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ON NON-RELATIVISTIC FIELD THEORY AND GEOMETRY

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Master's thesis

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ABSTRACT

New developments have recently led to an revival of the interest in non-relativistic theories in theoretical physics. The modern guise of these theories has departed considerably from their historical roots and now enjoys a field theoretical formulation close to that of their relativistic cousins. A principal ingredient is the suitable geometrical framework called Newton-Cartan geometry which allows for a general covariant formulation of physics. This has led to new insights in both relativistic and non-relativistic theories, often with the exposure of surprising features of the latter. There are substantial prospects in employing these methods to diverse areas of physics.

The intention of this thesis is to investigate field theoretic and geometric aspects of non-relativistic physics separate and together. We will provide all the necessary background material to understand the modern formulation Newton-Cartan geometry and field theories and develop powerful methods to find non-relativistic theories from relativistic ones.

One intriguing feature of Newton-Cartan geometry is that there exists no connection that can be expressed just in terms of the vielbeins unlike the Levi-Civita connection of Lorentzian geometry. As a consequence it is not at all clear how to define a minimal connection and what extra gauge field that comes with it, nor what this should couple to. The main question we want to address is on the general features of how non-relativistic fields couples to the background geometry.

An answer is given by applying the well-known Noether procedure to couple a non-relativistic field to the vielbeins and connection gauge fields of the geometry at linear level. With this original work we characterize the couplings in full generality and give a field theoretic argument that identifies the minimal connection that includes a background field M_μ. We find that the vielbeins couples to energy and momentum currents like in the Lorentzian case while M_μ couples to either a conserved symmetry current or a topological current depending on the symmetries of the theory.

The analysis will elucidate the origin of M_μ and its properties which turns out to be of significant importance in field theories coupled to Newton-Cartan geometry. We shall illustrate this by studying the properties of two concrete realizations, the Schrödinger model and Galilean electrodynamics, in detail.

An expert is a person who has found out by his own painful experience all the mistakes that one can make in a very narrow field.

— Niels Bohr [1]

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ACRONYMS

EOM Equation of Motion

EEP Einstein Equivalence Principle

EM Energy-Momentum

GCT General Coordinate Transformation

GED Galilean Electrodynamics

LLT Local Lorentz Transformation

MED Maxwellian Electrodynamics

QFT Quantum Field Theory

QM Quantum Mechanics

SCT Special Conformal Transformation

SUSY Supersymmetry

NOTATION AND CONVENTIONS

We will work in natural units unless otherwise stated.

We work generally in $D = d + 1$ spacetime dimensions, with d spatial directions and 1 temporal direction. We employ the Einstein summation convention unless otherwise stated. The time component is always the zeroth component. We use a, b, c, ... for spatial indices and they range from $1, \ldots, d$, and μ, ν, ρ, ... and A, B, C, ... are used for spacetime indices ranging from $0, \ldots, d$. If there is a hat on the indices (i.e. $\hat{A}$, $\hat{a}$, $\hat{\mu}$), then this index has one extra range.

We use a "mostly positive" Lorentzian signature. Bold symbols ($\boldsymbol{A}, \boldsymbol{\tau}, \ldots$ etc.) are coordinate free expression. A bar on objects, i.e. $\overline{\boldsymbol{A}}$, means that are linearized versions of previously defined objects unless otherwise stated. The Riemann curvature tensor $R_{\mu\nu\sigma}{}^{\rho}$ and the torsion tensor $T_{\mu\nu}{}^{\lambda}$ are both taken to be antisymmetric in the first two indices. The Levi-Civita tensor $\boldsymbol{\epsilon}$ of rank D is defined with $\epsilon_{012\cdots d} = 1$.

(Anti)symmetrization is with weight 1, i.e.

$$
\begin{aligned}
A_{[ij]} &\equiv \frac{1}{2!}\left(A_{ij} - A_{ji}\right) \\
A_{(ij)} &\equiv \frac{1}{2!}\left(A_{ij} + A_{ji}\right)
\end{aligned}
$$

and et cetera for more indices.

If not stated otherwise, the following symbols are used:

id Identity mapping.

I Identity operator.

$\doteq$ Equality up to a total derivative

$\mathcal{G}$ A Lie group.

$\mathfrak{g}$ The Lie algebra of $\mathcal{G}$.

$\ltimes$ Semi-direct product (of groups).

$\boxplus$ Semi-direct sum (of algebras).

$\simeq$ Isomorphic to.

$\boldsymbol{g}$, $g_{\mu\nu}$ General Lorentzian signature metric.

$\boldsymbol{\eta}$, $\eta_{\mu\nu}$, η_{AB} Flat Minkowski metric.

1 INTRODUCTION

1.1 APPETIZER AND HISTORICAL INTRODUCTION

The non-relativistic spacetime symmetries of Newtonian mechanics have now for over a hundred years been superseded by the Poincaré symmetry of special relativity as the best available description of our universe. One may wonder how it is not completely irrelevant to consider non-relativistic spacetime symmetries given this fact? The answer is manifold. Let us immediately state some persuasive reasons why the study of non-relativistic theories is of interest today:

- Many realistic systems are approximately non-relativistic and are often described well in terms of Newtonian mechanics or old Quantum Mechanics (QM). This includes many condensed matter systems, hydrodynamics and other non-elementary particle theories.

- There is often not a very clear-cut distinction in the literature between relativistic and non-relativistic effects. Working with a symmetry group that is inherently non-relativistic makes it possible to determine the true relativistic effects.

- It is actually possible to couple non-relativistic theories to gravity in the same general covariant fashion as general relativity which generalizes the foundations of Galilean relativity. This framework is called Newton-Cartan geometry and opens up for the application of differential geometric methods for these theories.

- The holographic boundary duals of some (non-)relativistic bulk geometries can be described by theories with non-relativistic symmetries, especially certain strongly coupled condensed matter systems. This gives hope of understanding for example high-temperature superconductivity using such holographic techniques.

After the formulation of special and general relativity a lot of the research in the last century has been centred around theories that were covariantly described in these frameworks. Since the end of the 1920s, the development of relativistic Quantum Field Theory (QFT) has led to the class of theories that so far fits experimental data from the real world the best. The foremost example is the Standard Model which has been very successful in describing the phenomenology at large particle accelerators since its finalization in the 1960s [2].

To get to this point, there has been a huge development in the understanding of relativistic field theories on the Minkowski manifold and their internal symmetries. The use of differential geometrical techniques has led to a way of formulating theories that is radically different from the mechanistic paradigms of the pre-Einstein and pre-QFT era. In particular the modern development of gauge theories as connections on relativistic manifold forms a very natural language to use for the aims of theoretical physics. Their relevance for particle physics can not be underestimated. As the current generation of physicists are all educated in this philosophy, it has become a part of the standard toolbox for solving problems in many different fields of physics.

It is hard to imagine how the modern statement of non-relativistic theories could have taken place if there have not been the development of their relativistic counterparts. At some point in the history in the 1960s starting with the work of Lévy-Leblond [3], some people in the community started to take a new look at the non-relativistic symmetries armed with the toolbox of differential geometry. This has given new insight to both relativistic and non-relativistic theories. Recently research on non-relativistic holographic aspects like those found in the works of Son [4], Taylor [5], Kachru et al. [6], Balasubramanian and McGreevy [7] has paved the way for some very exciting directions for future research. Because of this Newton-Cartan geometry describing non-relativistic gravity in a diffeomorphic invariant manner has recently experienced a surge in interest. The modern formulation is much more general than its original inception from 1923 by Élie Cartan [8] as a diffeomorphic invariant formulation of Newton's law of universal gravitation. Torsionless Newton-Cartan, of which Cartan's formulation of Newtonian gravitation is a special case, has subsequent been developed by among others Friedrichs [9], Trautman [10], Ehlers [11], Duval et al. [12]. In a sense this framework is too restricted as a framework for interesting theories because the torsionless condition is very restrictive for non-relativistic geometries as we shall see in chapter 3. The realization that in non-relativistic spacetimes torsion is a more natural feature unlike its relativistic cousins is responsible for a great deal of the newly found applications. This was first noticed by Christensen et al. [13, 14] in a specific holographic setting. Because of the natural focus on relativistic theories, there are still many questions of fundamental importance in non-relativistic theories that are left unanswered. Previously there has been discrepancy between various approaches to Newton-Cartan geometry and the interpretation of it, but they have now begun to converge. Another example is the structure of (non-abelian) non-relativistic gauge theories that is currently poorly understood (however see for example [15] for recent developments).

1.2 EINSTEIN, GEOMETRY AND THE EQUIVALENCE PRINCIPLE

The biggest philosophical consequence of introducing special relativity was the unification of space and time into a single quantity known as spacetime. These fundamental quantities are separate in non-relativistic theories, as is deeply rooted in the ideas and principles of absolute space and time in Galilean relativity. The idea that time might not be absolute can at least be traced back to Voigt [16] who derived the Voigt transformation that is very similar in structure to the later by Lorentz formulated transformation [17]. Voigt did however not realize the connection to spacetime symmetries. This connection was only made when Einstein found that it was necessary to replace the underlying Galilean spacetime symmetries of Newtonian mechanics with Lorentzian symmetries and relativistic mechanics, which eventually led to special relativity [18]. Soon after Minkowski gave a geometric formulation of Einstein's theory where spacetime is a flat manifold with a Lorentzian signature metric η [19].

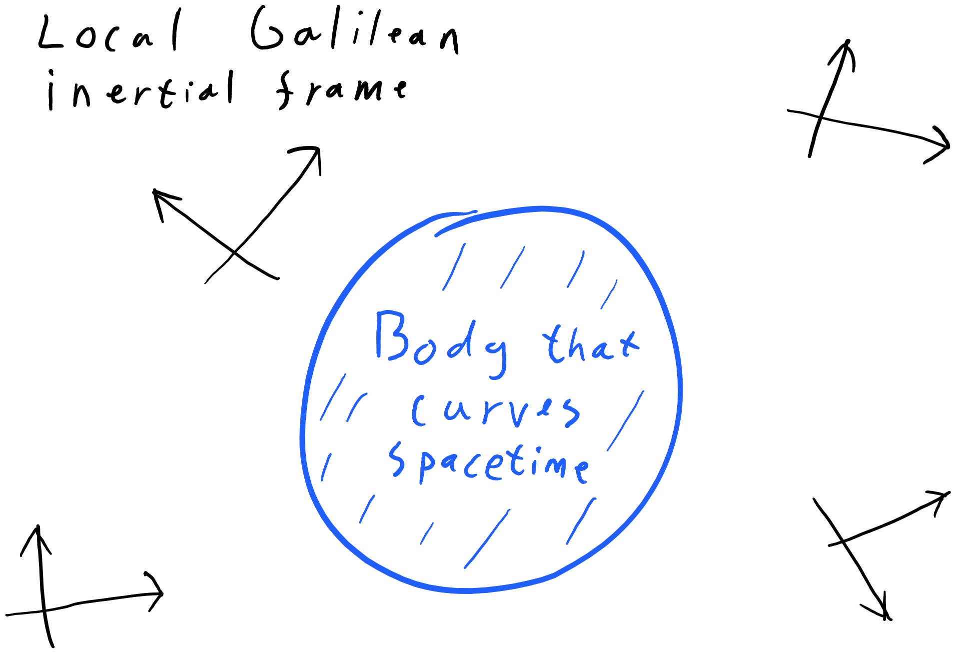

Figure 1: Illustration of the EEP applied to non-relativistic gravity: At any point in spacetime it is possible to choose a reference frame where the laws of physics obeys Newtonian mechanics locally.

A natural development from these ideas was to include gravitation which would later be known as general relativity. According to the legend, the way Einstein arrived at his field equations was by experiencing his "happiest thought" that in a nutshell states special relativity holds in small enough regions of spacetime. In reality not all the honor should be accredited to Einstein as a number of people including Hilbert was working on finding the correct formulation along the same lines. Einstein's reasoning was later formulated as the Einstein Equivalence Principle (EEP):

> The outcome of any local, non-gravitational test experiment is independent of the experimental apparatus' velocity relative to the gravitational field and is independent of where and when in the gravitational field the experiment is performed [20].

General relativity then arises as the consequence of the EEP where the laws of physics locally is taken to be special relativity. This led to the formulation of gravitation as a property of spacetime manifested through a Lorentzian metric g in the correct differential geometric framework.

A few years later in 1923 Cartan had his own "happiest thought". He essentially used the EEP, but assumed that locally the laws of Galilean relativity did hold. As described earlier, the result was a version of Newtonian gravitation in a completely differential geometric setting. This formulation did however not use a metric as the fundamental object. Gravitation did instead manifest itself through a more general affine connection, which determines the geodesics of point particles similar to in general relativity. To gain a bit intuition for the structure of such spacetimes, it is instructive to think about what one would expect from taking the non-relativistic limit of general relativity. This limit would have to "undo" the unification of space and time, which was one of the hallmarks of special and general relativities. This split-up means that effectively taking the non-relativistic limit should lead us to expect that the Lorentzian metric g of general relativity splits up into a part that "measures" time τ and a part that "measures" space h, i.e. [21]

$$g \underset{\text{NR}}{\rightarrow} (\tau, h)\ . \tag{1.1}$$

In a covariant fashion τ then is a tensor that gives the local direction of time, while h is a tensor that locally defines space. This is certainly what we are looking for. We will give the proper definitions of these objects in later chapters.

1.3 STRUCTURE OF THE THESIS

The main purpose of this thesis is to investigate various interesting aspects of non-relativistic field theories and Newton-Cartan geometry - separate and together. It can roughly be subdivided into two parts: Chapters 1 to 4 and 6 mostly review the existing literature on non-relativistic field theory and Newton-Cartan geometry while Chapters 5, 7 and 8 will contain independent work within these subjects. Parts of this will appear in [22].

We start this thesis by a thorough investigation of the relevant non-relativistic symmetry groups and their representations in chapter 2. Here we will continuously compare the Galilean group and its central extension known as the Bargmann group and discuss their relevance for non-relativistic physics. In chapter 3 we will give the proper definition of Newton-Cartan geometry taking a frame bundle approach. A major part of this chapter is concerned with the structure of connections on such spacetimes. Next we will in chapter 4 develop the relevant field representations and their properties. Again the difference between the representations of the Galilean and Bargmann group will be discussed in detail. In this chapter we will also discuss symmetry charges and currents along with the formulation of field theories on general Newton-Cartan backgrounds. We will in chapter 5 give an alternative construction of Newton-Cartan geometry at the linear level using the Noether procedure. This will give new insights to the developments of the previous chapters and we present a key result of this thesis. In chapter 6 we will review a convenient way to obtain non-relativistic theories from a reduction of the relativistic versions. Already in the following chapter 7 will we apply those results to study the Schrödinger model on both flat and curved spacetime as a concrete example. Here we will also study the various conservation equations, correlation functions and see how they shed light on the general results we have derived previously. Another example is given in the next chapter 8 where we study Galilean Electrodynamics (GED) in great detail. This will also be the chapter where we obtain the non-relativistic analog of scalar quantum electrodynamics coupled to gravity.

In parallel with the main text, we will in appendix A review the analog results for the more familiar relativistic theories and spacetimes. The general Noether theorem and the Noether procedure will be proven and discussed in appendix B. A short review of non-relativistic conformal symmetries can also be found in appendix C. Finally, we will in appendix D give details about selected calculations in the main text and in appendix E we list some useful formulas.

This is a thesis in the field of theoretical physics. Mathematical rigour is pursued when it makes sense, but we will in general refrain from giving formal proofs. The emphasis is on the development of the mathematical structure of the physical aspects. A special "feature" of this thesis is that we in the main text will give numerous examples that support and illustrate the ideas developed. All such examples are found in boxes for easy reference.

What is assumed to be known and will not be discussed further:

- Basic knowledge of (Lie) groups and algebras including structural theorems.
- Basic knowledge of representation theory and linear algebra.
- Basic knowledge of differential geometry. The structure of connections for fiber bundles will be reviewed to some extent.

2 NON-RELATIVISTIC GROUP THEORY

In this chapter we will review the relevant non-relativistic symmetry groups. The focus is on the structure of the groups, the Lie algebras and their representations. Most of our attention will be on the Galilean group and the Bargmann group, but we will also review the Euclidean and Lifshitz groups briefly.

2.1 GENERAL OVERVIEW AND THE EUCLIDEAN GROUP

Group $\mathcal{G}$	Transformations	$\dim \mathcal{G}$	Ref.
Euclidean $\mathrm{E}(d,1)$	Spatial rotations + translations $t' = t + a,\ x'^i = R^i{}_j x^j + b^i$	$\frac{1}{2}d(d+1)+1$	[23]
Lifshitz $\mathrm{Lif}(d,1)$	Euclidean + Lifshitz scaling $t' = \lambda^z t,\ x'^i = \lambda x^i$	$\frac{1}{2}d(d+1)+2$ for $d>1$	[24]
Galilean $\mathrm{Gal}(d,1)$	Euclidean + boosts $t' = t,\ x'^i = x^i + v^i t$	$\frac{1}{2}d(d+3)+1$	[25] [26]
Bargmann $\mathrm{Barg}(d,1)$	Galilean + internal $\mathrm{U}(1)$ central charge	$\frac{1}{2}d(d+3)+2$	[27] [25]
Schrödinger $\mathrm{Schr}(d,1)$	Bargmann + Lifshitz + 1 SCT $t' = \frac{t}{1-ct},\ x'^i = \frac{x^i}{1-ct}\ (z=2)$	$\frac{1}{2}d(d+3)+4$	[28] [25]
Conformal Galilean $\mathrm{CGal}(d,1)$	Galilean + dilatation + D SCTs $t' = t,\ x'^i = \frac{1}{2}a^i t^2 x^i$ (no sum)	$\frac{1}{2}d(d+5)+3$	[29] [30]

Table 1: Spacetime transformations and properties of the relevant non-relativistic symmetry groups in $D = d+1$ dimensions. See appendix C for a review of the non-relativistic conformal symmetry groups that will not be discussed in the main text.

The non-relativistic groups of interest in $D = d+1$ spacetime dimensions are shown in table 1. Their structures are more complicated than relativistic ones considered in appendix A.1, with more different types of generators, as we let time play a special role in the spirit of Galilean relativity. Let us now just emphasize some important general remarks and properties about the algebras before going in-depth. The Euclidean group $\mathrm{E}(d,1)$ is the most simple one which is contained as a subgroup of all the considered relativistic and non-relativistic symmetry groups. We will keep the presentation of this and the Lifshitz group a bit short, as they are not the main target of our investigation. The finite spacetime transformations of $\mathrm{E}(d,1)$ are translation along with spatial rotations

$$\begin{aligned} t' &= t + a^0 && \text{(2.1a)} \\ x'^i &= R^i{}_j x^j + a^i \,, && \text{(2.1b)} \end{aligned}$$

which are rotations parametrized by $R^i{}_j \in \mathrm{SO}(d)$, temporal translations $a^0 \in \mathbb{R}$ and spatial translations $a^i \in \mathbb{R}^d$ [23].

The pure spatial rotations $J_{ij} = -J_{ji}$ form a $\mathfrak{so}(d)$ subalgebra, under which the momentum P_i transforms as a vector and the Hamiltonian H as a scalar. The Lie algebra $\mathfrak{e}(d,1)$ spanned by the generators takes the defining commutation relations

$$\begin{aligned} [P_i, H] &= 0 && \text{(2.2a)} \\ [J_{ij}, H] &= 0 && \text{(2.2b)} \\ [P_i, P_j] &= 0 && \text{(2.2c)} \\ [P_k, J_{ij}] &= \delta_{ik} P_j - \delta_{jk} P_i && \text{(2.2d)} \\ [J_{ij}, J_{kl}] &= \delta_{jk} J_{il} - \delta_{ik} J_{jl} + \delta_{jl} J_{ik} - \delta_{il} J_{jk} \,. && \text{(2.2e)} \end{aligned}$$

Notice that the Hamiltonian H in this case actually is a central charge as it does not enter on the RHS of any of the commutators. The translations forms an abelian subgroup $\mathbb{R}^{d+1}$, while the rotations form a non-abelian subgroup $\mathrm{SO}(d)$. From the commutation relations we also see that $\mathrm{E}(d,1)$ is the semidirect product of rotations and translations, and so $\mathrm{E}(d,1)$ has the structure of an affine group and is thus in particular not semisimple. Hence we can write

$$\mathrm{E}(d,1) = \mathbb{R}^{d+1} \ltimes \mathrm{SO}(d) \,. \tag{2.3}$$

2.2 THE LIFSHITZ GROUP

A Lifshitz scaling with action

$$\begin{aligned} t' &= \lambda^z t && \text{(2.4a)} \\ x'^i &= \lambda x^i && \text{(2.4b)} \end{aligned}$$

can be added to the Euclidean transformations (2.1). Here $z \in \mathbb{R}$ is called the dynamical exponent and determines the anisotropy of the scaling. This transformation is only consistent with relativistic symmetries for $z = 1$ as time and space must be on the same footing. For non-relativistic theories general z can be allowed exactly because they need not be on the same footing. The additional commutation relations with the dilatation generator D for the algebra of the resulting Lifshitz group $\mathrm{Lif}\,(d,1)$ on top of the Euclidean algebra (2.2) are given by

$$[D,H] = zH \tag{2.5a}$$
$$[D,P_i] = P_i \tag{2.5b}$$
$$[D,J_{ij}] = 0\,. \tag{2.5c}$$

This is the same as saying in an algebraic language that H has dilatation weight z, P_i weight 1, and J_{ij} weight zero. The addition of D does not change the affine structure of the Euclidean group.

Interestingly enough a $D = 2$ field theory with $z = 0$ Lifshitz symmetry satisfying some technical assumptions about unitarity and the scaling spectrum is automatically invariant under an infinite dimensional symmetry group, which in the literature is known as the warped conformal group, first shown by Hofman and Strominger [31]. This group and its warped conformal field theory realizations are of relevance when using the holographic principle to calculate the energy of an extremal Kerr black hole, see for example [32, 33, 34].

2.3 THE GALILEAN GROUP

2.3.1 *The algebra*

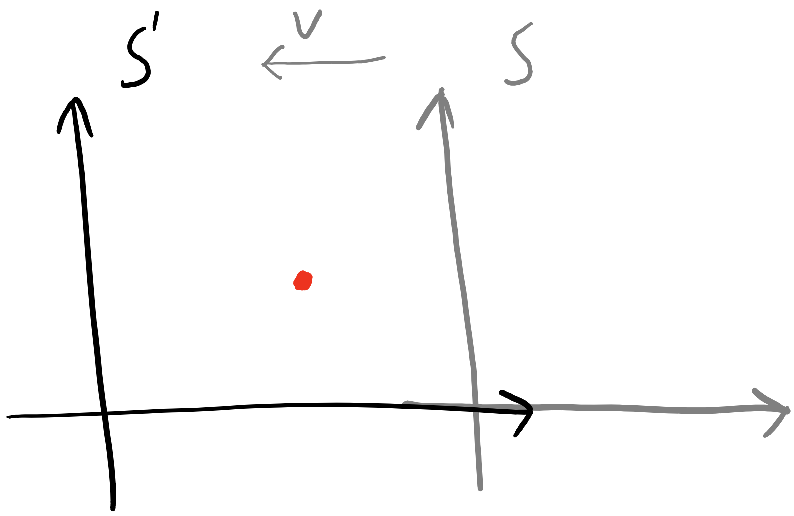

Figure 2: The action of the Galilean boost: The new reference frame S' moves with velocity $-v$ relative to the old S and the coordinates of the spacetime point (red dot) transforms according to (2.6).

On top of the Euclidean transformations (2.1) it is also possible to add the Galilean boost transformation

$$
\begin{aligned}
t' &= t && (2.6a)\\
x'^i &= x^i + v^i t\,. && (2.6b)
\end{aligned}
$$

This corresponds exactly to boosting one inertial to another inertial frame with a relative velocity $-\boldsymbol{v}$. When we add the d Galilean boosts to the Euclidean group, we obtain the Galilean group $\mathrm{Gal}\,(d,1)$ that describes the symmetries of non-relativistic theories of relevance in this thesis [35, 36]. The boost generator B_i associated to (2.6) now adds the following new commutation relations on top of (2.2) given by

$$
\begin{aligned}
[B_i, H] &= P_i && (2.7a)\\
[B_k, J_{ij}] &= \delta_{ik}B_j - \delta_{jk}B_i && (2.7b)\\
[B_i, P_j] &= 0 && (2.7c)\\
[B_i, B_j] &= 0\,. && (2.7d)
\end{aligned}
$$

The first structure to immediately read of from the commutation relations (2.7) is that the boost again transform as a vector under the spatial rotations. The fact that $[B_i, H] = P_i \neq 0$ while $[P_i, B_j] = 0$ also means that we have an asymmetry between space and time. The addition of B_i retains the semidirect sum of the Euclidean algebra, but now the boosts are also added through a semi-direct sum with rotations. For the corresponding group we can write its structure as

$$
\mathrm{Gal}\,(d,1) = \mathbb{R}^{d+1} \ltimes \left(\mathrm{SO}\,(d) \ltimes \mathbb{R}^d\right)\,. \tag{2.8}
$$

The boosts forms an abelian subgroup $\mathbb{R}^d$ on their own. We see that boosts and rotations combined forms a subgroup called the homogeneous Galilean group $\mathrm{HGal}\,(d,1) \equiv \mathrm{SO}\,(d) \ltimes \mathbb{R}^d$ of the Galilean group. In non-relativistic spacetimes this subgroup plays the role of the Minkowski group that is a subgroup of the Poincaré group in relativistic spacetimes.

2.3.2 *Galilei as a non-relativistic limit of Poincaré*

It is instructive to see how one obtains the Galilean group from the non-relativistic limit of the Poincaré group, which is discussed in appendix A.1. In natural units this is most easily done by performing an Inönü-Wigner contraction of the Poincaré group [37, 38]. This will leave us with the same number of generators, but a different group structure. We will motivate the correct contraction by studying the usual representation of the Poincaré group on spacetime with the generators represented as differential operators [39] (see also section 4.1). We then require that velocities are much smaller than the speed of light c, which is equivalent to sending $c \to \infty$. As velocities in some reference frame is given by $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}x^0}$, we see that we should really take the following rescaling of coordinates

$$x^0 \mapsto \alpha x^0 \tag{2.9}$$
$$x^i \mapsto x^i \tag{2.10}$$

and take the scaling parameter $\alpha \underset{\text{NR}}{\to} \infty$ to obtain the non-relativistic limit with velocities $v^i \underset{\text{NR}}{\to} 0$. Transferring this idea to the corresponding generators P_μ, $J_{\mu\nu}$ of the Poincaré algebra (A.2), we see that we should really leave $P_i = -i\partial_i$ and $J_{ij} = -ix_i\partial_j + ix_j\partial_i$ unchanged, while we must define new operators

$$H \equiv \lim_{\alpha\to\infty} \alpha P_0 \tag{2.11a}$$
$$B_i \equiv \lim_{\alpha\to\infty} \frac{1}{\alpha} J_{0i} \tag{2.11b}$$

in order to ensure that they stay finite and non-zero under the non-relativistic limit, as the rescaling of coordinates for the spacetime differential operators leads to

$$P_0 = i\partial_0 \mapsto \frac{1}{\alpha} i\partial_0 \tag{2.12a}$$
$$J_{0i} = -it\partial_i - ix_i\partial_0 \mapsto -\alpha it\partial_i - \frac{1}{\alpha} ix_i\partial_0 \,. \tag{2.12b}$$

These are going to be our new Hamiltonian H and Galilean boosts B_i for any representation of the Poincaré algebra when we take $\alpha \underset{\text{NR}}{\to} \infty$, and one sees that the resulting algebra is exactly the Galilean one.

Example 2.1 (Taking the non-relativistic limit)**.** It is straight-forward to take the non-relativistic limit using the above. One considers the Poincaré commutation relations with P_0 and J_{0i}, and multiply with α and α^{-1} appropriately to obtain the commutators with H and B_i. Doing this we will find it is consistent to take $\alpha \to \infty$. Four relations will change in total, for example

$$[P_i, J_{0j}] = \eta_{0i} P_j - \eta_{ji} P_0 \;\Rightarrow$$
$$\left[P_i, \frac{1}{\alpha} J_{0j}\right] = -\delta_{ij} \frac{1}{\alpha^2} (\alpha P_0) \underset{\text{NR}}{\to} [P_i, B_j] = 0 \tag{2.13}$$

and

$$[P_0, J_{0j}] = \eta_{00} P_j - \eta_{j0} P_0 \;\Rightarrow$$
$$\left[\alpha P_0, \frac{1}{\alpha} J_{0j}\right] = -P_j \underset{\text{NR}}{\to} [H, B_j] = -P_j \,. \tag{2.14}$$

The Galilean boosts B_i can thus be understood as the remnants of the Lorentz boosts J_{0i} in the non-relativistic limit. The boost transformation is now asymmetric in time and space, as we have the commutator (2.13) equal to zero, while (2.14) is non-zero. However, the time coordinate stays "absolute" and does not transform unlike for Lorentz boosts. The unification of space and time has been undone, which is just the spirit of Galilean relativity.

The Casimir invariants of Poincaré C_2 and C_4 (A.4) are investigated in appendix A.1 and are useful for classifying the representations. Taking the non-relativistic limit of the momentum squared gives us that one Casimir is

$$C_2 = \lim_{\alpha\to\infty} P^2 = \boldsymbol{P}^2 \,. \tag{2.15}$$

One can indeed verify that $\boldsymbol{P}^2$ commutes with all generators of the Galilean algebra $\mathfrak{gal}\,(d,1)$. For the other Casimir C_4 we need to rescale by α^{-2} to keep it finite (and put a sign for convenience) and find

$$\tilde{C}_4 \equiv \lim_{\alpha\to\infty} \frac{1}{\alpha^2}\left(\frac{1}{2}P^2 J_{\mu\nu}J^{\mu\nu} - J_{\mu\rho}P^{\rho}J^{\mu\sigma}P_{\sigma}\right) = \boldsymbol{P}^2\boldsymbol{B}^2 - (\boldsymbol{B}\cdot\boldsymbol{P})^2 \,. \tag{2.16}$$

There is no ambiguity in the ordering as P_i, B_j span an abelian subalgebra. In the special case of $d=3$ one sees that this can be written as $\tilde{C}_4 = (\boldsymbol{P}\times\boldsymbol{B})^2$. Together C_2 and $\tilde{C}_4$ span the center of the algebra and can be used to label representations.

2.3.3 *On unitary representations and little groups*

For a Hilbert space $\mathscr{H}$ of quantum mechanical states, spacetime symmetries are represented by (anti-)unitary operators[1]. Gal $(d,1)$ is non-compact because of the boosts and therefore it does not have any finite dimensional unitary representations just like what is well-known for representations of the Poincaré group [40, 39]. This is of course expected as there are infinitely many states with different momentum and one may boost and rotate the reference frame, with transformed states considered equivalent.

It is possible to use the method of induced representations that classifies all possible irreducible unitary representations by find various "little groups" of the homogeneous Galilean group HGal $(d,1)$ [36]. A little group is here a subgroup of HGal $(d,1)$ that leaves a certain standard energy E_0 and momentum $\boldsymbol{p}_0$ unchanged, which are dual to translations. The action of a homogeneous Galilean transformation $U(\boldsymbol{R},\boldsymbol{v},0,\boldsymbol{0})$ on a momentum eignenstate $|\boldsymbol{p},E,\sigma\rangle$, where σ is the remaining quantum numbers of relevance can then be written as

$$U(\boldsymbol{R},\boldsymbol{v},0,\boldsymbol{0})\,|E,\boldsymbol{p},\sigma\rangle = \sum_{\sigma} D_{\sigma\sigma'}(\boldsymbol{R},\boldsymbol{v};E_0,\boldsymbol{p}_0)\,|E',\boldsymbol{p}',\sigma'\rangle \,. \tag{2.17}$$

Here $D_{\sigma\sigma'}(\boldsymbol{R},\boldsymbol{v};E_0,\boldsymbol{p}_0)$ is a unitary representation of the little group corresponding to a standard standard energy E_0 and momentum $\boldsymbol{p}_0$ related to E, $\boldsymbol{p}$ by Galilean boosts and rotations that takes the form[2]

1 Up to a phase, which is very important for non-relativistic spacetime symmetries as we will discuss in detail later.

2 The energy and momentum together transforms as a covector under the fundamental representation (2.23).

$$\begin{aligned} E' &= E - \boldsymbol{v}^t \boldsymbol{R}\boldsymbol{p} & (2.18a)\\ \boldsymbol{p}' &= \boldsymbol{R}\boldsymbol{p}\,. & (2.18b)\end{aligned}$$

Since the states are assumed to be momentum eigenstates, translations will just give a simple phase as

$$U\left(\mathbf{0},\mathbf{0},a^0,\boldsymbol{a}\right)|E,\boldsymbol{p},\sigma\rangle = e^{-ia^0E+i\boldsymbol{a}\cdot\boldsymbol{p}}\,|E,\boldsymbol{p},\sigma\rangle\ . \tag{2.19}$$

There are now various choices of little groups relevant for physical representations. One immediate choice is to pick standard energy and momentum $E_0 > 0$ and $\boldsymbol{p}_0 = 0$ that is left invariant by spatial rotations in (2.18). This corresponds to a representation with Casimirs $C_2 = \tilde{C}_4 = 0$. Hence the little group is $\mathrm{SO}\,(d)$ which has the well-known $(2j+1)\times(2j+1)$-dimensional unitary irreducible representations, so states can be labeled as $\left|E,\boldsymbol{p},j,m_j\right\rangle$ (plus perhaps other quantum numbers). This gives particle representations of positive energy and spin-j, $j = 0$, $\frac{1}{2}$, 1, ... More representations and further discussions can be found in [41, 35].

A special feature of Galilean states is that they are not localizable: It turns out that it is not possible to construct a linear combination of the corresponding wave-functions $\langle t,\boldsymbol{x}\,|\,E,\boldsymbol{p},\sigma\rangle$ which is a δ-function in space and time so no state can be said to have a definite position at a given time, contrary to what is the case in standard QM. It is however possible to obtain approximate localizability where the amplitude decays with some distance. This shows that the Galilean states are not easily interpreted as particle states, see Inönü and Wigner [36] for further discussions.

2.3.4 *Finite dimensional non-unitary representations*

It is useful to have concrete finite dimensional matrix representations to work with, where the group product is represented by the usual matrix product. We will also use this for constructing the Galilean fiber bundles for our geometry later on. One useful non-unitary but finite representation is the faithful representation on $D+1\times D+1$ matrices given by

$$G^{\hat{A}}{}_{\hat{B}} = \begin{pmatrix} 1 & 0 & a^0\\ v^a & R^a{}_b & a^a\\ 0 & 0 & 1\end{pmatrix}, \tag{2.20}$$

where $v^a \in \mathbb{R}^d$ is the (finite) boost parameter and $R^a{}_b \in \mathrm{SO}\,(d)$ is the spatial rotation matrix and $\hat{A},\hat{B} = 1,2,\ldots,D+1$. Finally $a^0 \in \mathbb{R}$ is the parameter for temporal translations, and $a^a \in \mathbb{R}^d$ the spatial translation parameter. In this non-unitary representation the semi-direct product structure of the Galilean group is obvious, as are the various subgroups. One also sees that $\mathrm{Gal}\,(d,1) \subset \mathrm{GL}\,(D+1,\mathbb{R})$. Composing two transformations G, G' we have that this results in a transformation G'' given by

$$G''^{\hat{A}}{}_{\hat{B}} \equiv \left(GG'\right)^{\hat{A}}{}_{\hat{B}} = \begin{pmatrix} 1 & 0 & a^0 + a'^0 \\ v^a + R^a{}_b v'^b & R^a{}_c R'^c{}_b & a^a + v^a a'^0 + R^a{}_b a'^b \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.21}$$

One sees here that the temporal translations add up, while the spatial translation gets boosted and rotated.

HGal $(d, 1)$ furnishes from the above the fundamental $D \times D$ representation $\rho_D\left(\mathrm{HGal}\left(d, 1\right)\right)$ given by matrices of the following form [42, 12]

$$G^A{}_B = \begin{pmatrix} 1 & 0 \\ v^a & R^a{}_b \end{pmatrix} . \tag{2.22}$$

It is trivial to see that $\rho_D\left(\mathrm{HGal}\left(d, 1\right)\right) \subset \mathrm{GL}\left(D, \mathbb{R}\right)$. The inverse transformation is given by

$$\left(G^{-1}\right)^B{}_A = \begin{pmatrix} 1 & 0 \\ -\left(R^{-1}\right)^b{}_a v^a & \left(R^{-1}\right)^b{}_a \end{pmatrix} . \tag{2.23}$$

Notice that because $\mathbf{R}$ is an orthogonal matrix $\mathbf{R}^{-1} = \mathbf{R}^t$. The group action on components of (Galilean) vectors V^B in the vector space $\mathbb{R}^D$, and covectors U_A in the covector space $\mathbb{R}^{D*}$ is then defined by

$$V'^A = G^A{}_B V^B \tag{2.24a}$$

$$U'_A = U_B \left(G^{-1}\right)^B{}_A , \tag{2.24b}$$

which leaves any scalar $V^A U_A$ invariant under Galilean transformations as it should. It is then straight-forward to generalize this to obtain the transformation rules of general Galilean tensors by multi-linearity [43].

$\rho_D\left(\mathrm{HGal}\left(d, 1\right)\right)$ has two invariant symbols that are invariant under any Galilean transformation. These are in our representation $\rho_D\left(\mathrm{HGal}\left(d, 1\right)\right)$ easy to find and given by

$$\tau_A \equiv \left(1\, \mathbf{0}\right) \tag{2.25a}$$

$$h^{AB} \equiv \delta^A_a \delta^{Ba} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ab} \end{pmatrix} . \tag{2.25b}$$

Example 2.2 (Invariant symbols)**.** Invariance of τ_A is obvious, but we have to work a little to show that this is also the case for h^{AB}:

$$\begin{aligned} h^{AB} \to G^A{}_C G^B{}_D h^{CD} &= G^A{}_c G^B{}_d h^{cd} \\ &= \delta^A_a \delta^{Bb} R^a{}_c \left(R^{-1}\right)^c{}_b \\ &= h^{AB} \ \checkmark \,, \end{aligned} \tag{2.26}$$

where we used the orthogonality of the rotation matrix.

In fact, the preservation of these two objects characterizes the fundamental representation $\rho_D\,(\mathrm{HGal}\,(d,1)) = \rho_D\left(\mathrm{SO}\,(d) \rtimes \mathbb{R}^d\right)$ as a subgroup of $\mathrm{GL}\,(D,\mathbb{R})$, and we could equally well have defined the set of Galilean matrices [44]

$$\rho_D\left(\mathrm{SO}\,(d) \rtimes \mathbb{R}^d\right) \equiv \left\{ M^A{}_B \in \mathrm{GL}\,(D,\mathbb{R}) \mid M^A{}_C M^B{}_D h^{CD} = h^{AB},\, \tau_B M^B{}_A = \tau_A \right\} . \tag{2.27}$$

Finally it is also seen that the Levi-Civita tensor $\epsilon_{A_0 A_1 \cdots A_d}$ is an invariant symbol of $\mathrm{HGal}\,(d,1)$.

2.4 THE BARGMANN GROUP

2.4.1 *Algebraic structure*

There exists a central extension of the Galilean group, which is called the Bargmann group $\mathrm{Barg}\,(d,1)$. Adding a central charge extends the algebra, showing the impossibility of being the result of a Inönü-Wigner contraction of the Poincaré symmetry group of D spacetime dimensions $\mathrm{Poin}\,(d,1)$ as this preserves the number of generators. This does not mean that the Bargmann group cannot be the result of a reduction of the higher-dimensional Poincaré group which we consider in section 2.4.4.

$\mathrm{Barg}\,(d,1)$ is the relevant spacetime symmetry group for the typical non-relativistic theories considered, namely those with a notion of mass. The central charge corresponds to mass or particle number conservation, which is now non-trivially related to spacetime in such theories as we shall see demonstrated several times [27]. To see that this extension of the Galilean group exists and is unique, one can investigate the structure of the commutation relation: We can make an Ansatz for the central extension of (2.7) and then see what can be allowed by the Jacobi identities and redefined away. Unlike for the Poincaré group one finds that the Galilean group allows for a single central extension with central charge M [39]. In this unique extension we have to replace the $[P_i, B_j]$ commutator (2.7c) with

$$[P_i, B_j] = M\delta_{ij} \,. \tag{2.28}$$

With this replacement in (2.7) we obtain the Bargmann algebra. If we make the action of M trivial in some representation, then we are back in the Galilean case of section, so the Bargmann group can be considered the most general. The commutator (2.28) shows that on top of the structure of the Galilean group there is now another semi-direct product with the center $\mathbb{R}$ generated by M, which now goes with the translation subgroup exactly because it is an ideal of the translational subalgebra [42]. Hence the full group structure is

$$\mathrm{Barg}\,(d,1) = \left(\mathbb{R}^{d+1} \ltimes \mathbb{R}\right) \ltimes \left(\mathrm{O}\,(d) \ltimes \mathbb{R}^{d}\right) . \tag{2.29}$$

2.4.2 *Revisiting the unitary representations on states*

Because M is a central element it is by definition one of the invariant Casimir operators, so Bargmann has a total of three. The two Casimirs of the Galilean group (2.15), (2.16) are no longer Casimirs for the Bargmann group because of the non-zero commutation relation (2.28). We now need a new basis, which can conveniently be chosen as below [35]:

$$\begin{aligned}
M &\equiv mI \qquad &(2.30\text{a})\\
\mathring{C}_2 &\equiv H - \frac{1}{2m}\boldsymbol{P}^2 \qquad &(2.30\text{b})\\
\mathring{C}_4 &\equiv \frac{1}{2}J^{jk}J_{jk} - \frac{1}{m}\left(J^{jk}B_jP_k - B_jP_kJ^{jk}\right)\\
&\quad + \frac{1}{m^2}B^jP^k\left(B^jP^k - B^kP^j\right) . \qquad &(2.30\text{c})
\end{aligned}$$

After proper rescaling with m they all reduce to (2.15) and (2.16) when we take the $m \to 0$ limit. In $d = 3$ the last Casimir takes the more convenient expression as $\mathring{C}_4 = \left(\epsilon_{ijk}\left[\frac{1}{2}J^{jk} - \frac{1}{m}\left(B^jP^k\right)\right]\right)^2$, which we show in appendix D.2. The action of $\mathring{C}_2$ on a state $\left|E, \boldsymbol{p}, j, m_j\right\rangle$ in the spin-j representation of section 2.3.3 is seen to relate the eigenvalue of the Hamiltonian to that of momenta by the classical formula

$$E = \frac{\boldsymbol{p}^2}{2m} + E_0 \, , \tag{2.31}$$

where E_0 is the rest energy, the value of $\mathring{C}_2$ acting on the state. This demonstrates that we are then led to think of M as being the mass of some representation and shows that the Bargmann group is the one relevant for Newtonian physics. In particular (2.31) has no equivalent when the mass is zero. For $\mathring{C}_4$ we have with standard momentum $\boldsymbol{p}_0 = \boldsymbol{0}$ it reduces to

$$\mathring{C}_4 = \frac{1}{2}J^{jk}J_{jk} = j\,(j+1) \tag{2.32}$$

which is the invariant of the $\mathfrak{so}\,(d)$ algebra, which labels the spin representations. The states now being those of standard QM are localizable.

2.4.3 *Finite dimensional non-unitary representations*

Because of the direct product structure we have that $\mathrm{Barg}\,(d,1)$ has a natural faithful $(D+2) \times (D+2)$ matrix representation given by [12, 42]

$$B^{\tilde{A}}{}_{\tilde{B}} = \begin{pmatrix} 1 & 0 & 0 & a^0 \\ v^a & R^a{}_b & 0 & a^a \\ -\boldsymbol{v}^2/2 & -v_a R^a{}_b & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{2.33}$$

where $f \in \mathbb{R}$ is a parameter corresponding to the central transformation and $\tilde{A}, \tilde{B} = 0, 1, \ldots, D+1$. Notice that $\mathrm{Barg}\,(d,1)$ is not a subgroup of $\mathrm{GL}\,(D+1, \mathbb{R})$ like $\mathrm{Gal}\,(d,1)$, but it is a subgroup of $\mathrm{GL}\,(D+2, \mathbb{R})$ because of the different direct product structure. Composing two transformations B, B' we have that this results in a transformation $B'' = BB'$ given by

$$B''^{\tilde{A}}{}_{\tilde{B}} = \begin{pmatrix} 1 & 0 & 0 & a^0 + a'^0 \\ v^a + R^a{}_b v'^b & R^a{}_c R'^c{}_b & 0 & a^a + a'^0 v^a + R^a{}_b a'^b \\ \begin{pmatrix} -\frac{1}{2}\left(\boldsymbol{v}^2 + \boldsymbol{v}'^2\right) \\ -v_a R^a{}_b v'^b \end{pmatrix} & -\left(v'_c + v_a R^a{}_c\right) R'^c{}_b & 1 & \begin{pmatrix} f + f' - \frac{1}{2} a'^0 \boldsymbol{v}^2 \\ -v_a R^a{}_b a'^b \end{pmatrix} \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.34}$$

One sees that the structure of the composition for the Galilean case (2.21) is contained in this, but in addition to this one observes the rather complicated composition of the center of the group.

$\mathrm{HGal}\,(d,1) = \mathrm{O}\,(d) \ltimes \mathbb{R}^d$ now takes a $(D+1) \times (D+1)$-dimensional matrix representation $\rho_{D+1}\,(\mathrm{HGal}\,(d,1))$ with elements given by

$$B^{\hat{A}}{}_{\hat{B}} = \begin{pmatrix} 1 & 0 & 0 \\ v^a & R^a{}_b & 0 \\ -\boldsymbol{v}^2/2 & -v_a R^a{}_b & 1 \end{pmatrix} \tag{2.35}$$

$$\left(B^{-1}\right)^{\hat{B}}{}_{\hat{A}} = \begin{pmatrix} 1 & 0 & 0 \\ -\left(R^{-1}\right)^b{}_a v^a & \left(R^{-1}\right)^b{}_a & 0 \\ -\boldsymbol{v}^2/2 & v_a & 1 \end{pmatrix} . \tag{2.36}$$

Notice that this is an indecomposible but not irreducible representation, as it contains a submatrix with the fundamental representation (2.22). The representation (2.35) now naturally acts on (Bargmann) vectors $V^{\hat{A}}$ of $\mathbb{R}^{D+1}$ and covectors $U_{\hat{A}}$ of $\mathbb{R}^{D+1*}$. In a sense we now have (co)vectors with more components than we wished for, as would like to eventually define D-dimensional vectors on a D-dimensional spacetime.

We may define a projection $\mathscr{P} : \mathbb{R}^{D+1} \to \mathbb{R}^D$ by

$$\mathscr{P}^A{}_{\hat{B}} = \left(\delta^A_B, 0\right) . \tag{2.37}$$

Using the projector $\mathscr{P}^A{}_{\hat{B}}$ we may project contravariant Bargmann tensors of $\mathbb{R}^{D+1}$ to $\mathbb{R}^D$. We can also define a lift of covariant Galilean objects of $\mathbb{R}^{*D}$ to $\mathbb{R}^{*D+1}$ by contracting with the upper index of $\mathscr{P}^A{}_{\hat{B}}$.

Example 2.3 (Various projections and lift)**.** Using the projector $\mathscr{P}^A{}_{\hat{B}}$ we may project the representation of an element of $\rho_{D+1}\left(\mathrm{HGal}\left(d,1\right)\right)$ given by (2.35) to the one in (2.22). We have more specifically

$$\mathscr{P}^A{}_{\hat{C}}B^{\hat{C}}{}_{\hat{B}} = B^A{}_C\mathscr{P}^C{}_{\hat{B}} = G^A{}_C\mathscr{P}^C{}_{\hat{B}}\,, \tag{2.38}$$

which shows that the projection commutes with action of the $\rho_{D+1}\left(\mathrm{HGal}\left(d,1\right)\right)$ on vectors in the sense

$$\mathscr{P}^A{}_{\hat{B}}V'^{\hat{B}} = \mathscr{P}^A{}_{\hat{C}}B^{\hat{C}}{}_{\hat{B}}V^{\hat{B}} = G^A{}_CV^C = V'^A\,. \tag{2.39}$$

The two invariant objects $h^{\hat{A}\hat{B}}$, $\tau_{\hat{A}}$ of (2.35) are easily seen to be given by

$$\tau_{\hat{A}} \equiv (1,\mathbf{0},0) \tag{2.40a}$$

$$h^{\hat{A}\hat{B}} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta^{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.40b}$$

Notice that $h^{\hat{A}\hat{B}}$ is not degenerate. The relation to the same objects of the homogeneous Galilean group (2.25) may now be written nicely using $\mathscr{P}^A{}_{\hat{B}}$ to project or lift:

$$\mathscr{P}^A{}_{\hat{A}}\mathscr{P}^B{}_{\hat{B}}h^{\hat{A}\hat{B}} = h^{AB} \tag{2.41a}$$

$$\tau_{\hat{A}} = \tau_A\mathscr{P}^A{}_{\hat{A}}\,. \tag{2.41b}$$

2.4.4 *Bargmann as a null reduction of Poincaré*

While the Bargmann group $\mathrm{Barg}\left(d,1\right)$ cannot be the result of an Inönü-Wigner contraction as discussed in section 2.4.1, it is still possible that it may be the result of a reduction of a higher-dimensional Poincaré group. The $D+1$ spacetime dimensions symmetry group $\mathrm{Poin}\left(d+1,1\right)$ has $\frac{1}{2}\left(d+2\right)\left(d+3\right)$ generators which is $d+1$ more than the $\frac{1}{2}d\left(d+3\right)+2$ generators of $\mathrm{Barg}\left(d,1\right)$ for D spacetime dimensions according to tables 1, 4. This is a hint that a reduction might work if we can remove these extra generators consistently. It turns out that what we need to do is to perform a so-called null reduction which we will now explain.

Consider the $D+1$-dimensional Minkowski space where we make a change of coordinates to the usual light-cone coordinates

$$x^{\pm} \equiv \frac{1}{2}\left(x^0 \pm x^D\right)\,, \tag{2.42}$$

so that the coordinates used are ordered as $x^{\hat{\mu}} = \left(x^+, x^i, x^-\right)$. The Minkowski metric in these coordinates is

$$\eta_{\hat{\mu}\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta^{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{2.43}$$

Notice that this is exactly one of the invariant symbols (2.40b) of the extended homogeneous Galilean group. The generators of (A.2) that does not change in this basis are P_i and J_{ij} while the rest are given by an appropriate change of basis

$$\begin{aligned} H &\equiv \frac{1}{2}\left(P^0 - P^D\right) && (2.44a) \\ M &\equiv P^0 + P^D && (2.44b) \\ B_i &\equiv J_{0i} + J_{Di} && (2.44c) \\ K_i &\equiv J_{0i} - J_{Di} && (2.44d) \\ K &\equiv J_{0D} \,. && (2.44e) \end{aligned}$$

In this basis all generators that commute with M are P_i, H, M, B_i, J_{ij} and they precisely form a closed subalgebra which is the Bargmann algebra (2.2), (2.7), (2.28) [45, 46]. The commutation relations of M with the $d+1$ generators K_i, K are non-zero and all non-zero commutation relations with K_i, K are given by

$$\begin{aligned} [M, K_i] &= 2P_i && (2.45a) \\ [K_k, J_{ij}] &= \delta_{ik}K_j - \delta_{jk}K_i && (2.45b) \\ [P_i, K_j] &= 2H\delta_{ij} && (2.45c) \\ [B_i, K_j] &= 2\left(J_{ij} - \delta_{ij}K\right) && (2.45d) \\ [H, K] &= -H && (2.45e) \\ [M, K] &= M && (2.45f) \\ [B_i, K] &= B_i && (2.45g) \\ [K_i, K] &= -K_i \,. && (2.45h) \end{aligned}$$

This shows that Barg $(d, 1)$ can be embedded in Poin $(d+1, 1)$ as a subgroup[3]. Looking at the corresponding transformations in $D+1$-dimensional Minkowski space, we see that the Bargmann subalgebra is spanned by the generators that leaves out rotations in the (x^+, x^-) and (x^i, x^-) planes. An equivalent geometric statement is that we consider the symmetries of a hypersurface $x^- =$ constant orthogonal to the null x^--direction wrt. the Minkowski metric, which is why we call it a null reduction. $t \equiv x^+$ now acquires the interpretation as the non-relativistic time coordinate generated by H while the $u \equiv x^-$ translations generated by M are transformations of the center of the Bargmann group.

3 Interestingly enough the commutators involving just P_i, H, K_i, J_{ij} also span a closed subalgebra that can be identified with the so-called Carrollian algebra, where K_i are called Carrollian boosts and H is now a central charge [47, 48]. This to be thought of as the ultra-relativistic limit of the Poincaré algebra where we instead take $c \to 0$ and consider $x^+ =$ constant hypersurfaces. If we further include the generator K, this is seen to furnish an interpretation as some kind of dilatation operator since in this algebra H is no longer central.

3 NEWTON-CARTAN GEOMETRY

In this chapter we will construct the correct non-relativistic geometric framework. We take the approach of fiber bundles as an easy way of obtaining the relevant structures, relying heavily of the results of the previous chapter. A considerable effort is spent on interesting and subtle aspects of Galilean and Bargmann connections and covariant derivatives. Finally we consider the formulation of Newton-Cartan geometry on flat and linearized spacetimes.

3.1 GALILEAN FRAME BUNDLES

We now want to regard the entire Galilean group $\mathrm{Gal}(d, 1)$ as a frame bundle consisting of the $2D$-dimensional manifold FM and the bundle projection $\pi : FM \to M$. Intuitively what we want to do is to identify the Galilean boost and rotations acting as "internal transformations" of the vielbeins defined on the manifold M. This construction is naturally a fiber bundle where the D-dimensional base manifold is M and the fiber F consists of all bases of the (Galilean) vector space $\mathbb{R}^D$, i.e. all possible frames or vielbeins [49, 50]. This is a global trivialization $FM = F \times M$, and we can write it as the union

$$FM = \bigcup_{p \in M} (p, \{\boldsymbol{e}_A(p)\}) , \tag{3.1}$$

where $\{\boldsymbol{e}_A(p)\} \equiv F_p$ is the set of all vielbeins at point $p \in M$. The bundle projection π is now simply

$$\pi(p, \{\boldsymbol{e}_A(p)\}) = p \tag{3.2}$$

and we may naturally define the right-inverse of this map called the section $\sigma : M \to FM$ satisfying $\pi(\sigma(p)) = p$, so that we have $\sigma(p) = (p, \boldsymbol{e}_A(p))$. A particular vielbein $\boldsymbol{e}_A(p) \in F_p$ is then a section of FM as we illustrate in figure 3. By construction the action of the homogeneous Galilean group $\mathrm{HGal}(d, 1)$ is to transform the vielbein in representation (2.23) as

$$\boldsymbol{e}'_A(p) = \boldsymbol{e}_B(p) \left(G^{-1}(p)\right)^B{}_A . \tag{3.3}$$

$\mathrm{HGal}(d, 1)$ is here the structure group of the frame bundle, the group of all possible Galilean transformations of the frame. This shows that the frame bundle is a principal bundle, because the diffeomorphisms of the fiber F are the same as the action of $\mathrm{HGal}(d, 1)$ on it.

Similar reasoning goes for the coframe bundle

$$F^*M = \bigcup_{p \in M} \left(p, \left\{\boldsymbol{e}^A(p)\right\}\right) \tag{3.4}$$

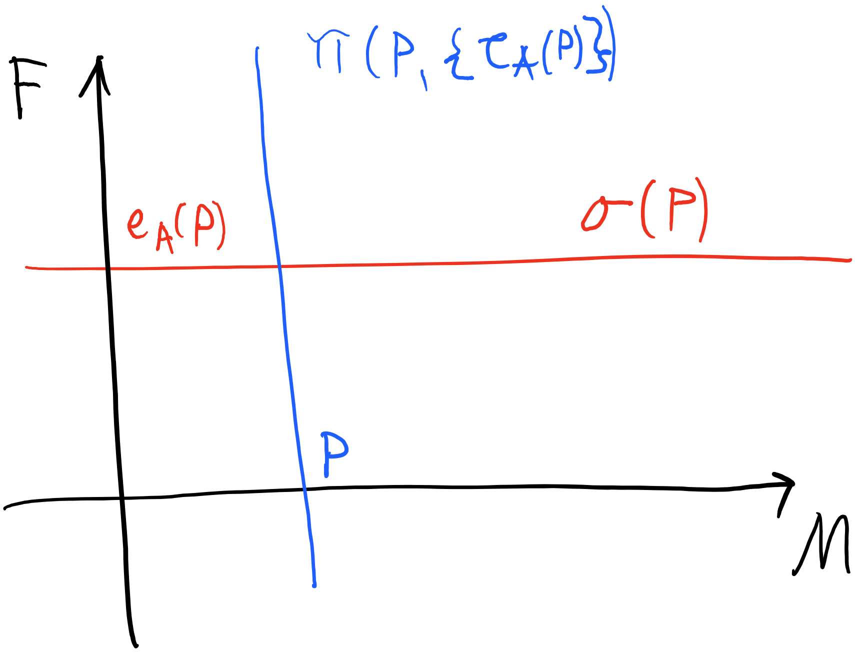

Figure 3: Illustration of the bundle projection π and section map σ for the fiber bundle $FM = F \times M$.

where coframes or inverse vielbeins $\{e^A(p)\} \equiv F_p^*$ transforms under the fundamental representation (2.22) as

$$e'^A(p) = G(p)^A{}_B \, e^B(p) \,. \tag{3.5}$$

This may be extended to tensor bundles by considering tensor products of frames. In the spirit of non-relativistic physics, we will single out the time and spatial component and write for a particular choice of (inverse) vielbeins

$$\begin{aligned} e^A &\equiv (\tau, e^a) && (3.6a) \\ e_A &\equiv (-v, e_a) \,. && (3.6b) \end{aligned}$$

The vielbeins must satisfy the completeness relations

$$\begin{aligned} \tau(v) &= -1 && (3.7a) \\ e^a(e_b) &= \delta^a_b && (3.7b) \\ e^a(v) = \tau(e_a) &= 0 && (3.7c) \\ e^a \otimes e_a &= \mathrm{id} + \tau \otimes v \,, && (3.7d) \end{aligned}$$

which defines the vielbeins in terms of the inverse vielbeins or vice versa. In simple terms, we should think of $A = \{0, a\}$ as a Galilean (frame) index and objects that are Galilean scalars and thus spacetime tensors must be fully contracted. These indices transform under the local boosts and rotations of the Galilean structure group, with a “down” index transforming oppositely to an "up" index. The local Galilean transformation laws (3.3), (3.5) of the (inverse) vielbeins can then be written in components as

$$\begin{aligned}
\boldsymbol{\tau}' &= \boldsymbol{\tau} && (3.8a)\\
\boldsymbol{e}'^a &= R^a{}_b \boldsymbol{e}^b + v^a \boldsymbol{\tau} && (3.8b)\\
\boldsymbol{v}' &= \boldsymbol{v} + \boldsymbol{e}_b \left(R^{-1}\right)^b{}_a v^a && (3.8c)\\
\boldsymbol{e}'_a &= \boldsymbol{e}_b \left(R^{-1}\right)^b{}_a , && (3.8d)
\end{aligned}$$

where $R^b{}_a$ is a local rotation and v^a is a local Galilean boost[1]. $\boldsymbol{\tau}$ is called the clock form because we see from (3.8) it is indeed a spacetime 1-form unlike the other objects that do transforms under Galilean transformations. This is of course clear since it is related to one of the two invariant objects of the structure group as described on page 14. The other Galilean invariant object we can construct is the (inverse) spatial metric constructed from a tensor product of the "inverse" spatial vielbeins[2] $\boldsymbol{e}_a$

$$\boldsymbol{h}^{-1} \equiv \delta^{ab} \boldsymbol{e}_a \otimes \boldsymbol{e}_b . \tag{3.9}$$

For a vector $\boldsymbol{V} \equiv V^\mu \partial_\mu$ given as a particular section of the tangent bundle TM and a covector $\boldsymbol{U} \equiv U_\mu \mathrm{d}x^\mu$ of the cotangent bundle T^*M, we can write them as contracted with their associated Galilean (co)vector components and (inverse) vielbeins as

$$\begin{aligned}
\boldsymbol{V} &\equiv V^A \boldsymbol{e}_A = -V^0 \boldsymbol{v} + V^a \boldsymbol{e}_a && (3.10a)\\
\boldsymbol{U} &\equiv U_A \boldsymbol{e}^A = U_0 \boldsymbol{\tau} + U_a \boldsymbol{e}^a , && (3.10b)
\end{aligned}$$

where the components transforms in the representation of $\mathrm{HGal}\,(d,1)$ discussed in section 2.3.4 so that $\boldsymbol{V}, \boldsymbol{U}$ are invariant under Galilean transformations. The mappings are bijective and isomorphisms $FM \simeq TM$ and $F^*M \simeq T^*M$. We can equivalently find the components of a given (co)tangent vector as

$$\begin{aligned}
V^A &= \boldsymbol{e}^A\left(\boldsymbol{V}\right) = e^A_\mu V^\mu && (3.11a)\\
U_A &= \boldsymbol{e}_A\left(\boldsymbol{U}\right) = e^\mu_A U_\mu . && (3.11b)
\end{aligned}$$

This shows that the vielbeins can be thought of objects that maps from a coordinate frame to a Galilean frame or the other way around. In a local coordinate basis we may write $e^A_\mu = \left(\tau_\mu, e^a_\mu\right)$ and $e^\mu_A = \left(-v^\mu, e^\mu_a\right)$, which can conveniently be used to map from one to another.

1 We apologize for the possible confusion the use of v^a for the boost and v^μ for the vielbein may cause.
2 We must stress that is not the inverse of anything, it is just poor but conventional terminology.

Example 3.1 (Contraction)**.** We have that the action $\boldsymbol{V}(\boldsymbol{U})$ of a vector $\boldsymbol{V}$ on a covector $\boldsymbol{U}$ (and vice versa) is a scalar by construction. Written in terms of both Galilean and spacetime components, we have

$$\begin{aligned} \boldsymbol{V}(\boldsymbol{U}) &= V^A U_A = -V^0 U_0 + V^a U_a \\ &= V^\mu U_\mu \,, \end{aligned} \tag{3.12}$$

where we in the last equality used (3.7) and (3.11).

3.2 GALILEAN SPACETIME GEOMETRY

We have here taken a specific point of view that highlights the origin of the basic properties of Newton-Cartan geometries as fiber bundle constructions. To calculate geometric quantities of interest, we often just need the spacetime tensors $\boldsymbol{\tau}$, $\boldsymbol{h}^{-1}$ that plays the role of a metric-like structure in these theories [21, 51]. These alone are sufficient to characterize what we may call Galilean spacetime or Newton-Cartan geometry:

Definition 3.2 (Galilean spacetime)**.** A Galilean spacetime consists of a triplet $\left(M, \boldsymbol{\tau}, \boldsymbol{h}^{-1}\right)$ where M is a D-dimensional manifold on which a non-vanishing 1-form $\boldsymbol{\tau}$ and a symmetric rank 2 contravariant tensor $\boldsymbol{h}^{-1}$ that satisfies $\boldsymbol{h}^{-1}(\boldsymbol{\tau}, \cdot) = \boldsymbol{0}$ are defined.

Here we can think of $\left(\boldsymbol{\tau}, \boldsymbol{h}^{-1}\right)$ as a degenerate metric structure, with $\boldsymbol{h}^{-1}$ having corank 1. It is of course useful to refer back to the frame bundle construction to interpret some of the properties of such Galilean spacetime. First of all, we may in this language then define projective inverses $(\boldsymbol{v}, \boldsymbol{h})$ as objects that satisfies

$$v^\mu \tau_\mu = -1 \tag{3.13a}$$
$$h_{\mu\lambda} h^{\lambda\nu} = \delta^\nu_\mu + \tau_\mu v^\nu \,. \tag{3.13b}$$

These are not uniquely determined and we may redefine them as [42]

$$v^\mu \mapsto v^\mu + h^{\mu\nu} b_\nu \tag{3.14a}$$
$$h_{\mu\nu} \mapsto h_{\mu\nu} - 2 b_{(\mu}\tau_{\nu)} + h^{\rho\sigma} b_\rho b_\sigma \tau_\mu \tau_\nu \,. \tag{3.14b}$$

These may be identified with the vielbein v^μ and $h_{\mu\nu} \equiv e^a_\mu e_{\nu a}$ of the frame bundle construction. The freedom in choosing the projective inverses can then be traced back to the transformation of the vielbeins under local boost transformations with $b_\rho \equiv v_a e^a_\rho$, though it is not obvious in this language.

Any vector $\boldsymbol{V}$ that satisfies $\boldsymbol{\tau}(\boldsymbol{V}) = 0$ is called a spatial vector, showing that the component $V^0 = 0$ of (3.10a), which is a geometric statement as $\boldsymbol{\tau}$ is a tensor. On the other hand there is no geometric definition of a spatial spacetime one-form $\boldsymbol{U}$ because the action $\boldsymbol{v}(\boldsymbol{U})$ is in general not Galilean invariant; e.g. it might be "spatial" in one frame,

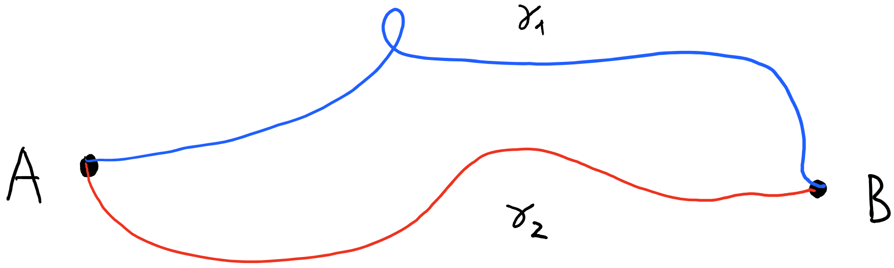

Figure 4: For two curves γ_1 and γ_2, the proper times T_{γ_1}, T_{γ_2} need not be equal.

but we can always boost to a frame where it is "non-spatial". We may however obviously define a time-like covector as any covector that satisfies $\boldsymbol{U}(\boldsymbol{e}_a) = 0$ or equivalently can be written as $\boldsymbol{U} = f(x)\,\boldsymbol{\tau}$.

With the Levi-Civita symbol $\epsilon_{A_0 A_1 \cdots A_D}$ of section 2.3.4 being an invariant symbol of the structure group, it may be used to define natural volume element on M as

$$\boldsymbol{\epsilon} \equiv \frac{1}{D!}\epsilon_{A_0 A_1 \cdots A_d}\boldsymbol{e}^{A_0} \wedge \boldsymbol{e}^{A_1} \wedge \cdots \wedge \boldsymbol{e}^{A_d}\,. \tag{3.15}$$

This will allow us to do integration on the manifold. In local coordinates we may simply write the measure as

$$\boldsymbol{\epsilon} = \det(\boldsymbol{\tau}, \boldsymbol{e}^a)\,\mathrm{d}^D x = e\mathrm{d}^D x\,. \tag{3.16}$$

As $\boldsymbol{\tau}$ is a one-form it may be integrated along some curve γ from a point $A \in M$ to a point $B \in M$. In Newton-Cartan geometry the integral $T_\gamma \equiv \int_\gamma \boldsymbol{\tau}$ is interpreted as the proper time that passes when going along the tangent vector $\dot{\boldsymbol{\gamma}}$ of the curve γ. Notice that the spatial part of $\dot{\boldsymbol{\gamma}}$ does not contribute to T_γ, which shows that if we have a spatial tangent vector, then $T_\gamma = 0$. In local coordinates we may write $\dot{\gamma}^\mu = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}$, and we thus have

$$T_\gamma = \int_\gamma \tau_\mu\left(x^\mu(\lambda)\right)\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda\,. \tag{3.17}$$

Notice that if it happens that if the clock form $\boldsymbol{\tau}$ is closed i.e. $\boldsymbol{\tau} = \mathbf{d}t$ which is generally not the case, then T_γ is independent of which curve γ connecting $A, B \in M$ one chooses by Stokes' theorem. In such spacetimes, we really have a notion of absolute time, which is a hallmark of Newtonian physics. This also shows on the contrary that general Newton-Cartan geometry is beyond Newtonian physics. We will call spacetimes with a closed clock form $\boldsymbol{\tau}$ and their corresponding constraint on Galilean spacetime Augustinian [44].

The inverse spatial metric $\boldsymbol{h}^{-1}$ may also be used to perform types of integration with, given that the Frobenius condition $\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} = 0$ or in local coordinates $\tau_{[\rho}\partial_\mu \tau_{\nu]} = 0$ holds. This is because the Frobenius' theorem states that there exists locally a nice foiliation of spacetime by $\boldsymbol{\tau}$ if and only if $\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} = 0$ [49]. In such cases we can really define absolute space as a spatial hypersurface at a specific time. The pullback φ^* of $\boldsymbol{h}$ to such a spatial hypersurface defines a Riemannian manifold with non-degenerate metric $\varphi^*\boldsymbol{h}$.

Name	Clock form	Absolute time	Absolute space
Leibnizian	$\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} \neq 0$	No	No
Aristotelian	$\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} = 0$	No	Yes
Augustinian	$\mathrm{d}\boldsymbol{\tau} = 0$	Yes	Yes

Table 2: Classification of Newton-Cartan geometries by the properties of the clock form $\boldsymbol{\tau}$.

Here we may apply all of the theorems about Riemannian manifolds, and we shall call such spacetimes and their corresponding structure Aristotelian. If there is no constraint on the clock form and we have the general case $\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} \neq 0$, then we shall call this class of spacetimes Leibnizian.

In total we thus see an obvious classification of Newton-Cartan geometry according to the constraints on $\boldsymbol{\tau}$, which is given in table 2. Now in order to define a proper derivative we must define a connection, which is the subject of the next section, where we will also have more to say about this classification.

3.3 GALILEAN CONNECTIONS

3.3.1 *Setting up the connection*

We now need to define a frame covariant derivative that transforms correctly, i.e. it must commute with the local Galilean transformations of the vielbeins [49, 52, 43]. We may define any Galilean frame covariant derivative on the coframe bundle F^*M by the action on the inverse vielbeins through [42]

$$\mathcal{D}_\mu \boldsymbol{e}^A \equiv -\omega_\mu{}^A{}_B \boldsymbol{e}^B \tag{3.18}$$

where we must have

$$\omega_\mu{}^A{}_B \equiv \begin{pmatrix} 0 & 0 \\ \Omega_\mu{}^a & \omega_\mu{}^a{}_b \end{pmatrix} \tag{3.19}$$

as we see from (3.8) that $\boldsymbol{e}^a$ transforms under both local boosts and rotations, while $\boldsymbol{\tau}$ is invariant. $\Omega_\mu{}^a \equiv \omega_\mu{}^a{}_0$ is the component of the connection associated with local boosts and $\omega_\mu{}^a{}_b$ is the component of the connection associated with local rotations. We must take $\omega_{\mu ab} = -\omega_{\mu ba}$ to have a metric compatible connection analog to the relativistic case in appendix A.2.2. The connection transforms under local Galilean transformations (2.22), (2.23) as

$$\omega_\mu{}^{A'}{}_{B'} = G^{A'}{}_A \omega_\mu{}^A{}_B \left(G^{-1}\right)^B{}_{B'} - \left(G^{-1}\right)^C{}_{B'} \partial_\mu G^{A'}{}_C\,, \tag{3.20}$$

while it transforms as a 1-form under a General Coordinate Transformation (GCT).

From the completeness relations (3.7) we find the action of the covariant derivative on the vielbeins from the above definition. Written in terms of the components we have

$$
\begin{aligned}
\mathcal{D}_\rho \boldsymbol{\tau} &= 0 && (3.21a)\\
\mathcal{D}_\rho \boldsymbol{e}^a &= -\Omega_\rho{}^a \boldsymbol{\tau} - \omega_\rho{}^a{}_b \boldsymbol{e}^b && (3.21b)\\
\mathcal{D}_\rho \boldsymbol{v} &= -\Omega_\rho{}^a \boldsymbol{e}_a && (3.21c)\\
\mathcal{D}_\rho \boldsymbol{e}_a &= \omega_\mu{}^b{}_a \boldsymbol{e}_b . && (3.21d)
\end{aligned}
$$

In a local coordinate basis the spacetime covariant derivative ∇_ρ is defined by its affine connection through the action on (co)tangent vectors as

$$
\nabla_\rho \partial_\mu \equiv \Gamma^\lambda_{\rho\mu} \partial_\lambda \, . \tag{3.22}
$$

One can then derive the transformation law under a GCT similar to (3.20) [53]

$$
\Gamma^{\lambda'}_{\rho'\mu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \Gamma^\lambda_{\rho\mu} + \frac{\partial x^{\lambda'}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^{\rho'} \partial x^{\mu'}} \, . \tag{3.23}
$$

We can then impose the vielbein postulate that identifies the connection of $\mathcal{D}_\rho$ with that of ∇_ρ through the isomorphisms $FM \simeq TM$ and $F^*M \simeq T^*M$ related to (3.10). The statement is then that (inverse) vielbeins are covariantly constant, i.e.

$$
\begin{aligned}
\nabla_\rho \tau_\mu &= \partial_\rho \tau_\mu - \Gamma^\lambda_{\rho\mu} \tau_\lambda \equiv 0 && (3.24a)\\
\nabla_\rho e^a_\mu &= \partial_\rho e^a_\mu - \Gamma^\lambda_{\rho\mu} e^a_\lambda - \Omega_\rho{}^a \tau_\mu - \omega_\rho{}^a{}_b e^b_\mu \equiv 0 && (3.24b)\\
\nabla_\rho v^\mu &= \partial_\rho v^\mu + \Gamma^\mu_{\rho\lambda} v^\lambda - \Omega_\rho{}^a e^\mu_a \equiv 0 && (3.24c)\\
\nabla_\rho e^\mu_a &= \partial_\rho e^\mu_a + \Gamma^\mu_{\rho\lambda} e^\lambda_a + \omega_\rho{}^b{}_a e^\mu_b \equiv 0 \, . && (3.24d)
\end{aligned}
$$

We then don't need to distinguish between the two covariant derivatives any more and we will denote both covariant derivatives by ∇_ρ.

The infinitesimal transformations of the vielbeins and connection are very useful and will be needed later. Performing the linearization of the transformation laws (3.8), (3.20) and including the transformation of the fields under general coordinate transformations we find [21, 26]

$$
\begin{aligned}
\delta \tau_\mu &= \mathcal{L}_\xi \tau_\mu && (3.25a)\\
\delta e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \lambda^a{}_b e^b_\mu + \Lambda^a \tau_\mu && (3.25b)\\
\delta v^\mu &= \mathcal{L}_\xi v^\mu + e^\mu_a \Lambda^a && (3.25c)\\
\delta e^\mu_b &= \mathcal{L}_\xi e^\mu_b + \lambda_b{}^a e^\mu_a && (3.25d)\\
\delta \Omega_\mu{}^a &= \mathcal{L}_\xi \Omega_\mu{}^a + \partial_\mu \Lambda^a + \lambda^a{}_b \Omega_\mu{}^b + \Lambda^b \omega_{\mu b}{}^a && (3.25e)\\
\delta \omega_\mu{}^{ab} &= \mathcal{L}_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} + 2\lambda^{[a}{}_c \omega_\mu{}^{|c|b]} \, , && (3.25f)
\end{aligned}
$$

where Λ_a is an infinitesimal boost parameter, $\lambda_{ab} = -\lambda_{ba}$ parametrizes the rotations and ξ is a vector that generates an infinitesimal coordinate transformation by following the flow along its Lie derivative $\mathcal{L}_\xi$.

Example 3.3 (∇_ρ and connections)**.** When the frame covariant derivative ∇_ρ acts on spacetime tensors, i.e. Galilean scalars, the action is familiar. It is however instructive to see how this works using the Leibniz rule when we write it as Galilean tensors contracted with the vielbeins. If we for example take $U_\mu = U_A e^A_\mu$ we see that the Leibniz rule gives us

$$\begin{aligned}\nabla_\rho U_\mu &= e^A_\mu\left(\nabla_\rho U_A\right) + U_A\left(\nabla_\rho e^A_\mu\right)\\ &= \left(\partial_\rho U_A + \omega_\rho{}^B{}_A U_B\right)e^A_\mu + U_A\left(\partial_\rho e^A_\mu - \Gamma^\lambda_{\rho\mu}e^A_\lambda - \omega_\rho{}^A{}_B e^B_\mu\right)\\ &= \partial_\rho\left(U_A e^A_\mu\right) - \Gamma^\lambda_{\rho\mu}e^A_\lambda U_A\\ &= \partial_\rho U_\mu - \Gamma^\lambda_{\rho\mu}U_\lambda \;\checkmark\,.\end{aligned} \tag{3.26}$$

We could also have used $\nabla_\rho e^A_\mu = 0$ directly to write

$$\begin{aligned}\nabla_\rho U_\mu &= e^A_\mu \nabla_\rho U_A\\ &= e^A_\mu\left(\partial_\rho U_A + \omega_\rho{}^B{}_A U_B\right)\,,\end{aligned} \tag{3.27}$$

which would then give the relation between $\Gamma^\lambda_{\rho\mu}$ and $\omega_\rho{}^B{}_A$ so that both expressions are identical.

3.3.2 *General results about curvatures*

We may solve the vielbein postulates (3.24) for the affine connection in terms of the vielbeins and the gauge connections. One finds

$$\Gamma^\lambda_{\mu\nu} = -v^\lambda\partial_\mu\tau_\nu + e^\lambda_a\left(\partial_\mu e^a_\nu - \Omega^a_\mu\tau_\nu - \omega_\mu{}^a{}_b e^b_\nu\right)\,. \tag{3.28}$$

We give the details of the calculation in appendix D.3.1. The torsion tensor $T_{\mu\nu}{}^\lambda$ is defined as usual as the antisymmetric part of the affine connection

$$T_{\mu\nu}{}^\lambda \equiv 2\Gamma^\lambda_{[\mu\nu]}\,. \tag{3.29}$$

As this is the difference of two connections it is indeed a true tensor. We may split this into a spatial and a temporal torsion in the spirit of Galilean relativity by projecting with the vielbeins as

$$T_{\mu\nu}{}^\lambda = -v^\lambda R_{\mu\nu}(H) + e^\lambda_a R_{\mu\nu}{}^a(P)\,, \tag{3.30a}$$

$$R_{\mu\nu}(H) \equiv \tau_\lambda T_{\mu\nu}{}^\lambda = 2\partial_{[\mu}\tau_{\nu]} \tag{3.30b}$$

$$R_{\mu\nu}{}^a(P) \equiv e^a_\lambda T_{\mu\nu}{}^\lambda = 2\partial_{[\mu}e^a_{\nu]} - 2\Omega_{[\mu}{}^a\tau_{\nu]} - 2\omega_{[\mu}{}^a{}_{|b|}e^b_{\nu]}\,. \tag{3.30c}$$

Name	Clock form	Abs. time	Abs. space	Torsion
TNC	$\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} \neq 0$	No	No	Yes
TTNC	$\boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} = 0$	No	Yes	Yes
TLNC	$\mathrm{d}\boldsymbol{\tau} = 0$	Yes	Yes	No

Table 3: Classification of Newton-Cartan geometries with a connection.

These are identical to the first Cartan structure equation discussed in appendix A.2.2, which for a given connection is an equivalent definition of the torsion tensor. Finally $R_{\mu\nu}(H)$ and $R_{\mu\nu}{}^{a}(P)$ are equal to the field strengths of the vielbeins in the Galilean gauge theory considered by Andringa et al. [21], Hartong et al. [54]. Notice that $R_{\mu\nu}(H)$ is indeed a spacetime tensor which is just the exterior derivative of $\boldsymbol{\tau}$.

If we want an entirely torsionless geometry a necessary but not sufficient condition is that $\partial_{[\mu}\tau_{\nu]} = 0$, i.e. the clock form must be closed and the spacetime Augustinian. This is a constraint on the vielbeins equivalent to the curvature constraint

$$R_{\mu\nu}(H) = 0\,. \tag{3.31}$$

Such a relation that does not have an analog in the case of Riemannian/Lorentzian geometry: Here taking zero torsion does not imply anything on the vielbein. Notice also that unless we take zero temporal torsion $\partial_{[\mu}\tau_{\nu]} = 0$, then vanishing spatial torsion is not a geometric statement as one can just boost to another local Galilean frame where in (3.30a) can see that there will be a new non-zero spatial torsion component.

This property allows us to extend table 2 with information about the connection we may have on such spacetimes, which we have done in table 3. We have here also included the standard nomenclature used in the literature when we have a connection defined. The most general case is also called "Torsional Newton-Cartan" (TNC), the Aristotelian case is also called "Twistless Torsional Newton-Cartan" (TTNC), and finally the Augustinian case is called "Torsionless Newton-Cartan" (TLNC).

The components of the Riemann curvature tensor may be defined conveniently through the usual relation [55] as the failure of two covariant derivatives to commute, i.e.

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] X^{\rho} = R_{\mu\nu\sigma}{}^{\rho} X^{\sigma} - T_{\mu\nu}{}^{\lambda}\nabla_{\lambda} X^{\rho} \tag{3.32a}$$

$$R_{\mu\nu\sigma}{}^{\rho} \equiv -2\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} - 2\Gamma^{\rho}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\sigma}\,. \tag{3.32b}$$

We may split this into a boost curvature a spatial rotation curvature by projecting with the vielbeins as

$$R_{\mu\nu}{}^{a}(B) \equiv -e^{a}_{\rho}v^{\sigma}R_{\mu\nu\sigma}{}^{\rho} = 2\partial_{[\mu}\Omega_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}\Omega_{\nu]b} \tag{3.33a}$$

$$R_{\mu\nu}{}^{a}{}_{b}(J) \equiv e^{a}_{\rho}e^{\sigma}_{b}R_{\mu\nu\sigma}{}^{\rho} = 2\partial_{[\mu}\omega_{\nu]}{}^{a}{}_{b} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]cb}\,. \tag{3.33b}$$

These are identical to the second Cartan structure equation that gives an equivalent definition of the Riemann curvature tensor as discussed in appendix A.2.2. Finally they are equal to the field strengths in the Galilean gauge theory.

We have already seen that if we take the entire torsion (3.29) to be zero, then we must have the surprising necessary constraint $\mathrm{d}\boldsymbol{\tau} = 0$. The next question is of course whether we actually can impose other curvature constraints that allows us to solve for $\omega_\mu{}^a{}_b$, $\Omega_{\mu a}$ in terms of the vielbeins like in the Lorentzian case.

Example 3.4 (Levi-Civita connection in Lorentzian spacetimes)**.** For the Lorentzian manifold of appendix A.2.2 we impose that the connection should be torsionless (i.e. $T_{\mu\nu}{}^\lambda = 0$) which together with the metric compatibility gives the Levi-Civita connection with Christoffel symbols expressed entirely in terms of the metric as the unique solution [52, 53].

We find in our Galilean case the somewhat surprising result that when we try to solve $\Omega_\mu{}^a$, $\omega_\mu{}^a{}_b$ from the torsionless condition $R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = 0$, it is not possible to write a solution in terms of the vielbeins alone. We will thus not be able to realize the geometry on just the vielbeins even in a torsionless geometry unlike for Lorentzian manifolds as in the example above [44, 56]. This can be traced back to the degenerate metric structure. As the connection and its covariant derivative is an additional structure on top of the Galilean manifold, it most be stressed that this does not imply any inconsistency, but it is an aspect that is unfamiliar from the relativistic point of view.

Example 3.5 (Newtonian spacetime)**.** It is of course possible to impose constraints on the Riemann curvature tensor also. One possibility in a torsionless geometry the so-called Duval-Künzle condition [12] given by

$$h^{\lambda[\mu}R_{\lambda(\nu\sigma)}{}^{\rho]} = 0\,, \tag{3.34}$$

which is the relevant condition to impose if one wants a covariantized version of Newtonian gravitation. Here we still need to bring in new fields, as this condition is not enough to realize the geometry on the vielbeins alone [44].

3.3.3 *Hartong-Obers parameterization*

One way to illuminate the degrees of freedom that are in the connection (3.28) is to solve the analog of metric compatibility, i.e.

$$\nabla_\rho \tau_\mu = \nabla_\rho h^{\mu\nu} = 0\,. \tag{3.35}$$

These are equivalent to the vielbeins postulates. The most general connection that solves this is easily found to have an expression for the affine connection given by [26]

$$\Gamma^\lambda_{\mu\nu} = -v^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + W^\lambda{}_{\mu\nu}\,, \tag{3.36}$$

and the object $W^\lambda{}_{\mu\nu}$ must satisfy some constraints

$$\tau_\lambda W^\lambda{}_{\mu\nu} = 0 \tag{3.37a}$$

$$h^{\nu(\rho} W^{\lambda)}{}_{\mu\nu} = 0\,. \tag{3.37b}$$

We shall denote "the affine (natural) pseudo-connection" $\hat{\Gamma}^{\lambda}_{\mu\nu}$ defined by the first terms in (3.36)

$$\hat{\Gamma}^{\lambda}_{\mu\nu} \equiv -v^{\lambda}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\lambda\sigma}\left(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right) . \tag{3.38}$$

This is because it does not define a good connection on its own exactly because it transforms under boosts, while the whole connection $\Gamma^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} + W^{\lambda}{}_{\mu\nu}$ must be boost invariant. This equivalent to stating that the covariant derivative must be independent of the chosen frame or a geometric object, which is a fundamental property of connections. We must thus require

$$\delta_B\Gamma^{\lambda}_{\mu\nu} = \delta_B\hat{\Gamma}^{\lambda}_{\mu\nu} + \delta_B W^{\lambda}{}_{\mu\nu} = 0\,, \tag{3.39}$$

This shows that $W^{\lambda}{}_{\mu\nu}$ is not a tensor[3] and must be non-zero even when the connection is torsionless, which in another way proves that the geometry cannot be realized on the vielbeins alone. $\hat{\Gamma}^{\lambda}_{\mu\nu}$ and its associated pseudo-gauge fields $\hat{\Omega}_{\mu}{}^{a}$, $\hat{\omega}_{\mu}{}^{a}{}_{b}$ are however useful quantities. Hartong and Obers [26] displays the structure of $W^{\lambda}{}_{\mu\nu}$ by writing it in terms of invariant non-tensorial objects $Y_{\sigma\mu\nu}$, $K_{\mu\nu}$ and $L_{\sigma\mu\nu}$ defined as

$$W^{\lambda}{}_{\mu\nu} = \frac{1}{2}h^{\lambda\sigma}Y_{\sigma\mu\nu} \tag{3.40a}$$

$$Y_{\sigma\mu\nu} \equiv \tau_{\nu}K_{\sigma\mu} + \tau_{\mu}K_{\sigma\nu} + L_{\sigma\mu\nu} \tag{3.40b}$$

$$K_{\mu\nu} = -K_{\nu\mu} \quad , \quad L_{\sigma\rho\lambda} = -L_{\lambda\rho\sigma}\,. \tag{3.40c}$$

$K_{\mu\nu}$, $L_{\sigma\rho\lambda}$ parametrizes all the possible "metric compatible" connections we may define on our Galilean manifold. Under an infinitesimal local boost with parameter $\Lambda_{\sigma} \equiv \Lambda_a e^a_{\sigma}$ we find

$$\begin{aligned}\delta_B\Gamma^{\rho}_{\mu\nu} = {} & \frac{1}{2}h^{\rho\sigma}\tau_{\mu}\left(\delta_B K_{\sigma\nu} + \partial_{\nu}\Lambda_{\sigma} - \partial_{\sigma}\Lambda_{v}\right) + \frac{1}{2}h^{\rho\sigma}\tau_{\nu}\left(\delta_B K_{\sigma\mu} + \partial_{\mu}\Lambda_{\sigma} - \partial_{\sigma}\Lambda_{\mu}\right) \\ & + \frac{1}{2}h^{\rho\sigma}\left(\delta_B L_{\sigma\mu\nu} - \Lambda_{\sigma}\left(\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}\right) + \Lambda_{\mu}\left(\partial_{\nu}\tau_{\sigma} - \partial_{\sigma}\tau_{\nu}\right) + \Lambda_{\nu}\left(\partial_{\mu}\tau_{\sigma} - \partial_{\sigma}\tau_{\mu}\right)\right) ,\end{aligned} \tag{3.41}$$

and thus to ensure $\delta_B\Gamma^{\lambda}_{\mu\nu} = 0$ we see that they must have the transformation property under boosts:

$$\begin{aligned}\delta_B K_{\sigma\nu} &= \partial_{\sigma}\Lambda_{v} - \partial_{\nu}\Lambda_{\sigma} && (3.42a)\\ \delta_B L_{\sigma\mu\nu} &= \Lambda_{\sigma}\left(\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}\right) - \Lambda_{\mu}\left(\partial_{\nu}\tau_{\sigma} - \partial_{\sigma}\tau_{\nu}\right) - \Lambda_{\nu}\left(\partial_{\mu}\tau_{\sigma} - \partial_{\sigma}\tau_{\mu}\right) . && (3.42b)\end{aligned}$$

Equating (3.28) and (3.36) we may derive expressions for the boost and rotation gauge fields given by

3 It still transforms correctly under GCTs, but it has additional local Galilean transformations.

$$\begin{aligned} \Omega_{\mu a} &= \hat{\Omega}_{\mu a} + C_{\mu a} && (3.43a) \\ \omega_{\mu ab} &= \hat{\omega}_{\mu ab} + C_{\mu ab}\,, && (3.43b) \end{aligned}$$

where $\hat{\Omega}_{\mu a}$ and $\hat{\omega}_{\mu ab}$ are the pseudo-gauge fields of the pseudo-connection (3.38) derived by using the linearity in the gauge fields of the affine connection (3.28)

$$\begin{aligned} \hat{\Omega}_{\mu a} &\equiv v^\nu \partial_{[\nu} e_{\mu]}^{\ a} + v^\nu e^{\sigma a} e_{\mu b} \partial_{[\nu} e_{\sigma]}^{\ b} && (3.44a) \\ \hat{\omega}_{\mu ac} &\equiv e^\lambda{}_{[a|} \partial_\lambda e_{\mu|c]} - e^\lambda{}_{[a|} \partial_\mu e_{\lambda|c]} - e_{\mu b} e^\sigma{}_{[a} e^\lambda{}_{c]} \partial_\lambda e_\sigma{}^b\,, && (3.44b) \end{aligned}$$

and we have defined

$$\begin{aligned} C_{\mu a} &\equiv -v^\nu e_{\lambda a} W^\lambda{}_{\mu\nu} && (3.45a) \\ C_{\mu ac} &\equiv e^\nu{}_c e_{\lambda a} W^\lambda{}_{\mu\nu}\,. && (3.45b) \end{aligned}$$

These objects are like the contortion tensor in Lorentzian manifolds, where it is well-known that any connection may be written as the sum of the Levi-Civita connection plus the contortion tensor as is reviewed in appendix A.2.2. For this reason we shall call $C_{\mu a}$, $C_{\mu ac}$ pseudo-contortions, which by their transformation properties derived from $W^\lambda{}_{\mu\nu}$ are not tensors, but yet play a similar role yielding any connection when added to the natural pseudo-connection. As $W^\lambda{}_{\mu\nu}$ may not vanish, neither may the pseudo-contortions.

3.3.3.1 *Connections linear in a background field M_μ*

There is a whole class of solutions for $K_{\mu\nu}$, $L_{\sigma\rho\lambda}$ expressed in terms of a background field M_μ that is defined to transform under Galilean transformations as

$$M'_\mu = M_\mu + v_a R^a{}_b e^b_\mu + \frac{1}{2} v^a v_a \tau_\mu\,. \qquad (3.46)$$

M_μ is sometimes called the graviphoton field in the literature because of some of its properties [57]. At this stage it might seem a bit arbitrary to introduce such a field to the geometry but we will see that it is relevant in many situations as we will see in section 3.4 and chapter 6. The infinitesimal version of this transformation law including diffeomorphisms on top of those of the vielbeins and gauge fields (3.25) is given by

$$\delta M_\mu = \mathcal{L}_\xi M_\mu + e^a_\mu \Lambda_a\,. \qquad (3.47)$$

From this transformation we see that its spacetime derivative must include the boost connection and be given by

$$\nabla_\rho M_\mu = \partial_\rho M_\mu - \Gamma^\lambda_{\rho\mu} M_\lambda - \Omega_{\rho b} e^b_\mu\,. \qquad (3.48)$$

If we take

$$\begin{aligned} K_{\sigma\rho} &= 2\partial_{[\sigma}M_{\rho]} && (3.49a)\\ L_{\sigma\mu\nu} &= 2M_\sigma\partial_{[\mu}\tau_{\nu]} - 2M_\mu\partial_{[\nu}\tau_{\sigma]} + 2M_\nu\partial_{[\sigma}\tau_{\mu]}\,, && (3.49b)\end{aligned}$$

then this is seen to define a good boost invariant connection [58, 59]. This choice actually corresponds to the unique connection linear in M_μ [54]. We will later see how this connection in the sense of being the physical relevant connection can be claimed to be the closest thing to a "Levi-Civita"-like connection we may have for general (torsionful) Newton-Cartan geometry.

We may construct some new boost invariant objects using M_μ:

$$\begin{aligned} \hat{v}^\mu &\equiv v^\mu - h^{\mu\lambda}M_\lambda && (3.50a)\\ \hat{e}^a_\mu &\equiv e^a_\mu - M_\mu e^{\mu a}\tau_\mu && (3.50b)\\ \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} + 2\tau_{(\mu}M_{\nu)}\,. && (3.50c)\end{aligned}$$

$\bar{h}_{\mu\nu}$ is non-degenerate now but still not the inverse of $h^{\mu\nu}$ and the new vielbeins-like objects still satisfy completeness relations analog to (3.7). We should not regard them as true vielbeins as they do not have the correct transformation properties under boosts any longer, but they are indeed useful objects in many ways as we shall see. As an aside we could also imagine breaking boost covariance in some way, which would reduce the symmetries of a given theory to that of the Lifshitz group or the Euclidean group studied in chapter 2 and make these objects good vielbeins and tensors.

The new tensorial quantities are useful to build connections with the required properties from, and one may see that the choice corresponding to (3.49) can be written as

$$\mathring{\Gamma}^\lambda_{\mu\nu} = -\hat{v}^\lambda\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\sigma}\left(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}\right)\,. \tag{3.51}$$

We shall call this connection[4] the "graviphotonic (Galilean) connection". Its torsion is given by

$$T_{\mu\nu}{}^\lambda = -2\hat{v}^\lambda\partial_{[\mu}\tau_{\nu]}\,, \tag{3.52}$$

and from this we see that the spatial torsion is given by

$$R_{\mu\nu}{}^a\,(P) = 2e^{\lambda a}M_\lambda\partial_{[\mu}\tau_{\nu]}\,. \tag{3.53}$$

The expression for $W^\lambda{}_{\mu\nu}$ and the pseudo-contortions (3.43) may be used to write the gauge fields (3.43) in terms of only M_μ and the vielbeins.

4 This connection, despite being well-known in the literature, has not been given a name. Calling it the "graviphotonic connection" is hence not standard terminology, but is motivated by the fact that it has a U(1) gauge transformation, while still being a non-trivial part of the geometry as we shall see.

3.3.3.2 *More general connections involving M_μ*

There are also more general choices, which however all are not linear in M_μ [26]. To see this, let us define two other useful objects as

$$\begin{aligned} \tilde{\Phi} &\equiv -v^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu && \text{(3.54a)} \\ M_a &\equiv M_\mu e^\mu_a && \text{(3.54b)} \end{aligned}$$

where $\tilde{\Phi}$ is indeed a true spacetime scalar, while M_a transforms under both rotations and boosts. It is easy to see that we may write

$$M_\mu = M_a e^a_\mu - \frac{1}{2} M_a M^a \tau_\mu + \tilde{\Phi} \tau_\mu \,. \tag{3.55}$$

We can define a whole class of Hartong-Obers connections parametrized by $\alpha \in \mathbb{R}$ by taking

$$\Gamma^\lambda_{\mu\nu} \equiv -\hat{v}^\lambda \partial_\mu \tau_\nu + \frac{1}{2} h^{\lambda\sigma} \left(\partial_\mu H_{\nu\sigma}(\alpha) + \partial_\nu H_{\mu\sigma}(\alpha) - \partial_\sigma H_{\mu\nu}(\alpha) \right) \tag{3.56a}$$

$$H_{\mu\nu}(\alpha) \equiv \bar{h}_{\mu\nu} + \alpha \tilde{\Phi} \tau_\mu \tau_\nu \tag{3.56b}$$

Notice that $\Gamma^\lambda_{\mu\nu}$ is boost and rotation invariant no matter how we choose α as it is build from tensorial quantities. If we take $\alpha = 0$, we obtain the connection (3.51), while other choices gives connections non-linear in M_μ. Another interesting choice is $\alpha = 2$, which is equivalent to taking $H_{\mu\nu}(2) = \hat{h}_{\mu\nu} \equiv \hat{e}^a_\mu \hat{e}_{\nu a}$.

This class certainly does not span all of the possible Galilean connections, as we may always add a tensor to the affine connection to obtain a new one.

3.3.4 *The Newton-Coriolis two-form*

There is also another way of presenting the degrees of freedom in the connection which is due to Geracie et al. [42]. This is equivalent to to the Hartong-Obers classification, and we may go from one formulation to another using some formulas given later. Starting again from the vielbein postulate (3.24), we may write the affine connection as

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} = -v^\lambda \partial_{(\mu} \tau_{\nu)} &+ \frac{1}{2} h^{\lambda\sigma} \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} \right) \\ &+ \frac{1}{2} \left(T_{\mu\nu}{}^\lambda - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu \right) + \tau_{(\mu} C^\lambda_{\nu)} \,, \end{aligned} \tag{3.57}$$

where one has the torsion tensor $T_{\mu\nu}{}^\lambda$ defined in (3.30a) and versions of it where the last two indices are raised by[5] $h^{\mu\nu}$ and lowered by $h_{\mu\nu}$ as

$$T_\mu{}^\sigma{}_\rho \equiv h_{\rho\lambda} h^{\sigma\nu} T_{\mu\nu}{}^\lambda \,. \tag{3.58}$$

5 We remind the reader that this is not a bijective mapping.

The other new object is the so-called Newton-Coriolis two-form $C_{\mu\nu}$ given by

$$C_{\mu\nu} \equiv 2\Omega_{[\mu}{}^{a} e_{\nu]a} \tag{3.59a}$$
$$C_{\nu}{}^{\lambda} \equiv C_{\nu\mu} h^{\mu\lambda} . \tag{3.59b}$$

The Newton-Coriolis two-form is not really a two-form as it transforms under finite local Galilean boosts parametrized by v^a in a non-trivial way:

$$C'_{\mu\nu} = -\partial_{[\mu} \left(v^a e_{\nu]a} + \frac{1}{2} v^2 \tau_{\nu]} \right) + \frac{1}{2} v^2 \partial_{[\mu} \tau_{\nu]} - v_a T_{\mu\nu}{}^{a} , \tag{3.60}$$

where $T_{\mu\nu}{}^{a}$ is the spatial torsion (3.30c). The connection (3.57) is invariant under boosts, but each term (except for the torsion tensor) will transform in exactly such a way that all non-invariances cancel as they should. The freedom in choosing the connection is parametrized by the Newton-Coriolis two-form $C_{\mu\nu}$ and the torsion. In this form it is easier to see what the various choices one can make do to the connection. Especially when the connection is torsionless, all of the ambiguity in the connection is parametrized by $C_{\mu\nu}$. For example, the Duval-Künzle condition (3.34) for a torsionless spacetime implies that $\mathrm{d}\boldsymbol{C} = 0$.

We may translate these results to those of section 3.3.3. Equating the two affine connections (3.57) and (3.36), and solving for $W^{\lambda}{}_{\mu\nu}$ gives after some work

$$W^{\lambda}{}_{\mu\nu} = v^{\lambda} \partial_{[\mu} \tau_{\nu]} + \frac{1}{2} \left(T^{\lambda}{}_{\mu\nu} - T_{\mu\nu}{}^{\lambda} - T_{\nu\mu}{}^{\lambda} \right) + \tau_{(\mu} C_{\nu)}^{\lambda} . \tag{3.61}$$

On the other hand we may also revert the formulas and write the Newton-Coriolis form in terms of $K_{\mu\nu}$, $L_{\mu\nu\lambda}$ as

$$C_{\mu\nu} = -v^{\lambda} e_{[\mu|a} \partial_{\lambda} e_{|\nu]}^{a} + \partial_{[\mu} v^{\lambda} e_{\nu]a} e_{\lambda}{}^{a} + v^{\lambda} L_{[\mu\nu]\lambda} - K_{\mu\nu} , \tag{3.62}$$

so we may translate between the two classifications at will. For our special graviphotonic connection (3.51), one sees in particular that the Newton-Coriolis 2-form is not closed, but neither does it have any simple expression, which underlines that the two classifications are not easily related.

3.4 BARGMANN SPACETIMES AND THE GRAVIPHOTON

3.4.1 *Extended Bargmann frame bundles*

We want to perform a fiber bundle construction for the Bargmann group (2.4) similar to what we did for the Galilean group in section 3.1. The relevant fiber bundle $\mathring{F}M$ will now be constructed as a $2D+1$-dimensional manifold because translations and the central transformation are given by the left coset $\mathbb{R}^{d+1} \ltimes \mathbb{R} \simeq \mathrm{Barg}\,(d,1)\,/\mathrm{HGal}\,(d,1)$ [42, 60]. One can then identify a set of $D+1$-dimensional "extended vielbeins" living on a D-dimensional manifold M again transforming under local Galilean boosts and rotations. There is a natural fiber bundle construction that is obviously going to be different from the Galilean case, but we will see that they are closely related.

The fiber $\mathring{F}_p = \left\{\mathring{e}_{\hat{A}}(p)\right\}$ now consists of all bases of the (Bargmann) vector space $\mathbb{R}^{D+1}$ that we discussed on page 17. A certain section is then a particular extended vielbein at a point $p \in M$, i.e. $\mathring{\sigma}(p) = \left(p, \mathring{e}_{\hat{A}}(p)\right)$. We can then write the extended fiber bundle $\mathring{F}M$ as a global trivialization

$$\mathring{F}M = \bigcup_{p\in M} \left(p, \{\mathring{e}_{\hat{A}}(p)\}\right) . \tag{3.63}$$

This construction is a principal bundle because the diffeomorphisms of the fiber are exactly given by the structure group $\mathrm{HGal}(d,1)$, albeit in the non-fundamental representation (2.35):

$$\mathring{e}_{\hat{A}}(p) = \mathring{e}_{\hat{B}}(p)\left(B^{-1}(p)\right)^{\hat{B}}{}_{\hat{A}} . \tag{3.64}$$

A similar construction can be performed for the extended cofiber bundle $\mathring{F}^*M$ of extended inverse vielbeins:

$$\mathring{F}^*M = \bigcup_{p\in M} \left(p, \left\{\mathring{e}^{\hat{A}}(p)\right\}\right) , \tag{3.65}$$

$$\mathring{e}'^{\hat{A}}(p) = B(p)^{\hat{A}}{}_{\hat{B}}\,\mathring{e}^{\hat{B}}(p) . \tag{3.66}$$

3.4.2 *The relation to the background field M_μ*

The extended (inverse) vielbeins does not constitute a frame because there are $D+1$ components of $\mathring{e}_{\hat{A}}(p)$ and $\mathring{e}^{\hat{A}}(p)$, which does not match with the D dimensions of M and its (co)tangent space. We may however consistently identify the first D components of the extended inverse vielbein with the true inverse vielbeins $\boldsymbol{e}^A = (\boldsymbol{\tau}, \boldsymbol{e}^a)$ of section 3.1. This is because the projector $\mathscr{P}^A{}_{\hat{A}}$ defined in (2.37) may be used to give $\boldsymbol{e}^A$ as the projection

$$\boldsymbol{e}^A = \mathscr{P}^A{}_{\hat{A}}\mathring{e}^{\hat{A}} , \tag{3.67}$$

and we showed in example 2.3 that this projection commutes with the local Galilean transformations. We can therefore write a particular section as

$$\mathring{e}^{\hat{A}} \equiv (\boldsymbol{\tau}, \boldsymbol{e}^a, -\boldsymbol{M}) . \tag{3.68}$$

The extra field $\boldsymbol{M}$ is a part of the extended coframe but not the true frame. We can interpret it as a background field on the manifold M that is a non-trivial part of the geometry. Using (2.35) it can be seen to transform under local Galilean transformations as

$$\boldsymbol{M}' = \boldsymbol{M} + \frac{1}{2}v^2\boldsymbol{\tau} + v_a R^a{}_b \boldsymbol{e}^b . \tag{3.69}$$

This is identical to the transformation background field M_μ (3.46), and we may thus regard this construction identical to the one in section 3.3.3.1. This explains the origin of M_μ, which was not clear at that point. From the gauging of the Bargmann group studied by Hartong and Obers [26] M_μ has a natural interpretation as the gauge field corresponding to the central charge, which we shall also see the equivalent of in our framework later. It is however not yet clear how they might result in the same graviphotonic connection (3.51), which will be the subject of the next section.

For $\mathring{e}_{\hat{A}}$ a projection that would allow it to serve as a true vielbeins does not exist because $\mathscr{P}^A{}_{\hat{A}}$ is not a bijective mapping. We can instead define the vielbeins $\boldsymbol{e}_A = (-\boldsymbol{v}, \boldsymbol{e}_a)$ as its inverse of a particular section $\boldsymbol{e}^A$ through the completeness relations (3.7). $\boldsymbol{e}_A$ may now be lifted to the extended frame as

$$\boldsymbol{e}_{\hat{A}} \equiv \boldsymbol{e}_A \mathscr{P}^A{}_{\hat{A}} = (-\boldsymbol{v}, \boldsymbol{e}_a, \boldsymbol{0}) \ . \tag{3.70}$$

This is clearly not a section of the extended frame bundle because it has a zero basis element.

We may then proceed as in section 3.1 and define the inverse spatial metric as $\boldsymbol{h} = \delta^{AB}\boldsymbol{e}_A\boldsymbol{e}_B$ and together with the clock form $\boldsymbol{\tau}$ this is by definition 3.2 a Galilean manifold.

3.5 BARGMANN CONNECTIONS

3.5.1 *Relation to the graviphotonic connection*

We may again define a covariant derivative on the fiber bundle similar to how we did in the Galilean case of section 3.3. We define a connection through its action on the extended vielbeins:

$$\hat{\mathcal{D}}_\mu \boldsymbol{e}^{\hat{A}} \equiv \omega_\mu{}^{\hat{A}}{}_{\hat{B}} \boldsymbol{e}^{\hat{B}} \tag{3.71}$$

and see that we must have

$$\omega_\mu{}^{\hat{A}}{}_{\hat{B}} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \Omega_\mu{}^a & \omega_\mu{}^a{}_b & 0 \\ 0 & -\Omega_{\mu b} & 0 \end{pmatrix} , \tag{3.72}$$

because the extra component $\boldsymbol{M}$ of the extended frame transforms under boosts as it is seen from (2.35). Written explicitly, the additional covariant derivative in addition to (3.21) is

$$\hat{\mathcal{D}}_\mu \boldsymbol{M} = \Omega_{\mu b} \boldsymbol{e}^b \,. \tag{3.73}$$

This will lead to the same spacetime covariant derivative of M_μ as given in (3.48), and it is in particular not related to any vielbein postulate in this case.

There is a related "graviphotonic torsion" defined from the extra component of the Cartan structure equation (A.10) for the extended vielbeins as

$$R_{\mu\nu}(M) \equiv 2\partial_{[\mu} M_{\nu]} - 2\Omega_{[\mu}{}^a e_{\nu]a} \,, \tag{3.74}$$

which is also the field strength in the Bargmann gauge theory extra to (3.30). Notice also that this can be written in terms of the Newton-Coriolis 2-form $C_{\mu\nu}$ defined in (3.62) as

$$R_{\mu\nu}(M) = 2\partial_{[\mu}M_{\nu]} + C_{\mu\nu}\,. \tag{3.75}$$

The spacetime torsion is independent of what the graviphotonic torsion is, but $R_{\mu\nu}(M)$ does transform under boosts, unless the spacetime torsion vanishes. Hence it is not a geometric statement to set $R_{\mu\nu}(M) = 0$ unless there is no torsion as we may always boost to another frame where it is non-zero.

We can certainly make curvature constraints, but they need to be boost covariant wrt. the transformation of $R_{\mu\nu}(M)$. An example of such curvature constraints are given by fixing the spatial and graviphotonic torsions as

$$R_{\mu\nu}{}^{a}(P) = 2e^{\lambda a}M_{\lambda}\partial_{[\mu}\tau_{\nu]} \tag{3.76a}$$
$$R_{\mu\nu}(M) = 2v^{\lambda}M_{\lambda}\partial_{[\mu}\tau_{\nu]}\,. \tag{3.76b}$$

Temporal torsion is already fixed by the general result (3.30b), the spatial torsion constraint is just (3.53) and thus we have fixed all of the torsions. The connection corresponding to this choice is exactly the graviphotonic connection (3.51). This shows that it is possible to realize the local Galilean symmetry on the extended vielbeins $\hat{\boldsymbol{e}}^{\hat{A}} = (\boldsymbol{\tau}, \boldsymbol{e}^{a}, -\boldsymbol{M})$ and $\boldsymbol{e}_{\hat{A}} = (\boldsymbol{v}, \boldsymbol{e}_{a}, \boldsymbol{0})$ that are natural objects of the extended Bargmann frames. With the simple curvature constraints (3.76) this shows that the origin of the graviphotonic connection and the field M_μ introduced there have natural interpretations. This also allows us to substitute the complicated expression for $C_{\mu\nu}$ (3.62) with the more simple one

$$C_{\mu\nu} = 2v^{\lambda}M_{\lambda}\partial_{[\mu}\tau_{\nu]} - 2\partial_{[\mu}M_{\nu]}\,. \tag{3.77}$$

In particular this shows that $C_{\mu\nu}$ is only closed when $\mathrm{d}\boldsymbol{\tau} = 0$.

3.6 PARALLEL TRANSPORT AND GEODESICS

The parallel transport equation for any rank (p,q) spacetime tensor $\boldsymbol{T}$ with components $T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q}$ is simply the requirement that it stays covariantly constant when moved along a curve γ parametrized by λ:

$$\frac{\mathrm{D}\boldsymbol{T}}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}x^{\mu}(\lambda)}{\mathrm{d}\lambda}\nabla_{\mu}T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} = 0\,. \tag{3.78}$$

For a general parallel transport the effect of torsion is measurable. The clock form and inverse spatial metric are always parallel transported along any curve as we see from (3.24) that

$$\nabla_{\rho}\tau_{\mu} = \nabla_{\rho}h^{\mu\nu} = 0\,. \tag{3.79}$$

The geodesic equation is found by considering autoparallel curves, i.e.

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}^2x^{\mu}(\lambda)}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{\mathrm{d}x^{\rho}(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\sigma}(\lambda)}{\mathrm{d}\lambda} = 0\,. \tag{3.80}$$

Here torsion does not play a role. The major difference compared to the Lorentzian case is that solutions of the geodesic equation no longer in general can be identified with the stationary point of some spacetime length functional as we do not have the analog of a Levi-Civita connection. In some cases special connections may be associated with the geodesic equation [61, 62].

3.7 FLAT NEWTON-CARTAN GEOMETRY

Flat Newton-Cartan geometry is an interesting and relevant special case of the theory derived so far. By a flat connection we mean one where there exists a (usually globally defined) set of vielbeins where the boost and rotation gauge fields in (3.18) vanishes, i.e. [49]

$$\omega_{\mu}{}^{A}{}_{B} = 0\,. \tag{3.81}$$

In this case the Riemann curvature $R_{\mu\nu\sigma}{}^{\rho} = 0$ and the covariant derivatives of the vielbeins (3.24) becomes

$$\begin{aligned}
\nabla_{\rho}\tau_{\mu} &= \partial_{\rho}\tau_{\mu} - \Gamma^{\lambda}_{\rho\mu}\tau_{\lambda} = 0 && (3.82a)\\
\nabla_{\rho}e_{\mu}{}^{a} &= \partial_{\rho}e_{\mu}{}^{a} - \Gamma^{\lambda}_{\rho\mu}e_{\lambda}{}^{a} = 0 && (3.82b)\\
\nabla_{\rho}v^{\mu} &= \partial_{\rho}v^{\mu} + \Gamma^{\mu}_{\rho\lambda}v^{\lambda} = 0 && (3.82c)\\
\nabla_{\rho}e^{\mu}{}_{a} &= \partial_{\rho}e^{\mu}{}_{a} + \Gamma^{\mu}_{\rho\lambda}e^{\lambda}{}_{a} = 0\,. && (3.82d)
\end{aligned}$$

In such a geometry this has the consequence that there is no change of vectors as they are parallel transported along any closed curve and for geodesics initially parallel, they will stay so. There could in principle still be non-zero torsion (3.29). Taking this to zero is seen from (3.82) to imply that $\partial_{[\mu}e^{A}_{\nu]} = 0$. A Galilean connection with no torsion and curvature is what we will define as a flat Newton-Cartan geometry.

Let us further assume that the flat frame has the property that the symmetric part of the affine connection is zero so we have $\Gamma^{\lambda}_{\rho\mu} = 0$ globally. Such a frame is a global inertial frame where the covariant derivatives (3.82) implies

$$\partial_{\mu}e^{A}_{\nu}(x) = 0 \quad \Leftrightarrow \quad e^{A}_{\mu}(x) = e^{A}_{\mu}\,. \tag{3.83}$$

It is now possible to do global Galilean transformations so that we may always choose the vielbeins as

$$\begin{aligned}
\tau_{\mu} &\equiv \delta^{0}_{\mu} && (3.84a)\\
v^{\mu} &\equiv -\delta^{\mu}_{0} && (3.84b)\\
e_{\mu}{}^{a} &\equiv \delta^{a}_{\mu} && (3.84c)\\
e^{\mu}{}_{a} &\equiv \delta^{\mu}_{a}\,. && (3.84d)
\end{aligned}$$

This is a global inertial (co)frame where the associated global inertial coordinates $x^{\mu} = (t, x^{i})$ are natural to use for non-relativistic field theories [63]. The spatial metric and its inverse then take the simple expressions

$$h_{\mu\nu} = \delta_{ab}\delta^a_\mu\delta^b_\nu \tag{3.85a}$$
$$h^{\mu\nu} = \delta^{ab}\delta^\mu_a\delta^\nu_b\,, \tag{3.85b}$$

which as we would expect for a Aristotelian connection gives a good Euclidean metric on spatial hypersurfaces.

For the graviphotonic connection (3.51) this does not imply that $M_\mu = 0$. Rather we see from the choice $K_{\sigma\rho} = 2\partial_{[\sigma}M_{\rho]}$ in the general Hartong-Obers classification of section 3.3.3 that for a flat geometry we can only say that

$$M_\mu = \partial_\mu M\,, \tag{3.86}$$

where M is an arbitrary scalar. The minimal choice of $M = 0$ is certainly valid in a particular global inertial frame. However, if we perform a local Galilean transformation that preserves global inertial observers (3.84) and (3.86) this may in principle transform $M \neq 0$ in some other frame. The various possibilities are called the orbits of M and are studied further in [54].

Example 3.6 (Uniformly accelerated frames)**.** The geometric framework makes it easy to formulate physics in arbitrary coordinates by the general covariant structure. Say we start in a flat Newton-Cartan in a global inertial frame (3.84) where the geodesic equation (3.80) is $\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = 0$, and now want to go to a uniformly accelerated frame, which is just the coordinate transformation

$$t' = t \tag{3.87a}$$
$$x'^i = x^i + \frac{1}{2}a^i t^2\,. \tag{3.87b}$$

The affine connection transforms non-tensorially as (3.23) which here is simply $\Gamma^{\lambda'}_{\mu'\nu'} = -\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu}$. The only non-zero components are

$$\Gamma^i_{00} = -a^i\,. \tag{3.88}$$

The geodesic equation with time as the parameter now becomes

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = a^i\,. \tag{3.89}$$

This shows that we have a fictitious force in an accelerated frame, completely equivalent to what we find in Newtonian mechanics.

3.8 LINEARIZED NEWTON-CARTAN GEOMETRY

3.8.1 *Linearizing the vielbeins*

It is useful to have a linearized version of the results derived above as the full non-linear theory can be rather complicated for some applications. We will consider small perturbations to global inertial frames of the flat geometry considered in the former section and keep everything at first order. We take

$$\begin{aligned} \tau_\mu &\equiv \delta^0_\mu + \bar{\tau}_\mu && (3.90a)\\ v^\mu &\equiv -\delta^\mu_0 - \bar{v}^\mu && (3.90b)\\ e_\mu{}^a &\equiv \delta^a_\mu + \bar{e}_\mu{}^a && (3.90c)\\ e^\mu{}_a &\equiv \delta^\mu_a - \bar{e}^\mu{}_a && (3.90d) \end{aligned}$$

where $\bar{e}_\mu{}^a$, $\bar{e}^\mu{}_a$, $\bar{\tau}_\mu$, $\bar{v}^\mu$ are the perturbations. The completeness relations 3.7 must still hold at first order, which implies some relations between the vielbeins and their inverses:

$$\begin{aligned} \bar{\tau}_0 &= -\bar{v}^0 && (3.91a)\\ \bar{e}^a_b &= \bar{e}_b{}^a && (3.91b)\\ \bar{e}^0{}_a &= \bar{\tau}_a && (3.91c)\\ \bar{e}_0{}^a &= -\bar{v}^a\,. && (3.91d) \end{aligned}$$

This shows that the linearized inverse vielbeins are completely determined in terms of the linearized vielbeins themselves. The spatial metric can then be expressed in terms of the vielbeins up to first order as

$$\begin{aligned} \bar{h}_{\mu\nu} &= \delta_{ab}\delta^a_\mu\delta^b_\nu + \bar{s}_{\mu\nu} = \begin{pmatrix} 0 & -\bar{v}_b \\ -\bar{v}_a & \delta_{ab} + \bar{s}_{ab} \end{pmatrix} && (3.92a)\\ \bar{h}^{\mu\nu} &= \delta^{ab}\delta^\mu_a\delta^\nu_b - \bar{s}^{\mu\nu} = \begin{pmatrix} 0 & -\bar{\tau}^b \\ -\bar{\tau}^a & \delta^{ab} - \bar{s}^{ab} \end{pmatrix}, && (3.92b) \end{aligned}$$

where we have defined the perturbations of the spatial metrics as

$$\begin{aligned} \bar{s}_{\mu\nu} &\equiv \bar{e}_\mu{}^a\delta_{\nu a} + \delta^a_\mu\bar{e}_{\nu a} = 2\delta^a_{(\mu}\bar{e}_{\nu)a} && (3.93a)\\ \bar{s}^{\mu\nu} &\equiv \bar{e}^\mu{}_a\delta^{\nu a} + \delta^\mu_a\bar{e}^{\nu a} = 2\delta^{(\mu}_a\bar{e}^{\nu)a}\,. && (3.93b) \end{aligned}$$

Because of the constraint (3.91b) we have that spatial components satisfy $s^{ab} = s_{ab}$, and all spatial indices may be raised or lowered by the flat spatial metrics (3.85).

3.8.2 *Linearizing the Connections*

The affine pseudo-connection (3.38) is straight-forward to linearize using (3.90) and we find:

$$\hat{\bar{\Gamma}}^{\lambda}_{\mu\nu} = \delta^{\lambda}_{0}\partial_{\mu}\bar{\tau}_{\nu} + \frac{1}{2}\delta^{\lambda a}\delta^{\sigma}_{b}\left(\partial_{\mu}\bar{s}_{\nu\sigma} + \partial_{\nu}\bar{s}_{\mu\sigma} - \partial_{\sigma}\bar{s}_{\mu\nu}\right) . \tag{3.94}$$

We can then write the linearized Hartong-Obers connection (3.36):

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \hat{\bar{\Gamma}}^{\lambda}_{\mu\nu} + \bar{W}^{\lambda}{}_{\mu\nu} \tag{3.95a}$$

$$\bar{W}^{\lambda}{}_{\mu\nu} = \frac{1}{2}\delta^{\lambda a}\left(\delta^{0}_{\mu}\bar{K}_{a\nu} + \delta^{0}_{\nu}\bar{K}_{a\mu} + \bar{L}_{a\mu\nu}\right) , \tag{3.95b}$$

where $\bar{K}_{\rho\nu}$, $\bar{L}_{\rho\mu\nu}$ are the linearized versions of the objects introduced in the Hartong-Obers parameterization in section 3.3.3.

For the boost and rotation gauge fields we linearize the pseudo-gauge fields (3.44), which after a small calculation gives

$$\hat{\bar{\Omega}}_{\mu a} = \begin{pmatrix} \hat{\bar{\Omega}}_{0a} \\ \hat{\bar{\Omega}}_{ba} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}\partial_0\bar{s}_{ba} - \partial_{(a}\bar{v}_{b)} \end{pmatrix} \tag{3.96a}$$

$$\hat{\bar{\omega}}_{\mu ac} = \begin{pmatrix} \hat{\bar{\omega}}_{0ac} \\ \hat{\bar{\omega}}_{bac} \end{pmatrix} = \begin{pmatrix} -\partial_0\bar{e}_{[ac]} - \partial_{[a}\bar{v}_{c]} \\ -\partial_b\bar{e}_{[ac]} + \partial_{[a}\bar{s}_{c]b} \end{pmatrix} . \tag{3.96b}$$

The details of the calculation can be found in appendix D.3.2. The full boost and rotations gauge fields (3.43) may then be written in their linearized forms as

$$\bar{\Omega}_{\mu a} = \hat{\bar{\Omega}}_{\mu a} + \bar{C}_{\mu a} \tag{3.97a}$$

$$\bar{\omega}_{\mu ac} = \hat{\bar{\omega}}_{\mu ac} + \bar{C}_{\mu ac} , \tag{3.97b}$$

where we here have the two linearized pseudo-contortions (3.45) given by

$$\bar{C}_{\mu a} = \delta_{\lambda a}\bar{W}^{\lambda}{}_{\mu 0} \tag{3.98a}$$

$$\bar{C}_{\mu ac} = \delta_{\lambda a}\bar{W}^{\lambda}{}_{\mu c} . \tag{3.98b}$$

3.8.2.1 *Transformation of gauge fields*

The non-linear transformations (3.25) were somewhat complicated. At linear order, this is going to simplify a great deal. The zeroth order part (the global inertial frame) does not transform under an infinitesimal local Galilean transformation, as the transformations must be considered a part of the linear order pieces. We find by substituting our linearization into the transformation laws that the perturbations at lowest order transforms as

$$\begin{aligned}
\delta\overline{\tau}_\mu &= \partial_\mu \xi^0 \,, && (3.99a)\\
\delta\overline{e}_\mu{}^a &= \partial_\mu \xi^a + \lambda^a{}_b \delta^b_\mu + \Lambda^a \delta^0_\mu && (3.99b)\\
\delta\overline{\Omega}_\mu{}^i &= \partial_\mu \Lambda^i && (3.99c)\\
\delta\overline{\omega}_\mu{}^{ij} &= \partial_\mu \lambda^{ij} \,, && (3.99d)
\end{aligned}$$

where all of the parameters are the same as those stated on page 27.

3.8.3 *Linearization of the graviphotonic connection*

The background field M_μ of the graviphotonic connection of section 3.3.3.1 in a flat geometry as discussed in section 3.7 is not necessarily zero but $M_\mu = \partial_\mu M$, and M is an arbitrary function. Up to a linear order piece we can therefore write

$$M_\mu = \partial_\mu M + \overline{M}_\mu \,. \tag{3.100}$$

For the graviphotonic connection (3.51), we had that the particular choice of $K_{\sigma\rho}$, $L_{\sigma\mu\nu}$ (3.49) at linear order becomes

$$\begin{aligned}
\overline{K}_{\sigma\rho} &= 2\partial_{[\sigma}\overline{M}_{\rho]} && (3.101a)\\
\overline{L}_{\sigma\mu\nu} &= 2\partial_\sigma M \partial_{[\mu}\overline{\tau}_{\nu]} - 2\partial_\mu M \partial_{[\nu}\overline{\tau}_{\sigma]} + 2\partial_\nu M \partial_{[\sigma}\overline{\tau}_{\mu]} \,. && (3.101b)
\end{aligned}$$

Any two choices of M should be equivalent for the geometry in the end even though it is not obvious at this point. If we choose $M = 0$, then we have $\overline{L}_{\sigma\mu\nu} = 0$ so it is entirely higher-order, while $\overline{K}_{\sigma\rho}$ is independent of how we choose M, so this is the simplest choice.

$$\overline{W}^\lambda{}_{\mu\nu} = \delta^{\lambda a}\left(\delta^0_\mu \partial_{[a}\overline{M}_{\nu]} + \delta^0_\nu \partial_{[a}\overline{M}_{\mu]}\right) \,. \tag{3.102}$$

The corresponding linearized pseudo-contortions $\overline{C}_{\mu a}$, $\overline{C}_{\mu ac}$ (3.98) take the expressions

$$\begin{aligned}
\overline{C}_{\mu a} &= -\begin{pmatrix} 2\partial_{[0}\overline{M}_{a]} \\ \partial_{[j}\overline{M}_{a]} \end{pmatrix} && (3.103a)\\
\overline{C}_{\mu ij} &= \begin{pmatrix} \partial_{[i}\overline{M}_{j]} \\ 0 \end{pmatrix} . && (3.103b)
\end{aligned}$$

These results of the linearization will be very useful in the following chapters, especially 7, 8.

4 NON-RELATIVISTIC FIELD THEORY

In this chapter we shall study non-relativistic field theories on both flat and curved Newton-Cartan geometry. We begin with the development of field representations and discuss some concrete models that can be realized. Next we will study the conserved spacetime symmetry currents of Galilean and Bargmann theories and how the currents may be improved. Finally the minimal coupling of field theories to Newton-Cartan geometry will be discussed.

4.1 FIELD REPRESENTATIONS

4.1.1 *Galilean theories*

Assume that we have a field theory covariant under global Galilean transformations. We assume further that it contains a field $\varphi_\ell(x)$, where the index ℓ is belongs to some representation of the homogeneous Galilean group $\mathrm{HGal}(d,1) = \mathrm{O}(d) \ltimes \mathbb{R}^d$, and the spacetime transforms as described (2.1), (2.6). We are especially interested in finite irreducible and indecomposable representations of the field components, which do not need to be unitary contrary to the states discussed in section 2.3.3. If we assume that the representations are linear, then we may represent the action of the Galilean group on the fields by the operator on field space $U_{\ell\bar{\ell}}(\xi,\Lambda,\lambda)$, where parameters are contracted with the generators, $\xi^\mu = (\xi^0,\xi^i)$ for translations, Λ^i for boosts and $\lambda^{ij} = -\lambda^{ji}$ rotations respectively [28]. We take the representation $U_{\ell\bar{\ell}}(\xi,\Lambda,\lambda)$ to be written as the direct product [3]

$$U_{\ell\bar{\ell}}(\xi,\Lambda,\lambda) = S_{\ell\bar{\ell}}(\Lambda,\lambda) \times T(\xi,\Lambda,\lambda)\ , \tag{4.1}$$

where $S_{\ell\bar{\ell}}(\Lambda,\lambda)$ is the representation matrix of $\mathrm{HGal}(d,1)$ that the field carries parametrized by the boost and rotation only, and the action on the spacetime is represented by a differential operator $T(\xi,\Lambda,\lambda)$. For the generators this product implies that they are represented by a commuting sum of a part that acts on the spacetime as differential operators, and a part that only acts on the field components. $T(\xi,\Lambda,\lambda)$ is an infinite dimensional and unitary representation of the form

$$\begin{aligned} T(\xi,\Lambda,\lambda) &\equiv \exp\left(-i\xi^0\hat{H} + i\xi^i\hat{P}_i + i\Lambda^i\hat{B}_i + i\frac{1}{2}\lambda^{ij}\hat{J}_{ij}\right) \\ &= \exp\left(\xi^\mu\partial_\mu + t\Lambda^i\partial_i + \frac{1}{2}\lambda^{ij}\left(x_i\partial_j - x_j\partial_i\right)\right) \end{aligned} \tag{4.2}$$

where the generators are represented as differential operators $\hat{H} = i\partial_0$, $\hat{P}_i = -i\partial_i$, $\hat{B}_i = -it\partial_i$ and $\hat{J}_{ij} = -ix_i\partial_j + ix_j\partial_i$ that are hermitian wrt. the standard $L^2(\mathbb{R}^D)$ inner product. The representation of $S_{\ell\bar{\ell}}(\Lambda,\lambda)$ may also be written as an exponentiated form with the generally non-anti-hermitian generators $(B_i)_{\ell\bar{\ell}}$ and $\left(J_{ij}\right)_{\ell\bar{\ell}}$ of the HGal $(d,1)$ representation as

$$S_{\ell\bar{\ell}}\left(\Lambda,\lambda\right) = \exp\left(\Lambda^i B_i + \frac{1}{2}\lambda^{ij}J_{ij}\right)_{\ell\bar{\ell}} . \tag{4.3}$$

We then require the passive transformation law that the new field at the new spacetime point is equal to the old field at the old spacetime point, i.e.

$$\varphi'_{\ell}\left(t',\boldsymbol{x}'\right) \equiv \varphi_{\ell}\left(t,\boldsymbol{x}\right) , \tag{4.4}$$

where at the infinitesimal level $t' \equiv t + \xi^0$, $x'^i \equiv \frac{1}{2}\lambda^i{}_j x^j + \Lambda^i t + \xi^i$ and $\varphi'_{\ell} \equiv S_{\ell\bar{\ell}}\left(\Lambda,\lambda\right)\varphi_{\bar{\ell}}$, which can also be written as the action of $U_{\ell\bar{\ell}}\left(\xi,\Lambda,\lambda\right)$ on the field as

$$\varphi'_{\ell}\left(t',\boldsymbol{x}'\right) = U_{\ell\bar{\ell}}\left(\xi,\Lambda,\lambda\right)\varphi_{\bar{\ell}}\left(t,\boldsymbol{x}\right) , \tag{4.5}$$

which is verified to be correct with the definition of the operators $S_{\ell\bar{\ell}}\left(\Lambda,\lambda\right)$, $T\left(\xi,\Lambda,\lambda\right)$ we have given. Two subsequent transformations must of course satisfy the group multiplication law of the Galilean group, which constrains the possible realizations.

4.1.2 *Bargmann theories and mass*

If we instead consider the Bargmann group the discussion above will be nearly identical except that we need to include the transformation of the field under the central charge M. The homogeneous Galilean group representations are the same as before and thus we can still use the same $S_{\ell\bar{\ell}}\left(\Lambda,\lambda\right)$ for Bargmann theories. The central charge is related to the spacetime part and not the homogeneous Galilean group as we have seen in section 2.4. As it is commutes with all generators, it may be introduced as another factor in (4.1) so we have

$$\mathring{U}_{\ell\bar{\ell}}\left(\xi,\Lambda,\lambda\right) \equiv S_{\ell\bar{\ell}}\left(\Lambda,\lambda\right) \times e^{-if(t,\boldsymbol{x})M} \times T\left(\xi,\Lambda,\lambda\right) , \tag{4.6}$$

where $f\left(t,\boldsymbol{x}\right)$ a priori is undetermined but constrained by the Galilean group structure, especially group closure under compositions [27]. It turns out that there is a unique and non-trivial solution given in terms of finite transformation parameters as

$$f\left(t,\boldsymbol{x}\right) = \frac{1}{2}\boldsymbol{v}^2 t + \boldsymbol{v}^t \boldsymbol{R}\boldsymbol{x} - \sigma . \tag{4.7}$$

These linear Bargmann representations are equivalent to the projective representation of the Galilean group we may study [39]. We have included a parameter $\sigma \in \mathbb{R}$ here, because we can see that doing first a boost with velocity $\boldsymbol{v}$, then a translation $\boldsymbol{a}$, then doing the inverse boost and finally the inverse translation is just a constant phase $e^{-i\boldsymbol{v}\cdot\boldsymbol{a}}$ leaving the spacetime and components unchanged, so the theory automatically has a $\mathrm{U}\left(1\right)$ symmetry, which is conveniently parametrized by σ. Following the discussion of section 2.4.2 we should here think of $(M)_{\ell\bar{\ell}} = mI_{\ell\bar{\ell}}$ as the of mass of the field, with the field φ_{ℓ} being an eigenvector of the mass generator. It is then clear that massless fields with $m = 0$ are exactly the Galilean theories considered in the former section.

4.1.3 *Realizations*

The spacetime part $T(\xi, \Lambda, \lambda)$ is the same for all Galilean or Bargmann theories, and only the representations $S_{\ell\bar{\ell}}(\Lambda, \lambda)$ may take different realizations. Besides this there is the question of whether the central charge M acts trivially on the fields. If it does (massless theories) we are to consider the linear Galilean representations, if the action is non-trivial (massive theories) we are to consider the linear Bargmann representations [64]. There are thus several possibilities and we will discuss some of them below.

4.1.3.1 *Massless (Galilean) scalar*

The simplest representation is the scalar representation where we simply take $S_{\ell\bar{\ell}}(\Lambda, \lambda) = 1$ and take the mass of the field $\varphi(t, \boldsymbol{x})$ to be zero. The field will then only have Galilean spacetime transformations under which it transforms linearly.

Example 4.1 (Massless free scalar)**.** The simplest lagrangian density we can construct is

$$\mathcal{L} = -\frac{1}{2}\partial_i\varphi\partial^i\varphi\,. \tag{4.8}$$

The Equation of Motion (EOM) is just the Poisson equation with no sources. This theory is a bit degenerate, as it does not involve time derivatives and thus there are no waves.

4.1.3.2 *Massive (Bargmann) scalar*

If we instead take $S_{\ell\bar{\ell}}(\Lambda, \lambda) = 1$ as before but now take non-zero mass m of the theory, we are considering a scalar under the Bargmann group. The field is necessarily complex because it gets a phase under the transformation (4.7) and it has a global $\mathrm{U}(1)$ symmetry corresponding to the central transformation.

Example 4.2 (Schrödinger scalar model)**.** A Lagrangian density

$$\mathcal{L} = (im\varphi^*\partial_t\varphi - im\varphi\partial_t\varphi^*) - \partial_i\varphi^*\partial^i\varphi \tag{4.9}$$

can be obtained from a null reduction of a free massless complex Klein-Gordan field, which we will show explicitly in chapter 7. The EOM is just the Schrödinger equation with no potential, but it is easy to add a Bargmann invariant potential of the form $V(|\varphi|)$ to the theory to make it interacting. Notice also that the massless scalar of example 4.1 can be obtained from the $m \to 0$ limit of this theory directly.

4.1.3.3 *Spinor fields*

$\mathrm{HGal}(d,1)$ contains a $\mathrm{SO}(d)$ subgroup, so we may identify the standard irreducible representations with various spin representations. Let us now restrict ourselves to $D = 4$. We may first take the representation of the boost to be trivial so that $S_{\ell\bar{\ell}}(\Lambda, \lambda) = S_{\ell\bar{\ell}}(\lambda)$ can be taken to be the usual $(2j+1) \times (2j+1)$ dimensional unitary and irreducible matrix representation of the $\mathrm{SO}(3)$. These are however not faithful representations of the homogeneous Galilean group. If we consider the defining spin-½ representation of $\mathrm{SU}(2)$ in terms of Pauli matrices [64], then the fields are 2-spinors $\psi_\ell(t, \boldsymbol{x})$ that simply transforms with a (4.3) here given by

$$S^{½}_{\ell\bar{\ell}}(\lambda) \equiv \exp\left(\frac{i}{2}\epsilon_{ijk}\sigma_i\lambda_{jk}\right)_{\ell\bar{\ell}} . \tag{4.10}$$

On the other hand, if we do not take (4.3) to be trivial in boosts, we find the only possible indecomposable but not irreducible spin-½ representation acts on 4-spinors $\psi(t, \boldsymbol{x}) = (\psi_+(t, \boldsymbol{x}), \psi_-(t, \boldsymbol{x}))$ with transformations of the form [3]

$$S(\Lambda, \lambda) = \begin{pmatrix} S^{½}(\lambda) & \mathbf{0} \\ -\frac{1}{2}\sigma_i\Lambda^i S^{½}(\lambda) & S^{½}(\lambda) \end{pmatrix} . \tag{4.11}$$

Example 4.3 (Free massive Fermi spinor)**.** It is possible to do a null reduction [65] of the massless Dirac Lagrangian density with Dirac 4-spinor $\Psi = (\psi_+, \psi_-)$, which results in

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\Big[&\partial_i\psi_-^\dagger\sigma^i\psi_+ - \psi_-^\dagger\sigma^i\partial_i\psi_+ + \partial_i\psi_+^\dagger\sigma^i\psi_- - \psi_+^\dagger\sigma^i\partial_i\psi_- \\ &- \sqrt{2}\left(\psi_+^\dagger\partial_0\psi_+ - \partial_0\psi_+^\dagger\psi_+ + i2m\psi_-^\dagger\psi_-\right)\Big] .\end{aligned} \tag{4.12}$$

This Lagrangian density describes a non-relativistic free spin-½ 4-spinor, which we may call the Fermi 4-spinor, that transforms under (4.11). The EOMs are easily derived from this and one sees that ψ_- enters without derivaties for the Euler-Lagrange equations wrt. $\psi_-^\dagger$. One may substitute this EOM into the other and the result is an equation for ψ_+ which is just the Schrödinger equation.
Interestingly enough the EOMs may be obtained by a similar line of reasoning as originally led Dirac to his equation by "taking the square root" of the Klein-Gordon equation. In the non-relativistic case we would try to "take the square root" of the Schrödinger equation by introducing a Clifford algebra which will eventually led us to (4.12) [3].

4.1.3.4 *Vector fields*

For the vector representations there are several ones of interest to be considered. It is of course possible to take the representation $\mathrm{HGal}(d, 1)$ be the fundamental irreducible representation (2.22)

$$S^A{}_B(v, R) = \begin{pmatrix} 1 & 0 \\ v^a & R^a{}_b \end{pmatrix} \tag{4.13}$$

which will act on Galilean D-vector fields. The Galilean D-covector fields can be constructed as fields that transform in the inverse transformation (2.23).

Besides this, there is also a $D+1$ vector representation, which will be relevant for Galilean electrodynamics that we will consider later. Here the $D+1$ components transforms under the representation (2.35), the extended representation of the homogeneous Galilean group, i.e.

$$S^{\hat{A}}{}_{\hat{B}}(v,R) = \begin{pmatrix} 1 & 0 & 0 \\ v^a & R^a{}_b & 0 \\ -v^2/2 & -v_a R^a{}_b & 1 \end{pmatrix} . \tag{4.14}$$

In this representation there is an extra component of the vector compared to the fundamental representation of before and it is indecomposable but not irreducible. On top of this comes also the $D+1$ covector representation that transforms under the inverse of (4.14).

Example 4.4 (Incompressible Euler fluid)**.** An example of a theory that furnishes the vector representation (4.13) is the incompressible Euler fluid [29, 66]. The EOM is

$$\partial_0 u_i(t,\boldsymbol{x}) + u_j(t,\boldsymbol{x})\,\partial^j u_i(t,\boldsymbol{x}) + \partial_i p(t,\boldsymbol{x}) = 0\,, \tag{4.15}$$

where the Galilean 4-vector is $(p(t,\boldsymbol{x}),\, u_i(t,\boldsymbol{x}))$.

4.2 CONSERVED GALILEAN SYMMETRY CURRENTS

4.2.1 *Canonical conserved Noether currents*

Noether's theorem states that every differentiable symmetry transformation of an action $S[\varphi] = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi)$ can be associated with a conserved symmetry current. We give the proof of the most general case for the field theoretical version in appendix B.

For Galilean spacetime symmetries the infinitesimal versions of the coordinate and field transformations (4.3), (4.5) are given by

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} \tag{4.16a}$$
$$\varphi'_\ell(x') = \varphi_\ell(x) + \delta\varphi_\ell(x) \tag{4.16b}$$

$$\delta t = \xi_0 \equiv \epsilon_0 \tag{4.16c}$$
$$\delta x_i = \xi_i \equiv \epsilon_i + \Lambda_i t + \lambda_{ij} x^j \tag{4.16d}$$
$$\delta\varphi_\ell(x) = \Lambda^i (B_i)_{\ell\bar{\ell}}\, \varphi_{\bar{\ell}}(x) + \frac{1}{2}\lambda^{ij} (J_{ij})_{\ell\bar{\ell}}\, \varphi_{\bar{\ell}}(x)\,, \tag{4.16e}$$

where $\lambda^{ij} = -\lambda^{ji}$ are just infinitesimal spatial rotation parameters, Λ^i boost parameters and ϵ^0, ϵ^i infinitesimal global translations that together are all of the parameters of the Galilean spacetime transformation.

Specializing to our case, the conserved canonical Noether currents (B.9) takes the expression

$$E^{\mu}_{\text{can}} \equiv \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\varphi_\ell\right]}\partial_0\varphi_\ell - \delta^{\mu}_{0}\mathcal{L} \tag{4.17a}$$

$$T^{\mu i}_{\text{can}} \equiv \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\varphi_\ell\right]}\partial^i\varphi_\ell - \delta^{\mu i}\mathcal{L} \tag{4.17b}$$

$$b^{\mu i}_{\text{can}} \equiv tT^{\mu i}_{\text{can}} + w^{\mu i} \tag{4.17c}$$

$$j^{\mu ij}_{\text{can}} \equiv x^i T^{\mu j}_{\text{can}} - x^j T^{\mu i}_{\text{can}} + s^{\mu ij} \tag{4.17d}$$

$$w^{\mu i} \equiv -\frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\varphi_\ell\right]}\left(B^i\right)_{\ell\bar{\ell}}\varphi_{\bar{\ell}} \tag{4.17e}$$

$$s^{\mu ij} \equiv -\frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\varphi_\ell\right]}\left(J^{ij}\right)_{\ell\bar{\ell}}\varphi_{\bar{\ell}} \tag{4.17f}$$

which are the canonical energy current E^{μ}_{can} and momentum current $T^{\mu i}_{\text{can}}$, the total spatial angular momentum current $j^{\mu ij}_{\text{can}}$, and the boost current $b^{\mu i}_{\text{can}}$. We call the non-conserved current $w^{\mu i}$ the lift-current and $s^{\mu ij}$ the spatial spin-current, which are related to the transformation of the field components under boosts and rotations. Notice from the above that in a non-relativistic theory, the would-be energy-momentum tensor of relativistic theories splits up into two independent currents. This demonstrates the general hallmark of non-relativistic theories with time and space not being on the same footing.

The conservation laws for the boost and rotation current implies the relations

$$T^{0i}_{\text{can}} = -\partial_\mu w^{\mu i} \tag{4.18a}$$

$$2T^{[ij]}_{\text{can}} = -\partial_\mu s^{\mu ij}\,. \tag{4.18b}$$

Hence the canonical stress tensor T^{ij}_{can} is only symmetric automatically if $s^{\mu ij}$ vanishes and T^{0i}_{can} is a total derivative with implications we shall study in the next section.

It is useful to write a variation of the action in terms of general local parameters. These corresponds to local translations, boosts and rotations at lowest possible order in the geometry and in this case (B.11) becomes

$$\delta S\left[\varphi\right] = -\int_M \mathrm{d}^D x\, \partial_\mu\xi_0 E^{\mu}_{\text{can}} + \partial_\mu\xi_0 T^{\mu i}_{\text{can}} + \partial_\mu\Lambda_i w^{\mu i} + \frac{1}{2}\partial_\mu\lambda_{ij}s^{\mu ij}\,. \tag{4.19}$$

This variation is non-zero for general transformations. We recover the on-shell conservation laws when we take the parameters to correspond to the global variations

$$\xi_0\left(x\right) = \epsilon_0 \tag{4.20a}$$

$$\xi_i\left(x\right) = \epsilon_i + \Lambda_i t + \lambda_{ij}x^j \tag{4.20b}$$

$$\Lambda_i\left(x\right) = \Lambda_i \tag{4.20c}$$

$$\lambda_{ij}\left(x\right) = \lambda_{ij}\,. \tag{4.20d}$$

In this way one recovers the symmetry currents (4.17) by grouping together in the same parameter.

4.2.2 *Generators of the Poisson algebra and $w^{\mu i}$*

The conserved Noether currents lead to conserved symmetry charges as described in appendix B. These charges are defined by an integration of the charge densities of the spacetime currents over a spatial hypersurface S:

$$H = \int_S \mathrm{d}^d x\, E^0_{\text{can}} \tag{4.21a}$$
$$P^i = \int_S \mathrm{d}^d x\, T^{0i}_{\text{can}} \tag{4.21b}$$
$$B^i = \int_S \mathrm{d}^d x\, t T^{0i}_{\text{can}} + w^{0i} \tag{4.21c}$$
$$J^{ij} = \int_S \mathrm{d}^d x\, x^i T^{0j}_{\text{can}} - x^j T^{0i}_{\text{can}} + s^{0ij}\,. \tag{4.21d}$$

These generators are the representation of the Galilean symmetry on the field theory. They furnish a representation of the Galilean algebra with the Poisson bracket defined as [67, 68]

$$\{F, G\} \equiv \int_S \mathrm{d}^d x \left[\frac{\delta F}{\delta \Pi^\ell}\frac{\delta G}{\delta \varphi_\ell} - \frac{\delta G}{\delta \Pi^\ell}\frac{\delta F}{\delta \varphi_\ell}\right], \tag{4.22}$$

where $\Pi^\ell \equiv \frac{\partial \mathcal{L}}{\partial[\partial_t \varphi_\ell]}$ is the conjugate field momentum transforming oppositely to the field.

If we now use the constraint from the conservation law of the boost and rotation currents (4.18), we see that this implies that we can write the corresponding momentum generator as

$$\begin{aligned} P^i &= -\int_S \mathrm{d}^d x \partial_\mu w^{\mu i} \\ &= -\partial_0 \int_S \mathrm{d}^d x w^{0i}\,, \end{aligned} \tag{4.23}$$

where we in the last step used Stokes' theorem and assumed that the boundary terms vanished. In the case that $w^{\mu i} = 0$, we see that this implies on the field generators (4.21) that

$$P^i = 0 \tag{4.24a}$$
$$B^i = 0 \tag{4.24b}$$
$$J^{ij} = \int_S \mathrm{d}x^d\, s^{0ij}\,. \tag{4.24c}$$

In such theories the fields the total momentum is zero and the fields transforms trivially under translations and boosts. This happens for several Galilean theories of relevance, namely all representations of $\mathrm{HGal}\,(d,1)$ that have $S_{\ell\bar{\ell}}\,(\Lambda,\lambda) = S_{\ell\bar{\ell}}\,(\lambda)$. These theories have no interesting dynamics, in particular no any wave solutions. If further $s^{\mu ij} = 0$, then the theory does not carry any angular momentum.

Example 4.5 (Galilean scalar)**.** For massless Galilean scalars studied in section 4.1.3.1, we have that $S_{\ell\bar{\ell}}(\Lambda,\lambda) = 1$, which means that the boost and rotations generators are trivially represented. Therefore $w^{\mu i} = s^{\mu ij} = 0$, and (4.18) implies that the momentum, boost and rotation charges charges are zero and that the stress tensor is symmetric.

4.2.3 *Improvement of the currents*

There is an ambiguity with defining the currents whose charges generate the symmetries on the fields as discussed in appendix B. They can be allowed to differ up to a total derivative, which can be used to get versions of the currents with simpler properties. We may define improved versions of the currents as

$$E^{\mu}_{\text{imp}} = E^{\mu}_{\text{can}} + \partial_\rho A^{\rho\mu} \tag{4.25a}$$
$$T^{\mu i}_{\text{imp}} = T^{\mu i}_{\text{can}} + \partial_\rho B^{\rho\mu i} \tag{4.25b}$$
$$j^{\mu ij}_{\text{imp}} = j^{\mu ij}_{\text{can}} + \partial_\rho D^{\rho\mu ij} \tag{4.25c}$$
$$b^{\mu i}_{\text{imp}} = b^{\mu i}_{\text{can}} + \partial_\rho E^{\rho\mu i} \tag{4.25d}$$

where the improvement terms all are antisymmetric in μ,ρ. The choice of improvements does not change the physics (i.e. the symmetry generators (4.21)) if there are no boundary terms, which we will always assume is the case. This is well-known in the case of relativistic theories, where for the Energy-Momentum (EM) tensor it is known as the Belinfante procedure for obtaining the symmetric EM tensor as discussed in appendix A.3 [69]. It is also worth noting that this discussion holds even for theories that does not have a Lagrangian formulation.

4.2.3.1 *Choosing the simplest improvements*

In our case we want to do something similar and put the currents in the simplest possible form. This is the same as saying that we want to make the constraints on the energy and momentum currents coming from the conservation equations $\partial_\mu j^{\mu ij}_{\text{imp}} = \partial_\mu b^{\mu i}_{\text{imp}} = 0$ as trivial as possible by removing as much of the lift- and spin-currents as possible. One can analyze how to maximally exploit the ambiguity to remove as much of them as possible, with an essentially unique answer. We give details of how to perform such an analysis in appendix D.4.1. The result one finds is

$$B^{\rho\mu i} = \delta_k^{[\mu}\delta_0^{\rho]}\left[2w^{(ki)} + s^{0ik}\right] + \frac{1}{2}\delta_j^{\mu}\delta_k^{\rho}\left[s^{kij} + s^{ikj} + s^{jki}\right] \tag{4.26a}$$
$$D^{\rho\mu ij} = x^i B^{\rho\mu j} - x^j B^{\rho\mu i} \tag{4.26b}$$
$$E^{\rho\mu i} = tB^{\rho\mu i} \tag{4.26c}$$

which gives the improved spatial angular momentum and boost currents given by

$$b^{\mu i}_{\text{imp}} = tT^{\mu i}_{\text{imp}} + \psi^{\mu i} \tag{4.27a}$$

$$j^{\mu ij}_{\text{imp}} \equiv x^i T^{\mu j}_{\text{imp}} - x^j T^{\mu i}_{\text{imp}} \tag{4.27b}$$

where we have defined the non-conserved current that we have not succeeded in removing by improvements as

$$\psi^{\mu i} \equiv \delta^\mu_0 \left[w^{0i} \right] + \delta^\mu_j \left[w^{[ji]} - \frac{1}{2} s^{0ij} \right] . \tag{4.28}$$

This current is non-conserved and as we see a combination of the lift and spin currents, and it is antisymmetric in the spatial indices $\psi^{ij} = -\psi^{ji}$. It will turn out to be of great significance later when we want to couple Galilean theories to Newton-Cartan geometry, but right now its interpretation is not clear. The new improved symmetry currents now give the following constraints from the conservation of the rotation and boost currents:

$$T^{0i}_{\text{imp}} = -\partial_\mu \psi^{\mu i} \tag{4.29a}$$

$$2T^{[ij]}_{\text{imp}} = 0 . \tag{4.29b}$$

The stress tensor can thus always be made symmetric, but T^{0i}_{imp} can in general only be made the total derivative of the object $\psi^{\mu i}$.

4.2.3.2 *Further improvements of the momentum current*

If we want to keep the stress current symmetric, there are still some improvements that can be done. We may write a new current as

$$\tilde{T}^{\mu i}_{\text{imp}} = T^{\mu i}_{\text{imp}} + \partial_\rho \tilde{B}^{\rho \mu i} . \tag{4.30}$$

Now, to still satisfy the properties (4.29), we must therefore have that $\partial_\rho \tilde{B}^{\rho ij}$ is symmetric but otherwise arbitrary.

4.2.3.3 *The improvements of the energy current*

The decoupling of time and space in a Galilean theory has the subtle consequence that energy and momentum are decoupled. In fact, contrary to the Poincaré case in appendix A.3 we have not specified the improvements of the energy current and they didn't have any influence on the analysis above. This is the reason why we couldn't define currents independent of the lift- and spin-current like in the relativistic case, as we simply didn't have enough improvements at our disposal because of the decoupling.

There might be other constraints that would determine the improvements uniquely. In gauge theories we can require it to be gauge invariant, which will lead to a unique choice of improvements.

4.3 CONSERVED BARGMANN SYMMETRY CURRENTS

4.3.1 *Canonical conserved Noether currents*

Take a theory with Bargmann symmetry described by an action $S[\varphi] = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi)$ where the fields carries some representation of the Bargmann group. The only difference to the Galilean case of section 4.2.1 is now that there are two extra terms in $\delta\varphi_\ell$ due to the representation of the central charge M on the field components. The infinitesimal versions of the coordinate and field transformations (4.7) and (4.6) are given by

$$x'^\mu = x^\mu + \delta x^\mu \tag{4.31a}$$
$$\varphi'_\ell(x') = \varphi_\ell(x) + \delta\varphi_\ell(x) \tag{4.31b}$$

$$\delta t = \xi_0 \equiv \epsilon_0 \tag{4.31c}$$
$$\delta x_i = \xi_i \equiv \epsilon_i + \Lambda_i t + \lambda_{ij} x^j \tag{4.31d}$$
$$\begin{aligned}\delta\varphi_\ell(x) &= +i\sigma(M)_{\ell\bar{\ell}}\,\varphi_{\bar{\ell}} - i\Lambda^i x_i (M)_{\ell\bar{\ell}}\,\varphi_{\bar{\ell}} \\ &\quad + \Lambda^i (B_i)_{\ell\bar{\ell}}\,\varphi_{\bar{\ell}}(x) + \frac{1}{2}\lambda^{ij}(J_{ij})_{\ell\bar{\ell}}\,\varphi_{\ell'}(x)\,.\end{aligned} \tag{4.31e}$$

The extra parameter compared to the Galilean case (4.16) is $\sigma \in \mathbb{R}$ corresponding to the global $\mathrm{U}(1)$ symmetry of the field. The corresponding conserved current is what we will call the mass current J^μ_{can}. There is an additional piece with a boost parameter from the linearization of the projective factor $\exp(-if(t,x)M)$ and this will change the boost current, but otherwise the rest is as in section 4.2.1. The two conserved canonical Noether currents that change compared to the Galilean currents (4.17) take the expression

$$J^\mu_{\text{can}} \equiv -i\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi_\ell]}(M)_{\ell\bar{\ell}}\,\varphi_{\bar{\ell}} \tag{4.32a}$$
$$b^{\mu i}_{\text{can}} \equiv tT^{\mu i}_{\text{can}} - x^i J^\mu_{\text{can}} + w^{\mu i}\,. \tag{4.32b}$$

We see that while the conservation law for the rotation current does not change, the one for the boost current now implies a new relation so that we have

$$T^{0i}_{\text{can}} = J^i_{\text{can}} - \partial_\mu w^{\mu i} \tag{4.33a}$$
$$2T^{[ij]}_{\text{can}} = -\partial_\mu s^{\mu ij}\,. \tag{4.33b}$$

There is now a relation between the momentum density and the mass flux, which we will exploit very soon to simplify the equation. It is again useful to write a variation of the action in terms of general local parameters which is done as

$$\delta S[\varphi] = -\int_M \mathrm{d}^D x\, \partial_\mu\xi_0 E^\mu_{\text{can}} + \partial_\mu\xi_0 T^{\mu i}_{\text{can}} + \partial_\mu\sigma J^\mu_{\text{var}} + \partial_\mu\lambda_i w^{\mu i} + \frac{1}{2}\partial_\mu\lambda_{ij}s^{\mu ij}\,. \tag{4.34}$$

where we recover the on-shell conservation when we take σ constant and the rest as in (4.20).

4.3.2 *Improvement of the currents*

The new mass current brings in new improvements besides (4.25). It can be improved as

$$J^{\mu}_{\text{imp}} = J^{\mu}_{\text{can}} + \partial_\rho C^{\rho\mu} \,. \tag{4.35}$$

We may now choose the extra improvement $C^{\rho\mu}$ of the mass current in a useful way to simplify the currents (4.33) as much as possible. The improvement $E^{\rho\mu i}$ for the boost current $b^{\mu i}$ must be chosen in a different way than in the Galilean case (4.26), but the remaining improvements are identical. The currents are maximally simplified when we chose

$$C^{\rho\mu} = 2\delta_i^{[\rho}\delta_0^{\mu]} w^{0i} + \delta_i^{\rho}\delta_j^{\mu} w^{[ij]} \tag{4.36}$$

$$E^{\rho\mu i} = tB^{\rho\mu i} - x^i C^{\rho\mu} \,. \tag{4.37}$$

We give the details of the argument in appendix D.4.2. With this choice we have for the boost currents that this leads to a complete cancellation of the lift current and we find

$$b^{\mu i}_{\text{imp}} = tT^{\mu i}_{\text{imp}} - x^i J^{\mu}_{\text{imp}} \,. \tag{4.38}$$

The conservation of this now implies that we have

$$T^{0i}_{\text{imp}} = J^{i}_{\text{imp}} \,, \tag{4.39}$$

which is to say that momentum density is the same as the mass flux. In a massive Newtonian theory this relation is to be expected.

Example 4.6 (Fluid)**.** To illustrate the physical relevance of (4.39), we consider a fluid with velocity field $\boldsymbol{u}\,(t, \boldsymbol{x})$ with mass density $\rho\,(t, \boldsymbol{x})$. The mass flux is then simply $J^i = \rho u^i$. On the other hand the momentum of the fluid is found by summing all contributions of the fluid particles, i.e.

$$P^i = \int_S \mathrm{d}^d x\, \rho u^i = \int_S \mathrm{d}^d x\, T^{0i} \,. \tag{4.40}$$

From this we see that $T^{0i} = J^i$, confirming the equation (4.39).

4.4 FIELD THEORIES ON NEWTON-CARTAN BACKGROUNDS

4.4.1 *(Minimal) coupling to gravity*

So far we have been considering theories on flat space. The Lagrangian densities we have discussed have all of the required spacetime symmetries, albeit only in global versions. In order to obtain a general covariant theory, we must replace the partial derivatives, expressions etc. with covariantized ones. It is relatively straight-forward to do this for known theories using the "minimal coupling principle" the same way we do it in general relativity. More precisely, we should substitute time derivatives of a field φ_ℓ with

$$\partial_0 \varphi_\ell \mapsto -v^\mu \nabla_\mu \varphi_\ell \,. \tag{4.41}$$

Spatial derivatives may be written contracted with e_i^μ along the lines of

$$\partial_i \varphi_\ell \mapsto e_i^\mu \nabla_\mu \varphi_\ell \,. \tag{4.42}$$

The covariant derivatives here contains the appropriate boost, rotation and affine connections that is required by the representation that φ_ℓ furnish. In principle any connection may be used as we do not have a notion of a Levi-Civita-like connection as discussed in section 3.3, which makes the minimal coupling principle more arbitrary here than in the relativistic case. The covariantized derivatives must be combined so that the action is invariant under local Galilean transformations. This is not as obvious as one would have wished for, as for example v^μ transforms under boosts, while $h^{\mu\nu}$ does not.

The graviphotonic connection of section 3.3.3.1 is of relevance here as it is particular simple, but it is not immediately clear what M_μ can be interpreted as or what it couples to. This connection is good for both Galilean and Bargmann theories, but in the Bargmann case M_μ has a natural interpretation as we will show soon. If we use the graviphotonic connection, then one may use $\hat{v}^\mu$ and $\hat{e}_i^\mu$ defined in (3.50) as they are boost invariant.

We will in the following use this connection exclusively. Let us suppose that we now have a theory that is properly coupled to some Newton-Cartan background with the "graviphotonic Galilean connection". The action is then in general expressed in terms of the (inverse) vielbeins and M_μ like

$$S\left[\varphi, \boldsymbol{\tau}, \boldsymbol{e}, \boldsymbol{M}\right] = \int_M \mathrm{d}^D x \, e \mathcal{L}\left(\varphi, \partial\varphi, \boldsymbol{\tau}, \boldsymbol{e}, \boldsymbol{M}\right) \,, \tag{4.43}$$

where $e = \det\left(\boldsymbol{\tau}, \boldsymbol{e}^a\right)$ so the Lagrangian density indeed is a real scalar. Varying with respect to the field will give us the EOMs through the Euler-Lagrange equations.

4.4.2 *Enhancement of M_μ to mass gauge field*

Consider a local transformation of the background defined by

$$\delta_\sigma M_\mu \equiv \partial_\mu \sigma\left(x\right) \tag{4.44}$$

together with leaving all vielbeins invariant, i.e. $\delta_\sigma e_b^\mu = 0$ et cetera. This is effectively a $\mathrm{U}\left(1\right)$ gauge transformation of M_μ. Such a transformation is actually the result of diffeomorphisms and local Galilean transformations in a special combination. To see this, we can write the δ_σ-transformation as defined above in terms of the general (infinitesimal) transformation rules for the vielbeins and M_μ given by (3.25), (3.47). We see that the kind of diffeomorphisms generated by the vector $\xi(\sigma)$, local boost $\Lambda_a(\sigma)$ and rotations $\lambda^a{}_b(\sigma)$ that produces such a transformation are of the kind

$$\mathcal{L}_{\xi(\sigma)} M_\mu = -e_\mu{}^a \Lambda_a(\sigma) + \partial_\mu \sigma \tag{4.45a}$$
$$\mathcal{L}_{\xi(\sigma)} \tau_\mu = 0 \tag{4.45b}$$
$$\mathcal{L}_{\xi(\sigma)} e_\mu{}^a = -\lambda^a{}_b(\sigma) e_\mu{}^b - \Lambda^a(\sigma) \tau_\mu \tag{4.45c}$$
$$\mathcal{L}_{\xi(\sigma)} v^\mu = -e^\mu{}_a \Lambda^a(\sigma) \tag{4.45d}$$
$$\mathcal{L}_{\xi(\sigma)} e^\mu{}_b = -\lambda_b{}^a(\sigma) e^\mu{}_a\,. \tag{4.45e}$$

We can actually find a solution that only involves a diffeomorphism and local boost [54]. This is completely analogous to the situation of section 4.1.2 where we saw that one can always make translations and boosts to effectively get a (global) $\mathrm{U}(1)$ transformation of Bargmann fields. For the geometry this shows that the δ_σ-transformation is always a local symmetry of the action (4.43) being just a particular diffeomorphism and boost, but it does in general not give rise to a conserved current since it is not necessarily a global symmetry to which the conserved currents are associated. This can only happen in certain special cases, namely those where M_μ can be enhanced to the gauge field associated with a local $\mathrm{U}(1)$ transformation. As Bargmann theories automatically have a global $\mathrm{U}(1)$ symmetry generated by the central charge it is always enhanced to a local $\mathrm{U}(1)$ symmetry with M_μ as a gauge field. We shall in this case often call it the mass gauge field because it is associated with the conserved mass current $\mathcal{J}^\rho$ to which it couples. It is not a usual gauge field since it is a non-trivial part of the geometry, in particular as it transforms under local Galilean boosts.

What M_μ couples to in Galilean theories is one of the main results of the thesis which we will give the answer to in section 5.2.

4.4.3 *Energy, momentum and the coupling of M_μ*

When coupling a field theory to a Newton-Cartan background, the background fields are now sources for energy, momentum et cetera of the theory [57, 54, 63]. We may write the variation of the action (4.43) wrt. the background in terms of the vielbeins and M_μ in a way that defines some currents:

$$\delta_{\mathrm{bgd}} S[\varphi, \boldsymbol{\tau}, \boldsymbol{e}, \boldsymbol{M}] \equiv \int_M \mathrm{d}^D x\, e \left(\mathcal{E}^\mu \delta\tau_\mu + \mathcal{T}^\mu{}_a \delta e^a_\mu + \mathcal{J}^\mu \delta M_\mu \right). \tag{4.46}$$

In general curved space this leads us to think of $\mathcal{E}^\mu$ as some kind of energy current, as it is the response of variation wrt. the local time direction $\boldsymbol{\tau}$. This current is invariant under local Galilean transformations because $\boldsymbol{\tau}$ is. This is not the case for the current $\mathcal{T}^\mu{}_a$, which should be thought of as some kind of momentum current, because it is the response to variation of the local spatial directions. The current $\mathcal{J}^\mu$ that couples to M_μ is likewise not invariant under local Galilean transformations and its physical interpretation is not clear in general. However when M_μ can be enhanced to the mass gauge field, we have by our previous analysis shown that this current should be thought of as the covariantized version of the mass current.

Notice that we may write the above variation in a way where the behavior under local Galilean transformations is more clear. We can identify the expression above as a contraction of the current

$$\mathcal{T}^{\mu}{}_{\hat{A}} \equiv \left(\mathcal{E}^{\mu}, \mathcal{T}^{\mu}{}_{a}, -\mathcal{J}^{\mu}\right) \tag{4.47}$$

with the extended (Bargmann) frame $\mathring{\boldsymbol{e}}^{\hat{A}} = (\boldsymbol{\tau}, \boldsymbol{e}^{a}, -\boldsymbol{M})$ of section 3.4. $\mathcal{T}^{\mu}{}_{\hat{A}}$ can be calculated from the functional derivative

$$\mathcal{T}^{\mu}{}_{\hat{A}} = \frac{\delta S\left[\varphi, \boldsymbol{\tau}, \boldsymbol{e}, \boldsymbol{M}\right]}{\delta \mathring{e}^{\hat{A}}_{\mu}}\,, \tag{4.48}$$

where the functional derivative is taken at some point at some particular section of the extended coframe bundle. We will in chapter 5 verify that if we expand around flat Newton-Cartan geometry, then the currents we obtain are exactly the improved ones of sections 4.2.3 and 4.3.2. This is similar to the relationship between the Belinfante-Rosenfeld EM tensor and the Hilbert EM tensor discussed in appendix A.3.2.

The transformation of $\mathcal{T}^{\mu}{}_{\hat{A}}$ is clear as it must leave the action invariant: It transforms under local inverse extended homogeneous Galilean group representation (2.36) as

$$\mathcal{T}'^{\mu}{}_{\hat{A}} = \mathcal{T}^{\mu}{}_{\hat{B}} \left(B^{-1}\right)^{\hat{B}}{}_{\hat{A}}\,. \tag{4.49}$$

The subcurrent $\mathcal{T}^{\mu}{}_{A} = \left(\mathcal{E}^{\mu}, \mathcal{T}^{\mu}{}_{a}\right)$ is Galilean covariant transforming under the homogeneous Galilean group representation (2.22) and the coupling $\mathcal{T}^{\mu}_{A}\delta e^{A}_{\mu}$ is Galilean invariant on its own, while $\mathcal{J}^{\mu}$ will mix with the other components of the current, but $\mathcal{J}^{\mu}\delta M_{\mu}$ stays invariant. As the action is invariant when we do GCTs and local Galilean transformations of (3.25), (3.47), there are now various off-shell Ward identities that relate the currents and state the covariant conservation equations [13, 54]. If we do diffeomorphic variations, i.e. $\delta\mathring{e}^{\hat{A}}_{\mu} = \mathcal{L}_{\xi}\mathring{e}^{\hat{A}}_{\mu}$, we obtain the covariantized version of the currents (4.17) and their covariant conservation equations:

$$\frac{1}{e}\partial_{\mu}\left(e\mathcal{T}^{\mu}{}_{\hat{A}}\mathring{e}^{\hat{A}}_{\lambda}\right) = \mathcal{T}^{\mu}{}_{\hat{A}}\partial_{\lambda}\mathring{e}^{\hat{A}}_{\mu}\,. \tag{4.50}$$

We can also perform the transformation (4.44) which then implies

$$\frac{1}{e}\partial_{\mu}\left(e\mathcal{J}^{\mu}\right) = 0\,. \tag{4.51}$$

Notice that this shows that we in general only have D conserved currents, but $\mathcal{T}^{\mu}{}_{\hat{A}}$ has $D+1$ Galilean components. This is a reminiscence of the fact that $\mathcal{J}^{\mu}$ only is a symmetry current when there is a local $\mathrm{U}(1)$ symmetry of the field theory. If this is the case, then the conservation (4.51) is non-trivial, which we see an example of in section 7.2.1. If there is no local $\mathrm{U}(1)$ symmetry of the field theory, then (4.51) is going to be trivial in one way or another as it cannot correspond to a symmetry current. This we see an example of in section 8.2.5.

If we instead do a Galilean boost (3.25), (3.47) we obtain:

$$\mathcal{J}^{\mu}e_{\mu a} = -\mathcal{T}^{\mu}{}_{a}\tau_{\mu}\,. \tag{4.52}$$

When $\mathcal{J}^\mu$ is a true symmetry current because of the local U(1) symmetry enhancement, then this is the covariant version of the statement that momentum density is equal to mass flux (4.39) that was found by considering improvements of the flat currents. Otherwise this is just relation that must be built into the theory.

If we do a spatial rotation (3.25) we obtain:

$$0 = e_{\mu[a}\mathcal{T}^{\mu}{}_{b]} \,. \tag{4.53}$$

This is the covariant version of the flat case version (4.29) where we showed that we can always make improvements so that the stress tensor is symmetric.

There is nothing stopping us from defining other energy and momentum current-like objects that encode the same information. Having such quantities in terms of spacetime tensors is particular useful. This is for example obtained by defining the variation of the action in terms of (3.50) like Hartong et al. [54] do it:

$$\begin{aligned}\delta_{\mathrm{bgd}} S\left[\varphi, \hat{\boldsymbol{v}}, \boldsymbol{h}^{-1}, \tilde{\boldsymbol{\Phi}}\right] \equiv \int_M \mathrm{d}^D x\, e\Big(&-\tau_\mu T^{\mu}{}_{\nu}\delta\hat{v}^\nu - \hat{e}^a_\mu \hat{v}^\nu T^{\mu}{}_{\nu}\hat{e}_{\sigma a}\tau_\rho \delta h^{\rho\sigma} \\ &+ \frac{1}{2}\hat{e}^b_\mu e^{\nu a} T^{\mu}{}_{\nu}\hat{e}_{\sigma b}\hat{e}_{\rho a}\delta h^{\rho\sigma} + \tau_\mu \mathcal{J}^\mu \delta\tilde{\Phi}\Big)\,, \end{aligned} \tag{4.54}$$

where we see that $T^{\mu}{}_{\nu}$ encodes information about both energy and momentum, so it may also claim to be a notion of an EM tensor for theories on Newton-Cartan geometry.

There is also yet another definition of an energy-momentum tensor-like object [70]. This one is one most convenient to use for the theories we consider in chapters 7 and 8, but it does of course encode the same information. We here define the variation wrt. v^μ, $h^{\mu\nu}$ and M_μ as

$$\delta_{\mathrm{bgd}} S\left[\varphi, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] \equiv \int_M \mathrm{d}^D x\, e\left(-\mathcal{S}_\mu \delta v^\mu + \mathcal{T}_{\mu\nu}\delta h^{\mu\nu} + \mathcal{J}^\mu \delta M_\mu\right) \,. \tag{4.55}$$

The current $\mathcal{T}_{\mu\nu}$ is here symmetric and invariant under local Galilean transformations, while this is not the case for $\mathcal{S}_\mu$, $\mathcal{J}_\mu$. We may solve for $T^{\mu}{}_{\nu}$, $\mathcal{J}^\mu$ or $\mathcal{S}_\mu$, $\mathcal{T}_{\mu\nu}$, $\mathcal{J}^\mu$ in terms of $\mathcal{T}^{\mu}{}_{\hat{A}}$ or vice versa equating (4.46) and (4.54) or (4.55). We find the relevant formulas to be:

$$T^{\mu}{}_{\nu} = -v^\mu \mathcal{S}_\nu + h^{\mu\rho}\mathcal{T}_{\rho\nu} - \mathcal{J}^\mu M_\nu \tag{4.56a}$$

$$\mathcal{E}^\mu = 2h^{\mu\rho}v^\sigma \mathcal{T}_{\rho\sigma} - v^\mu \mathcal{S}_\rho v^\rho \tag{4.56b}$$

$$\mathcal{T}^{\mu}{}_{a} = v^\mu \mathcal{S}_\rho e^\rho_a - 2h^{\mu\rho}\mathcal{T}_{\rho\sigma}e^\sigma_a \,. \tag{4.56c}$$

5 NEWTON-CARTAN FROM THE NOETHER PROCEDURE

In this chapter we see how we from the starting point of globally invariant non-relativistic field theories can construct Newton-Cartan geometry at the linear level. This is done following the Noether procedure for both Galilean and Bargmann field theories. The analysis will make the coupling to geometry clear so that we can see what the vielbeins and gauge fields couple to. We shall give an argument showing that the graviphotonic connection is the minimal connection of Newton-Cartan geometry and determine its coupling for both Galilean and Bargmann theories.

5.1 THE NOETHER PROCEDURE IN GALILEAN FIELD THEORY

The way we obtained Newton-Cartan geometry in chapter 3 was more-or-less simply figuring out how to define a geometry that had the correct properties imposed by a non-relativistic theory of gravitation. In a sense this was a very constructive procedure with little regard for the field theories that we want to live on these manifolds, as they entered at the last stage of our analysis. Only after coupling the field theories to Newton-Cartan geometry was it possible to figure out what could be inferred about the field theory from the geometric framework.

One might wonder whether it is possible to turn this whole thing around and take the field theories as the starting point and see what they would imply about the geometry. This is indeed possible and is well-known in the literature as the Noether procedure [71, 72]. The goal of the Noether procedure is to obtain an action $S[\varphi, A]$ invariant under local symmetries starting from an action $S^{(0)}[\varphi]$ invariant only under global symmetries. This is done by adding gauge fields A_μ to the theory and modifying the action by adding coupling of the gauge fields to matter currents and modifying transformation laws. There is a systematic way of doing this iteratively which we review in the general case in appendix B.2.

In our case the global symmetries are the Galilean transformations (4.16) of some field theory with action $S^{(0)}[\varphi] = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi)$. The relevance of the Noether procedure in coupling the theory to gravity stems from the observation that at lowest order this is equivalent to having the theory invariant under local Galilean transformations. Following the Noether procedure, we will obtain just that. We are not aware of a complete treatment applying this to theories with non-relativistic spacetime symmetries. The advantage of doing this is that it is easy to see what degrees of freedom couple to what and how to obtain a minimal geometric structure that realizes the coupling to gravity.

At lowest order the general results of the Noether procedure (B.15) are universal for any theory. We simply add the coupling of the flat conserved spacetime Noether currents coupled to gauge fields to the action, which will then provide the lowest order local invariance of the action. In our case that the gauge fields are going to be inverse vielbeins τ_μ, $e_{\mu i}$ coupling to the energy and momentum currents which gives us the local translations, and gauge fields $\Omega_{\mu i}$, $\omega_{\mu ij}$ that couples to the non-conserved lift- and spin-current to provide local boosts and rotations. The action up to first order is then given by

$$S\left[\varphi,\bar{e},\bar{\tau},\overline{\Omega},\overline{\omega}\right] \equiv S^{(0)}\left[\varphi\right] + S^{(1)}\left[\varphi,\bar{e},\bar{\tau},\overline{\Omega},\overline{\omega}\right] , \tag{5.1a}$$

$$S^{(1)} \equiv \int_M \mathrm{d}^D x \left[\bar{\tau}_\mu E^\mu_{\text{can}} + \bar{e}_{\mu i} T^{i\mu}_{\text{can}} - \overline{\Omega}_{\mu i} w^{\mu i} + \frac{1}{2}\overline{\omega}_{\mu ij} s^{\mu ij}\right] \tag{5.1b}$$

where we assign the local (first order) transformation law to the gauge fields that according to the general theory should be

$$\begin{aligned}
\delta^{(1)}\bar{\tau}_\mu &= \partial_\mu \epsilon_0 && (5.2a)\\
\delta^{(1)}\bar{e}_{\mu i} &= \partial_\mu \epsilon_i + \lambda_{ij}\delta^j_\mu + \Lambda_i \delta^0_\mu && (5.2b)\\
\delta^{(1)}\overline{\Omega}_{\mu i} &= \partial_\mu \Lambda^i && (5.2c)\\
\delta^{(1)}\overline{\omega}_{\mu ij} &= \partial_\mu \lambda_{ij} \,. && (5.2d)
\end{aligned}$$

In our case the vielbeins τ_μ, $e_{\mu i}$ must couple to E^μ_{can}, $T^{\mu i}_{\text{can}}$, while the boost and rotation gauge fields $\Omega_{\mu i}$, $\omega_{\mu ij}$ must couple to the non-conserved $w^{\mu i}$, $s^{\mu ij}$. This is in contrast with a general internal gauge theory where all gauge fields would couple to conserved currents and only tensors can couple to non-conserved currents. The coupling of gauge fields to non-conserved currents is a special feature of gauging spacetime symmetries and can be traced back to the fact that the background changes under the local transformations [73]. For instance one sees from (4.16) that a boost comes with an infinitesimal translation and only the total variation is proportional to a conserved current as described on page 50. In principle one could go to next order in the iteration as outlined in appendix B.2, but lowest order will be sufficient to study aspects of the coupling to Newton-Cartan geometry.

The gauge fields $\Omega_{\mu i}$, $\omega_{\mu ij}$ can in principle be chosen arbitrarily, but we would like to see what the "minimal choice" is. This is where the improvements (4.25), (4.26) that simplify the conservation equations maximally enter the picture. Because we have chosen the improvements so they were written entirely in terms of $w^{\mu i}$, $s^{\mu ij}$, we can do straightforward integration by parts to find new objects $C_{\mu a}$, $C_{\mu ab}$ that couple to these currents:

$$S^{(1)} = \int_M \mathrm{d}^D x \left[\bar{\tau}_\mu E^\mu_{\text{can}} + \bar{e}_{0i} T^{i0}_{\text{imp}} + \frac{1}{2}\bar{s}_{ij} T^{ij}_{\text{imp}} - \overline{C}_{\mu i} w^{\mu i} + \frac{1}{2}\overline{C}_{\mu ij} s^{\mu ij}\right] , \tag{5.3}$$

where we write $\bar{v}_a = -\bar{e}_{0a}$ and have defined the perturbation of the spatial metric $\bar{s}_{ij} \equiv 2\bar{e}_{(ij)}$ in agreement with section 3.8 and define

$$\begin{aligned}
\overline{C}_{\mu a} &\equiv \overline{\Omega}_{\mu a} - \hat{\overline{\Omega}}_{\mu a} && (5.4a)\\
\overline{C}_{\mu ab} &\equiv \overline{\omega}_{\mu ab} - \hat{\overline{\omega}}_{\mu ab} && (5.4b)
\end{aligned}$$

$$\hat{\overline{\Omega}}_{\mu a} \equiv -\delta^c_\mu \left(\frac{1}{2}\partial_0 s_{ca} + \partial_{(c}\overline{v}_{a)}\right) \tag{5.4c}$$

$$\hat{\overline{\omega}}_{\mu ab} \equiv -\delta^0_\mu \left(\partial_0 \overline{e}_{[ab]} + \partial_{[a}\overline{v}_{b]}\right) + \delta^0_\mu \left(\partial_{[a} s_{b]c} - \partial_c \overline{e}_{[ab]}\right) . \tag{5.4d}$$

The details of the calculation is given in appendix D.5.1. We give a demonstration of the above for Galilean electrodynamics in section 8.3. The objects $\hat{\overline{\Omega}}_{\mu a}$, $\hat{\overline{\omega}}_{\mu ab}$ arise from the integration by parts are not actual connections on their own, as one can check that their transformation properties under local Galilean transformations are not exactly those of gauge fields at lowest order given by (3.99). However, these objects are nonetheless well-known. They are identical to the linearized pseudo-gauge fields of the pseudo-connection (3.96).

The reason why it is expected that we obtain just this is that the improvements of the currents we chose simplifies them maximally. As it is the canonical currents that the vielbeins couple to and we simply substitute the improved currents and the improvements that simplified the conservation equations the most, we "extract" as much of a connection that can be expressed in terms of the vielbeins as possible. Contrary to the Noether procedure for field theories with global Poincaré symmetry studied in appendix A.4 where the same line of reasoning gave the linearized Levi-Civita connection, we here do not exactly get a connection, but only the well-known linearized pseudo-connection (3.96). It might be profitable to review how this works in a more familiar setting, where we also explain why it is expected that we in that case obtains a connection from the improvements.

This proves at the linear level the previous result that the Galilean algebra cannot be realized on the vielbeins alone that we discussed in section 3.3.2. The objects $\overline{C}_{\mu a}$, $\overline{C}_{\mu ab}$ have the exact same structure as the pseudo-contortions, being the difference between a true connection and the pseudo-connection that was introduced in section 3.3.3.

Hence the above shows that the Noether procedure reproduces a large part of the Newton-Cartan geometry studied in chapter 3. We still need to determine a connection in this language. To obtain a good connection, we thus need to add something (i.e. choose $\overline{C}_{\mu a}$, $\overline{C}_{\mu ab}$), and the big question is what must minimally be added.

5.2 THE COUPLING OF THE BACKGROUND FIELD M_μ

Let us now investigate what the minimal connection is that we can construct. To realize a connection in this field theoretical approach, we need to be able to build it from the coupling of some field to currents of the theory that are not coupled to anything, because otherwise they will modify the conserved currents of the theory. The only such current we have left after improvements is the current $\psi^{\mu i}$ defined in (4.28). A coupling to $\psi^{\mu i}$ directly will obviously not work as it does not have the right index structure to give pseudo-contortions. Also it cannot appear as a combination that gives a non-trivial conserved current, because then there is an additional symmetry generator which would enhance the theory. The unique choice that satisfies all of these requirements is given by defining the current Φ^ρ expressed in terms of $\psi^{\mu i}$ as

$$\Phi^\rho \equiv -\left(\partial_0 \psi^{0j} + \partial_i \psi^{ij}\right) \delta^\rho_j + \partial_j \psi^{0j} \delta^\rho_0 \,. \tag{5.5}$$

Notice that Φ^ρ as defined is automatically conserved by the antisymmetry of ψ^{ij} and thus is a topological current. In section 8.1.3 we shall investigate an example that sheds some light on what the origin of the topological current is.

Given the structure of Φ^ρ it is then obvious that we need to introduce a background field M_μ, with linear order piece $\overline{M}_\mu$ that is defined to transforms under boosts as

$$\delta^{(1)} \overline{M}_\mu = \delta^a_\mu \Lambda_a \,. \tag{5.6}$$

This is of course the linearized version of our graviphoton of section 3.3.3.1 as it have the same linearized transformation. By adding the coupling $\overline{M}_\rho \Phi^\rho$ to the action $S^{(1)}$ we have

$$S^{(1)} = \int_M \mathrm{d}^D x \left[\overline{\tau}_\mu E^\mu_{\text{can}} + \overline{e}_{0i} T^{i0}_{\text{imp}} + \frac{1}{2} \overline{s}_{ij} T^{ij}_{\text{imp}} + \overline{M}_\rho \Phi^\rho \right] . \tag{5.7}$$

The coupling $\overline{M}_\rho \Phi^\rho$ can actually be rewritten as pseudo-contortions by doing some integration by parts and using the definition of $\psi^{\mu j}$ in terms of lift- and spin-currents. One finds

$$\begin{aligned} \overline{C}_{\mu i} &= -2\delta^0_\mu \partial_{[0} \overline{M}_{i]} - \delta^j_\mu \partial_{[j} \overline{M}_{i]} && (5.8\text{a}) \\ \overline{C}_{\mu ij} &= \delta^0_\mu \partial_{[i} \overline{M}_{j]} && (5.8\text{b}) \end{aligned}$$

and one may check that this defines a good connection by adding these to $\hat{\overline{\Omega}}_{\mu a}$, $\hat{\overline{\omega}}_{\mu ab}$ as

$$\begin{aligned} \overline{\Omega}_{\mu a} &\equiv \hat{\overline{\Omega}}_{\mu a} + \overline{C}_{\mu a} && (5.9\text{a}) \\ \overline{\omega}_{\mu ab} &\equiv \hat{\overline{\omega}}_{\mu ab} + \overline{C}_{\mu ab} \,. && (5.9\text{b}) \end{aligned}$$

These are exactly the gauge fields of the graviphotonic connection (3.51) at the linear level discussed in section 3.8.3. This shows that the graviphotonic connection should be thought of as the closest to a "Levi-Civita"-like connection we can have in a torsional Newton-Cartan geometry in the sense that it realizes the geometry minimally.

Since (5.9) defines a connection, we can obtain any other Galilean connection by adding "contortion"-like tensors to it after restoring the proper tensorial index structure. This will evidently result in connections that couple non-minimally to the lift- and spin-currents $w^{\mu i}$, $s^{\mu ij}$ similar to the extension of general relativity to Einstein–Cartan–Sciama–Kibble theory with torsionful connections.

We can now also present the main contribution to Newton-Cartan geometry of this thesis from the analysis that solves the mystery about the field theoretic interpretation of M_μ when it does not become the mass gauge field:

> *When M_μ is not enhanced to a* $\mathrm{U}(1)$ *gauge field coupling to the mass current, it remains a background field that couples to the topological current Φ^ρ.*

By integrating over a spatial hypersurface we obtain the corresponding charge is $Q = \int_S \mathrm{d}^d x\, \partial_j \psi^{0j} = 0$ using Stokes' theorem as expected. This does not imply that the current is completely trivial. In particular it might have interesting correlation functions with other currents of the theory. This would imply that it in principle is measureable.

5.3 THE COUPLING OF M_μ IN BARGMANN THEORIES

We have in the last section solved the mystery of what the background M_μ couples to in Galilean theories with no central charge. We can understand more about the current Φ^ρ by examining the relationship to theories with Bargmann symmetries. Again we will use the Noether procedure starting with some globally invariant field theory that furnishes some representation of the Bargmann group. The extra global $\mathrm{U}(1)$ symmetry generated by M in the Bargmann transformations (4.31) introduces an extra coupling in the $S^{(1)}$ action of a new gauge field m_μ that couples to the mass current J^μ_{can} that we determined in (4.32). The action now becomes

$$S\left[\varphi,\overline{e},\overline{\tau},\overline{\Omega},\overline{\omega},\overline{m}\right] \equiv S^{(0)}\left[\varphi\right] + S^{(1)}\left[\varphi,\overline{e},\overline{\tau},\overline{\Omega},\overline{\omega},\overline{m}\right] , \tag{5.10a}$$

$$S^{(1)} \equiv \int_M \mathrm{d}^D x \left[\overline{\tau}_\mu E^\mu_{\mathrm{can}} + \overline{e}_{\mu i} T^{i\mu}_{\mathrm{can}} - \overline{m}_\mu J^\mu_{\mathrm{can}} + \frac{1}{2}\overline{\omega}_{\mu ij} s^{\mu ij} - \overline{\Omega}_{\mu i} w^{\mu i}\right] \tag{5.10b}$$

where, according to the general theory, we should give m_μ the first order transformation

$$\delta^{(1)}\overline{m}_\mu = \partial_\mu \sigma + \delta^a_\mu \Lambda_a , \tag{5.11}$$

and the remaining fields the same transformations as in the Galilean case (5.2).

Using the improvements of the boost and mass currents discussed in section 4.3.2 that simplified the conservation equations maximally, we may again express the canonical currents in terms of the improved ones together with their improvements. Comparing to the Galilean case it is only the extra improvements of the mass current that needs to be introduced. Doing integration by parts on this, we eventually find

$$S^{(1)} = \int_M \mathrm{d}^D x \left[\overline{\tau}_\mu E^\mu_{\mathrm{can}} + \overline{e}_{0i} T^{i0}_{\mathrm{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\mathrm{imp}} - \overline{m}_\mu J^\mu_{\mathrm{imp}} + \frac{1}{2}\overline{C}_{\mu ij} s^{\mu ij} - \overline{C}_{\mu i} w^{\mu i}\right] , \tag{5.12a}$$

$$\begin{aligned} \overline{C}_{\mu a} &\equiv \overline{\Omega}_{\mu a} - \mathring{\overline{\Omega}}_{\mu a} && (5.12b) \\ \overline{C}_{\mu ab} &\equiv \overline{\omega}_{\mu ab} - \mathring{\overline{\omega}}_{\mu ab} && (5.12c) \end{aligned}$$

$$\begin{aligned} \mathring{\overline{\Omega}}_{\mu a} &\equiv -\delta^c_\mu \left(\frac{1}{2}\partial_0 s_{ca} + \partial_{(c}\overline{v}_{a)}\right) - 2\delta^0_\mu \partial_{[0}\overline{m}_{a]} - \delta^b_\mu \partial_{[b}\overline{m}_{a]} && (5.12d) \\ \mathring{\overline{\omega}}_{\mu ab} &\equiv -\delta^0_\mu \left(\partial_0 \overline{e}_{[ab]} + \partial_{[a}\overline{v}_{b]}\right) + \delta^0_\mu \left(\partial_{[a} s_{b]c} - \partial_c \overline{e}_{[ab]}\right) + \delta^0_\mu \partial_{[a}\overline{m}_{b]} . \end{aligned}$$

The connection we obtain now is exactly the linearized version of the graviphotonic connection of section 3.8.3. We give the details of the calculation in appendix D.5.2. The difference compared to what we did for Galilean theories in section 5.1 is now that $\overline{m}_\mu$ immediately has the interpretation as the linearized mass gauge field while the linearized background field $\overline{M}_\mu$ of the previous section does not a priori couple to a symmetry current. Assigning to $\overline{M}_\mu$ the transformation $\delta_\sigma \overline{M}_\mu = \partial_\mu \sigma(x)$ as the result of a local translation and boost discussed in section 4.4.2 effectively makes them identical. This proves at the linear level that in Bargmann theories this enhancement of $\overline{M}_\mu$ to a $\mathrm{U}(1)$ gauge field is automatic.

The topological current Φ^ρ is exactly the same as those improvements of the mass current we choose above in order to simplify the conservation equations as much as possible. In general we can therefore write

$$J^\mu_{\text{imp}} = J^\mu_{\text{can}} + \Phi^\mu \,, \tag{5.13}$$

which $\overline{M}_\mu$ then couples to. A Galilean theory then has a zero canonical mass current J^μ_{can} as is familiar from the analysis of chapter 4.

6 DIMENSIONAL REDUCTION

In this chapter we see how to obtain non-relativistic theories from relativistic ones by performing a null reduction. Both theories on curved Lorentzian manifold and flat Minkowski spacetime will be considered. We will also discuss how to null reduce the higher-dimensional energy-momentum tensor and see how the improvements can shed light on the currents of non-relativistic theories.

6.1 NULL REDUCTION OF LORENTZIAN SPACETIMES

6.1.1 *Frames and metric*

In section 2.4.4 we analyzed how $\mathrm{Barg}\,(d,1)$ can be embedded in $\mathrm{Poin}\,(d+1,1)$ through a null reduction of the algebra. This can be carried over to the vielbeins $e^{\hat{\mu}}_{\ \hat{A}}$, $e_{\hat{\mu}}^{\ \hat{A}}$ and Lorentzian metric $g_{\hat{\mu}\hat{\nu}}$ of a $D+1$ dimensional spacetime that allows a null Killing vector. Null reductions are a very efficient way of obtaining non-relativistic theories coupled to Newton-Cartan geometry from higher dimensional relativistic ones coupled to Lorentzian geometry as we shall demonstrate this in several examples in the following chapters.

For a given $D+1$-dimensional Lorentzian manifold we take the coordinates to be $x^{\hat{\mu}} = \left(x^+, x^i, x^-\right) = \left(x^\mu, u\right)$ so that the null coordinate is $u \equiv x^-$ with null vector ∂_u. The $u =$ constant null hypersurface as illustrated in figure 5 is then a Bargmann spacetime of dimension D to which ∂_u is the normal vector [12, 47]. For general Lorentzian covariant tensors we can perform a pullback to the null hypersurface to define covariant tensors on Bargmann spacetime immediately. On the other hand, general contravariation Lorentzian tensors can only be consistently be pulled back to the null hypersurface if they are invariant in the u-direction.

To find the correct null reduction of the vielbeins, we first recall the analysis of section 2.4.4 that showed that we must require that the null reduction leaves out (Lorentz) rotations in (x^μ, u)-hyperplanes. This breaks the higher-dimensional structure group $\mathrm{SO}\,(d+1,1)$ to $\mathrm{HGal}\,(d,1) = \mathrm{SO}\,(d) \ltimes \mathbb{R}^d$. A convenient (co)frame for the reduction of the (inverse) vielbeins is found by choosing the sections of the higher-dimensional (co)frame bundles as [74]

$$e_{\hat{\mu}}^{\ \hat{A}} = \begin{pmatrix} \tau_\mu & e_\mu^{\ a} & -M_\mu \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1a}$$

$$e^{\hat{\mu}}_{\ \hat{A}} = \begin{pmatrix} -v^\mu & e^\mu_{\ a} & 0 \\ -M_\mu v^\mu & M_\mu e^\mu_{\ a} & 1 \end{pmatrix} . \tag{6.1b}$$

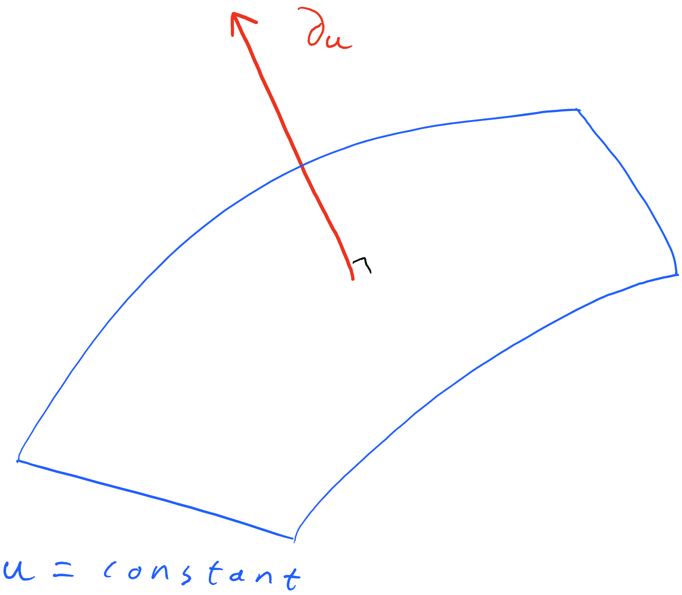

Figure 5: Geometry of the null reduction: The u = constant null hypersurface is the D dimensional Bargmann spacetime.

The naming of the various components is of course not arbitrary: They are in one-to-one correspondence with the objects we defined in chapter 3. It is then obvious that the vielbeins transform nicely under the reduction of SO$(d+1,1)$ to HGal$(d,1)$ with their usual transformation rules. This choice can be verified to satisfy $e_{\hat\mu}^{\hat A} e^{\hat\nu}_{\hat A} = \delta^{\hat\nu}_{\hat\mu}$ and $e_{\hat\mu}^{\hat A} e^{\hat\mu}_{\hat B} = \delta^{\hat A}_{\hat B}$ as they must being proper vielbeins, which makes it clear that a subset of the higher-dimensional vielbeins includes both the Galilean coframe $e_\mu{}^A = \left(\tau_\mu, e_\mu{}^a\right)$ of section 3.1 and the extended Bargmann coframe $\mathring{e}_\mu{}^{\hat A} = \left(\tau_\mu, e_\mu{}^a, -M_\mu\right)$ of section 3.5. In particular the null reduction gives a natural interpretation of the extended Bargmann frame bundle approach of section 3.4.1.

If we now write the Lorentzian metric $g_{\hat\mu\hat\nu} = \eta_{\hat A\hat B} e_{\hat\mu}{}^{\hat A} e_{\hat\nu}{}^{\hat B}$ and its inverse $g^{\hat\mu\hat\nu} = \eta^{\hat A\hat B} e^{\hat\mu}{}_{\hat A} e^{\hat\nu}{}_{\hat B}$ in terms of the vielbeins (6.1), we obtain [70]

$$g_{\hat\mu\hat\nu} = \begin{pmatrix} g_{\mu\nu} & g_{\mu u} \\ g_{u\nu} & g_{uu} \end{pmatrix} = \begin{pmatrix} \bar h_{\mu\nu} & \tau_\mu \\ \tau_\nu & 0 \end{pmatrix} \tag{6.2a}$$

$$g^{\hat\mu\hat\nu} = \begin{pmatrix} g^{\mu\nu} & g^{\mu u} \\ g^{u\nu} & g^{uu} \end{pmatrix} = \begin{pmatrix} h^{\mu\nu} & -\hat v^\mu \\ -\hat v^\nu & 2\tilde\Phi \end{pmatrix} \tag{6.2b}$$

$$\bar h_{\mu\nu} \equiv h_{\mu\nu} - 2\tau_{(\mu} M_{\nu)} \tag{6.2c}$$

$$\tilde\Phi \equiv -v^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu \tag{6.2d}$$

$$\hat v^\mu \equiv v^\mu - h^{\mu\nu} M_\nu\,, \tag{6.2e}$$

$$\hat e_\nu{}^a \equiv e_\nu{}^a - e^{\mu a} M_\mu \tau_\nu\,. \tag{6.2f}$$

We see that all of these quantities are known as we defined the exact same ones when considering the graviphotonic connection in section 3.3.3.1. Notice that ∂_u is a Killing vector of the metric in these coordinates. Had we done the simplest Kaluza-Klein reduction of a metric with such a null Killing vector ∂_u, we would have obtained the exact same result [75]. The difference compared to the usual reduction along a spatial Killing vector is that the nullness of ∂_u implies that the would-be dilaton scalar is zero as $g_{uu} = \boldsymbol{g}\left(\partial_u, \partial_u\right) = 0$. We can write the higher-dimensional metric in a component free form in some coordinate system as

$$g_{\hat{\mu}\hat{\nu}} \mathrm{d}x^{\hat{\mu}} \mathrm{d}x^{\hat{\nu}} = 2\tau_\mu \mathrm{d}x^\mu \otimes \left(\mathrm{d}u - M_\nu \mathrm{d}x^\nu\right) + h_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu . \tag{6.3}$$

Here it is easy to see that the most general transformation that preserves the form of the null Killing vector ∂_u is given by

$$u' = u + \sigma\left(x\right) \tag{6.4a}$$

$$M'_\mu = M_\mu + \partial_\mu \sigma\left(x\right) , \tag{6.4b}$$

where we see that the graviphoton M_μ transforms as a $\mathrm{U}\left(1\right)$ gauge field under such a coordinate transformation and u gets a local x^μ-dependent translation. This corresponds to a particular local translation along the null direction u , so it is the equivalent of the transformation (4.44) and what we discussed in section 4.4.2.

In the particular choice of frame (6.1) the measure reduces to

$$\sqrt{|g|} = e = \det\left(\boldsymbol{\tau}, \boldsymbol{e}^a\right) , \tag{6.5}$$

which is the same as the measure of D-dimensional Galilean spacetimes introduced in (3.16). When we write higher-dimensional actions, we will then be able to pull out the D-dimensional non-relativistic action with the correct measure.

6.1.2 *Dimensional reduction of the Hilbert EM tensor*

Assume now that we have an action $\hat{S}\left[\varphi, g\right] = \int \mathrm{d}x^{D+1} \sqrt{|g|} \mathcal{L}\left(\varphi, \nabla\varphi\right)$ for a $D+1$ dimensional field theory coupled to Lorentzian geometry. The response of the variation of the action wrt. the metric $g_{\hat{\mu}\hat{\nu}}$ is by definition the Hilbert energy-momentum tensor (A.23):

$$\delta\hat{S}\left[\varphi, g\right] = \frac{1}{2} \int \mathrm{d}x^{D+1} \sqrt{|g|} T^{\hat{\mu}\hat{\nu}}_{\text{Hil}} \delta g_{\hat{\mu}\hat{\nu}} . \tag{6.6}$$

The covariant conservation law for the current follows from the diffeomorphic Ward identity where we take $\delta g_{\hat{\mu}\hat{\nu}} = \mathcal{L}_\xi g_{\hat{\mu}\hat{\nu}}$ and the result is:

$$0 = T^{\hat{\mu}\hat{\nu}}_{\text{Hil}} \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} - 2\frac{1}{\sqrt{|g|}} \partial_{\hat{\mu}} \left(\sqrt{|g|} T^{\hat{\mu}\hat{\nu}}_{\text{Hil}} g_{\hat{\nu}\hat{\rho}}\right) . \tag{6.7}$$

In terms of the higher-dimensional Levi-Civita connection, which is only a part of the full connection we choose, this is equivalent to $\hat{\nabla}_{\hat{\mu}} T^{\hat{\mu}\hat{\nu}}_{\text{Hil}} = 0$. We can also write the components of the variation in terms of the reduced vielbeins using the variation of $\delta g_{\hat{\mu}\hat{\nu}}$ expressed in terms of the reduction (6.2a) [70]. For this we find

$$\begin{aligned}\frac{1}{2}T_{\text{Hil}}^{\hat{\mu}\hat{\nu}}\delta g_{\hat{\mu}\hat{\nu}} &= \frac{1}{2}T_{\text{Hil}}^{\mu\nu}\delta\bar{h}_{\mu\nu} + T_{\text{Hil}}^{\mu u}\delta\tau_\mu \\ &= \frac{1}{2}T_{\text{Hil}}^{\mu\nu}\left(\delta h_{\mu\nu} - 2\delta\tau_{(\mu}M_{\nu)} - 2\tau_{(\mu}\delta M_{\nu)}\right) + T_{\text{Hil}}^{\mu u}\delta\tau_\mu\,. \end{aligned} \tag{6.8}$$

This can easily be expressed in terms of the variations in section 4.4.3 where we gave three different definitions of objects that encoded the information about the EM tensor. In particular we can most easily convert the above to a variation of $\mathring{e}_\mu^{\hat{A}}$ which defined the current $\mathcal{T}^\mu{}_{\hat{A}}$ and using $\delta h_{\mu\nu} = 2e^a_{(\mu}\delta e^a_{\nu)}$ we find

$$\mathcal{E}^\mu = T_{\text{Hil}}^{\mu u} - T_{\text{Hil}}^{\mu\nu}M_\nu \tag{6.9a}$$

$$\mathcal{T}^\mu{}_a = T_{\text{Hil}}^{\mu\nu}e_{\mu a} \tag{6.9b}$$

$$\mathcal{J}^\mu = -T_{\text{Hil}}^{\mu\nu}\tau_\nu\,. \tag{6.9c}$$

The other currents defined in section can be found by converting the above ones using (4.56).

6.1.3 *Reduction of the Levi-Civita connection*

Using the formulas of the reduction we can null reduce the Levi-Civita connection as it is expressed entirely in terms of the vielbeine. For the Christoffel symbols we find using the formulas of (6.2)

$$\begin{aligned}\hat{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\hat{\sigma}}\left(\partial_\mu g_{\nu\hat{\sigma}} + \partial_\nu g_{\mu\hat{\sigma}} - \partial_{\hat{\sigma}}g_{\mu\nu}\right) \\ &= -\hat{v}^\lambda\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\lambda\sigma}\left(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}\right)\,. \end{aligned} \tag{6.10}$$

We see that the result is the graviphotonic connection (3.51) with no torsion. As we have not constrained $\boldsymbol{\tau}$ to be closed in the null reduction, this shows that doing the direct reduction of $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}}$ will only give a good connection for the Newton-Cartan hypersurface if we afterwards by hand enforce $\mathrm{d}\boldsymbol{\tau} = 0$.

It is actually possible to obtain a more general Galilean connection from the higher-dimensional Levi-Civita connection with $\boldsymbol{\tau}$ satisfying only the Frobenius condition $\boldsymbol{\tau}\wedge\mathrm{d}\boldsymbol{\tau} = \mathbf{0}$ and thus the spacetime being of the TTNC kind discussed in section 3.3.2. This is obtained through a different reduction of the Levi-Civita connection that preserves $\nabla_\rho\tau_\mu = \nabla_\rho h^{\mu\nu} = 0$, which is discussed in more details by Hartong [47]. In particular this shows that one can obtain a torsionful Galilean connection by null reduction of the Levi-Civita connection, but it is not possible to obtain completely general torsionful connections.

6.2 NULL REDUCTION OF MINKOWSKI SPACETIME

In the special case of flat Minkowski spacetime the null reduction is very simple and we can give general formulas for derivatives etc. [65]. The higher-dimensional vielbeins become identity matrices so the metric becomes (2.43) and the general action is of the form $\hat{S}[\varphi] = \int \mathrm{d}x^{D+1}\mathcal{L}(\varphi, \partial\varphi)$. The higher-dimensional partial derivative $\partial_{\hat{\mu}}$ decomposes to $(\partial_\mu, \partial_u)$. We can assume that the fields are mass eigenvectors satisfying $(M)_{\ell\bar{\ell}}\,\varphi_{\bar{\ell}} = m\varphi_\ell$ as discussed in section 4.1.2. Since mass is the momentum in u-direction this implies that they can always be written as

$$\varphi_\ell(t, \boldsymbol{x}, u) = e^{+imu}\phi_\ell(t, \boldsymbol{x}) \ . \tag{6.11}$$

If the corresponding non-relativistic fields are massless, then the higher-dimensional can have no dependence on u. The representation of the higher-dimensional Minkowski group that the field components carry reduce to one of the representations of $\mathrm{HGal}(d, 1)$ like those considered in section 4.1.

> **Example 6.1** (Vector representation)**.** When null reducing the defining $D+1$-vector representation of the Minkowski group, the representation becomes the extended Galilean vector representation (4.14).

With the higher-dimensional canonical energy-momentum tensor $\hat{T}^{\hat{\mu}\hat{\nu}}_{\mathrm{can}}$ and its improvements given in appendix A.3 there is now a short-cut to finding the corresponding non-relativistic symmetry currents. Specializing the equations (6.9) to flat space, we find

$$\hat{T}^{\mu u}_{\mathrm{can}} = E^{\mu}_{\mathrm{can}} \tag{6.12a}$$
$$\hat{T}^{\mu i}_{\mathrm{can}} = T^{\mu i}_{\mathrm{can}} \tag{6.12b}$$
$$\hat{T}^{\mu 0}_{\mathrm{can}} = J^{\mu}_{\mathrm{can}} \,. \tag{6.12c}$$

If it happens that the higher-dimensional field has no dependence on u, then this implies that there is no $\mathrm{U}(1)$ symmetry current and J^μ_{can} is zero. The symmetric (Belinfante-Rosenfeld) EM tensor $\hat{T}^{\hat{\mu}\hat{\nu}}_{\mathrm{can}}$ is obtained by choosing the improvements as (A.21). For the reduction this leads to improvements for all currents given by

$$E^{\mu}_{\mathrm{imp}} = E^{\mu}_{\mathrm{can}} + \partial_\lambda A^{\lambda\mu 0 u} \tag{6.13a}$$
$$T^{\mu i}_{\mathrm{imp}} = T^{\mu i}_{\mathrm{can}} + \partial_\lambda A^{\lambda\mu i} \tag{6.13b}$$
$$J^{\mu}_{\mathrm{imp}} = J^{\mu}_{\mathrm{can}} + \partial_\lambda A^{\lambda\mu 0} \,. \tag{6.13c}$$

The dimensional reduction is a good route to choosing the improvements of the energy current, that were otherwise completely undetermined by the analysis of section 4.2.1. Notice also that even if it happens that there is no conserved mass current, then it does still receive improvements. The current J^μ_{imp} contains in this case only total derivatives and in fact as we discussed in section 5.3 exactly the current Φ^μ. We will see an example of the relevance of this in section 8.1.3.

7 THE SCHRÖDINGER MODEL

In this chapter we will study the Schrödinger model on both flat and curved Newton-Cartan spacetime. We will see how it can be obtained from a null reduction of a relativistic theory. Its conserved currents are derived, and we will see how it fits into the general theory developed so far.

7.1 SCHRÖDINGER MODEL ON FLAT NEWTON-CARTAN SPACETIME

The D-dimensional free Schrödinger model is defined by the action on a flat Newton-Cartan spacetime as

$$S^{(0)} = \int_M \mathrm{d}x^D \mathcal{L}^{(0)} = \int_M \mathrm{d}x^D \, \left(im\phi^*\partial_0\phi - im\phi\partial_0\phi^*\right) - \partial_i\phi^*\partial^i\phi \,. \tag{7.1}$$

The EOMs are given by the Euler-Lagrange equations

$$i\partial_0\phi = -\frac{1}{2m}\partial_i\partial^i\phi \,. \tag{7.2}$$

This is of course the well-known free Schrödinger equation, which in quantum mechanics is interpreted as describing the time-evolution of the wave-function $\phi\left(t, \boldsymbol{x}\right)$ for a single particle [76]. As a classical field theory it is a scalar Bargmann field that we also considered briefly in example 4.2. Besides the Galilean spacetime symmetries, it also has a global $\mathrm{U}\left(1\right)$ symmetry $\phi' = e^{+im\sigma}\phi$ and the corresponding conserved currents are given by (4.17), (4.32):

$$\begin{aligned}
T^{ij} &= 2\partial^{(i}\phi^*\partial^{j)}\phi + \delta^{ij}\mathcal{L}^{(0)} && \text{(7.3a)}\\
T^{0i} &= im\phi\partial^i\phi^* - im\phi^*\partial^i\phi && \text{(7.3b)}\\
J^\mu &= \left[2m^2\phi\phi^*\right]\delta^\mu_0 + \left[im\phi\partial_i\phi^* - im\phi^*\partial_i\phi\right]\delta^{\mu i} && \text{(7.3c)}\\
E^\mu &= \left[im\phi\partial_0\phi^* - im\phi^*\partial_0\phi + \mathcal{L}^{(0)}\right]\delta^\mu_0 + \left[\partial_0\phi^*\partial^i\phi + \partial^i\phi^*\partial_0\phi\right]\delta^{\mu i} \,. && \text{(7.3d)}
\end{aligned}$$

There are no improvements like those of section 4.3.2 to be done because $s^{\mu ij} = w^{\mu i} = 0$. The mass current J^μ is in QM known as the probability current and the conservation of it guaranties the interpretation of $\left|\phi\left(t, \boldsymbol{x}\right)\right|^2 \propto J^0$ as probability density.

Besides these symmetries, the action is also invariant under a Lifshitz scaling with $z = 2$ and a so-called temporal special conformation transformation. The maximal symmetry group of the free Schrödinger model is the Schrödinger group which we consider in appendix C, which is one of the non-relativistic conformal groups [28, 77].

7.2 NULL REDUCTION OF MASSIVE SCALAR COUPLED TO GRAVITY

We now want to employ the methods of chapter 6 to obtain the Schrödinger model on a general Newton-Cartan background. Our stating point is the complex Klein-Gordon scalar field $\Psi(t, \boldsymbol{x}, u)$ with potential $V(|\Psi|)$ coupled to Lorentzian geometry as

$$\hat{S}_{\mathrm{KG}}[\Psi, g] = \int \mathrm{d}^{D+1}x\sqrt{-g}\left(-g^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\Psi^*\partial_{\hat{\nu}}\Psi - V(|\Psi|)\right) . \tag{7.4}$$

Since we want a Bargmann scalar of mass m, the higher-dimensional field must be of the form

$$\Psi(t, \boldsymbol{x}, u) = e^{+imu}\phi(t, \boldsymbol{x}) . \tag{7.5}$$

Using the results (6.2) we can easily perform the null reduction decomposing the metric and taking derivatives of the scalar field. The result is

$$\begin{aligned} S_{\mathrm{Sch}}\left[\phi, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] &= \int \mathrm{d}^D xe\left[imv^\nu\phi^* D_\nu\phi - imv^\mu\phi D_\mu\phi^*\right. \\ &\qquad \left. -h^{\mu\nu}D_\mu\phi^* D_\nu\phi - V(|\Psi|)\right] , \end{aligned} \tag{7.6}$$

where we have defined the covariant derivative

$$D_\mu\phi \equiv \partial_\mu\phi - imM_\mu\phi . \tag{7.7}$$

We give the details of this reduction in appendix D.7.1. This action thus describes the Schrödinger model coupled to Newton-Cartan geometry, which in flat space has the Schrödinger equation of quantum mechanics as EOM. This action would be the relevant one to use if we wanted to study the minimal coupling of quantum mechanics to non-relativistic gravity and is now completely covariantized [78]. Each term is not invariant under boosts etc. on their own as for example v^μ transforms under boosts, but the whole action is. The Bargmann scalar ϕ now transforms under the covariant generalization of the projective transformation (4.6).

We see that M_μ is the $\mathrm{U}(1)$ mass gauge field associated with the transformation (6.4a), which here gives a local $\mathrm{U}(1)$ transformation of the scalar field as

$$\phi(x) \rightarrow e^{+im\sigma(x)}\phi(x) \tag{7.8a}$$

$$M_\mu \rightarrow M_\mu + \partial_\mu\sigma(x) . \tag{7.8b}$$

7.2.1 *Energy, momentum, mass and Ward identities*

To determine the objects that encode information about energy, momentum and the mass current, we vary the background wrt. the fields v^μ, $h^{\mu\nu}$ and M_μ. This leads us directly to obtaining the currents $\mathcal{S}_\mu$, $\mathcal{T}_{\mu\nu}$, $\mathcal{J}^\mu$ defined in (4.55)

$$\mathcal{S}_\mu = -\mathcal{L}\tau_\mu - im\phi^* D_\mu\phi + im\phi D_\mu\phi^* \tag{7.9a}$$
$$\mathcal{T}_{\mu\nu} = -\frac{1}{2}\mathcal{L}h_{\mu\nu} - D_{(\mu}\phi^* D_{\nu)}\phi \tag{7.9b}$$
$$\mathcal{J}^\mu = -2m^2\phi\phi^* v^\mu + imh^{\mu\nu}\left(\phi^* D_\nu\phi - \phi D_\nu\phi^*\right). \tag{7.9c}$$

We give details of the calculation in appendix D.7.2. Here the current $\mathcal{J}^\mu$ satisfies the covariant conservation equation (4.51) non-trivially using the EOM. We can calculate the other components of the current $\mathcal{T}^\mu{}_{\hat{A}} = \left(\mathcal{E}^\mu, \mathcal{T}^\mu{}_a, -\mathcal{J}^\mu\right)$ using (4.56) and find

$$\begin{aligned}\mathcal{E}^\mu &= v^\mu\left(im\phi D_\rho\phi^* - im\phi^* D_\rho\phi\right)v^\rho \\ &\quad -h^{\mu\rho}v^\sigma D_{(\rho}\phi^* D_{\sigma)}\phi - \mathcal{L}v^\mu \end{aligned}\tag{7.10a}$$
$$\begin{aligned}\mathcal{T}^\mu{}_a &= v^\mu\left(im\phi D_\rho\phi^* - im\phi^* D_\rho\phi\right)e^\rho_a \\ &\quad +h^{\mu\rho}\left(\mathcal{L}h_{\rho\sigma} + 2D_{(\rho}\phi^* D_{\sigma)}\phi\right)e^\sigma_a\,. \end{aligned}\tag{7.10b}$$

It is instructive to check that with this more elaborate framework can obtain correct flat versions of the currents (7.3) using the results of section 3.7. With this we should also take $D_\mu \to \partial_\mu$ and after a small simplification we reproduce (7.3) with $\mathcal{T}^\mu{}_a = T^\mu{}_a$, $\mathcal{E}^\mu = E^\mu$ and $\mathcal{J}^\mu = J^\mu$ as expected.

7.3 LINEARIZATION

It is illuminating to see how this specific example reproduces the results predicted by the general theory of section 5.3. We use the general results of section 3.8 for how to linearize Newton-Cartan geometry. Keeping only zeroth and first-order terms, we then find that the action (7.6) (with $V(|\phi|) = 0$ for simplicity) linearizes as expected and we can read off the flat space currents in agreement with the correct flat currents (7.3). We can therefore write the linearized coupling as

$$\overline{S} = \int \mathrm{d}^{d+1}x\left[\mathcal{L}^{(0)} + \overline{\tau}_\rho E^\rho + \frac{1}{2}s_{ij}T^{ij} + \overline{e}_{i0}T^{0i} - \overline{M}_\rho J^\rho\right]. \tag{7.11}$$

One sees here that $\overline{M}_\rho$ couples to a conserved non-zero mass current as was predicted by the Noether procedure of section 5.3.

7.4 CORRELATION FUNCTIONS IN FLAT SPACETIME

7.4.1 *Dimensional reduction of relativistic propagator*

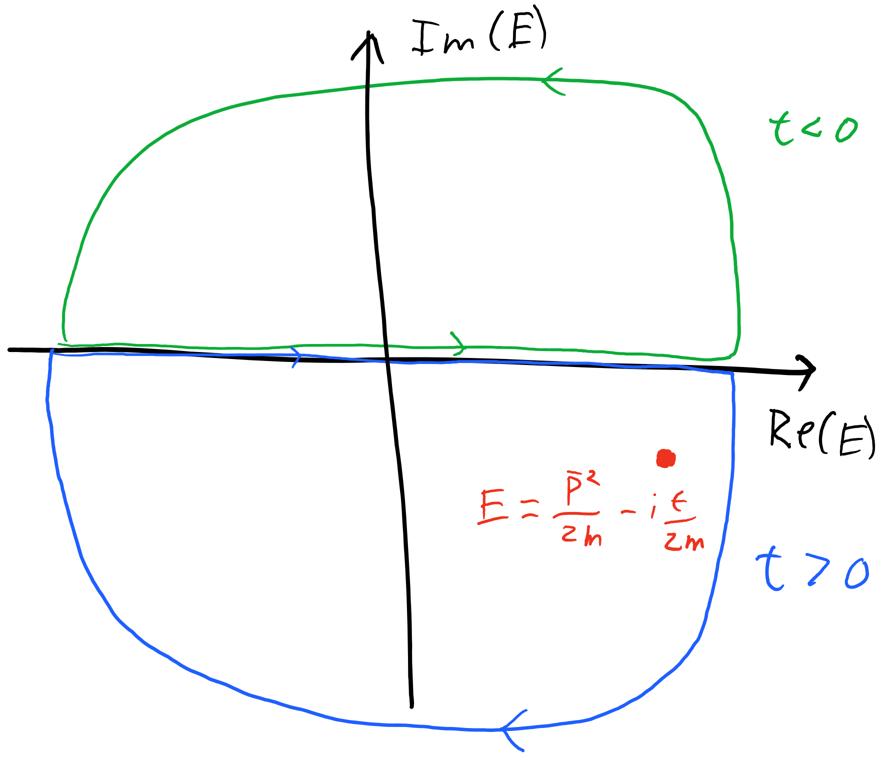

Figure 6: Deformation of the contour of integration for the Fourier transform of the propagator (7.13). Notice that there is only contributions from a single pole.

Propagators are essential to understand the dynamics of the theory. Moreover, they gives a simple way to solve inhomogeneous EOMs with sources being Green's functions. It would be useful if we could obtain the propagators of the non-relativistic theory by a null reduction of their relativistic cousins, which we will see is the case. The higher-dimensional momentum in the chosen null coordinates is given by

$$p^2 = -2p_+p_- + \boldsymbol{p}^2 = -2Em + \boldsymbol{p}^2 . \tag{7.12}$$

The massless Feynman propagator for the Klein-Gordon equation in $D+1$ dimensions in momentum space in the light-cone coordinates of the null reduction is given by

$$G(E, \boldsymbol{p}) = \frac{1}{-2Em + \boldsymbol{p}^2 - i\epsilon} . \tag{7.13}$$

The mass $m \neq 0$ is fixed by the corresponding mass of the field and hence it is not a variable of the propagator [79]. Unlike the relativistic case we see that here there is only a single pole at $E = \boldsymbol{p}^2/2m$ for $\epsilon \to 0$, which is exactly the on-shell Newtonian relation between energy, momentum and mass.

To obtain the position space propagator we have to perform a Fourier transformation as well. We can then perform the E integral using the residue theorem for $t > 0$ and $t < 0$ using the contours shown in figure 6, only the first one will give something non-zero. After performing the $\boldsymbol{p}$-integral, the result is

$$\begin{aligned}\langle \phi\left(t, \boldsymbol{x}\right) \phi^*\left(0, \boldsymbol{0}\right)\rangle &= \int_{\mathbb{R}^D} \frac{\mathrm{d}^d \boldsymbol{p} \mathrm{d}E}{(2\pi)^D} \frac{-i}{-2Em + \boldsymbol{p}^2 - i\epsilon} e^{iEt - i\boldsymbol{p}\cdot\boldsymbol{x}} \\ &= \theta\left(t\right) \frac{1}{2m} \left(-\frac{im}{2\pi t}\right)^{d/2} e^{-\frac{im\boldsymbol{x}^2}{2t}} . \end{aligned} \tag{7.14}$$

This is indeed the well-known free Schrödinger propagator which solves the Schrödinger equation (7.2) [80]. As the time-ordered massless relativistic propagators of theories with a standard kinetic term $\sim \partial^\mu \varphi^\ell \partial_\mu \varphi_\ell$ are proportional to the Feynman propagator this verifies that a null reduction will give the corresponding non-relativistic versions which is very useful. The appearance of the Heaviside step-function $\theta\left(t\right)$ shows that the non-relativistic propagator is causal in the sense that there is no correlation in the past direction $t < 0$.

Example 7.1 (Sources and potentials)**.** Given a source $J\left(t, \boldsymbol{x}\right)$ in the free Schrödinger equation (7.2)

$$i\partial_0 \phi + \frac{1}{2m} \partial_i \partial^i \phi = J , \tag{7.15}$$

the inhomogeneous solution is given by the integral with the free Schrödinger propagator as the kernel:

$$\phi\left(t, \boldsymbol{x}\right) = \int_{\mathbb{R}^D} \mathrm{d}^D x' \, \langle \phi\left(t, \boldsymbol{x}\right) \phi^*\left(t', \boldsymbol{x}'\right)\rangle \, J\left(t', \boldsymbol{x}'\right) . \tag{7.16}$$

This kind of interaction is not of relevance if we want to consider couplings to a potential $V\left(t, \boldsymbol{x}\right)$ where the relevant EOM is

$$i\partial_0 \phi + \frac{1}{2m} \partial_i \partial^i \phi = V\phi . \tag{7.17}$$

We can solve for the full propagator $G\left(t, \boldsymbol{x} | t', \boldsymbol{x}'\right)$ of this equation in terms of the free Schrödinger propagator, which has the well-known Born series as a formal solution [81]

$$\begin{aligned} G\left(t, \boldsymbol{x} | t', \boldsymbol{x}'\right) = \langle \phi\left(t, \boldsymbol{x}\right) \phi^*\left(t', \boldsymbol{x}'\right)\rangle + \int_{\mathbb{R}^D} \mathrm{d}^D x'' \, \langle \phi\left(t, \boldsymbol{x}\right) \phi^*\left(t'', \boldsymbol{x}''\right)\rangle \\ \times V\left(t'', \boldsymbol{x}''\right) \langle \phi\left(t'', \boldsymbol{x}''\right) \phi^*\left(t', \boldsymbol{x}'\right)\rangle + \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \dots . \end{aligned} \tag{7.18}$$

One could also be more general and add self-interactions of cubic, quartic etc. order, which we would have to study perturbatively as they make the EOMs non-linear in ϕ.

7.4.2 *Massless limit of correlator*

If we now take the $m = 0$ limit in the momentum space propagator (7.13), we will instead obtain

$$G(\boldsymbol{p}) = \frac{1}{\boldsymbol{p}^2 - i\epsilon}. \tag{7.19}$$

Performing the Fourier transformation to go to position space, the $\boldsymbol{p}$-integral can be done directly using the formula (E.2). For the E-integral we obtain a δ-function in time because only the complex phase e^{iEt} now depends on E, and in total we thus find

$$\begin{aligned} \langle \varphi(t, \boldsymbol{x}) \varphi^*(0, \boldsymbol{0}) \rangle &= \int_{\mathbb{R}^D} \frac{\mathrm{d}^d \boldsymbol{p} \mathrm{d}E}{(2\pi)^D} \frac{-i}{\boldsymbol{p}^2 - i\epsilon} e^{iEt - i\boldsymbol{p}\cdot\boldsymbol{x}} \\ &= \delta(t) \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} \frac{1}{\|\boldsymbol{x}\|^{d-2}}. \end{aligned} \tag{7.20}$$

The appearance of $\delta(t)$ is thus a feature of non-relativistic massless theories as we shall also verify for Galilean electrodynamics in the next chapter. It can be interpreted as the fact that in such theories interactions are instantaneous and there are no wave-phenomena that can propagate, which agrees with the intuition behind taking $c \to \infty$. We can also obtain the same from taking $m \to 0$ limit of the free Schrödinger propagator (7.14) directly, but it is more cumbersome to take this route.

8 GALILEAN ELECTRODYNAMICS

In this chapter we shall study Galilean electrodynamics as another interesting example of a non-relativistic theory. We will derive its action from a null reduction of the relativistic Maxwell electrodynamics in both flat and curved spacetime. The conserved currents are derived and we see how these and their couplings to the geometry fit with the general results derived for Galilean theories. Finally we shall study an interesting extension where we couple it to the Schrödinger model to obtain the non-relativistic analog of scalar QED on curved space.

8.1 GED ON FLAT NEWTON-CARTAN SPACETIME

8.1.1 *The non-relativistic limit of Le Bellac and Lévy-Leblond*

Galilean electrodynamics was first obtained by Le Bellac and Lévy-Leblond [82] as the non-relativistic limit(s) of Maxwellian Electrodynamics (MED) in $D = 4$ [83]. As discussed in section 1.2 it was realized before the appearance of special relativity that Maxwellian electrodynamics was not compatible with non-relativistic physics. It has thus taken more than 70 years to find a theory that in some respect was the theory physicists of the 19th century could reasonably have been expected to formulate.

Let us investigate how to take $c \to \infty$ in Maxwell's equations consistently following [82]. It turns out there are two limits to take, either the "electric" or the "magnetic", loosely corresponding to what type of phenomena are dominant. To see this intuitively, one realizes that the 4-current $J^\mu = (\rho, \boldsymbol{J})$ of MED can be said to be either "mostly electric" if $|\rho| \gg \|\boldsymbol{J}\|$ or "mostly magnetic" if $\|\boldsymbol{J}\| \gg |\rho|$. For the corresponding field strengths, this is then equivalent to having either electric or magnetic effects dominating, so our definition of the electric and magnetic limits may be taken more precisely to mean:

$$\text{Electric limit:} \qquad \|\boldsymbol{E}\| \gg c\,\|\boldsymbol{B}\| \tag{8.1a}$$

$$\text{Magnetic limit:} \qquad \|\boldsymbol{E}\| \ll c\,\|\boldsymbol{B}\|\ . \tag{8.1b}$$

Fixing a particular gauge of the 4-potential $A^\mu = \left(A^0, \boldsymbol{A}\right)$, we see that performing a Lorentz boost with velocity $v/c \ll 1$ gives in either limits the transformations:

$$\text{Electric limit:} \qquad A^{0\prime} = A^0 \tag{8.2a}$$
$$\boldsymbol{A}' = \boldsymbol{A} + A^0\boldsymbol{v} \tag{8.2b}$$

$$\text{Magnetic limit:} \qquad A^{0\prime} = A^0 - \boldsymbol{v}\cdot\boldsymbol{A} \tag{8.2c}$$
$$\boldsymbol{A}' = \boldsymbol{A} \tag{8.2d}$$

Notice that these transformations are Galilean (co)vector transformations, with 4-potential being a Galilean vector in the electric limit transforming as (2.22) and as a covector in the magnetic limit transforming as (2.23). This shows that electric and magnetic limits are really both non-relativistic limits. Taking the electric limit of Maxwell's equations gives

$$\partial_i B^i = 0 \tag{8.3a}$$
$$\partial_i E^i_e = \frac{1}{\epsilon_0}\rho \tag{8.3b}$$
$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i + \mu_0\epsilon_0\partial_t E^i \tag{8.3c}$$
$$\epsilon^i{}_{jk}\partial^j E^k = 0\,, \tag{8.3d}$$

and for the magnetic limit

$$\partial_i B^i = 0 \tag{8.4a}$$
$$\partial_i \tilde{E}^i = \frac{1}{\epsilon_0}\tilde{\rho} \tag{8.4b}$$
$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i \tag{8.4c}$$
$$\epsilon^i{}_{jk}\partial^j \tilde{E}^k = -\partial_t B^i\,, \tag{8.4d}$$

with fields defined as

$$cB^i \equiv \epsilon^i{}_{jk}\partial^j A^k \tag{8.5a}$$
$$E^i \equiv -\partial^i\varphi \tag{8.5b}$$
$$\tilde{E}^i \equiv -\partial^i\tilde{\varphi} - \frac{1}{c}\partial_t A^i\,. \tag{8.5c}$$

Here we put a tilde on electric fields in the magnetic limit to separate them from those in the electric limit, while the magnetic field is unchanged. However, the electric limit does not have the Faraday term that gives rise to electromagnetic induction in MED from a time-varying magnetic field. On the contrary in the magnetic limit we do not have the displacement current due to a time-varying electric field in Ampère's law with Maxwell's addition and there is only global charge conservation. This shows that in both limits electromagnetic waves are absent, and we essentially have electrostatics and magnetostatics respectively.

The Lorentz force law reduce in either limits to

$$\text{Electric limit:} \qquad \boldsymbol{F} = \int \mathrm{d}^3x\, \rho \boldsymbol{E} \tag{8.6a}$$

$$\text{Magnetic limit:} \qquad \tilde{\boldsymbol{F}} = \int \mathrm{d}^3x\, \boldsymbol{J} \times \boldsymbol{B}\,, \tag{8.6b}$$

which agrees with the initial definition of the limits.

8.1.2 *GED from null reduction*

We may also obtain GED in D spacetime dimensions from considering a flat null reduction of MED in $D+1$ spacetime dimensions in an elegant way [65]. All fields depends only on $(t, \boldsymbol{x})$ and we take $c = \epsilon_0 = \mu_0 = 1$ for simplicity. The starting point is to write the $D+1$-dimensional gauge field as

$$A_{\hat{\mu}} = (-\tilde{\varphi}, \boldsymbol{A}, \varphi)\ , \tag{8.7}$$

or with upper indices raising the indices with (2.43)

$$A^{\hat{\mu}} = (\varphi, \boldsymbol{A}, -\tilde{\varphi})\ . \tag{8.8}$$

Writing the higher-dimensional current $J^{\hat{\mu}} = (\rho, \boldsymbol{J}, -\tilde{\rho})$ we find that the flat Maxwell Lagrangian density

$$\hat{\mathcal{L}}^{(0)}_{\text{MED}} = -\frac{1}{4} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} + A_{\hat{\mu}} J^{\hat{\mu}} \tag{8.9}$$

using the method of null reduction from section 6.2 can be written as

$$\mathcal{L}^{(0)}_{\text{GED}} = -\frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} + \boldsymbol{E}\cdot\tilde{\boldsymbol{E}} + \frac{1}{2}a^2 - \rho\tilde{\varphi} + \boldsymbol{J}\cdot\boldsymbol{A} - \tilde{\rho}\varphi\,. \tag{8.10}$$

The field strengths are defined by (8.5) plus the scalar

$$a \equiv -\partial_0 \varphi_e\,, \tag{8.11}$$

and they may conveniently be written as the components of the higher-dimensional field strength tensor as

$$F_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} 0 & -\tilde{E}_1 & -\tilde{E}_2 & -\tilde{E}_3 & -a \\ \tilde{E}_1 & 0 & B_3 & -B_2 & -E_1 \\ \tilde{E}_2 & -B_3 & 0 & B_1 & -E_2 \\ \tilde{E}_3 & B_2 & -B_1 & 0 & -E_3 \\ a & E_1 & E_2 & E_3 & 0 \end{pmatrix} \tag{8.12a}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 & a \\ -E_1 & 0 & B_3 & -B_2 & \tilde{E}_1 \\ -E_2 & -B_3 & 0 & B_1 & \tilde{E}_2 \\ -E_3 & B_2 & -B_1 & 0 & \tilde{E}_3 \\ -a & -\tilde{E}_1 & -\tilde{E}_2 & -\tilde{E}_3 & 0 \end{pmatrix}. \tag{8.12b}$$

The Bianchi identities are trivially satisfied for the field strength, as it is the exterior derivative of a one-form and gives the homogeneous equations. The null reduction of the higher-dimensional gauge transformation $A_{\hat{\mu}} \to A_{\hat{\mu}} + \partial_{\hat{\mu}}\Lambda\,(t, \boldsymbol{x})$ is now equivalent to

$$\begin{aligned} \delta_{\mathrm{GT}} A^i &= \partial^i \Lambda && (8.13a) \\ \delta_{\mathrm{GT}} \tilde{\varphi} &= -\partial_0 \Lambda && (8.13b) \\ \delta_{\mathrm{GT}} \varphi &= 0\,. && (8.13c) \end{aligned}$$

The EOMs in terms of the potentials are now found to be

$$\begin{aligned} -\partial_i \left(\partial^i \tilde{\varphi} + \partial_0 A^i\right) &= \tilde{\rho} + \partial_0^2 \varphi && (8.14a) \\ -\partial_i \partial^i \varphi &= \rho && (8.14b) \\ \partial_i \partial_k A^k - \partial_k \partial^k A_i &= J^i - \partial_0 \partial^i \varphi\,. && (8.14c) \end{aligned}$$

There is no problem working with the theory without taking any of the limits, but it is instructive to see how they arise. To obtain the limits, we must set some of the sources equal to zero. We see that if we first choose $\tilde{\rho} = 0$ and then imposes the Lorenz-like gauge condition $\partial_i A^i + \partial_t \varphi = 0$, we should take $\tilde{\varphi} = 0$ to solve the EOMs trivially. The result obtained is exactly the electric limit of the MED EOMs (8.3). Had we chosen a different gauge we would still obtain the correct EOMs but we would now have to take $\tilde{\varphi} \neq 0$, so the Lorentz gauge is the most simple to work in.

On the other hand, if we impose $\rho = 0$, then we see that we must take $\varphi = 0$ to solve the EOMs trivially. This leads to the magnetic limit of the MED EOMs (8.4).

8.1.3 *Symmetry transformations and conserved currents*

It is obvious that the boost and rotation generators do not act trivially on the field components unlike for the Schrödinger model. Under a global Galilean transformation $A^{\hat{\mu}} = (\varphi, \boldsymbol{A}, -\tilde{\varphi})$ transforms as a vector under the extended homogeneous Galilean transformation (2.35), since it is the null reduction of the relativistic gauge potential as we discussed in example 6.1. The finite transformation rules of the components can thus be written as

$$\begin{aligned}
\varphi' &= \varphi & (8.15a)\\
A' &= \mathbf{R}A + v\varphi & (8.15b)\\
\tilde{\varphi}' &= \tilde{\varphi} + \frac{1}{2}v^2\varphi + v^t\mathbf{R}A\,. & (8.15c)
\end{aligned}$$

We see that the magnetic vector A transforms as a vector under rotations, but is shifted under boosts. While φ is a true scalar, the transformation of $\tilde{\varphi}$ is very non-trivial and assymmetric at the finite level. Infinitesimally the symmetry variations of the field components become

$$\begin{aligned}
\delta\varphi &= 0 & (8.16a)\\
\delta A^i &= \Lambda^i\varphi_e + 2\lambda_{jk}\delta^{i[j}A^{k]} & (8.16b)\\
\delta\tilde{\varphi} &= \Lambda^i A_i\,. & (8.16c)
\end{aligned}$$

With this we can then calculate the conserved canonical Noether currents using the results of section 4.2.1 in a straight-forward manner. We find

$$\begin{aligned}
E^0_{\text{can}} &= \frac{1}{2}\partial_0\varphi\partial_0\varphi - \partial^i\tilde{\varphi}\partial_i\varphi + \partial^j A^k\partial_{[j}A_{k]} & (8.17a)\\
E^j_{\text{can}} &= \partial^j\varphi\partial_0\tilde{\varphi} + \partial^j\tilde{\varphi}\partial_0\varphi + \partial_0 A^j\partial_0\varphi\\
&\quad -\partial^j A^k\partial_0 A_k + \partial^k A^j\partial_0 A_k & (8.17b)
\end{aligned}$$

$$\begin{aligned}
T^{0i}_{\text{can}} &= \partial_0\varphi\partial^i\varphi + \partial_k\varphi\partial^i A^k & (8.17c)\\
T^{ji}_{\text{can}} &= \partial^j\varphi\partial^i\tilde{\varphi} + \left(\partial^j\tilde{\varphi} + \partial_0 A^j\right)\partial^i\varphi\\
&\quad + \left(\partial^i A_k\partial^k A^j - \partial^i A_k\partial^j A^k\right) - \mathcal{L}^{(0)}\delta^{ij} & (8.17d)
\end{aligned}$$

$$\begin{aligned}
b^{\mu i} &= tT^{\mu i} + w^{\mu i} & (8.17e)\\
w^{0i} &= -\varphi\partial^i\varphi & (8.17f)\\
w^{ji} &= -A^i\partial^j\varphi + 2\varphi\partial^{[j}A^{i]} & (8.17g)
\end{aligned}$$

$$\begin{aligned}
j^{\mu ij}_{\text{can}} &= x^i T^{\mu j}_{\text{can}} - x^j T^{\mu i}_{\text{can}} + s^{\mu ij} & (8.17h)\\
s^{0ij} &= -2A^{[i}\partial^{j]}\varphi & (8.17i)\\
s^{kij} &= 2A^{[i}\partial^{|k|}A^{j]} - 2A^{[i}\partial^{j]}A^k\,. & (8.17j)
\end{aligned}$$

One can check that these currents are not invariant under the gauge transformation (8.13). The constraints (4.18) from the conservation equation for the boost and rotation current that we found for the general theory in section 4.2.1 is verified to be satisfied. Using the improvements that simplify the conservation equations the most as described in section 4.2.3 now leads to a set of improved currents given by:

$$T^{0i}_{\text{imp}} = \partial_0\varphi\partial^i\varphi + 2\partial_k\varphi\partial^{[i}A^{k]} \tag{8.18a}$$

$$T^{ji}_{\text{imp}} = 2\partial^{(i}\varphi\partial^{j)}\tilde{\varphi} + 2\partial_0 A^{(i}\partial^{j)}\varphi + 2\partial_k A^{(i}\partial^{j)}A^k \tag{8.18b}$$

$$-\partial^{(i}A_k\partial^{j)}A^k - \partial^k A^{(i}\partial_k A^{j)} - \mathcal{L}\delta^{ij} \tag{8.18c}$$

$$b^{\mu i}_{\text{imp}} = tT^{\mu i}_{\text{imp}} + \psi^{\mu i} \tag{8.18d}$$

$$\psi^{0i} = -\varphi\partial^i\varphi \tag{8.18e}$$

$$\psi^{ki} = 2\varphi\partial^{[k}A^{i]} \tag{8.18f}$$

$$j^{\mu ij}_{\text{imp}} = x^i T^{\mu j}_{\text{imp}} - x^j T^{\mu i}_{\text{imp}}\,. \tag{8.18g}$$

This confirms what we derived for general theories in section 4.2.3. In particular the symmetry of the stress tensor and $T^{0i}_{\text{imp}} = -\partial_\mu\psi^{\mu i}$ given (4.29) can be verified to hold using the EOMs. The energy current E^μ_{can} has still not received any improvements from our analysis, but they can be fixed uniquely by requiring gauge invariance as we do soon. One should also consider what happens in the two limits of GED. In the electric limit all currents are still non-trivial, but in the magnetic limit $\psi^{\mu i} = 0$, which then implies that $T^{0i}_{\text{imp}} = 0$ in agreement with the discussion in section 4.2.2.

An alternative is to use the formulas of section 6.2 for the null reduction of the well-known relativistic canonical Noether Maxwell EM tensor

$$\hat{T}^{\hat{\mu}\hat{\nu}}_{\text{can}} = -F^{\hat{\mu}\hat{\rho}}\partial^{\hat{\nu}}A_{\hat{\rho}} - \eta^{\hat{\mu}\hat{\nu}}\mathcal{L}_{\text{MED}}\,. \tag{8.19}$$

The reduction (6.12) can be seen to reproduce all of the currents (8.17). In addition we also have the would-have-been mass current $J^\mu_{\text{can}} = \hat{T}^{\mu 0}_{\text{can}}$ is here zero as expected since there is no $\mathrm{U}(1)$ symmetry.

From the higher-dimensional Lagrangian we can easily calculate the spin-current of MED using (A.16) and we find

$$s^{\rho\mu\nu} = -2F^{\rho[\mu}A^{\nu]}\,. \tag{8.20}$$

Choosing the improvements for the higher-dimensional EM tensor as (A.21) is seen to yield the required improvements to make the energy current gauge invariant, while the other components reproduce the improved currents above. The improved energy current is given by

$$E^0_{\text{imp}} = \mathcal{L} - \partial_0\varphi\partial_0\varphi - \partial^i\varphi\left(\partial_0 A_i + \partial_i\tilde{\varphi}\right) \tag{8.21a}$$

$$E^i_{\text{imp}} = 2\left(\partial_0 A_j + \partial_j\tilde{\varphi}\right)\partial^{[i}A^{j]} - \partial_0\varphi\left(\partial_0 A^i + \partial^i\tilde{\varphi}\right)\,. \tag{8.21b}$$

In addition to this, notice also that the zero would-be mass current receives improvements. We find that it is identical to the current defined in (5.5):

$$J^\mu_{\text{imp}} = \Phi^\mu\,. \tag{8.22}$$

This demonstrates the origin of the topological current that we discussed in section 6.2: It is the improvements of the would-be mass current in the higher-dimensional theory to make the conservation equations as simple as possible.

8.2 NULL REDUCTION OF MED ON CURVED SPACETIME

8.2.1 *Reduction of the gauge field*

The goal is to couple GED to an arbitrary torsionful Newton-Cartan background. This is conveniently done using the null reduction formalism developed in chapter 6. We can use the reduction of the vielbeins (6.1) to figure out how to properly reduce the $D+1$-dimensional U(1) gauge field $A_{\hat{\mu}}$ on an arbitrary Lorentzian spacetime. To do this, we first define the gauge field with flat frame indices inspired analog to the flat version (8.7) as

$$A_{\hat{A}} \equiv \left(-\tilde{\varphi}, \boldsymbol{A}, \varphi\right) . \tag{8.23}$$

The spacetime gauge field is then obtained by contracting with the inverse vielbeins:

$$A_{\hat{\mu}} = \begin{pmatrix} a_\mu - \tilde{\varphi}\tau_\mu - \varphi M_\mu \\ \varphi \end{pmatrix} , \tag{8.24}$$

where we have defined

$$a_\mu \equiv A_a e^a_\mu \, . \tag{8.25}$$

In the flat limit $A_{\hat{\mu}}$ agrees with the definition given in (8.7). Notice that a_μ is spatial in the sense that it satisfies $0 = v^\mu a_\mu$ in any frame as it is contracted with the spatial vielbein, so boosting to another frame does not destroy this property unlike if it had been a spacetime 1-form as discussed in section 3.2. The fields a_μ, $\tilde{\varphi}$, φ are invariant under transformations that preserves the null Killing vector ∂_u as they must not be charged under the $\mathrm{U}\left(1\right)$ central charge. They can thus be claimed to be the correct covariantized fields of GED. Their covariantized gauge transformations under the $D+1$-dimensional gauge transformation $A_{\hat{\mu}} \to A_{\hat{\mu}} + \partial_{\hat{\mu}}\Lambda\left(x\right)$ are found to be

$$\varphi' = \varphi \tag{8.26a}$$
$$\tilde{\varphi}' = \tilde{\varphi} + v^\mu \partial_\mu \Lambda \tag{8.26b}$$
$$a'_\mu = a_\mu + \partial_\mu \Lambda + \tau_\mu v^\lambda \partial_\lambda \Lambda \, . \tag{8.26c}$$

We see that the gauge transformation is essentially a covariant way of projecting along the time and spatial directions in agreement with what happens in the flat case. Under local Galilean transformations $A_{\hat{A}}$ transforms as (8.15) and along with the transformation of the vielbeins (3.8), we find that a_μ, $\tilde{\varphi}$, φ transforms like

$$\begin{aligned}
\varphi' &= \varphi && (8.27a)\\
a'_\mu &= a_\mu + a_\nu e^\nu_b \left(R^{-1}\right)^b{}_a v^a \tau_\mu + \varphi\left(v_a R^a{}_b e^b_\mu + v^a v_a \tau_\mu\right) && (8.27b)\\
\tilde{\varphi}' &= \tilde{\varphi} + \frac{1}{2} v^a v_a \varphi + v_a R^a{}_b A^b e^{\mu b} a_\mu\,. && (8.27c)
\end{aligned}$$

The transformation law for $a_\mu = A_a e^a_\mu$ is in particular complicated because both A_a and e^a_μ transform.

8.2.2 *Reduction of the 2-form field*

Taking the exterior derivative of the gauge field (8.24) defines the field strength tensor:

$$\begin{aligned}
F_{\hat{\mu}\hat{\nu}} &= \partial_{\hat{\mu}} A_{\hat{\nu}} - \partial_{\hat{\nu}} A_{\hat{\mu}}\\
&= \begin{pmatrix} \tilde{f}_{\mu\nu} + 2\tau_{[\mu}\partial_{\nu]}\tilde{\varphi} + 2M_{[\mu}\partial_{\nu]}\varphi & \partial_\mu\varphi \\ -\partial_\nu\varphi & 0 \end{pmatrix}, \qquad (8.28)
\end{aligned}$$

where we find it useful to define

$$\begin{aligned}
f_{\mu\nu} &\equiv 2\partial_{[\mu} a_{\nu]} && (8.29a)\\
\tilde{f}_{\mu\nu} &\equiv f_{\mu\nu} - 2\tilde{\varphi}\partial_{[\mu}\tau_{\nu]} - 2\varphi\partial_{[\mu} M_{\nu]}\,. && (8.29b)
\end{aligned}$$

However because of the non-linearity of the null reduction, this does not immediately give useable field strengths of dimensionally reduced theory. The "field strength" $\tilde{f}_{\mu\nu}$ includes besides what is the naive generalization of the magnetic field strength $f_{\mu\nu}$ also couplings of the fields to the curls of τ_μ and M_μ. It is obviously neither boost or gauge invariant, transforming under the gauge transformations (8.26) as

$$\tilde{f}'_{\mu\nu} = \tilde{f}_{\mu\nu} - 2\tau_{[\mu}\partial_{\nu]}\left(v^\lambda \partial_\lambda \Lambda\right)\,. \qquad (8.30)$$

To properly reduce the field strengths we identify the various fields of GED with the components of the Galilean field strength tensor (8.12) and then convert them to spacetime objects using the vielbeins (6.1). In general a $D+1$-dimensional 2-form can be written in Lorentzian frame indices as

$$F_{\hat{A}\hat{B}} = \begin{pmatrix} 0 & F_{+a} & F_{+-} \\ F_{b+} & F_{ab} & F_{b-} \\ F_{-+} & F_{-b} & 0 \end{pmatrix}. \qquad (8.31)$$

Contracting with the inverse vielbeins (3.6) we see that it is convenient to define components of the proper dimensional reduced field strengths as

$$F_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B_{\mu\nu} + 2\tilde{E}_{[\mu}\tau_{\nu]} + 2E_{[\mu}M_{\nu]} + 2a\tau_{[\mu}M_{\nu]} & -E_\mu - a\tau_\mu \\ E_\nu + a\tau_\nu & 0 \end{pmatrix}, \qquad (8.32)$$

where we have the properly covariantized components of the field strength tensor as

$$\begin{aligned} B_{\mu\nu} &\equiv F_{ab}e_\mu{}^a e_\nu{}^b = \tilde{f}_{\mu\nu} - 2\tau_{[\mu}v^\rho \tilde{f}_{\nu]\rho} && (8.33a)\\ \tilde{E}_\mu &\equiv F_{a+}e_\mu{}^a = v^\rho \tilde{f}_{\rho\mu} - \partial_\mu\tilde{\varphi} - \tau_\mu v^\nu \partial_\nu \tilde{\varphi} && (8.33b)\\ E_\mu &\equiv F_{-a}e_\mu{}^a = -\partial_\mu\varphi - \tau_\mu v^\lambda \partial_\lambda \varphi && (8.33c)\\ a &\equiv F_{-+} = v^\mu \partial_\mu \varphi\,. && (8.33d) \end{aligned}$$

These are the covariantized generalizations of the fields we have in the theory without taking the electric or magnetic limits. Their interpretations and relations to the original flat space field strengths can be confirmed by restricting to a flat Newton-Cartan geometry. The relation between the field strengths and the gauge fields is found from converting (8.28) to frame indices and then identifying the components of (8.31). This calculation is straight-forward but tedious. The field strengths are all spatial in the sense that they in any frame satisfy

$$v^\mu B_{\mu\nu} = v^\mu \tilde{E}_\mu = v^\mu E_\mu = 0\,. \qquad (8.34)$$

We can therefore define fields with indices raised using the inverse spatial metric $h^{\mu\nu}$ without loosing any information.

Example 8.1 (Raised indices)**.** The fields with lowered indices can be obtained from those with higher indices, showing that neither is more fundamental than the other. For example for $E^\mu \equiv h^{\mu\nu}E_\nu$ we see that the completeness relations (3.7) implies

$$\begin{aligned} h_{\mu\nu}E^\nu &= h_{\mu\nu}h^{\nu\lambda}E_\lambda \\ &= \delta_\mu^\lambda E_\lambda + \cancel{\tau_\mu v^\lambda E_\lambda} \\ &= E_\mu \end{aligned} \qquad (8.35)$$

which is only true because of (8.34).

The higher-dimensional field strength with upper indices is then found to be given by

$$F^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B^{\mu\nu} + 2E^{[\mu}h^{\nu]\lambda}M_\lambda & -M_\sigma B^{\sigma\mu} + \tilde{E}^\mu - av^\mu + 2M_\lambda E^{[\mu}v^{\lambda]} \\ M_\sigma B^{\sigma\nu} - \tilde{E}^\nu + av^\nu - 2M_\lambda E^{[\nu}v^{\lambda]} & 0 \end{pmatrix}. \qquad (8.36)$$

The non-gauge invariant "field-strength" $\tilde{f}_{\mu\nu}$ is seen to enter in many of the equations, but the field strengths $B_{\mu\nu}$, $\tilde{E}_\mu$, E_μ, a are as expected invariant under the gauge transformation (8.26) as one may check explicitly. It is also easy to see due to their definitions (8.33) that neither of the fields are tensors, but transform under local Galilean transformations in quite a complicated manner. One could simply use the transformation of the gauge fields (8.27) to derive the transformation laws directly, but they are not very enlightening and very complicated. Instead one can notice from the structure of (8.32) and (8.36) that certain combinations are indeed tensors as $F_{\hat{\mu}\hat{\nu}}$ and $F^{\hat{\mu}\hat{\nu}}$ are true spacetime tensors. We see for example directly that $E_\nu + a\tau_\nu$ is a tensor and thus in particular E^μ, when raising with $h^{\mu\nu}$. As the sum of a tensor with another tensor is still a tensor, we can also find some more tensorial objects that are convenient in the following.

Example 8.2 (Tensorial objects)**.** It is important to notice that $E^\mu \hat{v}^\nu$ is a tensor. We could then have added $2E^{[\mu}\hat{v}^{\nu]}$ to $B^{\mu\nu} + 2E^{[\mu}h^{\nu]\lambda}M_\lambda$ that is read directly from (8.36), which then gives another tensorial object

$$\begin{aligned} B^{\mu\nu} + 2E^{[\mu}h^{\nu]\lambda}M_\lambda + 2\hat{v}^\mu E^\rho &= B^{\mu\nu} + 2E^{[\mu}\left(h^{\nu]\lambda}M_\lambda + \hat{v}^{\nu]}\right) \\ &= B^{\mu\nu} + 2E^{[\mu}v^{\nu]} \,. \end{aligned} \tag{8.37}$$

The list of tensorial objects relevant for the construction below is found in this way by doing projections with $h^{\mu\nu}$, τ_ν and adding other tensorial objects:

$$E_\nu + a\tau_\nu \tag{8.38a}$$
$$B_{\mu\nu} + 2\tilde{E}_{[\mu}\tau_{\nu]} + 2E_{[\mu}M_{\nu]} + 2a\tau_{[\mu}M_{\nu]} \tag{8.38b}$$
$$M_\lambda E^\lambda + a \tag{8.38c}$$
$$B^{\mu\nu} + 2E^{[\mu}h^{\nu]\lambda}M_\lambda \tag{8.38d}$$
$$B^{\mu\nu} + 2E^{[\mu}v^{\nu]} \tag{8.38e}$$
$$\tilde{E}^\nu - av^\nu - M_\sigma\left(B^{\sigma\nu} + 2E^{[\sigma}v^{\nu]}\right) . \tag{8.38f}$$

Using the definitions of the field strengths (8.33) and the form of a general Bargmann connection defined in (3.72) and the covariant constance of the vielbeins (3.18), we can use the Leibniz rule to derive the expression of the covariant derivatives as

$$\nabla_\rho B_{\mu\nu} = \left(\partial_\rho\left(e_a^\lambda e_b^\sigma B_{\lambda\sigma}\right) + \omega_\rho{}^c{}_a e_c^\lambda e_b^\sigma B_{\lambda\sigma} + \omega_\rho{}^c{}_b e_a^\lambda e_c^\sigma B_{\lambda\sigma}\right) e_\mu^a e_\nu^b \tag{8.39a}$$
$$\nabla_\rho \tilde{E}_\mu = \left(\partial_\rho\left(e_a^\lambda \tilde{E}_\lambda\right) + \omega_\rho{}^b{}_a e_b^\lambda \tilde{E}_\lambda + \Omega_\rho{}^b e_a^\lambda e_b^\sigma B_{\lambda\sigma}\right) e_\mu^a \tag{8.39b}$$
$$\nabla_\rho E_\mu = \left(\partial_\rho\left(e_a^\lambda E_\lambda\right) + \omega_\rho{}^b{}_a e_b^\lambda E_\lambda - \Omega_\rho{}^b e_a^\lambda e_b^\sigma B_{\lambda\sigma}\right) e_\mu^a \tag{8.39c}$$
$$\nabla_\rho a = \left(\partial_\rho\partial_\mu\varphi - \Gamma^\lambda_{\rho\mu}\partial_\lambda\varphi\right) v^\mu . \tag{8.39d}$$

The expressions for the covariant derivatives are evidently not very simple for a general connection just like their transformation laws, but we see that they reduce to the correct flat derivatives. It is much more convenient to write covariant derivatives of the tensorial objects of (8.38) since the covariant derivative then only contains the affine connection as usual.

Example 8.3 (E_μ in flat spacetime)**.** When we take the geometry to be flat and restrict to global inertial frames, we find using the results of section 3.7 that E_μ reduces

$$\begin{aligned} E_\mu &= -\partial_\mu\varphi - \delta_\mu^0\left(-\delta_0^\lambda\right)\partial_\lambda\varphi \\ &= -\delta_\mu^i\partial_i\varphi \end{aligned} \tag{8.40}$$

which agrees with the flat definition given in (8.5). The covariant derivative of flat space-time with $\omega_\rho{}^b{}_a = \Omega_\rho{}^b = 0$ then becomes

$$\begin{aligned} \nabla_\rho E_\mu &= \left(\partial_\rho\left(\delta_a^\lambda E_\lambda\right) + 0 - 0\right)\delta_\mu^a \\ &= \partial_\rho\left(E_a\right)\delta_\mu^a\,. \end{aligned} \tag{8.41}$$

8.2.3 *Reduction of the action*

We are now ready to perform the null reduction of the action after defining all physically relevant fields in a consistent manner. The $D+1$ dimensional Lagrangian density for Maxwellian electrodynamics on a general Lorentzian spacetime is

$$\begin{aligned} \hat{S}_{\text{MED}} &\equiv -\frac{1}{4}\int \mathrm{d}^{D+1}x\sqrt{-g}\,F^{\hat{\mu}\hat{\nu}}F_{\hat{\mu}\hat{\nu}} \\ &= -\frac{1}{4}\int \mathrm{d}^{D+1}x\sqrt{-g}\,g^{\hat{\mu}\hat{\rho}}g^{\hat{\nu}\hat{\sigma}}F_{\hat{\mu}\hat{\nu}}F_{\hat{\rho}\hat{\sigma}}\,. \end{aligned} \tag{8.42}$$

It is now straight-forward to find the null reduced action using the formulas for the inverse metric (6.2) together with the proper decomposition of the field strength tensor (8.28) or (8.32). Depending on which version of the field strength tensor one chooses, the action will be written more-or-less convenient for some purposes, but they are of course equivalent. After pulling out the integral over the null coordinate u, the resulting action will be GED coupled to an arbitrary Newton-Cartan background.

We find, using either (8.28) or (8.32) to write the field strength tensor, that the action becomes:

$$\begin{aligned} S_{\text{GED}} &= \int \mathrm{d}^D xe\left[-\frac{1}{4}h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\tilde{f}_{\rho\sigma} + h^{\mu\rho}v^\nu\partial_\rho\varphi\tilde{f}_{\mu\nu}\right. \\ &\left.+h^{\mu\rho}\partial_\mu\varphi\partial_\rho\tilde{\varphi} + \frac{1}{2}v^\mu v^\nu\partial_\mu\varphi\partial_\nu\varphi\right] \end{aligned} \tag{8.43a}$$

$$\begin{aligned} S_{\text{GED}} &= \int \mathrm{d}^D xe\left[-\frac{1}{4}h^{\mu\rho}h^{\nu\sigma}\left(B_{\mu\nu} + 2E_{[\mu}M_{\nu]}\right)\left(B_{\rho\sigma} + 2E_{[\rho}M_{\sigma]}\right)\right. \\ &+\hat{v}^\rho h^{\nu\sigma}E_\nu\left(B_{\rho\sigma} + 2E_{[\rho}M_{\sigma]}\right) + h^{\nu\sigma}E_\nu\left(\tilde{E}_\sigma - aM_\sigma\right) \\ &\left.-\tilde{\Phi}h^{\nu\sigma}E_\nu E_\sigma - \frac{1}{2}\left(\hat{v}^\nu\hat{v}^\sigma E_\nu E_\sigma - 2a\hat{v}^\nu E_\nu + a^2\right)\right]. \end{aligned} \tag{8.43b}$$

These two actions are another of the main results of the thesis. The details of the calculation can be found in appendix D.8.1. One can check explicitly that the action has all of the required Galilean and gauge symmetries defined previously. The action (8.43a) is more convenient when we do variations of the action, but boost and gauge invariance is not obvious unlike for (8.43b). Restricting to flat Newton-Cartan geometry one can verify to reproduce the correct Lagrangian density (8.10).

8.2.4 *Equations of motion*

If we vary the action (8.43a) wrt. the fields φ, $\tilde{\varphi}$, a_μ we determine the EOMs to be

$$\begin{aligned} 0 &= h^{\mu\rho}h^{\nu\sigma}\partial_{[\mu}M_{\nu]}\tilde{f}_{\rho\sigma} - 2h^{\mu\rho}v^\nu\partial_\rho\varphi\partial_{[\mu}M_{\nu]} \\ &\quad -\frac{1}{e}\partial_\rho\left(eh^{\mu\rho}v^\nu\tilde{f}_{\mu\nu}\right) - \frac{1}{e}\partial_\mu\left(eh^{\mu\rho}\partial_\rho\tilde{\varphi} + ev^\mu v^\rho\partial_\rho\varphi\right) \qquad (8.44a) \\ 0 &= h^{\mu\rho}h^{\nu\sigma}\partial_{[\mu}\tau_{\nu]}\tilde{f}_{\rho\sigma} - 2h^{\mu\rho}v^\nu\partial_\rho\varphi\partial_{[\mu}\tau_{\nu]} - \frac{1}{e}\partial_\rho\left(eh^{\mu\rho}\partial_\mu\varphi\right) \qquad (8.44b) \\ 0 &= \frac{1}{e}\partial_\lambda\left[eh^{\rho[\lambda}h^{\mu]\sigma}\tilde{f}_{\rho\sigma} - 2eh^{\rho[\lambda}v^{\mu]}\partial_\rho\varphi\right]. \qquad (8.44c) \end{aligned}$$

They are the covariant generalized versions of the flat spacetime EOMs (8.14) presented in the same order as one can see by taking the flat limit. Expressed in terms of the gauge invariant field strengths with raised indices we can write the EOMs in a particular nice way:

$$\begin{aligned} 0 &= \partial_{[\mu}M_{\nu]}\left(B^{\mu\nu} + 2E^\mu v^\nu\right) + \frac{1}{e}\partial_\rho\left(e\tilde{E}^\rho - ev^\rho a\right) \qquad (8.45a) \\ 0 &= \partial_{[\mu}\tau_{\nu]}\left(B^{\mu\nu} + 2E^\mu v^\nu\right) + \frac{1}{e}\partial_\rho\left(eE^\rho\right) \qquad (8.45b) \\ 0 &= \frac{1}{e}\partial_\lambda\left[eB^{\lambda\mu} + 2eE^{[\lambda}v^{\mu]}\right]. \qquad (8.45c) \end{aligned}$$

We see that now it is not obvious in general how one would impose the covariant generalization of the electric/magnetic limits on the EOMs because neither $E^\mu = 0$ or $\tilde{E}^\mu = 0$ solves the EOMs. It is convenient to have EOMs that are written in terms of covariant derivatives. In order to do this, the easiest way is to rewrite (8.45) in terms of the tensorial objects of (8.38) and then identify the affine connection from the derivative of the measure e so that a covariant derivative may be formed. If we define the tensors

$$\begin{aligned} E^\mu &\equiv h^{\mu\nu}E_\nu \qquad (8.46a) \\ W^{\lambda\mu} &\equiv B^{\lambda\mu} + 2E^{[\lambda}v^{\mu]} \qquad (8.46b) \\ Z^\rho &\equiv \tilde{E}^\rho - v^\rho a - M_\sigma W^{\sigma\rho}, \qquad (8.46c) \end{aligned}$$

we find that the EOMs can be written as

$$\begin{aligned}
\mathring{\nabla}_\rho Z^\rho &= -M_\mu \mathring{\nabla}_\rho W^{\mu\rho} + \partial_{[\mu}\tau_{\nu]}\left(2M_\lambda \hat{v}^\lambda W^{\mu\nu} - M_\lambda \hat{v}^\mu W^{\nu\lambda} - 2\hat{v}^\mu Z^\nu\right) &\text{(8.47a)}\\
\mathring{\nabla}_\rho E^\rho &= -\partial_{[\mu}\tau_{\nu]} W^{\mu\nu} &\text{(8.47b)}\\
\mathring{\nabla}_\lambda W^{\lambda\rho} &= \partial_{[\mu}\tau_{\nu]}\left(2\hat{v}^\mu W^{\nu\rho} - \hat{v}^\rho W^{\mu\nu}\right), &\text{(8.47c)}
\end{aligned}$$

where the covariant derivative here is wrt. the graviphotonic connection (3.51). The analysis leading to this result can be found in appendix D.8.2. Notice that (8.47a) is not written in tensorial combinations because of M_μ appearing. We see that many terms on the RHS are proportional to $\mathrm{d}\boldsymbol{\tau}$ so when there is no temporal torsion they simplify substantially. Finally we can use the definition of Z^ρ (8.46c) to write the first EOM (8.47a) in a nicer albeit only Galilean covariant fashion by canceling the $M_\mu \mathring{\nabla}_\rho W^{\mu\rho}$ coming from its definition:

$$\begin{aligned}
\mathring{\nabla}_\rho\left(\tilde{E}^\rho - v^\rho a\right) &= \mathring{\nabla}_{[\rho} M_{\mu]} W^{\mu\rho} + \partial_{[\mu}\tau_{\nu]}\left(2M_\lambda \hat{v}^\lambda W^{\mu\nu} - M_\lambda \hat{v}^\mu W^{\nu\lambda} - 2\hat{v}^\mu Z^\nu\right)\\
&= -2\partial_{[\mu} M_{\rho]} W^{\mu\rho} + h^{\lambda\sigma} M_\lambda M_\sigma \partial_{[\mu}\tau_{\rho]} W^{\mu\rho}\\
&\quad + \partial_{[\mu}\tau_{\nu]}\left(2M_\lambda \hat{v}^\lambda W^{\mu\nu} - M_\lambda \hat{v}^\mu W^{\nu\lambda} - 2\hat{v}^\mu Z^\nu\right) \qquad\text{(8.48)}
\end{aligned}$$

where we in the equation used (3.48), (3.59a) and (3.77) to rewrite $\mathring{\nabla}_{[\rho} M_{\mu]}$.

8.2.5 *Energy, momentum and Φ^μ current*

If we vary the action (8.43a) wrt. the background fields $\mathring{e}^{\hat{A}} = (\boldsymbol{\tau}, \boldsymbol{e}^a, -\boldsymbol{M})$ we can determine the currents $\mathcal{T}^\mu{}_{\hat{A}} \equiv (\mathcal{E}^\mu, \mathcal{T}^\mu{}_a, -\mathcal{J}^\mu)$ defined in (4.46). We do the variation in appendix D.8.3 and find

$$\begin{aligned}
\mathcal{E}^\lambda &= -\mathcal{L} v^\lambda + h^{\nu\sigma} h^{\lambda\mu} v^\rho \tilde{f}_{\nu(\mu}\tilde{f}_{\rho)\sigma} + 2v^\nu v^\rho h^{\lambda\mu}\partial_{(\rho}\varphi \tilde{f}_{\mu)\nu}\\
&\quad + v^\nu v^\lambda h^{\mu\rho}\partial_\rho\varphi \tilde{f}_{\mu\nu} + 2h^{\lambda\mu} v^\rho \partial_{(\mu}\varphi\partial_{\rho)}\tilde{\varphi} + v^\lambda v^\mu v^\nu \partial_\mu\varphi\partial_\nu\varphi\\
&\quad + \frac{1}{e}\partial_\rho\left(e h^{\mu[\rho} v^{\lambda]}\tilde{\varphi}\partial_\mu\varphi - e h^{\mu[\rho} h^{\lambda]\nu}\tilde{\varphi}\tilde{f}_{\mu\nu}\right) &\text{(8.49a)}\\
\mathcal{T}^\lambda{}_c &= \mathcal{L} e^\lambda_c - h^{\nu\sigma}\tilde{f}_{\nu(\mu}\tilde{f}_{\rho)\sigma} h^{\lambda\mu} e^\rho_c - h^{\mu[\rho} h^{\lambda]\nu}\tilde{f}_{\mu\nu}\partial_\rho\left(a_{|\sigma|} e^\sigma_c\right)\\
&\quad - 2v^\nu \partial_{(\rho}\varphi\tilde{f}_{\mu)\nu} h^{\lambda\mu} e^\rho_c - h^{\mu\rho} v^\lambda e^\nu_c \tilde{f}_{\mu\nu}\partial_\rho\varphi\\
&\quad + 2h^{\rho[\mu} v^{\lambda]}\partial_\rho\varphi\partial_\mu\left(a_\sigma e^\sigma_c\right) - 2\partial_{(\mu}\varphi\partial_{\rho)}\tilde{\varphi} h^{\lambda\mu} e^\rho_c\\
&\quad - v^\nu \partial_\mu\varphi\partial_\nu\varphi v^\lambda e^\mu_c + \frac{1}{e}\partial_\rho\left(e h^{\mu[\rho} h^{\lambda]\nu}\tilde{f}_{\mu\nu} a_\sigma e^\sigma_c\right)\\
&\quad - \frac{1}{e}\partial_\rho\left(e 2h^{\mu[\rho} v^{\lambda]} a_\sigma e^\sigma_c \partial_\mu\varphi\right) &\text{(8.49b)}\\
\mathcal{J}^\lambda &= \frac{1}{e}\partial_\rho\left(e 2h^{\mu[\rho} v^{\lambda]}\varphi\partial_\mu\varphi - e h^{\mu[\rho} h^{\lambda]\nu}\varphi\tilde{f}_{\mu\nu}\right). &\text{(8.49c)}
\end{aligned}$$

These expressions might be simplified further, but we leave it like this for the future. What is of interest here is of course $\mathcal{J}^\mu$, which does not correspond to a symmetry current. It is the total derivative of something, and as it couples to M_μ, we see because of the antisymmetrization in the current that under the higher-dimensional transformation that preserves the null Killing vector (6.4), the variation of the action is trivial:

$$\begin{aligned}\delta_\sigma S &= \int_M \mathrm{d}^D x\, e \mathcal{J}^\lambda \delta_\sigma M_\lambda \\ &\doteq -\int_M \mathrm{d}^D x\, \partial_\rho \partial_\lambda \left(e 2 h^{\mu[\rho} v^{\lambda]} \varphi \partial_\mu \varphi - e h^{\mu[\rho} h^{\lambda]\nu} \varphi \tilde{f}_{\mu\nu}\right) \sigma \\ &= 0\,. \end{aligned} \tag{8.50}$$

This verifies the general analysis of section 4.4.3 that also on general Newton-Cartan backgrounds $\mathcal{J}^\mu$ is automatically conserved when there is no $\mathrm{U}(1)$ symmetry. It is also not hard to check that in the flat limit it is identical to the topological current Φ^λ as expected.

8.2.6 *Electric and magnetic limits for* $\mathrm{d}\boldsymbol{\tau} = 0$ *and comparison with other work*

GED on Newton-Cartan geometry was studied for the first time by Künzle [51], albeit restricted to Newtonian spacetime where the connection satisfies the Duval-Künzle condition (3.34). Bergshoeff et al. [84] has a slightly more general formulation but is still restricting to torsionless Newton-Cartan geometries with $\mathrm{d}\boldsymbol{M} = \mathrm{d}\boldsymbol{\tau} = 0$. Finally in the paper by van den Bleeken and Yunus [85] they generalize to torsionless spacetimes with just $\mathrm{d}\boldsymbol{\tau} = 0$ and closed Newton-Coriolis 2-form for the magnetic limit, but not in an entirely covariant fashion.

To the best of our knowledge, this is first time GED has been formulated on a completely general Newton-Cartan background. It is interesting to see how our results may reproduce the previous work when $\mathrm{d}\boldsymbol{\tau} = 0$. As noted on page 90 it is not obvious what the electric and magnetic limits are. However when $\mathrm{d}\boldsymbol{\tau} = 0$ the situation improves drastically. Imposing this on the EOMs (8.47), we simply obtain

$$\mathring{\nabla}_\rho Z^\rho = M_\mu \mathring{\nabla}_\rho W^{\mu\rho} \tag{8.51a}$$
$$\mathring{\nabla}_\rho E^\rho = 0 \tag{8.51b}$$
$$\mathring{\nabla}_\lambda W^{\lambda\mu} = 0\,. \tag{8.51c}$$

In particular (8.48) now simply becomes

$$\mathring{\nabla}_\rho \left(\tilde{E}^\rho - v^\rho a\right) = -2\partial_{[\mu} M_{\rho]} W^{\mu\rho}\,. \tag{8.52}$$

And using the expression for the Newton-Coriolis 2-form (3.77) we see that we can write this as

$$\mathring{\nabla}_\rho \left(\tilde{E}^\rho - v^\rho a\right) = C_{\mu\rho} W^{\mu\rho}\,. \tag{8.53}$$

Using (8.46) we can then write all the EOMs (8.51) in terms of the non-tensorial field strengths (8.33) in a particular frame where we have

$$\begin{aligned} \mathring{\nabla}_\rho \tilde{E}^\rho &= C_{\mu\rho}\left(B^{\mu\rho} + 2E^{[\mu}v^{\rho]}\right) + v^\rho \mathring{\nabla}_\rho a & (8.54a)\\ \mathring{\nabla}_\rho E^\rho &= 0 & (8.54b)\\ \mathring{\nabla}_\lambda B^{\lambda\mu} &= v^\lambda \mathring{\nabla}_\lambda E^\mu\,. & (8.54c)\end{aligned}$$

Notice that the first and last equation transforms covariantly under local Galilean transformations as we have split up the tensorial invariant expressions. The electric and magnetic limits are more clear now. We can set $E^\rho = a = 0$ corresponding to $\varphi = 0$, which gives the magnetic limit as one can verify by restricting to flat spacetime:

$$\begin{aligned} \mathring{\nabla}_\rho \tilde{E}^\rho &= C_{\mu\rho} B^{\mu\rho} & (8.55a)\\ \mathring{\nabla}_\lambda B^{\lambda\mu} &= 0\,. & (8.55b)\end{aligned}$$

This agrees with van den Bleeken and Yunus [85]. On the other hand the correct electric limit is imposed by choosing $\tilde{E}^\rho = a = 0$. In terms of the potentials on flat Newton-Cartan spacetime it reproduces the electric limit when imposing the Lorentz gauge and taking $\tilde{\varphi} = 0$ in some frame as discussed on page 82. If the connection is taken to satisfy the Duval-Künzle condition (3.34), then there exists a frame where $C_{\rho\sigma} = 0$ [42]. In this frame we obtain that the EOMs (8.51) simplifies to

$$\begin{aligned} \mathring{\nabla}_\rho E^\rho &= 0 & (8.56a)\\ \mathring{\nabla}_\lambda B^{\lambda\mu} &= v^\lambda \mathring{\nabla}_\lambda E^\mu\,, & (8.56b)\end{aligned}$$

which agrees with the result of Künzle [51].

8.3 LINEARIZATION

We have seen that the non-linear theory is very complicated in part due to the non-linear definitions of the field strengths. This simplifies drastically when we only keep first order terms. Using the general results of section 3.8 of linearized Newton-Cartan geometry we can write the action as couplings to conserved currents. The definitions of the fields we have used make the linearization a bit more involved than for the Schrödinger model of chapter 7. The definition of the a_μ (8.25) linearizes as

$$a_\mu = A_i\left(\delta^i_\mu + \bar{e}_\mu{}^i\right) \equiv a^{(0)}_\mu + a^{(1)}_\mu\,, \tag{8.57}$$

while $\tilde{f}_{\mu\nu}$ now becomes

$$\tilde{f}_{\mu\nu} = 2\partial_{[\mu}a^{(0)}_{\nu]} + 2\partial_{[\mu}a^{(1)}_{\nu]} - 2\tilde{\varphi}\partial_{[\mu}\bar{\tau}_{\nu]} - 2\varphi\partial_{[\mu}\overline{M}_{\nu]}\,, \tag{8.58}$$

After some straight-forward but tedious calculations, one finds that the linearized action can indeed be written as

$$\overline{S} = \int \mathrm{d}^{d+1}x\left[\mathcal{L}^{(0)} + \bar{\tau}_\rho E^\rho_{\text{imp}} + \frac{1}{2}s_{ij}T^{ij}_{\text{imp}} + \bar{e}_{i0}T^{0i}_{\text{imp}} + \overline{M}_\rho\Phi^\rho\right]\,, \tag{8.59}$$

where all of the currents that one can read of are identical to the improved flat spacetime currents given by (8.18), (8.21), (8.22). One sees here that $\overline{M}_\rho$ couples to the topological current Φ^ρ as predicted by the Noether procedure of section 5.1.

8.4 CORRELATION FUNCTIONS IN FLAT SPACETIME

8.4.1 *Dimensional reduction of relativistic propagator*

It is well-known that the photon propagator in Lorenz gauge for MED in $D+1$-dimensions in momentum space is given by [39]

$$\left\langle A_\mu(p)\, A_\nu(-p)\right\rangle = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon}\,. \tag{8.60}$$

From this one can derive that the (gauge-invariant) propagator for the field strength tensor in momentum space is given by

$$\begin{aligned}&\left\langle F_{\hat\mu\hat\nu}(p)\, F_{\hat\rho\hat\sigma}(-p)\right\rangle \\ &\qquad = \frac{-i}{p^2 - i\epsilon}\left(\eta_{\hat\nu\hat\sigma}p_{\hat\mu}p_{\hat\rho} - \eta_{\hat\nu\hat\rho}p_{\hat\mu}p_{\hat\sigma} - \eta_{\hat\mu\hat\sigma}p_{\hat\nu}p_{\hat\rho} + \eta_{\hat\mu\hat\rho}p_{\hat\nu}p_{\hat\sigma}\right)\,.\end{aligned} \tag{8.61}$$

We can obtain the propagators of the field strengths (8.5) of GED by performing a null-reduction of the above, taking the various components like in (8.12). In momentum space in $D=4$ we find

$$\left\langle \tilde E_i(p)\, \tilde E_j(-p)\right\rangle = \delta_{ij}\frac{-iE^2}{\boldsymbol{p}^2 - i\epsilon} \tag{8.62a}$$

$$\left\langle \tilde E_i(p)\, E_j(-p)\right\rangle = \frac{ip_ip_j}{\boldsymbol{p}^2 - i\epsilon} \tag{8.62b}$$

$$\left\langle E_i(p)\, E_j(-p)\right\rangle = 0 \tag{8.62c}$$

$$\left\langle B_i(p)\, B_j(-p)\right\rangle = \frac{-i}{\boldsymbol{p}^2 - i\epsilon}\left(\delta_{ij}\boldsymbol{p}^2 - p_ip_j\right) \tag{8.62d}$$

$$\left\langle B_i(p)\, \tilde E_j(-p)\right\rangle = \epsilon_{ijk}\frac{ip_kE}{\boldsymbol{p}^2 - i\epsilon} \tag{8.62e}$$

$$\left\langle B_i(p)\, E_j(-p)\right\rangle = 0 \tag{8.62f}$$

$$\left\langle a(p)\, \tilde E_i(-p)\right\rangle = \frac{-iEp_i}{\boldsymbol{p}^2 - i\epsilon} \tag{8.62g}$$

$$\left\langle a(p)\, E_i(-p)\right\rangle = 0 \tag{8.62h}$$

$$\left\langle a(p)\, B_i(-p)\right\rangle = 0 \tag{8.62i}$$

$$\left\langle a(p)\, a(-p)\right\rangle = 0\,. \tag{8.62j}$$

Notice that because of the mass $m=0$, the energy E only enters in the numerator and never in the denominator because $p^2 = \boldsymbol{p}^2$. Since many of these are zero and the energy drops out of several, it witnesses that the dynamics of the theory is quite limited.

Performing the Fourier transform of these propagators using (E.2) we will find derivatives of $\delta\left(t-t'\right)$-functions that will be the result when we have E in the nominator. In $D=4$ the result one easily obtains is

$$\begin{aligned}
\left\langle \tilde{E}_i\left(t,\boldsymbol{x}\right)\tilde{E}_j\left(t',\boldsymbol{x}'\right)\right\rangle &= \frac{i}{4\pi}\delta_{ij}\partial_t^2\delta\left(t-t'\right)\frac{1}{\|\boldsymbol{x}-\boldsymbol{x}'\|} && \text{(8.63a)}\\
\left\langle \tilde{E}_i\left(t,\boldsymbol{x}\right)E_j\left(t',\boldsymbol{x}'\right)\right\rangle &= \frac{i}{4\pi}\delta\left(t-t'\right)\left(\frac{4\pi}{3}\delta_{ij}\delta^d\left(\boldsymbol{x}-\boldsymbol{x}'\right)\right.\\
&\quad\left.+\frac{\delta_{ij}}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3}-3\frac{\left(x_i-x_i'\right)\left(x_j-x_j'\right)}{\|\boldsymbol{x}-\boldsymbol{x}'\|^5}\right) && \text{(8.63b)}\\
\left\langle E_i\left(t,\boldsymbol{x}\right)E_j\left(t',\boldsymbol{x}'\right)\right\rangle &= 0 && \text{(8.63c)}
\end{aligned}$$

$$\begin{aligned}
\left\langle B_i\left(t,\boldsymbol{x}\right)B_j\left(t',\boldsymbol{x}'\right)\right\rangle &= \frac{i}{4\pi}\delta\left(t-t'\right)\left(-\frac{8\pi}{3}\delta_{ij}\delta^d\left(\boldsymbol{x}-\boldsymbol{x}'\right)\right.\\
&\quad\left.+\frac{\delta_{ij}}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3}-3\frac{\left(x_i-x_i'\right)\left(x_j-x_j'\right)}{\|\boldsymbol{x}-\boldsymbol{x}'\|^5}\right) && \text{(8.63d)}
\end{aligned}$$

$$\begin{aligned}
\left\langle B_i\left(t,\boldsymbol{x}\right)\tilde{E}_j\left(t',\boldsymbol{x}'\right)\right\rangle &= -\frac{i}{4\pi}\epsilon_{ijk}\partial_t\delta\left(t-t'\right)\frac{x_k-x_k'}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3} && \text{(8.63e)}\\
\left\langle B_i\left(t,\boldsymbol{x}\right)E_j\left(t',\boldsymbol{x}'\right)\right\rangle &= 0 && \text{(8.63f)}
\end{aligned}$$

$$\begin{aligned}
\left\langle a\left(t,\boldsymbol{x}\right)\tilde{E}_i\left(t',\boldsymbol{x}'\right)\right\rangle &= \frac{i}{4\pi}\partial_t\delta\left(t-t'\right)\frac{x_i-x_i'}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3} && \text{(8.63g)}\\
\left\langle a\left(t,\boldsymbol{x}\right)E_i\left(t',\boldsymbol{x}'\right)\right\rangle &= 0 && \text{(8.63h)}\\
\left\langle a\left(t,\boldsymbol{x}\right)B_i\left(t',\boldsymbol{x}'\right)\right\rangle &= 0 && \text{(8.63i)}\\
\left\langle a\left(t,\boldsymbol{x}\right)a\left(t',\boldsymbol{x}'\right)\right\rangle &= 0\,. && \text{(8.63j)}
\end{aligned}$$

One can check that these satisfy the EOMs (8.14b) of the free theory at separated points and that they transform correctly under Galilean transformations. We notice again the feature that all propagators are proportional to $\delta\left(t-t'\right)$ (and derivatives) which we also saw considering the massless limit of the free Schrödinger propagator in section 7.4.2.

8.5 SCHRÖDINGER MODEL COUPLED TO ELECTRODYNAMICS

8.5.1 *Reduction of the action*

We now have all tools ready to write an interesting and interacting theory: The Schrödinger model coupled to Galilean electrodynamics. This theory will in particular have non-zero sources for the fields of GED unlike what we have studied so far. Again we obtain the non-relativistic action from a null reduction, this time of Klein-Gordon coupled to a U(1) gauge field to make the U(1) symmetry local like what we study in appendix B.2.2. The action coupled to Lorentzian geometry is given by

$$\begin{aligned}\hat{S} &= \hat{S}_{\text{KG}} + \hat{S}_{\text{MED}} + \hat{S}_{\text{int}} \\ &= \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \left[-g^{\hat{\mu}\hat{\nu}} D_{\hat{\mu}}\Psi^* D_{\hat{\nu}}\Psi - \frac{1}{4} g^{\hat{\mu}\hat{\rho}} g^{\hat{\nu}\hat{\sigma}} F_{\hat{\mu}\hat{\nu}} F_{\hat{\rho}\hat{\sigma}}\right] .\end{aligned} \tag{8.64}$$

We have already studied the null reduction of $\hat{S}_{\text{KG}}$ (7.4) in chapter 7 and $\hat{S}_{\text{MED}}$ (8.42) in this chapter. We thus only need to study the interaction term $\hat{S}_{\text{int}}$ which can be written as

$$\hat{S}_{\text{int}} = \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \left[g^{\hat{\mu}\hat{\nu}} \left(i\Psi\partial_{\hat{\mu}}\Psi^* - i\Psi^*\partial_{\hat{\mu}}\Psi\right) A_{\hat{\nu}} - \Psi^*\Psi g^{\hat{\mu}\hat{\nu}} A_{\hat{\mu}} A_{\hat{\nu}}\right] . \tag{8.65}$$

This can then be null-reduced using the methods of chapter 6 as we have done many times now. The result is

$$\begin{aligned}S_{\text{int}} = \int_M \mathrm{d}^D x\, e \Big[& ih^{\mu\nu} \left(\phi a_\mu \partial_\nu \phi^* - \phi^* a_\mu \partial_\nu \phi\right) \\ & -\phi^*\phi h^{\mu\nu} \left(a_\mu a_\nu + 2m a_\mu M_\nu\right) \\ & -i\phi \left(\varphi v^\mu \partial_\mu \phi^*\right) + i\phi^* \left(\varphi v^\mu \partial_\mu \phi\right) \\ & +\phi^*\phi \left(2\left(\varphi + m\right)\tilde{\varphi} + 2v^\mu M_\mu m\varphi\right) \Big] .\end{aligned} \tag{8.66}$$

The details of the reduction can be found in appendix D.8.4. In our theory the total action and its fields is given by

$$\begin{aligned}S\left[\phi, \varphi, \tilde{\varphi}, a, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] &= S_{\text{GED}}\left[\varphi, \tilde{\varphi}, a, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] \\ &\quad + S_{\text{Sch}}\left[\phi, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] + S_{\text{int}}\left[\phi, \varphi, \tilde{\varphi}, a, \boldsymbol{v}, \boldsymbol{h}^{-1}, \boldsymbol{M}\right] ,\end{aligned} \tag{8.67}$$

where S_{GED} is given by (8.43a) and S_{Sch} is given by (7.6). The fields of GED will now be dynamical and sourced by the Schrödinger field and themselves, which makes the EOMs rather complex and we shall refrain from writing them down while it is in principle straight-forward. We can write $S_{\text{Sch}} + S_{\text{int}}$ in a bit nicer way, which makes the structure more obvious:

$$\begin{aligned}S_{\text{Sch}} + S_{\text{int}} = \int_M \mathrm{d}^D x\, e \Big[& iv^\nu \left(m + \varphi\right)\phi^* \left(\partial_\nu - iA_\nu - iM_\nu\left(m + \varphi\right)\right)\phi \\ & - iv^\mu \left(m + \varphi\right)\phi \left(\partial_\mu + iA_\mu + iM_\mu\left(m + \varphi\right)\right)\phi^* \\ & - h^{\mu\nu} \left(\partial_\mu + iA_\mu + iM_\mu\left(m + \varphi\right)\right)\phi^* \left(\partial_\nu - iA_\nu - iM_\nu\left(m + \varphi\right)\right)\phi \Big] .\end{aligned} \tag{8.68}$$

The calculation is done in appendix D.8.5. The above invites us to define a new covariant derivative

$$\begin{aligned}
\mathcal{D}_\mu \phi &\equiv \left(\partial_\mu - iA_\mu - iM_\mu\left(m+\varphi\right)\right)\phi \\
&= \left(\partial_\mu - i\left(a_\mu - \tilde{\varphi}\tau_\mu - \cancel{\varphi M_\mu}\right) - iM_\mu\left(m+\cancel{\varphi}\right)\right)\phi \\
&= \left(\partial_\mu - ia_\mu - imM_\mu + i\tilde{\varphi}\tau_\mu\right)\phi
\end{aligned} \tag{8.69}$$

which allows us to write the action in a simple way

$$\begin{aligned}
S_{\mathrm{Sch}} + S_{\mathrm{int}} = \int_M \mathrm{d}^{D+1}x\, e\Big[& iv^\mu\left(m+\varphi\right)\phi^*\mathcal{D}_\mu\phi \\
& - iv^\mu\left(m+\varphi\right)\phi\mathcal{D}_\mu\phi^* - h^{\mu\nu}\mathcal{D}_\mu\phi^*\mathcal{D}_\nu\phi\Big]\,.
\end{aligned} \tag{8.70}$$

This covariant derivative commutes with both the electromagnetic $\mathrm{U}(1)$ gauge transformation and the $\mathrm{U}(1)$ transformation of the background. Under a local Galilean transformation $A_\mu = a_\mu - \tilde{\varphi}\tau_\mu - \varphi M_\mu$ is invariant, but $M_\mu\left(m+\varphi\right)$ transform to keep the entire action boost invariant like for the pure Schrödinger model.

It would now be straight-forward to derive the covariantly conserved currents $\mathcal{T}^\mu{}_{\hat{A}}$ for the entire theory by varying (8.66) wrt. the background and adding the contribution from the free GED currents (8.49) and the free Schrödinger currents (7.10). In particular $\mathcal{J}^\mu$ will now be a non-trivially conserved mass current that includes contributions from both the topological current Φ^μ of GED and the interaction.

8.5.2 *Flat spacetime*

The action on a general Newton-Cartan background is quite complicated, and it will be sufficent to understand more about the currents and the interaction on flat spacetime. It is also interesting to see how this compares with the usual coupling of the Schrödinger equation to the electromagnetic field [49]. Using the results of section 3.7, we find that (8.70) on a flat spacetime can be written as

$$S_{\mathrm{Sch}} + S_{\mathrm{int}} = \int_M \mathrm{d}^{D+1}x\, e\left[-i\left(m+\varphi\right)\phi^*\mathcal{D}_0\phi + i\left(m+\varphi\right)\phi\mathcal{D}_0\phi^* - \delta^{ij}\mathcal{D}_i\phi^*\mathcal{D}_j\phi\right], \tag{8.71}$$

where the covariant derivative now is

$$\mathcal{D}_\mu\phi = \left(\partial_\mu - iA_i\delta^i_\mu + i\tilde{\varphi}\delta^0_\mu\right)\phi\,. \tag{8.72}$$

We can then write $S_{\mathrm{Sch}} + S_{\mathrm{int}}$ more simple as

$$\begin{aligned}
S_{\mathrm{Sch}} + S_{\mathrm{int}} = \int_M \mathrm{d}^{D+1}x\, e\Big[& -i\left(m+\varphi\right)\phi^*\left(\partial_0 + i\tilde{\varphi}\right)\phi \\
& + i\left(m+\varphi\right)\phi\left(\partial_0 - i\tilde{\varphi}\right)\phi^* - \left(\partial_i + iA_i\right)\phi^*\left(\partial^i - iA^i\right)\phi\Big]\,.
\end{aligned} \tag{8.73}$$

The EOM for the Schrödinger scalar field is found to be

$$2i\left(m+\varphi\right)\left(\partial_0+i\tilde{\varphi}\right)\phi+i\partial_0\varphi\phi=\left(\partial_i-iA_i\right)\left(\partial^i-iA^i\right)\phi\,. \tag{8.74}$$

That φ always appears in together with the mass as $m+\varphi$ is a special feature of coupling to GED also seen when the Fermi model of example 4.3 is coupled to GED [65]. In a sense this is expected as φ is a scalar and such a coupling is the only possibility in the Schrödinger action.

We see that we reproduce the usual Schrödinger coupling to the electromagnetic field in the EOM if we take $\varphi=0$. However, it is now not a clear cut case what the electric or magnetic limits of the theory are because there are sources for all fields of GED. By rewriting (8.73) we determine the linear currents $\rho,\tilde{\rho},J^i$ that sources $\tilde{\varphi},\varphi,A_i$ respectively as in (8.10) and find

$$\rho = -2m\phi^*\phi \tag{8.75a}$$

$$\tilde{\rho} = i\phi^*\partial_0\phi-i\phi\partial_0\phi^* \tag{8.75b}$$

$$J^i = i\phi\partial^i\phi^*-i\phi^*\partial^i\phi\,. \tag{8.75c}$$

On top of these comes self-interactions of the GED fields that arises from the "Seagull term" in the dimensional reduction. One can here identify $\left(\rho,J^i\right)$ as being proportional the mass current (7.3c) of the free Schrödinger theory of section 7.1, which is no longer conserved. The full currents are covariantly conserved wrt. the derivative (8.72).

9 DISCUSSION AND CONCLUSION

In this thesis we have developed Newton-Cartan geometry and the field theories that lives on these manifolds. Using a great deal of differential geometry, group and representation theory we have accomplished formulating non-relativistic physics on a natural general covariant framework. The main challenge has been to understand the structure of Galilean connections. We have succeeded in elucidating their properties and found that there is a natural connection to use on these manifolds, namely the graviphotonic connection defined in section 3.3.3.1. The Schrödinger model and Galilean electrodynamics we have studied have served well to give concrete realizations of the general theory.

The main new contribution to this research area of this thesis has been to understand the coupling of M_μ in both Galilean and Bargmann theories. This has previously not been well-understood, but the Noether procedure approach we employed has cleared this up completely. We have showed that it couples to a topological current in Galilean theories and the conserved mass current in Bargmann theories. While the Schrödinger model has been studied extensively in the literature, we are as far as we know the first to study Galilean electrodynamics on a general Newton-Cartan background and give covariant equations of motion. The non-relativistic version of scalar QED on a Newton-Cartan background is a further contribution to the field, being an interesting example of an interacting theory.

The latter is one of the points where there is future work to be done. Another aspect is to understand the correlators of the Galilean theories, especially the δ-function in time that appears so that one can give meaning to correlators between currents which will tell us more about non-relativistic theories. In relation to this, it would be interesting to calculate correlators with the current Φ^μ to understand more about it properties. Finally, since the framework of non-relativistic theories is more-or-less understood, it would be interesting to see where the results of this thesis could be applied in holography. There are hints that for certain bulk geometries the theory on the holographic boundary is expected to be something along the lines of Galilean electrodynamics. This thesis has provided the tools that would help understand this better.

A RELATIVISTIC THEORIES

A.1 RELATIVISTIC SPACETIME SYMMETRY GROUPS

A.1.1 *Minkowski and Poincaré groups*

Group $\mathcal{G}$	Transformations	$\dim \mathcal{G}$	Ref.
Minkowski	Rotations + boosts	$\frac{1}{2}D\left(D-1\right)$	[86]
$\mathrm{SO}\left(d,1\right)$	$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$		
Poincaré	Minkowski + translations	$\frac{1}{2}D\left(D+1\right)$	[86]
$\mathrm{Poin}\left(d,1\right)$	$x'^{\mu} = x^{\mu} + a^{\mu}$		
Conformal	Poincaré + dilation + SCT	$\frac{1}{2}\left(D+2\right)\left(D+1\right)$ in $D>2$	[87]
$\mathrm{C}\left(d,1\right)$	$x'^{\mu} = \lambda x^{\mu}$, $x'^{\mu} = \frac{x^{\mu}-b^{\mu}x^{2}}{1-2b\cdot x+b^{2}x^{2}}$		

Table 4: Spacetime transformations and properties of relativistic symmetry groups in D spacetime dimensions.

Even though the main focus of this thesis is non-relativistic groups and their realizations, it will be useful to review the most important aspects of relativistic symmetry groups and compare them. There are 3 prominent relativistic symmetry groups of interest to theoretical physics as shown in table 4, but to these can of course be added various extensions which they are subgroups of.

All of the groups are provides rotational and boost invariance with the antisymmetric generator $J_{\mu\nu}$, which also spans the entire Minkowski Lie algebra through the commutation relations

$$\left[J_{\mu\nu}, J_{\rho\sigma}\right] = \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\sigma\nu}J_{\rho\mu} - \eta_{\sigma\mu}J_{\rho\nu}\,. \tag{A.1}$$

The Minkowski group can be defined as the group of matrix similarity transformations that leaves the Minkowski metric $\eta_{\mu\nu}$ unchanged, which is exactly $\mathrm{SO}\left(d,1\right)$. This is also the defining representation of $\mathrm{SO}\left(d,1\right)$.

One obtains the Poincaré group $\mathrm{Poin}\left(d,1\right)$ from the Minkowski group by adding translations generated by P_{μ} through a semi-direct product, where the momentum transforms as a vector under rotations and boosts. The corresponding new extra commutation relations of the algebra are [39]

$$\begin{aligned} \left[P_\mu, P_\nu\right] &= 0 && \text{(A.2a)} \\ \left[P_\rho, J_{\mu\nu}\right] &= \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu \,. && \text{(A.2b)} \end{aligned}$$

As it is a semi-direct product of Minkowski with the momentum generators, the structure is

$$\mathrm{Poin}\,(d, 1) = \mathbb{R}^D \rtimes \mathrm{SO}\,(d, 1) \;, \tag{A.3}$$

which makes it obvious that $\mathrm{Poin}\,(d,1)$ is an affine group. Its homogeneous subgrup is exactly the Minkowski group.

Neither the Minkowski or the Poincaré groups are compact. The algebraic structure of the Poincaré group is due to the semi-direct product decomposition not semisimple and so its representation theory is rather complicated, as is well-known due to Wigner [40]. Poincaré has two Casimir invariants given by [88]

$$\begin{aligned} C_2 &\equiv P^2 = P^\mu P_\mu && \text{(A.4a)} \\ C_4 &\equiv -\frac{1}{2} P^2 J_{\mu\nu} J^{\mu\nu} + J_{\mu\rho} P^\rho J^{\mu\sigma} P_\sigma \,. && \text{(A.4b)} \end{aligned}$$

In $D = 4$ we have that it is possible to write $C_4 = W_\mu W^\mu$, where $W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$ is the famous Pauli–Lubanski pseudovector. These labels the representations of interest.

A.1.2 *The conformal group*

If we do not require that the metric must stay invariant, but allow it to transform up to an overall scale $\eta \to \Lambda\,(x)^2\,\eta$, then the group of symmetries is of course larger and we obtain the conformal group $\mathrm{C}\,(d,1)$. In $D > 2$ one obtains after an analysis like that one found in [87], that the resulting group is finite dimensional with the extra generators compared to Poincaré being the dilatation D and the conformal boost K_μ. The dilatation generates scale transformations, while the conformal boosts generates the Special Conformal Transformation (SCT); an inversion of spacetime followed by a translation and yet another inversion. One finds that the new extra commutation relations of the algebra are

$$\begin{aligned} \left[D, P_\mu\right] &= P_\mu && \text{(A.5a)} \\ \left[D, J_{\mu\nu}\right] &= 0 && \text{(A.5b)} \\ \left[D, K_\mu\right] &= -K_\mu && \text{(A.5c)} \\ \left[P_\mu, K_\nu\right] &= 2\left(\eta_{\mu\nu} D - J_{\mu\nu}\right) && \text{(A.5d)} \\ \left[J_{\mu\nu}, K_\rho\right] &= \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu && \text{(A.5e)} \\ \left[K_\mu, K_\nu\right] &= 0 \,. && \text{(A.5f)} \end{aligned}$$

One should notice that $J_{\mu\nu}$ has zero weight under dilatations, while the boost and momentum have weights ± 1, and K_μ transforms as a vector under rotations. The introduction of the extra generators also simplifies the group structure, so it is now a semisimple Lie algebra due to (A.5d), which shows that there are now just the trivial ideals remaining.

In $D = 2$, one finds immediately doing the same analysis, yet still rather surprising, that the symmetry group is automatically infinite dimensional, and can be written as the direct product of two Witt algebras, a left one with generators L_n and a right one with generators $\overline{L}_n$, satisfying the algebra

$$\begin{aligned}
\left[L_m, L_n\right] &= (m-n)\, L_{m+n} && \text{(A.6a)}\\
\left[L_m, \overline{L}_n\right] &= 0 && \text{(A.6b)}\\
\left[\overline{L}_m, \overline{L}_n\right] &= (m-n)\, \overline{L}_{m+n}\,. && \text{(A.6c)}
\end{aligned}$$

The finite dimensional subgroup of this is the six-dimensional $\mathrm{SL}(2, \mathbb{C})$, which generates the dilation, SCT, translations and rotation. 2D conformal invariant theories are strongly constrained by the infinite number of symmetries and as a result have many nice properties. Further it is also true by an analysis first done by Polchinski [89], that any 2D scale- and Poincaré invariant theory under some very reasonable technical assumptions, automatically is conformally invariant.

One of the most interesting extensions that one can do of the above groups, is to add $\mathcal{N}$ fermionic supercharges to Poincaré or the conformal group, which is more-or-less the unique physical relevant extension of these groups. The result is that one gets $\mathbb{Z}_2$-graded Lie algebras, described in the framework of Lie superalgebras. These extensions can then be named Supersymmetry (SUSY) extensions of Poincaré or the conformal group, and they have a nice mathematical structure, that make them very interesting to study in their own right. Especially there is a non-trivial internal symmetry that can maximally be $\mathrm{U}(\mathcal{N})$ that the supercharges transform under some representation of along with some new central charges.

A.2 LORENTZIAN SPACETIMES

A.2.1 *Vielbein formalism and frame bundles*

The usual way to introduce vielbeins e^A_μ in a Lorentzian geometry with a metric $g_{\mu\nu}$ is as the "square-root" of the metric, i.e.

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu\,. \tag{A.7}$$

There are some computations that becomes significantly easier in this formalism, but in particular it also highlights the gauge theory aspect of Lorentzian geometry [55]. The (inverse) vielbein e^A_μ, e^ν_B are in this sense gauge fields transforming under the local Lorentz transformations $\Lambda^A{}_B \in \mathrm{SO}(d,1)$ in the (flat) Lorentz index of the vielbein. The metric $g_{\mu\nu}$ is invariant under these transformations because η_{AB} is the invariant symbol of $\mathrm{SO}(d,1)$.

This construction is quite naturally that of (co)frame bundles like those considered in the chapter 3 with $\mathrm{SO}(d,1)$ as the structure group. Contracting the (inverse) vielbein with a Lorentz vector or covector then defines a particular section of the (co)tangent bundles, and the mapping is bijective.

A.2.2 *Connections*

We can define a (frame) covariant derivative $\mathcal{D}_\mu$ for any connection on a Lorentzian geometry by its action on the inverse vielbein as[1]

$$\mathcal{D}_\mu \boldsymbol{e}^A \equiv -\omega_\mu{}^A{}_B \boldsymbol{e}^B \,. \tag{A.8}$$

The coefficients $\omega_\mu{}^A{}_B$ are called the spin-connection and transforms non-tensorial as (3.20). The associated spacetime derivative of the vielbein that satisfy the vielbein postulate is given by

$$\nabla_\rho e_\mu^A = \partial_\rho e_\mu^A - \Gamma^\lambda_{\rho\mu} e_\lambda^A - \omega_\rho{}^A{}_B e_\mu^B = 0 \,. \tag{A.9}$$

Given a connection, the Cartan structure equations defines the torsion $T_{\mu\nu}{}^A$ and Riemann curvature $R_{\mu\nu B}{}^A$ two-forms as

$$\partial_{[\mu} e_{\nu]}^A = -\omega_{[\mu}{}^{AB} e_{\nu]B} + \frac{1}{2} T_{\mu\nu}{}^A \tag{A.10a}$$

$$\partial_{[\mu} \omega_{\nu]}{}^A{}_B = -\omega_{[\mu}{}^{AC} \omega_{\nu]CB} + \frac{1}{2} R_{\mu\nu B}{}^A \,. \tag{A.10b}$$

The usual spacetime versions $T_{\mu\nu}{}^\lambda$, $R_{\mu\nu\sigma}{}^\rho$ are obtained contracting with the vielbeins. Taking the exterior derivative of the structure equations (A.10) and converting to spacetime tensors we obtain the usual Bianchi identities $R_{[\mu\nu\sigma]}{}^\rho = 0$ and $\nabla_{[\lambda} R_{\mu\nu]\sigma}{}^\rho = 0$. Until now this discussion has actually been completely general and in particular it also applies to the Galilean connections of section 3.3.2.

If we want a Lorentzian metric compatible connection, then we see that the connection must satisfy

$$\omega_{\rho AB} = -\omega_{\rho BA} \,. \tag{A.11}$$

The above procedure is equivalent to the gauge theory of the Poincaré group investigated in the appendix of the paper by Hartong and Obers [26]. Here we identify the inverse vielbein $e_\mu{}^A$ as the gauge field associated with the momentum generator P_A and the spin-connection $\omega_{\mu AB}$ as the gauge field associated with the rotation generator $J^{AB} = -J^{BA}$. The antisymmetry of the rotation generator J^{AB} implies that the spin-connection $\omega_{\mu AB}$ is also antisymmetric which again implies metric compatibility. After a deformation of the algebra one may also identify local translation gauge transformations with diffeomorphisms on the manifold M. The field strengths of Poincaré gauge theory are then seen to be the torsion and Riemann curvature tensors.

In Lorentzian spacetimes an important class of connections are the torsionless ones. If we also require metric compatibility, then we may solve the first Cartan structure

1 We here take opposite sign convention compared to what is usually chosen.

equation (A.10a) which yields a unique solution, the famous Levi-Civita connection [53, 52]

$$\hat{\omega}_\mu{}^{AB} \equiv e^{\nu A}\partial_{[\mu}e^B_{\nu]} - e^{\nu B}\partial_{[\mu}e^A_{\nu]} - e^{\nu[A}e^{B]\sigma}e_{\mu C}\partial_\nu e^C_\sigma\,. \tag{A.12}$$

Notice the similarity with the pseudo-rotation gauge field (3.44b).

It may also be proven that given any metric compatible connection $\omega_{\mu AB}$, we may write it as

$$\omega_{\mu AB} = \hat{\omega}_{\mu AB} + C_{\mu AB}\,, \tag{A.13}$$

where $C_{\mu AB}$ is called the contortion. Contracting with vielbeins, the corresponding contortion tensor $C_{\mu\nu\rho} = -C_{\mu\sigma\nu}$ transforms tensorial, and takes an expression in terms of torsion tensors

$$C_{\mu\nu\rho} = -\frac{1}{2}\left(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}\right)\,. \tag{A.14}$$

A.3 CONSERVED POINCARÉ SPACETIME SYMMETRY CURRENTS

A.3.1 *Canonical conserved Noether currents*

It is enlightening to see how one may use the Noether theorem investigated in appendix B works in the more familiar case of theories with Poincaré spacetime symmetries. We assume a Lagrangian description with action $S[\varphi] = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi)$ and infinitesimal transformations given by

$$x'^\mu = x^\mu + \delta x^\mu \tag{A.15a}$$
$$\varphi'_\ell(x') = \varphi_\ell(x) + \delta\varphi_\ell(x) \tag{A.15b}$$

$$\delta x^\mu \equiv \xi^\mu = \epsilon^\mu + \lambda^\mu{}_\nu x^\nu \tag{A.15c}$$
$$\delta\varphi_\ell(x) \equiv \frac{1}{2}\lambda_{\mu\nu}\left(J^{\mu\nu}\right)_{\ell\bar{\ell}}\varphi_{\bar{\ell}}(x) \tag{A.15d}$$

where $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ is just an infinitesimal Lorentz transformation and ϵ^μ an infinitesimal translation that together are all of the parameters of the transformation. The generators of rotations $(J^{\mu\nu})_{\ell\bar{\ell}}$ is in a suitable representation realized of the fields φ_ℓ. The corresponding conserved Noether currents are given by

$$T^{\rho\mu}_{\text{can}} \equiv \frac{\partial\mathcal{L}}{\partial\left[\partial_\rho\varphi_\ell\right]}\partial^\mu\varphi_\ell - \eta^{\rho\mu}\mathcal{L} \tag{A.16a}$$
$$j^{\rho\mu\nu}_{\text{can}} \equiv x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu} + s^{\rho\mu\nu} \tag{A.16b}$$

$$s^{\rho\mu\nu} \equiv -\frac{\partial\mathcal{L}}{\partial\left[\partial_\rho\varphi_\ell\right]}\left(J^{\mu\nu}\right)_{\ell\bar{\ell}}\varphi_{\bar{\ell}} \tag{A.16c}$$

where $T_{\mathrm{can}}^{\rho\mu}$ is the canonical energy-momentum tensor and $j_{\mathrm{can}}^{\rho\mu\nu}$ the total angular momentum with the non-conserved spin-current $s^{\rho\mu\nu}$. The conservation $\partial_\rho j_{\mathrm{can}}^{\rho\mu\nu} = 0$ implies

$$2T^{[\mu\nu]} = -\partial_\rho s^{\rho\mu\nu}\,, \tag{A.17}$$

so in general $T_{\mathrm{can}}^{\mu\nu} \neq T_{\mathrm{can}}^{\nu\mu}$. We can write a variation of the action in terms of arbitrary local parameters $\xi_\mu(x)\,,\ \lambda_{\mu\nu}(x)$ as

$$\delta S[\varphi] = -\int_M \mathrm{d}^D x \left[\partial_\rho \xi_\mu T^{\rho\mu} + \frac{1}{2}\partial_\rho \lambda_{\mu\nu} s^{\rho\mu\nu}\right]\,, \tag{A.18}$$

where we recover the conserved currents proportional to the parameters when we take

$$\begin{aligned} \xi_\mu(x) &= \epsilon_\mu + \lambda_{\mu\nu} x^\nu & \text{(A.19a)} \\ \lambda_{\mu\nu}(x) &= \lambda_{\mu\nu}\,. & \text{(A.19b)} \end{aligned}$$

A.3.2 *Improvements of currents*

The improvements of the EM tensor and angular momentum current takes the form

$$\begin{aligned} T_{\mathrm{imp}}^{\mu\nu} &= T_{\mathrm{can}}^{\mu\nu} + \partial_\lambda A^{\lambda\mu\nu}\,, & \text{(A.20a)} \\ J_{\mathrm{imp}}^{\rho\mu\nu} &= x^\mu T_{\mathrm{can}}^{\rho\nu} - x^\nu T_{\mathrm{can}}^{\rho\mu} + s^{\rho\mu\nu} + \partial_\lambda B^{\lambda\rho\mu\nu} & \text{(A.20b)} \end{aligned}$$

where $A^{\lambda\mu\nu} = -A^{\mu\lambda\nu}$ and $B^{\lambda\rho\mu\nu} = -B^{\rho\lambda\mu\nu}$ are the improvement terms. If we choose

$$\begin{aligned} A^{\lambda\mu\nu} &= \frac{1}{2}\left(s^{\mu\nu\lambda} + s^{\nu\mu\lambda} - s^{\lambda\mu\nu}\right) & \text{(A.21a)} \\ B^{\lambda\rho\mu\nu} &= x^\mu A^{\lambda\rho\nu} - x^\nu A^{\lambda\rho\mu}\,, & \text{(A.21b)} \end{aligned}$$

the improved angular momentum current becomes

$$J_{\mathrm{imp}}^{\rho\mu\nu} = x^\mu T_{\mathrm{imp}}^{\rho\nu} - x^\nu T_{\mathrm{imp}}^{\rho\mu}\,. \tag{A.22}$$

From this the conservation law for angular momentum implies that $T_{\mathrm{imp}}^{\mu\nu} = T_{\mathrm{imp}}^{\nu\mu}$. This improved EM tensor is also known in the literature as the Belinfante-Rosenfeld EM tensor, and it is identical to the Hilbert EM tensor of general relativity

$$T_{\mathrm{Hil}}^{\mu\nu} \equiv \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{A.23}$$

when we vary around flat space [90, 91].

A.4 NOETHER PROCEDURE FOR POINCARÉ THEORIES

If we want local Poincaré invariance of our theory at lowest order in the geometry, we need a first-order term in the action that cancels the non-invariance of (A.18) as described in general in section B.2. We define two gauge fields that couples to the currents in $S^{(1)}$ as

$$S^{(1)} = \int_M \mathrm{d}^D x \left[\bar{e}_{\mu\nu} T^{\mu\nu}_{\text{can}} + \frac{1}{2} \overline{\omega}_{\rho\mu\nu} s^{\rho\mu\nu} \right] \tag{A.24}$$

which are the linearization of the vielbein $e_{\mu\nu}$ and the spin-connection $\omega_{\rho\mu\nu}$ that according to the general Noether procedure should be taken to transforms at first order under infinitesimal local translations and rotations as

$$\delta^{(1)} \bar{e}_{\mu\nu} = \partial_\mu \xi_\mu + \lambda_{\mu\nu} \tag{A.25a}$$

$$\delta^{(1)} \overline{\omega}_{\rho\mu\nu} = \partial_\rho \lambda_{\mu\nu} , \tag{A.25b}$$

where $\lambda_{\mu\nu}$ is an infinitesimal Local Lorentz Transformation (LLT) and ξ_μ an infinitesimal local translation. Writing $T^{\mu\nu}_{\text{can}}$ in terms of the improved current (A.20) gives after some integration by parts that we may write the action as

$$S^{(1)} = \int_M \mathrm{d}^D x \left[\frac{1}{2} \bar{h}_{\mu\nu} T^{\mu\nu}_{\text{imp}} + \frac{1}{2} \overline{C}_{\rho\mu\nu} s^{\rho\mu\nu} \right] , \tag{A.26}$$

where we have defined

$$\overline{C}_{\rho\mu\nu} \equiv \overline{\omega}_{\rho\mu\nu} - \hat{\overline{\omega}}_{\rho\mu\nu} \tag{A.27a}$$

$$\hat{\overline{\omega}}_{\mu\nu\rho} \equiv \partial_{[\rho} \bar{h}_{\nu]\mu} + \partial_\mu \bar{e}_{[\nu\rho]} . \tag{A.27b}$$

$\bar{h}_{\mu\nu} \equiv 2\bar{e}_{(\mu\nu)}$ is here the perturbation of the Minkowski metric, and one may check that $\hat{\overline{\omega}}_{\rho\mu\nu}$ as defined is exactly the linearization of the Levi-Civita connection (A.12). What one should appreciate here is that contrary to the Galilean case of section 5.1, is that the calculation shows that it is possible to realize the geometry on the vielbeins alone (or equivalently the metric $g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$), with the Levi-Civita connection $\overline{\omega}_{\rho\mu\nu} = \hat{\overline{\omega}}_{\rho\mu\nu}$ as the "minimal choice" connection implied by the improvements. $\overline{C}_{\rho\mu\nu}$ is here the contortion tensor, which may be set to zero if we do not want a torsionful spacetime.

An explanation to why this is expected is based on the observation that connections must end up coupling to a conserved current. Here initially $\omega_{\rho\mu\nu}$ couple to a non-conserved current because the vielbeins transforms under a LLT. After using the improvements of the EM tensor the linearized metric occurs and this does not transform under LLTs. Therefore what is left of the spin connection must also be a tensor that couples to the non-conserved current to keep the action invariant under. Knowing that this only happens when the spin current $s^{\rho\mu\nu}$ couples to the difference of two connections, this implies that whatever comes from the improvements of the EM tensor must be a connection itself. As this connection then by the structure of the EM tensor coupling must be build from the vielbein explains why we get the linearized Levi-Civita connection and nothing else, as this is the unique connection build from the vielbein alone.

B THE NOETHER THEOREM AND PROCEDURE

B.1 Noether's theorem

B.1.1 *Statement and proof*

Theorem B.1 (Noether's theorem for fields)**.** *Let an action $S = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi, x)$ for a set of field(s) $\varphi_\ell(x)$ with at most first order derivatives*[1] *and perhaps explicit spacetime dependence be given. If the action is invariant under differentiable symmetry transformations, then there exists a corresponding conserved current for each independent transformation [73, 92].*

Proof. Take the action $S = \int_M \mathrm{d}^D x\, \mathcal{L}(\varphi, \partial\varphi, x)$ and let us restrict ourselves to infinitesimal symmetry transformations. We may without loss of generality assume that the possible symmetry transformations are described by (constant) infinitesimal independent parameters $\xi^A = \{\epsilon^\alpha, \lambda^a\}$, $A = 1, \ldots, N$, where λ_a are the subset of parameters that includes the variation of just the field φ_ℓ, while the parameters ϵ_α parametrizes just spacetime transformations. Both may give rise to infinitesimal spacetime variations $X^\mu_A(x) = \left\{\hat{X}^\mu_\alpha(x), \tilde{X}^\mu_a(x)\right\}$, and we may then write the infinitesimal transformation of spacetime and the field as the passive transformation

$$x'^\mu = x^\mu + \delta x^\mu \tag{B.1a}$$
$$\varphi'_\ell(x') = \varphi_\ell(x) + \delta\varphi_\ell(x) \tag{B.1b}$$

$$\delta x^\mu = \xi^A X^\mu_A(x) = \epsilon^\alpha \hat{X}^\mu_\alpha(x) + \lambda^a \tilde{X}^\mu_a(x) \tag{B.1c}$$
$$\delta\varphi_\ell(x) = \lambda_a U^a_{\ell\bar{\ell}} \varphi_{\bar{\ell}}(x)\ , \tag{B.1d}$$

where $U^a_{\ell\bar{\ell}}$ is a matrix that is a representation of the part of the infinitesimal symmetry transformation that acts on the field components. The structure of the type of symmetry transformations described is generally given by a Lie algebra $\mathfrak{g}$ and hence if $\dim\mathfrak{g} = N$ then there are going to be N conserved Noether currents as we will see.

As the Lagrangian density contains derivatives of the fields, we need to figure out what the symmetry variation of these are. Using the Leibniz and chain rules we find

$$\delta\left(\partial_\mu\varphi_\ell(x)\right) = \partial_\mu\delta\varphi_\ell(x) - \partial_\mu\left(\delta x^\nu\right)\partial_\nu\varphi_\ell(x)\ . \tag{B.2}$$

The variation (B.1d) of $\delta\varphi_\ell(x)$ consists of an internal transformation of the components given by $\lambda_a U^a_{\ell\bar{\ell}}$ together with a spacetime variation from changing the coordinates. From this we can more conveniently define a transformation $\bar{\delta}\varphi_\ell(x)$ that singles out the internal transformation part by calculating the difference of the transformed and original field at the old spacetime point, i.e.

1 It is not hard to generalize to more general cases.

$$\begin{aligned}
\bar{\delta}\varphi_\ell(x) &\equiv \varphi'_\ell(x) - \varphi_\ell(x) \\
&= \varphi'_\ell(x' - \delta x) - \varphi_\ell(x) \\
&= \left(\varphi_\ell(x' - \delta x) + \delta\varphi_\ell(x' - \delta x)\right) - \varphi_\ell(x) \\
&= \left(\cancel{\varphi_\ell(x)} - \delta x^\mu \partial_\mu \varphi_\ell(x) + \delta\varphi_\ell(x' - \delta x)\right) - \cancel{\varphi_\ell(x)} \\
&= \delta\varphi_\ell(x) - \delta x^\mu \partial_\mu \varphi_\ell(x) \ .
\end{aligned} \tag{B.3}$$

This is useful, as we see that we may now write

$$\delta\varphi_\ell(x) = \bar{\delta}\varphi_\ell(x) + \delta x^\mu \partial_\mu \varphi_\ell(x) \ , \tag{B.4}$$

and from the definition of $\bar{\delta}\varphi_\ell(x)$ (B.3) we see that it commutes with the derivative:

$$\begin{aligned}
\bar{\delta}\left(\partial_\mu \varphi_\ell(x)\right) &= \delta\left(\partial_\mu \varphi_\ell(x)\right) - \delta x^\nu \partial_\nu \left(\partial_\mu \varphi_\ell(x)\right) \\
&= \partial_\mu \delta\varphi_\ell(x) - \left(\partial_\mu \delta x^\nu\right) \partial_\nu \varphi_\ell(x) - \delta x^\nu \partial_\nu \left(\partial_\mu \varphi_\ell(x)\right) \\
&= \partial_\mu \delta\varphi_\ell(x) - \partial_\mu \left(\delta x^\nu \partial_\nu \varphi_\ell(x)\right) \\
&= \partial_\mu \left(\bar{\delta}\varphi_\ell(x)\right) \ .
\end{aligned} \tag{B.5}$$

We are now ready to consider the variation of the action $\delta S \equiv \int_{M'} \mathrm{d}^D x' \, \mathcal{L}' - \int_M \mathrm{d}^D x \, \mathcal{L}$, which in general transforms both the integration measure and the Lagrangian density. We may perform several useful manipulations and expansions, keeping only first order terms in ξ^A:

$$\begin{aligned}
\delta S &= \int_{M'} \mathrm{d}^D x' \, \mathcal{L}\left(\varphi'(x'), \partial\varphi'(x'), x'\right) - \int_M \mathrm{d}^D x \, \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right) \\
\text{(i)} &= \int_{M'} \mathrm{d}^D x' \, \mathcal{L}\left(\varphi'(x), \partial\varphi'(x), x\right) - \int_M \mathrm{d}^D x \, \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right) \\
&\quad + \delta x^\lambda \partial_\lambda \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right) + \mathcal{O}\left(\xi^2\right) \\
\text{(ii)} &= \int_M \mathrm{d}^D x \, \mathcal{L}\left(\varphi'(x), \partial\varphi'(x), x\right) - \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right) \\
&\quad + \partial_\lambda \left(\delta x^\lambda \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right)\right) + \mathcal{O}\left(\xi^2\right) \\
\text{(iii)} &= \int_M \mathrm{d}^D x \, \frac{\partial \mathcal{L}(\varphi, \partial\varphi, x)}{\partial \varphi_\ell} \bar{\delta}\varphi_\ell + \frac{\partial \mathcal{L}(\varphi, \partial\varphi, x)}{\partial \left[\partial_\lambda \varphi_\ell\right]} \partial_\lambda \left(\bar{\delta}\varphi_\ell\right) \\
&\quad + \partial_\lambda \left(\delta x^\lambda \mathcal{L}\left(\varphi(x), \partial\varphi(x), x\right)\right) + \mathcal{O}\left(\xi^2\right)
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} &= \int_M \mathrm{d}^D x \, \partial_\lambda \frac{\partial \mathcal{L}}{\partial \left[\partial_\lambda \varphi_\ell\right]} \bar{\delta}\varphi_\ell + \frac{\partial \mathcal{L}}{\partial \left[\partial_\lambda \varphi_\ell\right]} \partial_\lambda \left(\bar{\delta}\varphi_\ell\right) \\
&\quad + \partial_\lambda \left(\delta x^\lambda \mathcal{L}\right) + \mathcal{O}\left(\xi^2\right) \\
&= \int_M \mathrm{d}^D x \, \partial_\lambda \left[\frac{\partial \mathcal{L}}{\partial \left[\partial_\lambda \varphi_\ell\right]} \bar{\delta}\varphi_\ell + \mathcal{L} \delta x^\lambda\right] + \mathcal{O}\left(\xi^2\right) \ .
\end{aligned} \tag{B.6}$$

(i): We expand the transformed Lagrangian density at x^μ: $\mathcal{L}\left(\varphi'\left(x'\right),\partial\varphi'\left(x'\right),x'\right) = \delta x^\lambda \partial_\lambda \mathcal{L}\left(\varphi\left(x\right),\partial\varphi\left(x\right),x\right) + \mathcal{O}\left(\xi^2\right)$.

(ii): We change of coordinates from x'^μ to x^μ through $x'^\mu = x^\mu + \delta x^\mu$ and using $\mathrm{d}^D x' = \frac{\partial x'}{\partial x}\mathrm{d}^D x = \det\left(I_D + \frac{\partial \delta x^\mu}{\partial x^\nu}\right)\mathrm{d}^D x = \left(1 + \partial_\lambda\left(\delta x^\lambda\right)\right)\mathrm{d}^D x + \mathcal{O}\left(\xi^2\right)$. This extra piece $\partial_\lambda\left(\delta x^\lambda\right)\mathcal{L}\left(\varphi\left(x\right),\partial\varphi\left(x\right),x\right)$ may be combined with the term of the expansion of the previous line to give $\partial_\lambda\left[\delta x^\lambda \mathcal{L}\left(\varphi\left(x\right),\partial\varphi\left(x\right),x\right)\right]$.

(iii): We use that $\varphi'_\ell\left(x\right) = \varphi_\ell\left(x\right) + \bar{\delta}\varphi_\ell\left(x\right)$ from the definition of $\bar{\delta}\varphi_\ell\left(x\right)$ and expands $\mathcal{L}$ around $\left(\varphi\left(x\right),\partial\varphi\left(x\right),x\right)$, which kills of the zeroth order term and leaves the first order term in the variation.

(iv): Using Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \varphi_\ell} = \partial_\lambda \frac{\partial \mathcal{L}}{\partial\left[\partial_\lambda \varphi_\ell\right]}$ we can write a total derivative.

Now by the assumption of this variation with $\xi^A = \{\epsilon^\alpha, \lambda^a\}$ corresponding to a global symmetry transformation we have $\delta S = 0$ which implies that

$$J^\mu_{\text{can}}\left(\xi\right) \equiv \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\bar{\delta}\varphi_\ell + \mathcal{L}\delta x^\mu \tag{B.7}$$

is a conserved current when we use the EOMs of the theory, as we used the Euler-Lagrange equations in the derivation. We can write $J^\lambda\left(\xi\right)$ in a more useful way using (B.1) and (B.3):

$$\begin{aligned}
J^\mu_{\text{can}}\left(\xi\right) &\equiv \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\left(\delta\varphi_\ell - \delta x^\lambda \partial_\lambda \varphi_\ell\right) + \mathcal{L}\delta x^\mu \\
&= \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\left(\lambda_a U^a_{\ell\bar{\ell}}\varphi_{\bar{\ell}} - \left(\epsilon^\alpha \hat{X}^\lambda_\alpha + \lambda^a \tilde{X}^\lambda_a\right)\partial_\lambda \varphi_\ell\right) + \mathcal{L}\left(\epsilon^\alpha \hat{X}^\mu_\alpha + \lambda^a \tilde{X}^\mu_a\right) \\
&= -\lambda_a\left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\left(\tilde{X}^{\lambda a}\partial_\lambda \varphi_\ell - U^a_{\ell\bar{\ell}}\varphi_{\bar{\ell}}\right) - \mathcal{L}\tilde{X}^{\lambda a}\right) \\
&\quad -\epsilon_\alpha\left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\hat{X}^{\lambda\alpha}\partial_\lambda \varphi_\ell - \mathcal{L}\hat{X}^{\mu\alpha}\right).
\end{aligned} \tag{B.8}$$

This defines two sets of conserved currents which together gives N conserved (canonical) Noether currents corresponding to the N independent variations by

$$J^{\mu a}_{\text{can}} \equiv \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\left(\tilde{X}^{\lambda a}\partial_\lambda \varphi_\ell - U^a_{\ell\bar{\ell}}\varphi_{\bar{\ell}}\right) - \mathcal{L}\tilde{X}^{\lambda a} \tag{B.9a}$$

$$J^{\mu\alpha}_{\text{can}} \equiv \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \varphi_\ell\right]}\hat{X}^{\lambda\alpha}\partial_\lambda \varphi_\ell - \mathcal{L}\hat{X}^{\mu\alpha}. \tag{B.9b}$$

□

Notice that $J^{\mu\alpha}_{\text{can}}$ is the only current that contains the internal variation of the field, but both of them contains terms from the variation of spacetime. The internal part of $J^{\mu\alpha}_{\text{can}}$ does in general not define a conserved current on its own unless the symmetry transformation is purely internal, i.e. $\tilde{X}^{\lambda a} = 0$.

We may define conserved symmetry charges by in some frame where $\mu = 0$ corresponds to the time coordinate integrating over a d-dimensional spatial hypersurface S as

$$Q^{\alpha} \equiv \int_S \mathrm{d}x^d \, J^{\mu\alpha}_{\text{can}} \tag{B.10a}$$

$$Q^{a} \equiv \int_S \mathrm{d}x^d \, J^{\mu a}_{\text{can}} \,. \tag{B.10b}$$

These charges are then seen to furnish an representation of the Lie algebra $\mathfrak{g}$ of the symmetry group with the Poisson brackets representing the commutator as discussed in the main text.

We can take the parameters ξ^A to be local, i.e. $\xi^A = \xi^A(x)$ which then by assumption no longer corresponds to a symmetry of the action. Because ξ^A constant corresponds to a symmetry of the action, the variation of the action may be written proportional to the corresponding conserved canonical Noether current $J^{\mu A}$ as

$$\delta S[\varphi] = -\int_M \mathrm{d}^D x \, \partial_\mu \xi_A(x) \, J^{\mu A}_{\text{can}} \,, \tag{B.11}$$

which is seen to vanish when we again impose $\xi^A = \xi^A(x)$.

B.1.2 *Improvements*

The canonical Noether currents $J^{\mu\alpha}_{\text{can}}$, $J^{\mu a}_{\text{can}}$ are fixed uniquely by the calculation above, but they do not give rise to uniquely defined symmetry currents. From the conservation equations $\partial_\mu J^{\mu\alpha}_{\text{can}} = \partial_\mu J^{\mu a}_{\text{can}} = 0$ we see that we may define new currents by

$$J^{\mu\alpha}_{\text{imp}} \equiv J^{\mu\alpha}_{\text{can}} + \partial_\lambda A^{\lambda\mu\alpha} \quad , \quad A^{\lambda\mu\alpha} = -A^{\mu\lambda\alpha} \tag{B.12a}$$

$$J^{\mu a}_{\text{imp}} \equiv J^{\mu a}_{\text{can}} + \partial_\lambda B^{\lambda\mu a} \quad , \quad B^{\lambda\mu a} = -B^{\mu\lambda a} \tag{B.12b}$$

that are still obviously conserved because of the antisymmetry of the improvement terms $A^{\lambda\mu\alpha}$, $B^{\lambda\mu a}$. These improved currents does not change the charges (B.10) exactly because the improvement term is a total derivative. This shows that there are some freedom left in the symmetry currents that may be exploited to give more simple conservation equations. They may for example be used to make the currents gauge invariant, symmetric in some indices etc.

B.2 THE NOETHER PROCEDURE

B.2.1 *Generalities and setup*

The Noether procedure is a systematic way of making global symmetries of some action local by adding gauge fields to the theory [93, 94]. The starting point is a completely general action $S^{(0)}[\varphi]$ globally invariant under some representation of a symmetry Lie group $\mathcal{G}$. The goal of the procedure is to obtain a modified action

$$S[\varphi, A] = S^{(0)}[\varphi] + S^{(1)}[\varphi, A] + S^{(2)}[\varphi, A] + \dots \tag{B.13}$$

that is invariant under local symmetry transformations of $\mathcal{G}$. We want to obtain this by defining a set of gauge fields $gA_{\mu N}$, $N = 1, \dots, \dim \mathcal{G}$, in a systematic way by adding couplings with higher and higher orders of $A_{\mu N}$ until local symmetry is manifest. We find it useful to define a constant g so that all variations $\delta \propto g$. While this is usually absorbed into the generators, we prefer to write it explicitly in the following to keep track of orders in our procedure.

The big question is then how to construct the terms $S^{(1)}[\varphi, A]$, $S^{(2)}[\varphi, A]$, ... in a systematic manner. The central observation made in section B.1 is that if we have a global symmetry of any action $S^{(0)}[\varphi]$ but take the transformation parameter $\xi_N(x)$ to be local, then the variation of the action may be written proportional to the corresponding conserved canonical Noether current $J^{(0)\mu N}$ as

$$\delta S[\varphi] = -g \int_M \mathrm{d}^D x\, \partial_\mu \xi_N(x)\, J^{(0)\mu N}. \tag{B.14}$$

The non-invariance of the variation (B.14) may be canceled by defining and adding to the action the coupling of a gauge field $A_{\mu N}$ to the current $J^{(0)\mu N}$ and define the gauge field to transform (at first order) as

$$\begin{aligned} S^{(1)}[\varphi, A] &\equiv g \int_M \mathrm{d}^D x\, A^{(1)}_{\mu N} J^{(0)\mu N} && \text{(B.15a)} \\ S^{\Sigma(1)}[\varphi, A] &\equiv S^{(0)}[\varphi] + S^{(1)}[\varphi, A] && \text{(B.15b)} \\ A_{\mu N} &\to A_{\mu N} + \delta A_{\mu N} && \text{(B.15c)} \\ \delta A_{\mu N} &= \delta^{(1)} A_{\mu N} \equiv \partial_\mu \xi_N. && \text{(B.15d)} \end{aligned}$$

The variation of the new action is now easily seen to be

$$\delta S^{\Sigma(1)}[\varphi, A] = g \int_M \mathrm{d}^D x\, A_{\mu N} \delta J^{(0)\mu N} = \mathcal{O}\left(g^2\right), \tag{B.16}$$

so unless $\delta J^{(0)\mu n} = 0$ under the local symmetry transformation, the higher-order action $S^{\Sigma(1)}[\varphi, A]$ is only invariant under local transformations to zeroth order in g. We have then not achieved full local invariance of the action. However as we see that $\delta S^{\Sigma(1)}$ is of order $\mathcal{O}\left(g^2\right)$, as $\delta J^{(0)\mu N} \propto g$. The action $S^{\Sigma(1)}[\varphi, A]$ has a new set of conserved Noether currents that we can write as

$$J^{\Sigma(1)\mu N} = J^{(0)\mu N} + g J^{(1)\mu N}. \tag{B.17}$$

The current $J^{\Sigma(1)\mu N}$ now depends on $A_{\mu N}$, and the EOMs for the field φ_ℓ have also changed because of the new term.

To go to next order in (B.13), we proceed as above and thus must add a term to the action that we call $S^{(2)}[\varphi, A]$ which must be chosen so it has the property under a variation that it cancels $\delta S^{\Sigma(1)}[\varphi, A]$. Thus:

$$\delta S^{(2)}[\varphi, A] + \delta S^{\Sigma(1)}[\varphi, A] = g^2 \int_M \mathrm{d}^D x\, A_{\mu N} \delta J^{(1)\mu N} = \mathcal{O}\left(g^3\right). \tag{B.18}$$

It is now obvious how to perform the Noether procedure: For $n \in \mathbb{N}$ and a given $S^{\Sigma(n)}[\varphi, A]$, find a $S^{(n+1)}[\varphi, A]$ such that

$$\delta S^{(n+1)}[\varphi, A] + \delta S^{\Sigma(n)}[\varphi, A] = \mathcal{O}\left(g^{n+2}\right) \tag{B.19a}$$

$$S^{\Sigma(n+1)}[\varphi, A] \equiv S^{\Sigma(n)}[\varphi, A] + S^{(n+1)}[\varphi, A] \ , \tag{B.19b}$$

and continue until $\delta S^{\Sigma(K+1)} = 0$ for some $K \in \mathbb{N}$ or $K = \infty$. To determine $S^{(n+1)}[\varphi, A]$, we must not only figure out exactly what to couple to the product of n gauge fields $A_{\mu N}$, but also define higher order transformation laws. In general the representation of the symmetry group $\mathcal{G}$ on the coordinates of spacetime and the field φ_ℓ must be modified as we go higher order to make the iteration consistent. The transformation laws (B.1) when the symmetry is global along with that of the gauge field (B.15c) will each need to be corrected by additional terms of higher order in g in (B.1) in order to keep invariance to order $\mathcal{O}\left(g^{n+1}\right)$. We can write the generalization of (B.1) for the new local transformation law as

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \delta x^{\mu} && \text{(B.20a)} \\ \varphi'_{\ell}\left(x'\right) &= \varphi_{\ell}\left(x\right) + \delta\varphi_{\ell}\left(x\right) && \text{(B.20b)} \\ A'_{\mu N}\left(x'\right) &= A_{\mu N}\left(x\right) + \delta A_{\mu N} \end{aligned}$$

$$\begin{aligned} \delta x^{\mu}\left(x\right) &= \xi^{N}\left(x\right)\left(X_{N}^{(0)\mu}\left(x\right) + g X_{N}^{(1)\mu}\left(x\right) + \dots\right) && \text{(B.20c)} \\ \delta\varphi_{\ell}\left(x\right) &= \xi_{N}\left(x\right)\left(U_{\ell\bar{\ell}}^{(0)N} + g U_{\ell\bar{\ell}}^{(1)N}\left(x\right) + \dots\right)\varphi_{\bar{\ell}}\left(x\right) && \text{(B.20d)} \\ \delta A_{\mu N}\left(x\right) &= \partial_{\mu}\xi_{N} + \xi_{M}\left(x\right)\left(g W_{N\bar{N}}^{(1)M} + \dots\right) A_{\mu\bar{N}}\left(x\right) \ . && \text{(B.20e)} \end{aligned}$$

For $A_{\mu N}(x)$ the lowest order is an abelian gauge transformation, while higher order terms in general are non-abelian. This choice of how to define the higher order corrections to the transformation laws is generally not unique, as is neither the choice of the next order action $S^{(n+1)}[\varphi, A]$. We must require that the Poisson algebra of the generators closes at each order, which helps determine the transformation laws.

Assuming that we are able find the terms we need to add, define the local transformation laws and the series (B.13) converges, then we have obtained an action $S[\varphi, A]$ that is invariant under local $\mathcal{G}$-transformations. The result will be the same as if we had covarantized the theory by replacing partial derivatives with covariant derivatives associated with the connection of some fiber bundle. Depending on what choice of transformation laws and what terms we add to the action we will obtain different connections and there is typically a choice that corresponds to "minimal coupling".

If $\mathcal{G}$ is an internal symmetry group the series (B.13) will only contain terms up to $S^{(4)}[\varphi, A]$ corresponding to a general 4-gluon vertex with the gauge field $A_{\mu N}(x)$ being in the adjoint representation of $\mathcal{G}$. On the other hand, if $\mathcal{G}$ is a spacetime symmetry group the series will never terminate. This is because the local translations will exactly be general coordinate transformations on the manifold. The covariant derivative here will be that of some connection corresponding to curvature and torsion of the spacetime.

The procedure is straight-forward to follow, but in practice the calculations are too complicated because there is no systematic way to determine $S^{(n)}[\varphi, A]$. Further more, this will never be a practical way to obtain the full diffeomorphic invariance when $\mathcal{G}$ is a spacetime symmetry group.

However, even with spacetime symmetries we saw that the lowest order local invariance (B.15) is essentially unique and universal, up to rescaling of fields and currents. This is very useful as $S^{\Sigma(1)}[\varphi, A] = S^{(0)}[\varphi] + S^{(1)}[\varphi, A]$ then will be the coupling of the field theory described by $S^{(0)}[\varphi]$ to lowest order in the geometry, which should be enough to study the effects of coupling the theory to gravity. That is indeed the motivation for us to pursue this direction.

B.2.2 *Example: Noether Procedure for the complex relativistic scalar*

It is illustrative to see how the Noether procedure work in the case of a free massless complex scalar. We have $\mathcal{L} = -\partial^\mu \varphi^* \partial_\mu \varphi$, which is invariant under the purely internal U(1) transformation with parameter ϵ

$$\varphi \to e^{i\epsilon}\varphi \quad , \quad \varphi^* \to e^{-i\epsilon}\varphi^* , \tag{B.21}$$

$$\delta\varphi = i\epsilon\varphi \quad , \quad \delta\varphi^* = -i\epsilon\varphi^* \tag{B.22}$$

The globally conserved U(1) current is

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial\left[\partial_\lambda \varphi_\ell\right]} U^a_{\ell\ell'}\varphi_{\ell'} \\ &= -\frac{\partial \mathcal{L}}{\partial\left[\partial_\lambda \varphi^*\right]}(-i)\,\varphi^* + \frac{\partial \mathcal{L}}{\partial\left[\partial_\lambda \varphi\right]}(+i)\,\varphi \\ &= i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi\,. \end{aligned} \tag{B.23}$$

According to the Noether procedure we should then take

$$S^{(1)} = \int_M \mathrm{d}^D x\, \left[\left(i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi\right) A_\mu\right] \tag{B.24}$$

$$\delta A_\mu = \partial_\mu \epsilon + \text{(perhaps higher order terms)} \tag{B.25}$$

so

$$\begin{aligned} S^{\Sigma(1)} &= S^{(0)} + S^{(1)} \\ &= \int_M \mathrm{d}^D x\, \left[-\partial^\mu\varphi^*\partial_\mu\varphi + \left(i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi\right) A_\mu\right] \\ &= \int_M \mathrm{d}^D x\, \left[-\partial^\mu\varphi^*\partial_\mu\varphi + \partial^\mu\varphi^*\left(i\varphi A_\mu\right) - \partial_\mu\varphi\left(i\varphi^* A^\mu\right)\right] \\ &= \int_M \mathrm{d}^D x\, \left[-D^\mu\varphi^* D_\mu\varphi + |\varphi|^2 A^\mu A_\mu\right] , \end{aligned} \tag{B.26}$$

where we identified the covariant derivative as

$$D_\mu\varphi \equiv \partial_\mu\varphi - iA_\mu\varphi\,, \tag{B.27}$$

Let us check $\delta J^{(0)\mu}$ (using $[\delta,\partial] = \left[\overline{\delta},\partial\right] = 0$) under a local U(1) transformation:

$$\begin{aligned}
\delta J^{(0)\mu} &= i\delta\varphi\partial^\mu\varphi^* + i\varphi\partial^\mu\delta\varphi^* \\
&\quad -i\delta\varphi^*\partial^\mu\varphi - i\varphi^*\partial^\mu\delta\varphi \\
&= i\left[i\epsilon\varphi\right]\partial^\mu\varphi^* + i\varphi\partial^\mu\left[-i\epsilon\varphi^*\right] \\
&\quad -i\left[-i\epsilon\varphi^*\right]\partial^\mu\varphi - i\varphi^*\partial^\mu\left[i\epsilon\varphi\right] \\
&= \varphi\varphi^*\partial^\mu\epsilon + \varphi^*\varphi\partial^\mu\epsilon \\
&= 2\varphi\varphi^*\partial^\mu\epsilon \neq 0\,.
\end{aligned} \tag{B.28}$$

We thus have

$$\begin{aligned}
\delta S^{\Sigma(1)} &= \delta S^{(0)} + \delta S^{(1)} \\
&= \int_M \mathrm{d}^D x\, \delta J^{(0)\mu} A_\mu \\
&= \int_M \mathrm{d}^D x\, 2\varphi\varphi^* A_\mu\partial^\mu\epsilon\,,
\end{aligned} \tag{B.29}$$

which shows that we indeed have to add another term to the action, namely one of order $\mathcal{O}\left(A^2\right)$, but we do not need to put additional terms into the transformation law of δA_μ. The $S^{(2)}$ piece we guess from the above should be given by the "Seagull term":

$$S^{(2)} = -\int_M \mathrm{d}^D x\, \varphi^*\varphi A^\mu A_\mu\,, \tag{B.30}$$

because we then have

$$\begin{aligned}
\delta S^{(2)} &= -\delta\int_M \mathrm{d}^D x\, \varphi\varphi^* A^\mu A_\mu \\
&= -\int_M \mathrm{d}^D x\, \left[\delta\varphi\varphi^* A^\mu A_\mu + \varphi\delta\varphi^* A^\mu A_\mu + 2\varphi\varphi^* A_\mu\delta A^\mu\right] \\
&= -\int_M \mathrm{d}^D x\, \left[\cancel{(i\epsilon\varphi)\,\varphi^* A^\mu A_\mu} + \cancel{\varphi\left(-i\epsilon\varphi^*\right) A^\mu A_\mu} + 2\varphi\varphi^* A_\mu\left(\partial^\mu\epsilon\right)\right] \\
&= -\int_M \mathrm{d}^D x\, \left[2\varphi^*\varphi A_\mu\partial^\mu\epsilon\right]\,,
\end{aligned} \tag{B.31}$$

which then shows that

$$\begin{aligned}
\delta S^{\Sigma(2)} &= \delta S^{(0)} + \delta S^{(1)} + \delta S^{(2)} \\
&= \int_M \mathrm{d}^D x\, 2\varphi\varphi^* A_\mu\partial^\mu\epsilon - \int_M \mathrm{d}^D x\, \left[2\varphi\varphi^* A_\mu\partial^\mu\epsilon\right] \\
&= 0\,,
\end{aligned} \tag{B.32}$$

Thus the sequence in A terminates at order 2. We see that this can be assembled into the following local invariant action:

$$\begin{aligned} S &= S^{(0)} + S^{(1)} + S^{(2)} \\ &= \int_M \mathrm{d}^D x \left[-D^\mu \varphi^* D_\mu \varphi + \cancel{|\varphi|^2 A^\mu A_\mu} - \cancel{\varphi^* \varphi A^\mu A_\mu} \right] \\ &= \int_M \mathrm{d}^D x \left[-D^\mu \varphi^* D_\mu \varphi \right] . \end{aligned} \tag{B.33}$$

Hence the morale is: Making the theory locally U(1) invariant is equivalent to just taking $\partial_\mu \to D_\mu$ in the global invariant Lagrangian density. The original global U(1) current $J^\mu = i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi$ is no longer conserved. The new conserved current is the covariantized version of this

$$J^\mu = J^{\Sigma(2)\mu} = i\varphi D^\mu \varphi^* - i\varphi^* D^\mu \varphi \,. \tag{B.34}$$

We can write the interaction term as

$$S_{\text{int}} = \int_M \mathrm{d}^D x \left[\left(i\varphi\partial^\mu\varphi^* - i\varphi^*\partial^\mu\varphi \right) A_\mu - \varphi^* \varphi A^\mu A_\mu \right] . \tag{B.35}$$

C NON-RELATIVISTIC CONFORMAL GROUPS

C.1 THE SCHRÖDINGER GROUP

There are now various ways to define the analog of the relativistic conformal group. In any case we must add the Lifshitz scaling of section 2.2 to the Galilean or Bargmann groups, but the number of extra generators we need to add to make the algebra close depends strongly on the value of z. Adding the Lifshitz scaling with generator D to Bargmann for $z = 2$ leads to the necessity of adding a "temporal SCT" with generator C to the algebra for closure, which gives us the Schrödinger group $\mathrm{Schr}\,(d, 1)$. The new extra non-zero commutation relations [4] compared to Bargmann are given by

$$[D, H] = 2H \tag{C.1a}$$

$$[D, P_i] = P_i \tag{C.1b}$$

$$[D, B_i] = -B_i \tag{C.1c}$$

$$[D, C] = -2C \tag{C.1d}$$

$$[C, H] = D \tag{C.1e}$$

The corresponding symmetry group is the maximal symmetry group of the free Schrödinger equation of section 7.1, hence the name [28, 77]. The structure of the group is now a lot different than for Bargmann, but it is still a subgroup. In particular there is a $\mathfrak{sl}\,(2, \mathbb{R})$ subalgebra spanned by $\{D, H, C\}$.

C.2 THE CONFORMAL GALILEAN GROUP

If we instead for $z = 1$ add the Lifshitz scaling (dilatation) to the Galilean group, we see that we need to add the same temporal SCT generator C along with d spatial SCTs K_i for closure. The resulting group is the conformal Galilean group $\mathrm{CGal}\,(d, 1)$, which in many regards is the one that is closest to the relativistic conformal group, as space and time scale symmetrically. The resulting non-zero commutators [29] of the algebra are

$$[D, H] = H \tag{C.2a}$$

$$[D, P_i] = P_i \tag{C.2b}$$

$$[D, B_i] = -B_i \tag{C.2c}$$

$$[D, C] = -C \tag{C.2d}$$

$$[D, K_i] = -K_i \tag{C.2e}$$

$$[C, H] = 2D \tag{C.2f}$$
$$[C, P_i] = 2B_i \tag{C.2g}$$
$$[C, B_i] = K_i \tag{C.2h}$$
$$[K_i, H] = 2B_i \tag{C.2i}$$
$$\left[K_i, P_j\right] = 2J_{ij} + 2\delta_{ij}D \tag{C.2j}$$
$$\left[K_k, J_{ij}\right] = \delta_{ik}K_j - \delta_{jk}K_i \tag{C.2k}$$

Interestingly enough, it is easy to see from the Jacobi identity that it is not possible to add the previous central charge M. Again we have an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra spanned by $\{D, H, C\}$. CGal $(d,1)$ can also be obtained from an Inönü-Wigner contraction of the relativistic conformal group $\mathrm{C}(d,1)$ of appendix A.1.2 [29].

C.3 AN INFINITE DIMENSIONAL EXTENSION

It is not too hard to guess how to generalize the two above algebras to more general values of z, given that we have a $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in both cases: One could then try to extend it to a Witt algebra with generators L^n. It is then also natural to give the rotations and rotations a wight under this Witt algebra, which can be used to define two sets of generators T_i^n for generalized translations and J_{ij}^n for generalized rotations. After some work along these lines, which is straight-forward, one finds an infinite dimensional algebra that contains all of the non-relativistic algebras without central extension of table 1 exists and is given by

$$[L^m, L^n] = (m-n)\, L^{m+n} \tag{C.3a}$$
$$\left[M_{ij}^m, M_{kl}^n\right] = \delta_{jk}M_{il}^{m+n} - \delta_{ik}M_{jl}^{m+n} - \delta_{jl}M_{ik}^{m+n} + \delta_{il}M_{jk}^{m+n} \tag{C.3b}$$
$$\left[L^m, M_{ij}^n\right] = -nM_{ij}^{m+n} \tag{C.3c}$$
$$\left[T_i^m, T_j^n\right] = 0 \tag{C.3d}$$
$$[L^m, T_i^n] = \left(z^{-1}(m+1) - n\right) T_i^{m+n} \tag{C.3e}$$
$$\left[M_{ij}^m, T_k^n\right] = \delta_{kj}T_i^{m+n} - \delta_{ki}T_j^{m+n} \tag{C.3f}$$

This algebra consists of a Witt algebra spanned by L^n (may be centrally extended to a Virasoro immediately), a $\mathfrak{so}(d)$ Kac-Moody algebra spanned by M_{ij}^m, and T_i^m are $\mathfrak{so}(d)$ vectors and Witt primaries. These are known in the literature as the spin-$\frac{N}{2}$ extended Galilean algebras when we take $z = 2/N$, $N \in \mathbb{N}$ [95, 29]. One can work out the commutation relations to see that only if $z = 2/N$ do we have an finite dimensional subalgebra which is spanned by L^{-1}, L^0, L^1, M_{ij}^0 and $T_i^0, \ldots, T_i^N$. We may centrally extend this algebra for $z = 2, \frac{1}{2}, , \frac{1}{4}, \ldots$ by a central charge M so that it contains all of the algebras of table 1 with central charges as well. This is done by taking

$$\left[T_i^m, T_j^n\right] = I^{mn}\delta_{ij}M\,, \tag{C.4}$$

where I^{mn} is the invariant symbol of $\mathfrak{sl}(2,\mathbb{R})$ in the representations the generators T_i^m carry.

The corresponding finite transformations are obtained by integrating the infinitesimal spacetime transformations generated by (C.3). We find that the transformations are

$$t' = f(t) \,, \tag{C.5a}$$
$$x_i' = \left(f'(t)\right)^{1/z} \left(R_i{}^j(t)\, x_j + \lambda_i(t)\right) \tag{C.5b}$$

A theory with this symmetry would be invariant under arbitrary time redefinitions and time-dependent rotations and translations. The corresponding transformations of the finite subgroup are

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta} \,, \tag{C.6a}$$
$$x' = (\gamma t + \delta)^{-\frac{2}{z}} \left(R_i{}^j x_j + \sum_{a=0}^{N} \lambda_i^a t^a\right) \,. \tag{C.6b}$$

D CALCULATIONS

D.2 CHAPTER 2

D.2.1 $\mathring{C}_4$ in $d = 3$ dimensions

We show that $\mathring{C}_4 = \left(\epsilon_{ijk}\left[\frac{1}{2}J^{jk} - \frac{1}{m}\left(B^j P^k\right)\right]\right)^2$ in $d = 3$ by collecting terms in the formula (2.30c) and using (E.3):

$$
\begin{aligned}
\mathring{C}_4 &= \frac{1}{2}J^{jk}J_{jk} - \frac{1}{m}\left(J^{jk}B_jP_k - B_jP_kJ^{jk}\right) + \frac{1}{m^2}B^jP^k\left(B^jP^k - B^kP^j\right) \\
&= \frac{1}{2}J^{jk}J^{jk} - \frac{1}{2m}J^{jk}B^jP^k - \frac{1}{2m}B^jP^kJ^{jk} + \frac{1}{m^2}B^jP^kB^jP^k \\
&\quad + \frac{1}{2m}J^{jk}B^kP^j - \frac{1}{2m}B^jP^kJ^{jk} - \frac{1}{m^2}B^jP^kB^kP^j \\
&= \frac{1}{2}J^{jk}\left[\frac{1}{2}J^{jk} - \frac{1}{m}B^jP^k\right] - \frac{1}{m}B^jP^k\left[\frac{1}{2}J^{jk} - \frac{1}{m}B^jP^k\right] \\
&\quad - \frac{1}{2}J^{jk}\left[-\frac{1}{2}J^{jk} - \frac{1}{m}B^kP^j\right] + \frac{1}{m}B^jP^k\left[-\frac{1}{2}J^{jk} - \frac{1}{m}B^kP^j\right] \\
&= \left[\frac{1}{2}J^{jk} - \frac{1}{m}B^jP^k\right]\left[\frac{1}{2}J^{jk} - \frac{1}{m}B^jP^k\right] \\
&\quad - \left[\frac{1}{2}J^{jk} - \frac{1}{m}B^jP^k\right]\left[-\frac{1}{2}J^{jk} - \frac{1}{m}B^kP^j\right] \\
&= \left(\delta^{jn}\delta_{km} - \delta^{jm}\delta_{kn}\right)\left[\frac{1}{2}J^{jk} - \frac{1}{m}\left(B^jP^k\right)\right]\left[\frac{1}{2}J^{nm} - \frac{1}{m}\left(B^nP^m\right)\right] \\
&= \left(\epsilon_{ijk}\left[\frac{1}{2}J^{jk} - \frac{1}{m}\left(B^jP^k\right)\right]\right)^2 . \qquad \text{(D.1)}
\end{aligned}
$$

On the contrary one could also start from $\mathring{C}_4 = \left(\epsilon_{ijk}\left[\frac{1}{2}J^{jk} - \frac{1}{m}\left(B^jP^k\right)\right]\right)^2$ and by a careful analysis revert the calculation above to find the general d-dimensional expression for it as there is non explicit d-dependence in the expression. This is actually how we (2.30c) in the thesis generalizing the result of [35].

D.3 CHAPTER 3

D.3.1 Solving for the affine Galilean connection

An expression for the affine connection in terms of the gauge fields of the connection can be found from the vielbein postulates (3.24). We find using the completeness relation and the other vielbein postulate

$$
\begin{aligned}
0 &= e^{\lambda}{}_{a}\nabla_{\mu}e_{\nu}{}^{a} \\
&= e^{\lambda}{}_{a}\left(\partial_{\mu}e_{\nu}{}^{a} - \Gamma^{\rho}_{\mu\nu}e_{\rho}{}^{a} - \Omega_{\mu}{}^{a}\tau_{\nu} - \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b}\right) \\
&= e^{\lambda}{}_{a}\left(\partial_{\mu}e_{\nu}{}^{a} - \Omega_{\mu}{}^{a}\tau_{\nu} - \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b}\right) - \Gamma^{\rho}_{\mu\nu}\left(\delta^{\lambda}_{\rho} + v^{\lambda}\tau_{\rho}\right) \\
&= e^{\lambda}{}_{a}\left(\partial_{\mu}e_{\nu}{}^{a} - \Omega_{\mu}{}^{a}\tau_{\nu} - \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b}\right) - \Gamma^{\lambda}_{\mu\nu} - v^{\lambda}\Gamma^{\rho}_{\mu\nu}\tau_{\rho} \\
&= e^{\lambda}{}_{a}\left(\partial_{\mu}e_{\nu}{}^{a} - \Omega_{\mu}{}^{a}\tau_{\nu} - \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b}\right) - \Gamma^{\lambda}_{\mu\nu} - v^{\lambda}\partial_{\mu}\tau_{\nu} \quad \Rightarrow
\end{aligned}
$$

$$
\Gamma^{\lambda}_{\mu\nu} = -v^{\lambda}\partial_{\mu}\tau_{\nu} + e^{\lambda}{}_{a}\left(\partial_{\mu}e_{\nu}{}^{a} - \Omega_{\mu}{}^{a}\tau_{\nu} - \omega_{\mu}{}^{a}{}_{b}e_{\nu}{}^{b}\right). \tag{D.2}
$$

D.3.2 *Linearizing the Hartong-Obers connection*

We want to linearize the boost and rotation gauge fields (3.43). Starting with the boost gauge field we find

$$
\begin{aligned}
2\overline{\Omega}_{\mu a} &= 2v^{\nu}\partial_{[\nu}e_{\mu]a} + 2v^{\nu}e^{\sigma}{}_{a}e_{\mu b}\partial_{[\nu}e_{\sigma]}^{\ b} + 2\overline{C}_{\mu a} \\
&= -\delta^{\nu}_{0}\left(\partial_{\nu}\bar{e}_{\mu a} - \partial_{\mu}\bar{e}_{\nu a}\right) - \delta^{\nu}_{0}\delta^{\sigma}{}_{a}\delta_{\mu b}\left(\partial_{\nu}\bar{e}_{\sigma}{}^{b} - \partial_{\sigma}\bar{e}_{\nu}{}^{b}\right) + 2\overline{C}_{\mu a} \\
&= -\partial_{0}\bar{e}_{\mu a} + \partial_{\mu}\bar{e}_{0a} - \delta_{\mu b}\left(\partial_{0}\bar{e}_{a}{}^{b} - \partial_{a}\bar{e}_{0}{}^{b}\right) + 2\overline{C}_{\mu a}\,, \\
&\equiv -\partial_{0}\bar{e}_{\mu a} + \partial_{\mu}\bar{e}_{0a} - \delta_{\mu b}\left(\partial_{0}\bar{e}_{a}{}^{b} - \partial_{a}\bar{e}_{0}{}^{b}\right) + 2\overline{C}_{\mu a}
\end{aligned} \tag{D.3}
$$

We then find that the components become

$$
\begin{aligned}
2\overline{\Omega}_{0a} &= -\cancel{\partial_{0}\bar{e}_{0}{}^{a}} + \cancel{\partial_{0}\bar{e}_{0}{}^{a}} - \cancel{\delta_{0b}\left(\partial_{0}\bar{e}_{a}{}^{b} - \partial_{a}\bar{e}_{0}{}^{b}\right)} \\
&= 2\overline{C}_{0a}
\end{aligned} \tag{D.4}
$$

$$
\begin{aligned}
2\overline{\Omega}_{ba} &= -\partial_{0}\bar{e}_{ba} + \partial_{b}\bar{e}_{0a} - \left(\partial_{0}\bar{e}_{ab} - \partial_{a}\bar{e}_{0b}\right) + 2\overline{C}_{ba} \\
&= -2\partial_{0}\bar{e}_{(ba)} + \partial_{b}\bar{e}_{0a} + \partial_{a}\bar{e}_{0b} + 2\overline{C}_{ba} \\
&= -\partial_{0}s_{ba} + 2\partial_{(a}\bar{e}_{|0|b)} + 2\overline{C}_{ba} \\
&= -\partial_{0}s_{ba} - 2\partial_{(a}\overline{v}_{b)} + 2\overline{C}_{ba}\,.
\end{aligned}
$$

In total, after dividing by two, we find

$$
\overline{\Omega}_{\mu a} = \begin{pmatrix} \Omega_{0a} \\ \Omega_{ba} \end{pmatrix} = \begin{pmatrix} \overline{C}_{0a} \\ -\frac{1}{2}\partial_{0}s_{ba} - \partial_{(b}\overline{v}_{a)} + \overline{C}_{ba} \end{pmatrix}. \tag{D.5}
$$

Linearization of the rotation gauge field gives

$$\begin{aligned}
2\overline{\omega}_{\mu ac} &= 2e^{\lambda}{}_{[c|}\partial_{\mu}e_{\lambda|a]} + 2e^{\lambda}{}_{[a|}\partial_{\lambda}e_{\mu|c]} + 2e_{\mu b}e^{\sigma}{}_{[c}e^{\lambda}{}_{a]}\partial_{\lambda}e_{\sigma}{}^{b} + 2\overline{C}_{\mu ac} \\
&= 2\delta^{\lambda}{}_{[c|}\partial_{\mu}\overline{e}_{\lambda|a]} + 2\delta^{\lambda}{}_{[a|}\partial_{\lambda}\overline{e}_{\mu|c]} + 2\delta_{\mu b}\delta^{\sigma}{}_{[c}\delta^{\lambda}{}_{a]}\partial_{\lambda}\overline{e}_{\sigma}{}^{b} + 2\overline{C}_{\mu ac} \\
&= \left(\delta^{\lambda}_{c}\partial_{\mu}\overline{e}_{\lambda a} - \delta^{\lambda}_{a}\partial_{\mu}\overline{e}_{\lambda c}\right) + \left(\delta^{\lambda}_{a}\partial_{\lambda}\overline{e}_{\mu c} - \delta^{\lambda}_{c}\partial_{\lambda}\overline{e}_{\mu a}\right) \\
&\quad + \delta_{\mu b}\left(\delta^{\sigma}_{c}\delta^{\lambda}_{a} - \delta^{\sigma}_{a}\delta^{\lambda}_{c}\right)\partial_{\lambda}\overline{e}_{\sigma}{}^{b} + 2\overline{C}_{\mu ac} \\
&= \left(\partial_{\mu}\overline{e}_{ca} - \partial_{\mu}\overline{e}_{ac}\right) + \left(\partial_{a}\overline{e}_{\mu c} - \partial_{c}\overline{e}_{\mu a}\right) + \delta_{\mu b}\left(\partial_{a}\overline{e}_{c}{}^{b} - \partial_{c}\overline{e}_{a}{}^{b}\right) + 2\overline{C}_{\mu ac} \\
&= 2\partial_{\mu}\overline{e}_{[ca]} + 2\partial_{[a}\overline{e}_{|\mu|c]} + 2\delta_{\mu b}\partial_{[a}\overline{e}_{c]}^{\ b} + 2\overline{C}_{\mu ac} \\
&\equiv 2\partial_{\mu}\overline{e}_{[ca]} + 2\partial_{[a}\overline{e}_{|\mu|c]} + 2\delta_{\mu b}\partial_{[a}\overline{e}_{c]}^{\ b} + 2\overline{C}_{\mu ac}\,.
\end{aligned} \tag{D.6}$$

In components:

$$\begin{aligned}
2\overline{\omega}_{0ac} &= 2\partial_{0}\overline{e}_{[ca]} + 2\partial_{[a}\overline{e}_{|0|c]} + 2\overline{C}_{0ac} \\
&= -2\partial_{0}\overline{e}_{[ac]} - 2\partial_{[a}\overline{v}_{c]} + 2\overline{C}_{0ac}\,,
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
2\overline{\omega}_{bac} &= 2\partial_{b}\overline{e}_{[ca]} + 2\partial_{[a}\overline{e}_{|b|c]} + \delta_{bd}\partial_{[a}\overline{e}_{c]}^{\ d} + 2\overline{C}_{bac} \\
&= 2\partial_{b}\overline{e}_{[ca]} + 2\partial_{[a}\overline{e}_{|b|c]} + 2\partial_{[a}\overline{e}_{c]b} + 2K_{bac} \\
&= \left(\partial_{b}\overline{e}_{ca} - \partial_{b}\overline{e}_{ac}\right) + \left(\partial_{a}\overline{e}_{bc} - \partial_{c}\overline{e}_{ba}\right) + \left(\partial_{a}\overline{e}_{cb} - \partial_{c}\overline{e}_{ab}\right) + 2\overline{C}_{bac} \\
&= 2\partial_{b}\overline{e}_{[ca]} + 2\left(\partial_{a}\overline{e}_{(bc)} - \partial_{c}\overline{e}_{(ba)}\right) + 2\overline{C}_{bac} \\
&= 2\partial_{b}\overline{e}_{[ca]} + \partial_{a}s_{bc} - \partial_{c}s_{ba} + 2\overline{C}_{bac} \\
&= 2\partial_{b}\overline{e}_{[ca]} + 2\partial_{[a}s_{c]b} + 2\overline{C}_{bac} \\
&= -2\partial_{b}\overline{e}_{[ac]} + 2\partial_{[a}s_{c]b} + 2\overline{C}_{bac}
\end{aligned} \tag{D.8}$$

so we can finally write

$$\overline{\omega}_{\mu ac} = \begin{pmatrix} \Omega_{0ac} \\ \Omega_{bac} \end{pmatrix} = \begin{pmatrix} -\partial_{0}\overline{e}_{[ac]} - \partial_{[a}\overline{v}_{c]} + \overline{C}_{0ac} \\ -\partial_{b}\overline{e}_{[ac]} + \partial_{[a}s_{|b|c]} + \overline{C}_{bac} \end{pmatrix}. \tag{D.9}$$

From this we see that the pseudo-gauge fields (3.44) are linearized as

$$\hat{\overline{\Omega}}_{\mu a} = \begin{pmatrix} \hat{\overline{\Omega}}_{0a} \\ \hat{\overline{\Omega}}_{ba} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}\partial_{0}s_{ba} - \partial_{(a}\overline{v}_{b)} \end{pmatrix}, \tag{D.10a}$$

$$\hat{\overline{\omega}}_{\mu ac} = \begin{pmatrix} \hat{\overline{\omega}}_{0ac} \\ \hat{\overline{\omega}}_{bac} \end{pmatrix} = \begin{pmatrix} -\partial_{0}\overline{e}_{[ac]} - \partial_{[a}\overline{v}_{c]} \\ -\partial_{b}\overline{e}_{[ac]} + \partial_{[a}s_{c]b} \end{pmatrix}. \tag{D.10b}$$

We can then perform the linearization of the pseudo-contortions $\overline{C}_{\mu a}$ and $\overline{C}_{\mu ac}$ for the graviphotonic connection of section 3.3.3.1 in the main text. Let us first consider the linearization of $W^{\lambda}{}_{\mu\nu}$, as all pieces multiplied with this must be zeroth order:

$$\begin{aligned}
\overline{W}^{\lambda}{}_{\mu\nu} &= \frac{1}{2}h^{\lambda\sigma}\overline{Y}_{\sigma\mu\nu} \\
&= \frac{1}{2}\delta^{\lambda a}\delta^{\sigma}_{a}\overline{Y}_{\sigma\mu\nu} \\
&= \frac{1}{2}\delta^{\lambda a}\overline{Y}_{a\mu\nu} \\
&= \frac{1}{2}\delta^{\lambda a}\left(\tau_{\mu}K_{a\nu} + \tau_{\nu}K_{a\mu} + \cancel{L_{a\mu\nu}}\right) \\
&= \delta^{\lambda a}\left(\delta^{0}_{\mu}\partial_{[a}\overline{M}_{\nu]} + \delta^{0}_{\nu}\partial_{[a}\overline{M}_{\mu]}\right)
\end{aligned} \tag{D.11}$$

Where we used that $L_{\sigma\mu\nu}$ only contains higher-order pieces. The linearization of M_{μ} is independent of the orbits. We then find

$$\begin{aligned}
\overline{C}_{\mu a} &= -v^{\nu}e_{\lambda a}W^{\lambda}{}_{\mu\nu} \\
&= \delta^{\nu}_{0}\delta_{\lambda a}\delta^{\lambda b}\left(\delta^{0}_{\mu}\partial_{[b}\overline{M}_{\nu]} + \delta^{0}_{\nu}\partial_{[b}\overline{M}_{\mu]}\right) \\
&= \delta^{0}_{\mu}\partial_{[a}\overline{M}_{0]} + \partial_{[a}\overline{M}_{\mu]} \quad \Rightarrow
\end{aligned}$$

$$\overline{C}_{\mu a} = -\begin{pmatrix} 2\partial_{[0}\overline{M}_{a]} \\ \partial_{[j}\overline{M}_{a]} \end{pmatrix}, \tag{D.12}$$

and for the other one

$$\begin{aligned}
\overline{C}_{\mu ac} &= e^{\nu}{}_{c}e_{\lambda a}W^{\lambda}{}_{\mu\nu} \\
&= \delta^{\nu}{}_{c}\delta_{\lambda a}\delta^{\lambda b}\left(\delta^{0}_{\mu}\partial_{[b}\overline{M}_{\nu]} + \delta^{0}_{\nu}\partial_{[b}\overline{M}_{\mu]}\right) \\
&= \delta^{0}_{\mu}\partial_{[a}\overline{M}_{c]} + \cancel{\delta^{0}_{c}\partial_{[a}\overline{M}_{\mu]}} \quad \Rightarrow
\end{aligned}$$

$$\overline{C}_{\mu ij} = \begin{pmatrix} \partial_{[i}\overline{M}_{j]} \\ 0 \end{pmatrix}. \tag{D.13}$$

D.4 CHAPTER 4

D.4.1 *Improvements of Galilean spacetime currents*

Current	Improvement	
Time	$E^0_{\text{imp}} = E^0_{\text{can}} + \partial_i \alpha^i$	α^i
	$E^i_{\text{imp}} = E^i_{\text{can}} - \partial_0 \alpha^i + \partial_j \beta^{ji}$	$\beta^{ji} = -\beta^{ij}$
Translation	$T^{0i}_{\text{imp}} = T^{i0}_{\text{can}} + \partial_k \gamma^{ki}$	γ^{ki}
	$T^{ji}_{\text{imp}} = T^{ji}_{\text{can}} - \partial_0 \gamma^{ij} + \partial_k \zeta^{kji}$	$\zeta^{kji} = -\zeta^{jki}$
Rotation	$j^{0ij}_{\text{imp}} = x^i T^{0j}_{\text{can}} - x^j T^{0i}_{\text{can}} + s^{0ij} + \partial_k \kappa^{kij}$	κ^{kij}
	$j^{kij}_{\text{imp}} = x^i T^{kj}_{\text{can}} - x^j T^{ki}_{\text{can}} + s^{kij} - \partial_0 \kappa^{kij} + \partial_l \lambda^{lkij}$	$\lambda^{lkij} = -\lambda^{klij}$
Boost	$b^{0i}_{\text{imp}} = t T^{0i}_{\text{can}} + w^{0i} + \partial_k \xi^{ki}$	ξ^{ki}
	$b^{ji}_{\text{imp}} = t T^{ji}_{\text{can}} + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji}$	$\rho^{kji} = -\rho^{jki}$

Table 5: Improvements of currents and charges of a non-relativistic Galilean theory.

There is some freedom left in the currents defined in (4.17) as adding a divergence term will not change the generating charges. We have the following improvements which leaves the charges and transformation law unchanged

$$E^{\mu}_{\text{imp}} = E^{\mu}_{\text{can}} + \partial_\rho A^{\rho\mu} \tag{D.14a}$$
$$T^{\mu i}_{\text{imp}} = T^{i\mu}_{\text{can}} + \partial_\rho B^{\rho\mu i} \tag{D.14b}$$
$$j^{\mu ij}_{\text{imp}} = j^{\mu ij}_{\text{can}} + \partial_\rho D^{\rho\mu ij} \tag{D.14c}$$
$$b^{\mu i}_{\text{imp}} = b^{\mu i}_{\text{can}} + \partial_\rho E^{\rho\mu i} \tag{D.14d}$$

where all A, B, D, E are antisymmetric in the first two indices. In the spirit of Galilean relativity we can separate the components of the improvements as in table 5 which is more convenient for the following. The way to proceed is simply by expressing the canonical currents in terms of the improved ones of table 5 and substituting the results into the rotation and boost currents that are expressed in terms of the momentum current.

D.4.1.1 *Rotation current*

We start with the zero component of the rotation current:

$$\begin{aligned} j^{0ij}_{\text{imp}} &= x^i T^{0j}_{\text{can}} - x^j T^{0i}_{\text{can}} + s^{0ij} + \partial_k \kappa^{kij} \\ &= x^i \left(T^{0j}_{\text{imp}} - \partial_k \gamma^{kj}\right) - x^j \left(T^{0i}_{\text{imp}} - \partial_k \gamma^{ki}\right) + \partial_k \kappa^{kij} + s^{0ij} \\ &= x^i T^{0j}_{\text{imp}} - x^j T^{0i}_{\text{imp}} - x^i \partial_k \gamma^{kj} + x^j \partial_k \gamma^{ki} + \partial_k \kappa^{kij} + s^{0ij}\,. \end{aligned} \tag{D.15}$$

What is of interest is

$$-x^i\partial_k\gamma^{kj} + x^j\partial_k\gamma^{ki} + \partial_k\kappa^{kij} + s^{0ij}\,, \tag{D.16}$$

where we would like to choose γ^{ki} and κ^{kij} so that we can remove as much as s^{0ij} as possible. We can rewrite this using the product rule of differentiation:

$$\begin{aligned} &-\partial_k\left(x^i\gamma^{kj}\right) + \left(\partial_k x^i\right)\gamma^{kj} + \partial_k\left(x^j\gamma^{ki}\right) - \left(\partial_k x^j\right)\gamma^{ki} + \partial_k\kappa^{kij} + s^{0ij} \\ &\qquad = -\partial_k\left(x^i\gamma^{kj}\right) + \partial_k\left(x^j\gamma^{ki}\right) + \partial_k\kappa^{kij} + \gamma^{ij} - \gamma^{ji} + s^{0ij} \end{aligned} \tag{D.17}$$

This is useful, because we see that if we choose

$$\gamma^{[ji]} = \frac{1}{2}s^{0ij} \tag{D.18a}$$

$$\kappa^{kij} = x^i\gamma^{kj} - x^j\gamma^{ki}\,, \tag{D.18b}$$

then we have a unique solution that removes all of s^{0ij} in the improved current so that

$$j^{0ij}_{\text{imp}} = x^i T^{0j}_{\text{imp}} - x^j T^{0i}_{\text{imp}}\,, \tag{D.19}$$

like we postulated in the main text. Continuing with the spatial components in the same way, we have

$$\begin{aligned} j^{kij}_{\text{imp}} &= x^i T^{kj}_{\text{can}} - x^j T^{ki}_{\text{can}} + s^{kij} - \partial_0\kappa^{kij} + \partial_l\lambda^{lkij} \\ &= x^i\left(T^{kj}_{\text{imp}} + \partial_0\gamma^{kj} - \partial_l\zeta^{lkj}\right) - x^j\left(T^{ki}_{\text{imp}} + \partial_0\gamma^{ki} - \partial_l\zeta^{lki}\right) \\ &\quad -\partial_0\kappa^{kij} + \partial_l\lambda^{lkij} + s^{kij} \\ &= x^i T^{kj}_{\text{imp}} - x^j T^{ki}_{\text{imp}} + x^i\left(\partial_0\gamma^{kj} - \partial_l\zeta^{lkj}\right) \\ &\quad -x^j\left(\partial_0\gamma^{ki} - \partial_l\zeta^{lki}\right) - \partial_0\kappa^{kij} + \partial_l\lambda^{lkij} + s^{kij}\,. \end{aligned} \tag{D.20}$$

Now the terms we focus on are those of the last line. We see that given our previous choice of κ^{kij} in (D.18), we have three of the terms cancels, leaving some terms where we again use the product rule

$$\begin{aligned} &-x^i\partial_l\zeta^{lkj} + x^j\partial_l\zeta^{lki} + \partial_l\lambda^{lkij} + s^{kij} \\ &\quad = -\partial_l\left(x^i\zeta^{lkj}\right) + \left(\partial_l x^i\right)\zeta^{lkj} + \partial_l\left(x^j\zeta^{lki}\right) - \left(\partial_l x^j\right)\zeta^{lki} + \partial_l\lambda^{lkij} + s^{kij} \\ &\qquad = -\partial_l\left(x^i\zeta^{lkj}\right) + \partial_l\left(x^j\zeta^{lki}\right) + \zeta^{ikj} - \zeta^{jki} + \partial_l\lambda^{lkij} + s^{kij}\,, \end{aligned} \tag{D.21}$$

so we see that we can also make s^{kij} disappear in the improved currents by taking the only solution in terms of the spatial spin-current that satisfies $\zeta^{jki} = -\zeta^{kji}$

$$\begin{aligned} \zeta^{jik} &= \frac{1}{2}\left(s^{jki} + s^{ikj} + s^{kij}\right) && \text{(D.22a)} \\ \lambda^{lkij} &= x^i \zeta^{lkj} - x^j \zeta^{lki} \,. && \text{(D.22b)} \end{aligned}$$

We can still allow for extra terms that satisfies $\zeta^{jik} = \zeta^{jki}$, which are the further improvements described in section 4.2.3.2, but this solution is essentially unique.

Notice that while all freedom in κ^{kij}, λ^{lkij} have been used, we have still not fixed the symmetric part of γ^{ji}. The result is that we can always write

$$j^{kij}_{\text{imp}} = x^i T^{kj}_{\text{imp}} - x^j T^{ki}_{\text{imp}} \,, \tag{D.23}$$

as postulated in the main text.

D.4.1.2 *Boost current*

Let us now continue with the boost current in the same way. When we have no central charge and no associated mass current there are less freedom in choosing the improvements. We can start with the zero component of the current and see that we find

$$\begin{aligned} b^{0i}_{\text{imp}} &= tT^{0i}_{\text{can}} + w^{0i} + \partial_k \xi^{ki} \\ &= t\left(T^{0i}_{\text{imp}} - \partial_k \gamma^{ki}\right) + w^{0i} + \partial_k \xi^{ki} \\ &= tT^{0i}_{\text{imp}} - \partial_k\left(t\gamma^{ki}\right) + w^{0i} + \partial_k \xi^{ki} \,. \end{aligned} \tag{D.24}$$

Now, we have already fixed the antisymmetric part of γ^{ki} and have no consistent way of removing any components of w^{0i}. We can therefore only take

$$\xi^{ki} = t\gamma^{ki} \,, \tag{D.25}$$

which gives

$$b^{0i}_{\text{imp}} = tT^{0i}_{\text{imp}} + w^{0i} \,. \tag{D.26}$$

For the spatial components with the above choice we instead find

$$\begin{aligned} b^{ji}_{\text{imp}} &= tT^{ji}_{\text{can}} + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji} \\ &= t\left(T^{ji}_{\text{imp}} + \partial_0 \gamma^{ji} - \partial_l \zeta^{lji}\right) + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji} \\ &= tT^{ji}_{\text{imp}} + \cancel{\partial_0\left(t\gamma^{ji}\right)} - \gamma^{ji} - \partial_l\left(t\zeta^{lji}\right) + w^{ji} - \cancel{\partial_0 \xi^{ji}} + \partial_k \rho^{kji} \\ &= tT^{ji}_{\text{imp}} + w^{ji} - \gamma^{ji} - \partial_l\left(t\zeta^{lji}\right) + \partial_k \rho^{kji} \end{aligned}$$

We now see that since $\gamma^{[ji]}$ is already fixed, we can only remove the symmetric part of w^{ji} by choosing

$$\begin{aligned} \gamma^{(ji)} &= w^{(ji)} && \text{(D.27a)} \\ \rho^{kji} &= t\zeta^{kji} \,. && \text{(D.27b)} \end{aligned}$$

This is a unique solution. The antisymmetric part of w^{ji} and γ^{ji} are left in this expression and shows in total that we have

$$b^{0i}_{\text{imp}} = tT^{0i}_{\text{imp}} + \psi^{0i} \tag{D.28a}$$
$$b^{ki}_{\text{imp}} = tT^{ki}_{\text{imp}} + \psi^{ki} \tag{D.28b}$$

where we have defined the current $\psi^{\mu i}$ as in (4.28) by

$$\psi^{\mu i} \equiv \delta^{\mu}_{0}\left[w^{0i}\right] + \delta^{\mu}_{j}\left[w^{[ji]} - \frac{1}{2}s^{0ij}\right]. \tag{D.29}$$

This proves that the choices of improvements discussed in section 4.2.3.1 are the essentially unique (up to the further improvements of the stress tensor) that simplifies the Galilean symmetry currents as much as possible. The choice of improvements may then be converted to the expressions for B, D, E in the main text (4.26).

D.4.2 *Improvements of Bargmann spacetime currents*

Current	Improvement	
Time	$E^{0}_{\text{imp}} = E^{0}_{\text{can}} + \partial_i \alpha^i$	α^i
	$E^{i}_{\text{imp}} = E^{i}_{\text{can}} - \partial_0 \alpha^i + \partial_j \beta^{ji}$	$\beta^{ji} = -\beta^{ij}$
Translation	$T^{0i}_{\text{imp}} = T^{i0}_{\text{can}} + \partial_k \gamma^{ki}$	γ^{ki}
	$T^{ji}_{\text{imp}} = T^{ji}_{\text{can}} - \partial_0 \gamma^{ij} + \partial_k \zeta^{kji}$	$\zeta^{kji} = -\zeta^{jki}$
Mass	$J^{0}_{\text{imp}} = J^{0}_{\text{can}} + \partial_i \eta^i$	η^i
	$J^{i}_{\text{imp}} = J^{i}_{\text{can}} - \partial_0 \eta^i + \partial_j \theta^{ji}$	$\theta^{ji} = -\theta^{ij}$
Rotation	$j^{0ij}_{\text{imp}} = x^i T^{0j}_{\text{can}} - x^j T^{0i}_{\text{can}} + s^{0ij} + \partial_k \kappa^{kij}$	κ^{kij}
	$j^{kij}_{\text{imp}} = x^i T^{kj}_{\text{can}} - x^j T^{ki}_{\text{can}} + s^{kij} - \partial_0 \kappa^{kij} + \partial_l \lambda^{lkij}$	$\lambda^{lkij} = -\lambda^{klij}$
Boost	$b^{0i}_{\text{imp}} = tT^{0i}_{\text{can}} - x^i J^{0}_{\text{can}} + w^{0i} + \partial_k \xi^{ki}$	ξ^{ki}
	$b^{ji}_{\text{imp}} = tT^{ji}_{\text{can}} - x^i J^{j}_{\text{can}} + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji}$	$\rho^{kji} = -\rho^{jki}$

Table 6: Improvements of currents and charges of a non-relativistic Bargmann theory.

Like for the Galilean currents there is some freedom left in the currents defined in (4.17) and (4.32). We have the following improvements that leaves the charges and transformation law unchanged

$$E^{\mu}_{\text{imp}} = E^{\mu}_{\text{can}} + \partial_\rho A^{\rho\mu} \tag{D.30a}$$
$$T^{\mu i}_{\text{imp}} = T^{i\mu}_{\text{can}} + \partial_\rho B^{\rho\mu i} \tag{D.30b}$$
$$J^{\mu}_{\text{imp}} = J^{\mu}_{\text{can}} + \partial_\rho C^{\rho\mu} \tag{D.30c}$$
$$j^{\mu ij}_{\text{imp}} = j^{\mu ij}_{\text{can}} + \partial_\rho D^{\rho\mu ij} \tag{D.30d}$$
$$b^{\mu i}_{\text{imp}} = b^{\mu i}_{\text{can}} + \partial_\rho E^{\rho\mu i} \tag{D.30e}$$

where all A, B, C, D, E are antisymmetric in the first two indices and we can separate the components of the improvements as in table 6 which is more convenient for the following.

Now that we have the mass current, we can use its improvements in the boost current also. Nothing changes with the rotation if we fix this first in the exactly as for the Galilean case - starting fixing the freedom in the boost current would not give a different conclusion.

For the zero component of the boost current we now have improvements of the kind

$$\begin{aligned}
b^{0i}_{\text{imp}} &= tT^{0i}_{\text{can}} - x^i J^0_{\text{can}} + w^{0i} + \partial_k \xi^{ki} \\
&= t\left(T^{0i}_{\text{imp}} - \partial_k \gamma^{ki}\right) - x^i \left(J^0_{\text{imp}} - \partial_k \eta^k\right) + w^{0i} + \partial_k \xi^{ki} \\
&= tT^{0i}_{\text{imp}} - x^i J^0_{\text{imp}} - \partial_k \left(t\gamma^{ki}\right) + x^i \partial_k \eta^k + w^{0i} + \partial_k \xi^{ki} \\
&= tT^{0i}_{\text{imp}} - x^i J^0_{\text{imp}} - \partial_k \left(t\gamma^{ki}\right) + \partial_k \left(x^i \eta^k\right) - \left(\partial_k x^i\right) \eta^k + w^{0i} + \partial_k \xi^{ki} \\
&= tT^{0i}_{\text{imp}} - x^i J^0_{\text{imp}} - \partial_k \left(t\gamma^{ki}\right) + \partial_k \left(x^i \eta^k\right) - \eta^i + w^{0i} + \partial_k \xi^{ki} \,.
\end{aligned} \tag{D.31}$$

Contrary to the Galilean case there is now a unique solution that removes w^{0i} completely. This is done by choosing the improvements as

$$\xi^{ki} = t\gamma^{ki} - x^i \eta^k \tag{D.32a}$$
$$\eta^i = w^{0i} \,, \tag{D.32b}$$

which gives what we have postulated in the main text:

$$b^{0i}_{\text{imp}} = tT^{0i}_{\text{imp}} - x^i J^0_{\text{imp}} \,. \tag{D.33}$$

For the spatial components with the above choice we instead find

$$
\begin{aligned}
b^{ji}_{\text{imp}} &= tT^{ji}_{\text{can}} - x^i J^j_{\text{can}} + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji} \\
&= t\left(T^{ji}_{\text{imp}} + \partial_0 \gamma^{ji} - \partial_l \zeta^{lji}\right) - x^i \left(J^j_{\text{imp}} + \partial_0 \eta^j - \partial_k \theta^{kj}\right) \\
&\quad + w^{ji} - \partial_0 \xi^{ji} + \partial_k \rho^{kji} \\
&= tT^{ji}_{\text{imp}} - x^i J^j_{\text{imp}} + \cancel{\partial_0 \left(t\gamma^{ji}\right)} - \cancel{\partial_0 \left(x^i \eta^j\right)} - \cancel{\partial_0 \xi^{ji}} - \gamma^{ji} \\
&\quad - \partial_k \left(t\zeta^{kji}\right) + \partial_k \left(x^i \theta^{kj}\right) - \theta^{ij} + w^{ji} + \partial_k \rho^{kji} \\
&= tT^{ji}_{\text{imp}} - x^i J^j_{\text{imp}} - \gamma^{ji} \\
&\quad - \partial_k \left(t\zeta^{kji}\right) + \partial_k \left(x^i \theta^{kj}\right) - \theta^{ij} + w^{ji} + \partial_k \rho^{kji}
\end{aligned}
$$

We have already fixed the antisymmetric part of γ^{ki}, but we see that we can now remove w^{ji} completely by choosing

$$
\begin{aligned}
\gamma^{(ji)} &= w^{(ji)} && \text{(D.34a)} \\
\theta^{ij} &= w^{[ji]} && \text{(D.34b)} \\
\rho^{kji} &= t\zeta^{kji} - x^i \theta^{kj} \,. && \text{(D.34c)}
\end{aligned}
$$

This is a unique solution that shows we can always choose improvements so that

$$
b^{ji}_{\text{imp}} = tT^{ji}_{\text{imp}} - x^i J^j_{\text{imp}} \,. \tag{D.35}
$$

The choice of improvements may then be converted to the expressions for B, C, D, E in the main text section 4.3.2.

D.5 CHAPTER 5

D.5.1 *Noether procedure for Galilean theories*

The starting point is the expression (5.1b) for $S^{(1)}$ that is universally given by the the Noether procedure described in section B.2. The improvements of the energy current can using the notation of section D.4.1 be written as

$$
\begin{aligned}
S^{(1)} &\ni \int_M \mathrm{d}^D x \, \overline{\tau}_\mu E^\mu_{\text{can}} \\
&= \int_M \mathrm{d}^D x \, \overline{\tau}_0 E^0_{\text{can}} + \overline{\tau}_i E^i_{\text{can}} \\
&= \int_M \mathrm{d}^D x \, \overline{\tau}_0 \left(E^0_{\text{imp}} - \partial_i \alpha^i\right) + \overline{\tau}_i \left(E^i_{\text{imp}} + \partial_0 \alpha^i - \partial_j \beta^{ji}\right) \\
&= \int_M \mathrm{d}^D x \, \overline{\tau}_\mu E^\mu_{\text{imp}} - \overline{\tau}_0 \partial_i \alpha^i + \overline{\tau}_i \left(\partial_0 \alpha^i - \partial_j \beta^{ji}\right) \,, && \text{(D.36)}
\end{aligned}
$$

which does not give anything useful for our present purposes as we discussed in section 4.2.3.3 in the main text.

More interesting is the coupling of the momentum current with the linearized spatial vielbeins. We now use the improvements of section D.4.1 that was found to maximally simplify the conserved currents and write the canonical currents in terms of the improved ones. We find

$$
\begin{aligned}
S^{(1)} \ni & \int_M \mathrm{d}^D x\, \bar{e}_{\mu a} T^{\mu a}_{\text{can}} \\
= & \int_M \mathrm{d}^D x\, \bar{e}_{0i} T^{0i}_{\text{can}} + \bar{e}_{ji} T^{ji}_{\text{can}} \\
& \int_M \mathrm{d}^D x\, \bar{e}_{0i} \left(T^{0i}_{\text{imp}} - \partial_k \left(w^{(ki)} + \frac{1}{2} s^{0ik} \right) \right) \\
& + \bar{e}_{ji} \left(T^{ji}_{\text{imp}} + \partial_0 \left(w^{(ji)} + \frac{1}{2} s^{0ij} \right) - \frac{1}{2} \partial_k \left(s^{kji} + s^{ijk} + s^{jik} \right) \right) \\
= & \int_M \mathrm{d}^D x\, - \bar{v}_i \left(T^{0i}_{\text{imp}} - \partial_k \left(w^{(ki)} + \frac{1}{2} s^{0ik} \right) \right) \\
& + \bar{e}_{ji} \left(T^{ji}_{\text{imp}} + \partial_0 \left(w^{(ji)} + \frac{1}{2} s^{0ij} \right) - \frac{1}{2} \partial_k \left(s^{kji} + s^{ijk} + s^{jik} \right) \right) \\
= & \int_M \mathrm{d}^D x\, - \bar{v}_i \left(T^{0i}_{\text{imp}} - \partial_k \left(w^{(ki)} + \frac{1}{2} s^{0ik} \right) \right) \\
& + \left(\bar{e}_{(ji)} T^{ji}_{\text{imp}} + \bar{e}_{(ji)} \partial_0 w^{(ji)} + \frac{1}{2} \bar{e}_{[ji]} \partial_0 s^{0ij} - \bar{e}_{(ji)} \partial_k s^{(ij)k} - \frac{1}{2} \bar{e}_{[ji]} \partial_k s^{kji} \right) \\
= & \int_M \mathrm{d}^D x\, - \bar{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} + \bar{v}_i \partial_k \left(w^{(ki)} + \frac{1}{2} s^{0ik} \right) \\
& + \frac{1}{2} s_{ij} \partial_0 w^{ij} + \frac{1}{2} \bar{e}_{[ji]} \partial_0 s^{0ij} - s_{ij} \partial_k s^{ijk} - \frac{1}{2} \bar{e}_{[ji]} \partial_k s^{kji} \,.
\end{aligned}
\tag{D.37}
$$

We here wrote $\bar{e}_{(ji)} = \frac{1}{2} s_{ji} = \frac{1}{2} s_{ij}$ and use that $\bar{e}_{0i} = -\bar{v}_i$. Let us now collect all terms in the action by doing integration by parts on terms with derivatives of the lift and spin current:

$$
\begin{aligned}
S^{(1)} = & \int_M \mathrm{d}^D x \Big[\bar{\tau}_\mu E^\mu_{\text{can}} - \bar{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} + \bar{v}_i \partial_k \left(w^{(ik)} + \frac{1}{2} s^{0ik} \right) \\
& + \frac{1}{2} s_{ij} \left(\partial_0 w^{ij} - \partial_k s^{ijk} \right) + \frac{1}{2} \bar{e}_{[ji]} \left(\partial_0 s^{0ij} - \partial_k s^{kji} \right) + \frac{1}{2} \overline{\omega}_{\rho ab} s^{\rho ab} - \overline{\Omega}_{\rho a} w^{\rho a} \Big] \\
= & \int_M \mathrm{d}^D x \Big[\bar{\tau}_\mu E^\mu_{\text{can}} - \bar{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} + \frac{1}{2} s_{ij} \partial_0 w^{ij} + \bar{v}_{(i} \partial_{k)} w^{ik} \\
& - \frac{1}{2} s_{ij} \partial_k s^{ijk} + \frac{1}{2} \bar{v}_i \partial_k s^{0ik} + \frac{1}{2} \bar{e}_{[ji]} \partial_0 s^{0ij} - \frac{1}{2} \bar{e}_{[ji]} \partial_k s^{kji} + \frac{1}{2} \overline{\omega}_{\rho ab} s^{\rho ab} - \overline{\Omega}_{\rho a} w^{\rho a} \Big] \\
\doteq & \int_M \mathrm{d}^D x \Big[\bar{\tau}_\mu E^\mu_{\text{can}} - \bar{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} - \frac{1}{2} \left(\partial_0 s_{ij} + 2 \partial_{(j} \bar{v}_{i)} \right) w^{ij} \\
& - \frac{1}{2} \partial_{[i} s_{k]j} s^{jik} + \frac{1}{2} \partial_{[i} \bar{v}_{k]} s^{0ik} + \frac{1}{2} \left(\partial_0 \bar{e}_{[ij]} s^{0ij} + \partial_k \bar{e}_{[ij]} s^{kij} \right) + \frac{1}{2} \overline{\omega}_{\rho ab} s^{\rho ab} - \overline{\Omega}_{\rho a} w^{\rho a} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_M \mathrm{d}^D x \Big[\overline{\tau}_\mu E^\mu_{\text{can}} - \overline{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} \\
&\quad + \Big(0\Big) \times w^{0i} + \left(-\frac{1}{2}\partial_0 s_{ij} - \partial_{(i}\overline{v}_{j)}\right) w^{ij} \\
&\quad - \frac{1}{2}\left(-\partial_{[i}\overline{v}_{j]} - \partial_0 \overline{e}_{[ij]}\right) s^{0ij} - \frac{1}{2}\left(-\partial_k \overline{e}_{[ij]} + \partial_{[i} s_{j]k}\right) s^{kij} \\
&\quad + \frac{1}{2}\overline{\omega}_{\rho ab} s^{\rho ab} - \overline{\Omega}_{\rho a} w^{\rho a}\Big] \\
&= \int_M \mathrm{d}^D x \Big[\overline{\tau}_\mu E^\mu_{\text{can}} - \overline{v}_i T^{0i}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} \\
&\quad + \hat{\overline{\Omega}}_{0i} w^{0i} + \hat{\overline{\Omega}}_{ji} w^{ji} - \frac{1}{2}\hat{\overline{\omega}}_{0ij} s^{0ij} - \frac{1}{2}\hat{\overline{\omega}}_{kij} s^{kij} \\
&\quad + \frac{1}{2}\overline{\omega}_{\rho ab} s^{\rho ab} - \overline{\Omega}_{\rho a} w^{\rho a}\Big] \\
&= \int_M \mathrm{d}^D x \left[\overline{\tau}_\mu E^\mu_{\text{can}} - \overline{v}_i T^{i0}_{\text{imp}} + \frac{1}{2} s_{ij} T^{ij}_{\text{imp}} + \frac{1}{2}\overline{C}_{\rho ab} s^{ab\rho} - \overline{C}_{\rho a} w^{a\rho}\right].
\end{aligned}
\tag{D.38}
$$

In the second last equation we see that we can identify the linearized pseudo-gauge fields that we originally calculated in (3.96) and define pseudo-contortions as in (5.8).

D.5.2 *Noether procedure for Bargmann theories*

The first order action of the Noether procedure should according to the general theory of section B.2 be taken as

$$
S^{(1)} = \int_M \mathrm{d}^D x \left[\overline{\tau}_\mu E^\mu_{\text{can}} + \overline{e}_{\mu a} T^{\mu a}_{\text{can}} - \overline{m}_\mu J^\mu_{\text{can}} + \frac{1}{2}\overline{\omega}_{\mu ij} s^{\mu ij} - \overline{\Omega}_{\mu i} w^{\mu i}\right]. \tag{D.39}
$$

The linearized mass gauge field $\overline{m}_\mu$ is taken to transform at linear order as (5.11). For Bargmann we now have the improvements of the mass current of our disposal. In section D.4.2 we saw how to choose them to make the currents maximally simple. Using these improvements we find that its contribution to $S^{(1)}$ that are extra compared to the Galilean case in section D.5.1 can be rewritten as

$$
\begin{aligned}
S^{(1)} &\ni \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{can}} \\
&= \int_M \mathrm{d}^D x \overline{m}_0 \left(J^0_{\text{imp}} - \partial_i \eta^i\right) + \overline{m}_i \left(J^i_{\text{imp}} + \partial_0 \eta^i - \partial_j \theta^{ji}\right) \\
&= \int_M \mathrm{d}^D x \overline{m}_0 \left(J^0_{\text{imp}} - \partial_i w^{0i}\right) + \overline{m}_i \left(J^i_{\text{imp}} + \partial_0 w^{0i} - \frac{1}{2}\partial_j s^{0ij} + \partial_j w^{[ji]}\right) \\
&= \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{imp}} - \overline{m}_0 \partial_i w^{0i} + \overline{m}_i \partial_0 w^{0i} - \overline{m}_i \partial_j \left(\frac{1}{2} s^{0ij} - w^{[ji]}\right) \\
&\doteq \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{imp}} + \partial_i \overline{m}_0 w^{0i} - \partial_0 m_i w^{0i} + \partial_j \overline{m}_i \left(\frac{1}{2} s^{0ij} - w^{[ji]}\right) \\
&= \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{imp}} + 2\partial_{[i} \overline{m}_{0]} w^{0i} - \partial_{[j} \overline{m}_{i]} w^{ji} + \frac{1}{2}\partial_{[j} \overline{m}_{i]} s^{0ij} \\
&= \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{imp}} - 2\partial_{[0} \overline{m}_{i]} w^{0i} - \partial_{[j} \overline{m}_{i]} w^{ji} - \frac{1}{2}\partial_{[i} \overline{m}_{j]} s^{0ij} \\
&= \int_M \mathrm{d}^D x \overline{m}_\mu J^\mu_{\text{imp}} + \overline{C}_{\mu i} w^{\mu i} - \frac{1}{2}\overline{C}_{\mu i j} s^{\mu i j} \,.
\end{aligned}
\tag{D.40}
$$

where we in the last line identified the pseudo-contortions of the graviphotonic connection (3.8.3). This proves the claim of the main text.

D.7 CHAPTER 7

D.7.1 *Null reduction of Klein-Gordon field*

We simply use the methods of chapter 6 in the main text, in particular the expression for the inverse metric (6.2b). The calculation is then straight-forward:

$$
\begin{aligned}
\hat{S}_{\text{KG}} &= \int \mathrm{d}^{D+1} x \sqrt{-g} \Big[-g^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \Psi^* \partial_{\hat{\nu}} \Psi - V(|\Psi|) \Big] \\
&= \int \mathrm{d}^{D+1} x \sqrt{-g} \Big[-g^{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi - g^{u\nu} \partial_u \Psi^* \partial_\nu \Psi \\
&\quad - g^{\mu u} \partial_\mu \Psi^* \partial_u \Psi - g^{uu} \partial_u \Psi^* \partial_u \Psi - V(|\Psi|) \Big] \\
&= \int \mathrm{d}^{D+1} x \sqrt{-g} \Big[-h^{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi + \hat{v}^\nu \partial_u \Psi^* \partial_\nu \Psi \\
&\quad + \hat{v}^\mu \partial_\mu \Psi^* \partial_u \Psi - 2\tilde{\Phi} \partial_u \Psi^* \partial_u \Psi - V(|\Psi|) \Big] \\
&= \int \mathrm{d}^{D+1} x \sqrt{-g} \Big[-h^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + \hat{v}^\nu (-im) \phi^* \partial_\nu \phi \\
&\quad + \hat{v}^\mu \partial_\mu \phi^* (+im) \phi - 2\tilde{\Phi} (-im) \phi^* (+im) \phi - V(|\Psi|) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[-h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - im\hat{v}^\nu\phi^*\partial_\nu\phi \\
&\quad +im\hat{v}^\mu\phi\partial_\mu\phi^* - 2m^2\tilde{\Phi}\phi^*\phi - V\left(|\Psi|\right)\Big] \\
&= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[-h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - im\left(v^\nu - h^{\mu\nu}M_\mu\right)\phi^*\partial_\nu\phi \\
&\quad +im\left(v^\mu - h^{\mu\nu}M_\nu\right)\phi\partial_\mu\phi^* \\
&\quad -2m^2\left(-v^\mu M_\mu + \frac{1}{2}h^{\mu\nu}M_\mu M_\nu\right)\phi^*\phi - V\left(|\Psi|\right)\Big] \\
&= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[-h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi \\
&\quad -h^{\mu\nu}\left(imM_\nu\phi\partial_\mu\phi^* - imM_\mu\phi^*\partial_\nu\phi\right) \\
&\quad -imv^\nu\phi^*\partial_\nu\phi + imv^\mu\phi\partial_\mu\phi^* \\
&\quad -2m^2\left(-v^\mu M_\mu + \frac{1}{2}h^{\mu\nu}M_\mu M_\nu\right)\phi^*\phi - V\left(|\Psi|\right)\Big] \\
&= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[-h^{\mu\nu}\left(\partial_\mu\phi^* + imM_\mu\phi^*\right)\left(\partial_\nu\phi - imM_\nu\phi\right) \\
&\quad +h^{\mu\nu}\left(imM_\mu\right)\left(\cancel{\partial_\nu\phi} - \cancel{imM_\nu\phi}\right) - \cancel{h^{\mu\nu}\left(imM_\mu\phi^*\partial_\nu\phi\right)} \\
&\quad +imv^\nu\phi^*\left(\partial_\nu - imM_\nu\right)\phi - imv^\mu\phi\left(\partial_\mu + imM_\mu\right)\phi^* \\
&\quad -\cancel{m^2\phi^*\phi h^{\mu\nu}M_\mu M_\nu} - V\left(|\Psi|\right)\Big] \\
&= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[imv^\nu\phi^*\left(\partial_\nu - imM_\nu\right)\phi - imv^\mu\phi\left(\partial_\mu + imM_\mu\right)\phi^* \\
&\quad -h^{\mu\nu}\left(\partial_\mu + imM_\mu\right)\phi^*\left(\partial_\nu - imM_\nu\right)\phi - V\left(|\Psi|\right)\Big]\,.
\end{aligned} \tag{D.41}
$$

We can now define the covariant derivative

$$D_\mu\phi \equiv \partial_\mu\phi - imM_\mu\phi\,, \tag{D.42}$$

which allows us to write the action as

$$
\begin{aligned}
\hat{S}_{\mathrm{KG}} &= \int \mathrm{d}^{D+1}x\sqrt{-g}\Big[imv^\nu\phi^*D_\nu\phi - imv^\mu\phi D_\mu\phi^* \\
&\quad -h^{\mu\nu}D_\mu\phi^*D_\nu\phi - V\left(|\Psi|\right)\Big]\,,
\end{aligned} \tag{D.43}
$$

which then gives the Schrödinger action used in the main text.

D.7.2 *Variation of action wrt. background*

The response of the variation of the action $S\left[\varphi, \hat{v}^\mu, h^{\mu\nu}, \tilde{\Phi}\right]$ wrt. the background fields v^μ, $h^{\mu\nu}$ and M_μ is the currents $\mathcal{S}_\mu$, $\mathcal{T}_{\mu\nu}$, $\mathcal{J}^\mu$. We find directly, using (E.1) that

$$
\begin{aligned}
\delta S &= \int \mathrm{d}^{d+1}x\, \mathcal{L}\delta e + e\delta\mathcal{L} \\
&= \int \mathrm{d}^{d+1}xe\,\mathcal{L}\left(\tau_\mu \delta v^\mu - \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu}\right) \\
&\quad +\delta\left(-im\phi v^\mu D_\mu\phi^* + im\phi^* v^\mu D_\mu\phi - h^{\mu\nu}D_\mu\phi^* D_\nu\phi - \cancel{V(|\phi|)}\right) \\
&= \int \mathrm{d}^{d+1}xe\,\mathcal{L}\left(\tau_\mu \delta v^\mu - \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu}\right) \\
&\quad -im\phi\delta v^\mu D_\mu\phi^* + im\phi^*\delta v^\mu D_\mu\phi - \delta h^{\mu\nu}D_\mu\phi^* D_\nu\phi \\
&\quad -im\phi v^\mu \delta D_\mu\phi^* + im\phi^* v^\mu \delta D_\mu\phi - h^{\mu\nu}\delta D_\mu\phi^* D_\nu\varphi - h^{\mu\nu}D_\mu\phi^*\delta D_\nu\phi \\
&= \int \mathrm{d}^{d+1}xe\,\mathcal{L}\left(\tau_\mu \delta v^\mu - \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu}\right) \\
&\quad -im\phi D_\mu\phi^*\delta v^\mu + im\phi^* D_\mu\phi\delta v^\mu - D_\mu\phi^* D_\nu\phi\delta h^{\mu\nu} \\
&\quad -im\phi v^\mu\left(-im\phi^*\delta M_\mu\right) + im\phi^* v^\mu\left(im\phi\delta M_\mu\right) \\
&\quad -h^{\mu\nu}D_\nu\phi\left(-im\phi^*\delta M_\mu\right) - h^{\mu\nu}D_\nu\phi^*\left(im\phi\delta M_\mu\right) \\
&= \int \mathrm{d}^{d+1}xe\,\left(\mathcal{L}\tau_\mu + im\phi^* D_\mu\phi - im\phi D_\mu\phi^*\right)\delta v^\mu \\
&\quad +\left(-\frac{1}{2}\mathcal{L}h_{\mu\nu} - D_\mu\phi^* D_\nu\phi\right)\delta h^{\mu\nu} \\
&\quad +\left(-2m^2\phi^*\phi v^\mu + im\phi^* h^{\mu\nu}D_\nu\phi - im\phi h^{\mu\nu}D_\nu\phi^*\right)\delta M_\mu\,.
\end{aligned}
\tag{D.44}
$$

We use here that the variation of the covariant derivative is

$$
\delta D_\mu\phi = \delta\left(\partial_\mu\phi - imM_\mu\phi\right) = -im\phi\delta M_\mu \tag{D.45}
$$

$$
\delta D_\mu\phi^* = \delta\left(\partial_\mu\phi^* + imM_\mu\phi\right) = +im\phi^*\delta M_\mu \tag{D.46}
$$

Comparing to (4.55) we find the result given in the main text.

D.8 CHAPTER 8

D.8.1 *Null reduction of MED on a Lorentzian manifold*

Written in terms of the reduced gauge field $A_{\hat{\mu}}$ (8.24) and the field strength $F_{\hat{\mu}\hat{\nu}}$ we obtain directly using the methods of chapter 6 and in particular (6.2) that the reduction of the action can be performed straight-forwardly:

$$
\begin{aligned}
\hat{S}_{\text{MED}} &= -\frac{1}{4}\int \mathrm{d}^{D+1}x\sqrt{|g|}\, g^{\hat{\mu}\hat{\rho}}g^{\hat{\nu}\hat{\sigma}}F_{\hat{\mu}\hat{\nu}}F_{\hat{\rho}\hat{\sigma}} \\
&= -\frac{1}{4}\int \mathrm{d}^{D+1}x\sqrt{|g|}\, \Big[g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \\
&\quad +4g^{u\rho}g^{\nu\sigma}F_{u\nu}F_{\rho\sigma} + 2g^{uu}g^{\nu\sigma}F_{u\nu}F_{u\sigma}\Big] \\
&= -\frac{1}{2}\int \mathrm{d}^{D+1}\sqrt{|g|}\, \Big[\frac{1}{2}h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \\
&\quad -2h^{\mu\rho}\hat{v}^{\sigma}\partial_{\mu}\varphi F_{\rho\sigma} + \left(2h^{\mu\rho}\tilde{\Phi} - \hat{v}^{\mu}\hat{v}^{\rho}\right)\partial_{\mu}\varphi\partial_{\rho}\varphi\Big]
\end{aligned}
\tag{D.47}
$$

Now we use the reduction of the field strength tensor (8.28) to put it in a different form:

$$
\begin{aligned}
\hat{S}_{\text{MED}} &= -\frac{1}{2}\int \mathrm{d}^{D}xe\, \Big[\frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\left(\tilde{f}_{\mu\nu} + \cancel{2\tau_{[\mu}\partial_{\nu]}\tilde{\varphi}} + 2M_{[\mu}\partial_{\nu]}\varphi\right) \\
&\quad \times\left(\tilde{f}_{\rho\sigma} + \cancel{2\tau_{[\rho}\partial_{\sigma]}\tilde{\varphi}} + 2M_{[\rho}\partial_{\sigma]}\varphi\right) + \left(2h^{\mu\rho}\tilde{\Phi} - \hat{v}^{\mu}\hat{v}^{\rho}\right)\partial_{\mu}\varphi\partial_{\rho}\varphi \\
&\quad -2h^{\mu\rho}\hat{v}^{\sigma}\partial_{\mu}\varphi\left(\tilde{f}_{\rho\sigma} + 2\tau_{[\rho}\partial_{\sigma]}\tilde{\varphi} + 2M_{[\rho}\partial_{\sigma]}\varphi\right)\Big] \\
&= -\frac{1}{2}\int \mathrm{d}^{D}xe\, \Big[\frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\left(\tilde{f}_{\mu\nu} + 2M_{[\mu}\partial_{\nu]}\varphi\right)\left(\tilde{f}_{\rho\sigma} + 2M_{[\rho}\partial_{\sigma]}\varphi\right) \\
&\quad -2h^{\mu\rho}\hat{v}^{\sigma}\partial_{\mu}\varphi\left(\tilde{f}_{\rho\sigma} + \cancel{\tau_{\rho}\partial_{\sigma}\tilde{\varphi}} - \tau_{\sigma}\partial_{\rho}\tilde{\varphi} + 2M_{[\rho}\partial_{\sigma]}\varphi\right) \\
&\quad + \left(2h^{\mu\rho}\tilde{\Phi} - \hat{v}^{\mu}\hat{v}^{\rho}\right)\partial_{\mu}\varphi\partial_{\rho}\varphi\Big] \\
&= -\frac{1}{2}\int \mathrm{d}^{D}xe\, \Big[\frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\tilde{f}_{\rho\sigma} - \cancel{2h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}M_{\sigma}\partial_{\rho}\varphi} \\
&\quad +2h^{\mu\rho}h^{\sigma\nu}M_{[\mu}\partial_{\nu]}\varphi M_{[\rho}\partial_{\sigma]}\varphi \\
&\quad -2h^{\mu\rho}\left(v^{\nu} - \cancel{h^{\nu\sigma}M_{\sigma}}\right)\partial_{\rho}\varphi\tilde{f}_{\mu\nu} - 2h^{\mu\rho}\partial_{\mu}\varphi\partial_{\rho}\tilde{\varphi} \\
&\quad -4h^{\mu\rho}\left(v^{\sigma} - h^{\sigma\lambda}M_{\lambda}\right)\partial_{\mu}\varphi M_{[\rho}\partial_{\sigma]}\varphi \\
&\quad + \left(-2h^{\mu\rho}v^{\lambda}M_{\lambda} + h^{\mu\rho}h^{\lambda\nu}M_{\lambda}M_{\nu}\right)\partial_{\mu}\varphi\partial_{\rho}\varphi\Big]
\end{aligned}
$$

$$
\begin{aligned}
& - \left(v^\mu - h^{\mu\lambda} M_\lambda\right) \left(v^\rho - h^{\rho\lambda} M_\lambda\right) \partial_\mu \varphi \partial_\rho \varphi \Big] \\
= \quad & -\frac{1}{2} \int \mathrm{d}^D x e \left[\frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \tilde{f}_{\mu\nu} \tilde{f}_{\rho\sigma} - 2 h^{\mu\rho} v^\nu \partial_\rho \varphi \tilde{f}_{\mu\nu} \right. \\
& -2 h^{\mu\rho} \partial_\mu \varphi \partial_\rho \tilde{\varphi} - 2 h^{\mu\rho} v^\sigma \partial_\mu \varphi \left(\cancel{M_\sigma \partial_\rho \varphi} + \cancel{M_\rho \partial_\sigma \varphi} - \cancel{M_\sigma \partial_\rho \varphi} \right) \\
& \xcancel{+2 h^{\mu\rho} h^{\sigma\nu} \left(M_\mu \partial_\nu \varphi - M_\nu \partial_\mu \varphi + 2 M_\nu \partial_\mu \varphi_e \right) M_{[\rho} \partial_{\sigma]} \varphi} \\
& \left. + \left(\underline{h^{\mu\rho} h^{\lambda\nu} M_\lambda M_\nu} - v^\mu v^\rho + \cancel{2 v^\mu h^{\rho\lambda} M_\lambda} - \underline{h^{\mu\nu} h^{\rho\sigma} M_\nu M_\sigma} \right) \partial_\mu \varphi \partial_\rho \varphi \right] \\
= \quad & -\frac{1}{2} \int \mathrm{d}^D x e \left[\frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \tilde{f}_{\mu\nu} \tilde{f}_{\rho\sigma} - 2 h^{\mu\rho} v^\nu \partial_\rho \varphi \tilde{f}_{\mu\nu} \right. \\
& \left. -2 h^{\mu\rho} \partial_\mu \varphi \partial_\rho \tilde{\varphi} - v^\mu v^\rho \partial_\mu \varphi \partial_\rho \varphi \right] .
\end{aligned} \tag{D.48}
$$

Here we used that $M_\mu \partial_\nu \varphi - M_\nu \partial_\mu \varphi + 2 M_\nu \partial_\mu \varphi = M_\mu \partial_\nu \varphi + M_\nu \partial_\mu \varphi$, which contracted with $M_{[\rho} \partial_{\sigma]} \varphi_e$ is zero. This action is not obviously gauge invariant.

Alternatively In terms of the gauge invariant field strengths (8.32), we have instead

$$
\begin{aligned}
\hat{S}_{\mathrm{MED}} \quad = \quad & -\frac{1}{2} \int \mathrm{d}^D e \left[\frac{1}{2} h^{\mu\rho} h^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right. \\
& \left. -2 h^{\mu\rho} \hat{v}^\sigma \partial_\mu \varphi F_{\rho\sigma} + \left(2 h^{\mu\rho} \tilde{\Phi} - \hat{v}^\mu \hat{v}^\rho \right) \partial_\mu \varphi \partial_\rho \varphi \right] \\
= \quad & \int \mathrm{d}^D x e \left[-\frac{1}{4} h^{\mu\rho} h^{\nu\sigma} \left(B_{\mu\nu} + \cancel{2 \tilde{E}_{[\mu} \tau_{\nu]}} + 2 E_{[\mu} M_{\nu]} + \cancel{2 a \tau_{[\mu} M_{\nu]}} \right) \right. \\
& \times \left(B_{\rho\sigma} + \cancel{2 \tilde{E}_{[\rho} \tau_{\sigma]}} + 2 E_{[\rho} M_{\sigma]} + \cancel{2 a \tau_{[\rho} M_{\sigma]}} \right) \\
& + \hat{v}^\rho h^{\nu\sigma} \left(E_\nu + \cancel{a \tau_\nu} \right) \left(B_{\rho\sigma} + 2 \tilde{E}_{[\rho} \tau_{\sigma]} + 2 E_{[\rho} M_{\sigma]} + 2 a \tau_{[\rho} M_{\sigma]} \right) \\
& \left. - \left(\tilde{\Phi} h^{\nu\sigma} + \frac{1}{2} \hat{v}^\nu \hat{v}^\sigma \right) \left(E_\nu + a \tau_\nu \right) \left(E_\sigma + a \tau_\sigma \right) \right] \\
= \quad & \int \mathrm{d}^D x e \left[-\frac{1}{4} h^{\mu\rho} h^{\nu\sigma} \left(B_{\mu\nu} + 2 E_{[\mu} M_{\nu]} \right) \left(B_{\rho\sigma} + 2 E_{[\rho} M_{\sigma]} \right) \right. \\
& - \hat{v}^\rho h^{\nu\sigma} E_\nu \left(B_{\rho\sigma} + 2 E_{[\rho} M_{\sigma]} \right) - h^{\nu\sigma} E_\nu \left(\tilde{E}_\sigma - a M_\sigma \right) \\
& \left. - \tilde{\Phi} h^{\nu\sigma} E_\nu E_\sigma - \frac{1}{2} \left(\hat{v}^\nu \hat{v}^\sigma E_\nu E_\sigma - 2 a \hat{v}^\nu E_\nu + a^2 \right) \right] .
\end{aligned} \tag{D.49}
$$

This is as far as we can go without using some properties of the field strengths. This action is explicitly gauge invariant.

D.8.2 *Determining the EOMs in terms of covariant derivatives*

A special property of the graviphotonic connection (3.51) is that it is related to the derivatives of the measure in a simple way satisfying [26]

$$\mathring{\Gamma}^{\lambda}_{\rho\lambda} = \frac{1}{e}\partial_\rho e = \mathring{\Gamma}^{\lambda}_{\lambda\rho} - 2\hat{v}^\lambda \partial_{[\lambda}\tau_{\rho]} \,. \tag{D.50}$$

It is then possible to rewrite the EOMs in terms of the graviphotonic covariant derivative as we in can identify some of the combinations of the field strengths (8.38) that we know are spacetime tensors in (8.45). We notice first from (8.45b) that with E^ρ being a tensor, the derivative may be written as

$$\frac{1}{e}\partial_\rho \left(eE^\rho\right) = \mathring{\nabla}_\rho E^\rho + 2\hat{v}^\lambda \partial_{[\lambda}\tau_{\rho]} E^\rho \,. \tag{D.51}$$

The EOM (8.45b) may then be written as

$$\mathring{\nabla}_\rho E^\rho = -\partial_{[\mu}\tau_{\nu]}\left(B^{\mu\nu} + 2E^\mu h^{\nu\sigma} M_\sigma\right) , \tag{D.52}$$

which we see is written entirely in terms of the tensorial objects of (8.38).

As $B^{\lambda\mu} + 2E^{[\lambda}v^{\mu]}$ is a tensor, we can immediately write the EOM (8.45c) as a covariant derivative, where we find

$$\mathring{\nabla}_\rho \left(B^{\rho\mu} + 2E^{[\rho}v^{\mu]}\right) = \partial_{[\rho}\tau_{\lambda]}\left(2\hat{v}^\rho \left(B^{\lambda\mu} + 2E^{[\lambda}v^{\mu]}\right) + \hat{v}^\mu \left(B^{\lambda\rho} + 2E^{[\lambda}v^{\rho]}\right)\right) . \tag{D.53}$$

The last EOM doesn't look tensorial because $\partial_{[\mu}M_{\nu]}$ transforms under a boost, while we know that $B^{\rho\mu} + 2E^{[\rho}v^{\mu]}$ is a tensor. However, it is the EOM that is the result of variation wrt. the scalar φ, so it must be tensorial, which is useful to know in the following. We see by look at the last object of our list of tensorial objects (8.38f) that if we add and subtract $\frac{1}{e}\partial_\rho\left(eM_\sigma\left(B^{\sigma\nu} + 2E^{[\sigma}v^{\nu]}\right)\right)$, which is clearly non-tensorial, then the EOM (8.45a) can be written as

$$\begin{aligned} 0 = \partial_{[\mu}M_{\nu]}\left(B^{\mu\nu} + 2E^\mu v^\nu\right) + \frac{1}{e}\partial_\rho\left(eM_\sigma\left(B^{\sigma\rho} + 2E^{[\sigma}v^{\rho]}\right)\right) \\ + \frac{1}{e}\partial_\rho\left(e\tilde{E}^\rho - ev^\rho a - eM_\sigma\left(B^{\sigma\rho} + 2E^{[\sigma}v^{\rho]}\right)\right) \end{aligned} \tag{D.54}$$

The latter combination is then a tensor and we can write its covariant derivative as

$$\begin{aligned} &\frac{1}{e}\partial_\rho\left(e\tilde{E}^\rho - ev^\rho a - M_\sigma\left(B^{\sigma\rho} + 2E^{[\sigma}v^{\rho]}\right)\right) \\ &\qquad = \mathring{\nabla}_\rho\left(\tilde{E}^\rho - v^\rho a - M_\sigma\left(B^{\sigma\rho} + 2E^{[\sigma}v^{\rho]}\right)\right) \\ &\qquad\qquad + 2\hat{v}^\lambda\partial_{[\lambda}\tau_{\rho]}\left(\tilde{E}^\rho - v^\rho a - M_\sigma\left(B^{\sigma\rho} + 2E^{[\sigma}v^{\rho]}\right)\right) . \end{aligned} \tag{D.55}$$

The first line of (D.54) can be rewritten using

$$
\begin{aligned}
&\frac{1}{e}\partial_\rho\left(eM_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)\\
&\quad=\mathring{\Gamma}^{\lambda}_{\rho\lambda}\left(M_\mu\left(B^{\mu\rho}+2E^{[\sigma}v^{\rho]}\right)\right)-\partial_{[\mu}M_{\rho]}\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)+M_\mu\partial_\rho\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\\
&\qquad=\left(-\hat{v}^\lambda\partial_\rho\tau_\lambda+\frac{1}{2}h^{\lambda\sigma}\partial_\rho\bar{h}_{\lambda\sigma}\right)\left(M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)\\
&\qquad\quad-\partial_{[\mu}M_{\rho]}\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)+M_\mu\partial_\rho\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\,. \qquad \text{(D.56)}
\end{aligned}
$$

We see that inserting this into (D.54) it cancels the first term and we obtain that the EOM becomes

$$
\begin{aligned}
0=&\left(-\hat{v}^\lambda\partial_\rho\tau_\lambda+\frac{1}{2}h^{\lambda\sigma}\partial_\rho\bar{h}_{\lambda\sigma}\right)\left(M_\mu\left(B^{\mu\rho}+2E^{[\sigma}v^{\rho]}\right)\right)+M_\mu\partial_\rho\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\\
&+\mathring{\nabla}_\rho\left(\tilde{E}^\rho-v^\rho a-M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)+2\hat{v}^\lambda\partial_{[\lambda}\tau_{\rho]}\left(\tilde{E}^\rho-v^\rho a-M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)\\
&=M_\mu\left[\partial_\rho-\hat{v}^\lambda\partial_\rho\tau_\lambda+\frac{1}{2}h^{\lambda\sigma}\partial_\rho\bar{h}_{\lambda\sigma}\right]\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\\
&+\mathring{\nabla}_\rho\left(\tilde{E}^\rho-v^\rho a-M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)+2\hat{v}^\lambda\partial_{[\lambda}\tau_{\rho]}\left(\tilde{E}^\rho-v^\rho a-M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)\\
&=M_\mu\left[\partial_\rho-\hat{v}^\lambda\partial_{(\rho}\tau_{\lambda)}+\frac{1}{2}h^{\lambda\sigma}\partial_\rho\bar{h}_{\lambda\sigma}\right]\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\\
&+\mathring{\nabla}_\rho\left(\tilde{E}^\rho-v^\rho a-M_\mu\left(B^{\mu\rho}+2E^{[\mu}v^{\rho]}\right)\right)+2\hat{v}^\lambda\partial_{[\lambda}\tau_{\rho]}\left(\tilde{E}^\rho-v^\rho a\right)\,. \qquad \text{(D.57)}
\end{aligned}
$$

If we define spactime tensors as

$$
\begin{aligned}
E^\mu &\equiv h^{\mu\nu}E_\nu &\text{(D.58a)}\\
W^{\lambda\mu} &\equiv B^{\lambda\mu}+2E^{[\lambda}v^{\mu]} &\text{(D.58b)}\\
Z^\rho &\equiv \tilde{E}^\rho-v^\rho a-M_\sigma W^{\sigma\rho} &\text{(D.58c)}
\end{aligned}
$$

We see that the EOMs can be written more conveniently as

$$
\begin{aligned}
\mathring{\nabla}_\rho Z^\rho &= M_\mu\left(\partial_\rho-\hat{v}^\lambda\partial_\rho\tau_\lambda+\frac{1}{2}h^{\lambda\sigma}\partial_\rho\bar{h}_{\lambda\sigma}\right)W^{\rho\mu}\\
&\quad-2\hat{v}^\lambda\partial_{[\lambda}\tau_{\rho]}Z^\rho &\text{(D.59a)}\\
\mathring{\nabla}_\rho E^\rho &= -\partial_{[\mu}\tau_{\nu]}W^{\mu\nu} &\text{(D.59b)}\\
\mathring{\nabla}_\lambda W^{\lambda\mu} &= \partial_{[\lambda}\tau_{\rho]}\left(2\hat{v}^\lambda W^{\rho\mu}+\hat{v}^\mu W^{\rho\lambda}\right)\,. &\text{(D.59c)}
\end{aligned}
$$

(D.59a) can be rewritten in terms of the covariant derivative of $W^{\rho\mu}$.

$$
\begin{aligned}
\mathring{\nabla}_\rho Z^\rho &= -M_\mu\mathring{\nabla}_\rho W^{\mu\rho}+\partial_{[\mu}\tau_{\nu]}\left(2M_\lambda\hat{v}^\lambda W^{\mu\nu}-M_\lambda\hat{v}^\mu W^{\nu\lambda}-2\hat{v}^\mu Z^\nu\right) &\text{(D.60a)}\\
\mathring{\nabla}_\rho E^\rho &= -\partial_{[\mu}\tau_{\nu]}W^{\mu\nu} &\text{(D.60b)}\\
\mathring{\nabla}_\lambda W^{\lambda\mu} &= \partial_{[\lambda}\tau_{\rho]}\left(2\hat{v}^\lambda W^{\rho\mu}+\hat{v}^\mu W^{\rho\lambda}\right)\,. &\text{(D.60c)}
\end{aligned}
$$

D.8.3 *Vary action to find current* $\mathcal{T}^{\mu}{}_{\hat{A}}$

We first find the variation wrt. the background of the fields

$$\begin{aligned} \delta \tilde{f}_{\mu\nu} &= \delta \left(2\partial_{[\mu}\left(A_a e_{\nu]}{}^a\right) - 2\tilde{\varphi}\partial_{[\mu}\tau_{\nu]} - 2\varphi\partial_{[\mu}M_{\nu]}\right) \\ &= 2\partial_{[\mu}A_a\delta e_{\nu]}{}^a + 2A_a\partial_{[\mu}\delta e_{\nu]}{}^a - 2\tilde{\varphi}\partial_{[\mu}\delta\tau_{\nu]} - 2\varphi\partial_{[\mu}\delta M_{\nu]} \\ &= 2\partial_{[\mu}\left(a_{|\lambda|}e^{\lambda}_a\right)\delta e_{\nu]}{}^a + 2a_\lambda e^\lambda_a\partial_{[\mu}\delta e_{\nu]}{}^a - 2\tilde{\varphi}\partial_{[\mu}\delta\tau_{\nu]} - 2\varphi\partial_{[\mu}\delta M_{\nu]} \,. \end{aligned} \tag{D.61}$$

$$\begin{aligned} \delta h^{\mu\nu} &= \delta\left(e^\mu_a e^{\nu a}\right) \\ &= e^{\nu a}\left(-e^\lambda_a e^\mu_c \delta e^c_\lambda + e^\lambda_a v^\mu \delta\tau_\lambda\right) + e^{\mu a}\left(-e^\lambda_a e^\nu_c \delta e^c_\lambda + e^\lambda_a v^\nu \delta\tau_\lambda\right) \\ &= \left(-h^{\nu\lambda}e^\mu_c\delta e^c_\lambda + h^{\nu\lambda}v^\mu\delta\tau_\lambda\right) + \left(-h^{\mu\lambda}e^\nu_c\delta e^c_\lambda + h^{\mu\lambda}v^\nu\delta\tau_\lambda\right) \\ &= -2h^{\lambda(\mu}e^{\nu)}_c\delta e^c_\lambda + 2h^{\lambda(\mu}v^{\nu)}\delta\tau_\lambda \,. \end{aligned} \tag{D.62}$$

$$\delta v^\mu = -v^\lambda e^\mu_c \delta e^c_\lambda + v^\mu v^\lambda \delta\tau_\lambda \tag{D.63}$$

$$-\frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu} = e^\lambda_c \delta e^c_\lambda \tag{D.64}$$

We can then vary the action (8.43a)

$$\begin{aligned}
\delta S &= \int \mathrm{d}^{d+1}x\, \mathcal{L}\delta e + e\delta\Big[-\frac{1}{4}h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\tilde{f}_{\rho\sigma} + h^{\mu\rho}v^{\nu}\partial_{\rho}\varphi\tilde{f}_{\mu\nu} \\
&\quad + h^{\mu\rho}\partial_{\mu}\varphi\partial_{\rho}\tilde{\varphi} + \frac{1}{2}v^{\mu}v^{\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi\Big] \\
&= \int \mathrm{d}^{d+1}x\, \mathcal{L}\left(-v^{\mu}\delta\tau_{\mu} + e^{\lambda}_{c}\delta e^{c}_{\lambda}\right) \\
&\quad + e\Big[-\frac{1}{2}h^{\nu\sigma}\tilde{f}_{\mu\nu}\tilde{f}_{\rho\sigma}\delta h^{\mu\rho} - \frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\delta\tilde{f}_{\rho\sigma} \\
&\quad + v^{\nu}\partial_{\rho}\varphi\tilde{f}_{\mu\nu}\delta h^{\mu\rho} + h^{\mu\rho}\partial_{\rho}\varphi\tilde{f}_{\mu\nu}\delta v^{\nu} + h^{\mu\rho}v^{\nu}\partial_{\rho}\varphi\delta\tilde{f}_{\mu\nu} \\
&\quad + \partial_{\mu}\varphi\partial_{\rho}\tilde{\varphi}\delta h^{\mu\rho} + v^{\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi\delta v^{\mu}\Big] \\
&= \int \mathrm{d}^{d+1}x\, \mathcal{L}\left(-v^{\mu}\delta\tau_{\mu} + e^{\lambda}_{c}\delta e^{c}_{\lambda}\right) \\
&\quad + e\Big[h^{\nu\sigma}\tilde{f}_{\mu\nu}\tilde{f}_{\rho\sigma}\left(h^{\lambda(\mu}e^{\rho)}_{c}\delta e^{c}_{\lambda} - h^{\lambda(\mu}v^{\rho)}\delta\tau_{\lambda}\right) \\
&\quad - h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\left(\partial_{[\rho}\left(a_{|\lambda|}e^{\lambda}_{a}\right)\delta e_{\sigma]}{}^{a} + a_{\lambda}e^{\lambda}_{a}\partial_{[\rho}\delta e_{\sigma]}{}^{a}\right) \\
&\quad + h^{\mu\rho}h^{\nu\sigma}\tilde{f}_{\mu\nu}\left(\tilde{\varphi}\partial_{[\rho}\delta\tau_{\sigma]} + \varphi\partial_{[\rho}\delta M_{\sigma]}\right) \\
&\quad + v^{\nu}\partial_{\rho}\varphi\tilde{f}_{\mu\nu}\left(-2h^{\lambda(\mu}e^{\rho)}_{c}\delta e^{c}_{\lambda} + 2h^{\lambda(\mu}v^{\rho)}\delta\tau_{\lambda}\right) + h^{\mu\rho}\partial_{\rho}\varphi\tilde{f}_{\mu\nu}\delta v^{\nu} \\
&\quad + 2h^{\mu\rho}v^{\nu}\partial_{\rho}\varphi\left(\partial_{[\mu}\left(a_{|\lambda|}e^{\lambda}_{a}\right)\delta e_{\nu]}{}^{a} + a_{\lambda}e^{\lambda}_{a}\partial_{[\mu}\delta e_{\nu]}{}^{a}\right) \\
&\quad - 2h^{\mu\rho}v^{\nu}\partial_{\rho}\varphi\left[\tilde{\varphi}\partial_{[\mu}\delta\tau_{\nu]} + \varphi\partial_{[\mu}\delta M_{\nu]}\right] \\
&\quad + \partial_{\mu}\varphi\partial_{\rho}\tilde{\varphi}\left(-2h^{\lambda(\mu}e^{\rho)}_{c}\delta e^{c}_{\lambda} + 2h^{\lambda(\mu}v^{\rho)}\delta\tau_{\lambda}\right) \\
&\quad + v^{\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi\delta v^{\mu}\Big] \\
&= \int \mathrm{d}^{d+1}x\, \mathcal{L}\left(-v^{\mu}\delta\tau_{\mu} + e^{\lambda}_{c}\delta e^{c}_{\lambda}\right) \\
&\quad + e\Big[-h^{\nu\sigma}\tilde{f}_{\nu(\mu}\tilde{f}_{\rho)\sigma}\left(h^{\lambda\mu}e^{\rho}_{c}\delta e^{c}_{\lambda} - h^{\lambda\mu}v^{\rho}\delta\tau_{\lambda}\right) \\
&\quad - h^{\mu[\rho}h^{\sigma]\nu}\tilde{f}_{\mu\nu}\left(\partial_{\rho}\left(a_{|\lambda|}e^{\lambda}_{a}\right)\delta e^{a}_{\sigma} + a_{\lambda}e^{\lambda}_{a}\partial_{\rho}\delta e^{a}_{\sigma}\right) \\
&\quad + h^{\mu[\rho}h^{\sigma]\nu}\tilde{f}_{\mu\nu}\left(\tilde{\varphi}\partial_{\rho}\delta\tau_{\sigma} + \varphi\partial_{\rho}\delta M_{\sigma}\right) \\
&\quad + v^{\nu}\partial_{(\rho}\varphi\tilde{f}_{\mu)\nu}\left(-2h^{\lambda\mu}e^{\rho}_{c}\delta e^{c}_{\lambda} + 2h^{\lambda\mu}v^{\rho}\delta\tau_{\lambda}\right) \\
&\quad + h^{\mu\rho}\partial_{\rho}\varphi\tilde{f}_{\mu\nu}\left(-v^{\lambda}e^{\nu}_{c}\delta e^{c}_{\lambda} + v^{\nu}v^{\lambda}\delta\tau_{\lambda}\right) \\
&\quad + 2h^{\rho[\mu}v^{\nu]}\partial_{\rho}\varphi\left(\partial_{\mu}\left(a_{\lambda}e^{\lambda}_{a}\right)\delta e^{a}_{\nu} + a_{\lambda}e^{\lambda}_{a}\partial_{\mu}\delta e^{a}_{\nu}\right) \\
&\quad - 2h^{\rho[\mu}v^{\nu]}\partial_{\rho}\varphi\left[\tilde{\varphi}\partial_{\mu}\delta\tau_{\nu} + \varphi\partial_{\mu}\delta M_{\nu}\right] \\
&\quad + \partial_{(\mu}\varphi\partial_{\rho)}\tilde{\varphi}\left(-2h^{\lambda\mu}e^{\rho}_{c}\delta e^{c}_{\lambda} + 2h^{\lambda\mu}v^{\rho}\delta\tau_{\lambda}\right) \\
&\quad + v^{\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi\left(-v^{\lambda}e^{\mu}_{c}\delta e^{c}_{\lambda} + v^{\mu}v^{\lambda}\delta\tau_{\lambda}\right)\Big]
\end{aligned}$$

$$
\begin{aligned}
= & \int \mathrm{d}^{d+1}x e\, \delta\tau_\lambda \Big[-\mathcal{L}v^\lambda + h^{\nu\sigma}\tilde{f}_{\nu(\mu}\tilde{f}_{\rho)\sigma}h^{\lambda\mu}v^\rho - \frac{1}{e}\partial_\rho\left(eh^{\mu[\rho}h^{\lambda]\nu}\tilde{f}_{\mu\nu}\tilde{\varphi}\right) \\
& +2v^\nu v^\rho h^{\lambda\mu}\partial_{(\rho}\varphi\tilde{f}_{\mu)\nu} + v^\nu v^\lambda h^{\mu\rho}\partial_\rho\varphi\tilde{f}_{\mu\nu} + 2\frac{1}{e}\partial_\mu\left(eh^{\rho[\mu}v^{\lambda]}\tilde{\varphi}\partial_\rho\varphi\right) \\
& +2\partial_{(\mu}\varphi\partial_{\rho)}\tilde{\varphi}h^{\lambda\mu}v^\rho + v^\lambda v^\mu v^\nu\partial_\mu\varphi\partial_\nu\varphi \Big] \\
& +\delta e^c_\lambda\Big[\mathcal{L}e^\lambda_c - h^{\nu\sigma}\tilde{f}_{\nu(\mu}\tilde{f}_{\rho)\sigma}h^{\lambda\mu}e^\rho_c - h^{\mu[\rho}h^{\lambda]\nu}\tilde{f}_{\mu\nu}\partial_\rho\left(a_{|\sigma|}e^\sigma_c\right) \\
& +\frac{1}{e}\partial_\rho\left(eh^{\mu[\rho}h^{\lambda]\nu}\tilde{f}_{\mu\nu}a_\sigma e^\sigma_c\right) - 2v^\nu\partial_{(\rho}\varphi\tilde{f}_{\mu)\nu}h^{\lambda\mu}e^\rho_c - h^{\mu\rho}v^\lambda e^\nu_c\partial_\rho\varphi\tilde{f}_{\mu\nu} \\
& +2h^{\rho[\mu}v^{\lambda]}\partial_\rho\varphi\partial_\mu\left(a_\sigma e^\sigma_c\right) - 2\frac{1}{e}\partial_\mu\left(eh^{\rho[\mu}v^{\lambda]}\partial_\rho\varphi a_\sigma e^\sigma_c\right) \\
& -2\partial_{(\mu}\varphi\partial_{\rho)}\tilde{\varphi}h^{\lambda\mu}e^\rho_c - v^\nu\partial_\mu\varphi\partial_\nu\varphi v^\lambda e^\mu_c\Big] \\
& +\delta M_\lambda\left[\frac{1}{e}\partial_\rho\left(e2h^{\mu[\rho}v^{\lambda]}\varphi\partial_\mu\varphi - eh^{\mu[\rho}h^{\lambda]\nu}\tilde{f}_{\mu\nu}\varphi\right)\right]. \qquad \text{(D.65)}
\end{aligned}
$$

From this we read off the currents that we give in (8.49)

D.8.4 *Reduction of $\hat{S}_{\text{int}}$*

The null reduction is straight-forward:

$$
\begin{aligned}
\hat{S}_{\text{int}} = & \int_M \mathrm{d}^{D+1}x\sqrt{|g|}\left[g^{\hat{\mu}\hat{\nu}}\left(i\Psi\partial_{\hat{\mu}}\Psi^* - i\Psi^*\partial_{\hat{\mu}}\Psi\right)A_{\hat{\nu}} - \Psi^*\Psi g^{\hat{\mu}\hat{\nu}}A_{\hat{\mu}}A_{\hat{\nu}}\right] \\
= & \int_M \mathrm{d}^{D+1}x\sqrt{|g|}\Big[ih^{\mu\nu}\phi\partial_\mu\phi^*A_\nu - \hat{v}^\nu i\phi\left(-im\right)\phi^*A_\nu \\
& -\hat{v}^\mu i\phi\partial_\mu\phi^*\varphi + 2\tilde{\Phi}i\phi\left(-im\right)\phi^*\varphi \\
& -ih^{\mu\nu}\phi^*\partial_\mu\phi A_\nu + i\hat{v}^\nu\phi^*\left(+im\right)\phi A_\nu + i\hat{v}^\mu\phi^*\partial_\mu\phi\varphi \\
& -i2\tilde{\Phi}\phi^*\left(+im\right)\phi\varphi - \phi^*\phi\left(h^{\mu\nu}A_\mu A_\nu - 2\hat{v}^\nu\varphi A_\nu + 2\tilde{\Phi}\varphi^2\right)\Big] \\
= & \int_M \mathrm{d}^{D+1}x\sqrt{|g|}\Big[ih^{\mu\nu}\left(\phi A_\mu\partial_\nu\phi^* - \phi^*A_\mu\partial_\nu\phi\right) \\
& -i\phi\left(\varphi\hat{v}^\mu\partial_\mu\phi^*\right) + i\phi^*\left(\varphi\hat{v}^\mu\partial_\mu\phi\right) \\
& -\phi^*\phi\left(h^{\mu\nu}A_\mu A_\nu - 2\hat{v}^\nu\left(\varphi+m\right)A_\nu + 2\tilde{\Phi}\varphi^2 + 4\tilde{\Phi}m\varphi\right)\Big] \\
= & \int_M \mathrm{d}^{D+1}x\sqrt{|g|}\Big[ih^{\mu\nu}\left(\phi A_\mu\partial_\nu\phi^* - \phi^*A_\mu\partial_\nu\phi\right) \\
& -\phi^*\phi h^{\mu\nu}\left(A_\mu A_\nu + 2\left(\varphi+m\right)M_\mu A_\nu + M_\mu M_\nu\varphi^2 + 2M_\mu M_\nu m\varphi\right) \\
& -i\phi\left(\varphi\hat{v}^\mu\partial_\mu\phi^*\right) + i\phi^*\left(\varphi\hat{v}^\mu\partial_\mu\phi\right) \\
& -\phi^*\phi\left(-2\left(\varphi+m\right)v^\nu A_\nu - 2v^\mu M_\mu\varphi^2 - 4v^\mu M_\mu m\varphi\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
= & \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[i h^{\mu\nu} \left(\phi \left(a_\mu - \cancel{\tilde{\varphi}\tau_\mu} - \varphi M_\mu \right) \partial_\nu \phi^* - \phi^* \left(a_\mu - \cancel{\tilde{\varphi}\tau_\mu} - \varphi M_\mu \right) \partial_\nu \phi \right) \\
& - \phi^* \phi h^{\mu\nu} \left(\left(a_\mu - \cancel{\tilde{\varphi}\tau_\mu} - \varphi M_\mu \right) \left(a_\nu - \cancel{\tilde{\varphi}\tau_\nu} - \varphi M_\nu \right) \right) \\
& - \phi^* \phi h^{\mu\nu} \left(2 \left(\varphi + m \right) M_\mu \left(a_\nu - \cancel{\tilde{\varphi}\tau_\nu} - \varphi M_\nu \right) + M_\mu M_\nu \varphi^2 + 2 M_\mu M_\nu m \varphi \right) \\
& - i\phi \left(\varphi \left(v^\mu - h^{\mu\nu} M_\nu \right) \partial_\mu \phi^* \right) + i \phi^* \left(\varphi \left(v^\mu - h^{\mu\nu} M_\nu \right) \partial_\mu \phi \right) \\
& - \phi^* \phi \left(-2 \left(\varphi + m \right) v^\nu \left(\cancel{a_\nu} - \tilde{\varphi}\tau_\nu - \varphi M_\nu \right) - 2 v^\mu M_\mu \varphi^2 - 4 v^\mu M_\mu m \varphi \right) \Big] \\
= & \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[i h^{\mu\nu} \left(\phi \left(a_\mu - \varphi M_\mu \right) \partial_\nu \phi^* - \phi^* \left(a_\mu - \varphi M_\mu \right) \partial_\nu \phi \right) \\
& - \phi^* \phi h^{\mu\nu} \left(a_\mu a_\nu - \cancel{2 \varphi a_\mu M_\nu} + \cancel{\varphi^2 M_\mu M_\nu} \right) \\
& - \phi^* \phi h^{\mu\nu} \left(2 \varphi M_\mu \left(\cancel{a_\nu} - \cancel{\varphi M_\nu} \right) + 2 m M_\mu \left(a_\nu - \cancel{\varphi M_\nu} \right) + \cancel{M_\mu M_\nu \varphi^2} - \cancel{2 M_\mu M_\nu m \varphi} \right) \\
& - i\phi \left(\varphi \left(v^\mu - h^{\mu\nu} M_\nu \right) \partial_\mu \phi^* \right) + i \phi^* \left(\varphi \left(v^\mu - h^{\mu\nu} M_\nu \right) \partial_\mu \phi \right) \\
& - \phi^* \phi \left(-2 \left(\varphi + m \right) \left(\tilde{\varphi} - \varphi v^\nu M_\nu \right) - 2 v^\mu M_\mu \varphi^2 - 4 v^\mu M_\mu m \varphi \right) \Big] \\
= & \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[i h^{\mu\nu} \left(\phi \left(a_\mu - \cancel{\varphi M_\mu} \right) \partial_\nu \phi^* - \phi^* \left(a_\mu - \cancel{\varphi M_\mu} \right) \partial_\nu \phi \right) \\
& - \phi^* \phi h^{\mu\nu} \left(a_\mu a_\nu + 2 m a_\mu M_\nu \right) \\
& - i\phi \left(\varphi \left(v^\mu - \cancel{h^{\mu\nu} M_\nu} \right) \partial_\mu \phi^* \right) + i \phi^* \left(\varphi \left(v^\mu - \cancel{h^{\mu\nu} M_\nu} \right) \partial_\mu \phi \right) \\
& - \phi^* \phi \left(-2 \varphi \left(\tilde{\varphi} - \cancel{\varphi v^\nu M_\nu} \right) - 2 m \left(\tilde{\varphi} - \varphi v^\nu M_\nu \right) - \cancel{2 v^\mu M_\mu \varphi^2} - 4 v^\mu M_\mu m \varphi \right) \Big] \\
= & \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[i h^{\mu\nu} \left(\phi a_\mu \partial_\nu \phi^* - \phi^* a_\mu \partial_\nu \phi \right) \\
& - \phi^* \phi h^{\mu\nu} \left(a_\mu a_\nu + 2 m a_\mu M_\nu \right) \\
& - i\phi \left(\varphi v^\mu \partial_\mu \phi^* \right) + i \phi^* \left(\varphi v^\mu \partial_\mu \phi \right) \\
& + \phi^* \phi \left(2 \left(\varphi + m \right) \tilde{\varphi} + 2 v^\mu M_\mu m \varphi \right) \Big] \\
= & \int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[i h^{\mu\nu} \left(\phi a_\mu \partial_\nu \phi^* - \phi^* a_\mu \partial_\nu \phi \right) \\
& - \phi^* \phi h^{\mu\nu} \left(a_\mu a_\nu + 2 m a_\mu M_\nu \right) \\
& - i\phi \left(\varphi v^\mu \partial_\mu \phi^* \right) + i \phi^* \left(\varphi v^\mu \partial_\mu \phi \right) \\
& + \phi^* \phi \left(2 \left(\varphi + m \right) \tilde{\varphi} + 2 v^\mu M_\mu m \varphi \right) \Big] .
\end{aligned}
\tag{D.66}
$$

One can then pull out the D-dimensional integral, which gives the result in the main text.

D.8.5 *Simplification of* $S_{\mathrm{Sch}} + S_{\mathrm{int}}$

The easiest way to get is to null reduce $\hat{S}_{\mathrm{KG}} + \hat{S}_{\mathrm{int}}$ which has a simple expression in terms of the higher-dimensional gauge covariant derivative

$$D_{\hat{\mu}}\Psi = \partial_{\hat{\mu}}\Psi - iA_{\hat{\mu}}\Psi\,. \tag{D.67}$$

We then find directly

$$\begin{aligned}
&\hat{S}_{\text{KG}} + \hat{S}_{\text{int}} \\
= \;& \int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\,\left[-g^{\hat{\mu}\hat{\nu}}D_{\hat{\mu}}\Psi^* D_{\hat{\nu}}\Psi\right] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[g^{\mu\nu}D_\mu\Psi^* D_\nu\Psi + g^{u\nu}D_u\Psi^* D_\nu\Psi \\
& +g^{\mu u}D_\mu\Psi^* D_u\Psi + g^{uu}D_u\Psi^* D_u\Psi\Big] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[g^{\mu\nu}D_\mu\Psi^* D_\nu\Psi + g^{u\nu}\left(\partial_u + i\varphi\right)\Psi^* D_\nu\Psi \\
& +g^{\mu u}D_\mu\Psi^*\left(\partial_u - i\varphi\right)\Psi + g^{uu}\left(\partial_u + i\varphi\right)\Psi^*\left(\partial_u - i\varphi\right)\Psi\Big] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[h^{\mu\nu}\left(\partial_\mu + iA_\mu\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& -i\hat{v}^\nu\left(m+\varphi\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& +i\hat{v}^\mu\left(\partial_\mu + iA_\mu\right)\phi^*\left(m+\varphi\right)\phi + 2\tilde{\Phi}\left(m+\varphi\right)^2\phi^*\phi\Big] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[h^{\mu\nu}\left(\partial_\mu + iA_\mu\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& -i\left(v^\nu - h^{\mu\nu}M_\mu\right)\left(m+\varphi\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& +i\left(v^\mu - h^{\mu\nu}M_\nu\right)\left(m+\varphi\right)\phi\left(\partial_\mu + iA_\mu\right)\phi^* \\
& +\left(-2v^\mu M_\mu + h^{\mu\nu}M_\mu M_\nu\right)\left(m+\varphi\right)^2\phi^*\phi\Big] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[h^{\mu\nu}\left(\partial_\mu + iA_\mu\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& -iv^\nu\left(m+\varphi\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi + ih^{\mu\nu}M_\mu\left(m+\varphi\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& +iv^\mu\left(m+\varphi\right)\phi\left(\partial_\mu + iA_\mu\right)\phi^* - ih^{\mu\nu}M_\nu\left(m+\varphi\right)\phi\left(\partial_\mu + iA_\mu\right)\phi^* \\
& +\left(-2v^\mu M_\mu + h^{\mu\nu}M_\mu M_\nu\right)\left(m+\varphi\right)^2\phi^*\phi\Big] \\
= \;& -\int_M \mathrm{d}^{D+1}x\,\sqrt{|g|}\Big[h^{\mu\nu}\left(\partial_\mu + iA_\mu + iM_\mu\left(m+\varphi\right)\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& -iv^\nu\left(m+\varphi\right)\phi^*\left(\partial_\nu - iA_\nu\right)\phi \\
& +iv^\mu\left(m+\varphi\right)\phi\left(\partial_\mu + iA_\mu\right)\phi^* - ih^{\mu\nu}M_\nu\left(m+\varphi\right)\phi\left(\partial_\mu + iA_\mu\right)\phi^* \\
& +\left(-2v^\mu M_\mu + h^{\mu\nu}M_\mu M_\nu\right)\left(m+\varphi\right)^2\phi^*\phi\Big]
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[h^{\mu\nu} \left(\partial_\mu + iA_\mu + iM_\mu \left(m+\varphi\right)\right) \phi^* \\
& \times \left(\partial_\nu - iA_\nu - iM_\nu \left(m+\varphi\right)\right) \phi \\
& + h^{\mu\nu} \left(\partial_\mu + iA_\mu + iM_\mu \left(m+\varphi\right)\right) \phi^* \left(iM_\nu \left(m+\varphi\right)\right) \phi \\
& - iv^\nu \left(m+\varphi\right) \phi^* \left(\partial_\nu - iA_\nu\right) \phi \\
& + iv^\mu \left(m+\varphi\right) \phi \left(\partial_\mu + iA_\mu\right) \phi^* - ih^{\mu\nu} M_\nu \left(m+\varphi\right) \phi \left(\partial_\mu + iA_\mu\right) \phi^* \\
& + \left(-2v^\mu M_\mu + h^{\mu\nu} M_\mu M_\nu\right) \left(m+\varphi\right)^2 \phi^* \phi \Big] \\
= \quad & -\int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[h^{\mu\nu} \left(\partial_\mu + iA_\mu + iM_\mu \left(m+\varphi\right)\right) \phi^* \\
& \times \left(\partial_\nu - iA_\nu - iM_\nu \left(m+\varphi\right)\right) \phi \\
& + h^{\mu\nu} \left(\partial_\mu + iA_\mu\right) \phi^* \left(\cancel{iM_\nu \left(m+\varphi\right)}\right) \phi - h^{\mu\nu} \left(\cancel{M_\mu M_\nu \left(m+\varphi\right)^2}\right) \phi^* \phi \\
& - iv^\nu \left(m+\varphi\right) \phi^* \left(\partial_\nu - iA_\nu\right) \phi \\
& + iv^\mu \left(m+\varphi\right) \phi \left(\partial_\mu + iA_\mu\right) \phi^* \cancel{- ih^{\mu\nu} M_\nu \left(m+\varphi\right) \phi \left(\partial_\mu + iA_\mu\right) \phi^*} \\
& + \left(-2v^\mu M_\mu + \cancel{h^{\mu\nu} M_\mu M_\nu}\right) \left(m+\varphi\right)^2 \phi^* \phi \Big] \\
= \quad & -\int_M \mathrm{d}^{D+1}x \sqrt{|g|} \Big[h^{\mu\nu} \left(\partial_\mu + iA_\mu + iM_\mu \left(m+\varphi\right)\right) \phi^* \\
& \times \left(\partial_\nu - iA_\nu - iM_\nu \left(m+\varphi\right)\right) \phi \\
& - iv^\nu \left(m+\varphi\right) \phi^* \left(\partial_\nu - iA_\nu - iM_\nu \left(m+\varphi\right)\right) \phi \\
& + iv^\mu \left(m+\varphi\right) \phi \left(\partial_\mu + iA_\mu + iM_\mu \left(m+\varphi\right)\right) \phi^* \Big]
\end{aligned}
\tag{D.68}
$$

This then leads to the result (8.70) given in the main text.

E USEFUL FORMULAS

Variation of Newton-Cartan measure wrt. background:

$$\delta e = e\left(\tau_\mu \delta v^\mu - \frac{1}{2} h_{\mu\nu} \delta h^{\mu\nu}\right) . \tag{E.1}$$

Fourier-transform of $\frac{1}{\boldsymbol{p}^2}$:

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^D} \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d} \frac{1}{\boldsymbol{p}^2 + i\epsilon} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} \frac{1}{\|\boldsymbol{x}\|^{d-2}} . \tag{E.2}$$

Contraction of Levi-Civita symbols in $d = 3$:

$$\epsilon_{ijk} \epsilon^{inm} = 2\delta^n_{[j} \delta^m_{k]} . \tag{E.3}$$

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DECLARATION

I hereby declare that this thesis complies with good scientific practice and all university regulations.

Copenhagen, February 2016

Dennis Hansen