alpha_1$ messages to its contacts. Finally, we consider back-and-forth communication. For every message sent by i to j, a response is returned with a probability βk_i [Eq. (2)]. In the steady state, the number of messages sent is identical for all time steps and therefore i receives

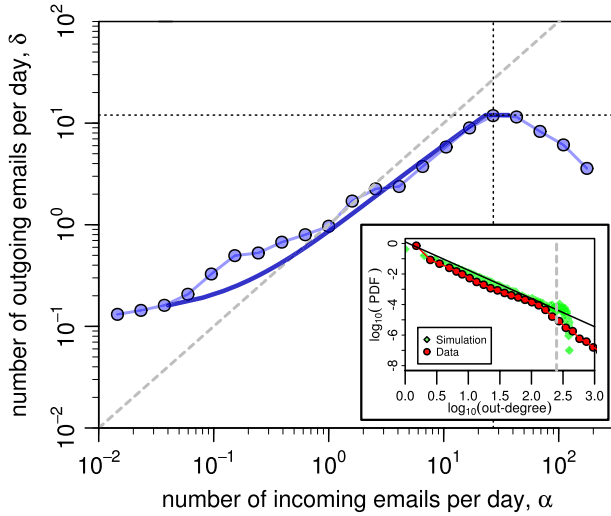$$\alpha_2 \equiv \beta k_i(\delta_0 + \delta_1 + \delta_2) \qquad (4)$$

FIG. 2 (color online).   Average number of messages sent per message received. Observational data is marked by "○". The solid line is a best fit by Eq. (6). The dotted lines mark the peak and the dashed diagonal line shows $\delta = \alpha$. Inset: out-degree distribution for model and empirical data. The dashed line denotes the scale break $k_{sb} \simeq 250$. Mean degree is 5.4 (Twitter data yields a mean degree of 8.8 and a similar exponent for the degree distribution [23]). Note the double-log scales.

FIG. 3 (color online).   Mean recipient degree as a function of degree ($\square$) and weighted by the number of messages sent to recipients ($\circ$). The horizontal line shows $\langle k^2 \rangle / \langle k \rangle$. The curves marked by "$\diamond$" and "$\triangle$" are analogous to the unweighted case but for half, respectively, and one quarter of the observational period. Dashed lines show projection of nodes with two values of k for a varying observation window. Note the double-log scale.

replies to messages sent in the previous time step. δ_2 is the number of messages i sends in response to messages received from others which again is a sum over contributions from the actions (a)–(c):

$$\delta_2 \equiv \beta(\alpha_0 \langle k \rangle_{\rho_0} + \alpha_1 \langle k \rangle_{\rho_1} + \alpha_2 \langle k \rangle_{\rho_{\alpha_2}}). \quad (5)$$

The terms on the right are, respectively, requests from any node in the network (distributed as ρ_0), messages from existing contacts (distributed as ρ_1), and back-and-forth messages (distributed as ρ_{α_2}). Each iteration of back-and-forth communication acts as a shift in the distribution of recipients relative to the distribution of senders $\mathcal{F}\rho_l \equiv \beta\rho_{l+1}$. The distribution ρ_{α_2} accounts for all high-order shifts. To close the equations for α_2 and δ_2, we use that the reply probability for each iteration is reduced by a factor β to approximate $\rho_{\alpha_2} \simeq \rho_2$. Inserting Eq. (4), α_0 and α_1 in Eq. (5) yields $\delta_2 = \beta[\alpha_0 \langle k \rangle + \alpha_1 k_c + \beta k_i k_c (\delta_0 + \delta_1)]/f(k_i)$ where we introduce $f(k_i) \equiv 1 - \beta^2 k_i k_c \leq 1$. Summing over δ_0, δ_1 and δ_2 we get

$$\delta = r_{\rm req} + r_{\rm ini} k_i + \frac{\beta k_i}{f(k_i)} [r_{\rm req} + r_{\rm ini} k_c + \beta k_c (r_{\rm req} + r_{\rm ini} k_i)].$$

$$(6)$$

Here the first three terms (referred to as $\delta_<$) are messages sent to recipients selected according to ρ_1 and with mean degree k_c. The other terms, $\delta_>$, are messages to recipients distributed according to the higher order distribution ρ_2 which has a mean $k_c^* \equiv \langle k^3 \rangle / \langle k^2 \rangle > k_c$ and contribute significantly only for large k_i. The mean of the weighted
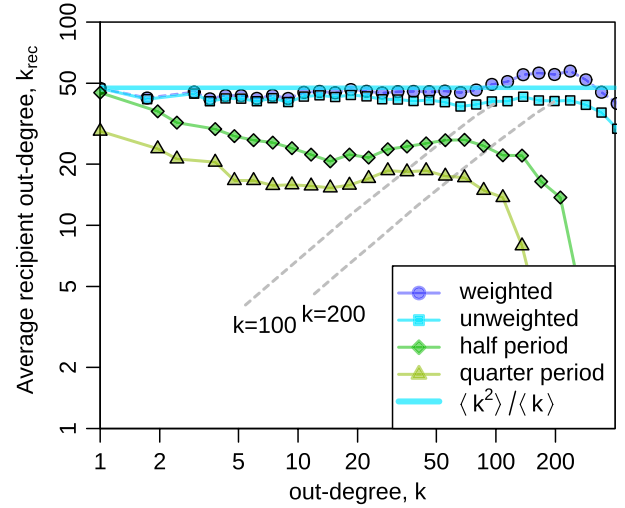
recipient degree (weighted by number of messages received) is $k_{\rm rec}^w \equiv k_c \delta_< / \delta + k_c^* \delta_> / \delta$, which departs from k_c when $\delta_>$ becomes appreciable (Fig. 3). For low k_i ($k_i = 1$), the ratio of sent to received messages becomes $\delta / \alpha \simeq (r_{\rm req} + r_{\rm ini}) / (r_{\rm req} / \langle k \rangle + r_{\rm ini}) > 1$. Conversely, $\delta / \alpha = 1$ when $k_i = \langle k \rangle$; hence, an average node has a "balanced" email account. When k_i becomes larger than $\langle k \rangle$, i will increasingly receive requests and responses to its messages (Fig. 2).

The Dunbar number k_D is the degree where δ reaches the capacity limit ($\delta = N_{\rm max}$) and δ / k is maximal. The scale break in the degree distribution ($k_{sb} \simeq 250$), Fig. 2 (inset), and $k_D \simeq 230$, Fig. 4, nearly coincide. In fact k_{sb} is related to k_D because nodes beyond k_D have a reduced probability to form new links. To determine k_{sb}, consider the evolution of the nodes' degree in the limit where all capacity is used for replying, hence $\delta_1 = 0$. Using that $\delta_0 \ll \delta_2$, we get $\delta \approx \delta_2 = N_{\rm max}$ which in turn yields $k_{sb} = \beta^{-1} N_{\rm max} f(k_{sb}) (r_{\rm req} + r_{\rm ini} k_c)^{-1}$. k_{sb} is found by solving this implicit equation. k_D then follows from Eq. (6).

The parameters $r_{\rm ini} = 0.023$, $r_{\rm req} = 0.13$ and $N_{\rm max} = 12$ are determined by the data in Fig. 2. From $r_{\rm req}$ and γ we obtain $\beta \approx 0.004$. Larger $N_{\rm max}$ increases the limit of δ. $r_{\rm req}$ is constrained by the offset at low α and $r_{\rm ini}$ effects the skewness of the curve which follows from analysis of Eqs. (4) and (6). Figure 4 shows the model prediction of δ / k_i and the corresponding email data. We complement our analysis with numerical computations. Using a large number of nodes, $\mathcal{N} = 10\,000$, we iterate actions (a)–(c) until the steady state is reached. While the mean-field
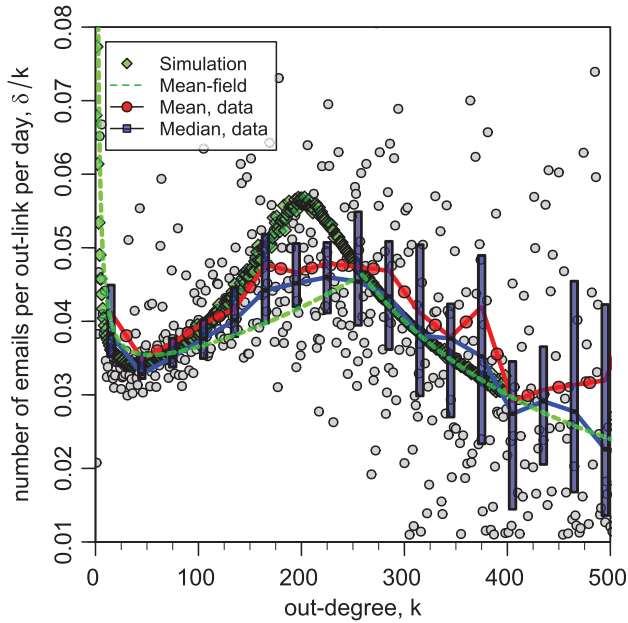
FIG. 4 (color). Average number of emails sent per link per day. Gray circles represent the average activity of all users of a certain out degree and the red (blue) lines represent coarse grained mean (median) values in the real communication network; boxes mark upper and lower quartiles. Best fit with the model (simulation) is shown by the green lines (diamonds). At small k, $f(k) \simeq 1$ [Eq. (6)] and δ/k is a superposition of a term $\sim k$ due to the final quadratic term and a decaying term $\sim k^{-1}$ from the constant. At $k > k_D$, nodes are limited to $N_{\max}$ messages per day, hence $\delta/k \sim N_{\max}/k$.

prediction (Figs. 3 and 4) is close to the numerical solution, some differences exist; e.g., at small k, $\rho(k)$ is not a strict power law in the numerical solution due to the discreteness of k. Further, the simulation gives a smooth peak in δ/k (Fig. 4) which is narrower than in the empirical data. This is due to slight overestimation of the repeated back-and-forth communication between well-connected nodes ($k \approx 200$) relative to the data. We have also simulated the information flow (Fig. 1) and achieve similar results. Finally, the average local clustering coefficient of the empirical and simulated networks is relatively small, ≈ 0.04 for both (similar clustering coefficient ≈ 0.06 [23] and $k_D \approx 150$ to 200 have been reported for other communication networks [11,15,24]). We further checked the robustness of the model to variations [16].

Discussion.—The data were recorded over three months and the communication network is therefore a finite-time projection of the real network. The projection reduces the number of links. More active links will more likely persist through the projection than less active links. Figure 3 shows the mean recipient degree k_{rec} as a function of the sender degree k_i for three observation time intervals. Consider again Eq. (6) and remember that recipients of the $\delta_<$ ($\delta_>$) messages are distributed as ρ_1 (ρ_2). When observing only a single day, the probability for an out-link

between i to j not to be active is $P_{ij}(\Delta t) \equiv 1 - \delta_< k_j / k_c k_i - \delta_> k_j / k_c^* k_i$. For d days we obtain $P_{ij}(d\Delta t) = P_{ij}(\Delta t)^d$. To produce the projected curves in Fig. 3, $P_{ij}(d\Delta t)$ is applied to both axes, k and $k_{\text{rec}}(k)$. Averaging with respect to all recipients j (distributed as ρ_1), the projected sender out degree becomes $k_i^{(d)} \equiv k_i \langle 1 - P_{ij}^d \rangle_{\rho_1}$. Similarly one can consider the projection of the mean recipient degree leading to a similar reduction in the degree for finite-time data. For example, consider the data for the quarter period ($d \approx 23$) in Fig. 3. We have $P_{ij}(\Delta t)^d \simeq (1 - r_{\text{ini}})^d$ and therefore $k_i^{(d)} / k_i < 1/2$; hence, less than half the links persist.

Concluding remarks.—The finite capacity of agents in social networks induces an upper limit on the number of possible interactions [11,13–15]. We propose a comprehensive model that reconciles structure and dynamics of networks with finite capacity agents that dynamically form or lose links. In agreement with a complete set of email data and results from other social networks [13,23], our model predicts a scale-free degree distribution up to a distinct scale break induced by the capacity limit. Further, as agents gain importance in the network, the per-link activity first increases with node degree, peaks at intermediate degrees, and declines at large degrees. The model and data therefore support the hypothesis of a general limit on the number (150–250) of active social relations that an individual can maintain [10] and is in agreement with empirical observations on social networks [11,24].

[1] D. Liben-Nowell and J. Kleinberg, Proc. Natl. Acad. Sci. U.S.A. **105**, 4633 (2008).
[2] J. Candia, M. C. González, P. Wang, T. Schoenharl, G. Madey, and A.-L. Barabási, J. Phys. A **41**, 224015 (2008).
[3] L. E. C. Rocha, F. Liljeros, and P. Holme, Proc. Natl. Acad. Sci. U.S.A. **107**, 5706 (2010).
[4] S. P. Borgatti, A. Mehra, D. J. Brass, and G. Labianca, Science **323**, 892 (2009).
[5] M. Kitsak, L. K. Gallos, S. Havlin, F. Liljeros, L. Muchnik, H. E. Stanley, and H. A. Makse, Nature Phys. **6**, 888 (2010).
[6] D. Centola, Science **329**, 1194 (2010).
[7] J. G. Oliveira and A.-L. Barabási, Nature (London) **437**, 1251 (2005).
[8] A.-L. Barabási, Nature (London) **435**, 207 (2005).
[9] R. D. Malmgren, D. B. Stouffer, A. S. L. O. Campanharo, and L. A. N. Amaral, Science **325**, 1696 (2009).

[10] R. I. M. Dunbar, J. Hum. Evol. **22,** 469 (1992); H. Kudo and R. I. M. Dunbar, Animal Behaviour **62,** 711 (2001).

[11] B. Gonçalves, N. Perra, and A. Vespignani, PLoS ONE **6,** e22 656 (2011).

[12] J. Ugander, B. Karrer, L. Backstrom, and C. Marlow, arXiv:1111.4503v1.

[13] R. Corten, PLoS ONE **7,** e34 760 (2012).

[14] C. Wilson, B. Boe, A. Sala, K. P. N. Puttaswamy, and B. Y. Zhao, in *EuroSys '09 Proceedings of the 4th ACM European Conference on Computer Systems* (ACM, New York, 2009), p. 205.

[15] B. Viswanath, A. Mislove, M. Cha, and K. P. Gummadi, in *Proceedings of the 2nd ACM SIGCOMM Workshop On Social Networks*, Barcelona, Spain (ACM, New York, NY, 2009).

[16] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.109.168701 for further details on the robustness of the model when various assumptions are relaxed and for comments on the filtering of bulk emails.

[17] H. Ebel, L. I. Mielsch, and S. Bornholdt, Phys. Rev. E **66,** 035103 (2002).

[18] R. Guimerà, L. Danon, A. Díaz-Guilera, F. Giralt, and A. Arenas, Phys. Rev. E **68,** 065103R (2003).

[19] M. E. J. Newman, S. Forrest, and J. Balthrop, Phys. Rev. E **66,** 035101 (2002).

[20] J. P. Eckmann, E. Moses, and D. Sergi, Proc. Natl. Acad. Sci. U.S.A. **101,** 14 333 (2004).

[21] J. Mathiesen, B. Jamtveit, and K. Sneppen, Phys. Rev. E **82,** 016104 (2010).

[22] B. Jamtveit, E. Jettestuen, and J. Mathiesen, Proc. Natl. Acad. Sci. U.S.A. **106,** 13 160 (2009).

[23] M. De Choudhury, Y.-R. Lin, H. Sundaram, K. S. Candan, L. Xie, and A. Kelliher, in *Proceedings of the 4th International AAAI Conference on Weblogs and Social Media* (Association for the Advancement of Artificial Intelligence, Palo Alto, CA, 2010), p. 34; http://konect.uni-koblenz.de/networks/munmun_twitterex_at.

[24] H. Chun *et al.*, *IMC'08* (ACM, New York, 2008).