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Master's thesis

Reproducing the Standard Model using strings on magnetized D-branes

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Abstract

In this thesis we present string theory on magnetized D-branes as a candidate for semi-realistic physics. By examining the spectrum of open superstring states that live on the branes, we find that all three generations of Standard Model particles can be reproduced without the emergence of unphysical extra particles. Furthermore, we construct a gauge group that contains that of the Standard Model and where the extra local symmetries can be made global through interactions with RR-sector superstrings. This implies baryon and lepton number conservation which ensures the stability of the proton.

Before going into the complicated process of reproducing the Standard Model, we derive the tools needed for this. We start out by quantizing the superstring and ensuring a stable vacuum through the GSO-projection. Secondly, we explain the presence of D-branes by using T-duality on closed and open strings in small compact dimensions and see how this lets us endow the strings with non-abelian gauge symmetries. Finally, we see how these can be broken in just the right way by magnetization. We also examine the low-energy limit of string theory and find that it agrees with the quantum mechanics of point particles.

Resumé

I dette speciale gennemgår jeg først bosonisk og superstrengteori for derefter at indføre D-braner og bruge dem til at konstruere en semi-realistisk partikelfysisk model. Jeg starter med en kort historisk gennemgang hvorefter jeg udfra Polyakovvirkningen udleder bevægelsesligninger og randbetingelser for både bosoniske og fermioniske strenge. Jeg løser bevægelsesligningerne under alle de relevante randbetingelser og indfører kanoniske (anti-)kommutatorrelationer hvilket giver en kvanteteori for strenge. Efter indførelsen af lyskeglekoordinater undersøger jeg de relevante mulige kvantetilstande for åbne og lukkede strenge, og finder eksistensen af fotoner, gravitoner og masseløse Majoranafermioner. Udover disse indeholder spektret også tachyoner som ved hjælp af GSO projektionen fjernes fra teorien. Jeg viser også at denne projektion sikrer at der ved alle masseniveauer er lige mange fermioner og bosoner.

Ved hjælp af T-dualitet indfører jeg kompakte dimensioner og D-braner i den udviklede superstrengteori. Jeg undersøger derefter spektret for strenge der har endepunkter på to forskellige braner der er separeret i rummet. Derpå forklarer jeg hvordan Chan-Paton indices og D-braner kan give strengteorier symmetri under klassiske gaugegrupper, noget som er nødvendigt for at konstruere en semi-realistisk partikelfysisk model. Herefter undersøger jeg hvad der sker hvis man i stedet for braner der er separeret i rummet betragter braner med samme position, men hvorpå der lever forksellige magnetfelter. Det viser sig at denne situation medfører mange ønskværdige træk. Når der er et påtrykt magnetfelt bliver de bosoniske strenge massive hvilket bryder supersymmetrien, de fermionske strenge mister chiralitet i de magnetiserede retninger hvilket betyder at jeg kan sikre fire dimensional chiralitet. Jeg finder desuden at to strenge der er strakt den modsatte vej mellem de samme braner er hinandens anti-strenge. Herefter kigger jeg på en ækvivalent situation fra en punktpartikels synspunkt og finder dens Hamiltonoperator. Jeg viser derpå at lav-energi grænsen af strengteori reproducerer punktpartikel udreningen.

Efter en kort gennemgang af gaugeteorier i almindelighed og Standardmodellen i særdeleshed går jeg herefter i detaljer med præcis hvad der skal til for at skabe en semi-realistisk model ud fra strengteori. Ved hjælp af fire stakke af braner med forskellig magnetisering og deres orientifold spejlbilleder konstruerer jeg derpå en model der har det samme partikelindhold som Standardmodellen. Dette gør jeg ved at fastsætte udartningen af de Landauniveauer der hører til hver mulig kombination af braner. For at sikre haletudseudligning viser det sig at være nødvendigt at indføre højrehåndede neutrini i teorien. De fire stakke af braner giver en samlet gaugegruppe som indeholder Standardmodellens, men også har tre yderligere $U(1)$ symmetrier. Pariklerne associaret med disse bliver dog massive når man medregner interaktioner fra lukkede strenge, hvilket gør symmetrierne globale. De globale symmetrier er baryon- og leptontalsbevarelse og Peccei-Quinn symmetri. Den første sikrer protonens stabilitet, og den sidste

hjælper med at løse det stærke CP problem.

Det viser sig altså at være muligt at konstruere en strengteoretisk model som indeholder de samme partikler og den samme gaugegruppe som Standardmodellen samt nogle af de vigtigste globale symmetrier. Desuden går modellen videre og forudsiger ting som ikke er i den basale Standardmodel.

Contents

List of symbols

J	Total angular momentum.
α'	Regge slope or string parameter.
M	Mass.
S	Action.
$G_{\mu\nu}$	Spacetime metric.
ξ^α	World-sheet coordinate.
X^μ	Bosonic string spacetime coordinate.
∂_α	$\frac{\partial}{\partial\xi^\alpha}$.
$g_{\alpha\beta}$	World-sheet metric.
$T_{\alpha\beta}$	Stress tensor.
$\bar{\alpha}_n^\mu$	Bosonic left-moving string oscillation mode.
α_n^μ	Bosonic right-moving string oscillation mode.
τ	Time-like world-sheet coordinate, $\tau = \xi^0$.
σ	Space-like world-sheet coordinate, $\sigma = \xi^1$.
p^μ	String centre-of-mass momentum.
L_m	Virasoro or super-Virasoro operator.
P^μ	Canonical string momentum.
$\eta^{\mu\nu}$	Metric of flat space with mostly positive signature (-1,1,1,...).
$\lvert\phi\rangle$	Arbitrary string state .
$\lvert p\rangle$	String ground state with momentum p.
a	Normal ordering constant. Equals 1 in the bosonic theory, $\frac{1}{2}$ in the Neveu-Schwarz sector and 0 in the Ramond sector.
$\Re(s)$	The real part of the complex variable s.
$\zeta(s)$	The Riemann zeta function.
g_s	String coupling strength.
ℓ_s	String length $\ell_s = \sqrt{\alpha'}$.
ℓ_P	Planck length $\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}\,\text{m}$.
M_P	Planck mass $M_P = \sqrt{\frac{\hbar c}{G}} \approx 2.2 \times 10^{-8}\,\text{kg} \approx 1.2 \times 10^{19}\,\text{GeV}/c^2$.
e_a^α	Two-dimensional world-sheet vielbein.
ψ^μ	Fermionic spacetime string coordinate.
ρ^a	Two-dimensional world-sheet Dirac matrices.
γ^a	Effective four-dimensional Dirac matrices.

Γ^a	10-dimensional spacetime Dirac matrices.
χ_α	Gravitino field.
J_α	Supercurrent.
$\bar{b}_k^\mu$	Fermionic left-moving string oscillation mode.
b_k^μ	Fermionic right-moving string oscillation mode.
G_k	Supercurrent modes.
$\vert p; i, j\rangle$	String ground state with momentum p stretched from brane i to brane j.
B_{mn}	Kalb-Ramond tensor field.
F_{mn}	Faraday (field strength) tensor.
$\mathcal{B}_{mn}$	Combined Kalb-Ramond and Faraday tensor $\mathcal{B}_{mn} = B_{mn} - 2\pi\alpha' q F_{mn}$.
$R^m{}_n$	Orthogonal matrix containing the corrections to string theory from toroidal compactifications and the $\mathcal{B}_{mn}$-field.
$\mathcal{E}^A{}_m$	Matrix that diagonalises $R^m{}_n$ to $\mathcal{R}^A{}_B$
ν_a	Shift in oscillator indices coming from toroidal compactifications and the $\mathcal{B}_{mn}$-field.
$A^a_{n\pm\nu_a}$	Shifted bosonic vibrational modes.
$B^a_{n\pm\nu_a}$	Shifted fermionic vibrational modes.
U	Complex structure of the two-torus, $U = U_1 + U_2 = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}}$.
T	Kähler structure of the two-torus, $T = T_1 + T_2 = -B_{12} + i\sqrt{G}$.
I	The first Chern class.
N_α	Number of branes in stack α.
Q_α	Charge associated with stack α.
$I_{\alpha\beta}$	Number of shared degenerate ground states of branes α and β.
Y	Weak hypercharge.

Chapter 1

Introduction

String theory is the theory of one-dimensional vibrating objects; classically these could be rubber bands or violin strings, not at all what we at first think of as something that would make a good fundamental theory of physics. However, another way of viewing a string is as something that can vibrate at an infinite number of frequencies, corresponding to musical notes in the case of the violin. This is exactly an infinite set of harmonic oscillators, and in physics, we find that harmonic oscillators are present everywhere and that they always play a fundamental role. Let us first take a look at the history of string theory and see how the theory became what it is today.

1.1 Regge trajectories

The quantum mechanical string entered the world of theoretical physics through the strong interactions of protons and neutrons, and more generally of all hadrons. In the 1960s, there was not yet a definite theory of these strong interactions, but only a lot of different and not yet well understood experimental facts. One of these facts is known as Regge trajectories (see figure 1.1); it was found that for several baryons and mesons there is a relation between their mass and their maximum spin

$$J = \alpha' M^2, \tag{1.1}$$

with the Regge slope $\alpha' \approx 1\,\mathrm{GeV}^{-2}$. This behaviour led Veneziano to develop the Dual resonance model for the strong interaction [1] and in particular to calculate the now famous Veneziano amplitude. It was later suggested independently by Nambu [2], Nielsen [3] and Susskind [4] that this amplitude came from the scattering of strings with tension $\frac{1}{2\pi\alpha'}$. However, while strings did have the Regge behaviour, they also came with a lot of baggage. First and foremost, the string ground state was found to be a tachyon; a superluminal particle indicating an inherent instability in the system. Secondly, the theory was found to contain a

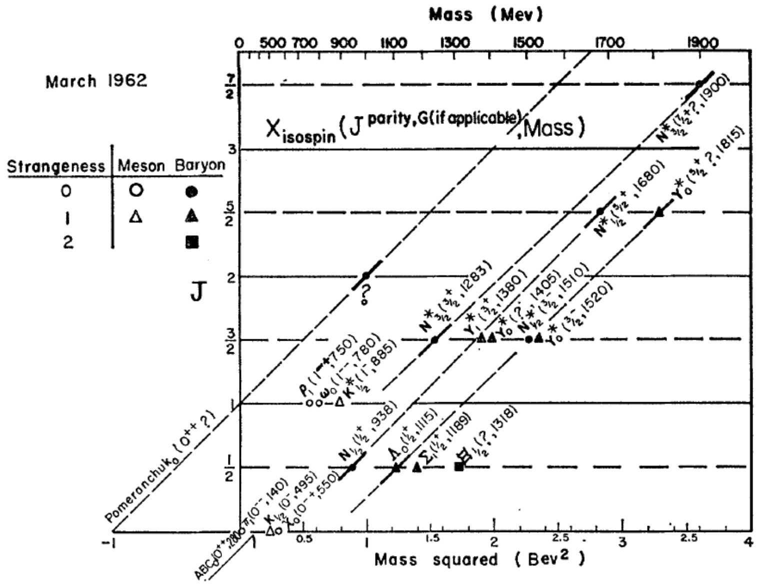


Figure 1.1: Regge trajectories of hadrons. Figure taken from [5].

massless spin two particle that was impossible to get rid of. Since no such particle had ever been seen, this was a prediction that seemed to be in contradiction with all experiments. Even though this was not, as the tachyon, an inherent problem with the theory, it was still a very significant issue. These were not the only problems with string theory. It also demanded 26 dimensions for quantum level Lorentz invariance and it contained only bosons. When these facts are considered, it is not hard to see why the idea of string theory as the fundamental theory of the strong interactions was rejected by the physics community. The strong interactions were instead understood in terms of the non-abelian gauge theory of quantum chromodynamics.

1.2 Quantum gravity

After its failure to explain the strong force, most of the few people who were interested in string theory gave it up and turned their attention elsewhere. But Schwarz and Scherk [6], and independently Yoneya [7], kept at it and found in 1974 that the persistent massless spin two boson could naturally be the graviton; the particle responsible for mediating gravity, if the size of α' was on the order of the Planck length squared instead of the Regge slope. Furthermore using the Kaluza-Klein method of compactifying dimensions, it was possible to construct a 4 dimensional theory of quantum gravity based on string theory. Something that is impossible using conventional methods of quantizing a classical theory since these give non-renormalizable divergences. Despite these promising features, only relatively few people worked on string theory, which only makes the discoveries made all the more impressive. Fermions were introduced through world-sheet supersymmetry and later Gliozzi, Scherk and Olive [8] found a consistent way to get rid of the tachyon and ensure spacetime supersymmetry.

1.3 The first superstring revolution

These things all lead up to what is now called the first superstring revolution when in 1984, Green and Schwarz [9] showed that a superstring theory free of gauge and gravitational anomalies could be constructed by endowing it with the gauge group $SO(32)$. They did this using a method developed much earlier by Chan and Paton [10] for endowing the dual model of Veneziano with any classical gauge group. During this time, it was quickly shown that the type I $SO(32)$ theory of Green and Schwarz was not the only anomaly free string theory, but four other theories were equally valid. A few years previously, the problem with superstring theory as a fundamental theory of physics had been that there was no anomaly free theory, and now the problem was that there were five. Having too many theories is a problem since we would want our fundamental theory to follow in a simple way from first principles, and now first principles implied five different things. This along with the fact that there was no simple way to get the $SU(3) \times SU(2) \times U(1)$ symmetry of the standard model in an anomaly-free way caused string theory to once more be put on the shelf as theoretical physicists everywhere turned to greener pastures.

Even though string theory was no longer as intensely investigated a subject as it had been, significant advancements were still made. Green and Schwarz showed that type IIA and type IIB superstring theory are related by a transformation called T-duality which inverts all lengths such that $\frac{\sqrt{\alpha'}}{R} \leftrightarrow \frac{R}{\sqrt{\alpha'}}$. Another transformation called S-duality was found to relate the original type I string to the $SO(32)$ heterotic string, and the type IIA string to the $E_8 \times E_8$ heterotic string. The seeds of the second superstring revolution were sown.

1.4 The second superstring revolution

These seeds came to fruition in 1995 when Edward Witten [11] proposed that all five known string theories and a hitherto unrelated quantum field theory called 11 dimensional supergravity were merely different limits of the same 11 dimensional theory. He called this theory M-theory without saying what the 'M' stands for. Though Witten could specify very little of this theory, not even its Lagrangian was known, he kindled the spark of interest in string theory once more. The second superstring revolution was kicked off when Polchinski showed that T-duality combined with Kaluza-Klein compactification leads to the existence of multidimensional D-branes. These were new dynamical objects in their own right and when stacked properly, they gave a simple visualization to the gauge groups introduced by Chan and Paton.

1.5 A semi-realistic model

In this thesis we try to construct a semi-realistic model for describing the world of particle physics with string theory. Such a model must be an effective four-dimensional theory, have the $SU(3) \times SU(2) \times U(1)$ local symmetry group of the Standard Model as well as the most important global symmetries, and it must have the three observed generations of chiral fermions.

To get three generations of chiral fermions, we cannot make do with simple stacked branes, but in 1995 Bachas [12] showed that if the branes are magnetized, these properties of the Standard Model also appear in string theory. An equivalent model was given in terms of branes intersecting at angles in 1996 by Berkooz, Douglas and Leigh [13]. When analysed in more detail, it turns out that this equivalence is not at all surprising since the two models are related by T-duality [14, pp. 14-16].

With respect to using string theory to reconstruct the Standard Model, this is largely where we stand today. So without further ado, let us begin!

Chapter 2

The superstring

We wish to build a semi-realistic model for extending the standard model using string theory. To do this we must first investigate the behaviour of the string itself. As we have mentioned above in the introduction, there are several different anomaly-free string theories, but we will primarily focus on the basic RNS superstring theory which is not necessarily anomaly-free. However, since even basic superstring theory is rather involved, we will start by considering the much simpler bosonic string.

2.1 The bosonic string

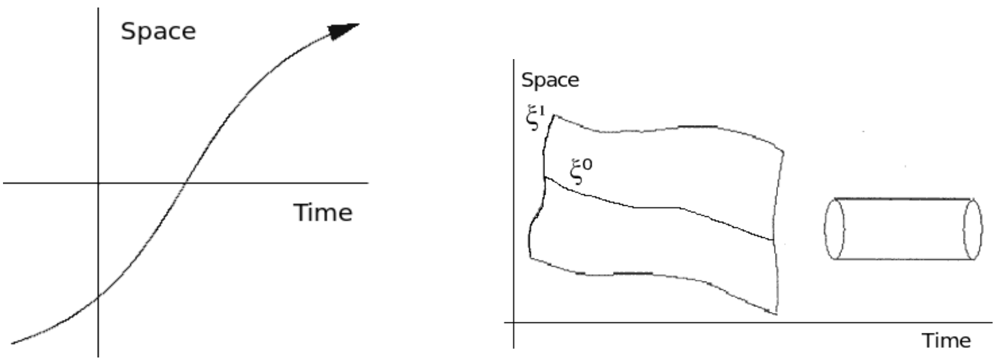
As is most often the case in physics, we draw upon our previous knowledge to get the inspiration required to describe a new system. When considering a point particle moving through space, it is found that it will follow the shortest possible path in d-dimensional spacetime. We call the path travelled by the point particle its world line (see figure 2.1(a)) and then we find its motion by demanding that the length of the world line is minimal. This is called the principle of least action, and generally the action can be any functional that has this property.

In the case of a string moving through space, we say that it traces out a world-sheet (see figure 2.1(b)). We call the area of the bosonic string world-sheet the Nambu-Goto action [15, p. 14]:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int \sqrt{-\det \hat{G}_{\alpha\beta}}\, \mathrm{d}^2\xi, \tag{2.1}$$

where ξ^α is the world-sheet coordinate, $X^\mu(\xi)$ is the spacetime coordinate and $\hat{G}_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ is the induced metric on the world-sheet.

However, while the Nambu-Goto action is intuitive, the presence of the square root of the spacetime coordinates makes it complicated to use in calculations. It would therefore be beneficial to find a simpler, but equivalent action. This has been done for the superstring [16] and [17], and it is a simple matter to remove



(a) A point particle tracing a worldline in spacetime.

(b) An open and a closed string each tracing a world-sheet in spacetime.

Figure 2.1: Worldlines and world-sheets.

the fermionic parts and one is then left with the Polyakov action* [15, p. 16]:

$$S_P = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\xi^2 \sqrt{-\det g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.2}$$

where $\det g = \det(g_{\alpha\beta})$ and $g^{\alpha\beta}$ is an intrinsic metric on the world-sheet which is independent of the spacetime coordinates.

Since this action is a functional of the world-sheet metric, $g^{\alpha\beta}$, as well as the spacetime coordinate X^μ, the metric must now also obey the Euler-Lagrange equations. We therefore vary the action with respect to the metric and obtain the stress tensor

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{-\det g}} \frac{\delta S_P}{\delta g^{\alpha\beta}} = \frac{1}{\alpha'} \left[\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \right]. \tag{2.3}$$

The equation of motion for the world-sheet metric is then $T_{\alpha\beta} = 0$. It is easy to check that by using this, we can regain the Nambu-Goto action from the Polyakov action.

2.1.1 Symmetries of the Polyakov action

In physics, we often find that a system is described as much by its symmetries as by any other traits. The simplest example is perhaps the unit sphere; it can be defined as a completely rotationally symmetric object with radius 1, and this alone tells us all there is to know about it. Similarly, the symmetries of an action tell us a great deal about the object described by it. We therefore consider the symmetries of the Polyakov action to better understand the bosonic string.

*It would perhaps make more sense to name it the Brink-Di Vecchia-Howe-Deser-Zumino action after the people who first derived it, but this is a very cumbersome name. The reason it is named for Polyakov is that he emphasized its virtues for doing path integrals [18].

Poincaré invariance

Poincaré transformations are combinations of translations and Lorentz transformations. Since the theory we are considering does not include interactions at any points in spacetime and since we have constructed it using the squares of Lorentz vectors, the action is in fact manifestly invariant under both translations and Lorentz transformations. Also, since we want a theory that can reproduce the Standard Model, which is Poincaré invariant, we would not consider a string theory that did not have this symmetry. Formally a Poincaré transformation has the form [15, p. 17]

$$\delta X^\mu = \omega^\mu{}_\nu X^\nu + \alpha^\mu, \quad \delta g_{\alpha\beta} = 0, \tag{2.4}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Local two-dimensional reparametrization invariance

We have chosen to describe the world-sheet by the parameters ξ^0 and ξ^1, but we could equally well have chosen functions of these. The action must therefore be invariant under reparametrizations of the form [15, p. 17]

$$\delta g_{\alpha\beta} = \xi^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \xi^\gamma g_{\beta\gamma} + \partial_\beta \xi^\gamma g_{\alpha\gamma}, \tag{2.5}$$

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu \tag{2.6}$$

$$\delta(\sqrt{-\det g}) = \partial_\alpha(\xi^\alpha \sqrt{-\det g}). \tag{2.7}$$

Weyl invariance

A Weyl (or conformal) transformation is a local rescaling of the metric. The fact that we are talking about a *local* rescaling means that there are many different, and seemingly unrelated metrics that turn out to be equivalent. It is immediately clear from the form of the Polyakov action (2.2) that it is invariant under the transformations,

$$\delta X^\mu = 0, \qquad \delta g_{\alpha\beta} = 2\Lambda(\xi^\alpha) g_{\alpha\beta}. \tag{2.8}$$

We shall later see that this transformation allows us to go to the very convenient conformal gauge [15, p. 17].

2.1.2 Equations of motion

Before finding the equations of motion by varying the Polyakov action with respect to the spacetime coordinate X^μ, we will use the symmetries of the action to put it on a simpler form. In particular, we would very much like for the world-sheet metric to be flat, and fortunately it can be shown [15, p. 17] that we can always choose a parametrization where the metric is conformally flat, such that:

$$g_{\alpha\beta} = e^{2\Lambda(\xi)}\eta_{\alpha\beta}. \tag{2.9}$$

It is easy to see that using Weyl symmetry, this metric can be further simplified to that of flat space. This choice is called the conformal gauge.

It is interesting to note that the world-sheet metric had $d(d+1)/2$ independent components, d of these were fixed using reparametrization invariance and one more using Weyl invariance. Since the string world-sheet is two-dimensional, all the independent components have been fixed, but had we been working with a higher-dimensional object, we could not so easily have done this.

Using this we can find the Polyakov action in the conformal gauge,

$$S_P = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\xi[-\partial_\tau X^\mu\partial_\tau X_\mu + \partial_\sigma X^\mu\partial_\sigma X_\mu]. \tag{2.10}$$

Varying this with respect to X^μ gives

$$\begin{aligned}\delta S_P = &-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\xi\delta X^\mu[\partial_\tau\partial_\tau X_\mu - \partial_\sigma\partial_\sigma X_\mu] \\ &+\frac{1}{2\pi\alpha'}\int_0^{\tau'} \mathrm{d}\tau\delta X^\mu\partial_\sigma X_\mu|_{\sigma=0,\sigma'} + \frac{1}{2\pi\alpha'}\int_0^{\sigma'} \mathrm{d}\sigma\delta X^\mu\partial_\tau X_\mu|_{\tau=0,\tau'},\end{aligned} \tag{2.11}$$

where $\sigma' = \pi$ in the case of an open string and 2π in the case of a closed one. From this we can easily read off the equation of motion and two boundary conditions. We satisfy the latter boundary condition by demanding that $\delta X^\mu = 0$ at $\tau = 0$ and $\tau = \tau'$. The former boundary condition, however, gives rise to several distinct and very interesting situations which we will return to at the end of this subsection.

We have found that the equation of motion is the well-known massless Klein-Gordon equation

$$\partial_\alpha\partial^\alpha X_\mu = 0, \tag{2.12}$$

but before we start working more with it, we wish to introduce a more convenient set of coordinates, namely the world-sheet light-cone coordinates:

$$\xi_+ = \tau + \sigma, \quad \xi_- = \tau - \sigma. \tag{2.13}$$

The equation of motion then takes the form

$$\partial_+\partial_- X^\mu = 0. \tag{2.14}$$

We have now fixed the gauge and chosen convenient coordinates to get the simplest possible equation of motion for the spacetime coordinate, but we still need to take into account that the equation of motion for the metric $g^{\alpha\beta}$, which tells us that stress tensor must vanish

$$T_{\alpha\beta} = \frac{1}{\alpha'}\left[\partial_\alpha X^\mu \partial_\beta X_\mu + 2g_{\alpha\beta}\partial_+ X^\mu \partial_- X_\mu\right] = 0. \tag{2.15}$$

This gives the following additional constraints

$$T_{++} = \frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = \frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu = 0, \quad T_{+-} = T_{-+} = 0. \tag{2.16}$$

These constraints are called the Virasoro-constraints and will be very important later. We will look more closely at them in section 2.1.4.

It is now time to explore in detail the boundary conditions for the string derived in (2.11).

Neumann boundary conditions

If we set

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0, \tag{2.17}$$

we have an open string where momentum cannot flow off the end points (see figure 2.2(a)). This means that the strings cannot normally interact with anything else since this would involve a momentum transfer of some form. Thus the Neumann strings are free open strings.

While we say that the strings do not interact, it is of course possible to later add interactions using perturbation theory.

Dirichlet boundary conditions

If we instead set

$$\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, \pi) = 0, \tag{2.18}$$

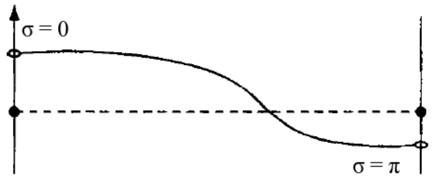
or equivalently

$$\partial_\tau X^\mu(\tau, 0) = \partial_\tau X^\mu(\tau, \pi) = 0, \tag{2.19}$$

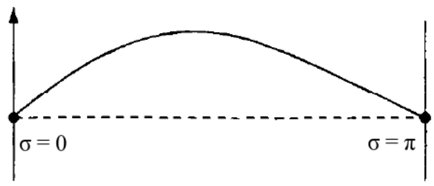
we are demanding that the end points of the open string do not move in space (see figure 2.2(b)). For the endpoints being stationary to make any kind of sense, they must be stuck on something, and this begs the question: In a theory of strings, what could strings be stuck on? It will later become apparent (see section 3) that multidimensional objects, called D(irichlet)-branes, enter the theory naturally and it is onto these the strings are stuck.

The Neumann and Dirichlet boundary conditions can be imposed independently on different coordinates, and it is therefore also possible to consider strings that have some Neumann boundary conditions and some Dirichlet. This is exactly what we will do when we start exploring how branes let us build a semi-realistic theory. Even though the final theory proposed (see chapters 5 and 7) will not use Dirichlet conditions, they are essential components for getting there.

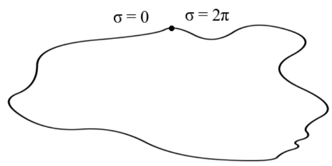
One could also impose the Neumann condition on one end and the Dirichlet condition on the other, but we will not go into this.



(a) A string with Neumann boundary conditions in one direction.



(b) A string with Dirichlet boundary conditions.



(c) A string with periodic boundary conditions.

Figure 2.2: Strings with different boundary conditions. Figures (a) and (b) taken from [19].

Periodic boundary conditions

Last, we can set

$$X^\mu(0) = X^\mu(2\pi), \tag{2.20}$$

such that the string is closed and thus no longer has end points (see figure 2.2(c)). This means that it can neither be stuck on other objects nor let momentum flow off its endpoints, but it is free to move around in the entire d-dimensional spacetime.

2.1.3 String motion

The equation of motion (2.14) can be solved in terms of functions of the world-sheet light-cone coordinates:

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma), \tag{2.21}$$

this solution can be examined by Fourier-expanding the (arbitrary) functions

$$X^\mu_L(\tau+\sigma) = \frac{x^\mu}{2} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}^\mu_0(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\bar{\alpha}^\mu_k}{k}e^{-ik(\tau+\sigma)}, \tag{2.22}$$

$$X^\mu_R(\tau-\sigma) = \frac{x^\mu}{2} + \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\alpha^\mu_k}{k}e^{-ik(\tau-\sigma)}, \tag{2.23}$$

where k is not necessarily an integer, but runs over integral steps.

To ensure the reality of $X^\mu(\tau,\sigma)$, we must demand that x^μ is real and that

$$(\alpha^\mu_k)^* = \alpha^\mu_{-k}, \quad (\bar{\alpha}^\mu_k)^* = \bar{\alpha}^\mu_{-k}. \tag{2.24}$$

Neumann boundary conditions

As mentioned above, the Neumann conditions (2.17) correspond to open strings moving freely in space. It seems clear that these will play a major part in any string theory. When we impose the Neumann conditions on the solution (2.21), we find that the motion of the string is given by:

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha^\mu_n}{n}e^{-in\tau}\cos(n\sigma). \tag{2.25}$$

Here it is worth noting a few things; first that the bars from (2.22) have disappeared, which means that the right and left moving waves on the string are not independent. This comes from the well known phenomenon from wave theory that waves are reflected at the endpoints. Second, we have introduced the momentum $p^\mu = \frac{\alpha^\mu_0}{\sqrt{2\alpha'}}$. It can be shown [15, pp. 22–23] that this is the total centre of mass momentum of the string.

Dirichlet boundary conditions

It seems less obvious to include Dirichlet strings, since these must be attached to objects that do not appear to be in the free string theory. However, as we mentioned at the end of the previous section, they are important for building a semi-realistic model. They give rise to the following motion:

$$X^\mu(\tau,\sigma) = x^\mu + \frac{y^\mu}{\pi}\sigma - \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha^\mu_n}{n}e^{-in\tau}\sin(n\sigma). \tag{2.26}$$

We see that the Dirichlet string has no momentum, but instead has a dependence on the space-like parameter σ given by the separation $y^\mu = \sqrt{2\alpha'}\pi\alpha_0^\mu$. To understand why this name is natural, we calculate the distance between the endpoints of the string

$$X^\mu(\tau,\pi) - X^\mu(\tau,0) = x^\mu + \frac{y^\mu}{\pi}\pi - x^\mu = y^\mu.$$

When we later consider situations where the separation of the string end points is significant, either because of winding around compact dimensions or distance between branes, y^μ is exactly this distance.

Periodic boundary conditions

Like the Neumann strings, free closed strings seem likely to be essential parts of any string theory, and as we will later see this is indeed the case, so we also consider these. The motion of the closed string is given by:

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\bar{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)} + \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)}. \tag{2.27}$$

We immediately see that closed strings are very different from the open strings in that the left and right moving oscillations are independent. When we impose quantization conditions, we will see that it is exactly this property that naturally introduces gravitation in string theory (see section 2.1.7). The momentum $p^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu$ is again the centre of mass momentum [15, p. 21].

2.1.4 Virasoro constraints

In section 2.1.2, we found the Virasoro constraints

$$T_{--} = \frac{1}{\alpha'}(\partial_- X)^2 = 0, \qquad T_{++} = \frac{1}{\alpha'}(\partial_+ X)^2 = 0. \tag{2.28}$$

It is convenient to consider the Fourier modes of the constraints, called the Virasoro operators. They are

$$L_m \propto \int_0^\pi \mathrm{d}\sigma [T_{--}e^{im(\tau-\sigma)} + T_{++}e^{im(\tau+\sigma)}] \tag{2.29}$$

for open strings and

$$L_m \propto \int_0^{2\pi} \mathrm{d}\sigma T_{--}e^{im(\tau-\sigma)}, \quad \bar{L}_m \propto \int_0^{2\pi} \mathrm{d}\sigma T_{++}e^{im(\tau+\sigma)} \tag{2.30}$$

for closed strings.

Since these will be sat equal to zero, we can choose whatever prefactor we wish. Using equations (2.25)-(2.27), it is a simple matter to calculate the Virasoro constraints L_m and $\bar{L}_m$ and, with the conventional choice of prefactor, they take the form

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha^\mu_{m-n} \alpha_{\mu\, n} = 0, \quad \bar{L}_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \bar{\alpha}^\mu_{m-n} \bar{\alpha}_{\mu\, n} = 0. \tag{2.31}$$

The zeroth Virasoro operator is particularly interesting since it contains an α_0^2-term. As seen above, this is related to the string centre of mass momentum, and using the mass-shell equation,

$$-p^2 = M^2, \tag{2.32}$$

we can get an expression for the string mass in terms of the oscillator modes. For the Neumann string it is:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = 0 \tag{2.33}$$

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n, \tag{2.34}$$

and for the closed string:

$$L_0 + \bar{L}_0 = \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n = 0 \tag{2.35}$$

$$M^2 = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n \right]. \tag{2.36}$$

Since both L_0 and $\bar{L}_0$ must also be zero individually, we can deduce that

$$\frac{4}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n, \tag{2.37}$$

which is known as the level-matching condition.

2.1.5 Quantization

We quantize the string by promoting position, momentum and vibration modes to operators. To determine the behaviour of these operators, we impose the canonical equal-time commutation relation:

$$[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{2.38}$$

where we have introduced

$$P^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X_\mu)} = \frac{\partial_\tau X^\mu}{2\pi\alpha'}. \tag{2.39}$$

Using (2.25)-(2.27), it is relatively simple to show that this leads to the following commutation relations for the fundamental operators:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}, \quad [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}, \tag{2.40}$$

where the barred relation of course only is relevant in the closed string case.

Classically, we had to demand that the string coordinate X^μ was a real-valued function, but quantum mechanically we have to demand that it is a hermitian operator. The reality condition on the vibrational modes (2.24) therefore becomes a hermiticity condition.

$$(\alpha_k^\mu)^\dagger = \alpha_{-k}^\mu, \quad (\bar{\alpha}_k^\mu)^\dagger = \bar{\alpha}_{-k}^\mu. \tag{2.41}$$

It is interesting to note that if we furthermore define $a_n^\mu = \frac{\alpha_n^\mu}{\sqrt{n}}$, we see that

$$[a_m^\mu, a_n^{\nu\dagger}] = \delta_{m,n}\eta^{\mu\nu}, \quad m, n > 0, \tag{2.42}$$

which is just the well-known harmonic oscillator commutation relation for an infinite set of oscillators.

We can now use our lowering operators to define a normalized vacuum state

$$\alpha_n^\mu|p\rangle = 0 \quad \forall \quad n > 0, \qquad \text{and} \quad \langle p|p\rangle = 1 \tag{2.43}$$

and our raising operators to define a Fock space of other states such as

$$\alpha_{-1}^\mu|p\rangle,\ \alpha_{-2}^\mu\alpha_{-5}^\nu|p\rangle \text{ and } \alpha_{-3}^\mu\alpha_{\mu,\,-3}|p\rangle. \tag{2.44}$$

There is, however, a problem with our newly defined Fock space; it contains ghosts, states with a negative norm. We see this explicitly by considering expectation value of the state

$$|\alpha_{-1}^0|p\rangle|^2 = \langle p|\alpha_1^0\alpha_{-1}^0|p\rangle = \langle p|(\eta^{00} + \alpha_{-1}^0\alpha_1^0)|p\rangle = -1. \tag{2.45}$$

Fortunately, we have not yet imposed the classical constraints $L_m = 0$ (see section 2.1.4), and these take care of the unphysical states [20, pp. 79–86]. Before we can impose the constraints, we must rewrite them as quantum operators. We define the quantum mechanical Virasoro operators as the normal-ordered version of the originals (2.31):

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}^\mu\alpha_{\mu n}:, \tag{2.46}$$

and since the commutator contains $\delta_{m,-n}$, the only operator that is affected by this is

$$L_0 = \frac{1}{2}\left[\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu n} + \sum_{n=1}^{\infty} \alpha_{n}^{\mu}\alpha_{\mu -n}\right] \tag{2.47}$$

$$= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu n}, \tag{2.48}$$

where we have suppressed the normal ordering constant, a. We will instead include it in the condition on the physical states (see equation (2.49)). We will consider a in detail in section 2.1.7.

The Virasoro constraints on the physical states are

$$L_{m>0}|\text{phys}\rangle = 0, \quad (L_0 - a)|\text{phys}\rangle = 0, \tag{2.49}$$

and we can find the quantum mechanical mass-shell condition

$$\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu n} - a = 0 \tag{2.50}$$

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu n} - a\right). \tag{2.51}$$

Let us pause for a moment and consider how this is changed from the classical expression (2.34). The new expression is quantized, meaning that in units of $1/\alpha'$ it is always an integer minus a as opposed to the continuum of possible classical masses. Another very important difference is that if a is a positive integer, the quantum string can have non-trivial solutions with negative or zero mass squared, whereas the classical string has manifestly non-negative mass and can only be massless in the trivial case $\alpha_n^{\mu} = 0$ for all n.

2.1.6 Light-cone quantization

We have previously had great success using world-sheet light-cone coordinates, $\xi^{\pm}$, and we will now see that spacetime light-cone coordinates can also be very useful. We define them as

$$X^{\pm} = X^0 \pm X^1. \tag{2.52}$$

When we went to the conformal gauge in section 2.1.2, we did not use up all of our gauge symmetries. In particular, if we choose a new parametrization of the form

$$\xi'^{+} = f(\xi^{+}), \quad \xi'^{-} = g(\xi^{-}), \tag{2.53}$$

the only change in the metric will be a rescaling that can be taken care of with Weyl symmetry. It is therefore possible to make sure the metric remains that of flat space. Using this residual invariance, we can set [15, p. 31]

$$X^+ = x^+ + \alpha' p^+ \tau. \tag{2.54}$$

Inserting this in the classical Virasoro constraints, we can calculate

$$\partial_\pm X^- = \frac{2}{\alpha' p^+} \partial_\pm X^i \partial_\pm X^i. \tag{2.55}$$

Equations (2.25) and (2.55) give us the α_n^--operators in terms of the transverse oscillations and p^+

$$\alpha_n^- = \frac{2}{p^+}\sqrt{\frac{2}{\alpha'}}\left[\sum_{m\in\mathbb{Z}} : \alpha_{n-m}^i \alpha_m^i : -2a\delta_{n,0}\right], \quad i = 2, \ldots, d-1, \tag{2.56}$$

where we have added the normal ordering constant introduced in equation (2.49). The α_n^+'s are obviously zero for $n \neq 0$, but it is convenient to express them generally:

$$\alpha_n^+ = \sqrt{\frac{\alpha'}{2}} p^+ \delta_{n,0}. \tag{2.57}$$

We have now expressed the vibrations in the light-cone directions entirely in terms of vibrations in the $(d-2)$ transverse directions.

A great advantage of the light-cone coordinates is that they make the theory explicitly ghost-free. It is obvious that norms like $|\alpha_{-n}^+|p\rangle|^2$ equal zero because of (2.57), and it is equally obvious that $\langle p|\alpha_n^- \alpha_{-n}^+|p\rangle$ and $\langle p|\alpha_n^+ \alpha_{-n}^-|p\rangle$ also equal zero. That $|\alpha_{-n}^-|p\rangle|^2$ vanishes comes from the fact that $[\alpha_m^-, \alpha_n^-] = 0$.

It should be noted that though we in this and the previous subsection have treated the Neumann string, entirely analogous calculations hold for the Dirichlet and closed strings.

2.1.7 Spectrum

We now return to the quantum mechanical mass-shell condition for the Neumann string

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - a\right). \tag{2.58}$$

By imposing light-cone coordinates before normal ordering, it is easy to see that only the transverse oscillations contribute to the mass.

Before going into detail with the spectrum, we wish to fix the normal ordering constant, a. It is not obvious how we should do this, but it turns out that demanding that the commutators of the Lorentz transformations vanish gives us an equation in which we can simply read off a. However, this is a very long and tedious calculation which is done rigorously elsewhere [21, pp. 59–63] and we will therefore merely state the end result,

$$[J^{i-}, J^{j-}] = \frac{1}{2(p^+)^2} \sum_{n=1}^{\infty} \left(\left[\frac{d-2}{12} - 2 \right] n + \frac{1}{n} \left[2a - \frac{d-2}{12} \right] \right) (\alpha_{-n}^{[i}\alpha_n^{j]} + \bar{\alpha}_{-n}^{[i}\bar{\alpha}_n^{j]}) = 0, \tag{2.59}$$

where $\alpha_{-n}^{[i}\alpha_n^{j]} = \alpha_{-n}^i\alpha_n^j - \alpha_{-n}^j\alpha_n^i$ and d is the number of dimensions. It is easy to see that this leaves only one possible choice of a and d, namely

$$a = 1, \qquad d = 26. \tag{2.60}$$

For this reason we call $d = 26$ the critical dimension of bosonic string theory. It is by no means impossible to do string theory in another number of dimensions, but if we do so, we lose Lorentz invariance and thus gain a gravitational anomaly.

Instead of going through the above-mentioned calculation, we will do another, much simpler, but less rigorous, calculation that gives the same result. We first recall that a comes from the commutator of α_{-n} and α_n and symbolically has the value

$$a = -\frac{1}{2} \sum_{n=1}^{\infty} [\alpha_n^i, \alpha_{-n}^i] = -\frac{1}{2} \sum_{n=1}^{\infty} n\eta^{ii} \tag{2.61}$$

which is clearly infinite. To give meaning to this quantity, we consider the Riemann ζ-function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{2.62}$$

This function is only well defined for $\Re(s) > 1$, but it has a unique analytical continuation that allows us to assign meaning to it for other values, and for $s = -1$ it is [22, p. 50]

$$\sum_{n=1}^{\infty} n \equiv \zeta(-1) = -\frac{1}{12}. \tag{2.63}$$

This method of giving meaning to otherwise divergent series is called Riemann ζ-function regularization, and by using it, we can find

$$a = -\frac{1}{2} \sum_{n=1}^{\infty} n\eta^{ii} = -\frac{1}{2}(-\frac{1}{12})(d-2) = \frac{d-2}{24}. \tag{2.64}$$

The problem with this expression is that we have merely traded one unknown constant for another, but if we consider the first excited state

$$\alpha^i_{-1}|p\rangle, \tag{2.65}$$

we can get the value of a and thus also fix the dimension.

The state (2.65) has $d-2$ independent vector components since the light-cone oscillators are completely given in terms of the transverse oscillators. This is exactly the property we, from group theory, expect of a massless spacetime vector particle.

It is therefore clear that the mass of the state (2.65) must be zero, and (2.51) then tells us that

$$a = 1 \tag{2.66}$$

and thus by (2.64) that

$$d = 26, \tag{2.67}$$

exactly as found in the result from the rigorous calculation.

To see, without using group theory, that a massless vector particle will only have $d-2$ independent components, consider the momentum of such a particle. It will a priori be a vector with d components, and the mass-shell condition tells us that

$$p^\mu p_\mu = -M^2 = 0, \tag{2.68}$$

$$p^\mu p_\mu = -p^0p^0 + p^1p^1 + p^ip^i = 0. \tag{2.69}$$

Since each of the squares must be non-negative, we can never apply a Lorentz-transformation that sets more than $d-2$ of them to zero. Therefore, for any non-trivial momentum, only $d-2$ of the components are truly independent.

Neumann string states

The mass-shell condition for the Neumann string is then

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n - 1\right), \tag{2.70}$$

and we can begin to consider the different states.

First there is the ground state

$$|p\rangle, \tag{2.71}$$

with mass

$$M^2 = -\frac{1}{\alpha'}. \tag{2.72}$$

We term states with negative M^2 tachyons and they will naturally move faster than the speed of light. Though superluminal velocities are unfamiliar and unlike anything we have observed in nature, there is another reason we should be worried about the presence of tachyons. In the language of quantum field theory [23, p. 40], we can consider the tachyons as excitations of a field, $T(X)$, and the mass-squared would be found as the second derivative of the corresponding potential

$$M^2 = \left.\frac{\partial^2 V(T)}{\partial T^2}\right|_{T=0} < 0. \tag{2.73}$$

In this case, the negative mass-squared tells us that we are expanding around a maximum of the potential, and thus that our theory is unstable. Attempts have been made to fix this problem by giving the tachyon a Higgs-like potential with a minimum away from $T = 0$, but they have so far been unsuccessful [23, p. 41]. When considering the corresponding state of the superstring, we will see that there exists a natural way of getting rid of the tachyon (section 2.3.3).

Secondly, we have the singly excited state

$$\alpha^i_{-1}|p\rangle, \tag{2.74}$$

which, as has already been established, is a massless vector boson. This single fact is enough for us to conclude that this state is in fact a photon. While the presence of the tachyon above was disheartening, the natural occurrence of a photon-like particle encourages us to carry on with string theory.

Thirdly, there are higher excited states in general, and

$$\alpha^i_{-2}|p\rangle, \quad \alpha^i_{-1}\alpha^j_{-1}|p\rangle, \tag{2.75}$$

in particular. These states are tensors of $SO(24)$, but can be combined to give a symmetric, traceless, massive $SO(25)$ rank 2 tensor [15, p. 33]. Higher excited states will be tensors of rank n and since the maximal spin of a tensor particle is $J = n$ (for information on why this is so, see the following discussion of the closed string states), we see that we can express this in terms of the mass

$$J = n = \alpha' M^2 + 1, \tag{2.76}$$

where, for historical reasons (see section 1.1), the string parameter α' is often called the *Regge slope*. This behaviour is part of what made string theory seem an attractive theory of the strong interaction [15, p. 33]. In modern string theory, α' is interpreted as the square of the string length, ℓ_s, which is closely related to the Planck length. In 10 dimensions (the critical dimension of superstring theory, see section 2.2.6) it can be shown (see [24, pp. 150 & 344] and [25]) that

$$\frac{1}{2}(2\pi)^7 g_s^2 \ell_s^8 = \frac{9\pi^{10}\ell_P^8}{2^{10}} \tag{2.77}$$

$$\ell_s = \frac{1}{4}\sqrt[8]{\frac{9\pi^3}{g_s^2}}\ell_P \approx 0.5\frac{\ell_P}{\sqrt[4]{g_s}} \tag{2.78}$$

We do not know the string coupling constant g_s, and thus cannot hope to give an exact result, but it is reasonable to assume a naive calculation of the mass of the second excited Neumann string state will not be far off. Using the four dimensional Planck length, such a calculation gives

$$M = \frac{1}{\sqrt{\alpha'}} \propto \ell_P^{-1} = M_P = \sqrt{\frac{\hbar c}{G}} = 1.22 \times 10^{19}\,\mathrm{GeV}/c^2. \tag{2.79}$$

To get a better feeling of this number, we consider that the heaviest known elementary particle, the top quark, has a mass of $M_t = 173.1\,\mathrm{GeV}/c^2$, that the LHC (as of October 1, 2010) has a center-of-mass energy of 7×10^3 GeV and that even the most energetic particles we know of, the ultra-high-energy cosmic rays, have a kinetic energy of only 10^{11} GeV [26]. It is thus clear that the massive string states are far more massive than anything we can expect to see in experiments.

Closed string states

We now move on to consider the closed string. It is similar to the open Neumann string in that the ground state is a tachyon and the massive states are far heavier than anything we have encountered in nature. We will therefore only consider the first excited states. Fortunately these have a very rich and exciting structure.

Imposing the level-matching condition (2.37), we see that the first excited state is

$$\alpha_{-1}^i \bar{\alpha}_{-1}^j |p\rangle. \tag{2.80}$$

Since $\alpha_{-1}^i \bar{\alpha}_{-1}^j$ is a general 24×24 two-tensor, it transforms in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$. However this representation is not irreducible, and thus we will not consider it a single fundamental particle. Instead, we decompose it into three parts that transform in irreducible representations:

$$\frac{1}{24}\delta^{ij}\alpha_{-1}^k \bar{\alpha}_{-1}^k|p\rangle \qquad \Phi \qquad \text{The dilaton} \tag{2.81}$$

$$\alpha_{-1}^{[i}\bar{\alpha}_{-1}^{j]}|p\rangle \qquad B^{ij} \qquad \text{The Kalb-Ramond field} \tag{2.82}$$

$$\left[\alpha_{-1}^{\{i}\bar{\alpha}_{-1}^{j\}} - \frac{1}{24}\delta^{ij}\alpha_{-1}^k \bar{\alpha}_{-1}^k\right]|p\rangle \qquad G^{ij} \qquad \text{The graviton.} \tag{2.83}$$

It may seem somewhat bold to claim that we have found the long sought-after graviton, and that it comes out naturally in our simple bosonic string theory, so let us take a moment to consider this. The fact that gravity in four dimensions goes as $\frac{1}{r^2}$ over long distances and is always attractive means that its carrier particle must be a massless spin 2 particle. Furthermore, it can be shown that any massless spin 2 particle will give rise to gravity (see [27, 2-3 and 3-1] and [23, pp. 43–44]). To see that our proposed graviton is in fact spin 2, we must consider what we, from the point of view of a theoretical physicist, mean by spin 2.

We say that an object has spin n if it transforms under Lorentz transformations as an irreducible tensor of rank n [28]. As already stated, all the proposed particles are written as irreducible representations of $SO(24)$. It is clear that since the dilaton only has a single component, it is a rank 0 tensor. Since the Kalb-Ramond field is anti-symmetric, it has fewer independent components than the graviton and therefore cannot be a tensor of higher rank than it. Furthermore since we started out with a rank 2 tensor, we cannot possibly have a rank 3 tensor. In 5 (3 transverse) dimensions, the Kalb-Ramond field has 3 independent components, the same number as a rank 1 tensor would have, in higher dimensions (such as the 26 of bosonic string theory), it has far more. It is therefore also clear that the proposed graviton is a spin 2 particle and thus able to play the role of the graviton.

A simpler way of seeing that a symmetric two-tensor field gives rise to gravity is to observe that the metric is a symmetric two-tensor. Any such field would then perturb the metric away from that of flat space, effectively curving spacetime and thus giving rise to gravitational effects.

The dilaton and the Kalb-Ramond field are very interesting objects in their own rights, but since we are mainly interested in using string theory to recreate the Standard Model, we will not consider them in any detail here. When we construct the beginnings of a semi-realistic model in chapter 5, we will include the Kalb-Ramond field as a background.

2.2 The superstring action

We have now quantized the simple bosonic string and examined its spectrum in detail. To move on, we need to examine what happens when we introduce a fermionic contribution. To do this we consider the superstring Lagrangian first derived by Brink, di Vecchia and Howe [16], and independently by Deser and Zumino [17]. In our notation, the action is:

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\xi^2 \sqrt{-\det g}\, \eta_{\mu\nu} \Big[& g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu + i e^\alpha_a \bar{\psi}^\mu \rho^a \partial_\alpha \psi^\nu \\ & - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu (\partial_\beta X^\nu - \frac{i}{4}\bar{\chi}_\beta \psi^\nu) \Big]. \end{aligned} \tag{2.84}$$

Here e_a^α is the two-dimensional vielbein which translates curved coordinates α to flat coordinates a, ρ^a is a two-dimensional Dirac matrix, $\rho^\alpha = e_a^\alpha \rho^a$, ψ^μ is the world-sheet fermion field, $\bar{\psi}^\mu = \psi^\mu \rho^0$ and χ_α is the supersymmetric partner of $g_{\alpha\beta}$ called the gravitino. It should be noted that both the gravitino χ_α and the fermion ψ^μ are anti-commuting Grassman variables to ensure that the Pauli principle is obeyed.

2.2.1 Symmetries of the action

Just as in the bosonic case, we wish to consider the symmetries of the action. Since the Polyakov action (2.2) is just a simplification of (2.84), they share Poincaré and reparametrization invariance, but the Weyl symmetry has been extended to super-Weyl symmetry [22, p. 118] and, as the name implies, the superstring exhibits world-sheet supersymmetry.

Supersymmetry

That the action (2.84) is world-sheet supersymmetric means that it is invariant under a transformation that turns fermionic coordinates into bosonic ones and vice-versa. This is a new kind of symmetry, and it has a lot of very appealing effects that we will see later. Formally, the world-sheet supersymmetry transformations are:

$$\begin{gathered}\delta g_{\alpha\beta} = i\bar{\varepsilon}(\rho_\alpha \chi_\beta + \rho_\beta \chi_\alpha), \quad \delta e_\alpha^a = i\bar{\varepsilon}\rho^a \chi_\alpha, \quad \delta\chi_\alpha = 2\nabla_\alpha \varepsilon \\ \delta X^\mu = i\bar{\varepsilon}\psi^\mu, \quad \delta\psi^\mu = \rho^\alpha\big(\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha \psi^\mu\big)\varepsilon,\end{gathered}$$

where ε is an arbitrary Majorana world-sheet spinor and $\bar{\varepsilon} = \varepsilon\rho^0$.

2.2.2 Equations of motion

As before, we want to find the equations of motion. To do this, we go to the superstring equivalent of the conformal gauge, naturally called the superconformal gauge

$$g_{\alpha\beta} = e^\phi \eta_{\alpha\beta}, \quad \chi_\alpha = \rho_\alpha \zeta, \tag{2.85}$$

where ζ is a constant Majorana spinor.

Using the identity $\rho_\alpha \rho^\beta \rho^\alpha = 0$ it is easy to show that this gauge, the action takes the much simpler form

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\xi^2 \eta_{\mu\nu} \Big[\partial^\alpha X^\mu \partial_\alpha X^\nu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu\Big], \tag{2.86}$$

and we see that the bosonic and fermionic parts decouple completely. Since the bosonic string has already been treated in detail, we will now only consider the fermionic string.

Using variational calculus gives us, unsurprisingly, the Dirac equation

$$\rho^\alpha \partial_\alpha \psi^\mu = 0, \tag{2.87}$$

and the boundary condition

$$(\eta_{\mu\nu} \delta \bar{\psi}^\mu \rho_1 \psi^\nu)|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.88}$$

As in the bosonic case, it is very convenient to introduce the light-cone coordinates (2.13) and their derivatives. The equations are further simplified if we consider left and right-handed world-sheet spinors

$$\psi_\pm^\mu = \frac{1 \mp \rho^3}{2} \psi^\mu, \quad \rho^3 = \rho^0 \rho^1. \tag{2.89}$$

The Dirac equation then splits up into the Weyl equations:

$$\partial_- \psi_+^\mu = \partial_+ \psi_-^\mu = 0, \tag{2.90}$$

and we can rewrite the boundary condition in terms of the Weyl coordinates:

$$(\psi_+ \delta \psi_+ - \psi_- \delta \psi_-)|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.91}$$

Though we have found the equations of motion for both the bosonic and fermionic fields, we are not finished since the addition of fermions complicates the stress tensor and we have introduced another new field, the gravitino. This new stress tensor can be calculated using the same method as before [22, p. 119],

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{-\det g}} \frac{\delta S_P}{\delta g^{\alpha\beta}} = \frac{1}{\alpha'} \left[\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{4} \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \frac{1}{4} \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (\text{trace}) \right], \tag{2.92}$$

which using light-cone coordinates can be rewritten in a simpler form

$$T_{++} = \frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2\alpha'} \psi_+^\mu \partial_+ \psi_{+\,\mu}, \tag{2.93}$$

$$T_{--} = \frac{1}{\alpha'} \partial_- X^\mu \partial_- X_\mu + \frac{i}{2\alpha'} \psi_-^\mu \partial_- \psi_{-\,\mu}. \tag{2.94}$$

From performing a variation of the gravitino field, we get the supercurrent [20, p. 231]:

$$J_\alpha = -\frac{\pi}{2\sqrt{-\det g}} \frac{\delta S}{\delta \chi^\alpha} = \frac{1}{2\alpha'} \rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu, \tag{2.95}$$

and again we can use the light-cone coordinates to write them more conveniently.

$$J_+ = \frac{1}{\alpha'}\psi_+^\mu \partial_+ X_\mu, \quad J_- = \frac{1}{\alpha'}\psi_-^\mu \partial_- X_\mu. \tag{2.96}$$

Since both the stress tensor and the supercurrent were found from a variation of the action, the principle of least action tells us that they must be zero, so it is easy to see that the super-Virasoro constraints are

$$T_{++} = T_{--} = J_+ = J_- = 0. \tag{2.97}$$

An alternate way of deriving the supercurrent is to use Noether's first theorem. This states that for every symmetry of the action there is a related conserved current which can be found by applying the symmetry transformations (see [22, pp. 37–38]). Had we done this, the supercurrent would have come from world-sheet supersymmetry.

Fermionic boundary conditions

With the equations of motion well in hand, we will return to the boundary condition found above. In the open string case, we see that the left and right-moving vibrational modes are dependent on each other since the conditions (2.91) are satisfied when

$$\begin{cases} \psi_-(\tau, 0) = \eta_1 \psi_+(\tau, 0), \\ \psi_-(\tau, \pi) = \eta_2 \psi_+(\tau, \pi), \end{cases} \tag{2.98}$$

where the η's can take the values ± 1. The overall sign is insignificant, and we therefore have two distinct cases. First, the Ramond (R) sector when $\eta_1 = \eta_2$ and secondly, the Neveu-Schwarz (NS) sector when $\eta_1 = -\eta_2$.

When considering the closed string we are delighted to find that the basic structure from the bosonic case reappears, since the left and right-moving functions are independent

$$\psi_-(\tau, 0) = \eta_3 \psi_-(\tau, 2\pi), \quad \psi_+(\tau, 0) = \eta_4 \psi_+(\tau, 2\pi). \tag{2.99}$$

Because the vibrations are independent, the overall sign matters and we are left with four distinct sectors

$$\begin{cases} \eta_3 = \eta_4 = 1 & \text{R-R}, \\ \eta_3 = -\eta_4 = 1 & \text{R-NS}, \\ \eta_3 = -\eta_4 = -1 & \text{NS-R}, \\ \eta_3 = \eta_4 = -1 & \text{NS-NS}. \end{cases} \tag{2.100}$$

2.2.3 Superstring motion

We once again use that the bosonic and fermionic parts of the action are independent and therefore only consider the fermionic motion here. As in the bosonic case, we consider the left and right-moving parts separately, but here it is very important to realise that while the total bosonic motion was given by the sum of the parts, the fermionic is given by the spinor composed of the different parts

$$\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}. \tag{2.101}$$

To get expressions for the left- and right-moving components, we will once again consider an oscillator expansion, this time with anti-commuting coefficients b^μ_k:

$$\psi^\mu_+(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_k \bar{b}^\mu_k e^{-ik(\tau+\sigma)} \tag{2.102}$$

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_k b^\mu_k e^{-ik(\tau-\sigma)}. \tag{2.103}$$

Open fermionic strings

We will first consider the open string case, where the boundary conditions are given by equation (2.98). As mentioned above, the overall sign is unimportant, and we will therefore start by imposing

$$\psi_-(\tau,0) = \psi_+(\tau,0), \tag{2.104}$$

from which we immediately see that the left- and right-moving oscillation modes b^μ_k and $\bar{b}^\mu_k$ are identical.

When imposing the Ramond condition,

$$\psi_-(\tau,\pi) = \psi_+(\tau,\pi), \tag{2.105}$$

we furthermore see that the summation index must be an integer, and thus the motion of the Ramond string is given by:

$$\psi_\pm = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} b^\mu_n e^{-in(\tau\pm\sigma)}. \tag{2.106}$$

Conversely, the Neveu-Schwarz condition,

$$\psi_-(\tau,0) = -\psi_+(\tau,0), \tag{2.107}$$

gives a half-integer summation index such that

$$\psi_\pm = \sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-ir(\tau\pm\sigma)}. \tag{2.108}$$

Closed fermionic strings

The calculations for the closed string are almost identical to the ones for the open string. The only real difference between the boundary conditions are that the open strings have one set of independent oscillator modes while the closed string has two. The closed string motion can thus be written as

$$\psi_+(\tau,\pi) = \sqrt{\frac{\alpha'}{2}} \sum_{k\in\mathbb{Z}+\frac{x}{2}} \bar{b}^\mu_k e^{-2ik(\tau+\sigma)} \tag{2.109}$$

$$\psi_-(\tau,\pi) = \sqrt{\frac{\alpha'}{2}} \sum_{k\in\mathbb{Z}+\frac{y}{2}} b^\mu_k e^{-2ik(\tau-\sigma)}, \tag{2.110}$$

with $x = 0$ ($y = 0$) if the left-moving (right-moving) oscillations are in the Ramond sector, and $x = 1$ ($y = 1$) if they are in the Neveu-Schwarz sector†.

2.2.4 Super-Virasoro constraints

As in the bosonic case, we now wish to consider the Fourier-modes of the super-Virasoro constraints found at the end of section 2.2.2. For the open superstring, the super-Virasoro operators are defined as [20, p. 199]

$$L_m \propto \int_0^\pi \mathrm{d}\sigma[e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}] \tag{2.111}$$

$$G_k \propto \int_0^\pi \mathrm{d}\sigma[e^{ik\sigma}J_+ + e^{-ik\sigma}J_-], \tag{2.112}$$

where we have also included the Fourier modes of the supercurrent since they can be used in a completely analogous way. For the closed superstring, there are completely equivalent expressions for the right-moving oscillation modes.

It is straightforward to calculate the super-Virasoro and supercurrent modes using the expressions for X^μ, (2.25), and ψ^μ, (2.106) and (2.108), as well as standard techniques for integrating exponentials and sums. The results are

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{-n}\alpha_{m+n\,\mu} + \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{x}{2}} \left(k + \frac{m}{2}\right) b^\mu_{-k} b_{m+k\,\mu}, \qquad m \in \mathbb{Z} \tag{2.113}$$

$$G_k = \sum_{n\in\mathbb{Z}} \alpha^\mu_{-n} b_{k+n\,\mu}, \qquad k \in \mathbb{Z} + \frac{x}{2}, \tag{2.114}$$

where $x = 0$ in the Ramond sector and $x = 1$ in the Neveu Schwarz sector.

†Note that for fermionic oscillators, we use index k if its value is undetermined, n if it is an integer and r if it is a half-integer.

2.2.5 Quantization

We have already quantized the bosonic string by imposing the canonical equal-time commutation relation,

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{2.115}$$

Using this we were able to calculate the commutation relations for the position, momentum and vibrational modes of the string

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\mu] = m\eta^{\mu\nu}\delta_{m,-n}. \tag{2.116}$$

The procedure for quantizing the fermionic part of the superstring is completely analogous, we impose the canonical anti-commutation relation

$$\{\psi_A^\mu(\tau,\sigma), \psi_B^\nu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta(\sigma-\sigma')\delta_{A,B}, \tag{2.117}$$

where A and B are the spinor indices. They can take the values $+$ and $-$, and we define the δ-function of them to be $\delta_{++} = \delta_{--} = 1$ and $\delta_{+-} = \delta_{-+} = 0$.

Imposing this condition and using equations (2.106) and (2.108), it is simple to show that the anti-commutators for the oscillation modes are just those of the a infinite number of fermionic oscillators,

$$\{b_k^\mu, b_l^\mu\} = \eta^{\mu\nu}\delta_{k,-l}, \quad k,l \in \mathbb{Z} + \frac{x}{2}, \tag{2.118}$$

where $x = 0$ ($x = 1$) in the Ramond (Neveu-Schwaz) sector. An equivalent relation of course holds for the barred modes in case of the closed string.

It is interesting to consider a particular case of these, namely the Ramond sector zero-mode operator, b_0^μ. It has the anti-commutator $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$, which is proportional to that of the spacetime Dirac matrices. We can therefore write

$$b_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu. \tag{2.119}$$

The Fock space vacuum we must introduce in this quantum theory is one that is annihilated by both the bosonic and fermionic modes,

$$\alpha_n^\mu|p\rangle = b_k^\mu|p\rangle = 0, \quad n,k > 0. \tag{2.120}$$

Excited modes can be created by acting with the creation operators α_{-n}^μ and b_{-k}^μ.

Let us start by checking if the superstring Fock space contains ghosts in the same way as the bosonic string did by calculating

$$|b_{-1}^0|p\rangle|^2 = \langle p|b_1^0 b_{-1}^0|p\rangle = \langle p|(\eta^{00} - b_{-1}^0 b_1^0)|p\rangle = -1, \tag{2.121}$$

and to our chagrin, they are present. Since the bosonic string theory also included ghosts, we now have two sets of such unphysical states. To make sure they are not part of our final spectrum, we must use two different symmetries to eliminate them. Fortunately, we have two such symmetries at our disposal since we have added supersymmetry to our system, so by imposing both the super-Virasoro constraint and that the supercurrent must vanish, we are left with a spectrum of non-negative norm states [20, pp. 202–205].

We now return to the super-Virasoro condition, and impose normal ordering on the super-Virasoro operators.

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha^{\mu}_{-n}\alpha_{m+n\,\mu} : +\frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{x}{2}} k : b^{\mu}_{-k}b_{m+k\,\mu} :, \qquad m \in \mathbb{Z} \tag{2.122}$$

$$G_k = \sum_{n\in\mathbb{Z}} \alpha^{\mu}_{-n}b_{k+n\,\mu}, \qquad k \in \mathbb{Z} + \frac{x}{2}, \tag{2.123}$$

where we have restated the supercurrent operators for completeness even though they are unchanged by normal ordering.

As in the bosonic case, it is easy to see that this is only important for L_0 which gains a normal ordering constant. The quantum super-Virasoro constraints are thus formally unchanged by supersymmetry, but we must remember to also include the supercurrent modes

$$G_{k\geq 0}|\text{phys}\rangle = 0, \quad L_{m>0}|\text{phys}\rangle = 0, \quad (L_0 - a)|\text{phys}\rangle = 0. \tag{2.124}$$

The mass formula is easily found from the last condition,

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha_{n\,\mu} + \sum_{k>0} kb^{\mu}_{-k}b_{k\,\mu} - a\right), \quad k \in \mathbb{Z} + \frac{x}{2}. \tag{2.125}$$

It should be noted that we expect that a will have different values in the Ramond and Neveu-Schwarz sectors.

2.2.6 Light-cone quantization

We have already seen how we can use leftover gauge symmetry to gauge away superfluous degrees of freedom in the motion of the bosonic string (section 2.1.6). When we do this, the light-cone coordinates, X^+ and X^-, are given completely in terms of the transverse motion X^i. We naturally want to do the same for the fermionic coordinates, therefore introduce[‡]

$$\psi^{\pm} = \psi^0 \pm \psi^1. \tag{2.126}$$

[‡]It should be noted that $\psi^{\pm}$ have nothing to do with $\psi_{\pm}$, the first are spacetime vector components and the second spinor components.

With this choice of coordinates, we can gauge away all the oscillation modes such that [20, p. 211]

$$\psi^+ = 0, \tag{2.127}$$

which is a consistent choice since the supersymmetry transformation $\delta X^+ = \bar{\varepsilon}\psi^+ = 0$ tells us that it does not change the bosonic light-cone coordinates.

It is now easy to solve the other light-cone coordinate in terms the transverse oscillations. Equations (2.93) and (2.96) tell us that

$$\partial_+ X^- = \frac{1}{\alpha' p^+}(\partial_+ X^i \partial_+ X^i + \frac{i}{2}\psi^i \partial_+ \psi^i) \tag{2.128}$$

$$\psi^- = \frac{2}{\alpha' p^+}\psi^i \partial_+ X^i. \tag{2.129}$$

From these equations it is now a simple, but bothersome, matter to extract expressions for the oscillation modes α_n^- and b_k^-. They take the form

$$\alpha_n^- = \frac{1}{2p^+}\left(\sum_{m\in\mathbb{Z}} :\alpha_{n-m}^i \alpha_m^i: + \sum_k (r - \frac{n}{2}) : b_{n-k}^i b_k^i : - a\delta_{n,0}\right), \quad k \in \mathbb{Z} + \frac{x}{2} \tag{2.130}$$

$$b_k^- = \frac{1}{p^+}\sum_l \alpha_{k-l}^i b_l^i, \qquad k, l \in \mathbb{Z} + \frac{x}{2}. \tag{2.131}$$

From this it is possible to construct the commutators of the Lorentz transformations, and by demanding that they vanish one can derive a result equivalent to equation (2.59). Using this one can show that the critical dimension of the superstring theory is $d = 10$ and that the Neveu-Schwarz normal ordering constant is $a_{NS} = \frac{1}{2}$ [20, pp. 212–213]. The Ramond sector normal ordering constant is in fact much simpler to find and we will do so in section 2.3.2.

2.3 Superstring spectrum

We now come to the culmination of this chapter, the superstring spectrum is what all the previous calculations have prepared us to find so that we may answer the question: Which states does superstring theory predict?

2.3.1 The Neveu-Schwarz sector

To answer this question, we first consider the Neveu-Schwarz sector. We recall the mass formula and normal ordering constant found above;

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2}\right). \tag{2.132}$$

The ground state is the one annihilated by all positive oscillation modes,

$$\alpha_n^i |p\rangle_{NS} = b_r^i |p\rangle_{NS} = 0, \quad n, r > 0. \tag{2.133}$$

This makes it very easy to calculate its mass, since this is simply the normal ordering constant

$$M^2 = -\frac{1}{2\alpha'}. \tag{2.134}$$

Just like in the bosonic case, we find that the ground state is a scalar tachyon. We have already dealt with why this is a very bad thing for our theory (see section 2.1.7) and will not do so again, but merely say that we will later see that spacetime supersymmetry saves us from the tachyon and ensures a stable vacuum (section 2.3.3).

To create the first excited state we have to choose between the two kinds of operators we have at our disposal. Looking at the mass formula, it is clear that acting with α_{-1}^i will produce a state of mass $M^2 = \frac{1}{2\alpha'}$ and $b_{-1/2}^i$ one of mass $M^2 = 0$. We therefore consider

$$b_{-1/2}^i |p\rangle_{NS} \tag{2.135}$$

and find it to be a spacetime vector, since the ground state was a scalar. Since we are in the light-cone gauge, we know that it has only $d-2$ independent degrees of freedom, which is consistent with what we expect of a massless vector.

In complete analogy with section 2.1.7, we could have used this argument to determine the Neveu-Schwarz normal ordering constant, and then by the same logic found that the critical dimension was $d = 10$.

2.3.2 The Ramond sector

Before going into detail with the Ramond sector, we must determine its normal ordering constant. To do this we recall heuristic calculation of the bosonic normal ordering constant (see section 2.1.7) and realise that

$$a_R = -\frac{1}{2}\sum_{n=1}^{\infty}\left([\alpha_n^i, \alpha_{-n}^i] - n\{b_n^i, b_{-n}^i\}\right) = -\frac{1}{2}\sum_{n=1}^{\infty}(n-n)\eta^{ii} = 0. \tag{2.136}$$

Note that this expression is sufficiently rigorous. Since we get the zero before taking the sum, we need not worry about how valid ζ-function regularization is.

The Ramond sector mass formula is therefore

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{n=1}^{\infty} n b_{-n}^i b_n^i\right). \tag{2.137}$$

Again we consider a ground state annihilated by any annihilation operator

$$\alpha_n^i|p\rangle_R = b_n^i|p\rangle_R = 0, \quad n > 0. \tag{2.138}$$

However, we cannot be sure that such a state obeys $G_0|p\rangle_R = 0$ and we must therefore also impose this,

$$\sum_{n\in\mathbb{Z}} \alpha_{-n}^\mu b_{n\,\mu}|p\rangle_R = \left[\sum_{n=1}^{\infty}(\alpha_{-n}^\mu b_{n\,\mu} + b_{-nu}^\mu \alpha_{n\,\mu}) + \alpha_0^\mu b_{0\,\mu}\right]|p\rangle_R = 0 \tag{2.139}$$

$$\Gamma^\mu p_\mu |p\rangle_R = 0. \tag{2.140}$$

which is just the 10-dimensional Dirac equation.

Since the normal ordering constant was zero, it is easy to see that the Ramond sector ground state is a massless spacetime spinor. A priori, such a spinor has $2^{d/2} = 32$ complex components corresponding to 64 physical degrees of freedom, but we will return to this number in section 2.3.4 and see how it is reduced to 8.

2.3.3 The GSO projection

We have now found the ground states in both the Neveu-Schwarz and Ramond sectors, but there are still significant issues with our theory. First and foremost is the presence of a tachyonic state. We therefore wish to truncate the spectrum by projecting out the tachyon. A second issue is that we have anti-commuting operators, $b_{-n/2}^i$ that can act on a bosonic state such as $b_{-1/2}^i|p\rangle_{NS}$ and produce a new bosonic state. This is contrary to our physical intuition since we associate bosons with commuting operators. A third cause for worry is that the state $\alpha_{-1}^i|p\rangle_{NS}$ is clearly massive, but has only 8 independent vector components.

Fortunately, there exists a way of removing these problems simultaneously. This is called the GSO projection [8] and to use it, we introduce the G-parity operators defined as

$$G_{NS} = (-1)^{F+1} = (-1)^{\sum_{r=1/2}^{\infty} b_{-r}^i b_r^i + 1} \tag{2.141}$$

$$G_R = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} b_{-n}^i b_n^i}, \tag{2.142}$$

where

$$\Gamma_{11} = \Gamma_0\Gamma_1\dots\Gamma_9 \tag{2.143}$$

is just the ten-dimensional equivalent of the fifth Dirac matrix γ_5 known from four-dimensional relativistic quantum mechanics.

If we now project out NS states that do not have even G-parity,

$$G_{NS}|\phi\rangle_{NS} = |\phi\rangle_{NS}, \tag{2.144}$$

we have made a truncation of the spectrum since all states without an odd number of $b^i_{-n/2}$ operators are projected out. It is easy to check that since the tachyonic ground state contains no b^i_{-r}-operators,

$$G_{NS}|p\rangle_{NS} = (-1)|p\rangle_{NS} = -|p\rangle_{NS}, \tag{2.145}$$

it is projected out. Furthermore, we see that if we act on an NS state with even G-parity, such as the first excited state, with an anti-commuting operator $b^i_{-n/2}$, it too is projected out

$$G_{NS}b^i_{-n/2}b^j_{-1/2}|p\rangle_{NS} = (-1)^3|p\rangle_{NS} = -|p\rangle_{NS}. \tag{2.146}$$

In the Ramond sector, it is a matter of convention if one chooses to project out the positive or negative states as both projections can be shown to give a consistent spectrum [22, pp. 135–136].

2.3.4 Space-time supersymmetry

With the GSO-projection, the Neveu-Schwarz and Ramond sectors only contain states of exactly the same masses since all states with half-integer masses are projected out. This, together with the fact that our Lagrangian (2.84) exhibits world-sheet supersymmetry, leads us to suspect that we may also find spacetime supersymmetry.

To look for spacetime supersymmetry, we consider again the number of independent degrees of freedom in the Ramond ground state. As mentioned above, the a priori number is $2^{D/2} = 32$ complex components, but when we impose the Majorana condition that these must be real, we have already removed half of them [20, pp. 218–220]. Furthermore, since demanding

$$G_R|p\rangle_R = |p\rangle_R \quad \text{or} \quad G_R|p\rangle_R = -|p\rangle_R, \tag{2.147}$$

is equivalent with

$$\Gamma_{11}|p\rangle_R = |p\rangle_R \quad \text{or} \quad \Gamma_{11}|p\rangle_R = -|p\rangle_R, \tag{2.148}$$

which is the Weyl condition of only accepting states with either positive or negative chirality, the GSO projection removes another half.

We have also seen how the Ramond sector ground state must obey the ten-dimensional massless Dirac equation (2.140) which relates half the remaining components with the other half [20, p. 221]. We are thus left with 8 independent degrees of freedom which is exactly the same number as we found in the Neveu-Schwarz sector. Further indication of spacetime supersymmetry is that it can be shown that the closed string sector contains a gravitino, the quantum of local supersymmetry [22, p. 134]. A theory that contains a gravitino is only consistent, if it exhibits spacetime supersymmetry.

Higher mass levels

We have thus far only shown that the number of bosons and fermions are equal at the lowest mass level, and to have a chance of supersymmetry, we must also do so generally. It is clear that in the Ramond sector we will see states of mass $\alpha' M^2 = n$ for all integer n since both α_n and b_n raise the mass of the state by one. In the Neveu-Schwarz sector, the same holds true because the GSO projection removes any state with an even number of b_r operators. We begin by considering the Neveu-Schwarz sector and call the number of degrees of freedom at each mass level $d_{NS}(n)$. We can now construct a function [20, p. 223]

$$f_{NS}(w) = \sum_{n=0}^{\infty} d_{NS}(n) w^n \tag{2.149}$$

$$= \frac{1}{\sqrt{w}} \mathrm{Tr}\left[\frac{1}{2}(1 + G_{NS} w^N)\right], \quad N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i. \tag{2.150}$$

This function has the convenient property that when written as a polynomial, we can easily read off the number of states at each mass level. The trace can be evaluated, and is

$$f_{NS}(w) = \frac{1}{2\sqrt{w}} \left[\prod_{m=1}^{\infty}\left(\frac{1 + w^{m-1/2}}{1 - w^m}\right)^8 - \prod_{m=1}^{\infty}\left(\frac{1 - w^{m-1/2}}{1 - w^m}\right)^8\right]. \tag{2.151}$$

The Ramond sector degeneracy function has a simpler form, namely [20, p. 224],

$$f_R(w) = \sum_{n=0}^{\infty} d_R(n) w^n \tag{2.152}$$

$$= \mathrm{Tr}\left[\frac{1}{2}(1 + G_R) w^N\right] = 8\,\mathrm{Tr} w^N, \quad N = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + n d_{-n}^i d_n^i). \tag{2.153}$$

This trace too can be evaluated and gives

$$f_R(w) = 8 \prod_{m=1}^{\infty}\left(\frac{1 + w^m}{1 - w^m}\right)^8. \tag{2.154}$$

Now we have two expressions that at first glance do not seem likely to be equal, and furthermore look horribly complicated to evaluate exactly. Fortunately, one of the greatest coincidences in the history of string theory is that in 1829, Jacobi actually proved that these two products are equal, and that

$$F(w) = f_{NS}(w) = f_R(w) = 8(1 + 16w + 144w^2 + \ldots). \tag{2.155}$$

While this in itself is not enough to prove spacetime supersymmetry at all levels, it is a necessary condition.

It turns out to be extremely involved to prove spacetime supersymmetry in this formulation of the superstring [24, pp. 25–29]. To have manifest spacetime supersymmetry, it is necessary to use the Green-Schwarz formulation of the superstring [20, Chapter 5], however, this formulation is very unwieldy to work with [24, p. 29].

2.3.5 The closed superstring

When working with the closed string, we have to consider both left- and right moving oscillators, each of which can satisfy either Ramond or Neveu-Schwarz conditions. It thus follows naturally that the closed string ground state is the tensor product of two open string ground states. The GSO projection in the NS sector removed the tachyon and left us with a massless ground state, and in the R sector it forced us to choose between positive and negative G-parity. In the case of the open string this choice was insignificant, but when we have to choose two such ground states, it matters if they are equal or opposite. We denote the Ramond state with positive G-parity $|+\rangle_R$ and the other $|-\rangle_R$, and see that the ground states of type IIA superstring theory are

$$|+\rangle_R \otimes |-\rangle_R \tag{2.156}$$

$$|+\rangle_R \otimes \bar{b}^j_{-1/2}|p\rangle_{NS} \tag{2.157}$$

$$b^i_{-1/2}|p\rangle_{NS} \otimes |-\rangle_R \tag{2.158}$$

$$b^i_{-1/2}|p\rangle_{NS} \otimes \bar{b}^j_{-1/2}|p\rangle_{NS}. \tag{2.159}$$

Since each of the open string states have 8 independent degrees of freedom, each of the closed string states will have 64. In particular, the R-R and NS-NS sectors gives us 128 bosonic degrees of freedom and the R-NS and NS-R sectors give us 128 fermionic degrees of freedom.

If we instead choose the R sector ground states to have equal G-parity, we get type IIB superstring theory with ground states

$$|+\rangle_R \otimes |+\rangle_R \tag{2.160}$$

$$|+\rangle_R \otimes \bar{b}^j_{-1/2}|p\rangle_{NS} \tag{2.161}$$

$$b^i_{-1/2}|p\rangle_{NS} \otimes |+\rangle_R \tag{2.162}$$

$$b^i_{-1/2}|p\rangle_{NS} \otimes \bar{b}^j_{-1/2}|p\rangle_{NS}. \tag{2.163}$$

Superficially, these states look very similar to those of type IIA theory, but it is very important to realise that the IIA theory is parity invariant and the IIB theory is not. Furthermore, they have different field content in the RR-sector which we will discuss below.

When the spectrum is analysed in detail [22, pp. 137–138], we find that the dilaton, Kalb-Ramond field and graviton from the bosonic theory appear in the

NS-NS sector and that their supersymmetric partners, the dilatino and the gravitino appear in the R-NS and NS-R sectors. However, in Type IIA theory, the gravitinos have opposite chirality due to the different chiralities of their Ramond sectors.

The R-R sector contains several bosonic p-form gauge potentials $C_{\mu_a \cdots \mu_p}$. And here the greatest difference between type IIA and type IIB shows up. In type IIA these fields are C_μ and $C_{\mu\nu\rho}$ whereas type IIB has the fields C, $C_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$, where the field-strength of the last field must be self-dual.

These fields are of great significance in string theory, particularly when considering branes in their own right. When constructing a semi-realistic model, they do not play the part of any Standard Model particles, but $C_{\mu\nu}$ turns out to be crucial for getting an anomaly-free theory and ensuring that we end up with only the gauge group $SU(3) \times SU(2) \times U(1)$ (see sections 7.2.2 and 7.3.2).

Chapter 3

D-branes

We have seen in the previous chapter that superstring theory is only Lorentz invariant in 10 dimensions. If we are to construct an even remotely realistic theory from this, we need an effective theory that lives in only 4 dimensions. There are two obvious ways of doing that. First, suppose we do not live in the entire universe, but merely on a 3-dimensional surface within it. Second, suppose that the surplus dimensions are small and compact. That a dimension is compact means that it is possible to move a finite distance in one of these dimensions, and return to one's origin.

A useful image to visualize small compact dimensions is to consider a ball moving on a plane. It is free to move in both horizontal directions. If we fold the plane, we turn it into a pipe and the ball now moves on its inside surface. If the diameter of the ball is much smaller than the diameter of the pipe, it can move forward and around the circumference and we call the dimension compact. But if we now reduce the pipe's diameter until it exactly matches the diameter of the ball, it can only move forwards and backwards, the compact dimension no longer seems to be there, and we have confined the ball to move in only one dimension. We call this process compactification.

The first way of reducing the number of dimensions requires us to add some 3-dimensional surface to the theory. It seems rather ad-hoc to just introduce new objects in the theory to make it match our expectations. We would much prefer it if we could get a 4-dimensional effective theory with what is already there. This is why the second way is better, since we do not need to introduce new things in the theory. Compactification is merely manipulation of the dimensions that are already there. Surprisingly, it turns out that small compact dimensions naturally give rise to lower-dimensional surfaces within string theory through a mechanism called T-duality.

Though the model we wish to construct in the end will be based on superstring theory, we will first consider how branes occur in bosonic string theory. This is to keep the emphasis on T-duality, the mechanism that gives rise to branes, and not on the complications superstring theory causes. Furthermore, in the following we will not consider large compact dimensions, so whenever compact dimensions are mentioned they are assumed to be small.

3.1 T-duality

3.1.1 Closed strings

We will first consider how compactified dimensions change the closed string. Since the compactified dimensions change the boundary conditions, we will use the general solution of the equation of motion given by equations (2.21)-(2.23) and impose that the summation index must be an integer.

We can then write

$$X^\mu(\tau,\sigma) = x^\mu + \bar{x}^\mu + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \bar{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \bar{\alpha}_0^\mu)\sigma \\ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \left[\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)} + \frac{\bar{\alpha}_n^\mu}{n}e^{-in(\tau+\sigma)}\right], \quad (3.1)$$

and define the physical centre of mass momentum

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \bar{\alpha}_0^\mu). \quad (3.2)$$

If we now compactify one or more dimension on circles, we have a periodicity condition for some of the spacetime coordinates:

$$X^a = X^a + 2\pi R^a. \quad (3.3)$$

This changes two things in relation to the non-compact case. First, we recall that the momentum is the generator of translations, and thus that

$$e^{ip_b q^b} X^a e^{-ip_b q^b} = X^a + q^a, \quad (3.4)$$

if we now set $q^a = 2\pi R^a$, the exponentials must equal unity and we see that

$$p_a = \frac{n}{R^a}, \quad n \in \mathbb{Z}, \quad (3.5)$$

these quantized momentum modes are called Kaluza-Klein modes [29, p. 19].

Secondly, since we now demand that $X^a(\tau, \sigma + 2\pi) = X^a(\tau,\sigma) + 2w\pi R^a$, we see that the requirement on the σ-prefactor is

$$2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^a - \bar{\alpha}_0^a) = 2\pi w R^a, \quad w \in \mathbb{Z}, \quad (3.6)$$

where w is the number of times the closed string is wound around the compact direction.

We can now write the α_0's in terms of the quantized momentum and the winding number

$$\alpha_0^a = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R_a} + \frac{wR^a}{\alpha'}\right), \quad \bar{\alpha}_0^a = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R_a} - \frac{wR^a}{\alpha'}\right). \tag{3.7}$$

If we have only one compact dimension, this gives the new Virasoro operators

$$L_0 = \frac{\alpha'}{4}\hat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{-n\mu} \tag{3.8}$$

$$\bar{L}_0 = \frac{\alpha'}{4}\hat{p}^2 + \frac{\alpha'}{4}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)^2 + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^{\mu}\bar{\alpha}_{-n\mu}, \tag{3.9}$$

where the $\hat{p}^2$ only includes the non-compact dimensions. The mass operator then becomes

$$M^2 = \frac{4}{\alpha'}\left[\sum_{n=1}^{\infty}(\alpha_{-n}^{\mu}\alpha_{-n\mu} + \bar{\alpha}_{-n}^{\mu}\bar{\alpha}_{-n\mu}) + \left(\frac{n\sqrt{\alpha'}}{R}\right)^2 + \left(\frac{wR}{\sqrt{\alpha'}}\right)^2\right]. \tag{3.10}$$

In this mass operator, there are two new terms. The first of these, $(n\sqrt{\alpha'}/R)^2$, comes from higher Kaluza-Klein modes and simply corresponds to a higher momentum. The second is less intuitive, it comes from the winding of the string. The reason this gives a mass contribution is that the string has a non-zero tension $T = \frac{1}{2\pi\alpha'}$, and so it takes energy, and thus mass, to stretch it.

We now see the appearance of T-duality, if we exchange the momentum mode with the winding number and invert the radius of compactification

$$w \leftrightarrow n \quad \text{and} \quad \frac{R}{\sqrt{\alpha'}} \leftrightarrow \frac{\sqrt{\alpha'}}{R} = \frac{\hat{R}}{\sqrt{\alpha'}}, \tag{3.11}$$

the spectrum is invariant. It can also be shown that also the partition function and the correlators are invariant under T-duality [29, p. 22].

We will now show the transformations of the string coordinates implied by the transformations of the oscillation modes.

Using (3.7), the action of T-duality on the zero modes is easily seen to be

$$\alpha_0^a \leftrightarrow \alpha_0^a, \quad \bar{\alpha}_0^a \leftrightarrow -\bar{\alpha}_0^a, \tag{3.12}$$

however, since this changes the $\bar{L}_m$-operators for $n \neq 0$ (see (2.31)), we must also insist on the more general transformation

$$\alpha_n^a \leftrightarrow \alpha_n^a, \quad \bar{\alpha}_n^a \leftrightarrow -\bar{\alpha}_n^a, \quad n \in \mathbb{Z}, \tag{3.13}$$

it is easy to see that both the L_n's and $\bar{L}_n$'s are invariant under this.

T-duality works only on the compact coordinates, we therefore consider

$$X^a(\tau,\sigma) = X_L^a + X_R^a, \tag{3.14}$$

where

$$X_L^a = \bar{x}^a + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^a(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{\bar{\alpha}_n^a}{n}e^{-in(\tau+\sigma)} \tag{3.15}$$

$$X_R^a = x^a + \sqrt{\frac{\alpha'}{2}}\alpha_0^a(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{\alpha_n^a}{n}e^{-in(\tau-\sigma)}. \tag{3.16}$$

We calculate

$$\partial_\tau X^a = \sqrt{\frac{\alpha'}{2}}(\bar{\alpha}_0^a + \alpha_0^a) + \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}(\bar{\alpha}_n^a e^{-in(\tau+\sigma)} + \alpha_n^a e^{-in(\tau-\sigma)}) \tag{3.17}$$

$$\partial_\sigma X^a = \sqrt{\frac{\alpha'}{2}}(\bar{\alpha}_0^a - \alpha_0^a) + \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}(\bar{\alpha}_n^a e^{-in(\tau+\sigma)} - \alpha_n^a e^{-in(\tau-\sigma)}). \tag{3.18}$$

It is easy to see that the T-duality transformations are equivalent with

$$\partial_\tau X^a \leftrightarrow \partial_\sigma \hat{X}^a, \quad \partial_\sigma X^a \leftrightarrow \partial_\tau \hat{X}^a, \tag{3.19}$$

which in turn is equivalent with

$$\hat{X}^a = X_L^a - X_R^a. \tag{3.20}$$

This means that T-duality only affects right moving part of the string motion. We could also have used this as the definition of T-duality. Doing so would have been less intuitive, but a lot simpler to work with, particularly if we have several compact dimensions.

3.1.2 Open strings

We saw above that T-duality worked by swapping the prefactor of the time-like coordinate τ with that of the space-like coordinate σ. However, Neumann strings do not have the periodicity condition that allowed the existence of winding, so at first glance it does not look like T-duality will be a symmetry of the open string. Though the Dirichlet string has a separation parameter, it needs to be stuck to some object, and we are still working in a free theory that only contains strings, and it is therefore not yet relevant to our discussion.

We first consider Neumann strings living in a d-dimensional space. We now compactify $d-p-1$ of them on circles with radii R^a. This means that the continuous momentum variable will be replaced by Kaluza-Klein modes. When we then take the limit

$$R^a \to 0, \quad a \in [p+1, p+2, \dots, d-1], \tag{3.21}$$

we see from (3.5) that the K-K modes become infinitely heavy. Since it is also impossible for the string to oscillate in directions of zero radius, these directions will decouple completely from the spectrum. The Neumann string is effectively living on a $p+1$-dimensional subspace.

Second, we consider the closed string in the same limit. Again the K-K modes become infinitely massive, but the winding modes become a continuum of states. This means that using T-duality, we can go back to the case where momentum is continuous and there is no winding, which is exactly the non-compact case. This is a major discrepancy between the open and closed cases and it is very unsatisfactory to have a theory that is so contradictory.

The solution to this problem is to demand that the T-dual of a Neumann string in a compact dimension is a Dirichlet string. In this picture, the endpoints of the strings can only move in the $p+1$ non-compact dimensions, but they can oscillate in all d. It does not require much reflection to realize that this is exactly the first case mentioned at the start of this chapter, strings living on a p-dimensional brane within the d-dimensional space. This lets the strings obey Dirichlet boundary conditions and is the reason we call these objects Dp-branes.

It is actually very natural to define the T-dual of a Neumann string to be a Dirichlet if we regard the T-dual of the string coordinate,

$$X^m = X_L^m + X_R^m, \quad \hat{X}^a = X_L^a - X_R^a. \tag{3.22}$$

Using the definitions (3.15) and (3.16) with $\bar{\alpha}_n^a = \alpha_n^a$, this gives

$$X^m = x^m + \bar{x}^a + \sqrt{2\alpha'}\alpha_0^m \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n^m}{n} e^{-in\tau}\cos(n\sigma), \quad m \in [0, p] \tag{3.23}$$

$$\hat{X}^a = x^a - \bar{x}^a + \sqrt{2\alpha'}\alpha_0^a \sigma - \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n^a}{n} e^{-in\tau}\sin(n\sigma), \quad a \in [p+1, d-1] \tag{3.24}$$

which are exactly the same as (2.25) and (2.26).

3.1.3 The superstring coordinates

In the beginning of this chapter, we said that we would only consider the bosonic string. However, we will need to know how the fermionic coordinates behave for later calculations. We therefore state them here. Supersymmetry tells us that fermionic coordinates must, in analogy with (3.20), transform as $\psi_\pm = \pm\psi_\pm$ under T-duality. We therefore get the following motion [30]

$$\psi^m_\pm(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{k\in\mathbb{Z}+\frac{x}{2}} b^m_k e^{-i(\tau\pm\sigma)}, \qquad m \in [0,p], \tag{3.25}$$

$$\psi^a_\pm(\tau,\sigma) = \pm\sqrt{\frac{\alpha'}{2}} \sum_{k\in\mathbb{Z}+\frac{x}{2}} b^a_k e^{-i(\tau\pm\sigma)}, \qquad a \in [p+1,d-1], \tag{3.26}$$

where $x = 0$ ($x = 1$) in the Ramond (Neveu-Schwarz) sector.

Chapter 4

Strings on parallel Dp-branes

In the previous chapter, we established that string theory naturally contains Dp-branes; multi-dimensional objects on which open strings can be attached, that give rise to Dirichlet boundary conditions. We will now move on to consider a very simple brane configuration in detail. We will consider two parallel p-dimensional D-branes, go through the quantization procedure, find the spectrum and see how this setup is different from the free strings we considered in chapter 2.

4.1 String motion on Dp-branes

We consider the brane configuration mentioned above (see figure 4.1) and the different kinds of boundary conditions this give rise to for open strings. Like when we had small compact dimensions and realised that it was equivalent with a single brane (see section 3.1.2), the string moves freely in some directions (Neumann conditions) and is bound in others (Dirichlet conditions). What is different is that it is now possible for the string to start on one brane and end on the other. Since the strings are oriented, we cannot be sure that a string going from brane 1 to brane 2 will behave exactly like one going from brane 2 to brane 1. Therefore, using equations (2.25) and (2.26), we consider the motion of a string going from brane i to brane j

$$X^m(\tau,\sigma) = x^m + 2\alpha' p^m \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n^m}{n} e^{-in\tau}\cos(n\sigma), \quad m \in [0,p], \tag{4.1}$$

$$X^a(\tau,\sigma) = y_i^a + \frac{y_j^a - y_i^a}{\pi}\sigma - \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\alpha_n^a}{n} e^{-in\tau}\sin(n\sigma), \quad a \in [p+1, d-1], \tag{4.2}$$

where y_i^a is the coordinate of i'th brane. Note that we have inserted the distance between the two branes as the separation.

The fermionic coordinates are unaffected by the presence of multiple branes and are therefore simply given by equations (3.25) and (3.26).

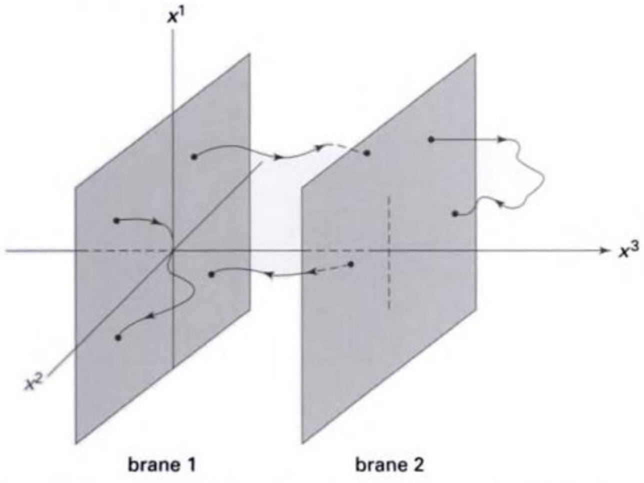


Figure 4.1: Two parallel D2-branes with examples of strings in all four sectors. Figure taken from [19].

When we go through the quantization procedure in this case, we find that it is almost identical to the one we did for the free particles in sections 2.1.5 and 2.2.5. The commutators are the same and we find the same ghosts, but we do get slightly different Virasoro operators. This is due to the fact that we now have a term in the string coordinate that is proportional to σ.

As we saw in sections 2.1.6 and 2.2.6, it is convenient to use the leftover gauge symmetry to go to the light-cone gauge. When dealing with strings on Dp-branes, we can only do this if $p > 0$, since we need at least one spatial Neumann coordinate to form the light-cone coordinates. This does, of course, not mean that branes with $p = 0$ do not exist, they merely need separate treatment and are of little interest to us, since they do not play a role in constructing a semi-realistic model. When we impose light-cone quantization, we can again express α_n^+ and α_n^- completely in terms of the transverse oscillations, but the different form of the string coordinate of course also affects these.

4.2 Neveu-Schwarz sector spectrum on Dp-branes

We want to examine how the presence of parallel Dp-branes changes the superstring spectrum and first consider the revised Neveu-Schwarz sector mass formula

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\left[\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^a\alpha_n^a\right] + \sum_{r=\frac{1}{2}}^{\infty} r\left[b_{-r}^i b_r^i + b_{-r}^a b_r^a\right] - \frac{1}{2}\right) + \left(\frac{y_i^a - y_j^a}{2\pi\alpha'}\right)^2, \quad i \in [2,p]. \quad (4.3)$$

We immediately see that we have a new and very different term, namely the square of the distance between the branes. Mathematically, this appears in place of momentum in the directions subject to Dirichlet boundary conditions. Physically, it comes from the energy required to stretch a string with non-zero tension, just like the winding described in section 3.1.1.

Since strings with both endpoints on the same brane behave almost entirely like free strings restricted to move in fewer dimensions, we will not go into detail with them here, but instead only consider strings stretching between the two branes.

It is immediately seen that the ground state mass is no longer given by the normal ordering constant alone. Indeed the state need not necessarily be a tachyon if the separation of the branes is $|y_i^a - y_j^a| \geq \sqrt{2\alpha'}\pi$. However, while this is an interesting fact, we know that the GSO-projection will remove the ground state from the spectrum, and we are therefore much more interested in what happens to the first excited state.

The first excited state is obviously massive since the separation gives a positive contribution. It also turns out that there are two distinct groups of states,

$$b^i_{-1/2}|p;1,2\rangle_{NS}, \qquad i \in [2,p], \qquad M^2 = \left(\frac{y_2^a - y_1^a}{2\pi\alpha'}\right)^2 \tag{4.4}$$

$$b^a_{-1/2}|p;1,2\rangle_{NS}, \qquad a \in [p+1,d-1], \qquad M^2 = \left(\frac{y_2^a - y_1^a}{2\pi\alpha'}\right)^2, \tag{4.5}$$

where $|p;i,j\rangle$ is the ground state of a string with momentum p stretched between branes i and j. At a glance, these kinds of states seem identical, but there is the very important difference that we are no longer working in the whole d-dimensional spacetime. By introducing Dp-branes we have restricted ourselves to working in their $p+1$-dimensional spacetime. This means that i is a Lorentz-index, while a is merely a counting label. Therefore, the states $b^a_{-1/2}|p;1,2\rangle_{NS}$ are massive scalars and it would seem obvious that $b^i_{-1/2}|p;1,2\rangle_{NS}$ is a massive vector. However, a massive Lorentz-vector in $p+1$ dimensions has p indices, while $b^i_{-1/2}|p;1,2\rangle_{NS}$ only has $p-1$. To get a Lorentz-vector, we must include one of the scalar states as well. To answer the question of which scalar field to use, we consider the system described (see figure 4.1) and look for any preferred directions. It is clear that there is only one direction that stands out, namely the direction given by the separation vector $(y_2^a - y_1^a)$, and the scalar field we add to $b^i_{-1/2}|p;1,2\rangle_{NS}$ in order to create a massless vector is therefore [19, p. 286]

$$\sum_a (y_2^a - y_1^a) b^a_{-1/2}|p;1,2\rangle_{NS}. \tag{4.6}$$

Note that since the two parallel branes are the only objects in an otherwise rotationally symmetric configuration, we can always place our coordinate system in such a way that the scalar field that takes the role of the last component in the Lorentz vector is

$$b^9_{-1/2}|p;1,2\rangle_{NS}. \tag{4.7}$$

As usual, higher excited states are so far removed from the (almost-)massless one that we will not consider them.

4.3 Ramond sector spectrum on Dp-branes

We can of course also find the revised mass formula in the Ramond sector

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \alpha^a_{-n}\alpha^a_n + nb^i_{-n}b^i_n + nb^a_{-n}b^a_n\right) + \left(\frac{y^a_i - y^a_j}{2\pi\alpha'}\right)^2, \quad i \in [2,p]. \tag{4.8}$$

We immediately see that the Ramond ground state has gained mass,

$$|p;1,2\rangle_R, \quad M^2 = \left(\frac{y^a_2 - y^a_1}{2\pi\alpha'}\right)^2, \tag{4.9}$$

but is otherwise unchanged. However, since massive fermions cannot be chiral, we have lost chirality. To have a semi-realistic model, we need 4 dimensional chirality, therefore this simple brane configuration is not clever enough to give us what we need. We will see in chapter 5 that it actually is possible to have a brane configuration that allows us to have 4 dimensional chirality.

Since it still holds that there is an equal number of physical degrees of freedom at each mass level and that all the mass levels match, supersymmetry remains unbroken.

4.4 Reversing the direction

If we, instead of considering a string stretching from brane 1 to brane 2, consider one directed from 2 to 1, we may ask; what changes?

Obviously, the (almost-)massless states are now

$$b^i_{-1/2}|p;2,1\rangle_{NS}, \qquad i \in [2,d-1], \qquad M^2 = \left(\frac{y^a_1 - y^a_2}{2\pi\alpha'}\right)^2 \tag{4.10}$$

$$b^a_{-1/2}|p;2,1\rangle_{NS}, \qquad a \in [p+1,d-1], \qquad M^2 = \left(\frac{y^a_1 - y^a_2}{2\pi\alpha'}\right)^2, \tag{4.11}$$

in the Neveu-Schwarz sector and

$$|p;2,1\rangle_R, \quad M^2 = \left(\frac{y^a_1 - y^a_2}{2\pi\alpha'}\right)^2, \tag{4.12}$$

in the Ramond-sector. We clearly see that the masses remain the same.

In the Neveu-Schwarz sector we can again form a massive $p+1$-dimensional Lorentz-vector by combining the longitudinally excited state with a superposition of the transversely excited modes given by

$$\sum_a (y_1^a - y_2^a) b_{-1/2}^a |p; 2, 1\rangle, \tag{4.13}$$

and in the Ramond sector, the ground state remains a massive d-dimensional Majorana spinor.

We thus see that all known properties of the strings are the same no matter if the string goes from brane 1 to brane 2 or vice-versa. Therefore it is clear that the physics of the two strings is exactly the same.

4.5 Chan-Paton indices

We have in the above simply labelled the branes 1 and 2, and used these designations when we wrote the different states. However, in themselves, these indices have nothing to do with the branes; one could imagine that the string endpoints merely had some integer label i without reference to anything else. This idea was examined by Chan and Paton [10] long before anyone had thought of branes in string theory.

When open strings live on branes, it is obvious that interactions can only happen when the end points are on the same brane. In the Chan-Paton picture the situation is the same; interactions can only happen between string endpoints with the same CP index.

If we consider the case of N different branes with zero separation, we have N^2 copies of each string state, described by oscillation modes, momentum and Chan-Paton indices. The ground state is

$$|p; i, j\rangle, \tag{4.14}$$

and after introducing the generators of the $U(N)$ Lie algebra, λ_{ij}^a,

$$\mathrm{Tr}[\lambda^a \lambda^b] = \delta^{ab}, \tag{4.15}$$

we can define an alternate basis for the states

$$|p; a\rangle = \sum_{ij} \lambda_{ij}^a |p; i, j\rangle. \tag{4.16}$$

Since only strings with end-points on the same brane (with the same CP index) can interact, the string amplitudes will be proportional to the trace of products of λ-matrices [15, p. 35]. This ensures that they are invariant under global $U(N)$ transformations. We have thus found a way of endowing strings with unitary gauge symmetries, exactly the kind of symmetry we need in order to reproduce the standard model!

However, this only holds true for zero separation, if we separate the branes into a stack of K branes and one of L, the symmetry group is broken into $U(K) \times U(L)$. This works to our advantage, since we can consider a brane configuration consisting of one stack of 3 and one of 2 which gives us $U(3) \times U(2)$ symmetry. This together with the fact that $U(N) = SU(N) \times U(1)$ gives us something very close to the standard model gauge group, namely

$$SU(3) \times SU(2) \times U(1) \times U(1). \tag{4.17}$$

To get the general features of the standard model, we just need a way to get rid of the surplus gauge group and introduce chirality and fermion generations. The second of these issues, we will address in the next chapter by adding magnetization to the branes.

Though $U(N)$ symmetry is the gauge group that appears in the simplest way, it is also possible to use this method to instead endow strings with one of the two other families of classical groups, namely $O(N)$ and $Sp(N)$. This happens when we are dealing with unoriented strings. As we have seen above, giving Chan-Paton indices to a string gives it a non-abelian gauge symmetry, and it makes sense to associate one end with the fundamental representation of the group and the other with the anti-fundamental representation [22, p. 196]. However, if the string is unoriented, these become indistinguishable and therefore the symmetry group must be one with a real fundamental representation. It can be shown that if the massless vectors correspond to symmetric states the symmetry group is $Sp(N)$ and if they are anti-symmetric, it is $O(N)$.

Chapter 5

Superstrings on magnetized D9-branes

We now wish to consider a case related to the one before, namely that of superstrings attached to two space-filling D9-branes on a space $M_4 \times \mathcal{M}_6$ where M_4 is our usual 4-dimensional flat spacetime and $\mathcal{M}_6$ is some 6-dimensional manifold with small, compact dimensions. We will furthermore let each of the branes carry some constant magnetic field given in terms of the field-strength, or Faraday, tensor $F^{(e)}_{mn}$ on the manifold and in these dimensions also consider the effect of the background Kalb-Ramond field B_{mn}. In this configuration, the magnetization will take on a role similar to that of the distance in the previous chapter. Furthermore, the fact that 6 of the dimensions are compact means that the effective low-energy theory will live in 4 dimensions.

Since the string will behave almost as if nothing had happened in the four flat dimensions, we will only consider the 6 dimensions on the manifold. The action on $\mathcal{M}_6$ has the form

$$S = S^X + S^\psi = S^X_{bulk} + S^X_{boundary} + S^\psi_{bulk} + S^\psi_{boundary}, \tag{5.1}$$

and we can once more treat the bosonic and fermionic coordinates independently.

5.1 The bosonic coordinates

We now consider the explicit expression for the bosonic part,

$$\begin{aligned} S^X = &-\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma[G_{mn}\partial_\alpha X^m \partial_\beta X^n \eta^{\alpha\beta} - B_{mn}\partial_\alpha X^m \partial_\beta X^n \varepsilon^{\alpha\beta}] \\ &+ \frac{q_1}{2}\int \mathrm{d}\tau F^{(1)}_{mn} X^n \partial\tau X^m|_{\sigma=0} - \frac{q_2}{2}\int \mathrm{d}\tau F^{(2)}_{mn} X^n \partial_\tau X^m|_{\sigma=\pi}, \end{aligned} \tag{5.2}$$

where we have gone to the gauge $A_m = -\frac{1}{2}F_{mn}X^n$.

5.1.1 Equations of motion

We find the equations of motion and the boundary conditions by performing a variation with respect to the spacetime coordinates. This is a rather tiresome, but very straightforward, calculation and the result is:

$$\begin{aligned}\delta S^X = \frac{1}{2\pi\alpha'}&\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma G_{mn}\partial_\alpha\partial^\alpha X^m \delta X^n \\ &- \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau [G_{mn}\partial_\sigma X^m\delta X^n - B_{mn}\partial_\tau X^m \delta X^n]_{\sigma=\pi} \\ &+ \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau [G_{mn}\partial_\sigma X^m\delta X^n - B_{mn}\partial_\tau X^m \delta X^n]_{\sigma=0} \\ &- q_1\int \mathrm{d}\tau F^{(1)}_{mn}\partial_\tau X^n\delta X^m|_{\sigma=0} + q_2\int \mathrm{d}\tau F^{(2)}_{mn}\partial_\tau X^n \delta X^m|_{\sigma=\pi}. \end{aligned} \tag{5.3}$$

We now have everything we need; the equations of motion,

$$\partial_\alpha\partial^\alpha X^m = 0 \tag{5.4}$$

and the boundary conditions

$$[G_{mn}\partial_\sigma + (B_{mn} - 2\pi\alpha' q_1 F^{(1)}_{mn})\partial_\tau]X^n|_{\sigma=0} = 0 \tag{5.5}$$

$$[G_{mn}\partial_\sigma + (B_{mn} - 2\pi\alpha' q_2 F^{(2)}_{mn})\partial_\tau]X^n|_{\sigma=\pi} = 0. \tag{5.6}$$

Before solving the equation of motion while taking the boundary conditions into account, we wish to write them in a simpler fashion. We therefore introduce*

$$\mathcal{B}_{e\,mn} = B_{mn} - 2\pi\alpha' q_e F^{(e)}_{mn},, \quad e = 1, 2, \tag{5.7}$$

and use it to rewrite the boundary condition slightly

$$[\partial_\sigma X^m + \mathcal{B}^m_{1\ n}\partial_\tau X^n]_{\sigma=0} = 0, \tag{5.8}$$

$$[\partial_\sigma X^m + \mathcal{B}^m_{2\ n}\partial_\tau X^n]_{\sigma=\pi} = 0. \tag{5.9}$$

To solve the equation of motion (5.4), we split up the string coordinate in a left-moving and a right-moving part,

$$X^m(\tau,\sigma) = X^m_L(\tau+\sigma) + X^m_R(\tau-\sigma). \tag{5.10}$$

We insert this expression in the $\sigma = 0$ boundary condition (5.8) and after some simple manipulations find

$$\partial_\tau X^p_R(\tau) = [(1-\mathcal{B}_1)^{-1}]^p{}_m[1+\mathcal{B}_1]^m{}_n\partial_\tau X^n_L(\tau). \tag{5.11}$$

*It is interesting to note that the Faraday tensor F_{mn} is not a gauge invariant quantity in string theory, but that this quantity $\mathcal{B}_{mn}$ is [19, p. 319].

To get a simpler expression now and for several calculations later, it is convenient to define

$$R_1{}^m{}_n = [(1 - \mathcal{B}_1)^{-1}]^m{}_p[1 + \mathcal{B}_1]^p{}_n. \tag{5.12}$$

We can now integrate (5.11) to find

$$X_R^m(\tau) = R_1{}^m{}_n X_L^n(\tau) + x^m, \tag{5.13}$$

where x^m is some constant.

Having satisfied the first boundary condition, we move on to the second one where we also use the newly found relation (5.13). Again we have to make some simple manipulations, but it is quite straightforward to find

$$\partial_\tau X_L^q(\tau + \pi) = [(1 + \mathcal{B}_2)^{-1}]^q{}_m[1 - \mathcal{B}_2]^m{}_n R_1{}^n{}_p \partial_\tau X_L^p(\tau - \pi). \tag{5.14}$$

It is now natural to introduce

$$R_2{}^m{}_n = [(1 - \mathcal{B}_2)^{-1}]^m{}_p[1 + \mathcal{B}_2]^p{}_n, \tag{5.15}$$

and

$$R^m{}_n = R_2^{-1m}{}_p R_{1n}^p, \tag{5.16}$$

so that we may write the boundary condition

$$\partial_\tau X_L^q(\tau + \pi) = R^q{}_p \partial_\tau X_L^p(\tau - \pi). \tag{5.17}$$

The motion of the string in the magnetized dimensions is thus

$$X^m(\tau, \sigma) = x^m + X_L^m(\tau + \sigma) + R_1{}^m{}_n X_L^n(\tau - \sigma), \tag{5.18}$$

with the further demand that $X_L^m(\tau)$ must satisfy (5.17).

5.1.2 The $R^m{}_n$ matrix

Let us now consider the matrix $R^m{}_n$. It is quite easy to show that both $R_1{}^m{}_n$ and $R_2{}^m{}_n$ are 6 dimensional, real, orthogonal matrices, and the product of two orthogonal matrices is a new orthogonal matrix. This has some very important implications for the eigenvalues and eigenvectors of $R^m{}_n$, specifically

$$R^m{}_n C_a^n = c_a C_a^m \tag{5.19}$$

$$(C^\dagger)_{am} C_b^m = \delta_{ab} \tag{5.20}$$

$$(C^\dagger)_{am}(R^\dagger)_n{}^m R^n{}_p C_a^p = |c_a|^2 = 1. \tag{5.21}$$

The first of these relations is just the eigenvalue equation, the second holds because the eigenvectors of an orthogonal matrix are orthonormal and the last because the hermitian conjugate of a real, orthogonal matrix is its inverse. The last of the above equations implies that the eigenvalues of $R^m{}_n$ are merely complex phases. Inspired by this, we consider

$$(R^*)^m{}_n (C_a^*)^n = c_a^* (C_a^*)^m \tag{5.22}$$
$$R^m{}_n (C_a^*)^n = c_a^* (C_a^*)^m \tag{5.23}$$

Therefore we have

$$R^m{}_n C_a^n = e^{2\pi i \nu_a} C_a^m; \quad R^m{}_n (C_a^*)^n = e^{-2\pi i \nu_a} (C_a^*)^m; \quad a = 1 \dots 3, \tag{5.24}$$

where we restrict ourselves to $0 \leq \nu_a \leq \frac{1}{2}$.

In order to get the eigenvalue equation in a convenient matrix-form, we define the matrix $(E^{-1})^m{}_A$ by joining together C_a^m and $(C^*)_a^m$,

$$(\mathcal{E}^{-1})^m{}_A = \left(C_a^m \;\; (C_a^*)^m \right), \qquad A = 1, \dots, 6, \tag{5.25}$$

The uninverted version is somewhat more complicated than this because it depends on the metric, but in flat space it is

$$\mathcal{E}^A{}_m = \begin{pmatrix} (C^*)^a_m \\ C^a_m \end{pmatrix}, \qquad A = 1, \dots, 6. \tag{5.26}$$

We can use this to write the eigenvalues in a diagonal matrix

$$\mathcal{E}^B{}_m R^m{}_n (\mathcal{E}^{-1})^n{}_A = \mathcal{R}^B_A = \begin{pmatrix} e^{2\pi i \nu_a} \delta^{ab} & 0 \\ 0 & e^{-2\pi i \nu_a} \delta^{ab} \end{pmatrix}. \tag{5.27}$$

5.1.3 Solving the equations of motion

We can use the results from the previous subsection to solve the equations of motion. To do this, we first introduce the linear combination

$$\mathcal{X}_A = \mathcal{E}_{Am} X_L^m = \begin{pmatrix} \mathcal{X}_a^- \\ \mathcal{X}_a^+ \end{pmatrix} = \begin{pmatrix} C^*_{am} X_L^m \\ C_{am} X_L^m \end{pmatrix} \tag{5.28}$$
$$X_L^m = (\mathcal{E}^{-1})^m{}_A \mathcal{X}_A = C_a^m \mathcal{X}_a^- + C_a^{*m} \mathcal{X}_a^+, \tag{5.29}$$

and then write the boundary condition (5.17) in the basis of $\mathcal{X}_A$

$$\partial_\tau (\mathcal{E}^{-1})^m{}_A \mathcal{X}_A(\tau + \pi) = R^m{}_n \partial_\tau (\mathcal{E}^{-1})^n{}_A \mathcal{X}_A(\tau - \pi) \tag{5.30}$$
$$\partial_\tau \mathcal{X}_A(\tau + \pi) = \mathcal{E}_{Am} R^m{}_n (\mathcal{E}^{-1})^n{}_B \partial_\tau \mathcal{X}_B(\tau - \pi) \tag{5.31}$$
$$= \mathcal{R}_{AB} \partial_\tau \mathcal{X}_B(\tau - \pi). \tag{5.32}$$

This simplifies considerably when we split it up in a plus and a minus part,

$$\partial_\tau \mathcal{X}_a^\pm(\tau+\pi) = e^{\mp 2\pi i \nu_a} \partial_\tau \mathcal{X}_a^\pm(\tau-\pi). \tag{5.33}$$

When using an oscillator expansion of X^μ similar to (2.22) to solve the equations of motion and recalling the definition of $\mathcal{X}^\pm$ (5.28), the boundary condition is satisfied by the following expression,

$$\mathcal{X}_a^\pm(\tau+\sigma) = \frac{i\sqrt{2\alpha'}}{2} \sum_n \frac{\alpha^a_{n\pm\nu_a}}{n\pm\nu_a} e^{-i(\tau+\sigma)(n\pm\nu_a)}. \tag{5.34}$$

Inserting this in the definition of X_L^m, we get

$$X_L^m(\tau+\sigma) = \frac{i\sqrt{2\alpha'}}{2} \sum_{n,a} \left[C_a^m \frac{\alpha^a_{n-\nu_a}}{n-\nu_a} e^{-i(\tau+\sigma)(n-\nu_a)} + (C_a^*)^m \frac{\alpha^a_{n+\nu_a}}{n+\nu_a} e^{-i(\tau+\sigma)(n+\nu_a)} \right]. \tag{5.35}$$

Using this and $z = e^{-i(\tau+\sigma)}$, $\bar{z} = e^{i(\tau+\sigma)}$, we can write the motion as

$$X^m(\tau,\sigma) = x^m + X^m(z) + (R_1)^m{}_n X^n(\bar{z}), \tag{5.36}$$

where

$$X^m(z) = \frac{i\sqrt{2\alpha'}}{2} \sum_{n,a} \left[C_a^m \frac{\alpha^a_{n-\nu_a}}{n-\nu_a} z^{-(n-\nu_a)} + (C_a^*)^m \frac{\alpha^a_{n+\nu_a}}{n+\nu_a} z^{-(n+\nu_a)} \right]. \tag{5.37}$$

Once again, we find it convenient to rewrite

$$X^m(z) = C_a^m \mathcal{Z}^a + (C_a^*)^m \overline{\mathcal{Z}}^a, \tag{5.38}$$

we get

$$\mathcal{Z}^a = \frac{i\sqrt{2\alpha'}}{2} \sum_{n\in\mathbb{Z}} z^{-(n-\nu_a)} \frac{\alpha^a_{n-\nu_a}}{n-\nu_a} \tag{5.39}$$

$$= \frac{i\sqrt{2\alpha'}}{2} \left[\sum_{n=1}^\infty z^{-(n-\nu_a)} \frac{\alpha^a_{n-\nu_a}}{n-\nu_a} - \sum_{n=0}^\infty z^{(n+\nu_a)} \frac{\alpha^a_{-n-\nu_a}}{n+\nu_a} \right], \tag{5.40}$$

and

$$\overline{\mathcal{Z}}^a = \frac{i\sqrt{2\alpha'}}{2} \sum_{n\in\mathbb{Z}} z^{-(n+\nu_a)} \frac{\alpha^a_{n+\nu_a}}{n+\nu_a} \tag{5.41}$$

$$= \frac{i\sqrt{2\alpha'}}{2} \left[\sum_{n=0}^\infty z^{-(n+\nu_a)} \frac{\alpha^a_{n+\nu_a}}{n+\nu_a} - \sum_{n=1}^\infty z^{(n-\nu_a)} \frac{\alpha^a_{-n+\nu_a}}{n-\nu_a} \right], \tag{5.42}$$

To simplify for later comparison with a point particle in a magnetic field (see section 6.3), we introduce new harmonic oscillators

$$A^a_{n+\nu_a} = \frac{\alpha^a_{n+\nu_a}}{\sqrt{n+\nu_a}}; \qquad A^{\dagger a}_{n+\nu_a} = \frac{\alpha^a_{-n-\nu_a}}{\sqrt{n+\nu_a}} \ ; \ n = 0, 1, \ldots, \tag{5.43}$$

$$A^a_{n-\nu_a} = \frac{\alpha^a_{n-\nu_a}}{\sqrt{n-\nu_a}}; \qquad A^{\dagger a}_{n-\nu_a} = \frac{\alpha^a_{-n+\nu_a}}{\sqrt{n-\nu_a}} \ ; \ n = 1, 2, \ldots, \tag{5.44}$$

and for completeness also the oscillators in the non-compact directions

$$A^i_n = \frac{\alpha^i_n}{\sqrt{n}} \quad ; \quad A^{\dagger i}_n = \frac{\alpha^i_{-n}}{\sqrt{n}}. \tag{5.45}$$

These will satisfy the usual harmonic oscillator commutators

$$[A^a_{n+\nu_a}, A^{\dagger b}_{m+\nu_b}] = \delta_{mn}\delta_{ab}; \quad [A^a_{n-\nu_a}, A^{\dagger b}_{m-\nu_b}] = \delta_{mn}\delta_{ab}. \tag{5.46}$$

We are now ready to write out the revised mass formula:

$$M^2 = \frac{1}{\alpha'}\left[N^X_4 + N^X_d - \frac{d-2}{24} - \frac{1}{2}\sum_{a=1}^{3}\nu_a(\nu_a - 1)\right], \tag{5.47}$$

where we have used a slightly more general form of ζ-function regularization (based on the Hurwitz ζ-function) to obtain the normal ordering constant than we did in (2.63),

$$\sum_{n=0}^{\infty}(n+\alpha) \equiv \frac{1}{2}\left(-\frac{1}{6} - \alpha^2 + \alpha\right). \tag{5.48}$$

The reason we have to do this is that the magnetization introduces a constant shift $\pm\nu_a$ in the oscillators (see equations (5.43) and (5.44)) which in turn shifts the normal ordering constant by $\frac{1}{2}\sum_{a=1}^{3}\nu_a(\nu_a - 1)$.

We have also defined

$$N^X_4 = \sum_{n=1}^{\infty} n A^{\dagger i}_n A^i_n \tag{5.49}$$

$$N^X_d = \sum_{a=1}^{3}\left[\sum_{n=0}^{\infty}(n+\nu_a)A^{\dagger a}_{n+\nu_a}A^a_{n+\nu_a} + \sum_{n=1}^{\infty}(n-\nu_a)A^{\dagger a}_{n-\nu_a}A^a_{n-\nu_a}\right]. \tag{5.50}$$

5.2 The fermionic coordinates

Having solved the motion and found the mass contribution of the bosonic coordinates, we will now move on to the fermionic one. From supersymmetry, we would expect the action to take the form

$$S_0^\psi = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma[iG_{mn}\bar{\psi}^m\rho_\alpha\partial_\beta\psi^n\eta^{\alpha\beta} + i\varepsilon^{\alpha\beta}B_{mn}\bar{\psi}^m\rho_\alpha\partial_\beta\psi^n] + \frac{q_1}{4}\int \mathrm{d}\tau F^{(1)}_{mn}\bar{\psi}^m\rho_0\psi^n|_{\sigma=0} - \frac{q_2}{4}\int \mathrm{d}\tau F^{(2)}_{mn}\bar{\psi}^m\rho_0\psi^n|_{\sigma=\pi}, \quad (5.51)$$

where ρ_α are 2-dimensional Dirac-matrices obeying $\{\rho_\alpha, \rho_\beta\} = 2\eta_{\alpha\beta}$, and ψ^n is a Grassman field with $\bar{\psi}^m = \psi^m\rho^0$. However, this turns out to give boundary conditions that are inconsistent with the ones from the bosonic part of the theory (see equation (5.8)) [31]. We must therefore add to the action a term

$$S_1^\psi = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma[i\eta^{\alpha\beta}B_{mn}\bar{\psi}^m\rho_\alpha\partial_\beta\psi^n - i\varepsilon^{\alpha\beta}B_{mn}\bar{\psi}^m\rho_\alpha\partial_\beta\psi^n], \quad (5.52)$$

$$(5.53)$$

resulting in a final action [31]

$$S^\psi = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma[i(G_{mn} + B_{mn})\bar{\psi}^m\rho_\alpha\partial_\beta\psi^n\eta^{\alpha\beta}] + \frac{q_1}{4}\int \mathrm{d}\tau F^{(1)}_{mn}\bar{\psi}^m\rho_0\psi^n|_{\sigma=0} - \frac{q_2}{4}\int \mathrm{d}\tau F^{(2)}_{mn}\bar{\psi}^m\rho_0\psi^n|_{\sigma=\pi}. \quad (5.54)$$

5.2.1 Equation of motion

We again use the principle of least action to find the equations of motion and the boundary conditions,

$$\delta S^\psi = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\delta\bar{\psi}^m\left[G_{mn}\rho_\alpha\partial_\beta\psi^n\eta^{\alpha\beta}\right] + \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\delta\bar{\psi}^m\left[(G_{mn} + B_{mn})\rho_\sigma\psi^n\right]_{\sigma=0,\pi} \frac{q_1}{2}\int \mathrm{d}\tau\delta\bar{\psi}^m F^{(1)}_{mn}\rho_0\psi^n|_{\sigma=0} - \frac{q_2}{2}\int \mathrm{d}\tau\delta\bar{\psi}^m F^{(2)}_{mn}\rho_0\psi^n|_{\sigma=\pi}. \quad (5.55)$$

Setting the variation to zero gives us the Dirac equation just like it did in the un-magnetized case,

$$\rho^\alpha\partial_\alpha\psi^n = 0, \quad (5.56)$$

but the boundary conditions are new:

$$\left[\delta\psi^m\rho^0(G + \mathcal{B}_e)_{mn}\rho_1\psi^n\right]_{\sigma=0,\pi} = 0, \quad e = 1, 2, \quad (5.57)$$

where $\mathcal{B}_e$ is defined in (5.7), just like when we considered the bosonic case.

We saw in section 2.2.2 that it is very convenient to write ψ^m in the Weyl representation. Doing this, we can write

$$\Big[\delta\psi_-^m(G+\mathcal{B}_e)_{mn}\psi_-^n - \delta\psi_+^m(G+\mathcal{B}_e)_{mn}\psi_+^n\Big]_{\sigma=0,\pi} = 0, \tag{5.58}$$

with $e=1$ ($e=2$) for $\sigma=0$ ($\sigma=\pi$).

To satisfy this boundary condition, we use the ansatz

$$\delta\psi_-^m = \pm R_e{}^m{}_n\delta\psi_+^n = \pm[(1-\mathcal{B}_e)^{-1}]^m{}_s[1+\mathcal{B}_e]^s{}_n\delta\psi_+^n, \quad e=1,2 \tag{5.59}$$

and find that this implies the revised boundary condition

$$\Big[\pm(G-\mathcal{B}_e)_{mn}\psi_-^n - (G+\mathcal{B}_e)_{mn}\psi_+^n\Big]_{\sigma=0,\pi} = 0. \tag{5.60}$$

This equation is very easy to solve and we get

$$\psi_-^m|_{\sigma=0,\pi} = \pm[(1-\mathcal{B}_e)^{-1}]^m{}_s[1+\mathcal{B}_e]^s{}_n\psi_+^n|_{\sigma=0,\pi} = \pm R_e{}^m{}_n\psi_+^n|_{\sigma=0,\pi}, \quad e=1,2. \tag{5.61}$$

Thus the ansatz is consistent and we will use this result in the following.

Since both the plus and minus version of the ansatz is consistent, we have to take both into account. As we have aready pointed out (see section 2.2.2) the only distinct sectors are when the signs are different at the end points. These sectors are

$$\psi_-^m(\tau) = R_1{}^m{}_n\psi_+^n(\tau) \tag{5.62}$$
$$\psi_-^m(\tau-\pi) = \pm R_2{}^m{}_n\psi_+^n(\tau+\pi), \tag{5.63}$$

which we term Ramond (+) and Neveu-Schwarz (-) in accordance with section 2.2.2. Using these, it is a simple matter to get the periodicity condition for ψ_+^m,

$$\psi_+^m(\tau+\pi) = \pm R^m{}_n\psi_+^n(\tau-\pi). \tag{5.64}$$

As in the bosonic case, we can use (5.26) to introduce

$$\Psi_+^A = \mathcal{E}_m^A\psi_+^m, \tag{5.65}$$

and write

$$\Psi_+^A(\tau+\pi) = \pm\mathcal{R}_B^A\Psi_+^B(\tau-\pi). \tag{5.66}$$

Because of the form of $\mathcal{R}$, we can split this up in a plus and a minus part like we did in the bosonic case (see equation (5.33))

$$\Psi_+^a(\tau+\pi) = \pm e^{2\pi i\nu_a}\Psi_+^a(\tau-\pi), \tag{5.67}$$
$$\tilde{\Psi}_+^a(\tau+\pi) = \pm e^{-2\pi i\nu_a}\tilde{\Psi}_+^a(\tau-\pi). \tag{5.68}$$

The solution to the equation of motion (5.56) under this boundary condition is

$$\Psi_+^a(\tau+\sigma) = \sqrt{2\alpha'} \sum_{k\in\mathbb{Z}+x} B^a_{n-\nu_a} e^{-i(\tau+\sigma)(n-\nu_a)} \tag{5.69}$$

$$\tilde{\Psi}_+^a(\tau+\sigma) = \sqrt{2\alpha'} \sum_{k\in\mathbb{Z}+x} B^a_{n+\nu_a} e^{-i(\tau+\sigma)(n+\nu_a)}, \tag{5.70}$$

with $x = 0$ ($x = \frac{1}{2}$) in the Ramond (Neveu-Schwarz) sector.

We can again introduce the coordinate $z = e^{i(\tau+\sigma)}$

$$\Psi_+^a(z) = \sqrt{2\alpha'} \sum_{k\in\mathbb{Z}+x} B^a_{n-\nu_a} z^{-(n-\nu_a)} \tag{5.71}$$

$$\tilde{\Psi}_+^a(z) = \sqrt{2\alpha'} \sum_{k\in\mathbb{Z}+x} B^a_{n+\nu_a} z^{-(n+\nu_a)}, \tag{5.72}$$

and find the final expression for ψ_+^m and ψ_-^m

$$\psi_+^m = \sqrt{2\alpha'} \sum_{k\in\mathbb{Z}+x} C_a^m B^a_{n-\nu_a} z^{-(n-\nu_a)} + \sum_{k\in\mathbb{Z}+x} C_a^{*m} B^a_{n+\nu_a} z^{-(n+\nu_a)}, \tag{5.73}$$

$$\psi_-^m = R_1{}^m{}_n \psi_+^n. \tag{5.74}$$

We now wish to calculate the revised mass formula for the superstring and therefore need to find the contribution from the fermionic coordinates.

In the Neveu-Schwarz sector it is

$$M^2_{NS} = \frac{1}{\alpha'}\left[N_4^\psi + N_d^\psi - \frac{d-2}{48} + \frac{1}{2}\sum_{a=1}^{3}\nu_a^2\right], \tag{5.75}$$

and in the Ramond sector

$$M^2_R = \frac{1}{\alpha'}\left[N_4^\psi + N_d^\psi + \frac{d-2}{24} + \frac{1}{2}\sum_{a=1}^{3}\nu_a(\nu_a-1)\right]. \tag{5.76}$$

Here we introduced

$$N_4^\psi = \sum_{k=x} k B_k^{\dagger i} B_k^i \tag{5.77}$$

$$N_d^\psi = \sum_{a=1}^{3}\left[\sum_{k=x}^{\infty}(k+\nu_a)B^{\dagger a}_{k+\nu_a}B^a_{k+\nu_a} + \sum_{k=1-x}^{\infty}(k-\nu_a)B^{\dagger a}_{k-\nu_a}B^a_{k-\nu_a}\right], \tag{5.78}$$

and once again used ζ-function regularization (5.48).

Adding these to the bosonic mass formula (5.47), we get

$$M^2 = \frac{1}{\alpha'}\left[N_4^X + N_4^\psi + N_d^X + N_d^\psi + x\left(\sum_{a=1}^{3}\nu_a - \frac{(d-2)}{8}\right)\right], \tag{5.79}$$

with $x = 0$ ($x = \frac{1}{2}$) in the Ramond (Neveu-Schwarz) sector.

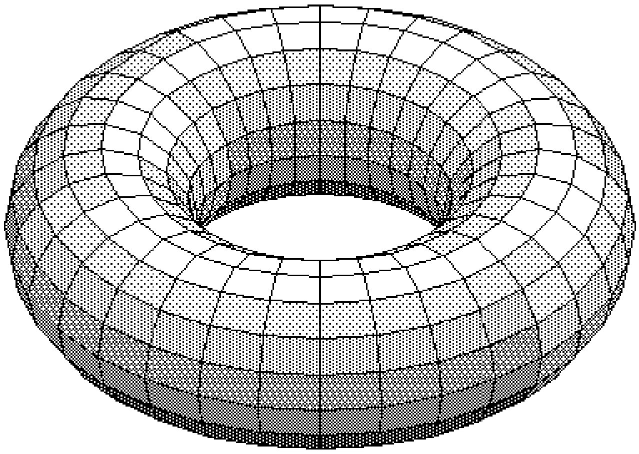


Figure 5.1: A torus such as this one is compact in two independent directions. This makes it very suited for compactification.

5.3 Toroidal geometry

We now return to the 6 dimensional manifold on which these magnetized branes are wrapped. In general this can take almost any form, but not all of them give realistic models, and not all of them are calculable. One choice that is both fairly simple to calculate and gives semi-realistic results is a factorizable six torus $\mathcal{M}_6 = T^2 \times T^2 \times T^2$. In this case, we only need to consider the two-torus (see figure 5.1) where the metric G_{mn} and the Kalb-Ramond field B_{mn} can be expressed as

$$G_{mn} = \frac{T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}; \quad G^{mn} \frac{1}{T_2 U_2} \begin{pmatrix} |U|^2 & -U_1 \\ -U_1 & 1 \end{pmatrix}; \quad B_{mn} = \begin{pmatrix} 0 & -T_1 \\ T_1 & 0 \end{pmatrix}, \tag{5.80}$$

in terms of the complex- and Kähler structures

$$U = U_1 + iU_2 = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{11}}; \quad T = T_1 + iT_2 = -B_{12} + i\sqrt{G}. \tag{5.81}$$

The changes brought on by introducing magnetized branes and toroidal compactification only affect the motion of the string through the matrix $R^m{}_n$, and through it the shift in the oscillator indices ν_a. It is therefore obvious that we need to calculate this matrix. Using the above expression for the metric and the well-known form of $\mathcal{B}_e$ (see equation (5.7)) it is a simple matter to perform the calculation. Note that in the following, we have suppressed the index e since it would clutter the expressions significantly.

$$R = (G - \mathcal{B})^{-1}(G + \mathcal{B}) \tag{5.82}$$

$$= \frac{1}{T_2^2 + \mathcal{B}_{12}^2} \begin{pmatrix} T_2^2 - \mathcal{B}_{12}^2 + 2\frac{U_1 T_2}{U_2}\mathcal{B}_{12} & 2|U|^2 \frac{T_2}{U_2}\mathcal{B}_{12} \\ -2\frac{T_2}{U_2}\mathcal{B}_{12} & T_2^2 - \mathcal{B}_{12}^2 - 2\frac{U_1 T_2}{U_2}\mathcal{B}_{12} \end{pmatrix}. \tag{5.83}$$

To calculate the shift, we find the eigenvalues

$$0 = \det(R - \lambda) \tag{5.84}$$

$$= \lambda^2 - 2\lambda \frac{T_2^2 - \mathcal{B}_{12}^2}{T_2^2 + \mathcal{B}_{12}^2} + 1 = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2. \tag{5.85}$$

By identifying the terms with the product and sum of the eigenvalues respectively, we can easily see that the two eigenvalues are the inverse of each other and that

$$\lambda_1 + \lambda_1^{-1} = \frac{T_2 + i\mathcal{B}_{12}}{T_2 - i\mathcal{B}_{12}} + \frac{T_2 - i\mathcal{B}_{12}}{T_2 + i\mathcal{B}_{12}}, \tag{5.86}$$

thus

$$\lambda_1 = \lambda_2^{-1} = \frac{T_2 - i\mathcal{B}_{12}}{T_2 + i\mathcal{B}_{12}}. \tag{5.87}$$

Since the eigenvalues are complex phases (see section 5.1.2), we relabel them $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^*$

To find the phase, we write

$$e^{2\pi i \nu} = \frac{T_2 - i\mathcal{B}_{12}}{T_2 + i\mathcal{B}_{12}} = \frac{\cos(\pi\nu) + i\sin(\pi\nu)}{\cos(\pi\nu) - i\sin(\pi\nu)}, \tag{5.88}$$

we identify

$$T_2 = \cos(\pi\nu), \quad \mathcal{B}_{12} = -\sin(\pi\nu), \tag{5.89}$$

and upon restoring the index e, we can now easily find

$$\nu_e = \frac{1}{\pi} \arctan\left(-\frac{\mathcal{B}_{e12}}{T_2}\right). \tag{5.90}$$

To find the actual shift, we recall that the total $R^m{}_n$ matrix can be put on a diagonal form (see (5.27)). Since $R_1^m{}_n$ and $R_2^m{}_n$ have exactly the same form, they can be diagonalized by the same matrices, and indeed the same matrices as $R^m{}_n$. This means that

$$\mathcal{E}^B{}_m R^m{}_n (\mathcal{E}^{-1})^n{}_A = \mathcal{E}^B{}_m R_2^{-1m}{}_n (\mathcal{E}^{-1})^n{}_C \mathcal{E}^C{}_o R_1{}^o{}_p (\mathcal{E}^{-1})^p{}_A = \mathcal{R}_2^{-1B}{}_C \mathcal{R}_1{}_A^C = \mathcal{R}_A^B. \tag{5.91}$$

Writing out the matrices of the last equality

$$\begin{pmatrix} e^{-2\pi i\nu_2} & 0 \\ 0 & e^{2\pi i\nu_2} \end{pmatrix}\begin{pmatrix} e^{2\pi i\nu_1} & 0 \\ 0 & e^{-2\pi i\nu_1} \end{pmatrix} = \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{-2\pi i\nu} \end{pmatrix}, \tag{5.92}$$

it is easy to see that

$$\nu = \nu_1 - \nu_2. \tag{5.93}$$

Note that when we introduced the phase ν, in equation (5.27), there were three of them corresponding to three pairs of compact directions. Now we are dealing with a two-dimensional subspace of the manifold and thus there is no need to specify which directions ν belongs to. The indices on the ν's instead refer to the branes they are attached to.

Let us, as an aside, consider the Faraday tensor in our toroidal manifold; it is expressed in units of $\sqrt{\alpha'}$, a quantity that only has a natural interpretation in string theory. For later comparison with a point particle, this is impractical. We would much rather work with a dimensionless number, such as the first Chern class, which can be calculated if we know the size of the torus. It is convenient to measure the radius of the torus in the same units as the string, and they are

$$R^m = r^m\sqrt{\alpha'}, \quad m \in [1,2], \quad r^m \in \mathbb{R}_+. \tag{5.94}$$

The first Chern class is then

$$I^{(e)} = \frac{q_e F_{12}^{(e)}}{2\pi}\int \mathrm{d}x^1 \wedge \mathrm{d}x^2 = 2\pi q_e F_{12}^{(e)} r^1 r^2 \alpha'. \tag{5.95}$$

We have now found the shift in the oscillation modes, if the branes live on a factorizable torus. However, we still need to find the eigenvectors, if we want to get an explicit expression for the string motion. These can be found using (5.83), (5.87) and Gaussian elimination

$$\lambda = \frac{T_2^2 - \mathcal{B}_{12}^2 - 2iT_2\mathcal{B}_{12}}{T_2^2 + \mathcal{B}_{12}^2}; \qquad (R^m{}_n - \lambda\delta^m{}_n)C^n = 0 \tag{5.96}$$

$$\begin{pmatrix} U_1 + iU_2 & |U|^2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C^1 \\ C^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.97}$$

The components C^1 and C^2 are now determined by the equations

$$C^1(U_1 + iU_2) + C^2|U|^2 = 0 \tag{5.98}$$

$$\frac{C^1}{C^2} = -\frac{|U|^2}{U_1 + iU_2} = -(U_1 - iU_2). \tag{5.99}$$

To solve these, we set

$$C^1 = U^* \qquad \text{and} \qquad C^2 = -1 \tag{5.100}$$

$$C^m = A \begin{pmatrix} U^* \\ -1 \end{pmatrix} \qquad \text{and} \qquad C^{*m} = A^* \begin{pmatrix} U \\ -1 \end{pmatrix}, \tag{5.101}$$

where A is a normalization constant.

To determine A, we impose that C^m and C^{*m} are orthonormal (see section 5.1.2) with respect to G_{mn}, the metric on the torus,

$$\delta_{ab} = C_a^{\dagger m} G_{mn} C_b^n \tag{5.102}$$

$$1 = |A|^2 \frac{T_2}{U_2} (U \;\; -1) \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \begin{pmatrix} U^* \\ -1 \end{pmatrix} \tag{5.103}$$

$$= |A|^2 \frac{T_2}{U_2} (2|U|^2 - U_1(U + U^*)) = 2|A|^2 T_2 U_2 \tag{5.104}$$

$$|A|^2 = \frac{1}{2T_2U_2} \tag{5.105}$$

$$A = \frac{e^{i\phi}}{\sqrt{2T_2U_2}}. \tag{5.106}$$

The phase $e^{i\phi}$ is arbitrary, and it is convenient to fix it to $\phi = \frac{\pi}{2}$. The final expressions for the eigenvectors are then

$$C^m = \frac{i}{\sqrt{2T_2U_2}} \begin{pmatrix} U^* \\ -1 \end{pmatrix}; \quad C^{*m} = -\frac{i}{\sqrt{2T_2U_2}} \begin{pmatrix} U \\ -1 \end{pmatrix}. \tag{5.107}$$

We can use this to get explicit expressions for (5.25) and (5.26)

$$\mathcal{E}^{-1m}{}_A = (C_a^m \; C_a^{*m}) = \frac{i}{\sqrt{2T_2U_2}} \begin{pmatrix} U^* & -U \\ -1 & 1 \end{pmatrix}, \tag{5.108}$$

$$\mathcal{E}^A{}_m = \sqrt{\frac{T_2}{2U_2}} \begin{pmatrix} 1 & U \\ 1 & U^* \end{pmatrix}. \tag{5.109}$$

5.4 The Neveu-Schwarz sector

As is our habit, we start by considering the Neveu-Schwarz sector. The ground state is projected out by the GSO-projection, and we therefore only consider the first excited state. Since the oscillator indices in the magnetized directions are shifted, it is no longer obvious which state we should consider. But fortunately, our guiding light, the prospect of a semi-realistic four dimensional theory, provides the answer. The relevant state is of course the one that lives in the four flat, non-magnetic dimensions

$$B^{\dagger\, i}_{\frac{1}{2}} |p; 1, 2\rangle_{NS}, \qquad i \in [2, 3], \qquad M^2 = \frac{1}{2\alpha'} \sum_{a=1}^{3} \nu_a, \tag{5.110}$$

where $|p;1,2\rangle$ just means that we are dealing with a string stretched between branes 1 and 2, and i runs over the flat coordinates transverse to the light-cone. This state is very similar to the one we found in the simple superstring case (see section 2.3.1), except for the very important facts that it has gained a mass and lives only in four dimensions. The fact that it has gained a mass means that it can no longer have only two degrees of freedom. For it to be a massive Lorentz vector, we need a third component. In the case of two separated branes (section 4.2) we found that the direction of the separation vector could be included, but here we are dealing with space-filling branes with zero separation.

To find the missing component, we must consider what happens to the excitations in the other directions. The answer is the same as when we considered separated branes; they become 6 independent scalars.

$$B^{\dagger\,a}_{\frac{1}{2}-\nu_a}|p;1,2\rangle_{NS}, \quad a \in [1,3], \quad M^2 = \frac{1}{\alpha'}\left[\frac{1}{2}\sum_{b=1}^{3}\nu_b - \nu_a\right], \quad 0 < \nu_m < \frac{1}{2}, \tag{5.111}$$

$$B^{\dagger\,a}_{\frac{1}{2}+\nu_a}|p;1,2\rangle_{NS}, \quad a \in [1,3], \quad M^2 = \frac{1}{\alpha'}\left[\frac{1}{2}\sum_{b=1}^{3}\nu_b + \nu_a\right], \quad 0 < \nu_m < \frac{1}{2}. \tag{5.112}$$

Generally, none of these states have the same mass as the vector particle, and thus no superposition of them can be a component of the massive vector. To get a state with the same mass, we must be clever and recall that the new content in the mass formula gives us new options. In particular, we can consider the state

$$\sum_{a=1}^{3} B^{\dagger\,a}_{\frac{1}{2}-\nu_a} A^{\dagger\,a}_{\nu_a}|p;1,2\rangle_{NS}, \quad M^2 = \frac{1}{2\alpha'}\sum_{a=1}^{3}\nu_a. \tag{5.113}$$

This is a scalar with the correct mass that is not projected out by the GSO-projection and we can therefore use it as the third component of the massive Lorentz-vector.

It is also possible to construct a myriad of tensor particles of the form

$$B^{\dagger\,a}_{\frac{1}{2}\pm\nu_a}\prod_{i=1}^{I} A^{\dagger\,a_i}_{\nu_{a_i}}|p;1,2\rangle_{NS}, \qquad a, a_i \in [1,3], \tag{5.114}$$

but they are generally not very interesting.

5.5 The Ramond sector

Moving on to the Ramond sector, it is a simple matter to find the ground state,

$$|p;1,2\rangle_R, \qquad M^2 = 0 \tag{5.115}$$

which is exactly the same result we found in the un-magnetized case. This fact that the Ramond sector ground state mass is unchanged, but the Neveu-Schwarz sector is not, means that supersymmetry is broken. This is a very appealing feature, since, when we construct a semi-realistic model, we want the Ramond sector ground state degrees of freedom to correspond to the fermions of the Standard Model and the Neveu-Schwarz sector degrees of freedom to play the role of gauge bosons. Since these are *not* supersymmetric partners, a semi-realistic theory must have broken supersymmetry.

It would, however, be possible to construct a theory with $\mathcal{N} = 1$ supersymmetry, but we are not interested in that.

5.5.1 Chirality

We saw in section 2.2.5 that the Ramond sector zero-mode oscillation operators obeyed the Clifford algebra

$$\{B_0^\mu, B_0^\nu\} = G^{\mu\nu} \tag{5.116}$$

However, when we have 6 magnetized directions, the above equation only holds for $\mu = 0, \dots, 3$, since there are no B_0^a-operators in magnetized dimensions. This means that we only have a 4-dimensional Clifford algebra and thus that the chirality operator becomes

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = 4B_0^0 B_0^1 B_0^2 B_0^3. \tag{5.117}$$

We then lose chirality in 10 dimensions, but gain it in 4, which is precisely what we were hoping for. It now becomes clear that magnetized branes are very promising objects indeed for constructing a semi-realistic model from string theory. However, this also means that we cannot simultaneously impose Majorana conditions on the effective 4 dimensional theory.

5.6 Reversing the direction

We will first see how reversing the string changes the shifts from magnetization. This is somewhat complicated because the endpoint charge enters in a very inconvenient way, namely through $\mathcal{B}_{emn}$. In the reversed case, we find

$$\hat{\mathcal{B}}_{1\,mn} = B_{mn} - 2\pi\alpha' q_2 F^{(1)}_{mn}, \tag{5.118}$$

$$\hat{\mathcal{B}}_{2\,mn} = B_{mn} - 2\pi\alpha' q_1 F^{(2)}_{mn}. \tag{5.119}$$

This means that there will be no simple relation between the R-matrix and the reversed version $\hat{R}$. However, we do know that $q_2 = nq_1$ where n is an integer since we know that the first Chern class (5.95) must be an integer. For simplicity, we will in the following assume that $q_1 = q_2$.

It is then clear from the definition of R, (5.16), that the new R-matrix is

$$\hat{R}^m{}_n = (R_1^{-1})^m{}_p R^p_{2n}, \tag{5.120}$$

and thus by the same definition that

$$\hat{R}^m{}_n = (R^{-1})^m{}_n. \tag{5.121}$$

It is therefore easy to see that the eigenvalues can be written in the diagonal matrix

$$\hat{\mathcal{R}}_A^B = (\mathcal{R}^{-1})_A^B = \begin{pmatrix} e^{-2\pi i\nu_a}\delta^{ab} & 0 \\ 0 & e^{2\pi i\nu_a}\delta^{ab} \end{pmatrix}. \tag{5.122}$$

When we consider the scalar that formed the last component of our Lorentz-vector, we see that it can be constructed in exactly the same way as before, namely as

$$\sum_{a=1}^{3} B^{\dagger\, a}_{\frac{1}{2}-\nu_a} A^{\dagger\, a}_{\nu_a} |p;2,1\rangle_{NS}. \tag{5.123}$$

Note that this linear combination of scalars and the state $B^{\dagger\, i}_{\frac{1}{2}}|p;2,1\rangle_{NS}$ are independent of our initial assumption that $q_1 = q_2$. This means that our effective 4-dimensional theory is independent of whether that assumption was correct.

5.6.1 String charge

In the same way that a point particle couples to a Maxwell field A_μ, a string couples to the Kalb-Ramond field $B_{\mu\nu}$. This is reflected by the inclusion of an interaction term in the Lagrangian (see equations (5.2) and (5.54)). Since point particles interact with the Maxwell field via a point charge, it is natural to think that strings interact with the Kalb-Ramond field via a string charge, and this is indeed the case [19, pp. 307-311]. Where the point charge is a simple number, the string charge is a vector, and it is at all points tangential to the string. This means that when we reverse the direction of the string, the string charge changes sign. Since a string and one that is reversed, but otherwise identical, share all traits except having opposite charge, they are the anti-strings of the each other [32, p. 14].

Chapter 6

Point particle on a magnetized torus

We now consider the case of a supersymmetric point particle in a constant magnetic field living on a two-dimensional torus. We do this because we want to compare string theory to point particle theory, and this is the simplest example of a low-energy limit of the model we developed in chapter 5 .

It is convenient to start out using the superfield formalism. Here the form of the initial Lagrangian is exactly what we would expect from our bosonic intuition and it is explicitly supersymmetric so we can be sure of getting the right expression also for the fermions.

The superfield is

$$X^i(t,\theta) = x^i(t) + i\theta\psi^i(t), \quad i = 1, 2, \tag{6.1}$$

where θ is a Grassman variable, and the covariant derivative is

$$DX^i(t,\theta) = i\theta\dot{x}^i(t) + i\psi^i(t). \tag{6.2}$$

We can then write the supersymmetric Lagrangian

$$\begin{aligned} L &= -i\int \mathrm{d}\theta \left[\frac{1}{2}DX^i G_{ij}\partial_t X^j + qA_i(X^j)DX^i\right] && (6.3)\\ &= \frac{1}{2}G_{ij}(\dot{x}^i\dot{x}^j - i\psi^i\dot{\psi}^j) + qA^i(x^j)\dot{x}^i + i\frac{q}{2}F_{ij}\psi^i\psi^j && (6.4)\\ &= L_B + L_F && (6.5)\end{aligned}$$

where we have chosen the gauge $A_i(x^j) = -\frac{1}{2}F_{ij}x^j$ and noted that the Lagrangian splits up into a bosonic and a fermionic part. Since these are completely independent, we will consider them one at a time.

6.1 Bosonic part

The bosonic Lagrangian is

$$L_B = \frac{1}{2}\dot{x}^i G_{ij}\dot{x}^j + qA_i\dot{x}^i \tag{6.6}$$
$$= \frac{T_2}{2U_2}\left(\dot{x}^1 + U\dot{x}^2\right)\left(\dot{x}^1 + \bar{U}\dot{x}^2\right) + qA_1\dot{x}^1 + qA_2\dot{x}^2, \tag{6.7}$$

where the metric G_{ij} is given in (5.80), and it should be noted that the position variable x^i is dimensionless and periodic such that $x^i + 1 = x^i$.*

It is convenient to define complex coordinates

$$z = x^1 + Ux^2, \qquad \bar{z} = x^1 + \bar{U}x^2, \tag{6.13}$$

which are periodic under the translations $(z, \bar{z}) \to (z+1, \bar{z}+1)$ and $(z, \bar{z}) \to (z+U, \bar{z}+\bar{U})$.

In this basis the metric takes the form

$$G_{ij}^{(z\bar{z})} = \frac{T_2}{2U_2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.14}$$

and the Lagrangian can be written as

$$L = \frac{T_2}{2U_2}\dot{z}\dot{\bar{z}} + qA_z\dot{z} + qA_{\bar{z}}\dot{\bar{z}}, \tag{6.15}$$

where

$$A_z = -\frac{\bar{U}A_1 - A_2}{U - \bar{U}}, \quad A_{\bar{z}} = \frac{UA_1 - A_2}{U - \bar{U}}. \tag{6.16}$$

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*x^i is related to the physical position variable y^i by

$$x^i = \frac{y^i}{2\pi R^i}, \tag{6.8}$$

where R^i are the radii of the torus, such that

$$y^i = y^i + 2\pi R^i \Rightarrow \tag{6.9}$$
$$x^i = \frac{y^i + 2\pi R^i}{2\pi R^i} = x^i + 1. \tag{6.10}$$

One could also define a third position variable $\tilde{y}^i$ such that

$$\tilde{y}^i = \tilde{y}^i + 2\pi R, \tag{6.11}$$

where R is the same in all directions. This is done by defining

$$\tilde{y}^i = \frac{R}{R^i}y^i = \frac{R}{R^i}(y^i + 2\pi R^i) = \tilde{y}^i + 2\pi R. \tag{6.12}$$

Note that there is no Einstein index summation in these expressions.

We can easily find the conjugate momenta

$$p_z = \frac{\partial L}{\partial \dot{z}} = \frac{T_2}{2U_2}\dot{\bar{z}} + qA_z, \quad p_{\bar{z}} = \frac{\partial L}{\partial \dot{\bar{z}}} = \frac{T_2}{2U_2}\dot{z} + qA_{\bar{z}}, \tag{6.17}$$

and the Hamiltonian

$$H = p_z\dot{z} + p_{\bar{z}}\dot{\bar{z}} - L \tag{6.18}$$
$$= \frac{2U_2}{T_2}(p_z - qA_z)(p_{\bar{z}} - qA_{\bar{z}}). \tag{6.19}$$

Since we want to compare the point particle states with those of string theory, we would like to express the Hamiltonian in terms of a dimensionless quantity. We have already introduced such an object in string theory (see equation (5.95)), and the first Chern class can also be calculated for a point particle,

$$I = \frac{q}{4\pi}\int F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{qF_{12}}{2\pi} = \frac{q(U-\bar{U})F_{z\bar{z}}}{2\pi}, \tag{6.20}$$

where we have expressed the Faraday tensor in terms of the complex coordiantes.

Using the above expression and $A_m = -\frac{1}{2}F_{mn}x^n$, we can write

$$A_z = \frac{\pi I}{q(U-\bar{U})}\bar{z} \qquad\qquad A_{\bar{z}} = -\frac{\pi I}{q(U-\bar{U})}z. \tag{6.21}$$

Before moving on to quantization, we should consider what it means that our system is restricted to a torus. It means that under a translation corresponding to moving around the torus once, nothing should change. However, we have a gauge field, A_i, and its physical properties are unchanged under gauge transformations, we should therefore allow it to transform as

$$(z,\bar{z}) \to (z+1, \bar{z}+1) \qquad \Rightarrow \qquad A_i \to A_i + \frac{1}{q}\partial_i\chi_1 \tag{6.22}$$
$$(z,\bar{z}) \to (z+U, \bar{z}+\bar{U}) \qquad \Rightarrow \qquad A_i \to A_i + \frac{1}{q}\partial_i\chi_U. \tag{6.23}$$

It is a simple matter to calculate explicit expressions for these functions and they are

$$\chi_1 = \pi I\frac{z-\bar{z}}{U-\bar{U}} \qquad\qquad \chi_U = \pi I\frac{z\bar{U}-\bar{z}U}{U-\bar{U}}. \tag{6.24}$$

With these matters made clear, we will now quantize the classical system. To do this, we impose the the canonical commutation relations

$$[x^i, p_j] = i\delta^i_j, \quad [x^i, x^j] = [p_i, p_j] = 0, \tag{6.25}$$

which imply

$$[z, p_z] = [\bar{z}, p_{\bar{z}}] = i, \quad [z, \bar{z}] = [p_z, p_{\bar{z}}] = 0, \tag{6.26}$$

where z can be thought of as having a superscript index and p_z as having a subscript index.

However, we would rather work with creation and annihilation operators satisfying

$$[a, a^\dagger] = 1. \tag{6.27}$$

It is quite simple to show that this relation is satisfied by

$$a = \sqrt{\frac{U_2}{\pi I}}(p_{\bar{z}} + \frac{\pi I}{U - \bar{U}} z), \quad a^\dagger = \sqrt{\frac{U_2}{\pi I}}(p_z - \frac{\pi I}{U - \bar{U}} \bar{z}). \tag{6.28}$$

In terms of these, the Hamiltonian is

$$H_B = \frac{2U_2}{T_2}\left[\frac{(p_z - qA_z)(p_{\bar{z}} - qA_{\bar{z}}) + (p_{\bar{z}} - qA_{\bar{z}})(p_z - qA_z)}{2}\right] \tag{6.29}$$

$$= 2\frac{\pi I}{T_2}\left[\frac{a^\dagger a + a a^\dagger}{2}\right] \tag{6.30}$$

6.2 Fermionic part

We recall from the beginning of this section that the fermionic part of the Lagrangian is

$$L_F = -i\frac{1}{2}\psi^i G_{ij}\dot{\psi}^j + i\frac{q}{2}\psi^i F_{ij}\psi^j \tag{6.31}$$

$$= -i\frac{T_2}{4U_2}(\psi^z\dot{\psi}^{\bar{z}} + \psi^{\bar{z}}\dot{\psi}^z) + iqF_{z\bar{z}}\psi^z\psi^{\bar{z}}, \tag{6.32}$$

where, in analogy with the bosonic case, we have introduced a complex fermionic coordinate

$$\psi^z = \psi^1 + U\psi^2 \qquad \psi^{\bar{z}} = \psi^1 + \bar{U}\psi^2. \tag{6.33}$$

We wish to proceed by calculating the conjugate momenta and finding the Hamiltonian. These are easily found

$$\pi_z = \frac{\partial L}{\partial\dot{\psi}^z} = \frac{T_2 i}{4U_2}\psi^{\bar{z}} \qquad \pi_{\bar{z}} = \frac{\partial L}{\partial\dot{\psi}^{\bar{z}}} = \frac{T_2 i}{4U_2}\psi^z, \tag{6.34}$$

and we see that each momentum is proportional to the other fermionic coordinate. Using this, it is a simple matter to calculate the Hamiltonian

$$H_F = \dot{\psi}^z\pi_z + \dot{\psi}^{\bar{z}}\pi_{\bar{z}} - L_F \tag{6.35}$$

$$= -iqF_{z\bar{z}}\psi^z\psi^{\bar{z}}. \tag{6.36}$$

The canonical anti-commutation relation that will let us quantize the system is

$$\{\psi^i, \psi^i\} = G^{ij} \tag{6.37}$$

$$\{\psi^z, \psi^{\bar{z}}\} = \frac{2U_2}{T_2}, \tag{6.38}$$

where the second expression follows from the inverse of the metric in the complex basis (6.14). However, like before we really want to deal with a set of raising and lowering operators with anti-commutator

$$\{b, b^\dagger\} = 1. \tag{6.39}$$

Using the canonical anti-commutator (6.38), it is easy to show that

$$b = \sqrt{\frac{T_2}{2U_2}}\psi^{\bar{z}}, \quad b^\dagger = \sqrt{\frac{T_2}{2U_2}}\psi^z. \tag{6.40}$$

We can now find the fermionic Hamiltonian

$$H_F = -\frac{iqF_{z\bar{z}}}{2}\left[\frac{\psi^z\psi^{\bar{z}} - \psi^{\bar{z}}\psi^z}{2}\right] \tag{6.41}$$

$$= \frac{2\pi I}{T_2}\left[\frac{b^\dagger b - bb^\dagger}{2}\right], \tag{6.42}$$

and add it to the bosonic Hamiltonian (6.30) to give us the total supersymmetric Hamiltonian

$$H = \frac{\pi I}{T_2}(a^\dagger a + aa^\dagger + b^\dagger b - bb^\dagger) \tag{6.43}$$

$$= \frac{2\pi I}{T_2}(a^\dagger a + b^\dagger b) \tag{6.44}$$

6.3 Comparison with string theory

We now wish to compare the Hamiltonian we have obtained for the point particle on a magnetic torus (6.44) to the one for string theory (5.79). We will work in the so-called field theory limit of string theory $\alpha' \to 0$, which means that only terms of order $\mathcal{O}(1)$ survive.

For simplicity, we will start by considering only the bosonic parts. For the string this is

$$-\alpha' p_{B,s}^2 = \frac{1}{2}\sum_{n=1}^{\infty} n(a_n^{\dagger\, i} a_n^i + a_n^i a_n^{\dagger\, i}) + \frac{1}{2}\sum_{a=1}^{3}\sum_{n=0}^{\infty}(n+\nu_a)(A_{n+\nu_a}^{\dagger\, a} A_{n+\nu_a}^a + A_{n+\nu_a}^a A_{n+\nu_a}^{\dagger\, a})$$
$$+ \frac{1}{2}\sum_{a=1}^{3}\sum_{n=1}^{\infty}(n-\nu_a)(A_{n+\nu_a}^{\dagger\, a} A_{n-\nu_a}^a + A_{n-\nu_a}^a A_{n+\nu_a}^{\dagger\, a}), \quad i = 2,3 \tag{6.45}$$

and for the point particle it is,

$$H_{B,pp} = \frac{qF_{12}}{T_2}\left[\frac{a^\dagger a + aa^\dagger}{2}\right]. \tag{6.46}$$

Since we have only done the point particle calculation for a single torus, and since the tori we have compactified on in the string theory case are factorisable, we only have to consider a single two-dimensional torus when taking the limit of string theory. Furthermore, we are considering the low energy limit of string theory, and therefore only consider the lowest excitation modes, that is the ones with $n = 0$. Before we can take the limit $\alpha' \to 0$, we must investigate how ν behaves for low α'. We have previously found ((5.90) and (5.7)) that

$$\nu = \frac{1}{\pi}\arctan\left(-\frac{B_{12} - 2\pi\alpha' qF_{12}}{T_2}\right), \tag{6.47}$$

where, for simplicity, we have set $\nu_2 = 0$ and only considered $\nu_1 = \nu$ In the limit where string theory correctly approximates a point particle, the Kalb-Ramond field cannot be of any significance, we therefore set $B_{12} = 0$ and Taylor-expand,

$$\nu \approx \frac{2\alpha' qF_{12}}{T_2}. \tag{6.48}$$

Inserting this in the mass formula for the bosonic string (6.45), we find

$$-p_{B,s}^2 \approx \frac{2qF_{12}}{T_2}\left(\frac{A_0^\dagger A_0 + A_0 A_0^\dagger}{2}\right), \quad \text{for } \alpha' \to 0. \tag{6.49}$$

Comparing this to the point particle result, we find

$$-p_{B,s}^2 = 2H_{B,pp}. \tag{6.50}$$

However, this factor 2 is actually not surprising since the point particle Hamiltonian is derived from a Lagrangian of the form

$$L_{B,pp} = \frac{1}{2}\dot{x}^i G_{ij}\dot{x}^j + \ldots, \tag{6.51}$$

and thus it is actually *needed* if the low-energy limit of string theory is to reproduce that of a point particle.

However, there is a subtlety; the point particle calculation is done in dimensionless units, whereas the string calculation is done in dimensionfull units. To find the relation between these, we consider the dimensionless first Chern class (see equations (5.95) and (6.20)), where we for convenience have sat the radii of the string theory torus equal to $\sqrt{\alpha'}$ and $F_{12}^{(1)} = F_{12}, F_{12}^{(2)} = 0$,

$$I^s = 2\pi q F_{12}^s \alpha' \tag{6.52}$$

$$I^{pp} = \frac{qF_{12}^{pp}}{2\pi}, \tag{6.53}$$

where the indices s and pp refer to the string and the point particle respectively. Equating the two Chern classes, we get the relation between the field strengths

$$I^s = I^{pp} = I \Rightarrow (2\pi\sqrt{\alpha'})^2 F^s_{12} = F^{pp}_{12}. \tag{6.54}$$

Recalling the footnote on page 65, we see that since the circumference of the torus in the point particle calculation is 1 as opposed to $2\pi\sqrt{\alpha'}$ in the string theory case. It is therefore natural to use units where $2\pi\sqrt{\alpha'} = 1$ such that the radius of compactification is the same in both cases, and when we do so we find that the bosonic string and the bosonic point particle have the same Hamiltonian in the field theory limit.

6.3.1 The supersymmetric case

Things get more complicated in the supersymmetric case, since in string theory we now have two different sectors against only one for the point particle. From our knowledge of the forms of the Ramond and Neveu-Schwarz sector mass formulas, we do not expect them to have the same low-energy limit. The fact that the Neveu-Schwarz Hamiltonian never acts on a vacuum state due to the GSO projection leads us to believe that it is more likely that the Ramond sector low energy limit matches that of the point particle.

To examine this we use the same approximations as above, and find the low-energy limit of the Ramond sector mass formula

$$\begin{aligned}
-\alpha' p^2_{R,s} = {} & \frac{1}{2}\sum_{n=1}^{\infty} n(B^{\dagger i}_n B^i_n - B^i_n B^{\dagger i}_n) + \frac{1}{2}\sum_{a=1}^{3}\sum_{n=0}^{\infty}(n+\nu_a)(B^{\dagger a}_{n+\nu_a} B^a_{n+\nu_a} - B^a_{n+\nu_a} B^{\dagger a}_{n+\nu_a}) \\
& + \frac{1}{2}\sum_{a=1}^{3}\sum_{n=1}^{\infty}(n-\nu_a)(B^{\dagger a}_{n-\nu_a} B^a_{n-\nu_a} - B^a_{n-\nu_a} B^{\dagger a}_{n-\nu_a})
\end{aligned} \tag{6.55}$$

$$-p^2_{R,s} \approx \frac{2qF_{12}}{T_2}\left(\frac{B^\dagger_0 B_0 - B_0 B^\dagger_0}{2}\right), \quad \text{for } \alpha' \to 0. \tag{6.56}$$

The fermionic part of the point particle Hamiltonian is (6.42)

$$H_{F,pp} = \frac{qF_{12}}{T_2}\left[\frac{b^\dagger b - b b^\dagger}{2}\right]. \tag{6.57}$$

We see that the Ramond sector Hamiltonian is exactly twice that of the fermionic point particle, and as mentioned above, this is just what we expect in the low-energy limit of the fermionic string.

In the Neveu-Schwarz sector, we cannot just take the simple expression for the Hamiltonian as we have done in the bosonic and Ramond sectors. The presence of the normal-ordering term arising from a combination of the bosonic and fermionic coordinates is an essential part of the Hamiltonian, and we must therefore consider the full normal-ordered Neveu-Schwarz sector Hamiltonian (5.79),

$$-\alpha' p_{NS,s}^2 = \sum_{n=1}^{\infty} n A_n^\dagger \cdot A_n + \sum_{a=1}^{3} \left[\sum_{n=0}^{\infty} (n+\nu_a) A_{n+\nu_a}^{\dagger a} A_{n+\nu_a}^{a} + \sum_{n=1}^{\infty} (n-\nu_a) A_{n-\nu_a}^{\dagger a} A_{n-\nu_a}^{a} \right]$$
$$+ \sum_{r=\frac{1}{2}} r B_r^\dagger \cdot B_r + \sum_{a=1}^{3} \left[\sum_{r=\frac{1}{2}}^{\infty} (r+\nu_a) B_{r+\nu_a}^{\dagger a} B_{r+\nu_a}^{a} + \sum_{r=\frac{1}{2}}^{\infty} (r-\nu_a) B_{r-\nu_a}^{\dagger a} B_{r-\nu_a}^{a} \right]$$
$$+ \frac{1}{2} \left(\sum_{a=1}^{3} \nu_a - 1 \right). \quad (6.58)$$

This simplifies considerably when we only consider the lowest excitation modes and only a single torus for the operators. Note that we must still consider the full normal ordering constant.

$$\begin{aligned} -\alpha' p_{NS,s}^2 &\approx \nu A_\nu^\dagger A_\nu + \frac{1}{2} B_{\frac{1}{2}}^{\dagger\, i} B_{\frac{1}{2}}^{i} \\ &\quad + \left(\frac{1}{2} + \nu \right) B_{\frac{1}{2}+\nu}^\dagger B_{\frac{1}{2}+\nu} + \left(\frac{1}{2} - \nu \right) B_{\frac{1}{2}-\nu}^\dagger B_{\frac{1}{2}-\nu} + \frac{1}{2} \left(\sum_{a=1}^{3} \nu_a - 1 \right), \end{aligned} \quad (6.59)$$

$$\begin{aligned} &= \frac{1}{2} \left(B_{\frac{1}{2}}^{\dagger\, i} B_{\frac{1}{2}}^{i} + B_{\frac{1}{2}+\nu}^\dagger B_{\frac{1}{2}+\nu} + B_{\frac{1}{2}-\nu}^\dagger B_{\frac{1}{2}-\nu} - 1 \right) \\ &\quad + \nu \left(A_\nu^\dagger A_\nu + B_{\frac{1}{2}+\nu}^\dagger B_{\frac{1}{2}+\nu} - B_{\frac{1}{2}-\nu}^\dagger B_{\frac{1}{2}-\nu} \right) + \frac{1}{2} \sum_{a=1}^{3} \nu_a. \end{aligned} \quad (6.60)$$

Due to the GSO projection and the fact that we are dealing with the low-energy limit, this operator will only act on states created from the vacuum by using exactly one of the operators $B_{\frac{1}{2}}^{\dagger\, i}$, $B_{\frac{1}{2}+\nu}^{\dagger a}$ or $B_{\frac{1}{2}-\nu}^{\dagger a}$. The anti-commutation relations of these operators tell us that the term proportional to $\frac{1}{2}$ will vanish for all relevant states. The final expression for the low-energy limit of the Neveu-Schwarz sector Hamiltonian is therefore

$$-p_{NS,s}^2 \approx \frac{2qF_{12}}{T_2} \left(A_\nu^\dagger A_\nu + B_{\frac{1}{2}+\nu}^\dagger B_{\frac{1}{2}+\nu} - B_{\frac{1}{2}-\nu}^\dagger B_{\frac{1}{2}-\nu} \right) + \sum_{a=1}^{3} \frac{q_a F_{12}^a}{T_2}, \quad \text{for } \alpha' \to 0. \quad (6.61)$$

This Hamiltonian is fundamentally different from the point particle one we saw above, and we do not currently know of any point particle description that accurately models it.

Chapter 7

A semi-realistic string theory model

We have now introduced the most important concepts needed for constructing a semi-realistic string theory model and are ready to put them together and make one. However, before we can begin comparing any string theory model to the firmly established Standard Model of particle physics, we need to examine it.

7.1 The Standard Model

The theory that most accurately describes the physics of sub-atomic particles as we know them today is called the Standard Model. It is a quantum field theory describing the fundamental particles and their interactions. What we later wish to do is to create a number of string states that have the same properties as these particles. To understand how we can identify a string state with a particle, we need to use the language of gauge theories.

Gauge theories were inspired by the early developments in constructing a theory of fundamental particles. Quantum electrodynamics (QED) developed from the theory of relativistic quantum mechanics first formulated by Dirac in the 1920s. QED was found to have a peculiar symmetry; it is invariant under the so-called local gauge transformations

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \tag{7.1}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x), \tag{7.2}$$

where $\alpha(x)$ is an arbitrary function. While this was interesting, it changed the entire field of theoretical physics when it was realised that one can derive the theory by demanding that the Lagrangian only contains terms that are invariant under the transformations (7.1) and (7.2) [32, pp. 482–483]. What had seemed like a curious symmetry turned out to be an immensely powerful and beautiful principle.

It seems natural that the physicists of the time wanted to generalize this concept of gauge invariance to more general cases. In particular, Yang and Mills [33] proposed that since the phase factor $e^{i\alpha(x)}$ is an element of the simplest continuous symmetry group, $U(1)$, more complicated symmetry groups could give rise to interesting physics. This turned out to be the case. In particular, the theory of the weak interactions is invariant under $SU(2)$ transformations* and the strong force is accurately described by an $SU(3)$ gauge theory. This is the reason we throughout this thesis have demanded that our string theory incorporates the gauge group of the Standard Model

$$SU(3) \times SU(2) \times U(1). \tag{7.3}$$

Gauge groups such as the ones described above are abstract mathematical concepts defined by their Lie algebra. For our purposes we can consider a Lie algebra to simply be a set of commutation relations. To use the gauge groups properly in physics, we need to choose a representation of the symmetry group. This is simply a set of matrices, T^a, whose commutation relations satisfy the Lie algebra. The infinitesimal elements of the group are then given by [32, p. 495]

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2). \tag{7.4}$$

In particular we are interested in the fundamental representations of the groups $SU(2)$ and $SU(3)$. These are sets of unitary 2×2 and 3×3 matrices with determinant 1. The fundamental representation of $SU(2)$ is the well-known set of Pauli spin matrices [32, p. 486]. For $SU(3)$, the fundamental representation is a less well-known set of eight 3×3 matrices. For this gauge group there also exists an anti-fundamental representation which is the complex conjugate of the fundamental one and is equally important for our purposes. The anti-fundamental representation of $SU(2)$ is the same as the fundamental one, since the complex conjugates of the Pauli matrices are equivalent with the non-conjugated ones under a group transformation [32, p. 499].

Before we can begin to uniquely classify the particles of the Standard Model in a way that can be easily reproduced in string theory, we need one last property. The weak hypercharge is, as the name implies, a generalization of the classical concept of electric charge. It comes from the fact that in the Standard Model, the electromagnetic and weak nuclear forces are not actually separate, but parts of the same electroweak force. Weak hypercharge can be found as a combination of electrical charge and weak isospin, the weak force equivalent of the spin of quantum mechanics.

*In the full Standard Model $SU(2)$ symmetry is broken by the Higgs mechanism, but we are not interested in that at this stage.

Electric charge was what gave rise to the $U(1)$ gauge theory QED, it is therefore not surprising that weak hypercharge is what gives rise to the $U(1)_Y$ part of the Standard Model gauge group. With this in mind we can write out the fermion content of the Standard Model conveniently in table 7.1.

Name	Label	Gauge group representation	B	L	Y
Left-handed quark	Q_L^i	$(\mathbf{3}, \mathbf{2})$	1/3	0	1/3
Right-handed up-like quark	U_R^i	$(\bar{\mathbf{3}}, \mathbf{1})$	-1/3	0	-4/3
Right-handed down-like quark	D_R^i	$(\bar{\mathbf{3}}, \mathbf{1})$	-1/3	0	2/3
Left-handed lepton	L_L^i	$(\mathbf{1}, \mathbf{2})$	0	1	-1
Right-handed electron-like lepton	E_R^i	$(\mathbf{1}, \mathbf{1})$	0	-1	2

Table 7.1: Fermion content of the Standard Model. The Gauge group representation in brackets refers to the which representations the particle transforms under. B is baryon number, a conserved quantity associated with quarks and L is the equivalent lepton number. Y is the weak hypercharge, a generalization of electric charge that is appropriate for the electroweak force. The index i refers to multiple generations.

It is important to point out a few things to properly understand table 7.1. First that all the particles are massless, since we have not introduced a gauge symmetry-breaking Higgs-mechanism, which would give masses to the particles. This means that the handedness referred to is the chirality of the particle which is a conserved quantity. Related to this fact is that the anti-particle of a left-handed particle is right-handed. Last, that the particles that transform according to **2**, the fundamental representation of $SU(2)$, are doublets. This means that though they share B, L and Y (see the caption of table 7.1), there are two separate particles for each such doublet. In particular Q_L^1 is a doublet containing the left-handed up and down quarks and L_L^1 is another doublet containing the left-handed electron and electron neutrino. However, we are not interested in this.

7.2 Brane configuration and complications

The first thing we need to find out in order to construct a model that will reproduce the particle content of the Standard Model is the brane configuration. In section 4.5 we found that the gauge group $SU(3) \times SU(2) \times U(1) \times U(1)$ could be constructed using two stacks of branes separated by some distance. However, in this model the Ramond sector ground state fermions became massive and lost chirality (see the beginning of chapter 4.3). We want the fermions to be massless and chiral in four dimensions, a property we found them to have, if we work with space-filling branes in a space of the form $M_4 \times \mathcal{M}_6$ (see section 5), when the branes are magnetized on $\mathcal{M}_6$. A stack of N magnetized branes will still have the gauge group $U(N)$, but since we are dealing with space-filling branes we cannot separate them to create a gauge group $U(A) \times U(B)$, and further more, such a separation would again give the fermions mass and thus cause us to lose four dimensional chirality. The solution then, is to have stacks of branes with different magnetizations. If we have one stack of three branes with magnetization F^a_{mn} and one of two branes with magnetization F^b_{mn} we can break the symmetry group $U(5)$ into $SU(3) \times SU(2) \times U(1) \times U(1)$.

While this seems very appealing; we have the gauge group with only one surplus $U(1)$ symmetry, it turns out that this configuration cannot be used to reproduce the particle content of the Standard Model [14, p. 66]. In order to get the proper transformations for the right-handed leptons of the standard model, we need to have two stacks of a single brane with different magnetizations. This leads to a total of four stacks of branes that in the notation of [14] can be seen in table 7.2 (see also figure 7.1). With this our model starts out with the gauge group

$$SU(3)_a \times SU(2)_b \times U(1)_a \times U(1)_b \times U(1)_c \times U(1)_d. \tag{7.5}$$

We immediately see that there are three $U(1)$'s too many. While this is a problem, it can be shown [14, pp. 62–64] that it is possible to give masses to the particles that are associated with up to three of the $U(1)$ symmetries (see section 7.3.2). This means that the local gauge symmetries will become global symmetries and thus that the gauge group is reduced to that of the Standard Model.

Label	Multiplicity	Gauge group	Name
stack a	$N_a = 3$	$SU(3)_a \times U(1)_a$	Baryonic brane
stack b	$N_b = 2$	$SU(2)_b \times U(1)_b$	Left brane
stack c	$N_c = 1$	$U(1)_c$	Right brane
stack d	$N_d = 1$	$U(1)_d$	Leptonic brane

Table 7.2: Brane content required to obtain the Standard Model particle content.

In general each of the states corresponding to a string stretched between two of the stacks described in table 7.2 will be degenerate and have a number of Landau levels given by their degeneracy numbers [34]

$$I_{\alpha\beta} = \prod_{i=1}^{3} \left[I^i_\beta N_\alpha - I^i_\alpha N_\beta \right], \tag{7.6}$$

where the index i refers to the i'th 2-torus, I^i_α is the first Chern class introduced in (5.95) and N_α is the stack size.

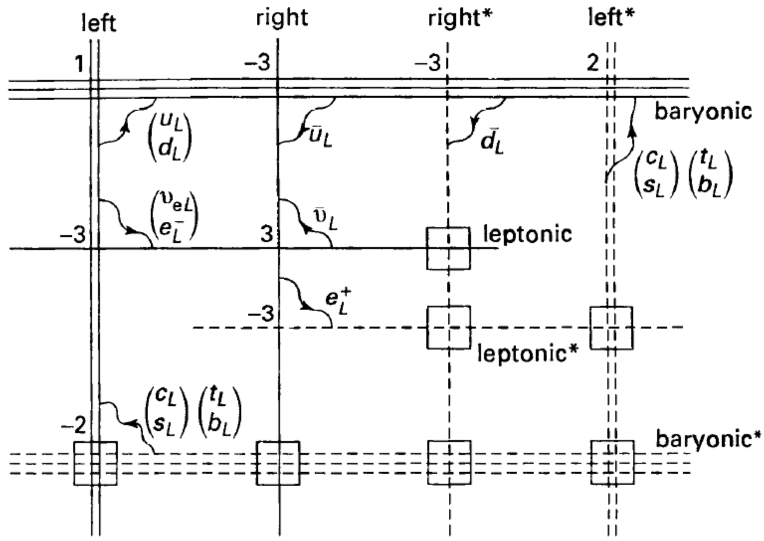


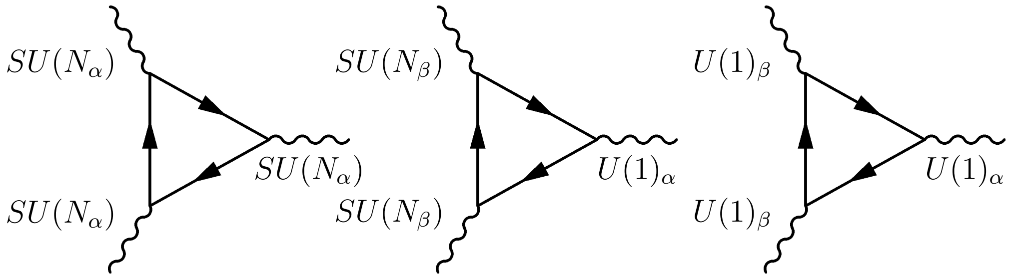
Figure 7.1: Illustration of the brane configuration. In this figure, the branes have intersections instead of shared Landau levels, but that picture is completely equivalent with what we have presented in this thesis. Figure taken from [19].

7.2.1 Orientifolds

As mentioned above, we have almost all the tools we need to turn this brane configuration into a fully fledged semi-realistic model. One of the things we do not yet have is an orientifold compactification. This is a slightly more general form of compactification than the simple toroidal one, we considered in section 5.3. The details of orientifold compactification go beyond the scope of this thesis, but there are several things from it that we need to take into account. First is the fact that it introduces mirror-images of our branes denoted a^*, b^*, c^* and d^*. These turn out to be essential for ensuring an anomaly-free theory (see section 7.2.2). Mirror-branes occur because of the presence of an orientifold plane O9. Op-planes are similar to the Dp-branes already introduced, except that they do not have strings attached to them and are not dynamical objects in their own right. For more information on orientifolds, see [14, pp. 17–23].

The degeneracy of a string attached to a brane and a mirror brane is not the same as that of one attached to two non-mirror branes, it is

$$I_{\alpha\beta^*} = -\prod_{i=1}^{3}\left[I_\beta^i N_\alpha + I_\alpha^i N_\beta\right]. \tag{7.7}$$



(a) Non-abelian diagram (b) Mixed diagram (c) Cubic abelian diagram

Figure 7.2: Triangular Feynman diagrams causing anomalies

7.2.2 Anomalies

Quantum anomalies occur when a symmetry of the classical Lagrangian does not hold for the quantum theory. We are most interested in gauge anomalies, which occur when a Feynman diagram gives a result that violates the gauge invariance that was the premise of the theory. If we have a theory with gauge anomalies, we find that it becomes impossible to get rid of ghosts (see equation (2.45)). The Standard Model is, of course, anomaly-free, and our semi-realistic string theory model must therefore also be anomaly-free.

Before going into gauge anomalies, we wish to consider tadpole cancellation conditions. These arise from the equations of motion of the closed string fields introduced in section 2.3.5 [14, p. 45]. The reason we want to consider these is that if we have tadpole cancellation, the gauge anomalies are simplified significantly. In the orientifold theory, the tadpole cancellation condition is†

$$\sum_\beta N_\beta(I_{\alpha\beta} + I_{\alpha\beta^*}) - 32I_{\alpha,\mathrm{O9}} = 0. \tag{7.8}$$

It also turns out that this condition implies cancellation of non-abelian gauge anomalies of the type $SU(N_\alpha)^3$ [14, p. 67] (see figure 7.2(a)).

We must also take care of mixed anomalies that include both abelian and non-abelian degrees of freedom. When the tadpole condition is satisfied, the mixed anomalies of the type $U(1)_\alpha; SU(N_\beta)^2$ (see figure 7.2(b)) are given by [34, p. 12]

$$\mathcal{A}^{\mathrm{mix}}_{\alpha\beta} = \frac{1}{2}N_\alpha(I_{\alpha\beta} + I_{\alpha\beta^*}). \tag{7.9}$$

The pure abelian anomalies of the type $U(1)_\alpha; U(1)^2_\beta$ (see figure 7.2(c)) are almost

†Strictly speaking, this exact expression only holds true for rectangular tori (ones with $U_1 = 0$ when following the notation of section 5.3). If the tori were skewed, $I_{\alpha,\mathrm{O9}}$ would get prefactors reflecting this. However, this is of little importance since, as we will see later, $I_{\alpha,\mathrm{O9}} = 0$.

the same as the mixed ones, namely [34, p. 11]

$$\mathcal{A}^{U(1)}_{\alpha\beta} = \frac{1}{2} N_\alpha N_\beta (I_{\alpha\beta} + I_{\alpha\beta^*}), \tag{7.10}$$

again assuming that the tadpole cancellation condition is satisfied.

7.3 Getting the Standard Model within string theory

In order to identify strings with the fundamental particle fields listed in table 7.1, we need to know under which gauge group representations they transform. Fortunately, this is very simple. First we need to consider the branes between which the string is stretched and their degeneracy number (see (7.16)), and secondly, if one of the branes is a mirror brane. This leads to four different situations, in which the string will transform according to different bifundamental representations,

$$\begin{aligned} I_{\alpha\beta} > 0 &\Rightarrow (N_\alpha, \bar{N}_\beta) \\ I_{\alpha\beta} < 0 &\Rightarrow (\bar{N}_\alpha, N_\beta) \\ I_{\alpha\beta^*} > 0 &\Rightarrow (N_\alpha, N_\beta) \\ I_{\alpha\beta^*} < 0 &\Rightarrow (\bar{N}_\alpha, \bar{N}_\beta). \end{aligned} \tag{7.11}$$

Before we go into the degeneracy numbers in general, we wish to consider a certain class of strings that does not fit easily into the above description; strings stretched between a brane and its mirror. It is easy to see that in this case, the degeneracy number, (7.7), simplifies to

$$I_{\alpha\alpha^*} = -2 \prod_{i=1}^{3} I^i_\alpha N_\alpha. \tag{7.12}$$

This class of strings does not transform according to the scheme (7.11), but instead according to both the two-index symmetric and two-index antisymmetric representation of $U(N_\alpha)$. No such particles have ever been observed, and we must therefore insist that $I_{\alpha\alpha^*} = 0$ for all α. Since N_α is the stack size, the only way of ensuring this is

$$\prod_{i=1}^{3} I^i_\alpha = 0. \tag{7.13}$$

This is a very interesting result, since we also have that [34, p. 9],

$$I_{\alpha,\mathrm{O9}} = -\prod_{i=1}^{3} I^i_\alpha. \tag{7.14}$$

We therefore find that the tadpole cancellation condition (7.8) simplifies to

$$\sum_{\beta} N_\beta (I_{\alpha\beta} + I_{\alpha\beta^*}) = 0, \tag{7.15}$$

which is also the final expression for cancellation of non-abelian anomalies.

Using this, and the prescription for which bifundamental representation a string will transform according to (see (7.11)), it is easy to find the degeneracies $I_{\alpha\beta}$ that will give the particle content of the Standard Model as described in table 7.1. They are [14, p. 67]

$$\begin{aligned} I_{ab} &= 1, & I_{ab^*} &= 2, \\ I_{ac} &= -3, & I_{ac^*} &= -3, \\ I_{bd} &= -3, & I_{bd^*} &= 0, \\ I_{cd} &= 3, & I_{cd^*} &= -3. \end{aligned} \tag{7.16}$$

Let us see how these degeneracy numbers satisfy the tadpole cancellation condition, (7.15),

$$\begin{array}{lccccccl} \text{Stack } a\text{:} & 2(1+2) & + & (-3-3) & + & 0 & = 0, & (7.17) \\ \text{Stack } b\text{:} & 3(-1+2) & + & 0 & + & (-3+0) & = 0, & (7.18) \\ \text{Stack } c\text{:} & 3(3-3) & + & 0 & + & (3-3) & = 0, & (7.19) \\ \text{Stack } d\text{:} & 0 & + & 2(3+0) & + & (-3-3) & = 0, & (7.20) \end{array}$$

where we have used that $I_{\alpha\beta} = -I_{\beta\alpha}$ and that $I_{\alpha\beta^*} = I_{\beta\alpha^*}$.

Now that we are sure of tadpole cancellation, it is a simple matter to use the degeneracy numbers to write out the string states our model contains in table 7.3.

Start stack	End stack	Label	Multi-plicity	Gauge group representation	Q_a	Q_b	Q_c	Q_d	Y
a	b	Q_L^i	1	$(\mathbf{3}, \mathbf{2})$	1	-1	0	0	1/3
a	b*	q_L^i	2	$(\mathbf{3}, \mathbf{2})$	1	1	0	0	1/3
a	c	U_R^i	3	$(\bar{\mathbf{3}}, \mathbf{1})$	-1	0	1	0	-4/3
a	c*	D_R^i	3	$(\bar{\mathbf{3}}, \mathbf{1})$	-1	0	-1	0	2/3
b	d	L_L^i	3	$(\mathbf{1}, \mathbf{2})$	0	-1	0	1	-1
c	d	N_R^i	3	$(\mathbf{1}, \mathbf{1})$	0	0	1	-1	0
c	d*	E_R^i	3	$(\mathbf{1}, \mathbf{1})$	0	0	-1	-1	2

Table 7.3: String content corresponding to the degeneracy numbers listed in (7.16). The Q's are the charges associated with the different stacks. The hypercharge has been calculated by defining $Y = \frac{1}{3}Q_a - Q_c - Q_d$ [14, p. 67] (see equation (7.24)). The multiplicity corresponds to the number of fermion generations, where in the case of Q_L^i and q_L^i the two must be added.

When considering table 7.3, we immediately notice a new field that was not there in the Standard Model (see table 7.1). This is the N_R^i field corresponding to three generations of right-handed neutrinos. These have never been observed in nature, but the experimental evidence for neutrino oscillations [35] implies that neutrinos have mass and thus that neutrinos of both left- and right-handed chirality must exist. Therefore, while experimentalists have not seen this particle, they have strong evidence that suggests that it actually does exist. In our model, it comes about to ensure that equation (7.15) holds true for stacks c and d.

Another thing that springs to mind when regarding table 7.3 is the fact that the left-handed quarks are made of two kinds of string. They transform according to the same bifundamental representation of the gauge group and have the same hypercharge, but have opposite values of Q_b. We have made this distinction between them to ensure anomaly cancellation just like when we added the right-handed neutrinos.

Notice that the tadpole cancellation for stack b relates the number of generations to the number of colours, and when we calculate the other anomaly cancellation conditions, we find the same relation. What we see is that when we have 3 colours, the number of generations must have 3 as a divisor, otherwise gauge anomalies will occur. The simplest choice that satisfied this is, of course, 3 which is exactly what is found in experiments.

7.3.1 $U(1)$ charges

Before going into detail with how we get rid of our surplus $U(1)$ symmetries, we wish to consider another consequence of their presence, namely their charges. When comparing table 7.3 to table 7.1, we immediately notice that there are similarities between some of the charges of the extra $U(1)$'s and the quantum numbers of the fundamental particles. In fact, we can, to an extent, identify them with each other. We see that

$$Q_a = 3B \tag{7.21}$$
$$Q_d = L, \tag{7.22}$$

where B is baryon number and L is lepton number. However, these turn out to be anomalous when we calculate the mixed and cubic $U(1)$ anomalies. Fortunately, it turns out that the combination

$$Q_a - 3Q_d = 3(B - L) \tag{7.23}$$

is anomaly-free [14, p. 68], and so is Q_c. The physical interpretation of Q_c is somewhat less obvious, since it is not related to one of the quantum numbers of the Standard Model, but it is instead twice the third component of the right-handed isospin, which is a central part of left-right symmetric models [14, p. 68].

We can ensure an anomaly-free expression for the hypercharge by building it from these parts, and the following linear combination gives exactly the expected results;

$$Y = \frac{1}{3}Q_a - Q_c - Q_d. \tag{7.24}$$

The only one of the string charges not yet identified is then Q_b. This charge is something that has no analogue in the Standard Model, it is identified with the Peccei-Quinn symmetry, proposed in 1977 [36] as a way of ensuring that QCD does not violate CP-symmetry. This symmetry has a mixed $SU(3)$ anomaly which is consistent with the Peccei-Quinn symmetry in QCD.

7.3.2 Massive $U(1)$ symmetries

We have seen in the above that the surplus $U(1)$ local gauge symmetries correspond to global symmetries of the Standard Model. We would therefore like to change these symmetries from local ones to global ones. To do this, we consider that there will, a priori, be a photon-like particle associated with each of these symmetry groups. It is this particle that ensures the symmetry locally. It can do this because it is massless, if it were massive it would propagate slowly and the symmetry would stop being local and only hold on the global level. Formally this is very easy to see as a massive particle would introduce terms in the action that are not gauge invariant.

As we have seen, some of the $U(1)$ symmetries have anomalies coming from Feynman diagrams of the types 7.2(b) and 7.2(c), but before we worry about their presence, we should consider if this is the whole picture. It is not. Since we are dealing with string theory, we have several new particles and they too give rise to Feynman diagrams. In particular, one of the fields from the RR-sector of the closed superstring that we mentioned in the end of section 2.3.5 gives rise to the Feynman diagrams 7.3(a) and 7.3(b). It can be shown [14, pp. 58–60] that these diagrams exactly cancel the anomalies from the triangle diagrams.

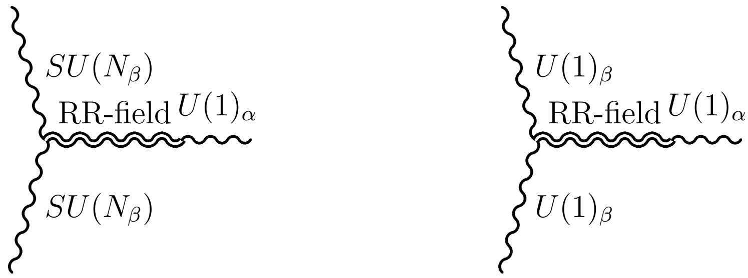
The field that gives us these interactions is the two-index tensor $C_{\mu\nu}$, and it must therefore be included in the low-energy Lagrangian. It enters both through its field strength $H_{\mu\nu\rho}$ and through a coupling with the $U(1)$ fields [14, p. 63],

$$\mathcal{L}_{le} = -\frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} - \frac{1}{4g_s^2}F^{\mu\nu}F_{\mu\nu} + \frac{c}{4}\varepsilon^{\mu\nu\rho\sigma}C_{\mu\nu}F_{\rho\sigma}, \tag{7.25}$$

where c is the strength with which the RR-field $C_{\mu\nu}$ couples to the $U(1)$ field A_μ.

Using the equation of motion for $H^{\mu\nu\rho}$ and the method of Lagrange multipliers, (7.25) can be rewritten

$$\mathcal{L}_A = -\frac{1}{4g_s^2}F^{\mu\nu}F_{\mu\nu} - \frac{c^2}{2}(A_\sigma + \partial_\sigma\eta)^2, \tag{7.26}$$



(a) Mixed RR-sector Feynman diagram

(b) Cubic abelian RR-sector Feynman diagram

Figure 7.3: Feynman diagrams with interactions with RR-sector fields cancelling the $U(1)$ anomalies.

where η is the Lagrange multiplier. At first, this Lagrangian looks unfamiliar, but it is actually very similar to one proposed by Proca [37] for a massive spin 1 vector boson. Comparing the above Lagrangian with Proca's, we see that the vector field A_μ has gained a mass of $M^2 = c^2 g_s^2$.

The $U(1)$ symmetries corresponding to the massive particle will be broken and thus decouple from the gauge group to become global symmetries. The only issue is that of what is left. It turns out that this mechanism only gives mass to three $U(1)$ particles [14, p. 64], so the fact that we were forced to choose a brane configuration with four stacks turns out to be an advantage. We have some choice in which gauge group that is left, so long as we make sure it is an anomaly-free one. To get the same physics as the Standard Mode, we choose to let the linear combination (7.24) remain, so that hypercharge stays a local gauge symmetry.

Since we have given mass to the particles with the charges corresponding to baryon and lepton number conservation, these are exact global symmetries of the theory. This means that the stability of the proton is ensured [14, p. 71]. This is a very appealing feature since proton stability is often hard to ensure.

7.4 Interaction with gravity

So far, this chapter has been all about recreating the Standard Model within string theory, but we have not really addressed why we want to do so. As mentioned in the introduction, the thing that sets string theory apart from quantum field theories is that it naturally incorporates gravity (see section 2.1.7). We should therefore ask how this model interacts with gravity, and if it can explain the apparent weakness of gravity. As seen in section 2.3.5, the graviton is an NSNS-sector closed string. This means that it is not attached to any of the D9-branes and can thus move completely freely in all 10 dimensions.

However, since the extra dimensions are compact, we must take contributions to its mass from winding and Kaluza-Klein modes into account (see section 3.1.1).

Their presence would mean that if the graviton were to have momentum in the compact dimensions, it would gain mass, and if it gained mass, gravity would not go as $\frac{1}{r^2}$ over long distances [27, 2-3 and 3-1]. Fortunately, this problem can be solved if the graviton is not wound around the extra dimensions. In this case, T-duality tells us that the graviton will behave as if the small compact dimensions were not small at all; it would have a continuum of available momenta and winding would be practically impossible.

Since the graviton can move in all 10 dimensions, we find that when particles interact via gravitation, a large number of the gravitons will escape into the extra dimensions where we cannot detect them. This would to us look as if gravity was very weak. However, it is important to realise that this is a general feature of models with extra dimensions, and not something unique to string theory.

Chapter 8

Concluding Remarks

8.1 Conclusions

In this thesis we have developed RNS superstring theory and seen how D-branes are naturally included in this theory through T-duality. Furthermore, we have expanded on these and seen how they can be used to construct an effective 4-dimensional theory. Using space-filling, magnetized branes compactified on a 6-torus, we were also able to break supersymmetry and ensure chirality in four dimensions. Having done this, we showed that the low-energy limit of such a Ramond sector superstring was an analogous point particle.

Finally, we have introduced the few remaining concepts necessary for reproducing the Standard model as a semi-realistic string theory using magnetized D-branes compactified on a 6-torus. This model includes the particle content of the Standard Model and the proper gauge group as well as several important global symmetries. It turns out that our model is a minor extension of the Standard Model, since it also predicts certain things that have not yet been observed. A new particle that this theory predicts is the right-handed neutrino. It is expected to exist since neutrino oscillations prove that neutrinos have mass, and thus that neutrinos do not have definite handedness. The extension proposed herein also predicts Peccei-Quinn symmetry, something that would help resolve the strong CP problem in QCD. Most importantly, since this is a string theoretical model, it naturally incorporates gravity through interactions between open and closed strings.

8.2 Outlook

Though we have put forth string theory as a serious alternative to the ordinary Standard Mode, in the hope that it might later become a candidate for Beyond the Standard Model physics, there remains much to be done. Though our model includes gravity through the NSNS-sector gravitons, the string states corresponding to fundamental particles remain massless and we have proposed no new ways of breaking electroweak symmetry and thus endowing them with mass. It is, of course, possible to construct a Higgs particle within the framework presented in this thesis [14, p. 74–76], but it would be much more satisfying, if there were a more natural way in which string theory gave the particles their observed masses.

Furthermore, the presented model needs to be more thoroughly tested. There could still be gravitational anomalies that are not cancelled in the configuration we have proposed. Also, the presence of the right particles and gauge groups mean nothing, if we find that the amplitudes for particle interactions are completely different from what we observe in nature and can calculate in quantum field theory. The primary reason we wish to calculate the amplitudes of particle interactions within this model is that these can be measured in experiments, and would give us a chance to make string theoretical predictions about particle physics experiments. This is the crucial test that has been the goal of string phenomenology for the last forty years.

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