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Master's Thesis

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The fluid/gravity correspondence

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Contents

Resumé

Den berømte AdS/CFT korrespondance, er en postuleret dualitet mellem en kvanteteori, indeholdende tyngdekraft defineret på Anti-de Sitter rum (en speciel type Einstein mangfoldighed), og en konform feltteori (CFT), der lever på overfladen af dette rum. Vi betragter en speciel grænse af AdS/CFT korrespondancen, hvor vi bruger fluid dynamik som en effektiv beskrivelse af CFT'en. Denne grænse giver anledning til den såkaldte fluid/gravitations korrespondance, der forudsiger, at der eksisterer en dualitet imellem tyngdekraft og fluid dynamik. Over de seneste par år har en ny tilgang til denne dualitet, mellem tyngdekraft og fluid dynamik, set dagens lys. Formålet med denne afhandling er at redegøre for denne nye tilgang. Vi redegør for hvordan fluid dynamik, som en effektiv beskrivelse af kvantefeltteori, naturligt kan skrives som en udvikling i kovariante afledte af de fluid dynamiske felter, og viser hvordan fluid dynamikken for konforme feltteorier kan udtrykkes til given orden i den omtalte udvikling vha. et sæt Weyl-invariante tensorer. Derefter analyserer vi dynamikken af fluktuerende braner (i anti-de Sitter) ud fra et holografisk synspunkt. Denne analyse går ud på, at løse Einsteins ligning (med en negativ kosmologisk konstant) perturbativt i bran-fluktuationen. Ved at bruge værktøjerne fra AdS/CFT korrespondancen ser vi, at dynamikken af branen præcis kan forstås via den duale fluid beskrivelse. Vi ser ydermere, at den før omtalte fluid dynamiske udvikling i de afledte af felterne reproduceres, hvilket lader os udtrække information om fluid dualen, og dermed den underliggende feltteori. Endelig introducerer vi en ny klasse af roterende sorte huller i anti-de Sitter, og redegør for deres termodynamik. Disse sorte huller giver os en ny baggrund at teste fluid/gravitations korrespondancen imod. Vi finder, at disse sorte huller passer perfekt ind i dualiteten og foreslår en mulig løsning på ladede sorte hullers fluid dynamik.

Abstract

The famous AdS/CFT correspondence is a conjectured equivalence between a quantum theory containing gravity defined on Anti-de Sitter space (AdS, a special type of Einstein manifold) and a conformal field theory (CFT) living on the boundary of this space. We consider a special limit of the AdS/CFT correspondence where we use fluid dynamics as an effective long-wave description of the CFT. The so-called fluid/gravity correspondence now predicts a duality between gravity and the effective fluid dynamic description. Over the recent couple of years a new approach to understanding this duality between gravity and fluid dynamics has emerged. The purpose of this thesis is to examine this new approach. We explain how fluid dynamics naturally can be written as a derivative expansion in the fluid dynamic fields and show how the fluid dynamics of conformal fluids, to a given order in the expansion, can be written in terms of a finite set of Weyl invariant tensors. We then move on to describing the dynamics of fluctuating branes (in anti-de Sitter). This analysis consists of solving Einstein's equation perturbatively in the brane fluctuation. By using the tools of the AdS/CFT correspondence we see that the dynamics of the brane exactly can be understood in terms of its dual fluid description. Moreover we see that the derivative expansion mentioned above is reproduced, which allows us to extract information about the dual fluid and thus the underlying field theory. Finally we introduce a new set of rotating black holes in anti-de Sitter and derive their thermodynamics. These black holes provide a new background against which the fluid/gravity correspondence can be (non-trivially) checked. We demonstrate how these black holes fit perfectly into the fluid/gravity correspondence and propose a possible solution to the fluid dynamics of charged rotating black holes.

1

Introduction

General introduction and overview

Strongly coupled field theories play an important role in many areas of modern physics with QCD being the most famous in particle physics. Over the last decades an enormous amount of research has therefore gone into understanding these theories. However, many aspects of strongly coupled theories are still poorly understood as the conventional methods pertaining to weakly coupled theories are not available in the strongly coupled regime. New non-perturbative methods are therefore needed to explore the properties of strongly coupled theories.

The AdS/CFT correspondence (discovered by Maldacena in 1997 [1]) has emerged as a very important tool for constructing such a non-perturbative framework. The AdS/CFT correspondence conjectures a deep connection between string theory on curved backgrounds and a certain class of interacting quantum field theories. More specifically, the remarkable AdS/CFT correspondence predicts a mathematical equivalence between string theory on a space of the type $\mathrm{AdS}_{d+1} \times X_I$ (where X_I is a compact manifold) and a d dimensional conformal field theory (CFT) defined on the boundary of AdS_{d+1} with internal symmetries of the CFT corresponding to the isometries of the manifold X_I. At generic values of parameters both sides are complicated quantum theories. However, the correspondence is a duality in the sense that the coupling constants on each side can be matched inversely to each other. This implies that the weakly coupled regime on the one side is equivalent to the strongly coupled regime on the other side. Especially if we consider the large N limit of the conformal boundary theory, the quantum corrections on the string theory side will be suppressed and the quantum theory on $\mathrm{AdS}_{d+1} \times X_I$ therefore reduces to classical string theory. Moreover, if we at the same time consider the strong 't Hooft coupling limit, all stringy α' corrections will also be suppressed and the quantum theory reduces to classical supergravity (SUGRA). Finally, by dimensional reduction, any two derivative theory on $\mathrm{AdS}_{d+1} \times X_I$ has a universal subsector consisting of gravity on AdS_{d+1}. This corner of the AdS/CFT correspondence therefore predicts an equivalence between gravity with a negative cosmological constant and a certain CFT in its planar limit.

Since we consider the strong 't Hooft coupling limit, the quantum dynamics on the gauge theory side is still very complicated and non-local. It is therefore useful to consider yet another limit. It is believed that any interacting quantum field theory admits an effective long-wave fluid dynamical description at sufficiently high temperatures and energy densities. By considering the fluid dynamical limit of the dual gauge theory, the effective dynamics of the gauge theory should therefore reduce to the equations governing relativistic fluid dynamics. The equations of fluid dynamics are simply the conservation equations expressing conservation of energy and momentum (and perhaps a set of currents) along with a set of constitutive equations that express the conserved currents in terms of the fluid dynamical variables. Since fluid dynamics is an effective long-wave description, the constitutive equations are naturally expressed as a derivative expansion in the fluid dynamical fields. The fluid thermodynamics and symmetries determine this expansion up to a finite set of unknown independent transport coefficients which in turn are determined by the underlying field theory. Moreover if the field theory possesses conformal symmetry (as is the case for the field theories relevant to the AdS/CFT correspondence), both the fluid thermodynamical degrees of freedom and the number of independent coefficients relevant to the derivative expansion reduce significantly.

These ideas were first implemented by Policastro, Son, and Starinets [2] using a "Kubo formula" approach to fluid dynamics, which relates the hydrodynamic pole behavior of certain correlators to the fluid dynamical transport coefficients. The authors of [2] were for example able to compute the viscosity of a general strongly coupled QFT with a gravitational dual. The gauge theory side equivalent of this computation is not known to this date since such a computation would require non-perturbative method yet to be discovered. This method was later generalized to extract information about the hydrodynamical behavior about more general field theories (see [3] for a nice review and an extensive collection of references.) Perhaps the most remarkable result was the viscosity bound conjecture which states that for any sensible relativistic quantum field theory [4]

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi} \tag{1.1}$$

where η is the viscosity and s is the entropy density. In general $\eta/s \gg 1$ for weakly coupled theories [3], however, as we enter the strongly coupled regime $\eta/s \to \mathcal{O}(1)$. The viscosity bound tells us that a liquid with a given volume density of entropy cannot be arbitrarily close to being a perfect fluid.

Recently a new approach to the duality between fluid dynamics and gravity has emerged. Since gravity on anti-de Sitter backgrounds is dual to an effective fluid description of certain QFTs, the dynamics of the fluid should directly be encapsulated in the equations of gravity. It should therefore, so to speak, be possible to directly read of the dynamical properties of the dual fluid by examining the equations of gravity, which in general would have to be solved in higher (than $D = 4$) dimensions. Moreover, it follows that the stationary solutions of gravity should be mapped to stationary boundary fluid configurations through the correspondence. This new approach was first implemented to examine stationary black holes by the paper [5]. This was later extended to include dynamics in the paper [6] which was generalized to different cases in a set of papers [7, 8, 9, 10] (and others.) The aim of this thesis is to explore this new framework for the fluid/gravity correspondence. We will explain how it is possible to extract informations about the transport coefficients directly from Einstein's equations and examine how the result $\eta/s = 1/4\pi$ is reproduced.

Outline of the thesis

The thesis is structured as follows:

- In chapter 2 we explain the fundamental concepts needed to understand the AdS/CFT correspondence. This includes basic introductions to conformal field theories in curved space, $\mathcal{N} = 4$ Super Yang-Mills, and anti-de Sitter spaces.
- In chapter 3 we account for the AdS/CFT correspondence and how it is possible to compute boundary CFT related quantities from the dual gravitational theory on AdS. Moreover we explain how the fluid/gravity correspondence is a natural corollary to the AdS/CFT correspondence.
- In chapter 4 we explain how it is possible to use conformal fluid dynamics as a long-wave effective description of a CFT. We explain how fluid dynamics can be expressed as a certain derivative expansion in various thermodynamical quantities and write down the most general expansion for a conformal fluid up to second order in the derivatives.
- In chapter 5 we utilize the AdS/CFT correspondence to compute the dual fluids of certain solutions to Einstein's equations. We show how Einstein's equations exactly are equivalent to the fluid dynamical equations governing the dual boundary fluid. We explain how the derivative expansion from chapter 4 is reproduced and show how it is possible to extract the coefficients of this expansion from a purely gravitational computation.
- In chapter 6 we introduce a class of exact (rotating) black hole solutions to Einstein's equation in anti-de Sitter. We will show how these black holes fit into the fluid/gravity scheme, both locally and globally, and demonstrate how the (special case of) five dimensional multi-charged rotating black hole also fits with the predictions of the fluid/gravity correspondence.
- Finally, in chapter 7 we conclude on the results obtained in the thesis and discuss some of the possible present and future applications of the fluid/gravity correspondence.

2

Conformal field theories and Anti-de Sitter spaces

2.1 Conformal Transformations

Conformal field theory (especially conformal field theory in two dimensions) is a *vast* subject. It is way beyond the scope of this thesis to go into the details of (quantum) conformal field theories. Here we shall be content with giving a brief review of the basic ideas of conformal field theory and the concepts needed for the thesis and in particular needed for understanding conformal fluids. We start by a review of the theory of conformal transformations in flat Minkowski space.

2.1.1 Conformal Symmetry in Flat Minkowski Space

Consider d dimensional Minkowski space M_d. The conformal transformations of Minkowski space (M_d, η_{ab}) are the (smooth) coordinate transformations $x^a \to \tilde{x}^a$ that leave the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,\cdots,+)$ invariant up to some positive non-vanishing function $\Omega^{-2}(x)$:

$$\tilde{g}_{ab} = \Omega^{-2}\eta_{ab} \tag{2.1}$$

Therefore, the conformal transformations are the transformations that leave the Minkowski flat space structure invariant up to a *local* change of scale. We immediately see that the Poincaré transformations (the semi direct product of translations and (pseudo)-rotations making up the Poincaré group $P(d)$, see e.g. [11]) of M_d are all conformal since they exactly correspond to $\Omega = 1$. As a trivial example of a conformal transformation which is not Poincaré, consider the (global) scale transformation

$$x^\mu \to \tilde{x}^\mu = \lambda x^\mu \ , \ \lambda \in \mathbb{R} \tag{2.2}$$

This is clearly a conformal transformation with $\Omega = \lambda$. Notice that the collection of conformal transformations make up a group denoted $C(1, d-1)$. This group is known as the *conformal group*. According to the above we have $P(d) \subset C(1, d-1)$. In order to work out the Lie algebra of $C(1, d-1)$ one must work out the infinitesimal form of the conformal transformations. By looking at the equation (2.1) for infinitesimal transformations, it is possible to work out the infinitesimal form of a general conformal transformation (we refer to [12] for the details). The infinitesimal conformal transformations fall in four types, two of them being Poincaré transformations (translations (T) and Lorentz rotations (R))

$$\text{(T)}\ \ x^\mu \to x'^\mu = x^\mu + a^\mu, \qquad \text{(R)}\ \ x^\mu \to x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \ \ (\omega_{\mu\nu} = -\omega_{\nu\mu}) \tag{2.3}$$

while the two remaining are new

$$\text{(D)} \quad x^\mu \to x'^\mu = (1+\beta)x^\mu, \quad \text{(SCT)} \quad x^\mu \to x'^\mu = x^\mu + 2(\boldsymbol{x}\cdot\boldsymbol{b})x^\mu - b^\mu x^2 \tag{2.4}$$

The first transformation is known as a dilation (D) and is recognized as being the infinitesimal form of the scale transformation (2.2). The last transformation is the (infinitesimal) special conformal transformation (SCT). Now let P_μ denote the generator of translations, $M_{\mu\nu}$ the generator of Lorentz rotations, D the generator of dilations and K_μ the generator of the SCTs. Using the infinitesimal transformations (2.3) and (2.4) it is straight forward to work out the algebra of $C(1,d-1)$: In addition to the usual commutators from the Poincaré algebra we find the commutators

$$[D, P_\mu] = iP_\mu, \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \tag{2.5}$$

$$[M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu), \qquad [D, K_\mu] = -iK_\mu \tag{2.6}$$

with the rest of the commutators $[D,*] = [K_\mu,*] = 0$.

By constructing a new set of generators J_{AB} from P_μ, $M_{\mu\nu}$, D and K_μ and using the above algebra, it possible to cast the algebra of the conformal group into the form [12]

$$[J_{AB}, J_{CD}] = i\big(\overline{\eta}_{AD}J_{BC} + \overline{\eta}_{BC}J_{AD} - \overline{\eta}_{AC}J_{BD} - \overline{\eta}_{BD}J_{AC}\big) \tag{2.7}$$

with $\overline{\eta}_{AB} \equiv \mathrm{diag}(-,+,+,\ldots,+,-)$. This algebra is exactly that of $SO(2,d)$ and we therefore conclude that $C(1,d-1)$ is isomorphic to SO$(2,d)$, C$(1,d-1) \cong SO(2,d)$.

2.1.2 Conformal Symmetry in Curved Space

Similarly a conformal transformation in curved space M is a transformation that leaves the (now generally non-flat) metric invariant up to a *local* scale. However, it turns out that it is more natural not to define conformal symmetry in terms of diffeomorphisms but instead in terms of (conformal) equivalence classes of the space of metrics on M. Two metrics $\tilde{\mathrm{g}}$ and g are said to be *conformally equivalent* if there exists a non-vanishing function $\Omega \equiv e^\phi$ on M so that

$$\tilde{\mathrm{g}}_{\mu\nu} = \Omega^{-2}\mathrm{g}_{\mu\nu} = e^{-2\phi}\mathrm{g}_{\mu\nu} \tag{2.8}$$

We emphasize that this equation does not need to be related to a diffeomorphism - this is a transformation of the metric field, not the coordinates. The two spacetimes (M,g) and $(M,\tilde{\mathrm{g}})$ therefore agree on "angles" between vectors but not on the "length" of vectors. Especially, the conformal transformation leaves the light cones of the two spacetimes (M,g) and $(M,\tilde{\mathrm{g}})$ invariant and they therefore have identical causal structure [13]. It is possible to express various geometrical quantities of respectively (M,g) and $(M,\tilde{\mathrm{g}})$ in terms of each other and the function Ω. If we let ∇_ν and $\tilde{\nabla}_\mu$ denote respectively the connections w.r.t. $\mathrm{g}_{\mu\nu}$ and $\tilde{\mathrm{g}}_{\mu\nu}$ it holds that [13] (in fact such a relation holds for any two connections)

$$\nabla_\mu t^\nu = \tilde{\nabla}_\mu t^\nu - C^\nu{}_{\mu\rho}t^\rho \tag{2.9}$$

for any vector t^μ and where the tensor $C^\rho{}_{\mu\nu}$ is given by

$$C^\rho{}_{\mu\nu} = \frac{1}{2}\tilde{\mathrm{g}}^{\rho\lambda}\big[\nabla_\mu\tilde{\mathrm{g}}_{\nu\lambda} + \nabla_\nu\tilde{\mathrm{g}}_{\mu\lambda} - \nabla_\lambda\tilde{\mathrm{g}}_{\mu\nu}\big] \tag{2.10}$$

Since ∇_μ is the connection w.r.t. $\mathrm{g}_{\mu\nu}$ it holds $\nabla_\mu\mathrm{g}_{\nu\rho} = 0$ so $\nabla_\mu\tilde{\mathrm{g}}_{\nu\rho} = -2\Omega^{-3}\mathrm{g}_{\nu\rho}\nabla_\mu\Omega$. This enables us to express $C^\rho{}_{\mu\nu}$ in terms of $\Omega = e^\phi$

$$\begin{aligned} C^\rho{}_{\mu\nu} &= -\Omega^{-3}\tilde{\mathrm{g}}^{\rho\lambda}\big[\mathrm{g}_{\nu\lambda}\nabla_\mu\Omega + \mathrm{g}_{\mu\lambda}\nabla_\nu\Omega - \mathrm{g}_{\mu\nu}\nabla_\lambda\Omega\big] \\ &= -2\delta^\rho_{(\mu}\nabla_{\nu)}\log\Omega + \mathrm{g}_{\mu\nu}\mathrm{g}^{\rho\lambda}\nabla_\lambda\log\Omega \\ &= -2\delta^\rho_{(\mu}\nabla_{\nu)}\phi + \mathrm{g}_{\mu\nu}\mathrm{g}^{\rho\lambda}\nabla_\lambda\phi \end{aligned} \tag{2.11}$$

Now the relation (2.9) directly shows how the Christoffel symbols of (M, g) are related to those of $(M, \tilde{\mathrm{g}})$. We simply have

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \tilde{\Gamma}^{\rho}{}_{\mu\nu} - C^{\rho}{}_{\mu\nu} \\ &= \tilde{\Gamma}^{\rho}_{\mu\nu} + 2\delta^{\rho}_{(\mu}\nabla_{\nu)}\phi - \mathrm{g}_{\mu\nu}\mathrm{g}^{\rho\lambda}\nabla_{\lambda}\phi\end{aligned} \tag{2.12}$$

Using this, it is straight forward to express the Ricci tensor of (M, g) in terms of that of $(M, \tilde{\mathrm{g}})$. We have

$$\begin{aligned}\tilde{R}_{\mu\nu} = R_{\mu\nu} &+ (d-2)\nabla_{\mu}\nabla_{\nu}\phi + \mathrm{g}_{\mu\nu}\mathrm{g}^{\rho\lambda}\nabla_{\rho}\nabla_{\lambda}\phi \\ &+ (d-2)(\nabla_{\mu}\phi)(\nabla_{\nu}\phi) - (d-2)\mathrm{g}_{\mu\nu}\mathrm{g}^{\rho\lambda}(\nabla_{\rho}\phi)(\nabla_{\lambda}\phi)\end{aligned} \tag{2.13}$$

From this we can calculate the transformed Ricci scalar. Here we must be a bit careful and remember that in order to to compute $\tilde{R}$ we must contract with the the metric $\tilde{\mathrm{g}}^{\mu\nu} = e^{2\phi}\mathrm{g}^{\mu\nu}$:

$$\tilde{R} = e^{2\phi}\big[R + 2(d-1)\mathrm{g}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi - (d-1)(d-2)\mathrm{g}^{\mu\nu}(\nabla_{\mu}\phi)(\nabla_{\nu}\phi)\big] \tag{2.14}$$

The relations between other conformally related geometrical quantities are derived in a similar manner see e.g. [14].

2.2 Conformal Theories

2.2.1 Flat space

In flat space, classical conformal field theories arise from the representations of the conformal algebra (2.5). Let $\Phi(x)$ denotes a generic (irreducible) field representation of the conformal group. Following the well-known method, it is possible to examine how the generators of the conformal group act on $\Phi(x)$. In addition to the usual Poincaré action

$$\begin{aligned}P_{\mu}\Phi(x) &= -i\partial_{\mu}\Phi(x) \\ M_{\mu\nu}\Phi(x) &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\Phi(x) + \Sigma_{\mu\nu}\Phi(x)\end{aligned} \tag{2.15}$$

we also have action from the dilation generator D and the generator of SCTs. They act like [12]

$$\begin{aligned}D\Phi(x) &= (-x^{\nu}\partial_{\nu} + i\Delta)\Phi(x) \\ K_{\mu}\Phi(x) &= \big\{i\Delta x_{\mu} - x^{\nu}\Sigma_{\mu\nu} - 2ix_{\mu}x^{\nu}\partial_{\nu} + ix^2\partial_{\mu}\big\}\Phi(x)\end{aligned} \tag{2.16}$$

Here $\Sigma_{\mu\nu}$ denotes the spin i.e. a matrix representation of the Lorentz group while Δ is a real number known as the scaling dimension of the field Φ. The general transformation of the field Φ under conformal transformations can now schematically be written as

$$\Phi'(x) = \exp(-\sum_{g} i\omega_g G_g)\Phi(x) \tag{2.17}$$

where G_g and ω_g respectively denote the generator and transformation parameter associated with the g symmetry. It is now possible to show that under a general finite conformal transformation the field Φ transforms as (assume that the field has no spin)

$$\Phi(x) \to \Phi'(x') = [\Omega(x)]^{\Delta}\Phi(x) \tag{2.18}$$

This explains the name for Δ. Fields with spin will of course also have their spin components transformed in a non-trivial manner. A conformal theory is then exactly a theory that takes the extra conformal symmetry into account i.e. scaling of fields.

Especially a conformally invariant theory is a theory whose action is left invariant by all conformal transformations (2.1), (2.15), (2.16).

In order to make this a bit more concrete, we consider conformal transformations of a rather general system. For simplicity we will only consider scale transformations.[1] We consider a physical theory consisting of a collection of fields ψ_i defined on d-dimensional flat space govern by the generic action

$$S = \int \mathrm{d}^d x \; \mathcal{L}\big(\psi_i(x), \partial_\mu \psi_i(x)\big) \tag{2.19}$$

Now suppose that we perform a scale transformation of the physical system by some factor λ. Assuming that the scaling dimension of the field ψ_i is denoted Δ_i, the scale transformation of the full physical system is given by

$$x^\mu \to x'^\mu = \lambda x^\mu \quad \text{and} \quad \psi_i(x^\mu) \to \psi_i'(x'^\mu) = \lambda^{-\Delta_i} \psi_i(x) \tag{2.20}$$

Computing the transformed action $S' \equiv S[\phi']$ is easy

$$S' = \lambda^d \int \mathrm{d}^d x \; \mathcal{L}\big(\lambda^{-\Delta_i} \psi_i(x), \lambda^{-\Delta_i - 1} \partial_\mu \psi_i(x)\big) \tag{2.21}$$

This expression is obtained by a change in integration variables, using the transformation rules (2.20) and finally that the form of the Jacobian $|\partial x'^\mu / \partial x^\nu| = \lambda^d$. As a simple example, let us consider a scalar field theory (here m is a parameter which can be thought of as a mass term)

$$S = \int \mathrm{d}^d x \; \frac{1}{2}\big(\partial_\mu \psi \partial^\mu \psi - m^2 \psi^2\big) \tag{2.22}$$

We see that theory is scale invariant only, $S = S'$, if $m = 0$ and $\Delta_\psi = 1 - d/2$. The massless condition $m = 0$ makes sense since the presence of a mass introduces a characteristic length scale into the system and the theory can thus not be scale invariant. Note that, it is of course not a coincidence that exactly $\Delta_\phi =$ *the natural length dimension of the field* leaves the (massless) scalar field action S invariant under scaling transformations - this is simply because the action has vanishing length dimension. These arguments clearly extend to more general field theories [15]. We conclude that, just like a vector field has a direction which correspondingly must transform under Lorentz transformations, a field also has a length (or scaling) dimension which means that the field must transform according to this scaling behavior under conformal transformations.

2.2.2 Conformal theories in curved backgrounds

Having explained how classical conformal field theories arise from representations of the conformal group and how they essentially correspond to theories that are invariant under rescalings of the coordinates and the fields, we will now explain how conformal field theories are defined on curved backgrounds. There are several more or less mathematical rigorous definitions of conformal field theories in curved space. Perhaps the best way to understand conformal field theories on curved spaces would be in terms of fiber bundles, much like we do for spinors on curved backgrounds [13]. Here we will be content with the definition given by [13] which is expressed in terms of the equations of motion.

[1] Of course invariance under scale transformations and the full conformal group is not the same. However, usually if a theory is scale invariant it will also conformally invariant.

Suppose that ψ is some matter field living on a space-time $(M, \mathrm{g}_{\mu\nu})$, possible coupled to gravity, with equation of motion $\mathcal{H}(\mathrm{g}_{\mu\nu}, \psi) = 0$. Now suppose that perform a conformal transformation:

$$\mathrm{g}_{\mu\nu} \to \tilde{\mathrm{g}}_{\mu\nu} = e^{-2\phi}\mathrm{g}_{\mu\nu} \quad \text{and} \quad \psi \to \tilde{\psi} = e^{s\phi}\psi \tag{2.23}$$

where s is some number $s \in \mathbb{R}$ called the *conformal weight* (conventionally denoted by Latin letters) of the field ψ. The conformal weight is the curved space generalization of the scaling dimension Δ. The theory described by $\mathrm{g}_{\mu\nu}$ and ψ is said to be conformally invariant if the equations of motion are left invariant under the conformal transformation i.e, $\mathcal{H}(\tilde{\mathrm{g}}_{\mu\nu}, \tilde{\psi}) = 0$ for all ϕ. Equivalently conformal invariance could be expressed on the level of invariance of the (matter) action under conformal transformations, see below.
As an example, consider a free scalar field conformally coupled to gravity with equation of motion

$$\mathcal{H}_{\text{scalar}}[\mathrm{g}^{\mu\nu}, \psi] = \mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu\psi - \frac{d-2}{4(d-1)}R\psi = 0 \tag{2.24}$$

where R is the Ricci scalar of the metric $\mathrm{g}_{\mu\nu}$. This theory is conformally invariant if and only if the scalar field ψ is assigned conformal weight $s = d/2 - 1$. Indeed, using the scaling relations (2.23) and (2.14), we can show that

$$\mathcal{H}_{\text{scalar}}[e^{-2\phi}\mathrm{g}^{\mu\nu}, e^{(d/2-1)\phi}\psi] = e^{(1+d/2)\phi}\,\mathcal{H}_{\text{scalar}}[\mathrm{g}^{\mu\nu}, \psi] = 0 \tag{2.25}$$

This equation is the curved space, conformally invariant generalization of the Klein-Gordon equation. It is also possible to show that the curved space Maxwell equations $\nabla_\nu F^{\mu\nu} = 0$ are conformally invariant in four spacetime dimensions [13].

Finally, we examine how the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.26}$$

behaves under conformal transformations. Suppose that $T^{\mu\nu}$ transforms with conformal weight w i.e. $T^{\mu\nu} \to \tilde{T}^{\mu\nu} = e^{w\phi}T^{\mu\nu}$. Notice that the weight w needs not to be the same as the conformal weight of the underlying field of fields. We then have

$$\begin{aligned}
\tilde{\nabla}_\mu\tilde{T}^{\mu\nu} &= \tilde{\nabla}_\mu(e^{w\phi}T^{\mu\nu}) \\
&= \partial_\mu(e^{w\phi}T^{\mu\nu}) + \tilde{\Gamma}^\mu_{\mu\lambda}(e^{w\phi}T^{\lambda\nu}) + \tilde{\Gamma}^\nu_{\mu\lambda}(e^{w\phi}T^{\mu\lambda}) \\
&= \nabla_\mu(e^{w\phi}T^{\mu\nu}) + C^\mu{}_{\mu\lambda}e^{w\phi}T^{\lambda\nu} + C^\nu{}_{\mu\lambda}e^{w\phi}T^{\mu\lambda} \\
&= e^{w\phi}\big\{\nabla_\mu T^{\mu\nu} - (d-w+2)T^{\mu\nu}\nabla_\mu\phi + T\nabla^\nu\phi\big\}
\end{aligned} \tag{2.27}$$

We therefore see that the conservation equation (2.26) is generally not conformal invariant. However, we see that if the trace $T = \mathrm{g}_{\mu\nu}T^{\mu\nu}$ of the stress tensor vanishes (this is clearly conformally invariant statement) then the equation (2.26) is conformal invariant if the stress tensor is assigned conformal weight $w = d + 2$. On the other hand, assume that the theory is conformally invariant. We then have $\tilde{S}_M \equiv S_M[\tilde{\mathrm{g}}^{\mu\nu}, \tilde{\psi}_i] = S_M \equiv S_M[\mathrm{g}^{\mu\nu}, \psi_i]$ under conformal transformations and we therefore have for the stress tensor

$$\tilde{T}^{\mu\nu} = -\frac{2}{\sqrt{\tilde{g}}}\frac{\delta\tilde{S}_M}{\delta\tilde{\mathrm{g}}_{\mu\nu}} = -e^{(d+2)\phi}\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta\mathrm{g}_{\mu\nu}} = e^{(d+2)\phi}T^{\mu\nu} \tag{2.28}$$

So, a conformally invariant theory gives rise to a stress tensor which transforms with conformal weight $w = d + 2$ under conformal transformations. Moreover, if the theory is conformally invariant, then the equation (2.26) must be conformally invariant, so from the equation (2.27) we see that conformal invariance of a theory must imply $T = 0$. Notice that we may also infer the tracelessness of the stress tensor directly from a variational argument: Consider an infinitesimal conformal transformation $g_{\mu\nu} \to e^{-2\delta\phi} g_{\mu\nu}$. Under such a transformation we clearly have $\delta g_{\mu\nu} = -2 g_{\mu\nu}\delta\phi$, so $T \equiv g_{\mu\nu}T^{\mu\nu} \sim g_{\mu\nu}\delta S_M/\delta g_{\mu\nu} = -\frac{d^2}{2}\delta S_M/\delta\phi = 0$.

Finally suppose that a conformal theory contains a conserved current J^μ and consider a conformal transformation $g_{\mu\nu} \to e^{-2\phi} g_{\mu\nu}$. Since the theory is conformal, the conservation equation

$$\nabla_\mu J^\mu = 0 \tag{2.29}$$

must also hold in the transformed metric i.e.

$$\begin{aligned}
\tilde{\nabla}_\mu \tilde{J}^\mu &= \tilde{\nabla}_\mu (e^{w\phi} J^\mu) \\
&= w e^{w\phi} (\nabla_\mu \phi) J^\mu + e^{w\phi} \tilde{\nabla}_\mu J^\mu \\
&= w e^{w\phi} (\nabla_\mu \phi) J^\mu + e^{w\phi} \nabla_\mu J^\mu + e^{w\phi} C^\mu{}_{\mu\lambda} J^\lambda \\
&= \nabla_\mu J^\mu + (w - d) e^{w\phi} (\nabla_\mu \phi) J^\mu
\end{aligned} \tag{2.30}$$

We therefore conclude that the current transforms as

$$J^\mu \to e^{d\phi} J^\mu \tag{2.31}$$

under conformal transformations.

2.2.3 Quantum conformal theories

As usual, in the quantum theory the fields are promoted to operators and the physical theory is then expressed in terms of the correlators of relevant operator products. Although there are special methods and theorems pertaining to quantum CFTs the general ideas are the same as with "normal" QFTs.

Many interesting field theories are scale invariant. Examples include four dimensional Yang-Mills theory [16]. However, this invariance is typically broken in the quantum theory. This can be understood directly from the method of renormalization. A renormalizable quantum field theory is *defined* in terms of some cutoff which explicitly brakes scale invariance. This procedure implies that the coupling constant g of the theory is running i.e. it depends directly on the scale μ of the system. Conventionally this scale dependence of the QFT is expressed in terms of the β-function

$$\beta(g, \mu) = g \frac{\mathrm{d}g}{\mathrm{d}\mu} \tag{2.32}$$

If the beta functions of a quantum field theory vanish, usually at particular values of the coupling parameters, then the full quantum theory is scale-invariant. Almost all scale-invariant QFTs are also conformally invariant [17].

2.3 Super conformal theories

It is natural to ask whether the conformal group can be combined with supersymmetry. This is indeed possible in dimensions $d \leq 6$ [16]. In addition to the conformal generators $(P_\mu, K_\mu, D, M_{\mu\nu})$ and the supersymmetry generators Q, the superconformal algebra contains a set of fermionic generators S and (sometimes) R-symmetry generators (R-symmetry is a symmetry transforming different supercharges of a supersymmetric theory into each other). Schematically the commutator relations of the superconformal algebra (in addition to those of the conformal algebra) are given by

$$\begin{aligned} &[D,Q] = -\frac{i}{2}Q; \quad [D,S] = \frac{i}{2}S; \quad [K,Q] \cong S; \quad [P,S] \cong Q \\ &\{Q,Q\} \cong P; \quad \{S,S\} \cong K; \quad \{Q,S\} \cong M + D + R \end{aligned} \tag{2.33}$$

The exact form of the algebra depends of the spacetime dimension of on the R-symmetry group.

2.3.1 $\mathcal{N} = 4$ SYM

Here we consider a specific superconformal theory, namely supersymmetric Yang-Mills (SYM) theory with four supersymmetries in $d = 4$ dimensions. This theory has been studied in great detail. The $\mathcal{N} = 4$ SYM theory in four dimensions contains one gauge boson A_μ, six scalars ϕ^I and four fermions $\chi_{\alpha i}$, $\chi_{\dot{\alpha}\bar{i}}$ (we refer to [18] for a much more detailed account of the field content and supersymmetry properties of $\mathcal{N} = 4$ SYM). Moreover, the theory has a global SO(6) R-symmetry.

The Lagrangian is schematically of the form [19]

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\left[F^2 + (D\phi)^2 + \overline{\chi}\not{D}\chi + \sum_{IJ}[\phi^I,\phi^J]^2 + \overline{\chi}\Gamma^I\phi^I\chi\right] + \theta\mathrm{Tr}[F \wedge F] \tag{2.34}$$

again we refer to [18] for the details. The $\mathcal{N} = 4$ SYM theory in four dimensions has the very special property that its β-function is zero to all orders [19]. It is therefore a conformally invariant theory.

2.4 Anti de-Sitter space

We will now introduce $D = d + 1$ dimensional anti-de Sitter space and important related concepts. Anti de-Sitter space is a maximally symmetric space of Lorentzian signature with constant negative scalar curvature. Moreover anti-de Sitter space is a vacuum solution of Einstein's field equations with negative cosmological constant Λ.

2.4.1 Defining Anti de-Sitter space

Anti de-Sitter space is conventionally defined as an embedding of $\mathbb{R}^{(2,d)}$, that is, $(Y^0, \cdots, Y^d, Y^{d+1}) \in \mathbb{R}^{D+1}$ equipped with pseudo-metric

$$\overline{\eta}_{AB} = \mathrm{diag}(-,+,+,\cdots,+,-) \tag{2.35}$$

The embedding space is naturally equipped with the "length" squared

$$Y^2 \equiv \overline{\eta}_{AB}Y^AY^B = -(Y^0)^2 + \sum_{i=1}^{d}(Y^i)^2 - (Y^{d+1})^2 \tag{2.36}$$

In this space, D-dimensional anti-de Sitter space (which we from now on will denote AdS_D) is now defined as being the "sphere" (equipped with the induced metric):

$$\mathrm{AdS}_D = \{Y \in \mathbb{R}^{(2,d)} \mid Y^2 = -L^2\} \tag{2.37}$$

The parameter $L \in \mathbb{R}$ is called the radius of the anti-de Sitter space. Of course anti-de Sitter spaces with different radii are diffeomorphic. Often we will therefore set L to unity since it is easily reintroduced into the equations if needed. In a similar fashion we would define ordinary de-Sitter space dS_D as the locus of $Y^2 = L^2$. Most of the manipulations below also apply to dS_D, however, we shall only be concerned with anti-de Sitter spaces in this thesis.

As we show below, AdS_D of radius L is a maximally symmetric solution to Einstein's vacuum equation with a negative cosmological constant given by $\Lambda = -\frac{1}{2}(D-1)(D-2)L^{-2}$. Before we do this, let us remind ourselves of the equations of gravity.

2.4.2 AdS_D as a gravitational vacuum solution with a negative cosmological constant

Gravity in D spacetime dimensions with a cosmological constant Λ is governed by the action

$$S = S_{H,\Lambda} + S_M \tag{2.38}$$

where S_M describes the action of non-gravitational matter and $S_{H,\Lambda}$ is the Einstein-Hilbert action with a cosmological term given by [14] (in principle there will also be a surface term in this action, such a term is, however, not important here and we will thus ignore it)

$$S_{H,\Lambda} = \frac{1}{16\pi G_D}\int \mathrm{d}^D x\sqrt{|G|}(R - 2\Lambda) \tag{2.39}$$

Here R is the Ricci scalar and G_{AB} is the bulk metric (we reserve $\mathrm{g}_{\mu\nu}$ for the field theory metric on the boundary). In vacuum $S_M = 0$ and variation of S_H leads to the Einstein equation

$$\mathcal{E}_{AB} \equiv R_{AB} - \frac{1}{2}G_{AB}R + \Lambda G_{AB} = 0 \tag{2.40}$$

This equation implies that $R = 2D/(D-2)\Lambda$ and especially

$$R_{AB} = \frac{2\Lambda}{D-2}G_{AB} \tag{2.41}$$

The metrics of the type $R_{AB} \propto \mathrm{G}_{AB}$ are collectively known as Einstein metrics. We will now show that AdS_D as it is defined in equation (2.37) solves (2.40).
First, we introduce a new set of coordinates (ρ, x^A) on $\mathbb{R}^{(2,d)}$ by

$$Y^0 = \rho\frac{1+x^2}{1-x^2}, \quad Y^A = \rho\frac{2x^A}{1-x^2}, \quad \text{for } A = 1,\cdots,d+1 \tag{2.42}$$

where we have defined[2]

$$x^2 \equiv (x^1)^2 + \cdots + (x^n)^2 - (x^{d+1})^2 \equiv \eta_{AB}x^Ax^B \tag{2.43}$$

These coordinates split up $\mathbb{R}^{(2,d)}$ in a radial ρ part and a x^A part describing AdS_D. In other words, the coordinates $(x^1,\cdots,x^{d+1})$ parametrizes AdS_D, this is simply because $y^2 = -\rho^2$, as is easy to check. Now, it is a straight forward exercise to express $\mathrm{d}y^0$ and $\mathrm{d}y^A$ in terms of $\mathrm{d}\rho$ and $\mathrm{d}x^A$ and substitute them into the metric $\mathrm{d}s^2 = -(\mathrm{d}y^0)^2 + \sum_{i=1}^d(\mathrm{d}y^i)^2 - (\mathrm{d}y^{d+1})^2$. In terms of the coordinates (ρ, x^A) the metric of the embedding space then takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\rho^2 + \frac{4\rho^2}{(1-x^2)^2}\mathrm{d}x^2 \tag{2.44}$$

[2] Notice that unconventionally the time direction is assigned to the last coordinate (so $\eta_{AB} = \mathrm{diag}(+,+,\cdots,-)$).

This means that the metric of AdS$_D$ with AdS-scale $\rho^2 = L^2$ is given by the very simple expression

$$G_{AB} = \frac{4L^2}{(1-x^2)^2}\,\eta_{AB} \tag{2.45}$$

We therefore conclude that Anti de-Sitter space AdS$_D$ is conformally flat meaning that the metric can be written

$$G_{AB} = e^{\phi}\,\eta_{AB} \tag{2.46}$$

for some function ϕ. In our case the function ϕ is takes the form

$$\phi = \log 4b^2 - 2\log(1-x^2) \tag{2.47}$$

It is straight forward to write down an expression for the Ricci tensor of a conformally flat metric using the usual expressions for the Christoffel symbols and the Ricci tensor

$$\begin{aligned}\Gamma^C_{AB} &= \frac{1}{2}G^{CD}\big\{\partial_A G_{BD} + \partial_B G_{DB} - \partial_D G_{AB}\big\}\\ R_{AB} &= \partial_C\Gamma^C_{AB} + \Gamma^C_{AB}\Gamma^D_{CD} - (B \longleftrightarrow D)\end{aligned} \tag{2.48}$$

The Christoffel symbols of the conformally flat metric (2.46) are given by

$$\Gamma^C_{AB} = \frac{1}{2}\big(\delta^B_A\partial_B\phi + \delta^C_B\partial_A\phi - \eta_{AB}\partial^C\phi\big) \tag{2.49}$$

and plugging this into the general expression for the Ricci tensor, we find that the Ricci tensor for the conformally flat metric (2.46) is given by the simple expression

$$R_{AB} = \Big[1 - \frac{D}{2}\Big]\big(\partial_A\partial_B\phi - \frac{1}{2}\partial_A\phi\partial_B\phi\big) + \frac{1}{2}\eta_{AB}\Big[\Big(1-\frac{D}{2}\Big)(\partial\phi)^2 - \partial^2\phi\Big] \tag{2.50}$$

For the function (2.47), we find

$$\partial_A\partial_B\phi = \frac{4}{1-x^2}\eta_{AB} + \frac{8x_Ax_B}{(1-x^2)^2} \quad \text{and} \quad \partial_A\phi\partial_B\phi = \frac{16x_Ax_B}{(1-x^2)^2} \tag{2.51}$$

This is readily plugged into (2.50). We find that

$$R_{\mu\nu} = -4\frac{(D-1)}{(1-x^2)^2}\,\eta_{\mu\nu} = -\left(\frac{D-1}{L^2}\right)\,G_{\mu\nu} \tag{2.52}$$

Comparing this to (2.41), we see that AdS$_D$ indeed is a solution to Einsteins vacuum equation with the negative cosmological constant

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2} \tag{2.53}$$

Having defined AdS$_D$ we can now move on to describing AdS$_D$ in various coordinate systems but first we will discuss the isometry group of AdS$_D$.

2.4.3 The symmetry group of AdS$_\mathrm{D}$

With the definition (2.36) of AdS$_D$, it is (almost) obvious what the symmetry group of AdS$_D$ is. Consider a general transformation of the type

$$Y^A \to Y'^A = \Lambda^A{}_B Y^B \tag{2.54}$$

The collection of matrices $\Lambda^A{}_B$ that leave the quadratic (2.36) invariant makes up the group $\mathrm{SO}(2,d)$. The $\mathrm{SO}(2,d)$ matrices therefore map AdS_D into itself. Note that the $\mathrm{SO}(2,d)$ matrices leave not only the quadratic $\overline{\eta}_{AB}Y^AY^B$ invariant but more generally leave the inner product $\overline{\eta}_{AB}Y_1^AY_2^B$ invariant. The isometry group of $(d+1)$-dimensional anti-de Sitter space is therefore exactly $\mathrm{SO}(2,d)$.
This especially means that a quantum theory on AdS_{d+1} must be $\mathrm{SO}(2,d)$ invariant. Since a d dimensional quantum conformal field theory is invariant under $C(1,d-1)$, the fact that $\mathrm{C}(1,d-1)$ is isomorphic to $SO(2,d)$ is the very first hint that a quantum theory on AdS_{d+1} could be equivalent to a CFT on $\partial\mathrm{AdS}_{d+1}$ (indeed if this were not the case, such an equivalence could not exist).

2.4.4 Coordinates on anti-de Sitter

Due to the high degree of symmetry, AdS_D has several nice coordinate descriptions. Here we look at some of the relevant for this thesis (and the most used in the literature).

Stereographic projective coordinates: These are the coordinates introduced in equation (2.42). The AdS_D metric in these coordinates is given by (2.45).

Static coordinates: We parametrize the "spatial" part of (2.36) with ordinary spherical coordinates (r,μ_i,ϕ_i) (for the explicit form of the coordinates see appendix A) i.e. so that $\sum_{i=1}^d (Y^i)^2 = r^2$. The resulting equation $(Y^0)^2+(Y^{d+1})^2 = L^2+r^2$ is now parametrized by setting

$$Y^0 = \sqrt{L^2+r^2}\ \cos(t/L) \quad \text{and} \quad Y^{d+2} = \sqrt{L^2+r^2}\ \sin(t/L) \tag{2.55}$$

This parametrization shows that AdS_{d+1} has topological structure $S^1\times\mathbb{R}^d$. [3] In the spherical coordinates we have $\sum_{i=1}^d \mathrm{d}(Y^i)^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-1}$, so all in all the metric takes the form

$$\mathrm{d}\hat{s}^2 = -\big(1+r^2/L^2\big)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2/L^2} + r^2\mathrm{d}\Omega_{d-1}^2 \tag{2.56}$$

This clearly shows the static nature of AdS_{d+1} with $\partial/\partial t$ being the Killing field generating time translations. Moreover this reveals that $\mathrm{SO}(2,d)$ has a (compact)

[3] In fact the coordinates (2.55) reveal an unpleasant feature of Anti de-Sitter space; it contains closed time-like curves, see fig. 2.1, and is therefore non-causal. There is, however, a solution to this problem: We simply unwrap the timelike circle S^1, that is, we take $-\infty < t < \infty$ with no identifications (this is the universal covering space of the space (2.36)). In this thesis, when referring to anti-de Sitter space AdS_{d+1}, we will always mean this unwrapped causal version of AdS_{d+1}.

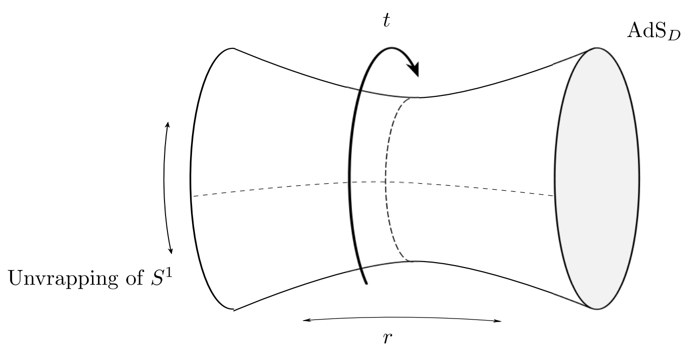

Figure 2.1: *The topological structure of anti de- Sitter.*

subgroup SO(2) $\times$ SO(d) (=rotations of S^1 times rotations of S^d). Some authors prefer to redefine the radial coordinate by setting $r = L\sinh\chi$. Here the metric is

$$\mathrm{d}\hat{s}^2 = -\cosh^2\chi\,\mathrm{d}t^2 + L^2\mathrm{d}\chi^2 + L^2\sinh^2\chi\,\mathrm{d}\Omega_d^2 \tag{2.57}$$

This is the form used in e.g. [16]. These coordinates cover the entire manifold and are therefore also known as global coordinates.

POINCARÉ COORDINATES: Define "lightcone" coordinates u and v by

$$u = \frac{1}{L^2}(Y^0 - Y^d) \quad \text{and} \quad v = \frac{1}{L^2}(Y^0 + Y^d) \tag{2.58}$$

Moreover define $x^i = Y^i/Lu$ and $t = Y^{d+1}/bu$. Here u, x^i and t are independent while v must be determined in terms of these. This is done by substituting them into (2.36):

$$L^4uv + L^2u^2(t^2 - x^2) = L^2\ ,\ x^2 \equiv \sum_{i=1}^{d}(x^i)^2 \tag{2.59}$$

which can now be solved for v. This yields $v = L^{-2}(u^{-1} - u(t^2 - x^2))$ and we obtain

$$Y^0 = \frac{1}{2u}(1 + u^2(L^2 + x^2 - t^2)), \qquad Y^i = Lux^i, \tag{2.60}$$

$$Y^{d+1} = \frac{1}{2u}(1 + u^2(-L^2 + x^2 - t^2)), \qquad Y^d = Lut \tag{2.61}$$

The coordinates u, x^i, t are known as Poincaré coordinates. The differential expressions for the Y^A's in terms of u, x^i, t are easily worked out and plugged into the metric induced from $\mathbb{R}^{(2,d)}$. This gives the metric in terms of the Poincaré coordinates

$$\mathrm{d}\hat{s}^2 = L^2\Big[\mathrm{d}u^2/u^2 + u^2(-\mathrm{d}t^2 + \mathrm{d}x^2)\Big] \tag{2.62}$$

Notice that as opposed to the global coordinates, Poincaré coordinates do not cover the entire manifold. This is seen by noting that the metric expressed in the Poincaré coordinate chart is singular at $u \to 0$. Thus the hyperboloid (fig. 2.1) is divided in two by the hyperplane $L^2u = Y^0 - Y^d = 0$, and one chart ($u > 0$) covers one half of the hyperboloid while the other ($u < 0$) covers the other.
A set of related coordinates are obtained by setting $z \equiv 1/u$. In these coordinates the metric takes the form

$$\mathrm{d}\hat{s}^2 = \frac{L^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2\right) \tag{2.63}$$

There are several other useful coordinate systems which we will not go through here.

Finally, we mention that it is possible to perform a Wick rotation $t \to it$ of anti-de Sitter and in this way obtain an Euclidean version of AdS$_D$ [16]. Effectively this results in replacing $-\mathrm{d}t^2$ with $+\mathrm{d}t^2$ in the various expressions. Of course, the physics and geometry of Euclidean AdS$_D$ is equivalent to that of ordinary AdS$_D$ (by analytical continuation). However, it is particular easy to see the boundary structure of AdS$_D$ in Euclidean signature (see below).[4]

[4]Moreover as usual a quantum theory (here on AdS$_D$) is better defined on spacetimes of Euclidean signature.

2.4.5 The Boundary of AdS$_D$

We will now discuss the boundary of AdS$_D$. We start by stating a well-known fact: The conformal compactification of Minkowski space M_d is given by $\mathbb{R} \times S^{d-1}$ [16]. Intuitively, conformal compactification results in adding a point "at spatial infinity" to Minkowski space. This point, so to speak, closes $\mathbb{R}^{d-1}$ (the spatial part of M_d) to a $(d-1)$-dimensional sphere. We will now explain how the boundary of AdS$_D$ exactly is the conformal compactification of M_d. Perhaps the easiest way to realize this is by considering AdS$_D$ the global coordinates (2.56). In this set of coordinates the boundary is located at $r = \infty$ and we see that it exactly has the topological structure $\mathbb{R}\times S^{d-1}$. This can also be seen by considering the Poincaré coordinates (2.62). Here the boundary corresponds to $u = \infty$ (the remaining coordinates parametrize M_d) and a single point "at infinity" $u = 0$ [5]. Again we therefore see that the boundary precisely corresponds to Minkowski space comformally compactified. Having made this identification, it is possible to show that the isometries of AdS$_D$ ($\sim$ SO$(2, d)$) exactly act as conformal transformations on the boundary [16, 20].

Notice that the metric blows up near the boundary. This is of course not a coincidence. It is because the boundary is located infinitely far away as measured by the AdS$_D$ metric. This fact also shows that a theory defined on the boundary of AdS$_D$ must be conformal (along with the above remark). Indeed, consider Euclidean AdS$_D$ in stereographic projective coordinates

$$G_{AB} = \frac{4}{(1 - x^2)^2}\, \delta_{AB} \tag{2.64}$$

with $x^2 \equiv \sum_{i=A}^{d+1} (x^A)^2 < 1$. Here the boundary consists of the points

$$\sum_{i=A}^{d+1} (x^A)^2 = 1 \tag{2.65}$$

The boundary is therefore S^d which is just the Euclidean version of the conformal compactification of Minkowski space. Again we see that the metric becomes singular on the boundary. This means that the metric G_{AB} does not extend to the boundary. It is therefore impossible to define a metric on the boundary directly from G_{AB}. If we want a metric which does extend to the boundary we can pick a non-vanishing function f which has first order zero on boundary and use $\tilde{G}_{AB} = f^2 G_{AB}$ (this is essentially the same as drawing Penrose diagrams for AdS$_D$). The metric $\tilde{G}_{AB}$ now restricts to a metric on the boundary S^d. However, there is no natural choice for the function f; the function $e^{-2\phi} f$ is just as good a choice of function as f. Such a change in function would induce a conformal transformation $\tilde{G}_{AB} \to e^{-2\phi}\tilde{G}_{AB}$. The boundary metric is therefore only defined modulo conformal transformations.

[5] The fact that $u = 0$ corresponds to a single point can be seen by converting to the alternative Poincaré coordinates (2.63). Here we see that at $u = 0$ ($z = \infty$) the metric vanishes in all directions and $u = 0$ must therefore correspond to a single point.

3

The AdS/CFT correspondence

3.1 String theoretical background

There are five different types of superstring theories: Type IIA, type IIB, type I, heterotic string theory with gauge group SO(32) and heterotic string theory with gauge group $E_8 \times E_8$. All five theories exist in ten spacetime dimensions. The Type IIA and type IIB string theories are collectively known as type II superstring theory. It is not the purpose of this thesis to describe the mathematical details of these theories, however, excellent reviews can be found in e.g. [21, 22, 23].

It has been shown that the five theories can in fact be related through various dualities (with S- and T-duality being the most "famous" of these dualities). This lead Witten (and others) to suggest that the five superstring theories in fact are different faces of one and the same theory known as M-theory. The fundamental interacting objects of M-theory are branes existing in 11 spacetime dimensions rather than the strings of 10 dimensional string theory. For a nice summary of the theories and their interconnecting dualities, see [22].

As described in the next section, all five string theories and M-theory reduce to a (corresponding) supergravity (SUGRA) description. We will focus on the type II string theory and M-theory.

3.1.1 Low energy actions

Since the massive tower of string excitations are not attainable in the low energy limit $\alpha' \to 0$ [1], the low energy string theory behavior is governed by the massless string excitations. As is well known, the massless type II string excitations in the bulk originates from the closed string sector. The massless excitations are described by the dilaton ϕ, the metric tensor G_{AB} (gravity), fields belonging to NS-NS sector (for example the antisymmetric Kalb-Ramond field B_{AB}), fields belonging to the RR sector fields along with their supersymmetric fermionic partners. The low energy effective supergravity action, here presented in the so-called string-frame, takes the

[1] In general the mass scale of the modes is set by $1/\alpha'$, therefore, in the limit of $\alpha' \to 0$ only the massless fields remain.

form [21]

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|G|}\Big[e^{-2\phi}\Big(R + 4G^{AB}\partial_A\phi\partial_B\phi - \frac{1}{12}(H^{(3)})^2\Big) \\ -\frac{1}{2}\sum_n \frac{1}{n!}(F^{(n)})^2 + \cdots\Big] \tag{3.1}$$

here R is the Ricci scalar associated with G_{AB}, $H^{(3)}$ is the field strength of the Kalb-Ramond field $B^{(2)}$

$$H^{(3)} = \mathrm{d}B^{(2)} \ , \tag{3.2}$$

the $F^{(n)}$ are the RR n-form field strengths while the dots represent fermionic terms along with Chern-Simons-like terms of the n-form potentials. It is worth mentioning that for IIA strings (IIB strings) we only have even (odd) values of n. Moreover, for type IIB strings the $n = 5$ field strength is self-dual, $F^{(5)} = *F^{(5)}$.

Since 10 dimensional SUGRA is the low energy limit of string theory, the SUGRA coupling constant κ_{10} is determined by the string length ℓ_s $(= \sqrt{\alpha'})$ and the closed string coupling constant g_s. Moreover 10 dimensional SUGRA theory gives rise to a 10 dimensional theory of gravity (see below) with Newton constant G_{10}. The relation between the various coupling constants is

$$2(\kappa_{10})^2 = 16\pi G_{10} = (2\pi)^7\ell_s^8 g_s^2 \tag{3.3}$$

Alternatively the action can be expressed in the Einstein-frame, obtained by the following Weyl rescaling[2]

$$G^s_{AB} \to G^E_{AB} = e^{-\phi/2}G^s_{AB} \tag{3.4}$$

In the Einstein-frame the action schematically takes the form [20]

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|G|}\left[R - \frac{1}{2}G^{AB}\partial_A\phi\partial_B\phi - \frac{1}{2}\sum_n \frac{1}{n!}e^{a_n\phi}(F^{(n)})^2 + \cdots\right] \tag{3.5}$$

where the factors a_n are easily worked out. The pure gravity part of the Lagrangian $\sqrt{|G|}R$ is exactly recognized as that of Einstein gravity. We therefore conclude that, as stated above, the low energy limit of string theory contains classical Einstein gravity, just as it should. Finally we present the low energy M-theory action which takes the form of (unique) 11 dimensional SUGRA. The bosonic fields of 11 dimensional SUGRA are the metric G_{AB} and a 3-form potential C with associated field strength $K = \mathrm{d}C$. The action has the form [20]

$$S = \frac{1}{(\kappa_{11})^2}\left(\int \mathrm{d}^{11}x\sqrt{|G|}\big(R - \frac{1}{48}K^2\big) - \frac{1}{6}\int C\wedge K\wedge K + \dots\right) \tag{3.6}$$

where the dots represents fermionic terms. Notice that 11 dimensional SUGRA does not contain the dilaton ϕ.

3.1.2 Dp-branes

Branes are an integral part of string theory/M-theory. Branes have many different descriptions and applications. It is not the purpose of this thesis to give a detailed account of brane physics, here we shall be content with a short review of the basic properties of p-branes. A nice reference on p-branes can be found in [24].

[2] We do not write the superscripts s and E in the expressions but the metric in the string-frame action (3.1) is G^s_{AB} and the metric in the Einstein-frame action (3.5) is G^E_{AB}.

The closed string sector description: The first description of p-branes is the so-called solitonic/gravitational brane description. A p-brane is a supergravity soliton-like object which extends over p spatial dimensions (and usually one time-like dimension). Since a p-brane is a $(p+1)$-dimensional object, it naturally couples (electrically) to a $n-1$ form gauge potential with $n=p+2$.[3] The relevant p-brane action from the supergravity actions (3.5), (3.6) therefore is

$$S = \frac{1}{16\pi G_D}\int \mathrm{d}^D x\sqrt{G}\left(R - \frac{1}{2}\partial_A\phi\partial^A\phi - \frac{1}{2n!}e^{a\phi}\left(F^{(n)}\right)^2\right) \tag{3.7}$$

where $F^{(n)}$ is the field strength corresponding to the gauge potential $A^{(n-1)}$, so $F^{(n)} = \mathrm{d}A^{(n-1)}$ (with $a \equiv a_n$). Again we have discarded the fermionic terms as they can be consistently set to zero. Moreover we have ignored the Chern-Simons-like terms as they will not be important for brane solutions which have a high degree of transverse symmetry [20]. Varying the action w.r.t. $G_{\mu\nu}$, $A^{(n)}$ and ϕ gives the equations of motion. We find by straight forward computation

$$\begin{aligned} R^A_{\ B} &= \frac{1}{2}\partial^A\phi\partial_B\phi \\ &\quad + \frac{1}{2n!}e^{a\phi}\left(nF^{AB_2\cdots B_n}F_{BB_2\cdots B_n} - \frac{n-1}{D-2}\delta^A_B(F^{(n)})^2\right) \\ \Box\,\phi &= \frac{a}{2n!}e^{a\phi}(F^{(n)})^2 \\ \partial_A(\sqrt{G}e^{a\phi}F^{AB_2\cdots B_n}) &= 0 \end{aligned}$$

where $\Box$ is the usual Laplacian on scalars. The solution to this set of equations was first worked out in [25]. A complete solution (and a nice derivation) can be found in [20]. Here we will be content with writing down the extremal brane solutions. The extremal branes are the branes where the BPS bound "$M \geq Q$" is saturated resulting in zero temperature and maximal supersymmetry (of the theory living on the brane, see below). They exhibit a $\mathrm{SO}(1,p)\times\mathrm{SO}(d)$ symmetry and therefore have the topological structure of $\mathbb{R}^{(1,p)}\times\mathbb{R}_+\times S^{d-1}$. We therefore introduce a set of p longitudinal coordinates $x^1,\cdots,x^p$, a time coordinate t and a radial coordinate r. The extremal brane solution is now given by

$$\mathrm{d}s^2 = H^{-\frac{d-2}{D-2}}\left(-\mathrm{d}t^2 + \sum_{i=1}^{p}(\mathrm{d}x^i)^2 + H\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{d-1}\right)\right) \tag{3.8}$$

where $\mathrm{d}\Omega_{d-1}$ is the metric on S^{d-1} and where H is a harmonic function in the radial variable r ($h = const.$)

$$H(r) = 1 + \left(\frac{h}{r}\right)^{d-2} \tag{3.9}$$

The $n-1$ gauge potential and dilaton are given by[4]

$$A^{(n-1)} = (H^{-1}-1)\,\mathrm{d}t\wedge\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^p\ , \quad e^{2\phi} = H^a \tag{3.10}$$

It can be shown that the tension of the extremal p-brane is given by [26]

$$T_p = \frac{1}{(2\pi)^p\ell_s^{p+1}g_s} \tag{3.11}$$

[3] There is also the possibility that the brane is magnetically coupled to an $n-1$ form potential with $n = D-p-2$.

[4] For the RR 5-form in IIB string theory we replace $F^{(5)} \to F^{(5)} + *F^{(5)}$ due to the self-duality constraint on $F^{(5)}$.

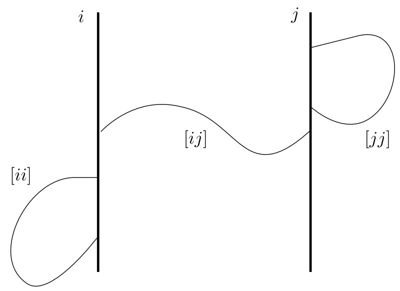

Figure 3.1: *The open string sectors.*

Since the tension goes as $\sim 1/g_s$, we conclude that branes are new dynamical non-perturbative (quantum mechanical) objects. The extremal brane (3.8) corresponds to the classical description of a p-brane in the ground state. Going away from the extremal condition ($Q = M, T = 0$) introduces a horizon in the transverse space (the brane becomes black) and the brane therefore acquires a finite temperature. The non-extremal black brane corresponds to the classical description of a p-brane in an exited state of some definite temperature.

The open string sector description: The second description of branes comes from string theory. Closed strings are not subject to any boundary conditions (since they have no boundaries). However, (the endpoints of) open strings are either subject to Neumann or Dirichlet boundary conditions [21]. Assume that a string is subject to Neumann boundary conditions in the directions $x^1, \cdots, x^p$ and Dirichlet boundary conditions in the remaining directions. The endpoints of the open string will then be confined to moving in the same hyperplane in the directions $x^1, \cdots, x^p$. Such a hyperplane is known as a Dp-plane (where D stands for Dirichlet). Generally a Dp-brane is defined as a hyperplane with p spatial dimensions where open strings can end. It is possible to study the behavior of open strings on Dp-branes. The (endpoints) of the open strings give rise to a massless U(1) gauge field, a set of massless Goldstone scalars and additional fermions + massive string excitations, all living on the Dp-brane. This is generalized to the case where we have more than one Dp-brane. Suppose that we have N parallel Dp-branes. The endpoints of open strings can now start and end on each of the different Dp-branes. Clearly this gives rise to N^2 sectors which can be labeled $[ij]$ (see fig. 3.1). As stated above, the open strings in the $[ii]$ sector give rise to a massless gauge field, however, the $[ij]$ $(i \neq j)$ sector generally contains no massless gauge fields (because of the string tension). However, in the limit where the separation between the N branes goes to zero and the N branes become coincident, the sector $[ij]$ $(i \neq j)$ will contain a massless gauge field. All in all the N coincident branes have N^2 massless gauge fields (along with scalars and fermions and, of course, massive fields) living in their collective world volume. The gauge fields are interacting. This can be understood in the string picture: An open string from the $[ij]$ sector can interact with an open string from the $[jk]$ sector and form a string belonging to the $[ik]$ sector (by joining start- and end-points). The effective interacting theory is determined by the Dirac-Born-Infeld action, schematically of the form [27]

$$S_{\text{DBI}} = -T_{\text{D}p}\text{Tr}\int \text{d}^{p+1}\xi\sqrt{\left|\det\Big(\text{g}_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\Big)\right|} \tag{3.12}$$

where $T_{\text{D}p}$ is the tension of the Dp-brane and $\text{g}_{\mu\nu}$ is the metric on the brane. The DBI action is an effective action where the massive modes of the open strings are integrated out. Notice that the DBI action also contains interaction terms with the massless bulk fields. This means that the DBI action does not only describe the gauge fields living on the brane, but also their interactions with the bulk fields. Physically these interactions correspond to scattering of closed strings on the Dp-brane (closed strings get "cut open" and form open strings ending on the Dp-brane and vice versa). If we write the total effective action of the (N coincident) brane(s) as

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \tag{3.13}$$

where S_{bulk} is the bulk action (closed strings), S_{brane} is the action of the fields living on the brane (open strings), and S_{int} describes the interaction between fields on the brane and bulk fields (open $\leftrightarrow$ closed strings), it can be shown that the open string sector decouples from the closed string sector in the limit $\alpha' \to 0$ with g_s fixed (i.e. the interaction part vanishes in a well-defined manner). Moreover, in this limit only the zero mass fields need to be considered, especially, as mentioned above, the bulk theory reduces to supergravity. It can be shown that the massless open string states are in the adjoint representation of SU(N) and in the limit $\alpha' \to 0$, with g_s fixed, that S_{brane} for the N coincident branes reduces to the action of SU(N) Yang-Mills theory with coupling constant

$$g_{YM}^2/4\pi = g_s(2\pi\ell_s)^{p-3} \tag{3.14}$$

The latter statement can be realized by expanding the DBI action to first order in α'. Especially in the limit $\alpha' \to 0$, with g_s fixed, the theory of the open strings on N coincident $D3$-branes reduces to $3+1$ dimensional $\mathcal{N}=4$ SYM theory with $g_{YM}^2/4\pi = g_s$ [16].

3.2 The AdS/CFT correspondence

The AdS/CFT correspondence [1] (also known as the Maldacena conjecture) comes from the dual description of Dp-branes. Here we consider type IIB string theory which is the setting of the original form of the correspondence. The AdS/CFT correspondence is in essence a duality between the open and closed string sectors pertaining to Dp-branes. The AdS/CFT correspondence is a vast subject, both in itself and its applications. Here we will explain the main points behind the motivation for the correspondence. For nice reviews on the correspondence we refer to e.g. [20, 17, 19].

3.2.1 Statement of the correspondence

Consider a stack of N coincident D3-branes in type IIB string theory. According to the brane description in the closed string sector, the brane is a supergravity background around which we can have quantum mechanical excitations (of the brane). A short computation shows that the excitations near the brane horizon have very small energies (due to an overall redshift factor corresponding to the throat geometry of the brane). This means that in the low energy limit only these excitations will survive. Moreover, at large distances from the brane, in the low energy limit, gravity becomes free [17]. We therefore have two decoupled systems: The low energy excitations near the brane and free SUGRA in the bulk of spacetime. As explained above, excitations of the brane can be understood in terms of open strings with endpoints on the brane. As also explained, for $D3$-branes we have that in the limit $\alpha' \to 0$, the following reductions hold $S_{\text{bulk}} \to S_{\text{SUGRA}}$, $S_{\text{int}} \to 0$ and $S_{\text{brane}} \to S_{\mathcal{N}=4\ \text{SYM}}$. In the open string sector we therefore also have two decoupled systems consisting of respectively free supergravity in the bulk and $\mathcal{N}=4$ SYM on the brane. Since these two viewpoints must describe the same physics, we conjecture that, in the low energy limit ($\alpha' \to 0$), the near horizon excitations of the N coincident branes in the closed string sector must be equivalent to $\mathcal{N} = 4$ Super Yang-Mills with gauge group SU(N). In order to understand the low energy, near horizon brane excitations in the closed string sector (gravity), we must simply examine the background metric (3.8) when $r \to 0$ and $\alpha' \to 0$ (and subsequently scale up the near horizon region in a well-defined manner). This is known as the near horizon limit. The near-horizon limit of the metric (3.8) is given by [20]

$$\mathrm{d}s^2 = \frac{U^2}{L^2}\left[-\mathrm{d}t^2 + \sum_{i=1}^{3}(\mathrm{d}x^i)^2\right] + L^2\frac{\mathrm{d}U^2}{U^2} + L^2\mathrm{d}\Omega_5^2 \tag{3.15}$$

where the L parameter can be related to the string parameters by $L^2 = 2\sqrt{\pi g_s N}\alpha'$. The metric (3.15) is immediately recognized as that of $\mathrm{AdS}_5 \times S^5$ with AdS_5 radius = radius of $S^5 = L$ (cf. equation (2.62)). The notion of near horizon brane excitations in the closed string sector should therefore be understood as string theory in its supergravity limit on the background (3.15). Since, in the near-horizon limit, we are examining spacetime as $r \to 0$, the gauge theory now lives at $U = \infty$ i.e. at the boundary of AdS_5 (we know that the boundary of AdS_5 exactly corresponds to four dimensional Minkowski space which is the space on which the gauge theory is defined). We therefore conclude/conjecture that string theory in its supergravity limit on AdS_5 compactified on S^5 is equivalent to $\mathcal{N} = 4$ SYM in four dimensions (= $\partial\mathrm{AdS}_5$). This is the AdS/CFT correspondence! As explained in §2.4.3 the isometry groups of the two sides of the correspondence are isomorphic, just as they should be. In fact, the AdS/CFT correspondence as it is stated above is in its weakest form. It is generally believed that the AdS/CFT correspondence holds completely in general, also away from the low energy (SUGRA) limit [17].

As mentioned above, going away from the extremal condition introduces a horizon (so the brane becomes black) and therefore a finite temperature. By arguing as above, in the non-extremal case (taking the relevant near-horizon limit), one can see that in the finite temperature case, the AdS/CFT correspondence becomes a duality between gravity on a black brane background in AdS_5 and thermal $\mathcal{N} = 4$ field theory on the boundary of anti de-Sitter [17, 20].

3.2.2 Regimes of validity

As explained above, we have that the common radius of $\mathrm{AdS}_5 \times S^5$ is given by

$$L^4 = \alpha'^2 4\pi g_s N = \alpha'^2 g_{YM}^2 N = \alpha'^2\lambda \tag{3.16}$$

where we have defined

$$\lambda = g_{YM}^2 N \tag{3.17}$$

According to general gauge dynamics lore, the effective coupling constant for large N SU(N) gauge theory is not g_{YM} but rather the 't Hooft coupling $\lambda = g_{YM}^2 N$ [16]. This means that if we consider the large N planar limit ($N \to \infty$) with λ fixed, the string coupling will go to zero, $g_s \to 0$. In other words, we are considering the classical (tree-level, no quantum loops) limit of string theory. In this regime the AdS/CFT correspondence therefore predicts that the full quantum mechanical behavior of the gauge theory in the planar limit can be obtained from string theory in its classical limit! Moreover if we consider the case where we hold the radius parameter L fixed and let $\lambda \to \infty$ (the extreme non-perturbative, strong coupling regime) we obtain the $\alpha' \to 0$ limit of string theory that we considered above. This is the reason why the AdS/CFT correspondence is called a duality; weak coupling on one side corresponds to the strongly coupled regime on the other side and vice versa. All in all we conclude that large N $\mathcal{N} = 4$ SYM in the extreme non-perturbative, strongly coupled regime is dual to classical supergravity on $\mathrm{AdS}_5 \times S^5$. This is indeed a remarkable prediction!

Finally we mention that there also exist AdS/CFT correspondences for the spaces (from M-theory) $\mathrm{AdS}_7(2L) \times S^4(L)$ and $\mathrm{AdS}_4(\frac{1}{2}L) \times S^7(L)$ (in obvious notation) [20] and even in more exotic cases [1, 28]. The most general form of the AdS/CFT correspondence is therefore between some complicated string/M-theory on an anti-de Sitter space [5] compactified on some internal compact manifold X_I and a gauge theory living on the boundary of the anti-de Sitter space. The exact form of the internal manifold X_I will not be important for most of this thesis, however, we emphasize that the form of X_I depends crucially on the boundary gauge theory. For example, the symmetry group of the $X_I = S^5$ internal manifold for the $\mathrm{AdS}_5 \times S^5$ AdS/CFT correspondence is SO(6) which exactly corresponds to the global R-symmetry of $\mathcal{N} = 4$ SYM. For more on the matching of symmetries see e.g. [20].

3.2.3 Some mathematical aspects of the correspondence

Following the discussion above, we consider the space $\mathrm{AdS}_{d+1} \times X_I$ where X_I is some compact manifold. According to the Maldacena conjecture string/M-theory on $\mathrm{AdS}_{d+1} \times X_I$ has a dual gauge theory living on the boundary of AdS_{d+1}. The question now is how the mapping between these two theories looks. This problem was addressed in several papers using highly detailed string theoretical methods, however, Witten suggested a surprisingly simple/beautiful explanation of how the mapping works in the famous paper [29]. This procedure is known as the Witten prescription which we will now review.

As usual, the fundamental objects in a quantum theory are the correlation functions. Assume that $\{\mathcal{O}_i\}$ are a set of boundary operators. We now wish to understand correlators of the type

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \tag{3.18}$$

in terms of the bulk theory. It turns out that the mapping is most naturally expressed on the level of partition functions: As usual the correlator functions $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ are obtained by taking functional derivatives of the partition function of the (boundary) field theory (with field content ψ_i)

$$\begin{aligned} Z_{\mathcal{O}}[\phi_0] &= \int [\mathfrak{D}\psi] \exp\Big(- S[\psi_i] + \int_{S^d} \phi_0(x)\mathcal{O}(x) \Big) \\ &= \Big\langle \exp \int_{S^d} \phi_0 \mathcal{O} \Big\rangle_{\mathrm{CFT}} \end{aligned} \tag{3.19}$$

Here we work in the Euclidean version of AdS_{d+1} (so $\partial \mathrm{AdS}_{d+1} = S^d$) and ϕ_0 is the current/field that couples to the operator $\mathcal{O}$. Notice that the field ϕ_0 must have conjugate quantum numbers of the operator $\mathcal{O}$ in order for the $\phi_0\mathcal{O}$ coupling term to form a singlet. The correlation functions are now obtained by

$$\langle \mathcal{O}(x) \rangle = \frac{\delta Z_{\mathcal{O}}[\phi_0]}{\delta \phi_0(x)}\bigg|_{\phi_0=0}, \quad \langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \frac{\delta^2 Z_{\mathcal{O}}[\phi_0]}{\delta \phi_0(x) \delta \phi_0(y)}\bigg|_{\phi_0=0}, \quad \text{and so on.} \tag{3.20}$$

These relations are straight forwardly generalized to an arbitrary number of operators.

The main point of Witten's prescription now is that the source ϕ_0 can be identified with a bulk field $\phi \leftrightsquigarrow \mathcal{O}$ dual to the boundary operator $\mathcal{O}$, via the AdS/CFT

[5] In the finite temperature case around an AdS_D black brane solution.

correspondence, in the way we will now explain. For simplicity assume that $\mathcal{O}$ is a scalar and therefore couples to a scalar field ϕ. Now let $Z_S[\phi_0]$ denote the partition function of bulk theory, calculated with the boundary condition $\phi = \phi_0$ on the boundary of AdS_{d+1}. The idea now is to identify the boundary partition function with the restricted bulk partition function i.e.,

$$Z_S[\phi_0] = Z_{\mathcal{O}}[\phi_0] \tag{3.21}$$

Especially in the large N limit and the limit where the 't Hooft coupling is large, the bulk theory reduces to classical supergravity. Here $Z_S[\phi_0]$ is simply calculated by finding a solution $\phi_{\text{classical}}$ to the classical equations of motion that fulfills $\phi_{\text{classical}} = \phi_0$ on the boundary and subsequently evaluating (minus the exponential of) the classical SUGRA action i.e.

$$Z_S[\phi_0] = \exp\big(-I_S(\phi_{\text{classical}})\big) \tag{3.22}$$

If the classical SUGRA approximation (= tree level action) is not valid, we must include stringy α' corrections and/or quantum loops, when the effective action is computed. In this thesis we shall only work in the regime where the tree-level SUGRA approximation is valid, so

$$\left\langle \exp \int_{S^d} \phi_0 \mathcal{O} \right\rangle_{\text{CFT}} = \exp\big(-I_S(\phi_{\text{classical}})\big) \tag{3.23}$$

This is Witten's suggestion for computing quantum correlator functions of the boundary theory from the bulk theory. It turns out that the statement (3.23) needs a slight modification when dealing with more complicated bulk fields ϕ, more specifically fields with spin, mass or indices. By analyzing the matter field AdS bulk equations of motion, it is realized that the above procedure will not work for more general fields since the boundary condition "$\phi = \phi_0$" is not attainable [29]. In the more general case the prescription is modified according to [29] (see also [20, 17])

$$\phi_0(x) = \lim_{z\to 0} \frac{\phi_{\text{classical}}}{z^{\Delta}} \tag{3.24}$$

Here z is the coordinate introduced in (2.63) and Δ is a constant which turns out to be identical to the conformal dimension of ϕ to which $\mathcal{O}$ couples. Witten's prescription tells us how to compute the boundary theory partition function for an operator $\mathcal{O}$ given the dual bulk field ϕ. It does, however, not tell us which operators couple to which bulk fields. Figuring out the operator $\leftrightarrow$ field mapping is in general a highly technical task and relies heavily on matching various (super)symmetries on each side. Here we will explain the most important concepts of the operator $\leftrightarrow$ field correspondence. A full account can be found in [16]. Again we will focus on our favorite example: The AdS/CFT correspondence for $\text{AdS}_5 \times S^5$. Since we have a correspondence between a theory on $\text{AdS}_5 \times S^5$ and ∂AdS_5, we need a way to control the field behavior on S^5. This is done by performing a Kaluza-Klein reduction of the type IIB string theory on $\text{AdS}_5 \times S^5$ over the five-sphere to obtain an effective theory on AdS_5: By expanding all the bulk fields in spherical harmonics on S^5, keeping only the lowest harmonics, we obtain a five-dimensional theory containing a metric G_{AB}, a massless dilaton ϕ and a set of SO(6) gauge bosons A^a_B governed by the action (where the dots represent additional massless fields)

$$S_{5D} = \frac{N^2}{8\pi^2 L^3} \int \mathrm{d}^5x \sqrt{|G|} \left[R_{5D} - 2\Lambda - \frac{1}{2} G^{AB} \partial_A \phi \partial_B \phi - \frac{L^2}{8} \mathrm{Tr}\big(F_{AB} F^{AB}\big) + \dots \right] \tag{3.25}$$

Here R_{5D} is the Ricci scalar associated with G_{AB}, Λ is the cosmological constant (2.53) and F_{AB} is the SO(6) field strength (not to be confused with the five-form

$F^{(5)}$ from the original theory). The operator $\leftrightarrow$ field mapping is now then between these reduced bulk fields and the gauge theory operators. Now taking the fields we introduced above, it holds that (see [3] and the references therein)

- The dilaton couples to the field theory Lagrangian density operator $\mathcal{O}_\phi = -\mathcal{L} = \frac{1}{4}\mathrm{Tr}F^2 + \cdots$.
- The SO(6) gauge fields A^a_B couples to the boundary theory conserved R-charge currents $\mathcal{O}_{A^a} = J^{a\mu}$
- The metric G_{AB} couples to the boundary theory stress tensor $\mathcal{O}_G = T^{\mu\nu}$.

This concludes our review of the AdS/CFT correspondence. We will now explain how the fluid/gravity correspondence is a natural corollary of the AdS/CFT correspondence.

3.3 The universal sector and the fluid/gravity conjecture

3.3.1 The gravitational sector

As explained above, it is believed that the most general gauge/gravity correspondence is between a two-derivative quantum theory (a quantum theory of gravity) on a background of the form $\mathrm{AdS}_D \times X_I$ (where X_I is a compact internal manifold) and a CFT on the boundary of AdS_D. Exactly as we did for type IIB string theory on $\mathrm{AdS}_5 \times S^5$, it is possible to perform a dimensional reduction of the theory on $\mathrm{AdS}_D \times X_I$ to obtain a theory on AdS_D. Heuristically this is done by setting all the Kaluza-Klein harmonics of the graviton modes on X_I along with all the matter degrees of freedom to zero (we set the matter fields to their background values on $\mathrm{AdS}_D \times X_I$ and thus remove their degrees of freedom). This reduction leads to D-dimensional Einstein gravity around AdS_D i.e. Einstein gravity with a negative cosmological constant

$$S = \frac{1}{16\pi G_D}\int \mathrm{d}^D x\,\sqrt{|G|}(R - 2\Lambda) \tag{3.26}$$

Here G_D is the D dimensional Newton constant which in principle can be related to the gauge theory parameters (as in (3.25)). As explained in §2.4 this action leads to the following equations of motion (we set the anti de-Sitter radius to unity and let $\mathcal{G}_{AB}$ denote the Einstein tensor)

$$\begin{aligned} \mathcal{E}_{AB} &= \mathcal{G}_{AB} - \frac{(D-1)(D-2)}{2}G_{AB} = 0 \Rightarrow \\ R_{AB} &= (1-D)G_{AB}, \quad R = -D(D-1) \end{aligned} \tag{3.27}$$

Such a reduction is also known in the literature as a *consistent truncation* of the two derivative theory on $\mathrm{AdS}_D \times X_I$ [6, 8]. In the gravitational subsector the dynamics therefore reduce to finding solutions to Einstein's equation with a negative cosmological constant that asymptotes to anti de-Sitter. By abuse of nomenclature we shall refer to such a solution as simply an AdS_D solution. This accounts for the gravitational universal subsector of gauge/gravity theories. Finally, we mention that in addition to gravity, it is also possible to consider excitations of matter fields. For example, as we saw (equation (3.25)) for $D = 5$, dimensional reduction of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ lead to gravity + a set of SO(6) gauge fields (along with other massless fields). Therefore, keeping the SO(6) excitations, the analog of the action (3.26) would then be the action of Einstein-Maxwell theory (with a Chern-Simons term [9]).

	Bulk	Boundary
AdS/CFT	Complicated gravitational quantum theory on $\text{AdS}_D \times X_I$ (for example type IIB string theory)	Boundary CFT on Einstein static universe $\mathbf{R} \times S^{d-1}$ (or alternatively the Poincaré patch)
Effective description	Einstein gravity with a negative cosmological constant	Relativistic fluid dynamics

Table 3.1: *Summery of the fluid/gravity correspondence.*

3.3.2 The fluid/gravity conjecture

The regime of validity of the AdS/CFT correspondence was discussed above, especially the regime where string/M-theory on $\text{AdS}_D \times X_I$ is reduced to supergravity. We also explained how a general two derivative quantum theory on $\text{AdS}_D \times X_I$ contains a universal subsector: Gravity on AdS_D. While only considering the universal gravitational subsector simplifies things a lot, the dynamics of the boundary stress tensor is still very complicated and non-local [8]. It is therefore useful to consider a further limit. We consider the limit where the stress tensor varies slowly compared to the local equilibration scale of the field theory ($\sim$ the mean free path). In this limit we expect the field theory to be locally thermalized and therefore suitable for a fluid dynamical effective description (more on this in §4.1.1). In this fluid dynamical limit we therefore expect the non-trivial dynamics of the stress tensor to reduce to that of fluid dynamics which can be formulated by a set of relativistic Navier-Stokes equations. It is expected that any quantum field theory admits such a fluid dynamical effective description at sufficiently high temperatures and energy densities. The fluid/gravity conjecture is therefore a conjectured equivalence between gravity (an effective description of string theory) and fluid dynamics (an effective description of the boundary CFT) at high temperatures. The main purpose of this thesis is to examine some of the many aspect of this correspondence between these two effective theories.

3.4 The holographic stress tensor

Here we examine dual operator that couples to gravity. As explained above, this conformal operator turns out to be the stress tensor of the boundary field theory.

3.4.1 The boundary stress tensor from AdS/CFT

In general the stress tensor $T^{\mu\nu}$ of a physical system acts as the source for the metric $\mathrm{g}_{\mu\nu}$ since

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_M}{\delta \mathrm{g}_{\mu\nu}} \tag{3.28}$$

Here S_M is the non-gravitational part of the action for the system. This means that a non-zero stress tensor introduces a coupling term $\mathrm{g}_{\mu\nu} T^{\mu\nu}$ into the (classical) Lagrangian. This especially means that the expectation value of the stress tensor in the quantum theory is given by

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{|g|}} \frac{\delta \log Z}{\delta \mathrm{g}_{\mu\nu}} \tag{3.29}$$

where $\log Z \equiv \Gamma$ is the effective action of the non-gravitational part of the theory. According to the AdS/CFT correspondence, the effective action Γ of the boundary gauge theory can be expressed in terms of the on-shell bulk theory action subject to certain well-defined boundary conditions. Suppose that the boundary is equipped with the boundary metric $\gamma_{\mu\nu}$.[6] Apart from a normalization factor which we discuss below, the expectation value of the stress tensor is given by functional differentiating Γ w.r.t. $\gamma_{\mu\nu}$, and therefore according to the correspondence

$$\langle T^{\mu\nu}\rangle \sim \frac{2}{\sqrt{|\gamma|}}\frac{\delta S_{\text{on-shell}}[G_{AB};\gamma_{\mu\nu}]}{\delta\gamma_{\mu\nu}} \tag{3.30}$$

where $S_{\text{on-shell}}[\gamma_{\mu\nu}]$ denotes the on-shell gravitational bulk action subject to the boundary condition that the boundary metric induced from G_{AB} should be $\gamma_{\mu\nu}$.

3.4.2 The quasi-local stress tensor

The stress tensor (3.30) is mathematically identical to the quasilocal stress tensor originally introduced by Brown and York [30], however the interpretation is different. The stress tensor (3.30) comes directly from the AdS/CFT correspondence while the quasilocal stress tensor of Brown and York was invented before the discovery of the AdS/CFT correspondence in order to give well-defined meanings to energy, mass etc. in rather arbitrary gravitational backgrounds. Naively we may write down the gravitational action using the usual expression [13]

$$S = \frac{1}{16\pi G_{d+1}}\left[\int_{\text{AdS}_{d+1}} \mathrm{d}^{d+1}x\sqrt{|G|}(R-2\Lambda) + 2\int_{\partial AdS_{d+1}} \mathrm{d}^d x\sqrt{|\gamma|}\Theta\right] \tag{3.31}$$

here $\Theta_{\mu\nu}$ is the extrinsic curvature (defined below) of the boundary of anti-de Sitter with *induced* metric $\gamma_{\mu\nu}$ from the metric G_{AB}. This action is clearly divergent: By virtue of Einstein's equations, the Ricci scalar is a constant and the on-shell bulk integral is therefore (proportional to) the volume of AdS_{d+1} which is of course infinite. Moreover, it is not clear how we should evaluate the surface integral since the induced metric blows up near the boundary. Divergent action is a general feature of the bulk actions in AdS. Such divergences are also expected to show up on the field theory side of the correspondence before renormalization of the boundary QFT. This implies that Witten's prescription should in fact be expressed in terms of renormalized actions on the gravity side and renormalized correlation functions on the field theory side. Indeed, such a formalism exists and is known as holographic renormalization, for a review see [31]. Here we will show how to renormalize the gravitational action above.

Following the usual renormalization procedure, we must first find a way to regularize the action (3.31). Following [32] we introduce a slicing of AdS in the following way: We foliate $d+1$ dimensional AdS near the boundary into a one-parameter set of d geometries homeomorphic to ∂AdS_{d+1}. We denote the coordinates on the timelike slicing surfaces by x^μ while the last spacelike direction is denoted by r where we demand that ∂AdS is located at $r=\infty$. Moreover denote the timelike surface at "radial" coordinate r by ∂AdS_r and let $\gamma_{\mu\nu}$ be the induced metric on ∂AdS_r. We now redefine the action (3.31) by adding a counter-term action S_{ct} of the type

$$S_{\text{ct}} = \int_{\partial\text{AdS}_r} \mathrm{d}^d x\,\mathcal{F}(L,\gamma_{\mu\nu}) \tag{3.32}$$

[6] As usual the proper way to think of the metric on the boundary of anti-de Sitter is as a conformal class of metrics.

where $\mathcal{F}$ is some functional of the AdS radius L $(= 1)$ and the boundary metric $\gamma_{\mu\nu}$. The renormalized action is now given by

$$S_{\text{ren}} = S + \frac{1}{8\pi G} S_{\text{ct}} \tag{3.33}$$

where the limit $r \to \infty$ is understood. The counter-term action must now be chosen so that the divergences in S and S_{ct} exactly cancel out as $r \to \infty$, leaving a finite renormalized gravitational action. Notice that we have assumed that the counter term action only depends on the intrinsic geometry of the surface ∂AdS_r i.e. local expressions in the boundary metric. This ensures that the full action S_{ren} leads to the same bulk equations of motion for gravity as the original action S. In order to write down the quasilocal stress tensor, we must now work out the variation of S_{ren} w.r.t. the boundary metric $\gamma_{\mu\nu}$. To this end introduce the extrinsic curvature of ∂AdS_r

$$\Theta^{\mu\nu} = -\nabla^{\mu} n^{\nu} \tag{3.34}$$

where n^{μ} is the outward pointing normal vector to ∂AdS_r. The extrinsic curvature has the associated trace

$$\Theta = \Theta^{\mu}{}_{\mu} = \gamma_{\mu\nu}\Theta^{\mu\nu} \tag{3.35}$$

Varying the boundary metric $\gamma_{\mu\nu}$ leads to an indirect variation of the bulk metric G_{AB} since we change the boundary conditions. As is well-known, variation of the bulk part of the action gives a term which is proportional to the equations of motion plus a boundary term. Since we are considering the on-shell action, the EOM term vanishes and we are therefore left with only contributions from surface terms. The variation of the renormalized action is then worked out to [32] (see also e.g. [13])

$$\delta S_{\text{ren}} = \int_{\partial\mathcal{M}} \mathrm{d}^d x \pi^{\mu\nu} \delta\gamma_{\mu\nu} + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \mathrm{d}^d x \frac{\delta S_{\text{ct}}}{\delta\gamma_{\mu\nu}} \delta\gamma_{\mu\nu} \tag{3.36}$$

where $\pi^{\mu\nu}$ is the conjugate momentum to $\gamma_{\mu\nu}$ given by the expression

$$\pi^{\mu\nu} = \frac{1}{16\pi G} \sqrt{-\gamma} \left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu}\right] \tag{3.37}$$

This leads to the following expression for the quasilocal stress tensor

$$T^{\mu\nu}_{BY} = \frac{1}{8\pi G} \left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + T^{\mu\nu}_{\text{ct}}\right] \tag{3.38}$$

where $T^{\mu\nu}_{\text{ct}} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{ct}}}{\partial\gamma_{\mu\nu}}$ is the counter-term stress tensor. In general the counter-term action depends on the spacetime dimension. It was computed for various dimensions in [32], here we present the result for $D = d + 1 = 5$ (here $S_{ct} = \int_{\partial\text{AdS}_r} \mathcal{L}_{\text{ct}}$)

$$L_{\text{ct}} = -\frac{3}{L}\sqrt{|\gamma|}\left(1 - \frac{L^2}{12}R\right) \quad \Rightarrow \quad T^{\mu\nu}_{\text{ct}} = -\frac{1}{16\pi G}\left[\frac{6}{L}\gamma^{\mu\nu} + L\mathcal{G}^{\mu\nu}\right] \tag{3.39}$$

where R and $\mathcal{G}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}R\gamma^{\mu\nu}$ refer to the boundary metric $\gamma_{\mu\nu}$. Using this counter-term stress tensor (in five dimensions), the stress tensor (3.38) yields a finite result.

3.4.3 The field theory stress tensor

As mentioned in the introduction to this section, the quasilocal stress tensor in the form (3.38) is not quite the boundary stress tensor we are looking for. This is because the metric has conformal dimension $\Delta = -2 \neq 0$ and the boundary condition for G_{AB} is therefore given according to (3.24).

Consider an asymptotic anti-de Sitter solution equipped with a set of coordinates (r, x^μ) in which the metric, near the boundary ($r = \infty$), takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + r^2 \mathrm{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{O}(r^{-2}) \tag{3.40}$$

Here the metric $\mathrm{g}_{\mu\nu}$ denotes the boundary metric while $\Lambda_r^2 \mathrm{g}_{\mu\nu}$ is the induced metric on $\partial\mathrm{AdS}_r$ where the scale factor Λ_r is given by $\Lambda_r = r$. These coordinates are exactly of the type we considered in the previous section. Near the boundary, the coordinate z is related to the r coordinate by $r = z^{-1}$. It therefore follows that the boundary condition should be chosen as $\gamma_{\mu\nu} = \lim_{r\to\infty} \Lambda_r^2 \mathrm{g}_{\mu\nu}$, therefore

$$\langle T^{\mu\nu}\rangle = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{on-shell}}[G_{AB}; \gamma_{\mu\nu}]}{\delta \mathrm{g}_{\mu\nu}} = \lim_{r\to\infty} \Lambda_r^{d+2} T^{\mu\nu}_{BY} \tag{3.41}$$

We conclude that the boundary field theory stress tensor is given by[7]

$$\langle T^{\mu\nu}\rangle = \lim_{\Lambda_r\to\infty} \frac{\Lambda_r^{d+2}}{8\pi G} \Big[\Theta^{\mu\nu} - \Theta \mathrm{g}^{\mu\nu} + \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{ct}}}{\delta \mathrm{g}_{\mu\nu}}\Big] \tag{3.43}$$

where the trace Θ is constructed w.r.t. the metric $\mathrm{g}_{\mu\nu}$. The stress tensor (3.43) coincides with the effective stress tensor obtained in [32] (using a slightly different argument for the prefactor Λ^{d+2}). It turns out that the exact form of $T^{\mu\nu}_{\text{ct}}$ will not be important for our purposes. Indeed, suppose that we for some arbitrary anti-de Sitter solution compute the combination $S^\mu_\nu = r^d(\Theta^\mu_\nu - \Theta\delta^\mu_\nu)$. The tensor S^μ_ν will have some divergent terms as $r \to \infty$ and we conclude that the counter-term stress tensor must exactly cancel out these divergences, up to a constant. The boundary stress tensor can therefore be given a finite value by simply ignoring/subtracting the divergent terms. In concrete computations we will therefore just use this naive subtraction method, keeping in mind that this is justified by the holographic renormalization procedure and that our stress tensor is only correct up to a (in general) non-zero constant. This constant turns out to have the interpretation of representing a zero-point Casimir energy of the boundary theory and will therefore just correspond to a shift in energy which does not affect the dynamics [32].

[7] Alternatively, we can write the stress tensor with one index up and one down. It is simply given by

$$\langle T^\mu_\nu\rangle = \mathrm{g}_{\nu\rho}\langle T^{\mu\rho}\rangle = \lim_{\Lambda\to\infty} \frac{\Lambda_r^d}{8\pi G} \Big[\Theta^\mu_\nu - \Theta\delta^\mu_\nu + T^\mu_{ct,\nu}\Big] \tag{3.42}$$

where $\Theta^\mu_\nu = \nabla_\nu n^\mu = \gamma_{\nu\rho}\Theta^{\mu\rho}$.

4

Conformal fluid dynamics

4.1 Fluid Dynamics

In this section we will introduce the theory of relativistic fluid dynamics needed for this thesis. In the following we will let $\mathcal{M}_d$ denote a generic d-dimensional spacetime manifold on which we assume there is some matter content suited for a fluid mechanical description.

4.1.1 Fluid dynamics as an effective description of QFTs

In this thesis we shall use fluid dynamics as a long wave effective description of certain quantum field theories. For a nice discussion of relativistic fluid models and their uses in other areas of modern physics see e.g. [33] In general quantum field theories are determined by their field content and its action which in turn (in principle) determines the stress tensor $T^{\mu\nu}$ and a set of conserved currents J_I^μ of the theory, see any quantum field theory text e.g. [34]. Intrinsic to the theory is the mean free path, ℓ_{mfp}, which indicates the overall interaction scale of the field theory. This means that if we consider the expectation values $\langle T^{\mu\nu}\rangle$ and $\langle J_I^\mu\rangle$ on scales $\gg \ell_{\text{mfp}}$ then we expect the short wave behavior of the field theory to be integrated out, leaving us with a long wave effective description of the field theory. Here the expectation values are taken over both the quantum and thermal fluctuations (which of course reduce to the vacuum ($|0\rangle$) expectation value at zero temperature). Here we will assume that our field theory have such an effective fluid dynamical description.

Moreover we assume that the system is in local thermal equilibrium. The appropriate thermodynamical description of a system with currents is the grand canonical ensemble. The grand canonical partition function $\mathcal{Z}_{\text{gc}}$ is determined by the temperature $\mathcal{T} = \beta^{-1}$, pressure p and chemical potentials μ_I associated with the currents. The assumption of local thermal equilibrium now is that thermodynamical quantities $\mathcal{T}, \mu_I, p, ...$ and so on, do not change on scales $\sim \ell_{\text{mfp}}$. This means that it makes sense to promote the thermodynamical quantities to smooth functions $\mathcal{T}(x^\mu), p(x^\mu), \mu_I(x^\mu), ...$

Finally we require that the scale ℓ_{mfp} is much smaller than the curvature and compactification scales of the manifold $\mathcal{M}_d$ on which the fluid propagates.

4.1.2 The fluid dynamical equations

We will now write down the fluid dynamical equations. As opposed to most effective theories, which are formulated by some action principle, fluid dynamics is described in terms of equations of motion, simply because fluid dynamics incorporates dissipation which has no action formulation (indeed an action formulation for special cases of the non-dissipative perfect fluid exists [35]). We start by identifying the fluid dynamical variables. They consist of the thermodynamical variables such as the temperature $\mathcal{T}$, energy density ρ (and so on) along with the fluid velocity u^μ which is used to write covariant expressions. The fluid velocity can be thought of as the velocity by which the fluid propagates on the background manifold $\mathcal{M}_d$. However there is an ambiguity in this definition since generally the flow on energy/momentum does not have to coincide with the flow of currents. This means that there is no preferred definition of the fluid 'velocity'. This 'gauge freedom' in u^μ can be fixed by introducing the so called *Landau frame* which is defined in the following way: We let u^μ be the (unique) future pointing, timelike, unit normalized $u_\mu u^\mu = -1$ eigenvector to the stress tensor $T^{\mu\nu}$. The corresponding eigenvalue will precisely be minus the energy density ρ [13, 14]. By considering the equation $T^{\mu\nu}u_\nu = -\rho u^\mu$ in the rest frame we see that the Landau frame exactly corresponds to defining u^μ to coincide with energy-momentum transport in the sense that in the rest frame $T^{0i} = 0$. This is how the Landau frame is defined in the original work of Landau & Lifshitz [36].

Having identified the fluid dynamical variables and fixed the ambiguity in u^μ we will now present the equations governing fluid dynamics. They consist of a set of constitutive equations and a set of dynamical equations:

- CONSTITUTIVE EQUATIONS: These consist of expressing the stress tensor $T^{\mu\nu}$ and currents J_I^μ in terms of the fluid dynamical variables.
- DYNAMICAL EQUATIONS: These are the conservation equations for the stress-tensor and the currents J_I^μ:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0 \\ \nabla_\mu J_I^\mu &= \partial_\mu J_I^\mu + \Gamma^\mu_{\mu\lambda} J_I^\lambda = 0 \end{aligned} \tag{4.1}$$

Together with the internal thermodynamical equations, these equations completely determine the motion of the fluid.

4.1.3 THE PERFECT FLUID

Here we will discuss the perfect fluid. The motivation for and derivation of the perfect fluid stress tensor can be found in any standard reference on general relativity see e.g [13, 14]. The stress-tensor of the perfect fluid is constructed out of terms that are at most zero-derivative in the fluid dynamical variables. This gives

$$\begin{aligned} T^{\mu\nu}_{(0)} &= (p+\rho)u^\mu u^\nu + p\mathrm{g}^{\mu\nu} \\ J^\mu_{I,(0)} &= q_I u^\mu \end{aligned} \tag{4.2}$$

Here ρ and p are respectively recognized as the energy density and the pressure. In the second equation q_I is the charge density associated with the current J_I^μ. Physically the second equation expresses nothing else than that the charge is being completely driven by advection. For future convenience we introduce the projector onto spatial directions perpendicular to u^μ

$$\Delta^{\mu\nu} = \mathrm{g}^{\mu\nu} + u^\mu u^\nu \tag{4.3}$$

In the same manner the tensor $-u^\mu u^\nu$ is recognized as the projector along u^μ. The stress tensor for the perfect fluid can therefore be written as

$$T^{\mu\nu}_{(0)} = \rho u^\mu u^\nu + p\Delta^{\mu\nu} \tag{4.4}$$

Writing the perfect fluid stress tensor in this form directly shows that the energy density is associated with the time-like vector u^μ while the pressure is associated with the spatial directions.

THE ENTROPY CURRENT: Now assume that the fluid has an associated entropy density s and therefore the entropy current

$$J^\mu_{S,(0)} = su^\mu \tag{4.5}$$

By use of simple thermodynamical relations it is possible to show that the entropy current is conserved [36]

$$\nabla_\mu J^\mu_{S,(0)} = 0 \tag{4.6}$$

The relation (4.6) is clearly the relativistic generalization of the classical equation $\mathrm{d}S/\mathrm{d}t = 0$ and we conclude that the perfect fluid describes a fluid in global equilibrium.

4.1.4 Non-equilibrium fluids

We will now look into how it is possible to model dissipation i.e. effects due to off-diagonal stress. In this section we assume that the are no currents. We saw that the perfect fluid stress tensor, containing only zeroth order derivatives in the fluid variables, leads to global equilibrium. It is therefore natural to expect that dissipation effects will show up in the stress tensor as higher order derivatives of the fluid variables. We therefore start by writing $T^{\mu\nu}$ as a derivative expansion:

$$\begin{aligned} T^{\mu\nu} &= \rho u^\mu u^\nu + p\Delta^{\mu\nu} + \Pi^{\mu\nu} \\ \Pi^{\mu\nu} &= \Pi^{\mu\nu}_{(1)} + \Pi^{\mu\nu}_{(2)} + \cdots \end{aligned} \tag{4.7}$$

where $\Pi^{\mu\nu}_{(n)}$ contains only n^{th} order derivative terms of the fluid fields. Notice that if we let L denote the typical scale of the flow of the fluid, then we expect the term $\Pi^{\mu\nu}_{(n+1)}$ to be suppressed by a factor $\ell_{\text{mfp}}/L \ll 1$ compared to $\Pi^{\mu\nu}_{(n)}$. The derivative expansion (4.7) therefore makes sense. Notice that the Landau frame condition means that

$$u_\mu \Pi^{\mu\nu} = 0 \tag{4.8}$$

The tensor $\Pi^{\mu\nu}$ is therefore build from terms which all orthogonal to u^μ - here the projector $\Delta^{\mu\nu}$ will come in handy. Also note that from the fluid equation of motion we have

$$-u_\mu \nabla_\nu T^{\mu\nu} = (p+\rho)\nabla_\mu u^\mu + u^\mu \nabla_\mu \rho + \Pi^{\mu\nu}\nabla_\nu u_\mu = 0 \tag{4.9}$$

We therefore conclude that whenever a singe derivative of the energy density ρ occurs, it can be replaced with a term which is at least first order of $\nabla_\mu u_\nu$. A similar result is easily derived for the pressure. Finally the fluid temperature $\mathcal{T}$ can be written in terms of ρ and p through an equation of state. We therefore see that the expansion (4.7) (no charges) can completely be written as an expansion in the fluid velocity derivative.

Constructing the most general form of the n^{th} order correction $\Pi^{\mu\nu}_{(n)}$ is of course a quite complicated task. However, we can do it for pretty easily for $\Pi^{\mu\nu}_{(1)}$. It is possible to decompose $\nabla_\mu u_\nu$ into irreducible representations, separating into components parallel or orthogonal to u^μ:

$$\nabla_\mu u_\nu = -a_\mu u_\nu + \sigma_{\mu\nu} + \varpi_{\mu\nu} + \frac{1}{d-1}\vartheta\Delta_{\mu\nu} \tag{4.10}$$

where the acceleration (as usual) is defined as $a_\mu = u^\nu \nabla_\nu u_\mu$ and the trace $\vartheta = \nabla_\nu u^\nu$ is totally contained in the last term $\frac{1}{d-1}\vartheta P_{\mu\nu}$. The *shear* $\sigma_{\mu\nu}$ and *vorticity* $\varpi_{\mu\nu}$ are given by the expressions

$$\begin{aligned}\sigma_{\mu\nu} &= \frac{1}{2}\big(\Delta^\rho{}_\mu \Delta^\lambda{}_\nu \nabla_\rho u_\lambda + \Delta^\rho{}_\mu \Delta^\lambda{}_\nu \nabla_\lambda u_\rho\big) - \frac{1}{d-1}\vartheta \Delta_{\mu\nu} \\ \varpi_{\mu\nu} &= \frac{1}{2}\big(\Delta^\rho{}_\mu \Delta^\lambda{}_\nu \nabla_\rho u_\lambda - \Delta^\rho{}_\mu \Delta^\lambda{}_\nu \nabla_\lambda u_\rho\big)\end{aligned} \tag{4.11}$$

It is straight forward to check that the RHS of (4.10) indeed equals $\nabla_\mu u_\nu$. We see that the first acceleration term $-a_\mu u_\nu$ is parallel with u^μ, moreover

$$u_\mu \sigma^{\mu\nu} = 0, \quad u_\mu \varpi^{\mu\nu} = 0, \quad a_\mu u^\mu = 0, \quad \sigma^\mu{}_\mu = 0, \quad \varpi^\mu{}_\mu = 0 \tag{4.12}$$

so the last three terms are all orthogonal to the velocity u^μ. The decomposition (4.10) shows how the different combinations of the velocity gradients transform under SO$(d-1)$ rotations around u^μ. Since $\Pi^{\mu\nu}_{(1)}$ must respect this rotational symmetry, we see that it must be built out of the terms $a^\mu u^\nu$, $\sigma^{\mu\nu}$, $\varpi^{\mu\nu}$ and $\vartheta P^{\mu\nu}$. However, since $a^\mu u^\nu$ is not orthogonal to u^μ and $\varpi^{\mu\nu}$ is not symmetric, they can not contribute. The only possible form of $\Pi^{\mu\nu}_{(1)}$ therefore is

$$\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta\vartheta\Delta^{\mu\nu} \tag{4.13}$$

where η and ζ respectively is the shear viscosity and the bulk viscosity (i.e., the friction coefficients associated with respectively fluid stress and expansion). The parameters η and ζ must for physical reasons be chosen to be non-negative. First of all if η and ζ are negative it will lead to unphysical instabilities in the fluid. Moreover the $\eta, \zeta < 0$ is in direct violation with the second law of thermodynamics.

THE ENTROPY CURRENT: It is possible to construct an entropy current associated with the dissipative fluid. As with the stress tensor, we write the entropy current as a derivative expansion

$$\begin{aligned}J^\mu_S &= s u^\mu + \Sigma^\mu \\ \Sigma^\mu &= \Sigma^\mu_{(1)} + \Sigma^\mu_{(2)} + \cdots\end{aligned} \tag{4.14}$$

By use of various thermodynamical relations it is possible to show that the divergence of the entropy current takes the form [36] (see also [33, 37])

$$\mathcal{T}\nabla_\mu J^\mu_S = 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta\vartheta^2 \tag{4.15}$$

We therefore see that η and ζ must be positive $\eta, \zeta \geq 0$ in order for the second law of thermodynamics

$$\nabla_\mu J^\mu_S \geq 0 \tag{4.16}$$

to be satisfied.

4.1.5 Charged fluids

We now consider the most general fluid with a set of conserved currents

$$\nabla_\mu J^\mu_I = 0 \tag{4.17}$$

As before we will write the current J^μ_I as a derivative expansion

$$\begin{aligned}J^\mu_I &= q_I u^\mu + \Upsilon^\mu_I \\ \Upsilon^\mu_I &= \Upsilon^\mu_{(1)I} + \Upsilon^\mu_{(2)I} + \cdots\end{aligned} \tag{4.18}$$

On physical grounds, we usually take the diffusion current Υ_I^μ to be transverse i.e. $u_\mu \Upsilon_I^\mu = 0$ [33]. Now there are two ways to write down the expression for $\Upsilon_{(1)}^\mu$. The first follows [28, 9] where we write down the must general contribution from various fluid dynamical derivatives and use the the zeroth order equation to express some thermodynamical derivatives in terms of $\nabla_\nu u_\mu$. The second approach follows that of Landau & Lifshitz [36] (see also [33, 37]) where the form of $\Upsilon_{(1)}^\mu$ is deduced from the entropy current, the first law of thermodynamics (here written in terms of covariant derivatives on the manifold)

$$\nabla_\mu \rho = \mathcal{T} \nabla_\mu s + \mu_I \nabla_\mu q_I \tag{4.19}$$

and the fact that there must be entropy production for *all* fluid configurations.

When presenting the general first order result we find it convenient *not* to assume the Landau frame condition $u_\mu \Pi^{\mu\nu} = 0$ (for more on fluid dynamics without fixing u^μ to the Landau frame see [33, 37]). This makes it easier comparing the relativistic laws of fluid dynamics to their well-known classical counterparts and accounts for the expressions found in [5]. The first order derivative corrections the stress-tensor and the currents are given by

$$T_{(0)}^{\mu\nu} = \rho u^\mu u^\nu + p \Delta^{\mu\nu} \ , \qquad \Pi_{(1)}^{\mu\nu} = -\zeta \vartheta \Delta^{\mu\nu} - 2\eta \sigma^{\mu\nu} + 2\kappa h^{(\mu} u^{\nu)} \tag{4.20}$$

$$J_{(0)I}^\mu = q_I u^\mu \ , \qquad \Upsilon_I^\mu = -\Delta^{\mu\nu} \sum_J \mathcal{D}_{IJ} \nabla_\nu (\mu_I / \mathcal{T}) \tag{4.21}$$

where $\zeta \geq 0$ is the bulk viscosity, $\eta \geq 0$ is the shear viscosity, $\kappa \geq 0$ is the the thermal conductivity and $\mathcal{D}_{IJ} \geq 0$ is the IJ'th diffusion coefficient. Here

$$h^\mu = -\Delta^{\mu\nu} (\nabla_\nu \mathcal{T} + a_\nu \mathcal{T}) \tag{4.22}$$

is the heat flux vector. By using the defining equation (4.8), we see that in the Landau frame this vector is "gauged out" i.e. $h^\mu = 0$. Indeed using the method of [36] (see also [33, 37]) along with the first law of thermodynamics, it is possible to show that the divergence of the entropy current associated with 4.20 is positive definite.

Finally we mention that in $d = 4$ it is possible to have an additional term

$$\mho_I \ell^\mu = \mho_I \epsilon^\mu{}_{\alpha\beta\gamma} u^\alpha \nabla^\beta u^\gamma \tag{4.23}$$

in the expression for $\Upsilon_{(1)I}$. Here $\mho_I$ is a new transport coefficient associated with the pseudo-vector ℓ^μ. Clearly such a term is theoretically possible in $d = 4$ moreover it is physically realized for certain field theories [9] (see also §6.3.8).

4.1.6 The Kubo formula and the viscosity/entropy ratio

It is possible to derive an expression for the viscosity η in terms of the correlators of the stress tensor $T^{\mu\nu}$. This is the famous Kubo formula. The Kubo formula for the viscosity is given by [3]

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}\, \mathrm{G}_{xy,xy}^R (\omega, \mathbf{0}) \tag{4.24}$$

Here $\mathrm{G}_{xy,xy}^R$ is the momentum space retarded Green's function of the xy component of the stress tensor (operator). Similar formulas can be derived for the other transport coefficients. The point is that the transport coefficients can in principle be derived directly from the underlying quantum theory. It should therefore also be possible to compute e.g. the viscosity η for weakly coupled theories by calculating

Feynman diagrams. Indeed, such a (highly non-trivial) computation is possible (see [38]). However, this Feynman diagram approach for computing the transport coefficients is completely unavailable in the strongly coupled regime. The reason that it is still possible to use the Kubo approach for computing the transport coefficients for strongly coupled theories is of course because of the duality provided by the AdS/CFT correspondence [3].

We now address the viscosity/entropy ratio. Given an effective fluid description of a field theory, the ratio η/s is a measure of how strongly the theory is coupled: For weakly coupled theories (more specifically our favorite field theory toy model, ϕ^4-theory), the ratio η/s can be estimated to be [3]

$$\frac{\eta}{s} \sim \frac{1}{\lambda^2} \tag{4.25}$$

where λ is the (weak) coupling. We therefore have $\eta/s \gg 1$ for weakly coupled theories. By extrapolating (4.25) to $\lambda \sim 1$, we see that $\eta/s \sim 1$ for strongly coupled theories. The viscosity/entropy bound mentioned in the introduction conjectures that the limit $\eta/s \to 0$ can not be obtained, even for extremely strong coupled theories.

4.2 Conformal Fluid Dynamics

4.2.1 Classification of the conformal fluid observables

We will now look at the fluid dynamics of a conformal field theory. As we have seen, conformal invariance demands that the trace of the stress tensor vanishes

$$\mathrm{g}_{\mu\nu} T^{\mu\nu} = 0 \tag{4.26}$$

Before we continue, we mention that in principle, on curved backgrounds, the underlying quantum theory can contain Weyl anomalies which break Weyl invariance and therefore lead to trace anomalies. However, such anomalies will be of high derivative nature and in order to see them in our fluid dynamic effective description, we would have to go beyond second order fluid dynamics [39]. We will therefore simply ignore Weyl anomalies.

Now, applying the tracelessness condition to the perfect fluid stress tensor we see that the energy density ρ and pressure p are related through the conformal equation of state $p = \frac{\rho}{d-1}$. We will now work out the transformation properties of the various fluid observables under conformal transformations:

$$\mathrm{g}_{\mu\nu} \to \tilde{\mathrm{g}}_{\mu\nu} = e^{-2\phi} \mathrm{g}_{\mu\nu}, \quad \mathrm{g}^{\mu\nu} \to \tilde{\mathrm{g}}^{\mu\nu} = e^{2\phi} \mathrm{g}^{\mu\nu} \tag{4.27}$$

First of all recall that the fluid velocity u^μ was defined so that $u_\mu u^\mu = -1$ which must hold both in the original metric and the transformed one. We therefore have $\mathrm{g}_{\mu\nu} u^\mu u^\nu = \tilde{\mathrm{g}}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu = -1$ and we conclude $\tilde{u}^\mu = e^\phi u^\mu$. Notice that this implies that the projection tensor $\Delta^{\mu\nu}$ transforms with conformal weight 2 i.e. $\tilde{\Delta}^{\mu\nu} = \tilde{u}^\mu \tilde{u}^\nu - \tilde{\mathrm{g}}^{\mu\nu} = e^{2\phi} \Delta^{\mu\nu}$. Also recall that the stress tensor of a conformal theory transforms homogeneously under Weyl transformations with weight $d + 2$

$$\tilde{T}^{\mu\nu} = e^{(d+2)\phi} T^{\mu\nu} \tag{4.28}$$

Using this we therefore see that the energy density ρ must transform as $\tilde{\rho} = e^{d\phi}\rho$. Now intensivity and conformal invariance implies that the energy density ρ of a conformal fluid can be written $\rho = \eta_0 \mathcal{T}^d$ where η_0 is a constant. This is realized using a simple dimensional analysis argument. Indeed, in a finite temperature CFT the only characteristic variables with non-vanishing length (energy^{-1}) dimension is

Conformal observable	Conformal weight
Metric, $\mathrm{g}_{\mu\nu}$	-2
Fluid velocity, u^μ	1
Projection tensor, $\Delta^{\mu\nu}$	2
Temperature, T	1
Entropy density, s	$d-1$
Charge density, q_I	$d-1$
Chemical potentials, μ_I	1

Table 4.1: Conformal observables

the temperature $\mathcal{T}$ and the manifold volume V and since ρ is intensive, it cannot depend on V. Since ρ has length dimension d, the result follows. The perfect stress tensor of a conformal fluid is therefore given by

$$T^{\mu\nu}_{(0)} = \eta_0 \mathcal{T}^d \Big(u^\mu u^\nu + \frac{1}{d-1} \Delta^{\mu\nu} \Big) \tag{4.29}$$

We conclude from the scaling behavior of the energy density that the temperature scales as $\tilde{\mathcal{T}} = e^\phi \mathcal{T}$. Finally we can work out the transformation properties of the chemical potentials. First of all the charge densities transform with conformal weight $d-1$. This is realized by considering e.g. (2.31). Now by the fundamental relation $\rho + p = \frac{d}{d-1}\rho = Ts + \mu_I q_I$ we then conclude that μ_I scale as $\tilde{\mu}_I = e^\phi \mu_I$ under Weyl transformations. The conformal weights of the conformal observables relevant for the zeroth order description of conformal fluid dynamics are summarized in the table 4.2.1. We now move on to describing the first order conformal properties of the first order corrections to the stress tensor (4.13).

4.2.2 Conformal properties of $\Pi^{\mu\nu}_{(1)}$

The Christoffel symbols transform as

$$\Gamma^\mu_{\nu\lambda} = \tilde{\Gamma}^\mu_{\nu\lambda} + \delta^\mu_\nu \partial_\lambda \phi + \delta^\mu_\lambda \partial_\nu \phi - \tilde{\mathrm{g}}_{\nu\lambda} \tilde{\mathrm{g}}^{\mu\sigma} \partial_\sigma \phi \tag{4.30}$$

Using this relation along with the transformation property of u^μ we can work out the transformation of the gradient of the fluid velocity

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda = e^{-\phi} \big(\tilde{\nabla}_\mu \tilde{u}^\nu + \delta^\nu_\mu \tilde{u}^\sigma \partial_\sigma \phi - \tilde{u}_\mu \tilde{\mathrm{g}}^{\nu\sigma} \partial_\sigma \phi \big) \tag{4.31}$$

We can now deduce the transformation properties of the various fluid quantities such as shear and vorticity

$$\begin{aligned}
\vartheta &\equiv \nabla_\mu u^\mu = e^{-\phi} (\tilde{\nabla}_\mu \tilde{u}^\mu + (d-1) \tilde{u}^\sigma \partial_\sigma \phi) = e^{-\phi} (\tilde{\vartheta} + (d-1) \tilde{\mathcal{D}} \phi) \\
a^\mu &\equiv \mathcal{D} u^\mu = e^{-2\phi} (\tilde{a}^\mu + \tilde{\Delta}^{\mu\sigma} \partial_\sigma \phi) \\
\sigma^{\mu\nu} &\equiv \Delta^{\lambda(\mu} \nabla_\lambda u^{\nu)} - \frac{1}{d-1} \Delta^{\mu\nu} \nabla_\lambda u^\lambda = e^{-3\phi} \tilde{\sigma}^\mu \\
\varpi^{\mu\nu} &\equiv \Delta^{\lambda[\mu} \nabla_\lambda u^{\nu]} = e^{-3\phi} \tilde{\varpi}^\mu
\end{aligned} \tag{4.32}$$

where we have introduced a fluid directional derivative

$$\mathcal{D} \equiv u^\mu \nabla_\mu \tag{4.33}$$

We therefore see that the trace ϑ and the acceleration a^μ do *not* transform homogeneously under Weyl transformations while the shear $\sigma^{\mu\nu}$ and vorticity $\varpi^{\mu\nu}$ both transform homogeneously with weight 3. From the tracelesness of the stress tensor we have that for a conformal fluid $\Pi^{\mu}_{(1)\mu} = 0$. It therefore follows from (4.13) that the bulk viscosity ξ must vanish for conformal fluids. We conclude that the first order derivative correction to the stress tensor of a conformal fluid with non-vanishing shear is given by

$$\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu}, \quad \eta > 0 \tag{4.34}$$

Since this term must have conformal weight $d+2$, we see that the shear viscosity of a conformal fluid transforms as $\tilde{\tilde{\eta}} = e^{(d-1)\phi}\tilde{\eta}$.

Working out which terms are allowed in the stress tensor up to first order for a conformal theory along with the Weyl scaling of the transport coefficients was pretty straight forward. However it should be clear that already at second order, this becomes quite involved. It would therefore be nice to have a general formalism that allows us to write down higher order derivative terms that transform homogeneously under Weyl transformations. Such a construction would be reminiscent of the construction of the covariant derivative ∇_μ well-know from general relativity.

4.2.3 The Weyl-covariant derivative

Indeed such a formalism exists. In this section we follow the beautiful work of [40].

Consider a conformal object $\mathcal{Q}^{\mu\cdots}_{\nu\cdots}$ with conformal weight w. Under Weyl transformations we have

$$\mathrm{g}_{\mu\nu} \to \tilde{\mathrm{g}}_{\mu\nu} = e^{-2\phi}\mathrm{g}_{\mu\nu} \ ; \quad \mathcal{Q}^{\mu\cdots}_{\nu\cdots} \to \tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots} = e^{w\phi}\mathcal{Q}^{\mu\cdots}_{\nu\cdots} \tag{4.35}$$

We wish to construct a derivative operator $\mathcal{D}$ which, in addition to being coordinate transformation covariant, has the property that $\mathcal{D}\mathcal{Q}$ transforms homogeneously with the same weight w as $\mathcal{Q}$ under Weyl transformations. To this end introduce a one-form $\mathcal{A}_\mu$ with the following "gauge-like" transformation properties

$$\mathcal{A}_\mu \to \tilde{\mathcal{A}}_\mu = \mathcal{A}_\mu - \partial_\mu\phi \tag{4.36}$$

under Weyl transformations (4.35). Notice that we have not yet specified the explicit form of $\mathcal{A}_\mu$, only how it should transform. Recall that for a general tensor $T^{\mu\cdots}_{\nu\cdots}$ we have

$$\begin{aligned}\nabla_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} = \partial_\lambda T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} &+ \Gamma^{\mu_1}_{\alpha\lambda}T^{\alpha\mu_2\cdots}{}_{\nu_1\nu_2\cdots} + \Gamma^{\mu_2}_{\alpha\lambda}T^{\mu_1\alpha\cdots}{}_{\nu_1\nu_2\cdots} + \cdots \\ &- \Gamma^{\beta}_{\nu_1\lambda}T^{\mu_1\mu_2\cdots}{}_{\beta\nu_2\cdots} - \Gamma^{\beta}_{\nu_2\lambda}T^{\mu_1\mu_2\cdots}{}_{\nu_1\beta\cdots} - \cdots\end{aligned} \tag{4.37}$$

This means that using equation (4.30)

$$\begin{aligned}\tilde{\nabla}_\lambda\tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots} &= \nabla_\lambda\tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots} + (\mathrm{g}_{\alpha\lambda}\partial^\mu\phi - \delta^\mu_\alpha\partial_\lambda\phi - \delta^\mu_\lambda\partial_\alpha\phi)\tilde{\mathcal{Q}}^{\alpha\cdots}_{\nu\cdots} + \cdots \\ &\qquad - (\mathrm{g}_{\nu\lambda}\partial^\beta\phi - \delta^\beta_\nu\partial_\lambda\phi - \delta^\beta_\lambda\partial_\nu\phi)\tilde{\mathcal{Q}}^{\mu\cdots}_{\beta\cdots} - \cdots \\ &= e^{w\phi}\nabla_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} + w(\mathcal{A}_\lambda e^{w\phi}\mathcal{Q}^{\mu\cdots}_{\nu\cdots} - \tilde{\mathcal{A}}_\lambda\tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots}) \\ &\quad + e^{w\phi}(\mathrm{g}_{\alpha\lambda}\mathcal{A}^\mu - \delta^\mu_\alpha\mathcal{A}_\lambda - \delta^\mu_\lambda\mathcal{A}_\alpha)\mathcal{Q}^{\alpha\cdots}_{\nu\cdots} - (\tilde{\mathrm{g}}_{\alpha\lambda}\tilde{\mathcal{A}}^\mu - \delta^\mu_\alpha\tilde{\mathcal{A}}_\lambda - \delta^\mu_\lambda\tilde{\mathcal{A}}_\alpha)\tilde{\mathcal{Q}}^{\alpha\cdots}_{\nu\cdots} + \cdots \\ &\quad - \Big(e^{w\phi}(\mathrm{g}_{\nu\lambda}\mathcal{A}^\beta - \delta^\beta_\nu\mathcal{A}_\lambda - \delta^\beta_\lambda\mathcal{A}_\nu)\mathcal{Q}^{\mu\cdots}_{\beta\cdots} - (\tilde{\mathrm{g}}_{\nu\lambda}\tilde{\mathcal{A}}^\beta - \delta^\beta_\nu\tilde{\mathcal{A}}_\lambda - \delta^\beta_\lambda\tilde{\mathcal{A}}_\nu)\tilde{\mathcal{Q}}^{\mu\cdots}_{\beta\cdots}\Big) - \cdots\end{aligned} \tag{4.38}$$

where we used that $\mathrm{g}_{\mu\nu}\mathcal{A}^\lambda = \tilde{\mathrm{g}}_{\mu\nu}\tilde{\mathcal{A}}^\lambda + \mathrm{g}_{\mu\nu}\partial^\lambda\phi$. This means that we can define a Weyl-covariant derivative by

$$\begin{aligned}\mathcal{D}_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} &= \nabla_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} + w\mathcal{A}_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} \\ &\quad + (\mathrm{g}_{\alpha\lambda}\mathcal{A}^\mu - \delta^\mu_\alpha\mathcal{A}_\lambda - \delta^\mu_\lambda\mathcal{A}_\alpha)\mathcal{Q}^{\alpha\cdots}_{\nu\cdots} + \cdots \\ &\quad - (\mathrm{g}_{\nu\lambda}\mathcal{A}^\beta - \delta^\beta_\nu\mathcal{A}_\lambda - \delta^\beta_\lambda\mathcal{A}_\nu)\mathcal{Q}^{\mu\cdots}_{\beta\cdots} - \cdots\end{aligned} \tag{4.39}$$

By moving all the tilded terms in (4.38) to the RHS, we see that this derivative has the sought property $\tilde{\mathcal{D}}_\lambda \tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots} = e^{w\phi}\mathcal{D}_\lambda \mathcal{Q}^{\mu\cdots}_{\nu\cdots}$. We will now work out the explicit expression of the one-form $\mathcal{A}_\mu$. We have that

$$\mathcal{D}_\lambda u^\mu = \nabla_\lambda u^\mu + u_\lambda \mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha u^\alpha \tag{4.40}$$

The gauge field $\mathcal{A}_\mu$ is uniquely determined by requiring that the Weyl-covariant fluid derivative $\mathcal{D}_\lambda u^\mu$ is traceless and transverse i.e. $\mathcal{D}_\mu u^\mu = 0$ and $u^\mu \mathcal{D}_\mu u^\nu = 0$. Indeed, from the first condition $\mathcal{D}_\mu u^\mu = 0$ we find that $\mathcal{A}_\mu u^\mu = \frac{1}{d-1}\nabla_\mu u^\mu$ while the second condition gives $\mathcal{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \mathcal{A}_\lambda u^\lambda u_\mu$, so

$$\mathcal{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \frac{1}{d-1}\nabla_\lambda u^\lambda u_\mu = a_\mu - \frac{\vartheta}{d-1}u_\mu \tag{4.41}$$

As a consistency check notice that using the relations (4.32) we see that

$$\mathcal{A}_\mu = \tilde{\mathcal{A}}_\mu + \partial_\mu \phi \tag{4.42}$$

So everything works as it should. We are now ready to cast conformal fluid dynamics into manifestly Weyl-covariant form. With $\mathcal{A}_\mu$ defined as in (4.41) the derivative of the fluid velocity takes the form

$$\mathcal{D}^\mu u^\nu = \nabla^\mu u^\nu + u^\mu a^\nu - \frac{\vartheta}{d-1}\Delta^{\mu\nu} = \sigma^{\mu\nu} + \varpi^{\mu\nu} \tag{4.43}$$

where the shear $\sigma^{\mu\nu}$ and vorticity $\varpi^{\mu\nu}$ were defined in (4.11). This especially means that

$$\begin{aligned} \sigma^{\mu\nu} &= \mathcal{D}^{(\mu}u^{\nu)} \\ \varpi^{\mu\nu} &= \mathcal{D}^{[\mu}u^{\nu]} \end{aligned} \tag{4.44}$$

This accounts for the first order terms in the Weyl-covariant formalism. It is also possible to express the fluid dynamical conservation equations $\nabla_\mu T^{\mu\nu}$, $\nabla_\mu J^\mu = 0$ in terms of the derivative $\mathcal{D}_\mu$. We have

$$\begin{aligned} \mathcal{D}_\mu T^{\mu\nu} &= \nabla_\mu T^{\mu\nu} + (w-d-2)\mathcal{A}_\mu T^{\mu\nu} + \mathcal{A}^\nu T^\mu_{\ \mu} \\ \mathcal{D}_\mu J^\mu &= \nabla_\mu J^\mu + (w-d)\mathcal{A}_\mu J^\mu \end{aligned} \tag{4.45}$$

Now since $T^{\mu\nu}$ is traceless transforming with weight $w = d+2$ and J^μ has conformal weight $w = d$, we see that the fluid dynamical conservation equations (4.1) can be written in the manifestly Weyl-covariant form

$$\begin{aligned} \mathcal{D}_\mu T^{\mu\nu} &= 0 \\ \mathcal{D}_\mu J^\mu &= 0 \end{aligned} \tag{4.46}$$

The entropy current also has conformal weight $w = d$ and the second law of thermodynamics can therefore be cast into manifestly Weyl-covariant form as

$$\mathcal{D}_\mu J^\mu_S \geq 0 \tag{4.47}$$

4.2.4 The Weyl curvature tensors

Here we introduce the various curvature tensors associated with the Weyl-covariant derivative. These tensors, originally introduced in [40], are important when considering fluid dynamics to second order.

We start by defining a field strength associated with the field $\mathcal{A}_\mu$ by

$$\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu \tag{4.48}$$

Notice that this tensor is 'gauge invariant' under the transformation (4.36) i.e, $\mathcal{F}_{\mu\nu} = \tilde{\mathcal{F}}_{\mu\nu}$. We will now define a Riemann curvature tensor associated with the Weyl-covariant derivative $\mathcal{D}_\mu$. We do this in the usual way, the curvature tensor arises from the commutator of covariant derivatives, $[\mathcal{D}_\mu, \mathcal{D}_\nu]V_\lambda$. Since the Weyl curvature ${\mathcal{R}_{\mu\nu\lambda}}^\rho$ should only encapsulate the conformal geometry of the underlying manifold, the object on which we evaluate the commutator should be invariant under Weyl transformations. Therefore let V^μ be a vector with weight $w = 0$ and define

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]V_\lambda = -{\mathcal{R}_{\mu\nu\lambda}}^\rho V_\rho \tag{4.49}$$

Now if V^μ is instead a vector of weight w then (using (4.39) on a vector)

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]V_\lambda = w\mathcal{F}_{\mu\nu}V_\lambda - {\mathcal{R}_{\mu\nu\lambda}}^\rho V_\rho \tag{4.50}$$

Notice that the first term comes from the $w\mathcal{A}_\mu$-term in (4.39). We can check that

$$\begin{aligned} {\mathcal{R}_{\mu\nu\lambda}}^\rho = {R_{\mu\nu\lambda}}^\rho &+ \left[\nabla_\mu(\mathrm{g}_{\lambda\nu}\mathcal{A}^\rho + \delta^\rho_\lambda\mathcal{A}_\nu - \delta^\rho_\nu\mathcal{A}_\lambda) \;-\; (\mu \leftrightarrow \nu)\right] \\ &+ \left[(\mathrm{g}_{\lambda\nu}\mathcal{A}^\sigma - \delta^\sigma_\lambda\mathcal{A}_\nu - \delta^\sigma_\nu\mathcal{A}_\lambda)(\mathrm{g}_{\sigma\mu}\mathcal{A}^\rho - \delta^\rho_\sigma\mathcal{A}_\mu - \delta^\rho_\mu\mathcal{A}_\sigma) \;-\; (\mu \leftrightarrow \nu)\right] \end{aligned} \tag{4.51}$$

Lowering the last index we get

$$\mathcal{R}_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \delta^\alpha_{[\mu}\mathrm{g}_{\nu][\lambda}\delta^\beta_{\sigma]}(\nabla_\alpha\mathcal{A}_\beta + \mathcal{A}_\alpha\mathcal{A}_\beta - \frac{\mathcal{A}^2}{2}\,\mathrm{g}_{\alpha\beta}) - \mathcal{F}_{\mu\nu}\mathrm{g}_{\lambda\sigma} \tag{4.52}$$

Now this tensor, being a curvature tensor, fulfills various Bianchi identities which can be found in [40]. The curvature tensor $\mathcal{R}_{\mu\nu\lambda\sigma}$ does not have the same symmetry properties as the Riemann tensor $R_{\mu\nu\lambda\sigma}$ under interchange of indices. We see from the equation (4.52) that

$$\begin{aligned} \mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu\sigma\lambda} &= -2\mathcal{F}_{\mu\nu}\mathrm{g}_{\lambda\sigma} \\ \mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{R}_{\lambda\sigma\mu\nu} &= \delta^\alpha_{[\mu}\mathrm{g}_{\nu][\lambda}\delta^\beta_{\sigma]}\mathcal{F}_{\alpha\beta} - \mathcal{F}_{\mu\nu}\mathrm{g}_{\lambda\sigma} + \mathcal{F}_{\lambda\sigma}\mathrm{g}_{\mu\nu} \end{aligned} \tag{4.53}$$

Having defined a Riemann Weyl invariant curvature tensor we can now proceed to constructing the Weyl invariant Ricci tensor and scalar: [1]

$$\begin{aligned} \mathcal{R}_{\mu\nu} = {\mathcal{R}_{\mu\alpha\nu}}^\alpha &= R_{\mu\nu} - (d-2)\big(\nabla_\mu\mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu - \mathcal{A}^2\mathrm{g}_{\mu\nu}\big) - \mathrm{g}_{\mu\nu}\nabla_\lambda\mathcal{A}^\lambda - \mathcal{F}_{\mu\nu} \\ \mathcal{R} &= R - 2(d-1)\nabla_\lambda\mathcal{A}^\lambda + (d-2)(d-1)\mathcal{A}^2 \end{aligned} \tag{4.54}$$

Finally we will take a look at the Weyl tensor $C_{\mu\nu\lambda\sigma}$. The Weyl tensor is a well known example of a conformal tensor (see e.g. [13, 14]) i.e. a tensor that transforms homogeneously and is independent of the background fluid velocity field (in other words, it is a purely geometrical quantity). It is not surprising that these tensors will also be present in our manifestly conformal invariant formalism. We define the conformal Weyl tensor $\mathcal{C}_{\mu\nu\lambda\sigma}$ in the same way as the ordinary Weyl tensor $C_{\mu\nu\lambda\sigma}$ is defined in terms of the ordinary curvature tensors $R_{\mu\nu\lambda\sigma}, S_{\mu\nu}$ (see e.g. [13, 14]). We therefore define $\mathcal{C}_{\mu\nu\lambda\sigma}$ by

$$\mathcal{R}_{\mu\nu\lambda\sigma} = \mathcal{C}_{\mu\nu\lambda\sigma} - \delta^\alpha_{[\mu}\mathrm{g}_{\nu][\lambda}\delta^\beta_{\sigma]}\mathcal{S}_{\alpha\beta} \tag{4.55}$$

where $\mathcal{S}_{\mu\nu}$ is the conformal generalization of the the Schouten tensor i.e. the ordinary Schouten tensor $S_{\mu\nu}$ (see e.g [13] for its explicit form) with $R_{\mu\nu}$ replaced with $\mathcal{R}_{\mu\nu}$. It is now possible to show that [40]

$$\mathcal{C}_{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu}\mathrm{g}_{\lambda\sigma} \tag{4.56}$$

[1] Since $\mathcal{R}_{\mu\nu\lambda\sigma}$ does not have the same symmetry properties as $R_{\mu\nu\lambda\sigma}$ it is important which indices are contracted to construct the Ricci tensor and scalar. Here we use the same convention as [40].

which shows how the Weyl tensor emerges from the conformally invariant formalism. Note how (4.56) directly shows that $C_{\mu\nu\lambda\sigma}$ transforms homogeneously with weight $w=-2$ (recall that $\mathcal{F}_{\mu\nu}$ has weight 0 and the metric has weight -2). Moreover recall that $C_{\mu\nu\lambda\sigma}$ has the same symmetry properties as $R_{\mu\nu\lambda\sigma}$, $C_{\mu\nu\lambda\sigma}=-C_{\nu\mu\lambda\sigma}$, $C_{\mu\nu\lambda\sigma}=-C_{\nu\mu\sigma\lambda}$, $C_{\mu\nu\lambda\sigma}=+C_{\lambda\sigma\nu\mu}$ and is traceless

$$C_{\mu\alpha\nu}{}^{\alpha}=0 \tag{4.57}$$

4.2.5 Second Order Fluid Dynamics

We are now ready to write down the most general stress tensor of conformal fluid dynamics to second order. Notice that if κ is some transport coefficient with conformal weight w we can define

$$\kappa=\mathcal{T}^{w}\kappa_0 \tag{4.58}$$

where $\kappa_0=\mathcal{T}^{-w}\kappa=\tilde{\kappa}_0$ is invariant under Weyl transformations. In this way we can absorb the conformal weight of κ into $\mathcal{T}$ and trade κ_0 in for κ. We will especially define a new variable ν_I in terms of the chemical potentials μ_I by $\nu_I=\mu_I/\mathcal{T}=\tilde{\nu}_I$.

From the above analysis we conclude that, up to second order, any conformal fluid observable can be written in terms of the following 14 quantities:

$$\begin{aligned}
\mathcal{O}(1):&\quad \nu_I,\ \mathcal{T},u_\mu,\ \mathrm{g}_{\mu\nu},\ \varepsilon_{\mu\nu\cdots\sigma}\\
\mathcal{O}(\mathcal{D}):&\quad \mathcal{D}_\mu\nu_I,\ \sigma_{\mu\nu},\ \varpi_{\mu\nu}\\
\mathcal{O}(\mathcal{D}^2):&\quad \mathcal{D}_\mu\mathcal{D}_\nu\nu_I,\ \mathcal{D}_\mu\mathcal{T},\ \mathcal{D}_\lambda\sigma_{\mu\nu},\ \mathcal{D}_\lambda\varpi_{\mu\nu},\ \mathcal{F}_{\mu\nu},\ \mathcal{R}_{\mu\nu\lambda}{}^{\sigma}
\end{aligned} \tag{4.59}$$

Here $\varepsilon_{\mu\nu\cdots\sigma}$ is the Levi-Civita tensor i.e. the totally antisymmetric symbol defined with a factor $\sqrt{g}$, it therefore transforms as $\varepsilon_{\mu\nu\cdots\sigma}=e^{d\phi}\tilde{\varepsilon}_{\mu\nu\cdots\sigma}$ under Weyl rescalings. It is from these quantities that the corrections $\Pi_{(1)},\Pi_{(2)},\Upsilon_{(1)},\Upsilon_{(2)},\Sigma_{(1)}$ and $\Sigma_{(2)}$ are built. Note that $\mathcal{D}_\mu\mathcal{T}$ is second order in $\mathcal{D}_\mu u_\nu$. This follows from the conformal equation of motion $u^\lambda\mathcal{D}_\lambda\rho=-\Pi^{\mu\nu}\sigma_{\mu\nu}$ (see (4.72)). Since $\rho\sim\mathcal{T}^d$ we therefore conclude that whenever a singe derivative in the temperature occurs, it can be replaced with a term containing two or more derivatives of the fluid velocity. Also note that there is no equation relating the derivative of the chemical potential to that of the fluid derivative. This means that $\mathcal{D}_\mu\nu_I$ can not be replaced by derivatives of the fluid velocity in the derivative expansion (4.59).

We will now focus on the second order stress tensor correction $\Pi^{\mu\nu}_{(2)}$ in the case where there are no charges, $\nu_I=0$. Since $\Pi^{\mu\nu}_{(2)}$ is symmetric, Weyl-covariant, transverse and traceless, it must be built out of terms with the same properties. For a general tensor $A^{\mu\nu}$ of rank two it is therefore convenient to introduce the operation $\langle\cdot,\cdot\rangle$ defined by

$$A^{\langle\mu\nu\rangle}\equiv\Delta^{\mu\alpha}\Delta^{\nu\beta}A_{(\alpha\beta)}-\frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta} \tag{4.60}$$

When acting on $A^{\mu\nu}$, this operation exactly subtracts the antisymmetric part, the longitudinal part, and the trace part of $A^{\mu\nu}$, leaving the symmetric, transverse and traceless part of $A^{\mu\nu}$. Clearly $\sigma^{\langle\mu\nu\rangle}=\sigma^{\mu\nu}$ and $\varpi^{\langle\mu\nu\rangle}=0$. This means that by using the $\langle\cdot,\cdot\rangle$ operation on combinations of the terms (4.59) with two free indices and recalling that the derivative of the temperature can be written in terms of fluid gradients through the equation of motion, we can write down possible contributions to $\Pi^{(2)}_{\mu\nu}$. They are

$$\begin{aligned}
&u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu},\ \varpi^{\langle\mu}{}_{\lambda}\sigma^{\lambda\nu\rangle},\ \mathcal{D}_\lambda u^{\langle\mu}\mathcal{D}^\lambda u^{\nu\rangle},\\
&\sigma^{\langle\mu}{}_{\lambda}\sigma^{\lambda\nu\rangle},\ \varpi^{\langle\mu}{}_{\lambda}\varpi^{\lambda\nu\rangle},\ C^{\langle\mu\ \nu\rangle}_{\ \ \alpha\ \beta}u^\alpha u^\beta,\\
&\mathcal{R}^{\langle\mu\nu\rangle},\ \mathcal{R}^{\langle\mu\ \nu\rangle}_{\ \ \alpha\ \beta}u^\alpha u^\beta
\end{aligned} \tag{4.61}$$

Note that using the properties of $C_{\mu\nu\lambda\rho}$ mentioned below equation (4.56), we have $C^{\langle\mu\ \nu\rangle}{}_{\alpha\ \beta}u^\alpha u^\beta = C^{\mu\ \nu}{}_{\alpha\ \beta}u^\alpha u^\beta$. Moreover, using the symmetry properties (4.53), the last two terms can be written

$$\begin{aligned} \mathcal{R}^{\langle\mu\nu\rangle} &= \Delta^{\mu\alpha}\Delta^{\nu\beta}(\mathcal{R}_{\alpha\beta} + \frac{d}{2}\mathcal{F}_{\alpha\beta}) - \frac{\Delta^{\mu\nu}}{d-1}\Delta^{\alpha\beta}\mathcal{R}_{\alpha\beta} \\ \mathcal{R}^{\langle\mu\ \nu\rangle}{}_{\alpha\ \beta}u^\alpha u^\beta &= \Delta^{\mu\lambda}\Delta^{\nu\rho}(\mathcal{R}_{\lambda\alpha\rho\beta}u^\alpha u^\beta - \frac{1}{2}\mathcal{F}_{\lambda\rho}) - \frac{\Delta^{\mu\nu}}{d-1}\Delta^{\lambda\rho}\mathcal{R}_{\lambda\alpha\rho\beta}u^\alpha u^\beta \end{aligned} \tag{4.62}$$

Notice that the terms in (4.61) are not all independent, for example

$$\mathcal{D}_\lambda u^{\langle\mu}\mathcal{D}^\lambda u^{\nu\rangle} = \sigma^{\langle\mu}{}_\lambda\sigma^{\lambda\nu\rangle} + 2\varpi^{\langle\mu}{}_\lambda\sigma^{\lambda\nu\rangle} + \varpi^{\langle\mu}{}_\lambda\varpi^{\lambda\nu\rangle} \tag{4.63}$$

It is also possible to show that the curvature terms (4.62) can be written in the terms of the other tensors in (4.61), see appendix B of [40]. The possible independent contributions to $\Pi^{\mu\nu}$ up to second order therefore are

$$\begin{aligned} &\sigma^{\mu\nu},\ \mathfrak{T}_1^{\mu\nu} = u^\lambda\mathcal{D}_\lambda\sigma^{\mu\nu},\ \mathfrak{T}_2^{\mu\nu} = \varpi^{\langle\mu}{}_\lambda\sigma^{\lambda\nu\rangle} \\ &\mathfrak{T}_3^{\mu\nu} = \sigma^{\langle\mu}{}_\lambda\sigma^{\lambda\nu\rangle},\ \mathfrak{T}_4^{\mu\nu} = \varpi^{\langle\mu}{}_\lambda\varpi^{\lambda\nu\rangle},\ \mathfrak{T}_5^{\mu\nu} = C^{\mu\ \nu}{}_{\alpha\ \beta}u^\alpha u^\beta \end{aligned} \tag{4.64}$$

All the second order tensors are seen to have conformal weight $w = 4$. The expressions for $\mathfrak{T}_2$, $\mathfrak{T}_3$, and $\mathfrak{T}_4$ are easily written down

$$\begin{aligned} \mathfrak{T}_2 &= \varpi^\mu{}_\lambda\sigma^{\lambda\nu} + \varpi^\nu{}_\lambda\sigma^{\lambda\mu} \\ \mathfrak{T}_3 &= \sigma^\mu{}_\lambda\sigma^{\lambda\nu} - \frac{1}{d-1}\Delta^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta} \\ \mathfrak{T}_4 &= \varpi^\mu{}_\lambda\varpi^{\lambda\nu} + \frac{1}{d-1}\Delta^{\mu\nu}\varpi^{\alpha\beta}\varpi_{\alpha\beta} \end{aligned} \tag{4.65}$$

It therefore follows that the stress tensor can be written (here $\eta_1, \eta_2, \eta_3, \eta_4$ and η_5 are all defined to be Weyl invariant)

$$\begin{aligned} T^{\mu\nu} &= \eta_0\mathcal{T}^d\Big(u^\mu u^\nu + \frac{1}{d-1}\Delta^{\mu\nu}\Big) - 2\eta\sigma^{\mu\nu} \\ &\quad - \mathcal{T}^{d-2}(\eta_1\mathfrak{T}_1^{\mu\nu} + \eta_2\mathfrak{T}_2^{\mu\nu} + \eta_3\mathfrak{T}_3^{\mu\nu} + \eta_4\mathfrak{T}_4^{\mu\nu} + \eta_5\mathfrak{T}_5^{\mu\nu}) + \mathcal{O}(\partial^3) \end{aligned} \tag{4.66}$$

where the unit temperature energy density η_0, shear viscosity η and higher order transport coefficients depend on the field and interaction content of the underlying field theory. For later purposes we will find it convenient to introduce a slightly different parameterization of the stress tensor. We define

$$\mathcal{T}^d\eta_0 = (d-1)p, \qquad \eta_1\mathcal{T}^{d-2} = -2\eta\tau_1, \qquad \mathcal{T}^{d-2}\eta_2 = 2\eta\tau_2, \tag{4.67}$$

$$\mathcal{T}^{d-2}\eta_3 = -\xi_\sigma, \qquad \mathcal{T}^{d-2}\eta_4 = -\xi_\varpi, \qquad \mathcal{T}^{d-2}\eta_5 = -\xi_C \tag{4.68}$$

so that (here p is recognized has the fluid pressure)

$$T^{\mu\nu} = p\left(g^{\mu\nu} + du^\mu u^\nu\right) - 2\eta\left[\sigma^{\mu\nu} - \tau_1\mathfrak{T}_1^{\mu\nu} + \tau_2\mathfrak{T}_2^{\mu\nu}\right] + \xi_\sigma\mathfrak{T}_3^{\mu\nu} + \xi_\varpi\mathfrak{T}_4^{\mu\nu} + \xi_C\mathfrak{T}_5^{\mu\nu} \tag{4.69}$$

4.2.6 Discussion of the entropy current

Here we will elaborate on the entropy current in conformal fluid dynamics discussed in the papers [40, 8]. The thermodynamical fields satisfy the first and second law of thermodynamics

$$\mathcal{T}\mathcal{D}_\mu s = \mathcal{D}_\mu\rho - \mu_I\mathcal{D}_\mu q_I\ , \quad \mathcal{D}_\mu J^\mu_S \geq 0 \tag{4.70}$$

The first of these two equations is the the first law (4.19) written for a conformal fluid. Along with the equations of motion $\mathcal{D}_\mu T^{\mu\nu} = \mathcal{D}_\mu J^\mu = 0$, these relations allow us to write down an expression for the entropy current following essentially the procedure of Landau & Lifshitz [36, 33].

As with the relativistic fluid we assume that the conformal fluid has an associated conformal entropy current J^μ_S. Again we write the entropy current as a part coming from the perfect fluid plus some correction Σ^μ due to dissipation

$$J^\mu_S = su^\mu + \Sigma^\mu \tag{4.71}$$

Using the expressions for the stress tensor and the currents (4.18) along with the conditions $u_\mu \Pi^{\mu\nu} = u_\mu \Upsilon^\mu_I = 0$, we easily obtain

$$\begin{aligned} 0 &= -u_\nu \mathcal{D}_\mu T^{\mu\nu} = u^\mu \mathcal{D}_\mu \rho + \Pi^{\mu\nu}\sigma_{\mu\nu} \\ 0 &= \mathcal{D}_\mu J^\mu_I = u^\mu \mathcal{D}_\mu q_I + \mathcal{D}_\mu \Upsilon^\mu_I \end{aligned} \tag{4.72}$$

Now using the first law (4.70) we then see

$$\mathcal{D}_\mu J^\mu_S = \frac{1}{\mathcal{T}}\left[-\Pi^{\mu\nu}\sigma_{\mu\nu} + \mu_I \mathcal{D}_\mu \Upsilon^\mu_I + \mathcal{T}\mathcal{D}_\mu \Sigma^\mu\right] \tag{4.73}$$

This means that if we know the form of the dissipative stress tensor $\Pi^{\mu\nu}$ and the dissipative currents Υ^μ_I, we can work out the dissipative correction to the entropy current Σ^μ since $\mathcal{D}_\mu J^\mu_S \geq 0$ for all field configurations. However such a construction is clearly not unique. We therefore emphasize that the entropy current presented in this section is a construction which is consistent with the laws of thermodynamics and which, in principle, should be checked from the underlying quantum theory. We now proceed as [40] and substitute the expression for the second order stress tensor (4.66) into the expression for the entropy divergence. Again we will assume that there are no charges. We find

$$\begin{aligned} \mathcal{T}\mathcal{D}_\mu J^\mu_S = {} & \mathcal{D}_\mu \Sigma^\mu + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \eta_1 \mathcal{T}^{d-2}\sigma_{\mu\nu}u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\ & + \eta_3 \mathcal{T}^{d-2}\sigma_{\mu\nu}\sigma^\mu{}_\lambda \sigma^{\lambda\nu} + \eta_4 \mathcal{T}^{d-2}\sigma_{\mu\nu}\varpi^\mu{}_\lambda \varpi^{\lambda\nu} + \eta_5 \mathcal{T}^{d-2}\sigma^{\mu\nu}C_{\mu\alpha\nu\beta}u^\alpha u^\beta \end{aligned} \tag{4.74}$$

For completeness notice that the second term in this equation is recognized as the divergence of the (*constructed*) entropy current in the uncharged case, see equation (4.15) (with $\vartheta = 0$). Using the properties of the Weyl covariant derivative, this expression can be rewritten as

$$\begin{aligned} \mathcal{T}\mathcal{D}_\mu J^\mu_S = {} & 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \mathcal{D}_\mu\Sigma^\mu + (\eta_3+\eta_5)\mathcal{T}^{d-2}\sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu} \\ & + \mathcal{T}^{d-2}\mathcal{D}_\lambda \Bigg[\left(\frac{2(\eta_1+\eta_5)\sigma^{\mu\nu}\sigma_{\mu\nu} + (\eta_4+\eta_5)\varpi^{\mu\nu}\varpi_{\mu\nu}}{4}\right)u^\lambda \\ & \qquad - \frac{\eta_5 u_\mu(\mathcal{G}^{\mu\nu} + \mathcal{F}^{\mu\lambda})}{d-2} - \frac{(\eta_4+3\eta_5)}{2(d-3)}\mathcal{D}_\nu \varpi^{\lambda\nu}\Bigg] \end{aligned}$$

We refer to [40] for the details of this computation. Now since we are working with the divergence of a quantity which is correct up to second order in the derivatives, this expression is correct up to third order in the derivatives. However, it is hard to see how this should give something positive definite, however, [40] suggests taking

$$\begin{aligned} \Sigma^\lambda_{(\leq 2)} = {} & \left(\frac{2\mathcal{T}^{d-3}(\eta_1+\eta_5)\sigma^{\mu\nu}\sigma_{\mu\nu} + \mathcal{T}^{d-3}(\eta_4+\eta_5)\varpi^{\mu\nu}\varpi_{\mu\nu}}{4}\right)u^\lambda \\ & + \frac{\mathcal{T}^{d-3}\eta_5 u_\mu(\mathcal{G}^{\mu\nu} + \mathcal{F}^{\mu\lambda})}{d-2} + \frac{\mathcal{T}^{d-2}(\eta_4+3\eta_5)}{2(d-3)}\mathcal{D}_\nu\varpi^{\lambda\nu} \end{aligned} \tag{4.75}$$

To second order this gives for the divergence of the entropy current

$$\begin{aligned}\mathcal{T}\mathcal{D}_\mu J_S^\mu &= 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + (\eta_3+\eta_5)\mathcal{T}^{d-3}\sigma_{\mu\nu}\sigma^\mu{}_\lambda\sigma^{\lambda\nu} \\ &= 2\eta\left(\sigma^{\mu\nu} + \frac{\eta_3+\eta_5}{4\eta}\sigma^\mu{}_\lambda\sigma^{\lambda\nu}\right)^2 \geq 0\end{aligned} \tag{4.76}$$

since a single derivative of the temperature $\mathcal{T}$ counts as two u^μ derivatives. This shows that to second order, the entropy current (4.75) is consistent with the laws of thermodynamics.

We will now look at a more general expression for the entropy current, originally proposed in [8]. The paper [8] takes a slightly different approach than [40], however, the principles are the same. We start by writing down the most general expression for the entropy current, correct up to second order, and consistent with Weyl covariance:

$$\begin{aligned}J_S^\mu = su^\mu &+ \mathcal{T}^{d-3}u^\mu\left[a_1\sigma_{\mu\nu}\sigma^{\mu\nu} + a_2\varpi_{\mu\nu}\varpi^{\mu\nu} - a_3\mathcal{R}\right] \\ &+ \mathcal{T}^{d-3}\left[b_1\mathcal{D}_\lambda\sigma^{\mu\nu} + b_2\mathcal{D}_\lambda\varpi^{\mu\lambda}\right] + \cdots\end{aligned} \tag{4.77}$$

where $a_1, a_2, \ldots$ are Weyl invariant constants and the dots represent higher order terms. A slightly different notation is used by [8], first of all they define a new variable (which we shall also find convenient to introduce in the subsequent chapter)

$$b = \frac{d}{4\pi\mathcal{T}} \tag{4.78}$$

Moreover they use that

$$s = \alpha\mathcal{T}^{d-1} = \alpha\left(\frac{d}{4\pi}\right)^{d-1} b^{1-d} \tag{4.79}$$

which is realized using the same dimensional-type argument as for the energy density. The constant α is identified with $\alpha = \left(\frac{4\pi}{d}\right)^{d-1}\frac{1}{G_{\mathrm{AdS}}}$ using a holographic computation (this is derived in the next chapter). Using this and the following identifications $a_1 = \alpha\left(\frac{d}{4\pi}\right)^2 A_1,\ a_2 = \alpha\left(\frac{d}{4\pi}\right)^2 A_2, \ldots,$ the entropy current (4.77) coincides with the one used in [8]. It is now possible to compute the divergence of (4.77). The result correct up to third order is [8]

$$\begin{aligned}4G_{\mathrm{AdS}}b^{d-1}\mathcal{D}_\mu J_S^\mu = \frac{2b}{d}\Bigg[&\sigma_{\mu\nu} - \frac{bd(d-2)}{2}\left(A_3 - \frac{2A_2}{d-2}\varpi_{\mu\lambda}\varpi^\lambda{}_\nu\right) \\ &- \frac{bd(d-2)}{2}\left(A_3 + \frac{1}{d(d-2)}\right)\left(\sigma_\mu{}^\lambda\sigma_{\lambda\nu} + u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right) \\ &+ \frac{1}{2}(A_1 bd + \tau_\varpi)u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}\Bigg]^2 \\ &+ b^2(B_1 + 2A_3)\mathcal{D}_\mu D_\nu\sigma^{\mu\nu} + \cdots\end{aligned} \tag{4.80}$$

The second law of thermodynamics therefore puts the following simple condition on the coefficients

$$B_1 + 2A_3 = 0 \tag{4.81}$$

Moreover [8] shows that the entropy current (4.75) can be obtained by a certain choice of the coefficients A_1, A_2, A_3, B_1, B_2 fulfilling the condition (4.81).

5

Fluid dynamics of fluctuating branes

5.1 Introduction

We will now implement the ideas behind the fluid/gravity conjecture. In the following we will follow the original work of [6] which was later generalized in [7, 9, 8, 10]. Notice that the authors of [6] carried out their computations for $D = 5$. Here will generalize the (first order) computations of [6] to an arbitrary number of spacetime dimensions and thereby verify the results of [8]. Before carrying out the calculations we will explain the main idea behind this chapter.

The essence of the fluid/gravity correspondence was explained in §3.3 and is summarized in the table 3.1. The dynamics in the gravitational subsector is completely determined by the equation

$$\mathcal{E}_{AB} = \mathcal{G}_{AB} - \frac{(D-1)(D-2)}{2} G_{AB} = 0 \tag{5.1}$$

Assuming that the field theory is in the hydrodynamics regime, this means that we can map gravitational solutions to fluid dynamic solutions through the AdS/CFT correspondence. We will construct such a map by solving Einstein's equations perturbatively around a well-understood stationary solution. The starting point of this construction is therefore to find a suitable stationary gravitational background around which we can do perturbation theory. The dual description of such a stationary gravitational configuration must also be stationary i.e. in global equilibrium, in other words, a perfect fluid. As we explain below (and have motivated in chapter 3) these stationary gravitational solutions will exactly be the so-called boosted black branes. These solutions can be thought of as ordinary black brane solutions which

	Einstein Gravity	Fluid dynamics
Stationary solutions	Boosted black branes	Perfect fluid dynamics
Perturbations	Non-uniformly evolving black branes	Dissipative fluid flows

Table 5.1: *According to the fluid/gravity correspondence, the two columns are dual. By understanding the theory of one column we therefore get an understanding of the other.*

are uniformly moving in their transverse directions. The idea now is to consider the effect on Einstein's equations if we slightly perturb the background black brane metric. Using the tools of the AdS/CFT correspondence, the perturbed gravitational equations should then translate into the equations of relativistic viscous fluid mechanics, especially the derivative expansion (4.66) and the conservation equation (4.1). By solving Einstein's equations in AdS_D we can therefore extract information about the dual fluid (= compute its transport coefficients) and thereby extract information about the underlying dual field theory.

It can be shown that the fluid dynamics of a $d = 3-1 = 2$ dimensional conformal field theory essentially is trivial (see appendix B of [8]). In the following analysis we will therefore exclude the $D = 2+1 = 3$ dimensional case.

5.2 The boosted black brane as an ideal fluid

In this section we introduce the boosted black branes in anti-de Sitter background and show that these solutions are exactly dual to a perfect conformal boundary fluid. In the dual picture these solutions therefore corresponds to an equilibrium state of the thermal field theory on Minkowski space. These boosted black branes will serve as the background spacetime which we will slightly perturb in order to derive the properties of the dual fluid.

5.2.1 Preliminaries and the black brane temperature

Here we introduce a class of AdS_{d+1} solutions known as boosted black branes. Boosted black branes are simply boosted versions of the well-known ordinary black brane solutions which are the geometry duals to thermal field theory on Minkowski space (the finite temperature AdS/CFT correspondence). Consider the ordinary black brane[1] metric given by the well-known expression

$$G_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{r^2}{L^2}\left[-f(br)\mathrm{d}t^2 + \sum_{i=1}^{d-1}(\mathrm{d}x^i)^2\right] + \frac{L^2}{r^2 f(br)}\mathrm{d}r^2 \tag{5.2}$$

with $f(r) = 1 - r^{-d}$ and where A, B as usual denotes the bulk indices. Here $x^\mu \equiv (t, x^i)$ denotes the transverse boundary coordinates while r is a "radial" AdS coordinate. Indeed, it is straight forward to show that the Ricci tensor of the metric G_{AB} fulfills $R_{AB} \propto G_{AB}$ with $R = -d(d+1)/L^2$. The metric (5.2) is therefore an (asymptotic) AdS_{d+1} solution. Note that the black brane solution indeed is black: The metric (5.2) contains an event horizon located at $r \equiv r_+ = 1/b$. This is why the metric (5.2) becomes singular at $r = r_+$, however, as usual this singularity is not related to the spacetime but rather to the choice of coordinates - more on this below.

Now suppose that we perform a "boost" of the black brane in the boundary directions with boost velocity u^μ, $u_\mu u^\mu = -1$. The resulting metric is easily found by covariantizing the metric (5.2). We have that $\mathrm{d}t = -u_\mu \mathrm{d}x^\mu$ and $\mathrm{d}x^i = \Delta^i{}_\mu \mathrm{d}x^\mu$ where $u^\mu = (1, 0, \cdots)$ and where $\Delta^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$ is the usual projector onto the spatial boundary directions with $\eta_{\mu\nu}$ being the Minkowski boundary metric. This especially means that $\mathrm{d}t^2 = u_\mu u_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu$ and $\sum_i (\mathrm{d}x^i)^2 = \Delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. The boosted version of (5.2) is therefore given by

$$G^{\mathrm{boost}}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{r^2}{L^2}\big[-f(br)u_\mu u_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu + \Delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\big] + \frac{L^2}{r^2 f(br)}\mathrm{d}r^2 \tag{5.3}$$

[1] The black brane (5.2) also is known as the planar Schwarzschild black hole (in AdS).

where u^μ is a general four vector fulfilling that $u_\mu u^\mu = -1$. Since boosts in the boundary directions belong to the anti-de Sitter isometry group, the class of solutions (5.3) are all solutions to Einstein's equation (5.1). This fact is also easily verified by simply plugging (5.3) into (5.1). It is possible to parameterize the velocity u^μ in terms of a set of parameters $\beta^i \in \mathbb{R}$ in the usual manner

$$u^v = \frac{1}{\sqrt{1-\beta^2}}, \quad u^i = \frac{\beta^i}{\sqrt{1-\beta^2}} \tag{5.4}$$

Finally, by performing a Wick rotation and compactifying the time direction, we can show that the temperature of the black brane is given by

$$\mathcal{T} = \frac{dr_+}{4\pi L^2} = \frac{d}{4\pi L^2 b} \tag{5.5}$$

We refer to the appendix B for the details of this short computation. From now on we will set the AdS radius L to unity $L \equiv 1$.

5.2.2 The dual fluid of the boosted black brane

In this section we demonstrate that the fluid dual to the boosted black brane is a conformal perfect fluid. As mentioned in the introduction, such a result must be expected since the black brane is a stationary gravitational solution. Computing the boundary stress tensor from the bulk metric field relies on the gravity part of the AdS/CFT directory. We refer to §3.4 for the mathematical details. Using the standard method of the AdS/CFT directory (equation (3.43)), it is possible to show that the stress tensor dual to the boosted black brane metric (5.3) is given by (we refer to the appendix B for the details of this computation)

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{16\pi G_D}\left(\frac{4\pi\mathcal{T}}{d}\right)^d [\eta^{\mu\nu} + du^\mu u^\nu] \\ &= \frac{d-1}{16\pi G_D}\left(\frac{4\pi\mathcal{T}}{d}\right)^d [u^\mu u^\nu + \frac{1}{d-1}\Delta^{\mu\nu}] \end{aligned} \tag{5.6}$$

where $\mathcal{T}$ is the black brane temperature. The stress tensor (5.6) is exactly that of a perfect fluid fulfilling the conformal equation of state with

$$\rho = \frac{d-1}{16\pi G_D}\left(\frac{4\pi\mathcal{T}}{d}\right)^d, \quad p = \frac{\rho}{d-1} \tag{5.7}$$

We refer to §4.2 on perfect conformal fluid dynamics for the details. We conclude that the boosted black brane has a dual description in terms of a zeroth order (i.e, perfect) conformal fluid with fluid velocity corresponding to the boost velocity of the brane. However, as also mentioned in the introduction, the dual fluid is expected to have a set non-zero higher order transport coefficients. The dual fluid configuration of the boosted black brane should therefore be thought of as a viscous fluid that has reached hydrostatic equilibrium. Indeed, since all the u^μ derivatives vanish, we conclude that all viscous tensors such as the shear $\sigma_{\mu\nu}$ vanish.

5.2.3 The boosted black brane in generalized Gaussian null coordinates

Here we introduce a set ingoing Eddington-Finkelstein-like coordinates which are suited for solving Einstein's equation perturbatively. Again consider the black brane

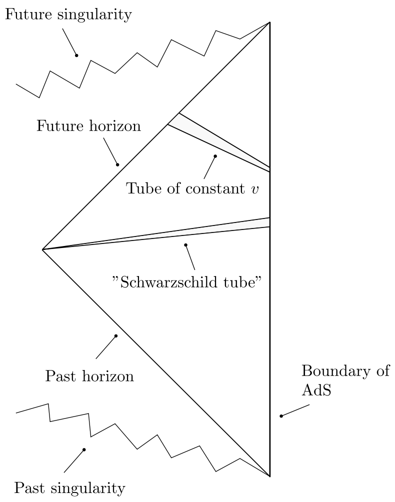

Figure 5.1: *Penrose diagram of the black brane metric with a null tube, $(v, x^i) =$ const., and a Schwarzschild tube, $(t, x^i) =$ const., illustrated.*

metric (5.2). Now introduce a new set of coordinates (r, v, x^i) by the following equations

$$v = t + r^*, \quad \mathrm{d}r^* = \frac{\mathrm{d}r}{r^2 f(br)} \tag{5.8}$$

We shall refer to these coordinates as Gaussian null coordinates. It is straight forward to show that the in these coordinates, the metric takes the form

$$G_{AB}\mathrm{d}x^A\mathrm{d}x^B = 2\mathrm{d}v\mathrm{d}r - r^2 f(br)\mathrm{d}v^2 + r^2 \sum_{i=1}^{d-1}(\mathrm{d}x^i)^2 \tag{5.9}$$

The Gaussian null coordinates encapsulates the causal structure of the spacetime in a very nice way. Indeed, suppose that we slightly disturb the the gravitational field near the boundary. This disturbance will propagate towards the horizon along the null geodesic ($\mathrm{d}s^2 = 0$) which exactly corresponds to the line of constant v, $\mathrm{d}v = 0$. This property is essential when we will solve the bulk Einstein equations using a perturbative expansion. Moreover we see that the metric (5.9) is completely regular at the horizon $r = r_+$. In fact (5.9) only becomes singular at $r = 0$, which was of course expected. It is again straight forward to write down the expression for the boosted black brane by covariantizing the expression (5.9). We get (where now $x^\mu = (v, x^i)$)

$$G_{AB}^{\text{boost}}\mathrm{d}x^A\mathrm{d}x^B = -2u_\mu\mathrm{d}x^\mu\mathrm{d}r - r^2 f(br)u_\mu u_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu + r^2\Delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{5.10}$$

5.3 The non-uniformly evolving black branes as a viscous fluid

In the above section we verified that the fluid/gravity correspondence is valid to to 0^{th} order. However, the first real check/implementation of the fluid/gravity correspondence comes at 1^{th} order fluid dynamics. In order to see viscous effects we must break the uniformity of the black brane so that e.g. the shear becomes non-zero. We will do this by taking take the uniform black brane (5.10) and promote the brane velocity and (inverse) temperature to *slowly* varying fields in the transverse boundary coordinates $u^\mu \to u^\mu(x^\alpha), b \to b(x^\alpha)$:

$$G_{AB}^{(0)}\mathrm{d}x^A\mathrm{d}x^B = -2u_\mu(x^\alpha)\mathrm{d}x^\mu\mathrm{d}r - r^2 f(b(x^\alpha)r)u_\mu(x^\alpha)u_\nu(x^\alpha)\mathrm{d}x^\mu\mathrm{d}x^\nu + r^2\Delta_{\mu\nu}(x^\alpha)\mathrm{d}x^\mu\mathrm{d}x^\nu \quad (5.11)$$

Of course this metric will in general *not* be a solution to Einstein's equation (5.1) anymore. However, it is still regular for all r well-separated from $r = 0$ and we expect it to be a good approximation to the true solution since locally in x^μ it can be "tubewise" well approximated by a boosted black brane.[2]

5.3.1 Setting up the perturbative expansion

Following the work of [6] we will now solve Einstein's equation perturbatively in the number of field theory x^μ derivatives.[3] First we will perform a rescaling of the equations: By a simple scaling transformation it is always possible to set $b = 1$ at any given boundary point. The condition for the validity of the perturbation expansion therefore reduces to $L \gg 1$. Now introduce a parameter $\varepsilon = \mathcal{O}(1/L)$. We will now consider a rescaled field theory where the functions b and β_i are considered as functions of the rescaled field coordinates εx^μ. We therefore see that every derivative of b or β_i produces a power of ε, it follows that ε counts the number of derivatives. In the rescaled field theory the derivatives are therefore of $\mathcal{O}(1)$. However a term containing n-derivatives will always have an associated factor ε^n. The parameter ε should therefore be thought of as a book-keeping parameter which keeps track of the field theory derivatives and which should eventually be set to unity (along with reintroducing the length dimension).

We will now write the bulk metric as a power series in ε. We write

$$G_{AB} = G_{AB}^{(0)}\big(\beta_i, b\big) + \varepsilon G_{AB}^{(1)}\big(\beta_i, b\big) + \varepsilon^2 G_{AB}^{(2)}\big(\beta_i, b\big) + \mathcal{O}(\varepsilon^3) \quad (5.12)$$

Here G_{AB} is the metric that solves the full set of Einstein equations while $G_{AB}^{(0)}$ is the metric (5.11) that solves Einstein's equation for b and β_i constant. Notice that $G_{AB}^{(0)}$ also will contain $\varepsilon, \varepsilon^2, \ldots$ terms coming from Taylor expanding the functions b and β_i around the point which we solve (5.1) (see below). Similarly for $G_{AB}^{(1)}$, $G_{AB}^{(2)}, \ldots$. Having corrected the metric, we are forced to correct the temperature and boost velocity:

$$\begin{aligned}\beta_i(x^\mu) &= \beta_i^{(0)}(\varepsilon x^\mu) + \varepsilon\beta_i^{(1)}(\varepsilon x^\mu) + \mathcal{O}(\varepsilon^2)\\ b(x^\mu) &= b^{(0)}(\varepsilon x^\mu) + \varepsilon b^{(1)}(\varepsilon x^\mu) + \mathcal{O}(\varepsilon^2)\end{aligned} \quad (5.13)$$

Notice that if we correct the metric up to n^{th} order in ε, consistent perturbation theory only requires us to correct the temperature and velocities up to $(n-1)^{\text{th}}$ order. This is because the Einstein equation will not contain terms that have no derivatives (and a derivative produces an additional factor of ε) since $G_{AB}^{(0)}$ with b and β_i constant solves the Einstein equation.

As is well known in gravitational problems, gravity has a huge gauge group consisting of all the possible spacetime diffeomorphisms. In essence this means that the same gravitational physics can be described in an infinite set of different

[2] This is where our generalized Gaussian coordinates come to use: The tube of constant r lies along a null geodesic.

[3] It is possible to show that the field theory x^μ derivatives of either $\log b(x^\mu)$ or $\beta_i(x^\mu)$ always appear with a factor of $b \sim 1/T$. This means that the contribution of an n-derivative term always is suppressed by a factor $(b/L)^n \sim 1/(TL)^n$, where L is the typical transverse variation scale of the temperature and velocity fields. It therefore follows that provided that $LT \gg 1$ it is sensible to solve Einstein's equation perturbatively in the number of field theory derivatives.

coordinate systems. For computational reasons it is therefore useful (however not necessary) to fix a gauge. We use the following gauge choice

$$G_{rr} = 0, \quad g_{r\mu} \propto u_\mu, \quad \mathrm{Tr}\left(\left(G^{(0)}\right)^{-1} G^{(n)}\right) = 0 \ \ \forall n > 0 \tag{5.14}$$

This is also the gauge used in [6, 8]. For other gauge choices and physical interpretations see [7]. Having discussed some of the general features of the perturbation expansion, we will now write down an expression for G^0_{AB} to 1^{th} order in ε.

Take some boundary point p. Having promoted the constants b and β_i to functions of the rescaled boundary coordinates εx^μ, we may now do a Taylor expansion in the boundary coordinates around the point p. By a simple coordinate transformation we may choose $u^\mu = (1, 0, 0, 0)$ and (as mentioned above) $b = 1$ in p. Notice that we are assuming nothing about the derivatives in p. Also note that $\partial_\mu u^i = \partial_\mu \beta^i$, $\partial_\mu u^v = 0$ for $u^\mu = (1, 0, 0, 0)$. We therefore have that around p (chosen to be in the origin of $\mathbb{R}^{3,1}$ by a simple translation)

$$\begin{aligned} G^{(0)}_{AB}\, \mathrm{d}x^A \mathrm{d}x^B &= 2\mathrm{d}v\mathrm{d}r - r^2 f(r)\mathrm{d}v^2 + r^2 \mathrm{d}x_i \mathrm{d}x^i \\ &- \varepsilon\Big(2x^\mu \partial_\mu \beta^{(0)}_i \mathrm{d}x^i \mathrm{d}r - 2x^\mu \partial_\mu \beta^{(0)}_i r^2 (1 - f(r)) \mathrm{d}x^i \mathrm{d}v - d\frac{x^\mu \partial_\mu b^{(0)}}{r^{d-1}} \mathrm{d}v^2\Big) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{5.15}$$

It is to this expression we will add the 1^{th} order correction $G^{(1)}_{AB}$ so that $G^{(0)} + \varepsilon G^{(1)}$ solves (5.1) (to first order in ε).

5.3.2 The SO$(d-1)$ sectors

Even though the Einstein equation is non-linear, it is often possible, in perturbation theory, to exploit the symmetries of the background metric to separate the resulting equations into different sectors of the background symmetries. This decouples the equations and makes them much easier to solve. The background metric (the first line of (5.15)) has a clear SO$(d-1)$ symmetry in which we will decompose the correction tensor G_1.

A generic two-tensor $S^{\mu\nu}$ can be split up into irreducible representations of SO$(d-1)$. Clearly the components S^{vv}, S^{vr}, S^{rv} and S^{vr} all transform as scalars ($\mathbf{0}$) under SO$(d-1)$. The components S^{ir} and S^{iv} transform like vectors ($\mathbf{1}$) while the components S^{ij} transform as a SO$(d-1)$ tensor ($\mathbf{1} \otimes \mathbf{1}$). As usual the tensor S^{ij} can be split up in a trace part ($\mathbf{0}$), an antisymmetrical part ($\mathbf{1}$) and a traceless symmetrical part ($\mathbf{2}$).

It is now straight forward to write down the transformation properties of the metric tensor (= symmetric tensor). We must simply have $\mathrm{G}_{AB} \in (\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0}) \oplus (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{2})$. The collection of scalar representations (the first parentheses) is referred to as the *scalar sector*, the collection of vector representations (the second parentheses) is referred to as the *vector sector* while the symmetric traceless tensor representation (the third parentheses) is referred to as the *tensor sector*. We will now write down the $G^{(1)}$ correction according to this SO$(d-1)$ symmetry.

The scalar sector: According to our gauge choice (5.14) we have $G^{(1)}_{rr} = 0$. Moreover, we see that gauge condition (5.14) implies that $\mathrm{Tr}\{G_{ij}\} + 2r^2 G_{vr} = 0$. The components in the scalar sector in the particular gauge (5.14) can therefore be parameterized by two independent functions. We parameterize the part of metric fluctuation that transform in the **0** rep. in the following way

$$\begin{aligned} \mathrm{Tr}\{\, G^{(1)}_{ij}\} &= (d-1)r^2 h^{(1)}(r) \\ G_{vr} &= -\frac{d-1}{2} h^{(1)}(r) \\ G^{(1)}_{vv} &= \frac{k^{(1)}(r)}{r^{d-2}} \end{aligned} \tag{5.16}$$

THE VECTOR SECTOR: With the particular choice of u^μ and gauge we have that $G_{ri} = 0$. One of the vectors is therefore gauged to zero. We parameterize the remaining vector according to

$$(G^{(1)}_V)_{AB} \mathrm{d}x^A \mathrm{d}x^B = 2r^2 \left(1 - f(r)\right) j^{(1)}_i(r) \mathrm{d}x^i \mathrm{d}v \tag{5.17}$$

THE TENSOR SECTOR: Finally we parameterize the tensor sector in the following way

$$(G^{(1)}_T)_{AB} \mathrm{d}x^A \mathrm{d}x^B = r^2 \alpha_{ij}(r) \mathrm{d}x^i \mathrm{d}x^j \tag{5.18}$$

where α_{ij} is a symmetric traceless matrix.

5.4 Results of the first order computation

Plugging this into the Einstein equations $\mathcal{E}_{AB} = 0$ yields a set of equations which we now solve to first order in ε. The full set of equations can be found in the appendix B.

5.4.1 The structure of the equations

A priori the Einstein equation $\mathcal{E}_{AB} = 0$ contains $\frac{D(D+1)}{2}$ independent equations. It will be useful to split these equations up into two classes of equations known respectively as constraint equations and dynamical equations.

CONSTRAINT EQUATIONS: These are the equations that are first order in r-derivatives. These equations are obtained by dotting $\mathcal{E}_{AB} = 0$ with ξ^B where $\xi_B = (\mathrm{d}r)_B$. The constraint equations are therefore

$$\mathcal{E}_{AB}\xi^B = \mathcal{E}_{AB} G^{Br} = 0 \tag{5.19}$$

As we shall see below, the set of equations (5.19) are referred to as constraint equations since they constrain the fluctuating fields b and β_i in a certain way.

DYNAMICAL EQUATIONS: The rest of the $\frac{D(D-1)}{2}$ equations $\mathcal{E}_{AB} = 0$ are referred to as dynamical equations. These equations determine the dynamics of G_{AB} given the constraint equations.

5.4.2 Scalar sector

The scalar sector contains four equations, namely $\mathcal{E}_{rr} = 0$, $\mathcal{E}_{vr} = 0$, $\mathcal{E}_{vv} = 0$ and $\mathrm{Tr}\{\mathcal{E}_{ij}\} = 0$. The constraint equations relevant for the scalar sector are

$$\mathcal{E}_{rB} G^{Br} = r^2 f(r) \mathcal{E}_{rr} + \mathcal{E}_{vr} = 0 \quad \text{and} \quad \mathcal{E}_{vB} G^{Br} = r^2 f(r) \mathcal{E}_{rv} + \mathcal{E}_{vv} = 0 \tag{5.20}$$

The first constraint yields

$$d(d-1)r^{d-1}h_1 + \left((d-1)r^d - \frac{d-2}{2}\right)\frac{\mathrm{d}h_1}{\mathrm{d}r} - \frac{\mathrm{d}k_1}{\mathrm{d}r} = -2(d-1)r^{d-2}\left(\frac{\partial_i \beta^{(0)}_i}{d-1}\right) \tag{5.21}$$

while the second constraint equation is found to be completely independent of r. It yields

$$\partial_v b^{(0)} = \frac{\partial_i \beta_i^{(0)}}{d-1} \tag{5.22}$$

This equation turns out to have a very nice physical interpretation in terms of the dual boundary theory. It is the temporal component of the energy-momentum conservation equation for the dual stress tensor! This relationship if further examined in §5.6. For now, we will only think about equation (5.21) as a constraint on the fields b and β_i.

In addition to the constraint equations, the scalar sector contains a set of dynamical equations. These two equations can freely be chosen as any linear combination of $\mathcal{E}_{rr} = 0$ and $\mathcal{E}_{vv} = 0$ (we choose $\mathcal{E}_{vv} = 0$) and $\mathrm{Tr}\{\mathcal{E}_{ij}\} = 0$. The first dynamical equation $\mathcal{E}_{vv} = 0$ is found to be equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{d+1}\frac{\mathrm{d}h^{(1)}(r)}{\mathrm{d}r}\right] = 0 \tag{5.23}$$

It turns out that the second dynamical equation is automatically satisfied once (5.20) and (5.23) are satisfied. Indeed we have that

$$\mathrm{Tr}\{\mathcal{E}_{ij}\} = r^{d-4}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{d-1}\mathcal{E}_{rB}G^{Br}\right] + \left(r^d + \frac{d-2}{2}\right)\mathcal{E}_{vv}\right] \tag{5.24}$$

We therefore conclude that the scalar sector contains d equations where one of the equations is redundant.

The equations (5.21) and (5.23) can now be solved. Integration of the dynamical equation (5.23) shows that

$$h^{(1)}(r) = \gamma + \frac{\beta}{r^d} \tag{5.25}$$

where γ and β are two integration constants in r (i.e. $\gamma \equiv \gamma(x^\mu)$ and $\beta \equiv \beta(x^\mu)$). Plugging this solution into the constraint (5.21) yields a differential equation for $k^{(1)}$ which can now be solved. We find that

$$k^{(1)}(r) = (d-1)\gamma r^d + \frac{2}{d-1}r^{d-1}\partial_i\beta_i^{(0)} - \frac{d-2}{2}\frac{\beta}{r^d} + \pi_v \tag{5.26}$$

with $\pi_v \equiv \pi_v(x^\mu)$. The solutions in the scalar sector are therefore given by a three-parameter family of solutions. However, these solutions can effectively be reduced to a unique solution by virtue of the boundary conditions, gauge invariance in the bulk and gauge invariance on the boundary (the final point was discussed in §4.1.4). First of all, as demonstrated in the appendix B, the parameter γ multiplies a non-normalizable mode meaning that the field theory stress tensor will blow up in the large r limit. The bulk term multiplying γ simply grow to fast for them to give rise to a meaningful dual field theory (our boundary condition). The parameter γ is therefore forced to $\gamma = 0$. Moreover, it is easy to show that with a certain choice of a the action of the coordinate transformation $r \to r(1 + \varepsilon a/r^d)$ removes the β/r^d terms in the expressions for $h^{(1)}$ and $k^{(1)}$ (and adds a constant to $k^{(1)}$, so effectively the action is $(\beta, \pi_v) \to (0, \pi_v'))$. Since the terms multiplying β are produced by coordinate transformations, they are "pure gauge" and β may be set to zero with out loss of generality. Finally, as we also show in the appendix B, the constant term π_v appearing in (5.26) is fixed to zero by requiring the Landau gauge condition $u_\mu \Pi^{\mu\nu} = 0$ for the boundary fluid.

All in all, we find that the solution in the scalar sector, which we denote G_S, is given by

$$(G_S^{(1)})_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = \frac{2}{d-1}r\partial_i\beta_i\,\mathrm{d}v^2 \tag{5.27}$$

5.4.3 Vector sector

Having solved the equations in the scalar sector we now move on to solving the vector sector. The equations relevant to the vector sector are the $2(d-1)$ equations $\mathcal{E}_{ri} = 0$ and $\mathcal{E}_{vi} = 0$. The constraint equation takes the form

$$\mathcal{E}_{iB} G^{Br} = r^2 f(r) \mathcal{E}_{ri} + \mathcal{E}_{vi} = 0 \tag{5.28}$$

This contraint is again found to be independent of r and yields

$$\partial_v \beta_i^{(0)} = \partial_i b^{(0)} \tag{5.29}$$

Once again, as explained in §5.6, this equation has a very nice interpretation in terms of the boundary fluid theory. Having worked out the constraint, we can now write down the dynamical equation. Again this equation can be chosen as any linear combination of $\mathcal{E}_{ri} = 0$ and $\mathcal{E}_{vi} = 0$, we choose the latter. The equation $\mathcal{E}_{vi} = 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{r^{d-1}} \frac{\mathrm{d}j_i^{(1)}(r)}{\mathrm{d}r} \right] = -\frac{d-1}{r^2} \partial_v \beta_i \tag{5.30}$$

This equation may now be integrated to yield an expression for the metric fluctuation in the vector sector. We get

$$j_i^{(1)} = \partial_v \beta_i^{(0)} r^{d-1} + \gamma_i r^d + \pi_i \tag{5.31}$$

Notice that we also use the vector constraint equation (5.29) to obtain this expression. As in the case of the scalar sector the γ_i's multiply non-normalizable modes and are, by virtue of the boundary conditions, thus forced to zero. Moreover the Landau frame condition again forces the constants π_i parameters to zero. The solution in the vector sector is therefore

$$(G_V^{(1)})_{AB}\, \mathrm{d}x^A \mathrm{d}x^B = 2r \partial_v \beta_i^{(0)}\, \mathrm{d}v \mathrm{d}x^i \tag{5.32}$$

5.4.4 Tensor sector

We are now ready to solve the tensor sector. It is important to realize that at this point we have already solved the scalar and the vector sector. In particular this means that we have solved $\mathrm{Tr}\{\mathcal{E}_{ij}\} = 0$. The dynamical equations in the stress sector therefore reduce to $\mathcal{E}_{ij} = 0$. These dynamical equations give us the following set of equations

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^{d+1} f(r) \frac{\mathrm{d}\alpha_{ij}^{(1)}(r)}{\mathrm{d}r} \right) = -2(d-1) r^{d-2} \sigma_{ij}^{(0)} \tag{5.33}$$

where we have defined the traceless $\mathrm{SO}(d-1)$ tensor $\sigma_{ij}^{(0)}$ by

$$\sigma_{ij}^{(0)} = \partial_{(i} \beta_{j)}^{(0)} - \frac{1}{d-1} \delta_{ij} \partial_m \beta_m^{(0)} \tag{5.34}$$

The solution the equation (5.33) is again found by direct integration. We will demand two things of the solution $\alpha_{ij}^{(1)}$: First of all it must be regular everywhere (especially at the horizon $r = 1$). Secondly we require that $\alpha_{ij}^{(1)} \to 0$ for $r \to \infty$ by virtue of the boundary conditions. These two conditions fix the integration constants relevant for (5.33) and the solution is therefore $\alpha_{ij}(r) = 2\sigma_{ij}^{(0)} F(r)$, where

$$F(r) \equiv (d-1) \int_r^\infty \frac{\mathrm{d}\xi}{\xi^{d+1} f(\xi)} \int_1^\xi \mathrm{d}y\, y^{d-2} = \int_r^\infty \mathrm{d}\xi\, \frac{\xi^{d-1} - 1}{\xi(\xi^d - 1)} \tag{5.35}$$

Notice that by choosing the lower limit in the inner integral to 1, the overall solution will be regular at the horizon $r = 1$ even though the factor $1/(r^{d+1}f(r)) \to \infty$ for $r \to 1$. Finding an analytical expression for the integral (5.35) is quite easy, however, since it is only the asymptotic form of the bulk metric that is used to obtain the boundary stress tensor, we shall be content with recording the large r behavior of $F(r)$:

$$F(r) = \frac{1}{r} - \frac{1}{dr^d} + \mathcal{O}(1/r^{d+1}) \tag{5.36}$$

The solution in the tensor sector is therefore given by the expression

$$(G_T^{(1)})_{AB}\,\mathrm{d}x^A x^B = 2r^2 F(r)\sigma^{(0)}_{ij}\,\mathrm{d}x^i\mathrm{d}x^j \tag{5.37}$$

This concludes the computation in the tensor sector.

5.4.5 The first order metric

Collecting the results from the previous section, we are now ready to write down the metric that solves Einstein's equation to first order in ε.

With the function F and the traceless SO$(d-1)$ tensor $\sigma^{(0)}_{ij}$ defined by respectively (5.35) and (5.34), the metric

$$\begin{aligned}(G)_{AB}\,\mathrm{d}x^A\mathrm{d}x^B &= 2\mathrm{d}v\mathrm{d}r - r^2 f(r)\mathrm{d}v^2 + r^2\sum_{i=1}^{d-1}(\mathrm{d}x^i)^2 \\ &- \varepsilon\Big(2x^\mu\partial_\mu\beta^{(0)}_i\mathrm{d}x^i\mathrm{d}r - 2x^\mu\partial_\mu\beta^{(0)}_i r^2(1-f(r))\mathrm{d}x^i\mathrm{d}v - d\frac{x^\mu\partial_\mu b^{(0)}}{r^{d-2}}\mathrm{d}v^2 \\ &+\frac{2}{d-1}r\partial_i\beta^{(0)}_i\mathrm{d}v^2 + 2r\partial_v\beta^{(0)}_i\mathrm{d}v\mathrm{d}x^i + 2r^2F(r)\sigma^{(0)}_{ij}\mathrm{d}x^i\mathrm{d}x^j\Big)\end{aligned} \tag{5.38}$$

solves Einstein's equation in a neighborhood of $x^\mu = 0$ to first order in ε, provided that

$$\partial_v b^{(0)} = \frac{\partial_i\beta^{(0)}_i}{d-1} \quad \text{and} \quad \partial_v\beta^{(0)}_i = \partial_i b^{(0)} \tag{5.39}$$

We will now write down the global solution to Einstein's equation correct up to first order in the derivatives. Since the expression for the metric is ultra-local in the boundary coordinates (it only depends on the field values in $x^\mu = 0$) it can now be extended to all boundary points: The metric (5.38) was written under the assumption that $u^\mu = \delta^\mu_0$ and $b = 1$ in the particular point $x^\mu = 0$. In order to write down the global solution we must therefore simply covariantize (5.38). Moreover we must re-introduce the length parameter and set $\varepsilon = 1$ (i.e. so that now $\mathcal{O}(\partial^n) = 1/(\mathcal{T}L)^n \ll 1$). It is easy to see that for $b = 1$, $u^\mu = \delta^\mu_0$ and to first order in the derivatives (recall that in this case $\partial_\mu u^0 = 0$, $\partial_\mu u^i = \partial_\mu\beta^i$), the metric

$$\begin{aligned}G_{AB}\,\mathrm{d}x^A\mathrm{d}x^B &= -2u_\mu\mathrm{d}x^\mu\mathrm{d}r - r^2 f(br)u_\mu u_\nu\mathrm{d}x^\nu\mathrm{d}x^\mu + \Delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &2r^2bF(br)\sigma_{\mu\nu}\mathrm{d}x^\nu\mathrm{d}x^\mu + \frac{2}{d-1}ru_\mu u_\nu\partial_\lambda u^\lambda\mathrm{d}x^\nu\mathrm{d}x^\mu - ru^\lambda\partial_\lambda(u_\mu u_\nu)\mathrm{d}x^\nu\mathrm{d}x^\mu\end{aligned} \tag{5.40}$$

reduces to the metric (5.38). Here $\sigma_{\mu\nu}$ is the viscosity tensor introduced in §4.1 given by $\sigma_{\mu\nu} = \mathcal{D}_{(\mu}u_{\nu)}$. Provided that the fields u^μ and b fulfill the constraints (5.39) (which we will write a covariant expression for below), it follows that the metric (5.40) is the global solution to Einstein's equation correct up to first order in the derivatives.

5.5 The boundary stress tensor

Having determined the global metric correct up to first order in the derivatives, we may now compute the boundary stress tensor. The computation relies on the standard method from the AdS/CFT directory and can be found in the appendix B. We find that the stress tensor dual to the metric (5.40) is given by

$$T^{\mu\nu} = \frac{d-1}{16\pi G_{d+1}}\left(\frac{4\pi\mathcal{T}}{d}\right)^d\left[u^\mu u^\nu + \frac{1}{d-1}\Delta^{\mu\nu}\right] - \frac{1}{8\pi G_{d+1}}\left(\frac{4\pi\mathcal{T}}{d}\right)^{d-1}\sigma^{\mu\nu} \quad (5.41)$$

This concludes the first order black brane computations.

5.6 The dual view of the gravitational computation

We saw in §5.2.2 that the dual stress tensor of the boosted black brane is that of a perfect fluid. We interpreted this through the fluid/gravity correspondence: The boosted black brane is dual to a certain stationary configuration of a viscous fluid on the boundary. In §5.3 we promoted the black brane velocity and temperature to be slowly varying and solved Einstein's equations by introducing correction terms. We found that the equations of gravity split up in a dynamical part and a constraint part. The dynamical equations were completely solved to first order in the derivatives and led to the metric (5.40). This in turn determined the boundary stress tensor (5.41). The fluid dynamical interpretation of the boundary stress tensor is clear: According to our discussion on conformal fluid dynamics, the near-equilibrium stress tensor of a conformal fluid is given by

$$T^{\mu\nu} = \eta_0\mathcal{T}(u^\mu u^\nu + \frac{1}{d-1}\Delta^{\mu\nu}) - 2\eta\mathcal{T}^{d-1}\sigma^{\mu\nu} + \text{higher order corrections.} \quad (5.42)$$

This expression is immediately compared to the dual stress tensor (5.41). Just as predicted by the fluid/gravity conjecture, introducing dynamics on the gravity side reveals the viscous nature of the dual fluid. Clearly, the second part of the stress tensor (5.41) is the first order dissipative correction to the conformal fluid stress tensor with the viscosity $\eta = \frac{1}{16\pi G_{d+1}}\left(\frac{4\pi\mathcal{T}}{d}\right)^{d-1}$. It is possible to re-write the expression for η in terms of the gauge theory variables. We for example have for the $D = 5$ dimensional truncation (3.25) that $\eta = \pi N^2\mathcal{T}^3/8$ which is in accordance with the result of [2]. Note that our computation found that that the stress tensor contains no bulk viscosity ϑ term. This is of course a key feature of a conformal fluid. Also notice that even though the first order shear correction is universal (i.e. it is of the same form for all fluids), the specific temperature dependence in (5.41) is unique for a conformal fluid. This concludes the interpretation of the dynamical part of gravity to first order in the derivatives.

The remaining Einstein equations were not solved directly, instead they constrained the velocity and temperature derivatives through the constraint equations (5.39). This suggest that the constraint equations should be understood as the equations of motion of the boundary theory. Indeed, it is easy to convince oneself that the covariant version of the equations (5.39) is equivalent to (as usual in the section, this expression is only correct up to first order in the derivatives)

$$\nabla_\nu T^{\mu\nu} = 0 \quad (5.43)$$

This is exactly the equations of motion for the boundary dual fluid! The seemingly arbitrary constraint equations on the gravity side therefore have an extremely simple interpretation in terms of the dual boundary theory: They simply represent energy/momentum conservation of the dual fluid.

Finally we address the viscosity to ratio mentioned in the introduction. The (shear) viscosity to entropy ratio of a fluid is simply the local ratio between the entropy density and the viscosity η/s. From conformal thermodynamics we find that the viscosity to entropy ratio is given by

$$\eta/s = \frac{1}{4\pi} \tag{5.44}$$

This is in accordance with the famous result of [2]. As mentioned in the previous chapter, the property $\eta/s = \mathcal{O}(1)$ is characteristic of a strongly coupled fluid. As mentioned in the introduction, it has been conjectured that $\eta/s \geq \frac{1}{4\pi}$ for any sensible quantum field theory [2]. The theories we are considering therefore exactly saturate this bound. We will not go further into this very interesting discussion but merely state that there seems to be some doubt on the validity of the viscosity/entropy bound (see [41] and the references therein).

5.7 The general structure of the perturbation theory

Here we explain how the perturbative procedure presented above generalizes to arbitrary order following [6]. As usual in perturbation theory, the perturbative expansion is solved iteratively. Therefore, assume that we have solved the perturbative problem to $(n-1)^{\text{th}}$ order, in other words, we have solved $G^{(m)}$ for $m \leq n-1$ and $\beta_i^{(m)}$, $b^{(m)}$ for $m \leq n-2$ (cf. the expansions (5.12) and (5.13)). Moreover, assume that the stress tensor dual to G up to $\mathcal{O}(\varepsilon^{n-1})$ obeys the Landau frame condition $u_\mu T^{\mu\nu}_{(n-1)} = 0$.

In general, $G^{(m)}$ is an expression in β_i and b containing m derivatives. Inserting the expansions for the velocities and temperature, Taylor expanding to the relevant order and plugging the expansion $G = \sum_{i=0}^{n} \varepsilon^m G^{(m)}$ into Einstein's equation and finally extracting the coefficient for ε^n, we obtain an equation around $x^\mu = 0$, schematically of the form

$$\mathbb{H}\left[G^{(0)}(\beta_i^{(0)}, b^{(0)})\right] G^{(n)}(r, x^\mu) = s_n \tag{5.45}$$

Here $\mathbb{H}$ is a second order differential operator in only the radial coordinate r and s_n is a "source term" for the operator. It is important to realize that $\mathbb{H}$ contains no boundary derivatives. This follows from the fact that $G^{(n)}$ is already of order n and that each boundary derivative produces an additional power of ε ($\mathbb{H}$ is therefore also referred to as an ultralocal operator in the boundary directions). As indicated by the notation, the operator $\mathbb{H}$ only depends on the values of $\beta_i^{(0)}$ and $b^{(0)}$ in the distinguished point $x^\mu = 0$ but not on their derivatives. Finally, the operator $\mathbb{H}$ is independent of the order n. The last point is crucial: The only dependence on the order n comes from the source term s_n. This means that the structure of the perturbation theory is in fact extremely simple. As in the first order computation, we may now utilize the $\mathrm{SO}(d-1)$ background symmetry to decouple the equations into a set of first order operators (which are easily solved by direct integration). For example, the left hand side of the equations (5.20), (5.21) and (5.23) are the restriction of the operator $\mathbb{H}$ to the scalar sector while the right hand side of the same equations are the scalar parts of the source term s_1. Therefore, going to e.g. second order in the scalar sector, the equations would have the same form but with a different source term.

As we saw, the system of equations (5.45) contains a constraint sector and a dynamical sector. As in the first order case, the constraint equations are obtained by considering the equation $\mathcal{E}_{AB}\xi^B = 0$. Also as in the first order case, this set of equations will be equivalent to energy-momentum conservation of the boundary stress tensor

$$\nabla_\mu T^{\mu\nu}_{(n-1)} = 0 \tag{5.46}$$

where $T^{\mu\nu}_{(n-1)}$ is the boundary stress tensor dual to G_{AB} up to order $\mathcal{O}(\varepsilon^{n-1})$ (the form of which we already know from $(n-1)^{\text{th}}$ order perturbation theory). The constraint equations are used to determine (= constraint) the corrections $\beta_i^{(n-1)}$ and $b^{(n-1)}$. The solution in the dynamical sector can be written (assuming that we have fixed some appropriate gauge)

$$G^{(n)} = \text{particular}(s_n) + \text{homogeneous}(\mathbb{H}) \tag{5.47}$$

Exactly as we saw in the first order case, there is a certain non-uniqueness associated with this solution, however, (modulo the constraint equations) this does not correspond to a physical non-uniqueness. The solution to (5.45) is uniquely fixed by demanding that G asymptotes to AdS_{d+1} and that $G^{(n)}$ is regular at all $r \neq 0$, especially at $br = 1$. There is still a residual freedom in the solution, this freedom is removed by requiring the Landau frame condition fulfilled

$$u_\mu \Pi^{\mu\nu}_{(n)} = 0 \tag{5.48}$$

This accounts for the structure of higher order perturbation theory. As is hopefully clear, the methods used for going to e.g. 2^{nd} order are the same as the one used for solving 1^{st} order perturbation theory. In essence, the only thing that changes is the source term which becomes more complicated.

5.7.1 Including a weakly curved boundary metric

The computation outlined above allows a slight generalization. Consider the stationary black brane (5.10). The main idea in the above computation was to let the constant temperature and boost velocity to be slowly varying fields in the boundary directions and then solve the resulting equations according to this perturbation. Similarly it is also possible to promote the constant boundary metric to be slowly varying, that is, in addition to the expansions (5.12) and (5.13), we take

$$\eta_{\mu\nu} \to \mathrm{g}_{\mu\nu} = \mathrm{g}^{(0)}_{\mu\nu} + \varepsilon \mathrm{g}^{(1)}_{\mu\nu} + \mathcal{O}(\varepsilon^2) \tag{5.49}$$

Since Einstein's equation (5.1) contains only zero-derivative and two-derivative terms (from respectively the metric part and the curvature part of the generalized Einstein tensor $\mathcal{E}_{AB}$), we see that effects from a weakly curved boundary metric will not show up at the 1^{st} order derivative level (i.e. at $\mathcal{O}(\varepsilon)$). In order to see the effects of a weakly curved boundary metric we must therefore go to 2^{nd} order. From a fluid dynamic point of view this is because the boundary fluid couples to the boundary metric $\mathrm{g}_{\mu\nu}$ through the 2^{nd} order curvature terms introduced in §4.2.4. Before presenting the results of the full second order computation (including this weakly curved boundary metric generalization), we will explain how the Weyl invariance of the boundary fluid can be understood in terms of a class of diffeomorphisms in the bulk.

5.8 Manifest Weyl invariance of the bulk solutions

As should be clear from the discussion of the AdS boundary, the Weyl invariance of the boundary theory is related to the re-parametrization invariance of the radial coordinate in the bulk: The boundary theory does not depend on how we parameterize the radial bulk direction and re-parametrizations of the radial coordinate exactly generates Weyl transformations of the boundary theory. To see how this works in the fluid dynamic regime, consider the uniformly boosted black brane solution

$$\mathrm{d}s^2 = -2u_\mu \mathrm{d}x^\mu \mathrm{d}r - r^2 f(br) u_\mu u_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + r^2 \Delta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{5.50}$$

Now consider a coordinate transformation of the radial coordinate of the type $r \to \tilde{r} = e^\phi r$, where $\phi \equiv \phi(x^\mu)$ is a function of only the boundary coordinates. The boosted brane metric now takes the form

$$\mathrm{d}s^2 = -2\tilde{u}_\mu \mathrm{d}x^\mu (\mathrm{d}\tilde{r} + \tilde{r}\tilde{\mathcal{A}}_\nu \mathrm{d}x^\nu) - \tilde{r}^2 f(\tilde{b}\tilde{r}) \tilde{u}_\mu \tilde{u}_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + \tilde{r}^2 \tilde{\Delta}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{5.51}$$

where $\tilde{\mathcal{A}}^\mu$ is the vector field defined by (4.41) and ($\mathrm{g}_{\mu\nu} = \eta_{\mu\nu}$)

$$\tilde{\mathrm{g}}_{\mu\nu} = e^{-2\phi} \mathrm{g}_{\mu\nu}, \quad \tilde{u}^\mu = e^\phi u^\mu, \quad \tilde{b} = e^{-\phi} b \;\Rightarrow\; \tilde{\mathcal{T}} = e^\phi \mathcal{T} \tag{5.52}$$

We can now proceed as in §5.2.2 to compute the boundary stress tensor of the metric (5.51). We do this in the $(\tilde{r}, x^\mu)$ coordinates, that is, we evaluate the boundary stress tensor $\tilde{T}^{\mu\nu}$ on the surface $\tilde{r} = \mathit{const.} \to \infty$ with the boundary metric $\tilde{\mathrm{g}}_{\mu\nu}$. The result of this computation is

$$\tilde{T}^{\mu\nu} = \frac{1}{16\pi G_{d+1} \tilde{b}^d} \left(\tilde{\mathrm{g}}^{\mu\nu} + d\tilde{u}^\mu \tilde{u}^\nu \right) = e^{(d+2)\phi} T^{\mu\nu} \tag{5.53}$$

where $T^{\mu\nu}$ is the dual stress tensor computed via the metric (5.50). We conclude that there is a one-to-one correspondence between the bulk transformations of the type $r \to \tilde{r} = e^\phi r$ and the boundary Weyl transformations (4.27). This means that performing a Weyl transformation on the boundary induces a transformation of the dual bulk metric. Let us see what this implies for the fluctuating brane metric G_{AB} (b and u^μ are slowly varying functions of the boundary coordinates) that solves Einstein's equation. First we must choose a gauge.[4] Here (and below) we will use a slightly different gauge choice, and more "physical", than the one we used in the first order analysis. We will choose the gauge $G_{rr} = 0$ together with $G_{r\mu} = -u_\mu$. This gauge has a very nice geometrical interpretation: In this gauge the lines of constant x^μ are null-geodesics along each of which r is an affine parameter (for more details on this gauge choice, see [7]). In this gauge the general metric G_{AB} can be written

$$G_{AB} \mathrm{d}x^A \mathrm{d}x^B = -2u_\mu \mathrm{d}x^\mu (\mathrm{d}r + \mathcal{V}_\nu(r,x) \mathrm{d}x^\nu) + \mathfrak{G}_{\mu\nu}(r,x) \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{5.54}$$

where $\mathfrak{G}_{\mu\nu}$ is the transverse part of $G_{\mu\nu}$, i.e., $u^\mu \mathfrak{G}_{\mu\nu} = 0$. Here the tensors $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ are functions of u^μ and b (and their derivatives). For example, to first order (5.40) we have $\mathfrak{G}_{\mu\nu} = 2r^2 b F(br) \sigma_{\mu\nu}$ and a similar expression for $\mathcal{A}_\mu$. The tensors $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ go like $\sim r^2$ and the boundary metric is therefore defined by [5]

$$\mathrm{g}_{\mu\nu} = \lim_{r\to\infty} r^{-2} \left[\mathfrak{G}_{\mu\nu} - u_{(\mu} \mathcal{V}_{\nu)} \right] \tag{5.55}$$

Now suppose that we perform a bulk diffeomorphism $r \to \tilde{r} = e^\phi r$ ($\phi \equiv \phi(x^\mu)$) along with a rescaling of the temperature $b \to \tilde{b} = e^{-\phi} b$. The metric (5.54) now transforms according to

$$\tilde{u}^\mu = e^\phi u^\mu, \quad \tilde{\mathcal{V}}_\mu = e^\phi \left(\mathcal{V}_\mu - r\partial_\mu \phi \right), \quad \tilde{\mathfrak{G}}_{\mu\nu} = \mathfrak{G}_{\mu\nu}, \quad \tilde{\mathrm{g}}_{\mu\nu} = e^{-2\phi} \mathrm{g}_{\mu\nu} \tag{5.56}$$

[4] As mentioned, Weyl invariance of the boundary theory follows from "radial" diffeomorphism invariance in the bulk. Roughly these transformations fall in two groups: Redefinition of the radial coordinate in the "boundary direction", $r \to e^{\phi(x^\mu)} r$ (these transformations correspond to Weyl transformations on the boundary) and redefinition in the "radial direction", $r \to e^{\psi(r)} r$. In order to get rid of the redundancy in the radial direction we must fix a gauge.

[5] All boundary indices are raised and lowered with the boundary metric $\mathrm{g}_{\mu\nu}$.

Now since the tensors $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ are functions of u^μ and b, which both respectively pick up a factor of e^ϕ and $e^{-\phi}$ under Weyl transformations, we conclude that $\mathfrak{G}_{\mu\nu}$ must be invariant under Weyl transformations while $\mathcal{V}_\mu - r\mathcal{A}_\mu$ must transform as a Weyl covariant vector with conformal weight $w = 1$ (using the transformation property (4.42) of $\mathcal{A}_\mu$). This means that the expression for $\mathfrak{G}_{\mu\nu}$, correct up to second order, must be given by a linear combination of (transverse, however, not traceless) terms of the type $\Delta_{\mu\nu}, b^{-1}\sigma_{\mu\nu}, \sigma_\mu{}^\lambda\sigma_{\lambda\nu}, u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}, \ldots,$ where the coefficients are functions of $rb = \tilde{r}\tilde{b}$ (cf. the section on second order fluid dynamics). Similarly we conclude that $\mathcal{V}_\mu$ must be given by a linear combination of terms of the type $r\mathcal{A}_\mu, b^{-2}u_\mu, \mathcal{R}u_\mu, \mathcal{D}_\lambda\varpi^\lambda{}_\mu, \ldots,$ where the coefficients again are functions of rb. This is the best we can do using Weyl covariance. In order to determine the precise form of the different coefficients, one must perturbatively solve the equations of gravity to second order in the derivatives.

5.9 The second order results

We are now ready to understand and present the results of the second order computation. Following the discussion above, we write the metric according to

$$G_{AB}\mathrm{d}x^A\mathrm{d}x^B = -2u_\mu\mathrm{d}x^\mu(\mathrm{d}r + \mathcal{V}_\nu\mathrm{d}x^\nu) + \mathfrak{G}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{5.57}$$

As explained in the above section, Weyl covariance determines the structure of the tensors $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$. In order to find the coefficients multiplying the Weyl invariant terms of $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$, we must invoke the perturbative procedure accounted for in §§5.3-5.7. Using this procedure the vector $\mathcal{V}_\mu$ and the tensor $\mathfrak{G}_{\mu\nu}$ are determined to respectively [8, 6] (for the definitions of the various tensors, see §4.2.5)

$$\begin{aligned}\mathcal{V}_\mu = r\mathcal{A}_\mu &+ \frac{1}{d-2}\left[\mathcal{D}_\lambda\varpi^\lambda{}_\mu - \mathcal{D}_\lambda\sigma^\lambda{}_\mu + \frac{\mathcal{R}}{2(d-1)}u_\mu\right] - \frac{2L(br)}{(br)^{d-2}}\Delta^\nu_\mu\mathcal{D}_\lambda\sigma^\lambda{}_\nu \\ &+ \frac{u_\mu}{2(br)^d}\left[r^2(br)^d f(r) + \frac{1}{2}\varpi_{\alpha\beta}\varpi^{\alpha\beta} + (br)^2K_2(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\right]\end{aligned} \tag{5.58}$$

and

$$\begin{aligned}\mathfrak{G}_{\mu\nu} &= r^2\Delta_{\mu\nu} - \varpi_\mu{}^\lambda\varpi_{\lambda\nu} \\ &+ 2(br)^2F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right] - 2(br)^2K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\Delta_{\mu\nu} \\ &- 2(br)^2H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}\Delta_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] \\ &+ 2(br)^2H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \varpi_\mu{}^\lambda\sigma_{\lambda\nu} + \varpi_\nu{}^\lambda\sigma_{\mu\lambda}\right]\end{aligned} \tag{5.59}$$

The tensors have the predicted structure. The functions f, F, H_1, H_2, K_1, K_2, and L are given by [8] (we of course already know the form of f and F which followed from respectively the 0^{th} and 1^{st} order computation)

$$f(br) = 1 - \frac{1}{(br)^d} \tag{5.60}$$

$$F(br) = \int_{br}^\infty \mathrm{d}\xi\,\frac{\xi^{d-1}-1}{\xi(\xi^d-1)} \tag{5.61}$$

$$H_1(br) = \int_{br}^\infty \mathrm{d}\xi\,\frac{\xi^{d-2}-1}{\xi(\xi^d-1)} \tag{5.62}$$

$$H_2(br) = \frac{1}{2}F(br)^2 - \int_{br}^\infty \frac{\mathrm{d}\xi}{\xi(\xi^d-1)}\int_1^\xi \mathrm{d}\zeta\,\frac{\zeta^{d-2}-1}{\zeta(\zeta^d-1)} \tag{5.63}$$

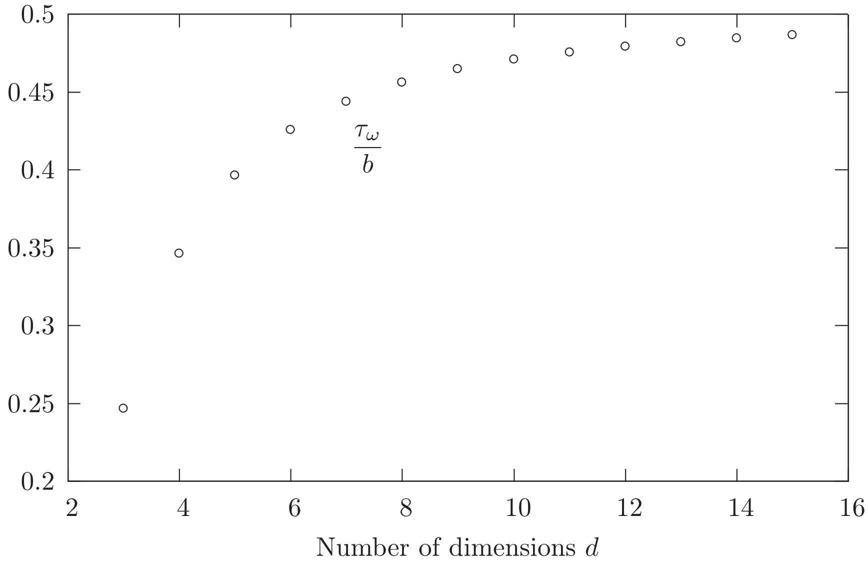

Figure 5.2: *The ratio* τ_ω/b *given by the integral* (5.69) *for boundary spacetime dimension up to* 15. *The ratio is bounded from above by* 1/2.

$$K_1(br) = \int_{br}^{\infty} \frac{\mathrm{d}\xi}{\xi^2} \int_{\xi}^{\infty} \mathrm{d}\zeta\, \zeta^2 F'(\zeta)^2 \tag{5.64}$$

$$K_2(br) = \int_{br}^{\infty} \frac{\mathrm{d}\xi}{\xi^2} \Big[1 - \xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1} \tag{5.65}$$

$$+ \big(2(d-1)\xi^d - (d-2)\big) \int_{\xi}^{\infty} \mathrm{d}\zeta\, \zeta^2 F'(\zeta)^2 \Big] \tag{5.66}$$

$$L(br) = \int_{br}^{\infty} \mathrm{d}\xi\, \xi^{d-1} \int_{\zeta}^{\infty} \mathrm{d}\zeta\, \frac{\zeta - 1}{\zeta^3(\zeta^d - 1)} \tag{5.67}$$

The metric (5.57) with the above functions solves Einstein's equation to second order in the derivatives. As usual only the asymptotic form of the above functions is needed to compute the boundary stress tensor dual to (5.57). By using the asymptotic form of the functions F, H_1, H_2, K_1, K_2, and L (which can be found in [8]), it is possible to show that the boundary stress tensor dual to (5.57) takes the form

$$\begin{aligned} T^{\mu\nu} &= p\Bigg[(g^{\mu\nu} + d u^\mu u^\nu) - \frac{2\eta}{p}\sigma^{\mu\nu} - 2b^2 \mathcal{I}\left[u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \varpi^{\mu}{}_{\lambda}\sigma^{\lambda\nu} + \varpi^{\nu}{}_{\lambda}\sigma^{\mu\lambda}\right] \\ &+ 2b^2 \left[u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + \sigma^{\mu}{}_{\lambda}\sigma^{\lambda\nu} - \frac{1}{d-1}\Delta^{\mu\nu}\sigma_{\alpha\beta}\sigma^{\alpha\beta} + C^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} u^\alpha u^\beta \right]\Bigg] \\ &= p\,(g^{\mu\nu} + d u^\mu u^\nu) - 2\eta\left[\sigma^{\mu\nu} - \tau_1 \mathfrak{T}_1^{\mu\nu} + \tau_2 \mathfrak{T}_2^{\mu\nu}\right] + \xi_\sigma \mathfrak{T}_3^{\mu\nu} + \xi_\varpi \mathfrak{T}_4^{\mu\nu} + \xi_C \mathfrak{T}_5^{\mu\nu} \end{aligned} \tag{5.68}$$

where the quantities pertaining to the first equality are given by

$$b = \frac{d}{4\pi\mathcal{T}}, \qquad p = \frac{\rho}{(d-1)} = \frac{1}{16\pi G_{d+1} b^d}, \tag{5.69}$$

$$\eta = \frac{s}{4\pi} = bp, \qquad \mathcal{I} \equiv \int_1^\infty \mathrm{d}\zeta \, \frac{\zeta^{d-2} - 1}{\zeta(\zeta^d - 1)} \tag{5.70}$$

In the last equality in (5.69) we recorded the most general expression for the Weyl covariant stress tensor (cf. §4.2.5). We see that the second order results are in perfect agreement with the second order derivative expansion under the identifications

$$\xi_\sigma = \xi_C = 2\eta(\tau_1 + \tau_2) = 2\eta b, \quad \tau_2 = \tau_\omega = b\mathcal{I}, \quad \xi_\varpi = 0 \tag{5.71}$$

A few comments are in order. First of all, as illustrated by the graph 5.2, the ratio $\tau_\omega/b = \mathcal{I}$ is well behaved and $\mathcal{O}(1)$ for arbitrary spacetime dimension (we can show that $\tau_\omega/b < 1/2$ for arbitrary d). Moreover notice that the transport coefficient τ_ϖ associated with (only) the fluid vorticity vanishes. This observation is important when we apply the AdS/CFT correspondence to the rotating black holes introduced in the next chapter. Finally, much like the viscosity/entropy ratio $\eta/s = 1/4\pi$, the relations (5.71) are universal in the sense that they hold for all *uncharged* fluids with a gravity dual in arbitrary dimensions. This is a quite interesting result. It would be interesting to understand the underlying gravitational structure leading to this result. Moreover it would be even more interesting to understand this universality only in terms of the gauge theory.

Finally we mention that it is possible to construct a boundary entropy current associated with fluctuating black branes by using a geometric argument (roughly a local version of the area theorem) [42]. By performing such a computation for the second order fluctuating black brane (5.57), the authors of [8] were able to show that the form of the "geometric" entropy current is in perfect agreement with the expansion (4.77) and thus gives a method for working out the coefficients relevant to this expansion. Moreover [8] beautifully explains how the the second law of thermodynamics on the fluid side is dual to the Hawking's area theorem (= the second law of thermodynamics) on the gravity side.

6

Black holes in the fluid/gravity correspondence

6.1 The Kerr solution in anti-de Sitter

The solutions to Einstein's vacuum equation are the metrics that fulfill [1]

$$R_{AB} = (D-1)\lambda G_{AB} \tag{6.2}$$

One might ask for the most general stationary, regular solution to this equation. How are the Einstein metrics classified? Apart from this being a mathematically very interesting problem, this question is clearly also physically relevant (for string theory $\sim$ higher dimensional gravity, theories with large extra dimensions, the AdS/CFT correspondence etc.) The answer to this question in $D = 4$ with $\Lambda = 0$ is well-known; the vacuum solutions to Einstein's equation are exactly the two parameter (M, J) family of Kerr solutions (see for example [43], [13]). However, in spacetimes with dimension $D > 4$, the family of vacuum solutions becomes much richer and finding them is an active area of research, both from a purely theoretical point of view (see for example the reviews [44] and [45]) but also from a practical/experimental point of view [46], [47]. Although in $D > 4$ the vacuum solutions branch out in a wealth of qualitatively different solutions, it turns out, for reasons explained below, that the four-dimensional Minkowski rotating black hole (Kerr) solution generalizes to all dimensions and cosmological constants.

6.1.1 The maximally symmetric Einstein metrics

Consider the maximally symmetric, non-compact solutions to Einstein's vacuum equation. That is, consider the infinite volume solutions to (6.2) that fulfill (details on this can be found in e.g. [14])

$$R_{ABCD} = \frac{R}{D(D-1)}(G_{AC}G_{BD} - G_{AD}G_{BC}) \tag{6.3}$$

[1] Here λ is a constant parameter that is related to the cosmological constant Λ by

$$\lambda(D-1)(D-2) = 2\Lambda \tag{6.1}$$

The form of these solutions is well-known. For $\lambda > 0$ the solution is de-Sitter space dS_D, for $\lambda = 0$ the solution is flat Minkowski space M_D while $\lambda < 0$ corresponds to anti-de Sitter space AdS_D. These solutions are completely regular and contain no horizons (indeed the presence of a singularity or a horizon would destroy maximal symmetry). The three types of metrics can collectively be written in spherical-like coordinates (t, r, μ_i, ϕ_i) by the following simple expression

$$\mathrm{d}\hat{s}^2 = \hat{G}_{AB}\mathrm{d}x^A\mathrm{d}x^B = -(1-\lambda r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\lambda r^2} + r^2\mathrm{d}\Omega^2_{D-2} = \left\{\begin{array}{ll} \mathrm{dS}_D, & \lambda > 0. \\ M_D, & \lambda = 0. \\ \mathrm{AdS}_D, & \lambda < 0. \end{array}\right\} \tag{6.4}$$

where $\mathrm{d}\Omega^2_{D-2}$ is the metric on S^{D-2} which is given by (see appendix A)

$$\mathrm{d}\Omega^2_{D-2} = \sum_{i=1}^{N+\epsilon} \mathrm{d}\mu_i^2 + \sum_{i=1}^{N} \mu_i^2 \mathrm{d}\phi_i^2 \tag{6.5}$$

Here ϕ_i denotes the $N = [(D-1)/2]$ azimuthal angles on S^{D-2} and μ_i are the $N + \epsilon = [D/2]$ 'directional cosines'. As explained in the appendix, the coordinates μ_k are not 'real' coordinates since they are not independent as they are subject to the following constraint

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1 \tag{6.6}$$

This equation may be solved for e.g. $\mu_{N+\epsilon}$ leaving $N+\epsilon-1$ independent coordinates. This must be taken into account if one wishes to do calculations (such as integration) on the sphere S^{D-2} in the coordinates (μ_i, ϕ_i). Note that the number of independent coordinates (one time coordinate, one radial coordinate, N azimuthal coordinates and $N + \epsilon - 1$ directional cosines) fulfill $1 + 1 + N + (N + \epsilon - 1) = D$, just as it should.

6.1.2 The Kerr solution

It is a well-known fact that the original four dimensional Kerr solution in flat space can be written in the so-called Kerr-Schild form (in fact, this was how the Kerr solution was derived [48]). This realization allowed Myers and Perry in their celebrated paper [49] to generalize the Kerr solution to an arbitrary number of (flat) space-time dimensions. Following the work of [50] we will now review how this method is generalized to not only an arbitrary number of dimensions but also a non-zero cosmological constant.

In this section we set $G_D = 1$. If needed, the D dimensional Newton constant is easily reintroduced into the equations.

The Kerr-Schild ansatz: Inspired by the original Kerr solution, the Kerr-Schild ansatz for the generalized Kerr solution is

$$G_{AB} = \hat{G}_{AB} + H_{AB} \tag{6.7}$$

where $\hat{G}_{AB}$ is a maximally symmetric solution to $R_{AB} = (D-1)\lambda G_{AB}$ i.e. one of the solutions (6.4) and H_{AB} can be written as the following tensor product

$$H_{AB} = 2H\boldsymbol{\theta}_A\boldsymbol{\theta}_B \tag{6.8}$$

Here H is a scalar function and $\boldsymbol{\theta}^A$ is a null vector w.r.t. $\hat{G}_{AB}$ and therefore also w.r.t. the full metric G_{AB}, that is

$$\boldsymbol{\theta}^A G_{AB}\boldsymbol{\theta}^B = \boldsymbol{\theta}^A \hat{G}_{AB}\boldsymbol{\theta}^B = 0 \tag{6.9}$$

Moreover, the vector field $\boldsymbol{\theta}^A$ is tangent to a null geodesic w.r.t. both the background metric and the full metric which may be stated as $\boldsymbol{\theta}^B \nabla_A \boldsymbol{\theta}_B = \boldsymbol{\theta}^B \hat{\nabla}_A \boldsymbol{\theta}_B = 0$ where $\hat{\nabla}$ (∇) denotes the Levi-Civita connection w.r.t. the (full) metric $\hat{G}_{AB}$ (G_{AB}). The null condition on the vector $\boldsymbol{\theta}^A$ implies that the inverse of the full metric is given by $G^{AB} = \hat{G}^{AB} - H^{AB}$.

Recall the linearized gravity scheme: Given some background metric $\hat{G}_{AB}$ we can consider *small* perturbations of this metric $\hat{G}_{AB} \to \hat{G}_{AB} + h_{AB}$. To linear order, the inverse perturbed metric is given by $\hat{G}^{AB} \to \hat{G}^{AB} - h^{AB}$. Using this, one can now work out the field equations to linear order in the perturbation h_{AB}

$$\mathcal{H}[h_{AB}; \hat{G}_{AB}] = 0 \tag{6.10}$$

These are the linearized equations of gravity w.r.t. the background metric $\hat{G}_{AB}$. Needless to say, the linearized equations of gravity are much simpler than the full set of non-linear Einstein equations. It now turns out that given a solution $\hat{G}_{AB}$ to Einstein's vacuum equation with a cosmological constant ($\hat{R}_{AB} = const. \times \hat{G}_{AB}$) and the Kerr-Schild ansatz (6.8), one can show that (see [50] in the references therein) that the full metric fulfills the Einstein vacuum equation with the same cosmological constant ($R_{AB} = const. \times G_{AB}$) *if* $H_{AB} = 2H\boldsymbol{\theta}_A\boldsymbol{\theta}_B$ fulfills the linearized system of equations i.e. if

$$\mathcal{H}[H_{AB}; \hat{G}_{AB}] = 0 \tag{6.11}$$

So, given the Kerr-Schild ansatz, the problem of finding vacuum solutions with a cosmological constant reduces to solving linear gravity in the background $\hat{G}_{AB}$. This simplifies the mathematical problem significantly, moreover the linearity of (6.10) allows us to superimpose different vacuum solution to obtain new very non-trivial solution [51]. In the case of anti-de Sitter ($\lambda < 0$) we see that $|\lambda| = L^{-1}$, where L is the usual AdS radius. Since the results below hold for all λ we shall for completeness keep it as a free parameter, however, we will only be concerned with the case where $\lambda < 0$ (usually we just take $\lambda = -1$).

THE GENERALIZED SPHEROIDAL COORDINATES: Recall that the four dimensional Minkowski Kerr solution is most naturally presented in a set of 'Boyer-Lindquist coordinates' since these coordinates respect the symmetry of the spacetime. This is also true for the general D dimensional Kerr solution. Also recall that the four-dimensional Boyer-Lindquist coordinates do not reduce to ordinary spherical coordinates in the zero mass limit but rather to a set of spheroidal coordinates on M_4. This can be understood from the Kerr-Schild form of the metric: If one expresses M^4 in spheroidal coordinates the vector, $\boldsymbol{\theta}^A$ and the scalar function H (which is proportional to the Kerr mass) both take very simple forms.

We will now explain how these spheroidal coordinates are generalized to cover all the D dimensional maximally symmetric Einstein metrics. To this end introduce N $(+\epsilon)$ parameters a_i (in the case where $\epsilon = 1$ we introduce $a_{N+1} \equiv 0$) associated which each of the N planes of rotation introduced in appendix A. These parameters will be identified with the rotational parameters of the rotating Kerr black hole below. Now consider the spacetime (6.4) with coordinates (t, r, μ_i, ϕ_i) and introduce a new set of 'spheroidal coordinates' (t, y, ν_i, ϕ_i) by the following defining equation

$$(1 + \lambda a_i^2) r^2 \mu_i^2 = (y^2 + a_i^2) \nu_i^2 \tag{6.12}$$

where the ν_i's are a new set of directional cosines i.e.

$$\sum_{i=1}^{N+\epsilon} \nu_i^2 = 1 \tag{6.13}$$

Together these two equations imply that the Euclidean distance r^2 can be expressed in terms of (y, ν_i) by

$$r^2 = \sum_{i=1}^{N+\epsilon} \left\{ \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \right\} \nu_i^2 \tag{6.14}$$

It is now possible to express pure AdS in terms of this new set of coordinates. After some calculation we find (see appendix A)

$$\begin{aligned} \mathrm{d}\hat{s}^2 = &-W(1-\lambda y^2)\mathrm{d}t^2 + F\mathrm{d}y^2 + \sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \mathrm{d}\nu_i^2 \\ &+ \sum_{i=1}^{N} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i^2 \mathrm{d}\phi_i^2 + \frac{\lambda}{W(1-\lambda y^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i \mathrm{d}\nu_i \right)^2 \end{aligned} \tag{6.15}$$

where the two scalar functions W and F are given by

$$W \equiv \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{1 + \lambda a_i^2}, \quad F \equiv \frac{y^2}{1 - \lambda y^2} \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{y^2 + a_i^2} \tag{6.16}$$

Using the coordinates (t, y, ν_i, ϕ) on the background $\mathrm{d}\hat{s}^2$ we will now write down the expression for the general D dimensional Kerr solution. In accordance with the Kerr-Schild ansatz the Kerr anti-de Sitter metric is given by

$$ds^2 = \mathrm{d}\hat{s}^2 + 2H(\boldsymbol{\theta}_A \mathrm{d}x^A)^2, \quad H \equiv M/U \tag{6.17}$$

where the one-form $\boldsymbol{\theta}_A$ and the scalar function U are given by [50]

$$\boldsymbol{\theta}_A \mathrm{d}x^A = W\mathrm{d}t + F\mathrm{d}y - \sum_{i=1}^{N} \frac{a_i \nu_i^2}{1 + \lambda a_i^2} \mathrm{d}\phi_i, \quad U = y^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{y^2 + a_i^2} \prod_{j=1}^{N} (y^2 + a_j^2) \tag{6.18}$$

The parameter M is a real constant and it has a mass-like interpretation (in curved backgrounds the notion of mass is even more cumbersome than in flat space).

THE KERR SOLUTION IN BOYER-LINDQUIST FORM: The metric (6.17), (6.18) is not quite the Boyer-Lindquist form we are looking for. In order to cast the Kerr metric into Boyer-Lindquist form we introduce a new coordinate transformation defined by the (exact) differential expressions

$$\mathrm{d}t = \mathrm{d}\tau + \frac{2M\mathrm{d}y}{(1-\lambda y^2)(V - 2M)}, \quad \mathrm{d}\phi_i = \mathrm{d}\varphi_i + \frac{2Ma_i\mathrm{d}y}{(y^2 + a_i^2)(V - 2M)} \tag{6.19}$$

Here the function V is defined by

$$V \equiv \frac{U}{F} = y^{\epsilon - 2}(1 - \lambda y^2) \prod_{i=1}^{N} (y^2 + a_i^2) \tag{6.20}$$

Using this coordinate transformation, it is straight forward to show that the Kerr AdS metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = &-W(1-\lambda y^2)\mathrm{d}\tau^2 + \frac{2M}{VF}\left(W\mathrm{d}\tau - \sum_{i=1}^{N} \frac{a_i \nu_i^2 \mathrm{d}\varphi_i}{1 + \lambda a_i^2} \right)^2 + \sum_{i=1}^{N} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i^2 \mathrm{d}\varphi_i^2 \\ &+ \frac{VF\mathrm{d}y^2}{V - 2M} + \sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \mathrm{d}\nu_i^2 + \frac{\lambda}{W(1-\lambda y^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i \mathrm{d}\nu_i \right)^2 \end{aligned} \tag{6.21}$$

This is the generalized Kerr solution in Boyer-Lindquist coordinates. Notice that the last term is not present in the flat space case and makes the general Kerr solution somewhat more involved than its flat space counterpart.

It is useful to record the asymptotic form of the generalized AdS Kerr solution. Of course, the form of the asymptotic metric depends on the coordinates in which it is presented. Here we will use coordinates such that the metric near the boundary takes the form

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2\mathrm{d}\Omega^2_{d-1} + \mathcal{O}(1/r^2) \tag{6.22}$$

This is noting but the AdS metric in standard form (6.4) plus some small perturbation (we have set $-\lambda = L^{-1} = 1$). Following the above discussion, it is clear how we should cast the Kerr solution into this form: We simply perform the inverse coordinate transformation (6.12) of the Boyer-Lindquist metric (6.21) (from 'spheroidal coordinates' to ordinary spherical coordinates) and find the form of the Kerr metric in the $r \to \infty$ limit. We find (for details, see appendix A)

$$\begin{aligned}\mathrm{d}s^2 = -(1+r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\sum_{i=1}^{N+\epsilon}\mathrm{d}\mu_i^2 + r^2\sum_{i=1}^{N}\mu_i^2\mathrm{d}\phi_i^2 \\ + \frac{2M}{r^{d-2}}\gamma^{d+2}\mathrm{d}t^2 + \frac{2M}{r^{d+2}}\gamma^{d}\mathrm{d}r^2 - \sum_{i=1}^{N}\frac{4Ma_i\mu_i^2}{r^{d-2}}\gamma^{d+2}\mathrm{d}t\mathrm{d}\phi_i \\ + \sum_{i,j=1}^{N}\frac{2Ma_ia_j\mu_i^2\mu_j^2}{r^{d-2}}\gamma^{d+2}\mathrm{d}\phi_i\mathrm{d}\phi_j + \cdots\end{aligned} \tag{6.23}$$

with

$$\gamma^{-2} = 1 - \sum_{i=1}^{N} a_i^2\mu_i^2 \tag{6.24}$$

and where $d = D - 1$ as usual denotes the boundary dimension. Notice that the first three terms correspond to pure AdS while the terms proportional to M come from the AdS black hole as desired. The dots represent terms that are $\mathcal{O}(1/r^D)$ *compared* to the pure AdS metric where the (vanishing) cross terms are given the natural order in r; for example, since the ϕ_i direction has an associated r factor, the (vanishing) $\mathrm{d}\phi_i\mathrm{d}\phi_j$ component of the pure AdS metric is assigned order $\mathcal{O}(r^2)$ and so on.

6.1.3 Horizons

We will now examine the geometrical properties of the generalized Kerr solution. This analysis follows the same steps as the analysis of the well-known four dimensional Minkowski Kerr black hole, see e.g. [13, 52, 43].

The Kerr solution contains an event horizon. In the Boyer-Lindquist coordinates the event horizon is located at (the largest possible) y value y_+ where the yy metric component becomes singular [14], i.e

$$1/G_{yy} \sim V(y_+) - 2M = 0 \tag{6.25}$$

Even though the metric (6.21) is only valid outside $y > y_+$, the fact that this equation has, in general, many solutions *suggests* that the AdS Kerr spacetime contains a wealth of event horizons. This implies a non-trivial global spacetime structure which would be very interesting to examine. However, since this thesis is mainly concerned with the well-understood asymptotic behavior of the Kerr solution, we will not go further into this.

Angular velocities: In order to find the angular velocities Ω_i (one for each rotational degree of freedom) of the Kerr solution, we must determine the timelike Killing vector field K_Ω^A which becomes null on the horizon. Since K_Ω^A is Killing we write it as a linear combination according to

$$K_\Omega^A \partial_A = \frac{\partial}{\partial t} + \sum_{i=1}^{N} \Omega_H^i \frac{\partial}{\partial \phi^i} \tag{6.26}$$

Here t and ϕ_i are the Kerr-Schild coordinates from equation (6.15). By solving $K_{\Omega A} K_\Omega^A = 0$ on the horizon we find that the parameters Ω_i are given by

$$\Omega_H^i = \frac{a_i(1-\lambda y_+^2)}{y_+^2 + a_i^2} \tag{6.27}$$

The existence of such a Killing vector is in fact a more convincing proof that $y = y_+$ indeed is an event horizon than the condition (6.25) (which is clearly rather coordinate dependent). Notice that the authors of [50] reach a different set of values for the angular velocities. They find

$$\tilde{\Omega}_H^i = \Omega_H^i + \lambda a_i \tag{6.28}$$

The reason for the discrepancy can be found in the coordinates in which [50] expresses K_Ω^A. They find that $K_\Omega^A \partial_A = \partial/\partial\tau + \tilde{\Omega}_H^i \partial/\partial\tilde{\varphi}_i$, where the $\tilde{\varphi}_i$'s are related to our φ_i coordinates by $\mathrm{d}\tilde{\varphi}_i = \mathrm{d}\varphi_i + \lambda a_i \mathrm{d}\tau$. The essential point now is that at infinity the vectors $\partial/\partial\tau$ and $\partial/\partial\tilde{\varphi}_i$ reduce to respectively $\partial/\partial\tau \to \partial/\partial t - \lambda a_i \partial/\partial\phi_i$ while $\partial/\partial\tilde{\varphi}_i \to \partial/\partial\phi_i$. The parameters $\tilde{\Omega}_H^i$ of [50] are therefore the angular velocities of the Kerr black hole, as measured by a rotating frame at infinity, while the parameters (6.27) are the angular velocities of the black hole, measured by a frame that is non-rotating at infinity. As usual the latter set of angular velocities are the proper ones to use.

Event horizon surface area: The metric (6.21) induces the following metric on the event horizon $y = y_+$ (formally by setting $\mathrm{d}\tau = \mathrm{d}y = 0$, $y = y_+ \Rightarrow V = 2M$)

$$\begin{aligned} \mathrm{d}r^2 = {} & \frac{1}{F(y_+)} \left(\sum_{i=1}^{N} \frac{a_i \nu_i^2 \mathrm{d}\varphi_i}{1+\lambda a_i^2} \right)^2 + \sum_{i=1}^{N} \frac{y_+^2 + a_i^2}{1+\lambda a_i^2} \nu_i^2 \mathrm{d}\varphi_i^2 \\ & + \sum_{i=1}^{N+\epsilon} \frac{y_+^2 + a_i^2}{1+\lambda a_i^2} \mathrm{d}\nu_i^2 + \frac{\lambda}{W(y_+)(1-\lambda y_+^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{y_+^2 + a_i^2}{1+\lambda a_i^2} \nu_i \mathrm{d}\nu_i \right)^2 \end{aligned} \tag{6.29}$$

The topology of this space is clearly S^{D-2}. Having determined the induced metric, the area of the event horizon is then given by

$$\mathcal{A} = \int_{S^{D-2}} \sqrt{\tilde{\Omega}} \prod_{i=1}^{N+\epsilon-1} \mathrm{d}\nu_i \prod_{j=1}^{N} \mathrm{d}\varphi_j \tag{6.30}$$

where $\tilde{\Omega}$ is the determinant of the above metric. Computing the determinant of the induced metric is tedious but straight forward. We find that[2]

$$\sqrt{\tilde{\Omega}} = \left[\frac{1}{y_+^{1-\epsilon}} \prod_{i=1}^{N} \frac{y_+^2 + a_i^2}{1+\lambda a_i^2} \right] \frac{\prod_{j=1}^{N} \nu_j}{\nu_{N+\epsilon}} \tag{6.31}$$

[2]We have showed this in dimensions up to $D = 8$, which seems to include most physical relevant cases. It is not clear whether [50] managed to prove this for arbitrary D. However, since no special features distinguish the $D \leq 8$ cases, we can be confident that our expression is valid in all dimensions.

The last piece multiplying this expression is recognized as the square root of the usual metric determinant Ω of S^{D-2} in the given coordinates (see appendix A). Notice that the leading y_+ behavior of this determinant is y_+^{D-2}, just as it should. The fact that $\tilde{\Omega} = const. \times \Omega$ renders the integration trivial. In accordance with [50], we find that for $D = 2N + 1 + \epsilon$ even, the horizon area is given by

$$\epsilon = 1: \quad \mathcal{A} = \mathrm{Vol}(S^{D-2}) \prod_{i=1}^{N} \frac{y_+^2 + a_i^2}{1 + \lambda a_i^2} \tag{6.32}$$

while for D odd, the horizon area is given by the expression

$$\epsilon = 0: \quad \mathcal{A} = \frac{\mathrm{Vol}(S^{D-2})}{y_+} \prod_{i=1}^{N} \frac{y_+^2 + a_i^2}{1 + \lambda a_i^2} \tag{6.33}$$

SURFACE GRAVITY: The surface gravity κ is defined by the equation

$$K_\Omega^A \nabla_A K_{\Omega B} = -\frac{1}{2} \nabla_B (K_\Omega^A K_{\Omega A}) = \kappa K_{\Omega B} \tag{6.34}$$

This is the geodesic equation in a non-affine parameterization. The equation (6.34) therefore just expresses that K_Ω^A is a non-affine parameterized tangent to the null-geodesic generators of the horizon. Recall that the surface gravity is constant on (each connected component) of the horizon. Using similar reasoning as in e.g. [13], we can show that the surface gravity is given by the expression [50]

$$\kappa = \frac{1 - \lambda y_+^2}{2V(y_+)} \frac{\mathrm{d}V}{\mathrm{d}y}\bigg|_{y=y_+} \tag{6.35}$$

Carrying out the y differentiation of the function V is trivial. We find that the surface gravity for even dimensional spacetimes is given by

$$\epsilon = 1: \quad \kappa = y_+(1 - \lambda y_+^2) \sum_{i=1}^{N} \frac{1}{y_+^2 + a_i^2} - \frac{1 + \lambda y_+^2}{2y_+} \tag{6.36}$$

In a similar way it is possible to show that for odd dimensional spacetimes we have

$$\epsilon = 0: \quad \kappa = y_+(1 - \lambda y_+^2) \sum_{i=1}^{N} \frac{1}{y_+^2 + a_i^2} - \frac{1}{y_+} \tag{6.37}$$

THE KOMAR ANGULAR MOMENTA: Let J_i^A be the Killing vector associated with the rotational symmetry in the ϕ_i direction (i.e. $J_i^A \partial_A = \partial_{\phi_i}$) and let J_i denote the corresponding Killing 1-form (i.e, $J_{iA} = G_{AB} J_i^B$). The angular momentum L_i of the Kerr spacetime is then given by the usual Komar integral associated with the symmetry J_i which in form notation is given by the expression (see [13, 53], see also appendix A)

$$L_i = -\frac{1}{8\pi} \int_{S^{D-2}} *\mathrm{d}J_i \tag{6.38}$$

Here we have chosen the spatial boundary at infinity to be S^{D-2}. Notice that by virtue of Killing's equation $\nabla_A J_{iB} = \partial_{[A} J_{iB]}$, so this expression coincides with the one found in e.g. [13]. By manipulating this expression, it is possible to show that the Komar integral can be written

$$L_i = -\frac{1}{8\pi} \int_{S^{D-2}} n^A \sigma^B \left[\partial_A J_{iB} - \Gamma_{AB}^C J_{iC} \right] \tag{6.39}$$

Here n^A and σ^A are respectively the future pointing ($\sim$ time direction) and outward pointing ($\sim$ radial direction) normal vectors to the boundary surface S^{D-2}. Notice that since the L_i's vanish for pure AdS$_D$, this definition of the angular momenta is rather unambiguous (as opposed to, for example, the Komar mass of a black hole in an AdS$_D$ background [53]).
By using the asymptotic form of the metric (6.23) along with the Christoffel symbols (A.34), it is possible to show that the Komar angular momenta are given by (see appendix A)

$$L_i = \frac{M\text{Vol}(S^{D-2})}{8\pi\prod_{j=1}^{N}(1+\lambda a_j^2)}\left[\frac{2a_i}{1+\lambda a_i^2}\right] \tag{6.40}$$

6.2 Black hole thermodynamics

We will now review the thermodynamics of the AdS Kerr black hole solution presented in the previous section. The first law of black hole thermodynamics, when applied to a charged, rotating black hole, is expected to take the form

$$\mathrm{d}E = T\mathrm{d}S + \Omega_i \mathrm{d}L_i + \mu_I \mathrm{d}R_I \tag{6.41}$$

where L_i is the i'th angular momentum and R_I is the I'th charge with associated chemical potential μ_I. Here we employed the fact that the conjugate variable to the angular momentum L_i exactly is the angular velocity Ω_i computed above. Since we are considering uncharged black holes the first law reduces to

$$\mathrm{d}E = T\mathrm{d}S + \Omega_i \mathrm{d}L_i \tag{6.42}$$

As is well-known we have already calculated the black hole entropy and temperature once we have computed the area and surface gravity pertaining to the event horizon. It holds that (the Hawking-Bekenstein formulæ)

$$S = \frac{\mathcal{A}}{4} \quad \text{and} \quad T = \frac{\kappa}{2\pi} \tag{6.43}$$

This means that once we have worked out the area, surface gravity, and angular momenta, we can determine the energy $E(S, L_i)$ of the black hole by integrating up the first law (6.42), assuming that the expression for $\mathrm{d}E$ is an exact differential. In fact the latter point gives a non-trivial check of the correctness of the expressions (6.40) [53].

There is another argument for obtaining the energy of a black hole once the values of T, S and L_i are known. The argument, which was first employed in [54], relies on the Euclidean formulation of field theory. The thermodynamic partition function $\mathcal{Z}$ can be expressed in terms of the Euclidean path integral in the following way [54]

$$\begin{aligned}\mathcal{Z} &= \text{Tr}\,\exp\left[-\beta\big(H-\Omega_i L_i\big)\right] \\ &= \int_{\text{PBC}} \mathfrak{D}[G]\,\exp\left(-I[G]\right)\end{aligned} \tag{6.44}$$

where I is the Euclidean action and where the subscript 'PBC' indicates that the path integral is subject to periodic boundary conditions in the Euclidean time direction (the period is as usual identified with the inverse temperature β). A standard thermodynamic argument shows that the partition function (in its original form, the first line of (6.44)) can be written in terms of the grand potential W by

$$\ln\mathcal{Z} = -\beta W \quad \text{with} \quad W = E - TS - \Omega_i L_i \tag{6.45}$$

Now, the path integral in the second line of (6.44) is, in the classical limit relevant for the corner of the AdS/CFT correspondence considered in this thesis, nothing but (minus the exponential of) the Euclidean action evaluated on the classical (black hole) solution. Therefore

$$I_{\text{classical}} = \beta W \tag{6.46}$$

As usual, the bare gravitational action in AdS_D is divergent and must therefore be regularized in a well-defined manner (see [53, 55]. For a discussion between the methods presented in [53, 55] and the counter-term prescription discussed in §3.4, see [56]). The relation (6.46) therefore holds for the renormalized Euclidean action. By using the relation (6.46) it is therefore possible to obtain a non-trivial check of the energy computation described in the previous paragraph.

Using the relations (6.40), (6.43), and the first law (6.41), the energy is computed to [53]

$$E = \frac{M \mathrm{Vol}(S^{D-2})}{8\pi \prod_{i=1}^{N}(1+\lambda a_i^2)} \left[D - 2 - \lambda \sum_{j=1}^{N} \frac{2a_i^2}{1+\lambda a_i^2} \right] \tag{6.47}$$

Here the integration constant is fixed by requiring that $E = 0$ for $M = 0$. By explicitly evaluating the Euclidean action of the Kerr solution, the authors of [53] were able to show that the quantum statistical relation (6.46) is satisfied, thus providing a consistency check of the Kerr thermodynamic expressions.

6.3 AdS Kerr black holes in the AdS/CFT correspondence

The rotating Kerr solutions in anti-de Sitter provide us with a new classical gravitational configuration in the bulk which we can check the fluid/gravity correspondence against. Moreover, using the deep gravitational theorems of Hawking and Bekenstein we have determined the full thermodynamic behavior of this system which should be comparable to the thermodynamics of the dual fluid. According to the fluid/gravity correspondence, the rotating black holes of D dimensional supergravity are (dual to) certain thermal fluid states of the gauge theory on $\mathbb{R} \times S^{D-2}$, provided that the dual theory is suited for a fluid dynamical effective description. Notice that due to the structure of the Kerr solution we here take $\mathbb{R} \times S^{D-2}$ (conformally = Minkowski) to represent the boundary of AdS_D. Before proceeding we will first derive the condition for the Kerr black holes to be suited for a fluid description on $\mathbb{R} \times S^{D-2}$. It can be argued that the condition for a fluid dynamical effective description on $\mathbb{R} \times S^{D-2}$ to be valid is [5]

$$\ell_{\text{mfp}} \ll 1 \tag{6.48}$$

Now the mean free path of a generic fluid can be estimated from kinetic theory by $\ell_{\text{mfp}} \sim \eta/\rho$ [3]. As explained, for (at least) the class of gauge theories we are considering, it holds that $\eta = s/4\pi$. This means that the mean free path of a rotating conformal fluid on $\mathbb{R} \times S^{D-2}$ can be estimated as

$$\ell_{\text{mfp}} \sim \frac{s}{4\pi\rho} = \frac{d}{4\pi(D-2)} \frac{1}{T} \tag{6.49}$$

where we substituted in the expressions for the entropy and energy densities which will be derived below. Here T is the (global) fluid temperature, see below. We therefore see that the fluid dynamical description is applicable for “hot” fluids, $T \gg 1$, this is just as expected. As usual we identify the dual fluid temperature with the temperature of the bulk which is given by $T = \kappa/2\pi$. For large horizon radius y_+ we see that $y_+ \sim T$. The condition for the rotating black holes to be suited for a fluid dynamical dual description is therefore directly translated into the condition that the horizon radius should be large, $y_+ \gg 1$. [3]

6.3.1 THE BOUNDARY STRESS TENSOR

Here we compute the boundary stress tensor dual to the rotating AdS Kerr black hole. According to the general prescription for calculating the boundary stress tensor, we must perform a coordinate transformation of the Kerr metric (6.21), so that it takes the form (6.22) in a neighborhood of the boundary. The result of such a coordinate transformation to leading order was recorded in the expression (6.23). Since, to leading order, the metric contains no $r\mu$ cross-terms, we conclude that to leading order $n^A \partial_A = n^r \partial_r$ with

$$n^r = \frac{1}{G_{rr}^{1/2}} \left(1 + \frac{\delta G_{rr}}{G_{rr}}\right)^{-1/2} = r\sqrt{1 + 1/r^2} \left(1 - \frac{M\gamma^d}{r^d}\right) \tag{6.50}$$

Since the normal vector n^A is only a function of the radial coordinate, we conclude that the extrinsic curvature is given by

$$\Theta^\mu_\nu = -\Gamma^\mu_{\nu r} n^r \tag{6.51}$$

Now according to the general formula (3.43), we see that in order to obtain the boundary stress tensor, we only need to keep the leading order terms in the metric since higher order contributions will lead to terms of at least order $\mathcal{O}(1/r^{d+1})$ in the extrinsic curvature.[4] The relevant Christoffel symbols were computed in appendix A. We write

$$\Gamma^\mu_{\nu r} = \hat{\Gamma}^\mu_{\nu r} + \delta\Gamma^\mu_{\nu r} + \mathcal{O}\left(1/r^{2(d+1)}\right), \quad n^r = \hat{n}^r + \delta n^r + \mathcal{O}\left(1/r^{d+1}\right) \tag{6.52}$$

here $\hat{\Gamma}^\mu_{\nu r}$ and $\hat{n}^\mu$ respectively denotes the Christoffel symbols and normal vector of pure AdS and $\delta\Gamma^\mu_{\nu r}$ and δn^r are the leading order contributions. The extrinsic curvature is then given by (ignoring the terms independent of the M parameter corresponding to the counter-term stress tensor)

$$\Theta^\mu_\nu = -\hat{\Gamma}^\mu_{\nu r}\delta n^r - \delta\Gamma^\mu_{\nu r}\hat{n}^r + \mathcal{O}\left(1/r^{d+1}\right) \tag{6.53}$$

We may now compute the extrinsic curvature. We find

$$\Theta^t_t = -\frac{M\gamma^{d+2}}{r^d}\left(d - \gamma^{-2}\right), \quad \Theta^t_{\phi_i} = \frac{Md\gamma^{d+2}a_i\mu_i^2}{r^d} \tag{6.54}$$

$$\Theta^{\phi_i}_{\phi_j} = \frac{M\gamma^{d+2}}{r^d}\left(\delta^i_j\gamma^{-2} + da_ia_j\mu_j^2\right), \quad \Theta^{\phi_i}_t = -\frac{Md\gamma^{d+2}a_i}{r^d}, \quad \Theta^{\theta_i}_{\theta_j} = \frac{M\gamma^d}{r^d}\delta^i_j \tag{6.55}$$

Here we have used the manifestly independent coordinates $\left(\{\theta_i\}, \{\phi_j\}\right)$ on S^{d-1} in which the metric (A.14) is diagonal. It is straight forward to check that $\Theta = 0$. Having computed the extrinsic curvature and its trace, we can now compute the boundary stress tensor (3.43). Taking the large r limit and raising the index with the boundary metric $\mathrm{g}_{\mu\nu}dx^\mu x^\nu = \mathrm{d}t^2 + \sum_i \mathrm{d}\mu_i^2 + \sum_i \mu_i^2 \mathrm{d}\phi_i^2$ (using (A.13)), we obtain

$$T^{tt} = \frac{M\gamma^{d+2}}{8\pi G_{d+1}}\left(d - \gamma^{-2}\right), \quad T^{t\phi_i} = \frac{Md\gamma^{d+2}a_i}{8\pi G_{d+1}} \tag{6.56}$$

[3] This result holds for black holes sitting in AdS of unit radius. We can easily reintroduce the AdS radius. The condition now is that the black holes radius must be large compared to the radius of AdS, $y_+/L \gg 1$.

[4] The leading order terms in the Christoffel symbols $\Gamma^\mu_{\nu r}$ will be of order one lower that the ones of the metric since the Christoffel symbols contain r derivatives.

$$T^{\phi_i\phi_j} = \frac{M\gamma^{d+2}}{8\pi G_{d+1}}\left(1/\mu_j^2\delta^{ij}\gamma^{-2} + da_ia_j\right), \quad T^{\theta_i\theta_j} = \frac{M\gamma^d}{8\pi G_{d+1}}\left(\prod_{k=1}^{i-1}\sec^2\theta_k\right)\delta^{ij} \tag{6.57}$$

This completes the computation of the boundary stress tensor.

As we will see at the end of this section, the stress tensor obtained above is in complete agreement with the (perfect) stress tensor of a rigidly rotating fluid on the boundary. In order to demonstrate this we will therefore now turn to examining fluid dynamics on the boundary manifold $\mathbb{R} \times S^m$.

6.3.2 Rotating conformal fluid configurations on $\mathbb{R} \times S^m$

Here we examine the rigidly rotating (conformal) fluid configurations on $\mathbb{R} \times S^m$. Therefore consider a conformal fluid on $\mathbb{R} \times S^m$ with energy density $\rho = \eta_0 \mathcal{T}^{m+1}$, viscosity η and second order transport coefficients $\eta_1, \eta_2, \ldots$. For now, assume that the fluid has no charges.

Since we are looking for fluid configurations which are dual to the Kerr black hole, we are seeking stationary fluid configurations, that is, fluids is global equilibrium. There exists a nice way to determine a subset of these stationary solutions to the equations of fluid dynamics. The following argument is based on [57]. Assume that the fluid is in equilibrium and thus described by a perfect fluid stress tensor. Using the equations of motion (i.e. conservation of the stress tensor, $\mathcal{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$), the first law of thermodynamics (4.19), and imposing stationarity, it is easy to show that

$$a_\mu = -\nabla_\mu \ln \mathcal{T} \tag{6.58}$$

where a^μ is the acceleration of the fluid. Now, since the fluid is stationary it must have vanishing shear $\sigma_{\mu\nu}$. Moreover, assume that the expansion ϑ of the velocity also vanishes. It then follows from (4.10) that

$$\nabla_\mu u_\nu = \varpi_{\mu\nu} - u_\mu a_\nu \tag{6.59}$$

Now using the equation (6.58) we conclude that (recall that $\varpi_{(\mu\nu)} = 0$)

$$\nabla_{(\mu}(\alpha u_{\nu)}) = \alpha u_{(\mu}\nabla_{\nu)}\ln(\alpha\mathcal{T}) \tag{6.60}$$

for any function α. We therefore see that if we choose the function α to be proportional to the inverse temperature field, i.e. $\alpha = \tau/\mathcal{T}$, the vector field αu_μ is Killing. A solution to (6.60) therefore is

$$\frac{\tau u^\mu \partial_\mu}{\mathcal{T}} = \partial_t + \omega_i \partial_{\phi_i} \tag{6.61}$$

where we introduced a set of rotational parameters ω_i. The temperature field is now determined by the defining equation $u_\mu u^\mu = -1$. We have

$$\mathcal{T} = \gamma\tau, \quad u^\mu = \gamma(1, 0, \cdots, 0, \omega_1, \cdots, \omega_N) \tag{6.62}$$

with

$$\gamma = \left(1 - \sum_{i=1}^N \mathrm{g}_{\phi_i\phi_i}\omega_i^2\right)^{-1/2} = \left(1 - \sum_{i=1}^N \left(\prod_{j=1}^{i-1}\cos^2\theta_j\right)\sin^2\theta_i\omega_i^2\right)^{-1/2} \tag{6.63}$$

The fluid (6.62) is in global equilibrium to first order. What about to second order? Since $\sigma_{\mu\nu} = 0$, the second order contributions $\mathfrak{T}_1^{\mu\nu}$, $\mathfrak{T}_2^{\mu\nu}$ and $\mathfrak{T}_3^{\mu\nu}$ all vanish.

Moreover, since $\mathbb{R}\times S^m$ is conformally flat, the Weyl tensor is identically equal to zero, $C_{\mu\nu\rho\lambda}=0$. The dissipative correction $\mathfrak{T}_5^{\mu\nu}$ therefore also vanishes. This means that to second order

$$\Pi^{\mu\nu} = -\eta_4 \mathcal{T}^{m-1}\mathfrak{T}_4 = -\eta_4 \mathcal{T}^{m-1}\varpi^{\langle\mu}{}_{\lambda}\varpi^{\lambda\nu\rangle} \tag{6.64}$$

However, the vorticity $\varpi^{\mu\nu}$ (and therefore the second order correction $\mathfrak{T}_4^{\mu\nu}$) does in general not vanish for the rotating fluid (6.62). This means that if we want the rotating fluid to be in thermodynamic and dynamic equilibrium, we must impose the condition $\eta_4\sim\xi_\varpi=0$ on the transport coefficient associated with $\mathfrak{T}_4^{\mu\nu}$. Assuming this, the fluid stress tensor is then given by

$$T^{\mu\nu} = \rho\left(u^\mu u^\nu + \frac{1}{m}\Delta^{\mu\nu}\right) = \eta_0\mathcal{T}^{m+1}\left(u^\mu u^\nu + \frac{1}{m}\Delta^{\mu\nu}\right) \tag{6.65}$$

It is of course comforting that the second order analysis of the fluctuating brane in §5.9, exactly found that $\xi_\varpi=0$. We elaborate on this below.

Now using the expressions for the fluid velocity and the metric on $\mathbb{R}\times S^m$, we can write down the expression for the perfect fluid stress tensor in terms of the function γ and the constant τ. We find

$$\begin{aligned}
T^{tt} &= \frac{\eta_0\tau^{m+1}}{m}\gamma^{m+1}\left((m+1)\gamma^2-1\right), \quad T^{t\phi_i} = \frac{\eta_0(m+1)\tau^{m+1}}{m}\gamma^{m+3}\omega_i, \\
T^{\phi_i\phi_i} &= \frac{\eta_0\tau^{m+1}}{m}\gamma^{m+1}\left[(m+1)\gamma^2\omega_i^2 + \left(\prod_{j=1}^{i-1}\sec^2\theta_j\right)\csc^2\theta_i\right], \\
T^{\phi_i\phi_j} &= \frac{\eta_0(m+1)\tau^{m+1}}{m}\gamma^{m+3}\omega_i\omega_j, \quad T^{\theta_i\theta_i} = \frac{\eta_0\tau^{m+1}}{m}\gamma^{m+1}\left(\prod_{j=1}^{i-1}\sec^2\theta_j\right)
\end{aligned} \tag{6.66}$$

where we have explicitly written the $\mathrm{g}_{\phi_i\phi_i}=\mu_i^2$ angular θ_i dependence.

Now let us consider the charged case. Here we will restrict ourselves to first order. Therefore, assume that the fluid has a set of conserved charges (which we will from now on call 'R-charges', since they will exactly correspond to the R-charges of the underlying field theory) with corresponding currents

$$r_I^\mu = r_I u^\mu + \Upsilon_I^\mu \tag{6.67}$$

Here r_I is the charge density corresponding to the I'th R-charge. According to the discussion in §4.1.5, the most general first order current dissipation term is given by $\nabla_\mu(\mu_I/\mathcal{T})=\mathcal{D}_\mu(\mu_I/\mathcal{T})$ (in the conformal case this follows directly from Weyl invariance). Here we have ignored the subtle case of $d=4$ (see below). This means that in order for the current dissipation Υ_I^μ to vanish (to first order), we must take

$$\mu_I = \nu_I\mathcal{T} = \gamma\tau\nu_I \tag{6.68}$$

where ν_I is a constant. We have now solved the fluid dynamics of the R-charged fluid to first order which is also easy to check by direct computation [5]. The fluid velocity and temperature are given by (6.62) and the R-charge current is $r_I^\mu=r_Iu^\mu$. Finally, since the fluid is described by perfect fluid dynamics, the entropy current is given by

$$J_S^\mu = su^\mu \quad\Rightarrow\quad \mathcal{D}_\mu J_S^\mu = 0 \tag{6.69}$$

6.3.3 Conserved quantities

We will now write down the expressions for the total energy, entropy, angular momenta, and R-charges of the solution (6.66). Recall that a Killing vector K^μ on an arbitrary manifold has an associated Komar current

$$J^\mu[K] = K_\nu \left[T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right] \tag{6.70}$$

It can been shown that this current is conserved, see e.g. [14]. The conserved quantity (the Komar charge) associated with this current is therefore given by [5]

$$Q[K] = -\int_\Sigma n_\mu J^\mu[K] \tag{6.71}$$

here n^μ is the normal vector to the timelike surface Σ. The expression (6.71) is independent of the surface Σ. It follows that it is independent of time i.e. conserved.

Returning to our fluid on $\mathbb{R} \times S^m$ we will take Σ to be the surface of constant t. Since the fluid is conformal and the metric is diagonal we have

$$Q[K] = -\int_{S^m} g_{\mu\nu} K^\mu T^{t\nu} \tag{6.72}$$

where evaluation of the expression for constant t is understood. Now, the manifold $\mathbb{R} \times S^m$ has the Killing vectors $\partial/\partial t$ and $\partial/\partial\phi_1, \cdots, \partial/\partial\phi_N$. The vector $\partial/\partial t$ is the generator of time translations and the associated conserved quantity $Q(\partial_t)$ is therefore naturally identified with the energy of the fluid solution. The vector $\partial/\partial\phi_i$ is the generator of the i'th angular momentum and we therefore associate $Q(\partial_{\phi_i})$ with the ϕ_i angular momentum of the solution. We therefore have

$$E = -\int_{S^m} g_{tt} T^{tt}, \quad L_i = -\int_{S^m} g_{\phi_i\phi_i} T^{t\phi_i} \tag{6.73}$$

Finally, since the (perfect) entropy and R-charge currents are conserved, we can write down the total conserved entropy and R-charge of the solution:

$$S = -\int_{S^m} n_\mu J^\mu_S = \int_{S^m} J^t_S, \quad R_I = -\int_{S^m} n_\mu r^\mu_I = \int_{S^m} r^t_I \tag{6.74}$$

6.3.4 Evaluation of conserved quantities

Here we compute the total energy, entropy, angular momenta, and R-charges of the solution (6.66). First, using conformal invariance we will parametrize the full thermodynamic behavior of the system in terms of a single function $H(\{\nu_I\})$, where $\nu_I = \mu_I/\mathcal{T}$ are the reduced chemical potentials. Let $\mathcal{E}$, $\mathcal{S}$, $\mathcal{R}_I$ respectively denote the energy, entropy, and charge contained in a small volume V and consider the grand potential

$$\Phi = \mathcal{E} - \mathcal{T}S - \mu_I\mathcal{R}_I \quad \text{with first law} \quad d\Phi = -\mathcal{S}d\mathcal{T} - p dV - \mathcal{R}_I d\mu_I \tag{6.75}$$

It now follows from extensivity and conformal invariance that

$$\Phi = -V\mathcal{T}^{m+1}H\big(\{\nu_I\}\big) \tag{6.76}$$

[5] As usual when writing expressions like $\int_M f$ we mean $\int d^D x\sqrt{|g|}\, f(x)$ where g is the metric of the D-dimensional manifold M.

where the function H is only a function of the reduced potentials ν_I. Therefore using this simple expression for Φ and the relation (6.75), we obtain

$$\rho = mp = mA\mathcal{T}^{m+1}, \quad s = B\mathcal{T}^m, \quad r_I = C_I\mathcal{T}^m \tag{6.77}$$

where the Weyl invariant functions A, B, and C_I are given by (as usual a tilde denotes a Weyl transformed quantity)

$$A \equiv H = \tilde{A}, \quad B \equiv (m+1)H - \nu_I \partial_{\nu_I} H = \tilde{B}, \quad C \equiv \frac{\partial H}{\partial \nu_I} = \tilde{C} \tag{6.78}$$

Using the equations (6.65), (6.67), and (6.69), we therefore see that

$$\eta_0 = mA, \quad J_S^t = B\gamma^{m+1}\tau^m, \quad r_I^t = C_I \gamma^{m+1}\tau^m. \tag{6.79}$$

We can now carry out the integrations. The relevant integrals can be derived from the identity [5]

$$\int_{S^m} \gamma^{m+1}[\{\omega_i\}] = \frac{\mathrm{Vol}(S^m)}{\prod_{j=1}^N (1-\omega_j^2)} \tag{6.80}$$

where γ is the function introduced in equation (6.63). By parameterizing S^m using the directional cosines $\big(\{\mu_i\}, \{\phi_j\}\big)$, this identity is easily verified for odd m by a simple rescaling of the variables. Suppose that m is odd i.e. $\epsilon = 0$. From the above identity it follows that (identical results hold when m is even)

$$\frac{\mathrm{d}}{\mathrm{d}(1-\omega_i^2)} \left(\frac{\mathrm{Vol}(S^m)}{\prod_{j=1}^N (1-\omega_j^2)} \right) = \int_{S^m} \frac{\mathrm{d}\gamma^d[\{\omega_i\}]}{\mathrm{d}(1-\omega_i^2)} = \int_{S^m} \frac{m+1}{2} \gamma^{m+3}\mu_i^2 \tag{6.81}$$

therefore

$$\int_{S^m} \gamma^{m+3}\mu_i^2 = \frac{2}{(m+1)(1-\omega_i^2)} \frac{\mathrm{Vol}(S^m)}{\prod_{j=1}^N (1-\omega_j^2)} \tag{6.82}$$

By using the defining relation $\sum_i \mu_i^2 = 1$, we therefore obtain that

$$\int_{S^m} \gamma^{m+3} = \frac{\mathrm{Vol}(S^m)}{(m+1)\prod_{j=1}^N (1-\omega_j^2)} \left(2\sum_{i=1}^N \frac{\omega_i^2}{1-\omega_i^2} + m + 1 \right) \tag{6.83}$$

The relations (6.80), (6.82) and (6.83) are precisely the ones needed to evaluate the conserved quantities. We find

$$E = \frac{A\tau^{m+1}\mathrm{Vol}(S^m)}{\prod_i (1-\omega_i^2)} \left[2\sum_j \frac{\omega_j^2}{1-\omega_j^2} + m \right], \qquad S = \frac{B\tau^m \mathrm{Vol}(S^m)}{\prod_i (1-\omega_i^2)} \tag{6.84}$$

$$L_i = \frac{A\tau^{m+1}\mathrm{Vol}(S^m)}{\prod_j (1-\omega_j^2)} \left[\frac{2\omega_i}{1-\omega_a^i} \right], \qquad R_I = \frac{C_I \tau^m \mathrm{Vol}(S^m)}{\prod_i (1-\omega_i^2)} \tag{6.85}$$

Having determined the global conserved charges of the fluid solution we can now examine its (global) thermodynamics. Physically such an analysis makes sense since the fluid is in global thermodynamic equilibrium.

6.3.5 Thermodynamics of the fluid solution

Since the rotating fluid is in (global) thermodynamic equilibrium, we expect that its thermodynamics can be encapsulated by a grand canonical partition function Z_{gc} of the form

$$\mathcal{Z}_{\mathrm{gc}} = \mathrm{Tr}\left\{ \exp\left[\frac{1}{T}\Big(-H + \Omega_i L_i + \xi_I R_I \Big) \right] \right\} \tag{6.86}$$

where the trace is taken over the globally defined fluid states. Using a standard thermodynamic argument, the partition function can be written as

$$\mathcal{Z}_{\text{gc}} = \exp\left[-\frac{1}{T}\Big(E - TS - \Omega_i L_i - \xi_I R_I\Big)\right] \tag{6.87}$$

Here T is the global temperature, Ω_i is the global "chemical potential" associated with the i'th angular momentum, and ξ_I is the chemical potential associated with the I'th R-charge. It is important to stress out that the globally defined temperature T must not be confused with the (local) fluid temperature $\mathcal{T}$. In the same way the chemical potentials ξ_I must not be confused with the fluid chemical potentials μ_I. The thermodynamic system (6.86) has the associated first law

$$\mathrm{d}E = T\mathrm{d}S + \Omega_i \mathrm{d}L_i + \xi_I \mathrm{d}R_I \tag{6.88}$$

so

$$T = \left(\frac{\partial E}{\partial S}\right)_{\Omega_i, R_I} \qquad \Omega_i = \left(\frac{\partial E}{\partial L_i}\right)_{S, \Omega_{j\neq i}, R_I} \qquad \xi_I = \left(\frac{\partial E}{\partial R_I}\right)_{S, \Omega_i, R_{J\neq I}} \tag{6.89}$$

Using these relations we can now compute the global temperature, "angular velocities", and chemical potentials of the fluid configuration. In order to do this, notice that the energy can be written

$$E = \omega_i L_i + \frac{m}{m+1}\tau\nu_I R_I + \frac{m}{m+1}\tau S \tag{6.90}$$

and therefore

$$\mathrm{d}E = \omega_i \mathrm{d}L_i + L_i \mathrm{d}\omega_i + \frac{m}{m+1}\big(\tau\nu_I \mathrm{d}R_I + \tau R_I \mathrm{d}\nu_I\big) + \frac{m}{m+1}\big(\tau\mathrm{d}S + (S + \nu_I R_I)\mathrm{d}\tau\big) \tag{6.91}$$

A straight forward computation reveals that (just use the expressions for S and R_I)

$$(S + \nu_I R_I)\mathrm{d}\tau = \frac{1}{m}\tau\mathrm{d}S - \tau(S + \nu_J R_J)\sum_i \frac{\omega_i \mathrm{d}\omega_i}{1-\omega_i} - \tau R_I \mathrm{d}\nu_I + \frac{1}{m}\tau\nu_I \mathrm{d}R_I \tag{6.92}$$

Substituting this into (6.91) and using the relations (6.84), we finally obtain

$$\mathrm{d}E = \tau\mathrm{d}S + \Omega_i \mathrm{d}L_i + \tau\nu_I \mathrm{d}R_I \tag{6.93}$$

We therefore conclude

$$T = \tau, \quad \Omega_i = \omega_i, \quad \xi_I = \tau\nu_I \tag{6.94}$$

The result is just as expected: The global temperature T equals the (constant) local rest frame temperature i.e. $T = \mathcal{T}/\gamma$ and so forth. Having computed the temperature and potentials we can now finally compute the partition function from (6.87). It is easy to show that $\mathcal{Z}_{\text{gc}}$ takes the simple form

$$\ln \mathcal{Z}_{\text{gc}} = \frac{\mathrm{Vol}(S^m)T^m H(\{\xi_I/T\})}{\prod_i (1-\Omega_i^2)} = \frac{\ln \mathcal{Z}_{\text{gc}}^{(\Omega=0)}}{\prod_i(1-\Omega_i^2)} \tag{6.95}$$

Notice that the function $H(\{\xi_I/T\})$ is rather arbitrary, it depends on the properties of the conformal "stuff" making up the fluid. Also notice that the partition function $\mathcal{Z}_{\text{gc}}$ is completely fixed once the form of $H(\{\xi_I/T\})$ is specified. This means that the fluid dynamical analysis of the conformal fluid on $\mathbb{R}\times S^m$ tells us nothing about the thermodynamic properties of the charged fluid at rest. This was of course pretty much expected. However, once the thermodynamics of the fluid at rest is given, the thermodynamics of the corresponding rotating fluid is known. All in all, summarizing the above computation, the thermodynamics of the conformal rotating fluid is completely determined by the equations (6.94), (6.84), and (6.85).

6.3.6 Comparison between AdS_D Kerr black holes and perfect fluid dynamics on $\mathbb{R} \times \mathrm{S}^{D-2}$

We are now ready to compare the physics on the gravitational side with a perfect rotating conformal fluid on $\mathbb{R} \times \mathrm{S}^{D-2}$. As we have argued above, the fluid/gravity correspondence is expected to be valid for Kerr black holes in the large horizon limit. We will therefore start by working out the thermodynamics of the rotating Kerr black hole in the large horizon limit. This is then readily compared to the fluid thermodynamic formulæ derived above.

Thermodynamics of Kerr black holes in the large horizon limit: We will now relate the horizon radius y_+ to the M parameter in the large horizon limit. The black hole horizon is located at $y = y_+$ where y_+ is the largest root of the equation (here we as usual take $\lambda = -1$, cf. §6.1.3)

$$V - 2M = y^{\epsilon-2}(1 + y^2)\prod_{i=1}^{N}(y^2 + a_i^2) - 2M = 0 \tag{6.96}$$

This means that if $y_+ \gg 1$, the horizon radius is related to the M parameter by

$$2M = y_+^{D-1}\left(1 + \mathcal{O}(1/y_+^2)\right) \tag{6.97}$$

where as usual D denotes the dimension of the bulk. Notice that this relation holds for both D even and odd. Now it is straightforward to express the temperature $T = \kappa/2\pi$ in terms of the horizon radius in the large horizon limit. We use the expressions (6.36) and (6.37) in the limit $y_+ \gg 1$. Notice how the extra term in (6.36) exactly ensures that the expressions are identical for even and odd D dimensions. We find

$$T = \left[\frac{(D-1)y_+}{4\pi}\right]\left(1 + \mathcal{O}(1/y_+^2)\right) \tag{6.98}$$

This especially means that, in the large horizon limit, the M parameter is related to the temperature in the following way

$$2M = T^{D-1}\left[\frac{4\pi}{D-1}\right]^{D-1}\left(1 + \mathcal{O}(1/T^2)\right) \tag{6.99}$$

This expression can now be substituted into the thermodynamic formulæof §6.2. First of all we see that, to leading order, the angular velocity can be identified with the corresponding rotational parameter, that is, $\Omega_i = a_i$. We then find to leading order

$$\begin{aligned}
E &= \frac{\mathrm{Vol}(S^{D-2})T^{D-1}}{16\pi G_D \prod_{j=1}^N (1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1}\left[\sum_{i=1}^N \frac{2a_i^2}{1-a_i^2} + D - 2\right] \\
L_i &= \frac{\mathrm{Vol}(S^{D-2})T^{D-1}}{16\pi G_D \prod_{j=1}^N (1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1}\left[\frac{2a_i}{1-a_i^2}\right] \\
S &= \frac{(D-1)\mathrm{Vol}(S^{D-2})T^{D-2}}{16\pi G_D \prod_{j=1}^N (1-a_j^2)}\left[\frac{4\pi}{D-1}\right]^{D-1} \\
R_i &= 0
\end{aligned} \tag{6.100}$$

where we have re-introduced the D dimensional gravitational constant.

Thermodynamics of uncharged rotating fluids: The thermodynamics of the most general R-charged rotating conformal fluid on $\mathbb{R} \times S^m$ was derived in §§6.3.4-6.3.5. Here we will focus on the case where the fluid is uncharged, $\nu_I = 0$. In this case the thermodynamic function H is simply a constant

$$H = h \tag{6.101}$$

And it follows that $A = h$, $B = (m+1)h$ and $C = 0$. According to the equations (6.84), the energy, angular momenta, and entropy of the uncharged conformal rigidly rotating fluid is then given by

$$E = \frac{h\mathrm{Vol}(S^m)T^{m+1}}{\prod_{i=1}^{N}(1-\Omega_j^2)}\left[\sum_{j=1}^{N}\frac{2\Omega_i^2}{1-\Omega_i^2} + m\right], \quad S = \frac{(m+1)h\mathrm{Vol}(S^m)T^m}{\prod_{j=1}^{N}(1-\Omega_j^2)} \tag{6.102}$$

$$L_i = \frac{h\mathrm{Vol}(S^m)T^{m+1}}{\prod_{j=1}^{N}(1-\Omega_j^2)}\left[\frac{2\Omega_i}{1-\Omega_i^2}\right], \quad R_i = 0 \tag{6.103}$$

These expressions can now directly be compared to those of the large rotating black holes (6.100). We see that by identifying D with $m+2$, the Kerr rotation parameters a_i with Ω_i (this amounts to identifying the Kerr rotation parameters with the fluid rotation parameters, cf. equation (6.94)) and the constant h with

$$h = \frac{1}{16\pi G_D}\left[\frac{4\pi}{D-1}\right]^{D-1} \tag{6.104}$$

the thermodynamics of the two systems are in complete agreement. Moreover by comparing the dual boundary stress tensor (computed in §6.3.1) with the perfect fluid stress tensor (6.66) and making the identifications above, we see that there is complete agreement between the two expressions.

Moreover we see that the (unit temperature) energy density of the fluid exactly is the same as we found for the black brane (cf. equation (5.41)). We have therefore demonstrated that the fluid/gravity correspondence, developed for the fluctuating black brane, applies directly to the rotating black holes of Einstein gravity. Furthermore we saw that the fluid dynamic analysis of the boosted black brane yielded $\xi_\varpi = 0$. In the rotating Kerr black hole scheme this condition on ξ_ϖ is a consistency requirement. This, as far as we see, provides a non-trivial check of the fluid/gravity correspondence. Following the same line of thought, it is possible that the fact that the boundary stress tensor of the rotating Kerr black hole is that of a perfect fluid, could be used (on grounds of consistency) to deduce that certain higher order transport coefficients (containing only vorticity terms) must vanish. We will now briefly discuss the rotating $\mathcal{N} = 4$ Yang-Mills plasma on $\mathbb{R} \times S^3$.

6.3.7 The rotating $\mathcal{N} = 4$ Yang-Mills plasma on $\mathbb{R} \times S^3$

We will now show how it is possible to consider rotating plasma configurations of $\mathcal{N} = 4$ SYM, using the above fluid dynamical analysis. To this end consider the consistent truncation of supergravity on $\mathrm{AdS}_5 \times S^5$, given by (3.25). Instead of only considering pure gravity, we now wish also to consider excitations of the SO(6) gauge fields (and scalar fields which were represented by the dots in (3.25)). However, we will only consider a subgroup of these excitations (this amounts to performing yet another truncation of the theory). In the following we will only consider the abelian part of SO(6): The maximal abelian subgroup of SO(6) is $\mathrm{U}(1)^3$. With the rest of the gauge degrees of freedom set to zero, we then obtain the so-called STU model, which is a consistent truncation of supergravity on $\mathrm{AdS}_5 \times S^5$ with $\mathrm{U}(1)^3$ which is dual to $\mathcal{N}=4$ SYM with 3 Cartan $\mathrm{U}(1)_R^3$'s inside $SU(4)_R \cong \mathrm{SO}(6)$ [58]. The truncated action takes the form ($I=1,2,3$) [58, 59]

$$S = \frac{1}{16\pi G_5}\int\sqrt{|G|}\Big[R + 2\mathcal{V}(X) - \frac{1}{2}E_{IJ}(X)(F^I)_{AB}(F^J)^{AB} - E_{IJ}(X)\partial_A X^I\partial^A X^J \\ + \frac{1}{24\sqrt{|G|}}C_{IJK}\epsilon^{ABCDE}(F^I)_{AB}(F^J)_{CF}(A^K)_E\Big] \quad (6.105)$$

where is C_{IJK} is totally symmetric with $C_{123}=1$ and where the scalar potential and the metric on the scalar manifold are given by (notice that our notation differs from that of [58, 59], since we reserve G for the bulk metric).

$$\mathcal{V}(X) = 2\sum_{I=1}^{3}\frac{1}{X^I}\ , \quad E_{IJ} = \frac{1}{2}\mathrm{diag}\left(\frac{1}{(X^I)^2}\right) \quad (6.106)$$

Moreover, the three scalars are subject to the constraint $X^1X^2X^3=1$. Notice that the last term in the action (6.105) is immediately recognized as the Chern-Simons term

$$C_{IJK}\int F^I\wedge F^J\wedge A^K \quad (6.107)$$

One can now straightforwardly work out the equations of motion pertaining to the STU action (6.105). Several solutions to the EOM are known, here we write down the three-charge non-extremal black brane STU solution. The metric part of the solution is given by [60]

$$\mathrm{d}s^2 = -\mathcal{H}^{-\frac{2}{3}}f(r)\mathrm{d}t^2 + \mathcal{H}^{\frac{1}{3}}\left(\frac{\mathrm{d}r^2}{f(r)} + r^2\sum_{i=1}^{3}\mathrm{d}x_i^2\right) \quad (6.108)$$

$$f(r) = \frac{M}{r^2} - r^2\mathcal{H}, \quad H_I = 1 + \frac{q_I}{r^2}, \quad \mathcal{H} = H_1H_2H_3 \quad (6.109)$$

while the gauge field part takes the form

$$X^I = \frac{\mathcal{H}^{\frac{1}{3}}}{H_I}, \quad A_t^I = \frac{\sqrt{Mq_I}}{r^2+q_I} \quad (6.110)$$

This SO(6) charged black brane, or equivalently SO(6) charged *non-rotating* black hole (see appendix A of [59] for a discussion of the relationship between the black brane and black hole solution), is dual to the $\mathcal{N}=4$ SYM R-charged plasma on $\mathbb{R}\times S^3$ at rest. By working out the thermodynamics of the STU black brane (6.108), we can then obtain an equation of state for the $\mathcal{N}=4$ R-charged plasma at rest, using the AdS/CFT correspondence. The equation of state is determined by the following partition function [5, 59]

$$\mathcal{Z}^{\mathcal{N}=4\ \mathrm{SYM}}_{\mathrm{gc,\ rest}} = \frac{2\pi^2 N\mathrm{Vol}(S^3)T^3\prod_I(1+\kappa_I)^3}{\left(2+\sum_J\kappa_J - \prod\kappa_J\right)^4} \quad (6.111)$$

Here the auxiliary parameters κ_I are directly related to the R-charge density/entropy density ratio in the following way[6]

$$\kappa_I = \frac{4\pi^2 r_I^2}{s^2} \quad (6.113)$$

[6] Although not important for our purposes, we mention that the κ_I parameters are constrained by $\kappa_I\geq 0$ and

$$\sum_I\frac{1}{1+\kappa_I} - 1 \geq 0 \quad (6.112)$$

which is obtained by requiring thermodynamic stability. The latter constraint amounts to requiring non-negative temperature $T\geq 0$.

Having extracted the equation of state for the $\mathcal{N}=4$ SYM R-charged plasma at rest, we can immediately write down the partition function for the rotating $\mathcal{N}=4$ plasma configuration. According to the equation (6.95), it is given by

$$\mathcal{Z}_{\rm gc}^{\mathcal{N}=4 \text{ SYM}} = \frac{2\pi^2 N \mathrm{Vol}(S^3) T^3 \prod_I (1+\kappa_I)^3}{(1-\Omega_1^2)(1-\Omega_2^2)\left(2+\sum_J \kappa_J - \prod \kappa_J\right)^4} \tag{6.114}$$

Here we used that S^3 is parametrized by two azimuthal angles ϕ_1, ϕ_2 and one directional cosine θ. In these coordinates the function γ, defined by (6.63), takes the form

$$\gamma^{-2} = 1 - \Omega_1^2 \sin^2\theta - \Omega_2^2 \cos^2\theta \tag{6.115}$$

Here we used equation (6.94) to identify the rotational parameters ω_1 and ω_2 with the conjugate variables to the angular momenta (respectively Ω_1 and Ω_2).

Much like we have black brane (transverse planar symmetry) and black hole (transverse spherical symmetry) solutions to Einstein gravity, there also exist known five dimensional rotating SO(6) charged black hole solutions. Since these rotating black holes are expected to be dual to rotating $\mathcal{N}=4$ SYM plasma configurations, these black holes will provide a check of the validity of (6.114). This analysis was carried out in the paper [5]. The analysis is essentially same as the one we carried out in §6.3.6, however, the thermodynamic expressions are of course more complicated (as they now also contain (three) R-charge(s)). Here we summarize the results of [5]. For the original references on the R-charged rotating black hole solutions and their thermodynamics, we refer to [5].

- The thermodynamics predicted from (6.114) is in complete agreement, to order $\mathcal{O}(1/y_+^2)$, with the thermodynamics of the large horizon rotating R-charged black hole with two R-charges set equal to zero.

- The thermodynamics predicted from (6.114) is in complete agreement, to order $\mathcal{O}(1/y_+)$, with the thermodynamics of the large horizon rotating R-charged black hole with all R-charges equal.

- Finally the same produce can be carried out for respectively the rotating fluid of the M5 and the M2 brane conformal field theory on respectively $\mathbb{R}\times S^2$ and $\mathbb{R}\times S^5$. The thermodynamics predicted from (the equivalent of) (6.114) is in complete agreement, to order $\mathcal{O}(1/y_+^2)$, with the thermodynamics of the large horizon rotating R-charged black hole with arbitrary R-charges.

Notice that the difference between the large horizon black hole thermodynamics and the thermodynamics of the dual fluid is expected to show up at $\mathcal{O}(1/y_+^2)$. This is because, as argued above, $\ell_{\rm mfp} \sim 1/y_+$ and we have only solved the fluid dynamics to first order in $\ell_{\rm mfp}$. It is, however, quite a surprise that we see that there already is a discrepancy at $\mathcal{O}(1/y_+) \sim \mathcal{O}(\ell_{\rm mfp})$ for black holes with all R-charges non-zero.

6.3.8 Resolution to the $\mathcal{N}=4$ fluid/gravity discrepancy

The simplest explanation for the discrepancy is that, in the case of the most general R-charged fluid, the velocity field (constructed in the previous section), does not solve the equations of fluid dynamics to first order in $\ell_{\rm mfp}$ (only to zeroth order). This indeed seems to be the case. As mentioned in §4.1.5, in four dimensions, it is possible to construct a pseudo-vector containing only one derivative by

$$\ell^\mu = \epsilon^{\mu\nu\rho\lambda} u_\nu \mathcal{D}_\rho u_\lambda \tag{6.116}$$

This (Weyl covariant) term can then be used to alter the equation for the current dissipation (4.18) according to

$$\Upsilon_I = D_I \Delta^{\mu\nu} \mathcal{D}_\nu \nu_I + \mho_I \ell^\mu \tag{6.117}$$

Notice that the vector ℓ^μ cannot be used to construct corrections to the stress tensor to first order (in the Landau frame), so only the R-charge currents get corrected. The pseudo-vector term was not considered by the authors of [5], however, as was first realized in [9], such a term can be physically realized (the authors of [9] considered an Einstein-Maxwell + Chern-Simons term type system). More recently the fluid dynamics of the fluctuating STU brane was worked out (conceptually in the same way as presented in chapter 5) in the paper [58]. The authors of [58] were able to compute the $\mho_I$ transport coefficient associated with ℓ^μ, they found

$$\mho_I \sim C_{IJK} \frac{\sqrt{Mq_J}\sqrt{Mq_K}}{(y_+^2 + q_J)(y_+^2 + q_K)} - \frac{\sqrt{Mq_I}}{3M} C_{JKL} \frac{\sqrt{Mq_J}\sqrt{Mq_K}\sqrt{Mq_L}}{(y_+^2 + q_J)(y_+^2 + q_K)(y_+^2 + q_L)} \tag{6.118}$$

We see that the form of this transport coefficient accounts for the above observations. Indeed, for a general R-charged black hole we have non-zero $\mho_I$'s and we expect the thermodynamics to get corrections of order $\mathcal{O}(\ell_{\mathrm{mfp}})$. However, when two of the R-charges are zero, we see that all the $\mho_I$'s vanish and the thermodynamics should only get corrections at $\mathcal{O}(\ell_{\mathrm{mfp}}^2)$. Finally we emphasize that a one-derivative pseudo-vector term can only be constructed in four dimensions. Of course similar pseudo-vector terms can be constructed in higher dimensions, but they will contain more than one derivative. First order pseudo-vector corrections are therefore only seen in four dimensional theories. This explains why we do not see a first order discrepancy in the M2- and M5-brane conformal field theories.

In the four dimensional case, it should be possible to include these correction terms and solve fluid dynamics to first order. This would introduce $\mathcal{O}(\ell_{\mathrm{mfp}})$ corrections to the temperature and therefore the global thermodynamic charges. Hopefully such an analysis would account for the $\mathcal{O}(\ell_{\mathrm{mfp}})$ corrections to the large horizon black hole thermodynamic expressions. We will leave this analysis for future work.

7

Concluding remarks

Conclusion

In this thesis we have motivated the existence of a correspondence between gravity and fluid dynamics (as an effective long-wave description of a boundary CFT), using the AdS/CFT correspondence. It was explicitly shown how the finite temperature AdS/CFT correspondence gives a mapping between the solutions to the relativistic Navier-Stokes equation, governing fluid dynamics, to a set of "slowly varying" solutions to Einstein's equation with a negative cosmological constant. More specifically we have demonstrated how the equations of gravity can be used to directly extract information about a given strongly coupled CFT (in the planar limit) with a gravitational dual. It was shown how the transport coefficients, determining the kinematic properties of the fluid, in principle can be found by perturbatively solving Einstein's equation. We emphasize that, with the technology available today, such an analytical computation is not possible, using only gauge theoretical methods. Since fluid dynamics is a long-wave effective description of a given CFT, knowing the properties of the fluid (both transport coefficients but perhaps also the phase structure), can presumably tell us a lot about the fundamental properties of the underlying field theory. The fluid/gravity correspondence thus provides us with an extremely powerful tool for understanding strongly coupled theories.

Outlook

The fluid/gravity correspondence has several "obvious" generalizations: In addition to studying gravity systems coupled to matter, in the fluid/gravity correspondence (which we touched upon in §6.3.7), it would be very interesting to study the fluid/gravity correspondence away from the different limits considered in the thesis. For example, how does the fluid/gravity correspondence look if we consider stringy effects on the gravitational side? Moreover, we could consider the effect of quantum corrections on the gravity side or, equivalently, go away from the large N limit on the gauge theory side. It is possible that this could teach us important lessons in gravity, string theory and strongly coupled theories. These fascinating issues are currently being studied by the fluid/gravity community.

As mentioned in the introduction, one of the ultimate goals of the fluid/gravity correspondence is to understand the Quark Gluon Plasma state, studied experimentally in RHIC and LHC. The QGP is believed to be a phase of QCD which is completely locally thermalized and thus suitable for an effective fluid dynamic description. The fluid/gravity correspondence therefore seems to be a promising tool for getting a better understanding of the QGP. Although QCD is a strongly coupled theory, it is obviously a long way from the conformal field theories considered in this thesis. First of all, QCD does not exhibit supersymmetry and is, more importantly, obviously not a conformal field theory. Moreover $N = 3 \ll \infty$ for QCD , which posses an even greater problem, since taking the planar limit was vital for doing computations in the AdS/CFT correspondence. However, there do exist methods for breaking SUSY and conformal invariance of the original boundary theory (we refer to the reviews [17, 20] and the references therein). Moreover there has also been developed methods for going away from the $N = \infty$ limit for certain theories. Finally models that mimic some of the properties of QCD have been constructed, for example the STU model (6.105), which was an example of a multi-charged plasma [58]. Needless to say, directly obtaining QCD from a theory with a known gravitational dual (e.g. $\mathcal{N} = 4$ SYM) is still far away, however, it is the hope that these methods in the future can be implemented in the fluid/gravity scheme and thus provide a non-perturbative framework for understanding the QGP.

Recently the fluid/gravity correspondence has found applications in a completely different type of physical theories, namely condensed matter systems. More specifically holographic methods seem to be a promising tool (it already is) for understanding quantum critical systems. A quantum critical system is a condensed matter system containing a quantum critical point which is a special class of continuous phase transition that takes place at zero temperature. As usual a condensed matter system is effectively described in terms of quasi particles which, at quantum criticality, interact in a non-weak manner. Moreover, the system at a quantum critical point exhibits scale invariance. These are exactly the essential properties of the boundary theory in the AdS/CFT correspondence. For an excellent introduction to these concepts and ideas, see [61]. When dealing with condensed matter systems, there is a subtlety which we have not mentioned. The theory of condensed matter systems is non-relativistic, one therefore needs to find a way of reducing the (relativistic) fluid/gravity correspondence to a non-relativistic fluid/gravity duality. Indeed, such a formalism exists, see [28] for a nice introduction.

So far we have only discussed what the equations of gravity can teach us about the fluid dynamics of strongly coupled theories. We have, however, not discussed the opposite point. What can we learn about gravity from the equations of hydrodynamics? Since the equations of fluid dynamics have been studied in great detail over the last couple of centuries, it is possible that the understanding of fluid dynamics could lead to new insights into gravitational physics. For example, the fluid/gravity correspondence seems to be a natural setting for understanding the holography of turbulence [62].

A

Various AdS-Kerr black hole computations

A.1 Coordinates on S^m

We will now show how it is possible to construct coordinates on the m dimensional sphere. To this end consider S^m as embedded in $m+1$ dimensional Euclidean space $\mathbb{R}^{m+1}$ and introduce the two numbers $n \equiv [(m+2)/2]$ and $N \equiv [(m+1)/2]$ along with an evenness integer $\epsilon \equiv n - N$. Here $[x]$ denotes the integer part of the number x. All in all, if m is even i.e. $m = 2q$ then $n = q+1$, $N = q$, $\epsilon = 1$ and if m is odd i.e. $m = 2q+1$ then $n = N = q+1$ and $\epsilon = 0$. Usually in D dimensional spacetimes we consider $D-2$ dimensional spheres. In this case the numbers n and N are given by $n = [D/2]$ and $N = [(D-1)/2]$. In both cases we have $D = 2m+2 = 2N+1+\epsilon$.

Euclidean $m+1$ dimensional space has N commuting generators of rotation corresponding to N orthogonal two-dimensional planes. Of course, each of these planes have an $U(1)$ symmetry seen from $\mathbb{R}^{m+1}$ but also seen from the embedded m-sphere. Indeed, consider the embedding

$$\begin{aligned} \epsilon = 0: &\quad (\overbrace{x_1, y_1}^{U(1)}, \overbrace{x_2, y_2}^{U(1)}, \cdots, \overbrace{x_N, y_N}^{U(1)}) \in S^m \subset \mathbb{R}^{m+1} \\ \epsilon = 1: &\quad (\underbrace{x_1, y_1}_{U(1)}, \underbrace{x_2, y_2}_{U(1)}, \cdots, \underbrace{x_N, y_N}_{U(1)}, x_n) \in S^m \subset \mathbb{R}^{m+1} \end{aligned} \tag{A.1}$$

We will now exploit the $U(1)$ symmetry on each of the N two-planes span(x_i, y_i). Equip each of the planes with "polar coordinates"

$$z_i = x_i + \mathrm{i}y_i = \mu_i e^{\mathrm{i}\phi_i}, \quad i = 1, \cdots, N \tag{A.2}$$

where $0 \leq \mu_i < \infty$ and $0 \leq \phi_i < 2\pi$. In the case where $\epsilon = 1$ set $z_n \equiv x_n \equiv \mu_n$ ($n = N+1$) where $-\infty < \mu_n < \infty$. We have now equipped $\mathbf{R}^{m+1}$ with a new set of coordinates. The points on S^m are now exactly those that fulfill

$$\sum_{i=1}^{N+\epsilon} z_i z_i^* = \sum_{i=1}^{N+\epsilon} \mu_i^2 = 1 \tag{A.3}$$

In this way we have equipped S^m with $2N+\epsilon$ coordinates. These coordinates are, however, not independent due to the constraint (A.3). The constraint leaves $2N+\epsilon-1 = m$ (= the dimension of S^m) *independent* coordinates. Computing the metric on S^m in these coordinates is straight forward

$$\mathrm{d}\Omega_m^2 = \sum_{i=1}^{N+\epsilon} \mathrm{d}z_i^* \mathrm{d}z_i = \sum_{i=1}^{N+\epsilon} \mathrm{d}\mu_i^2 + \sum_{i=1}^{N} \mu_i^2 \mathrm{d}\phi_i^2 \tag{A.4}$$

where one of the μ_i's has to be expressed in terms of the rest through (A.3). This is essential if we want to e.g. integrate on the sphere. As an example let us calculate the volume of the m-sphere (that is, the volume of the space S^m, not to be confused with the volume contained in S^m as an embedding of $\mathbb{R}^{m+1}$). We choose the variables $\mu_1, \cdots, \mu_{N+\epsilon-1}$ and $\phi_1, \cdots, \phi_N$ as the coordinates on S^m while $\mu_{N+\epsilon}$ is expressed completely in terms of $\mu_1, \cdots, \mu_{N+\epsilon-1}$. We start by computing the form of the metric (A.4) in terms of the genuine independent coordinates. To this end notice that the constraint (A.3) implies

$$\mathrm{d}\mu_{N+\epsilon}^2 = \frac{1}{\mu_{N+\epsilon}^2} \left[\sum_{i=1}^{N+\epsilon-1} \mu_i \mathrm{d}\mu_i \right]^2 \tag{A.5}$$

We therefore obtain the following expression for the metric

$$\mathrm{d}\Omega_m^2 = \sum_{i,j=1}^{N+\epsilon-1} \Big(\delta_{ij} + \frac{\mu_i \mu_j}{\mu_{N+\epsilon}^2} \Big) \mathrm{d}\mu_i \mathrm{d}\mu_j + \sum_{i=1}^{N} \mu_i^2 \mathrm{d}\phi_i^2 \tag{A.6}$$

The expression for the volume of the m-sphere is then given by

$$\mathrm{Vol}(S^m) = \int_{S^m} \mathrm{d}V = \int_{S^m} \prod_{i=1}^{N+\epsilon-1} \mathrm{d}\mu_i \prod_{i=1}^{N} \mathrm{d}\phi_i \, \sqrt{\Omega} \tag{A.7}$$

where $\mathrm{d}V$ is the volume element on S^m which has been expressed in terms of the determinant Ω of the metric and the coordinate one-forms in the usual way. The determinant of the metric is given by

$$\Omega = \det(M) \prod_{i=1}^{N} \mu_i^2 \quad \text{with} \quad M = 1 + \frac{\vec{\mu} \otimes \vec{\mu}}{\mu_{N+\epsilon}^2} \tag{A.8}$$

where $\vec{\mu}$ denotes the $(N+\epsilon-1)$-component vector $\vec{\mu}_i \equiv \mu_i$. Having identified M as the identity plus a dyadic product, it is straight forward to compute $\det(M)$.[1] We have

$$\det(M) = 1 + \frac{1}{\mu_{N+\epsilon}^2} \sum_{i=1}^{N+\epsilon-1} \mu_i^2 = \frac{1}{\mu_{N+\epsilon}^2} \tag{A.9}$$

where we used the defining relation $\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1$. We therefore have

$$\sqrt{\Omega} = \frac{\prod_{i=1}^{N} \mu_i}{\mu_{N+\epsilon}} \tag{A.10}$$

Let us record the result for the volume of the m-sphere. It holds that

$$\mathrm{Vol}(S^m) = \frac{2\pi^{(m+1)/2}}{\Gamma\big((m+1)/2\big)} \tag{A.11}$$

which can be showed using (A.7) and the expression for the volume element (however, this computation is easier in standard "angular coordinates", see below).

[1] It holds that for any invertible matrix $\mathbf{A}$ and any two vectors $\vec{v}$ and $\vec{q}$ that $\det(\mathbf{A} + \vec{u} \otimes \vec{v}) = (1 + \vec{v}^{\mathrm{T}} \mathbf{A}^{-1} \vec{u}) \det(\mathbf{A})$. For a proof, see for example [63].

Finally it is possible to directly equip S^m with m independent coordinates by parameterizing the equation (A.3) in the following way

$$\begin{aligned}\mu_j &= \Big(\prod_{i=1}^{i-1}\cos\theta_i\Big)\sin\theta_j, \quad j=1,\dots,(N+\epsilon)-1 \\ \mu_{N+\epsilon} &= \prod_{i=1}^{N+\epsilon-1}\cos\theta_i\end{aligned} \tag{A.12}$$

That this indeed is a parametrization of (A.3) is straight forward to check. Notice that if m is odd then (on S^m) all the μ_i's fulfill $0\le\mu_i\le 1$ and therefore $\theta_i\in[0,\pi/2]$ for all i while if m is even then $0\le\mu_i\le 1,\ i=1,\dots,N$ and $-1\le\mu_{N+1}\le 1$ so $\theta_i\in[0,\pi/2]$ for $i=1,\dots,N$ and $\theta_{N+1}\in[0,\pi]$. A quick computation reveals that

$$\sum_{i=1}^{N+\epsilon}\mathrm{d}\mu_i^2 = \sum_{a=1}^{N+\epsilon-1}\Big(\prod_{b=1}^{a-1}\cos^2\theta_b\Big)\mathrm{d}\theta_a^2 \tag{A.13}$$

This means that the metric on the even dimensional sphere S^{2N} in the coordinates $(\theta_1,\cdots,\theta_N,\phi_1,\cdots,\phi_N)$ is given by

$$\mathrm{d}\Omega_{2q}^2 = \sum_{a=1}^{N}\Big(\prod_{b=1}^{a-1}\cos^2\theta_b\Big)\mathrm{d}\theta_a^2 + \sum_{a=1}^{N}\Big(\prod_{b=1}^{a-1}\cos^2\theta_b\Big)\sin^2\theta_a\mathrm{d}\phi_a^2 \tag{A.14}$$

In order to obtain the metric on the odd dimensional sphere S^{2N-1}, we simply set the last coordinate constant to $\theta_N=\pi/2$.

A.2 Pure AdS metric in spheroidal form

In this subappendix we derive the expression for the pure anti de-Sitter metric expressed in the spheroidal Kerr-Schild coordinates introduced in §6.1.2. In ordinary D dimensional 'spherical' coordinates anti-de Sitter takes the form

$$\mathrm{d}\hat{s}^2 = -(1-\lambda r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\lambda r^2} + r^2\left(\sum_{i=1}^{N+\epsilon}\mathrm{d}\mu_i^2 + \sum_{i=1}^{N}\mu_i^2\mathrm{d}\phi_i^2\right) \tag{A.15}$$

where the directional cosines μ_i are subject to the constraint $\sum_{i=1}^{N+\epsilon}\mu_i = 1$. We now introduce a new set of 'spheroidal' coordinates (Boyer-Lindquist coordinates in the zero mass limit $M=0$) by the following two equations

$$(1+\lambda a_i^2)r^2\mu_i^2 = (y^2+a_i^2)\nu_i^2, \quad \sum_{i=1}^{N+\epsilon}\nu_i^2 = 1 \quad \Rightarrow \quad r^2 = \sum_{i=1}^{N+\epsilon}\left\{\frac{y^2+a_i^2}{1+\lambda a_i^2}\right\}\nu_i^2 \tag{A.16}$$

We see that

$$\begin{aligned}1-\lambda r^2 &= \sum_{i=1}^{N+\epsilon}\left(1-\lambda\left\{\frac{y^2+a_i^2}{1+\lambda a_i^2}\right\}\right)\nu_i^2 \\ &= W(1-\lambda y^2)\end{aligned} \tag{A.17}$$

where we defined W by the equation

$$W = \sum_{i=1}^{N+\epsilon}\frac{\nu_i^2}{1+\lambda a_i^2} \tag{A.18}$$

Now using the relations $\sum_{i=1}^{N+\epsilon} \mu_i^2 = \sum_{i=1}^{N+\epsilon} \nu_i^2 = 1$, $\sum_{i=1}^{N+\epsilon} \mu_i \mathrm{d}\mu_i = \sum_{i=1}^{N+\epsilon} \nu_i \mathrm{d}\nu_i = 0$, it is straight forward to show

$$r^2 \mathrm{d}r^2 = \left[\sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i \mathrm{d}\nu_i\right]^2 + W^2 y^2 \mathrm{d}y^2 + 2W \left(\sum_{i=1}^{N+\epsilon} \frac{(y^2 + a_i^2)\nu_i \mathrm{d}\nu_i}{1 + \lambda a_i^2}\right) y \mathrm{d}y \quad \text{(A.19)}$$

and

$$r^2 \sum_{i=1}^{N+\epsilon} \mathrm{d}\mu_i^2 = -\mathrm{d}r^2 + \left(\sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{(y^2 + a_i^2)(1 + \lambda a_i^2)}\right) y^2 \mathrm{d}y^2 + \sum_{i=1}^{N+\epsilon} \left(\frac{y^2 + a_i^2}{1 + \lambda a_i^2}\right) \mathrm{d}\nu_i^2 + 2 \left(\sum_{i=1}^{N+\epsilon} \frac{\nu_i \mathrm{d}\nu_i}{1 - \lambda a_i^2}\right) y \mathrm{d}y \quad \text{(A.20)}$$

Using these two expressions and the form of the metric in the original coordinates, we obtain

$$\begin{aligned} \mathrm{d}\hat{s}^2 = &-W(1 - \lambda y^2)\mathrm{d}t^2 + \frac{\lambda}{W(1 - \lambda y^2)} \left[\sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i \mathrm{d}\nu_i\right]^2 \\ &+ \frac{y^2}{1 - \lambda y^2} \left\{\lambda W + (1 - \lambda y^2) \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{(y^2 + a_i^2)(1 + \lambda a_i^2)}\right\} \mathrm{d}y^2 \\ &+ \frac{2y}{1 - \lambda y^2} \left\{\sum_{i=1}^{N+\epsilon} \left(\frac{\lambda(y^2 + a_i^2)\nu_i \mathrm{d}\mu_i}{1 + \lambda a_i^2} + (1 - \lambda y^2) \frac{\nu_i \mathrm{d}\nu_i}{1 + \lambda a_i^2}\right)\right\} \mathrm{d}y \\ &+ \sum_{i=1}^{N+\epsilon} \left(\frac{y^2 + a_i^2}{1 + \lambda a_i^2}\right) \mathrm{d}\nu_i + \sum_{i=1}^{N} \left(\frac{y^2 + a_i^2}{1 + \lambda a_i^2}\right) \nu_i^2 \mathrm{d}\phi_i^2 \end{aligned} \quad \text{(A.21)}$$

The first curly bracket is evaluated to

$$\lambda W + (1 - \lambda y^2) \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{(y^2 + a_i^2)(1 + \lambda a_i^2)} = \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{y^2 + a_i^2} \quad \text{(A.22)}$$

while the second curly bracket vanishes since $\sum_{i=1}^{N+\epsilon} \nu_i \mathrm{d}\nu_i = 0$. We therefore obtain

$$\begin{aligned} \mathrm{d}\hat{s}^2 = &-W(1 - \lambda y^2)\mathrm{d}t^2 + F\mathrm{d}y^2 + \sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \mathrm{d}\nu_i^2 + \sum_{i=1}^{N} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i^2 \mathrm{d}\phi_i^2 \\ &+ \frac{\lambda}{W(1 - \lambda y^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{y^2 + a_i^2}{1 + \lambda a_i^2} \nu_i \mathrm{d}\nu_i\right)^2 \end{aligned} \quad \text{(A.23)}$$

where F is given by

$$F \equiv \frac{y^2}{1 - \lambda y^2} \sum_{i=1}^{N+\epsilon} \frac{\nu_i^2}{y^2 + a_i^2} \quad \text{(A.24)}$$

A.3 The asymptotic form of the Kerr metric

Here we derive the asymptotic form of the Kerr metric (6.23) (relevant for computing the boundary stress tensor) in the coordinates (t, r, μ_i, ϕ_j), so that the metric takes the form (6.22). To this end consider the Kerr-Schild form of the metric (6.17) and simply perform the transformation $(y, \nu_i) \to (r, \mu_i)$. We know that $\mathrm{d}\hat{s}^2$ takes the desired form under such a coordinate transformation. We see that near the boundary

$$y^2 = \gamma^{-2} r^2 \big(1 + \mathcal{O}(r^{-2})\big) \ , \ \gamma^{-2} = 1 - \sum_{i=1}^{N} a_i^2 \mu_i^2 \tag{A.25}$$

So all we now have to determine is the asymptotic form of the functions V, F and W in the coordinates (t, r, μ_i, ϕ_j). We find

$$\begin{aligned} V &= y^{2N+\varepsilon}(1 + \mathcal{O}(1/y^2) = \gamma^{-(D-1)} r^{D-1} \left(1 + \mathcal{O}(1/r^2)\right) \\ F &= y^{-2}(1 + \mathcal{O}(1/y^2)) = \gamma^2 r^{-2} \left(1 + \mathcal{O}(1/r^2)\right) \\ W &= \gamma^2 \left(1 + \mathcal{O}(1/r^2)\right) \end{aligned} \tag{A.26}$$

Using these relations it is straight forward to show that the metric takes the form (6.23). Here it is important to note that we only retain terms that are subleading up to order $\mathcal{O}(1/r^{D-1})$ *compared* to the pure AdS metric.

A.4 The Christoffel symbols of the asymptotic Kerr metric

Here we compute the (relevant) Christoffel symbols of the asymptotic metric (6.23). As we have explained, the relevant Christoffel symbols are those of the type $\Gamma^{\mu}_{\nu r}$. In general the Christoffel symbols are given by the usual expression

$$\Gamma^A_{BC} = \frac{1}{2} G^{AD} \big\{ \partial_B G_{CD} + \partial_C G_{BD} - \partial_D G_{BC} \big\} \tag{A.27}$$

here G_{AB} denotes the Kerr metric in the (t, r, μ_i, ϕ_j) coordinates. Near the boundary we write

$$G_{AB} = \hat{G}_{AB} + \delta G_{AB} \tag{A.28}$$

where $\hat{G}_{AB}$ is pure AdS metric (6.4) and δG_{AB} is the small perturbation due to the rotating black hole. From (6.23) we have

$$\delta G_{tt} = \frac{2M}{r^{d-2}} \gamma^{d+2} \qquad \delta G_{rr} = \frac{2M}{r^{d+2}} \gamma^{d} \tag{A.29}$$

$$\delta G_{t\phi_i} = -\frac{2M a_i \mu_i^2}{r^{d-2}} \gamma^{d+2} \qquad \delta G_{\phi_i \phi_j} = \frac{2M a_i a_j \mu_i^2 \mu_j^2}{r^{d-2}} \gamma^{d+2} \tag{A.30}$$

Now to leading order in $1/r$, the inverse metric is given by

$$\tilde{G}^{AB} = G^{AB} - \delta G^{AB} \tag{A.31}$$

Since the unperturbed metric G_{AB} is diagonal, the non-vanishing components of δG^{AB} are exactly the same as those of δG_{AB}. To leading order we find

$$\delta G^{tt} = \frac{2M}{(1 + r^2) r^d} \gamma^{d+2} \qquad \delta G^{rr} = \frac{2M(1 + r^2)}{r^d} \gamma^d \tag{A.32}$$

$$\delta G^{t\phi_i} = \frac{2M a_i}{r^{d+2}} \gamma^{d+2} \qquad \delta G^{\phi_i \phi_j} = \frac{2M a_i a_j}{\tilde{r}^{d+2}} \gamma^{d+2} \tag{A.33}$$

We can now compute the Christoffel symbols using (A.27). We find (retaining only terms that are at most of order $\mathcal{O}(1/r^{d+1})$ as higher order terms will be irrelevant for computing the boundary stress tensor)

$$
\begin{aligned}
\Gamma^t_{tr} &= \frac{1}{2}G^{tt}\partial_r G_{tt} + \frac{1}{2}G^{tt}\partial_r \delta G_{tt} - \frac{1}{2}\delta G^{tt}\partial_r G_{tt} \\
&= \frac{r}{1+r^2}\left[1 + \frac{dM}{r^d}\gamma^{d+2}\right] \\
\Gamma^t_{\phi_i r} &= \frac{1}{2}\tilde{g}^{tt}\partial_r \delta G_{\phi_i t} - \frac{1}{2}\delta G^{t\phi_i}\partial_{\tilde{r}} G_{\phi_i\phi_i} \\
&= -\frac{dMa_i\mu_i^2}{r^{d+1}}\gamma^{d+2} \\
\Gamma^{\phi_i}_{tr} &= \frac{1}{2}G^{\phi_i\phi_i}\partial_r \delta G_{t\phi_i} - \frac{1}{2}\delta G^{\phi_i t}\partial_r G_{tt} \\
&= \frac{dMa_i}{r^{d+1}}\gamma^{d+2} \\
\Gamma^{\phi_i}_{\phi_j r} &= \frac{1}{2}G^{\phi_i\phi_j}\partial_r G_{\phi_i\phi_j} + \frac{1}{2}G^{\phi_i\phi_i}\partial_r \delta G_{\phi_i\phi_j} - \frac{1}{2}\delta G^{\phi_i\phi_j}\partial_r G_{\phi_j\phi_j} \\
&= \frac{1}{r}\left[\delta_{ij} - \frac{dMa_ia_j\mu_j^2}{r^d}\gamma^{d+2}\right] \\
\Gamma^{\theta_i}_{\theta_j r} &= \frac{1}{2}\delta_{ij}G^{\theta_i\theta_i}\partial_r G_{\theta_i\theta_i} \\
&= \frac{\delta_{ij}}{\tilde{r}}
\end{aligned}
\tag{A.34}
$$

The Christoffel symbols Γ^t_{tr}, $\Gamma^{\phi_i}_{\phi_j r}$, and $\Gamma^{\theta_i}_{\theta_i r}$ contain terms which are non-zero even when $M = 0$ (pure AdS), these terms are important when computing the boundary stress tensor.

A.5 The Kerr angular momentum

We will now look at the different, equivalent, expressions for the Komar charge associated with some symmetry $\mathcal{J}$. To this end let J^A be the Killing vector corresponding to the symmetry $\mathcal{J}$ and let J denote the associated Killing 1-form (i.e, $J_A = G_{AB}J^B$). The Komar charge associated with the symmetry $\mathcal{J}$ is then given by

$$Q[\mathcal{J}] = \int_{\partial M} *\nabla J \tag{A.35}$$

where $*$ denotes the Hodge star operator and ∇J denotes the 2-form $\nabla_A J_B = \partial_{[A}J_{B]}$. Here ∂M is a timelike surface at spatial infinity of co-dimension 2 (usually taken to be S^{D-2}). In order to cast this expression into the form of [13], simply recall that the volume element ϵ is given by

$$\epsilon = \sqrt{g}\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \tag{A.36}$$

or in components $\epsilon_{A_1\cdots A_D} = \sqrt{g}\ \tilde{\epsilon}_{A_1\cdots A_D}$, where $\tilde{\epsilon}_{A_1\cdots A_D}$ is the totally antisymmetric *symbol*. In abstract index notation (as in [13]) we then have $*(\nabla J)_{A_1\cdots A_{D-2}} = \epsilon_{A_1\cdots A_{D-2}BC}\nabla^B J^C$. The Komar charge can therefore be written as

$$Q[\mathcal{J}] = \int_{\partial M} \epsilon_{A_1\cdots A_{D-2}BC}\nabla^B J^C \tag{A.37}$$

which is obviously the expression from [13] generalized from 4 to D dimensions. In order to cast this expression into the pure component/coordinate notation as in e.g.

[14], recall that the volume element of the surface ∂M, $\hat{\epsilon}$, can be written in terms of ϵ by

$$\hat{\epsilon}_{A_1 \ldots A_{D-2}} = n^B \sigma^C \epsilon_{BCA_1 \ldots A_{D-2}} \tag{A.38}$$

where n^A and σ^A are two (respectively future and outward pointing) unit normal vectors to ∂M. When restricted to ∂M, $*\nabla J$ is a differential form of full dimensionality since $\dim(\partial M) = D-2$. This implies that on ∂M we have a relation $*\nabla J = f\hat{\epsilon}$, where f is some scalar function. Therefore

$$* \nabla J = f\hat{\epsilon} = *f \tag{A.39}$$

It is important to realize that the $*$ on the LHS of this equation denotes the Hodge star operator on M, where the resulting $(D-2)$-form is restricted to ∂M, while the $*$ on the RHS denotes the Hodge star operator on the $(D-2)$-dimensional manifold ∂M. Now applying the Hodge star operator (on ∂M) to this equation, we obtain

$$\begin{aligned} f &= (-1)^s * *f \\ &= (-1)^s * (*\nabla J) \\ &= (-1)^s \frac{1}{(D-2)!} \hat{\epsilon}^{A_1 \cdots A_{D-2}} \left(\frac{1}{2!} \epsilon_{A_1 \cdots A_{D-2} BC} \nabla^B J^C \right) \\ &= (-1)^s \frac{1}{(D-2)!} n_E \sigma_F \epsilon^{EFA_1 \cdots A_{D-2}} \left(\frac{1}{2!} \epsilon_{A_1 \cdots A_{D-2} BC} \nabla^B J^C \right) \\ &= (-1)^{2s} (D-2)! 2! \frac{1}{(D-2)!2!} n_E \sigma_F \delta_B^{[E} \delta_C^{F]} \nabla^B J^C \\ &= n^A \sigma^B \nabla_A J_B \end{aligned} \tag{A.40}$$

The Komar charge can therefore be written

$$\begin{aligned} Q[\mathcal{J}] = \int_{\partial M} \hat{\epsilon}\, n^A \sigma^B \nabla_A J_B &= \int_{\partial M} \mathrm{d}^{D-2} y \sqrt{\gamma}\, n^A \sigma^B \nabla_A J_B \\ &= \int_{\partial M} \mathrm{d}^{D-2} y \sqrt{\gamma}\, n^A \sigma^B \left[\partial_A J_B - \Gamma^C_{AB} J_C \right] \end{aligned} \tag{A.41}$$

where y denotes coordinates on ∂M and γ is the induced metric.

Evaluation of the angular momentum: The Killing vector associated with the angular momentum (in the ϕ_i'th direction) is ∂_{ϕ_i}. In pure AdS$_D$, the corresponding Komar integral is therefore given by $L_i \sim \int_{S^{D-2}} n^t \sigma^r \Gamma^{\phi_i}_{rt} V_{\phi_i}$. Since the Christoffel symbol $\Gamma^{\phi_i}_{rt}$ vanishes in pure AdS$_D$, the corresponding angular momentum is zero, just as it should. Now suppose that we consider a rotating Kerr black hole spacetime (with non-vanishing M parameter) in the ordinary spherical coordinates $(t, r, \{\mu_i\}, \{\phi_i\})$. The asymptotic metric now takes the form (6.23). The presence of a rotating mass introduces a small perturbation of the Christoffel symbol $\Gamma^{\phi_i}_{rt}$ which we found in (A.34). It is easy to convince one self that only the leading order r behavior of $\sqrt{\gamma} n^t \sigma^r \Gamma^{\phi_i}_{\tilde{r}t} V_{\phi_i}$ will be relevant for computing the Komar integral, since it is evaluated on a $(D-2)$-sphere which is located at $r = \infty$. This gives

$$\begin{aligned} L_i &\sim \int_{S^{D-2}} \sqrt{\Omega} \prod \mathrm{d}\mu_j \prod \mathrm{d}\phi_k r^D g_{\phi_i \phi_i} \Gamma^{\phi_i}_{rt} \\ &\sim a_i \int_{S^{d-1}} \sqrt{\Omega} \prod \mathrm{d}\mu_j \prod \mathrm{d}\phi_k \mu_i^2 \gamma^{d+2} \end{aligned} \tag{A.42}$$

Here $\sqrt{\Omega}$ is the measure of S^{D-2} in the coordinates $(\{\mu_i\}, \{\phi_i\})$ introduced in §A.1. We see that the integral is exactly of the type considered in §6.3.4 and the result (6.40) follows.

B

Various black brane computations

B.1 The black brane temperature

Here we compute the temperature of the black brane (5.2) using the usual Euclidean approach [52]. Consider a QFT in thermal equilibrium living on the spacetime (5.2). As usual the thermal properties of a field theory is examined by consider the corresponding Euclidean field theory with the time dimension compactified $\tau \sim \tau + \beta$. The period β is exactly identified with the inverse temperature of the field theory. We therefore perform a Wick rotation $t \to i\tau$ of (5.2). This changes the signature of the metric to Euclidean and gives

$$G^E_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = \frac{r^2}{L^2}\left[f(br)\mathrm{d}\tau^2 + \sum_{i=1}^{d-1}(\mathrm{d}x_i)^2\right] + \frac{L^2}{r^2 f(br)}\mathrm{d}r^2 \tag{B.1}$$

Clearly this metric is singular at $r = r_+$ (again a coordinate singularity). Moreover we see that it only makes sense to have a thermal theory for $r \geq r_+$, since inside the horizon the signature will not be Euclidean anymore. In order to examine the behavior of the metric near that horizon we introduce a variable ρ by the equation

$$r - r_+ = \frac{dr_+}{4L^2}\rho^2 \tag{B.2}$$

It is straight forward to show that, near the horizon, the metric takes the form

$$G^E_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\left[\frac{dr_+\tau}{2L^2}\right]^2 + \frac{r_+^2}{L^2}\sum_{i=1}^{d-1}(\mathrm{d}x_i)^2 \tag{B.3}$$

We will now require that the metric is free of a singularity at $\rho = 0$ and that the metric can not be continued inside $\rho < 0$. The geometry in the (τ, ρ) directions is recognized as that of an ordinary cone. The above metric is only free of a conical singularity at $\rho = 0$ if we make the periodic identification (the cone simply reduces to $\mathbb{E}^2$ - the ordinary two dimensional plane of Euclidean signature, see fig. B.1):

$$\frac{dr_+\tau}{2L^2} \sim \frac{dr_+\tau}{2L^2} + 2\pi \tag{B.4}$$

Moreover, we see that by construction, this metric cannot be continued for $\rho < 0$. This means that a consistent thermal field theory living on the metric (5.2) requires

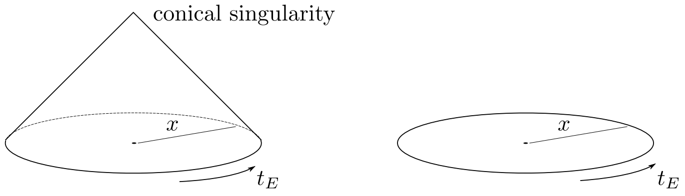

Figure B.1: *The conical singularity is only avoided by choosing the period β for τ.*

that the period of the Euclidean time coordinate must be chosen as $\beta = \frac{4\pi L^2}{dr_+}$. It follows that the temperature of the thermal field theory is given by

$$\mathcal{T} = \frac{dr_+}{4\pi L^2} = \frac{d}{4\pi L^2 b} \tag{B.5}$$

B.2 Dual stress tensor to the D-dimensional black brane

In this section we compute the stress tensor dual to the metric (5.40) up to first order in the derivatives.

B.2.1 The 0th order stress tensor

Here we compute the dual stress tensor of the boosted black brane in an arbitrary number of spacetime dimensions. The $D = d + 1$ dimensional boosted black brane is given by

$$G_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = r^2\big[-f(br)u_\mu u_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu + \Delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\big] + \frac{1}{r^2 f(br)}\mathrm{d}r^2 \tag{B.6}$$

with $f(r) = 1 - 1/r^d$ and $\Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$. The method for obtaining the boundary stress tensor dual to a generic bulk metric is explained in §3.4. The first step in this construction consists of selecting a set of coordinates in which the metric takes the appropriate asymptotic form. However, the coordinates (r, x^μ) already meet this requirement. Indeed, in a neighborhood of the boundary $r = \infty$ the metric (B.6) takes that form

$$G_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = \frac{1}{r^2}\mathrm{d}r^2 + r^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathcal{O}(1/r^2) \tag{B.7}$$

Having made sure that we are working in the proper set of coordinates, we can now compute the boundary stress tensor from the usual formula (cf. equation (3.43) and the discussion below)

$$T^\mu_\nu = \lim_{r\to\infty}\frac{\Lambda_r^d}{8\pi G_{d+1}}S^\mu_\nu, \quad S^\mu_\nu = \Theta^\mu_\nu - \delta^\mu_\nu\Theta \tag{B.8}$$

where Λ_r denotes the surface of constant r, Θ^μ_ν is the extrinsic curvature of the surface Λ_r and as usual we use Greek letters $\mu, \nu, ...$ to denote the boundary (field theory) directions. We may now proceed to compute the extrinsic curvature of Λ_r. The extrinsic curvature is given by

$$\Theta^\mu_\nu = -\nabla_\nu n^\mu \tag{B.9}$$

where n^μ is the unit normal vector to Λ_r. Clearly we have $n^\mu = (n^r, 0, 0, ...)$ with $n^r = r\sqrt{f(rb)}$. The extrinsic curvature can now be computed. After some computation we find

$$\Theta^\mu_\nu = K\left(\frac{d}{2}u^\mu u_\nu - \left((br)^d - 1\right)\delta^\mu_\nu\right) \tag{B.10}$$

where the factor K only depends on the radial coordinate r and is given by

$$K = \frac{1}{(br)^d\sqrt{1-1/(br)^d}} \to \frac{1}{(br)^d}\left(1 + \frac{1}{2}\frac{1}{(br)^d} + \mathcal{O}\left((br)^{-2d}\right)\right) \tag{B.11}$$

where we also recorded the large r asymptotic form of K. The trace of the extrinsic curvature is easily computed using $u_\mu u^\mu = -1$. Contracting the indices gives

$$\Theta = -\frac{dK}{2}(2(br)^d - 1) \tag{B.12}$$

This gives the following expression for the combination S^μ_ν

$$S^\mu_\nu = K\left(\frac{d}{2}u^\mu u_\nu + \left[(d-1)(br)^d + \frac{2-d}{2}\right]\delta^\mu_\nu\right) \tag{B.13}$$

We can now compute the boundary stress tensor through equation (B.8). Notice that S^μ_ν contains terms (independent of u^μ and b) that will lead to divergences in the stress tensor when we take the limit $\Lambda_r \to \infty$. These terms are removed through the holographic renormalization procedure and we will therefore simply ignore such divergent terms, again we refer to the discussion below (3.43) for the details. Normalizing S^μ_ν appropriately, taking the large r limit, and raising the indices with the boundary metric $\eta^{\mu\nu}$, we finally find

$$T^{\mu\nu} = \frac{1}{16\pi G_{d+1}}\left(\frac{4\pi\mathcal{T}}{d}\right)^d \left[\eta^{\mu\nu} + d u^\mu u^\nu\right] \tag{B.14}$$

where we used the relation $4\pi\mathcal{T}b = d$. Comparing this to the expression (4.29) we see that this is exactly the stress tensor of the a perfect conformal fluid in $d = D-1$ dimensions.

B.2.2 The 1th order stress tensor

We now compute the first order stress tensor dual to the metric (5.40). We can now straight forwardly proceed as above and carry out this computation for general boost velocity u^μ, however, as explained in §5.3 we are free to set $u^\mu = \delta^\mu_0$ and $b = 1$ in any given point. This freedom simplifies the computations significantly. We will therefore compute the boundary stress tensor of the metric

$$\begin{aligned} G_{AB}\,\mathrm{d}x^A\mathrm{d}x^B &= -2u_\mu \mathrm{d}x^\mu \mathrm{d}r - r^2 f(br) u_\mu u_\nu \mathrm{d}x^\nu \mathrm{d}x^\mu + \Delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &\quad + \varepsilon\left(r\mathfrak{G}_v(r)\,\mathrm{d}v^2 + 2r\mathfrak{G}_i(r)\,\mathrm{d}v\mathrm{d}x^i + 2r^2F(r)\mathfrak{S}_{ij}(r)\,\mathrm{d}x^i\mathrm{d}x^j\right) \end{aligned} \tag{B.15}$$

with $u^\mu = \delta^\mu_0$ and $b = 1$ in $x^\mu = 0$ and

$$\begin{aligned} \mathfrak{G}_v(r) &= \gamma r + \frac{2}{d-1}\partial_i\beta_i + \frac{\pi_v}{r^{d-1}} \\ \mathfrak{G}_i(r) &= \gamma_i r + \partial_v\beta_i + \frac{\pi_i}{r^{d-1}} \\ \mathfrak{S}_{ij}(r) &= (1 + \gamma_{ij}/F(r))\sigma_{ij} \quad \text{(no sum over } i \text{ and } j) \end{aligned} \tag{B.16}$$

The above metric solves Einstein's equation in a neighborhood of $x^\mu = 0$ for *all* r. Here the term multiplying γ_{ij} is included since the function F was in principle only defined up to a constant for each γ_{ij}. We start by arguing that the γ- and π-constants must all be chosen to zero in order for the boundary and normalization conditions to be fulfilled.

First, we see that the γ terms multiply modes that go like r^2. By comparing these to the original metric (the zeroth order term from the Taylor expansion of the first line of (B.15)) we see that in order for the metric (B.15) to asymptote AdS_{d+1}, all the γ factors must be chosen to zero. Moreover non-vanishing γ terms give rise to uncontrollable infinite terms in the boundary stress tensor. All the γ terms must therefore be set to zero in order for the metric (B.15) to fit into the AdS/CFT scheme.

Now computing the extrinsic curvature is easy. First of all, it is not hard to convince oneself that the above method still works in our Gaussian normal coordinates, simply because they only involve a coordinate transformation of the boundary coordinates. For this computation it is enough to use the asymptotic form of the function F which was recorded in the expression (5.36). Both the normal vector n^μ and the Christoffel symbols get a small ε-correction. The extrinsic curvature is therefore modified according to

$$\Theta^\mu_\nu = (\Theta^{(0)})^\mu_\nu + \varepsilon(\Theta^{(1)})^\mu_\nu + \mathcal{O}(\varepsilon^2) \tag{B.17}$$

where $(\Theta^{(0)})^\mu_\nu$ is the extrinsic curvature from above, here evaluated for the boost velocity $u^\mu = \delta^\mu_0$. Having computed the extrinsic curvature we may now compute the tensor S^μ_ν as above. If we let $(S^{(0)})^\mu_\nu$ denote the large r zeroth order term of the combination S^μ_ν, we find for large r

$$\begin{aligned} S^v_v &= (S^{(0)})^v_v - \frac{d-1}{2}\varepsilon\pi_v + \mathcal{O}(1/r^d) + \mathcal{O}(\varepsilon^2) \\ S^v_i &= (S^{(0)})^v_i - \frac{d}{2}\varepsilon\pi_i + \mathcal{O}(1/r) + \mathcal{O}(\varepsilon^2) \\ S^i_j &= (S^{(0)})^i_j + \frac{1}{2}\varepsilon\pi_0\delta^i_j - \varepsilon\sigma_{ij} + \mathcal{O}(1/r) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{B.18}$$

The stress tensor is now given by the usual expression

$$T^{\mu\nu} = \frac{1}{8\pi G_{d+1}} \lim_{r\to\infty} S^{\mu\nu} = T^{\mu\nu}_{(0)} + \varepsilon\Pi^{\mu\nu}_{(1)} + \mathcal{O}(\varepsilon^2) \tag{B.19}$$

where $T^{(0)}_{\mu\nu}$ is the perfect fluid stress tensor (B.14). We therefore see that the Landau frame condition $u^\nu\Pi_{\mu\nu} = 0$ (which here evaluates to $\Pi_{v\mu} = 0$) forces all the π_μ parameters to zero. Taking the large r limit and raising the indices with the boundary metric, we finally find the following simple expression for the first order correction for the boundary metric

$$\begin{aligned} \Pi^{v\mu}_{(1)} &= 0 \\ \Pi^{ij}_{(1)} &= -\frac{2}{16\pi G_{d+1}}\sigma^{ij} \end{aligned} \tag{B.20}$$

We now re-introduce the length dimension. On dimensional grounds the correction $\Pi^{ij}_{(1)}$ is therefore modified to $\Pi^{ij}_{(1)} = -\frac{2b^{1-d}}{16\pi G_{d+1}}\sigma^{ij}$ (setting $L \equiv 1$ renders G_{d+1} a dimensionless coupling constant in the boundary theory). Moreover, finding the covariant version of the above equations is easy. We must simply have

$$\Pi^{\mu\nu}_{(1)} = -\frac{2}{16\pi G_{d+1}}\left(\frac{4\pi\mathcal{T}}{d}\right)^{d-1}\sigma^{\mu\nu} \tag{B.21}$$

This concludes our first order computation of the boundary stress tensor of the fluctuating black brane.

B.3 The first order expression for $\mathcal{E}_{AB}$

Here we record the raw first order expressions for the generalized Einstein tensor $\mathcal{E}_{AB}$ evaluated on the metric $G^{(0)} + \varepsilon G^{(1)}$. All the expressions are valid in a neighborhood of $x^\mu = (0, \mathbf{0})$. Especially all the derivatives are evaluated in the point p chosen to be in the origin of $\mathbb{R}^{(1,3)}$. As explained in §5.4, most of the equations below can be re-written as covariant expressions.

B.3.1 Scalar sector

$$
\begin{aligned}
\mathcal{E}_{rr} &= -\frac{d-1}{2r}\left[(d-1)\frac{\mathrm{d}h^{(1)}}{\mathrm{d}r} + r\frac{\mathrm{d}^2h^{(1)}}{\mathrm{d}r^2}\right]\varepsilon + \mathcal{O}(\varepsilon^2)\\
\mathcal{E}_{vr} &= \frac{d-1}{2r^{d-1}}\Bigg[d(d-1)r^{d-1}h^{(1)} - d\Big(\frac{3}{2} + 2r^d\Big)\frac{\mathrm{d}h^{(1)}}{\mathrm{d}r}\\
&\qquad - r^{d-1}\frac{\mathrm{d}^2h^{(1)}}{\mathrm{d}r^2} - \frac{\mathrm{d}k^{(1)}}{\mathrm{d}r} + 2\sum_{i=1}^{d-1}\partial_i\beta_i^{(0)}\Bigg]\varepsilon + \mathcal{O}(\varepsilon^2)\\
\mathcal{E}_{vv} &= -\frac{d-1}{2r^{2d-3}}\Bigg[d(d-1)r^{d-2}(r^{d-1}-1)h^{(1)}\\
&\qquad + \Big(2dr^{2d} - \frac{7d}{2}r^d + \frac{3d}{2}\Big)\frac{\mathrm{d}h^{(1)}}{\mathrm{d}r} + r(1-2r^d+r^{2d})\frac{\mathrm{d}^2h^{(1)}}{\mathrm{d}r^2} + (1-r^d)\frac{\mathrm{d}k^{(1)}}{\mathrm{d}r}\\
&\qquad + \Big(2r^d - \frac{3d-2}{d-1}r^{d-2}\Big)\sum_{i=1}^{d-1}\partial_i\beta_i^{(0)} + dr^{d-2}\partial_v b^{(0)}\Bigg]\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{B.22}
$$

B.3.2 Vector sector

$$
\begin{aligned}
\mathcal{E}_{ri} &= -\frac{1}{2r^{d-1}}\left[r^{d-1}(d-1)\partial_v\beta_i^{(0)} - (d-1)\frac{\mathrm{d}j_i^{(1)}}{\mathrm{d}r} + r\frac{\mathrm{d}^2j_i^{(1)}}{\mathrm{d}r^2}\right]\varepsilon + \mathcal{O}(\varepsilon^2)\\
\mathcal{E}_{vi} &= -\frac{1}{2r^{2d-3}}\Bigg[r^{d-2}\big((d-1)r^d+1\big)\partial_v\partial_i - dr^{d-2}\partial_i b^{(0)}\\
&\qquad + (d-2)(1-r^d)\frac{\mathrm{d}j_i^{(1)}}{\mathrm{d}r} + r(r^d-1)\frac{\mathrm{d}^2j_i^{(1)}}{\mathrm{d}r^2}\Bigg]\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{B.23}
$$

B.3.3 Tensor sector

$$
\begin{aligned}
\mathcal{E}_{ii} - \frac{1}{d-1}\mathrm{Tr}\mathcal{E}_{ij} &= -\frac{1}{2r^{d-3}}\Bigg[2(d-2)r^{d-2}\partial_i\beta_i^{(0)} - 2r^{d-2}\sum_{j\neq i}\partial_j\beta_j^{(0)}\\
&\qquad \big((d+1)r^d-1\big)\frac{\mathrm{d}\alpha_{ii}^{(1)}}{\mathrm{d}r} + r(r^d-1)\frac{\mathrm{d}^2\alpha_{ii}^{(1)}}{\mathrm{d}r}\Bigg]\varepsilon + \mathcal{O}(\varepsilon^2)\\
\mathcal{E}_{ij} &= -\frac{1}{2r^{d-3}}\Bigg[(d-1)r^{d-2}(\partial_i\beta_i^{(0)} + \partial_j\beta_i^{(0)})\\
&\qquad + \big((d+1)r^d-1\big)\frac{\mathrm{d}\alpha_{ij}^{(1)}}{\mathrm{d}r} + (r^{d+1}-r)\frac{\mathrm{d}^2\alpha_{ij}^{(1)}}{\mathrm{d}r}\Bigg]\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{B.24}
$$

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200`.

[2] G. Policastro, D. T. Son, and A. O. Starinets, “The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma,” *Phys. Rev. Lett.* **87** (2001) 081601, `arXiv:hep-th/0104066`.

[3] D. T. Son and A. O. Starinets, “Viscosity, Black Holes, and Quantum Field Theory,” *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95–118, `arXiv:0704.0240 [hep-th]`.

[4] P. Kovtun, D. T. Son, and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” *Physical Review Letters* **94** (2005) 111601. `doi:10.1103/PhysRevLett.94.111601`.

[5] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, “Large rotating ads black holes from fluid mechanics,” *JHEP* **0809** (2008) 054. `doi:10.1088/1126-6708/2008/09/054`.

[6] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, “Nonlinear fluid dynamics from gravity,” *JHEP* **0802** (2008) 045. `doi:10.1088/1126-6708/2008/02/045`.

[7] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi, and S. R. Wadia, “Forced fluid dynamics from gravity,” *JHEP* **0902** (2009) 018. `doi:10.1088/1126-6708/2009/02/018`.

[8] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, “Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions,” *JHEP* **0812** (2008) 116. `doi:10.1088/1126-6708/2008/12/116`.

[9] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, “Fluid dynamics of r-charged black holes,” *JHEP* **0901** (2009) 055. `doi:10.1088/1126-6708/2009/01/055`.

[10] N. Banerjee *et al.*, “Hydrodynamics from charged black branes,” `arXiv:0809.2596 [hep-th]`.

[11] M. Maggiore, *A Modern Introduction to Quantum Field Theory*. Oxford Master Series in Physics. Oxford University Press, 2005.

[12] P. Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory.* Graduate Texts in Contemporary Physics. Springer, 1996.

[13] R. M. Wald, *General Relativity.* The University of Chicago Press, 1984.

[14] S. M. Carroll, *Spacetime and Geometry, An Introduction to General Relativity.* Addison-Wesley publising company, 2004.

[15] M. Blagojevic, *Gravitation & Gauge Symmetries.* Series in High Energy Physics, Cosmology and Gravitation. Taylor & Francis, 1 ed., 2001.

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `arXiv:hep-th/9905111`.

[17] H. Nastase, "Introduction to AdS-CFT," `arXiv:0712.0689 [hep-th]`.

[18] S. Kovacs, "N=4 supersymmetric Yang-Mills theory and the AdS/SCFT correspondence," `arXiv:hep-th/9908171`.

[19] J. M. Maldacena, "Lectures on AdS/CFT," `arXiv:hep-th/0309246`.

[20] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," *Int. J. Mod. Phys.* **A14** (1999) 3597–3672, `arXiv:hep-th/9902131`.

[21] E. Kiritsis, *String Theory in a Nutshell.* Princeton University Press, 2007.

[22] R. J. Szabo, "BUSSTEPP lectures on string theory: An introduction to string theory and D-brane dynamics," `arXiv:hep-th/0207142`.

[23] S. Mukhi, "Recent developments in string theory: A brief review for particle physicists," `arXiv:hep-ph/9710470`.

[24] C. V. Johnson, *D-Branes.* Cambridge University Press, 2006.

[25] G. T. Horowitz and A. Strominger, "Black strings and P-branes," *Nucl. Phys.* **B360** (1991) 197–209.

[26] J. Polchinski, "Dirichlet Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) no. 26, 4724–4727.

[27] A. A. Tseytlin, "Born-Infeld action, supersymmetry and string theory," `arXiv:hep-th/9908105`.

[28] M. Rangamani, "Gravity & hydrodynamics: Lectures on the fluid-gravity correspondence," *Class. Quant. Grav.* **26** (2009) 224003, `arXiv:0905.4352 [hep-th]`.

[29] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `arXiv:hep-th/9802150`.

[30] J. Brown and J. York, "Quasilocal energy and conserved charges derived from the gravitational action," *Phys. Rev.* **D47** (1993) 1407.

[31] K. Skenderis, "Lecture notes on holographic renormalization," *Class. Quant. Grav.* **19** (2002) 5849–5876, `arXiv:hep-th/0209067`.

[32] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, `arXiv:hep-th/9902121`.

[33] N. Andersson and G. L. Comer, "Relativistic fluid dynamics: Physics for many different scales," *LIVINGREV.REL* **1** (2005) . `http://www.citebase.org/abstract?id=oai:arXiv.org:gr-qc/0605010`.

[34] S. Weinberg, *The Quantum Theory of Fields*, vol. 1 of *The Quantum Theory of Fields*. Cambridge Univ. Press, Cambridge, 1995.

[35] D. Brown, "Action functionals for relativistic perfect fluids," *Classical and Quantum Gravity* **10** (1993) 1579. `http://www.citebase.org/abstract?id=oai:arXiv.org:gr-qc/9304026`.

[36] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, vol. 6. England Pergamon Press, 2 ed., 1987.

[37] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. Wiley, New York, NY, 1972.

[38] P. B. Arnold, G. D. Moore, and L. G. Yaffe, "Transport coefficients in high temperature gauge theories: (I) Leading-log results," *JHEP* **11** (2000) 001, `arXiv:hep-ph/0010177`.

[39] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, "Relativistic viscous hydrodynamics, conformal invariance, and holography," *JHEP* **04** (2008) 100, `arXiv:0712.2451 [hep-th]`.

[40] R. Loganayagam, "Entropy current in conformal hydrodynamics," *JHEP* **0805** (2008) 087. `doi:10.1088/1126-6708/2008/05/087`.

[41] A. Jakovac, "Viscosity to entropy density ratio: violation of the lower bound at small temperatures," `arXiv:0901.2802 [hep-th]`.

[42] S. Bhattacharyya *et al.*, "Local Fluid Dynamical Entropy from Gravity," *JHEP* **06** (2008) 055, `arXiv:0803.2526 [hep-th]`.

[43] S. W. Hawking and G. F. R. Ellis, *The Large scale structure of space-time*. Cambridge University Press, Cambridge, 1973.

[44] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," *Living Rev. Rel.* **11** (2008) 6, `arXiv:0801.3471 [hep-th]`.

[45] T. Harmark, V. Niarchos, and N. A. Obers, "Instabilities of black strings and branes," *Class. Quant. Grav.* **24** (2007) R1–R90, `arXiv:hep-th/0701022`.

[46] M. Cavaglia, "Black hole and brane production in TeV gravity: A review," *Int. J. Mod. Phys.* **A18** (2003) 1843–1882, `arXiv:hep-ph/0210296`.

[47] P. Kanti, "Black holes in theories with large extra dimensions: A Review," *Int. J. Mod. Phys.* **A19** (2004) 4899–4951, `arXiv:hep-ph/0402168`.

[48] R. P. Kerr and A. Schild, "Some algebraically degenerate solutions of einstein's gravitational field equation," *Proc. Symp. Appl. Math.* **17** (1965) 199.

[49] R. C. Myers and M. J. Perry, "Black holes in higher dimensional space-times," *Annals of Physics* **172** (1986) 304–347.

[50] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope, "The general Kerr-de Sitter metrics in all dimensions," *J. Geom. Phys.* **53** (2005) 49–73, `arXiv:hep-th/0404008`.

[51] T. Dereli and M. Gurses, “The generalised Kerr-Schild tranform in eleven-dimensional supergravity,” *Phys. Lett.* **B171** (1986) 209.

[52] P. K. Townsend, “Black holes,” `arXiv:gr-qc/9707012`.

[53] G. W. Gibbons, M. J. Perry, and C. N. Pope, “The first law of thermodynamics for Kerr - anti-de Sitter black holes,” *Class. Quant. Grav.* **22** (2005) 1503–1526, `arXiv:hep-th/0408217`.

[54] G. W. Gibbons and S. W. Hawking, “Action integrals and partition functions in quantum gravity,” *Phys. Rev. D* **15** (1977) no. 10, 2752–2756.

[55] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2** (1998) 505–532, `arXiv:hep-th/9803131`.

[56] R. Emparan, C. V. Johnson, and R. C. Myers, “Surface terms as counterterms in the AdS/CFT correspondence,” *Phys. Rev.* **D60** (1999) 104001, `arXiv:hep-th/9903238`.

[57] M. M. Caldarelli, O. J. C. Dias, R. Emparan, and D. Klemm, “Black Holes as Lumps of Fluid,” *JHEP* **04** (2009) 024, `arXiv:0811.2381 [hep-th]`.

[58] M. Torabian and H.-U. Yee, “Holographic nonlinear hydrodynamics from AdS/CFT with multiple/non-Abelian symmetries,” *JHEP* **08** (2009) 020, `arXiv:0903.4894 [hep-th]`.

[59] D. T. Son and A. O. Starinets, “Hydrodynamics of R-charged black holes,” *JHEP* **03** (2006) 052, `arXiv:hep-th/0601157`.

[60] K. Behrndt, M. Cvetic, and W. A. Sabra, “Non-extreme black holes of five dimensional N = 2 AdS supergravity,” *Nucl. Phys.* **B553** (1999) 317–332, `arXiv:hep-th/9810227`.

[61] C. P. Herzog, “Lectures on Holographic Superfluidity and Superconductivity,” *J. Phys.* **A42** (2009) 343001, `arXiv:0904.1975 [hep-th]`.

[62] V. Jejjala, D. Minic, Y. J. Ng, and C.-H. Tze, “Turbulence and Holography,” *Class. Quant. Grav.* **25** (2008) 225012, `arXiv:0806.0030 [hep-th]`.

[63] M. Brookes, “The Matrix Reference Manual,” `http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/intro.html`.