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Thermodynamics of Blackfolds in String Theory

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Abstract

Using the blackfold approach to construct higher dimensional black holes in string theory, we study the thermodynamics of higher dimensional D0-Dp and F1-Dp -brane charged black holes with different horizon topologies. Different alignments of the fundamental string in the probe brane are considered and analyzed. Furthermore we study the thermodynamics of the six dimensional D1-D5-P-brane charged thin black ring, and studied its extremal behaviour.

To my family and to the loving memory of my grandparents.

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Introduction

General relativity has been one of the deepest physical theories of the 20th century. The novel insight of this theory is that the gravitational force is just a reflection of the geometry of space time. This arises from the intuition that the source of the gravitational field, deflects the sorrounding spacetime so that particles following their own trajectory will not perceive any external force, but rather they are just following the shortest path between points in this curved background.

The action governing pure gravity in four dimensions is the Einstein action,

$$S = \frac{1}{8\pi G} \int dx^4 R\sqrt{-g}\,, \tag{1}$$

whose equation of motion are the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2}$$

The solutions of (2) are the components of the metric tensor. The inertial trajectories of particles in the resulting geometry are then calculated using the geodesic equation.

One of the main novelty of looking at the nature of the gravitational force in this way is that now also photons "feel" gravity and in following their own geodesic they follow curved paths, which has no counterpart in the Newtonian theory. In this new theory of gravity there is nothing which could prevent geometries where regions of spacetime are as much curved that even photons start to follow a future directed close path, so that they are confined in it. We refer to these solutions as black hole solutions, and the surface where the geodesics of photons close off define an event horizon. Furthermore black hole solutions posses a singularity, where spacetime curvature becomes infinite, and where general relativity fails to provide a proper description. Indeed to study the intimate nature of black holes, we need a quantum description of gravity.

Nowadays there are several observations which show that black holes really exist in our universe. Anyway the possible black hole solutions in four dimensional general relativity is very limited. They are characterized indeed at most by three parameters ("No-Hair theorem"), namely mass, charge (if one adds the Maxwell term into the action (1)), and angular momenta, and their horizon is always approximately a round S^2 sphere.

In recent years there has been an increasing attention to study general relativity in more than four spacetime dimensions, and in particular its black hole solutions. Among the main reasons why we should be interested in studying higher dimensional gravity we may mention:

- Reducing five dimensional Einstein pure gravity into four dimensions gives rise to the Einstein equations and the Maxwell field equations (plus a scalar field, the radion), as Kaluza Klein noticed in the early twenties. This opened the scenario of the possible utility in studying higher dimensional gravity and how to get rid of the fact that our universe has only four extended dimensions.
- String theory, which nowadays is the most prominent theory of quantum gravity, requires more than four dimensions.
- If extra dimensions exist in nature, higher-dimensional black holes may be produced in future colliders.
- As mathematical objects, black hole spacetimes are among the most important Lorentzian Ricci-flat manifolds in any dimension.
- Spacetime dimensions can be viewed as a tunable parameter, so that we can study which properties of black holes are universal and which ones are dependent on the dimension.
- Black holes in higher dimensions show a richer phase structure, and the existence of critical dimensions has been observed, above which some properties of black holes can change drastically.

In recent years our understanding of the dynamic of black hole solutions in 5 dimensions has been greatly developed. In addition to Myers-Perry black holes [1], different exact solutions have been constructed using different techniques [2] [3] [4] [5], thus five dimensional rotating black rings [6] [7] and other non trivial geometries like multiple black hole solutions (i.e. black saturns and multi black rings [8] [9] [10] [11]) are now well known . Furthermore a class of algebraic classification of spacetime theorems on how to determine uniquely the black holes solutions with two symmetry axes [12] [13], has been shown to exist. It seems indeed possible that the entire zoo of five dimensional black holes with two axials Killing vectors have been found by now.

Unfortunately these techniques have not been successfully extended to more than five dimensions and, a part from some exact solutions [1], black holes in $D \geq 6$ remain *terra incognita*. Basically these difficulties arise due to the fact that the most general black holes in $D \geq 6$ are not contained in the generalized Weyl ansatz ([14] [15] [16]), since they do not posses $D-2$ Killing symmetries as their lower-dimensional counterpart. As a consequence it is not possible to apply the inverse scattering technique of [2] [3] [4] to asymptotically flat black objects in $D \geq 6$.

Within this scenario, a new method has been recently proposed, called *blackfold approach*, to study the physics and the dynamics of a certain classes of higher dimensional black holes, namely the ones with ultra-spinning regimes. This approach started in [17] with the construction of thin black rings in $D \geq 5$, then in [18] a general framework has been presented, and for the first time the term "blackfolds" has been used. It was then further developed in [19] and in [20], and it has been fully reviewed in [21].

The aim of this thesis is to study the thermodynamics of different classes of black hole systems in higher dimensions, using the blackfold approach in string theory.

In ch. 1 we will briefly review some relevant black hole solutions in string theory, and how this theory answers to some unsolved questions arising from black hole when treated at a classical or semi-classical level. This chapter is far from be exhaustive, but it is aimed at introducing some basic concepts and ideas needed for this work. In ch. 2 we will introduce the blackfold approach, and we will study the thermodynamics of systems whose exact solutions are well known. These are recovered in an exact manner by this approach. Furthermore novel classes of black holes with new horizon topologies and novel isometry group have been found in [20] and here reviewed. In ch. 3 we will analyze the thermodynamics of a blackfold when it has electric charges, or dipole charges or p-brane charges dissolved in its worldvolume as studied in [22] and [23].

The core of this thesis work starts in ch.4 where we will introduce the blackfold approach in supergravity and string theory as developed in [23]. Then we will apply it to study the thermodynamics of black holes with different horizon topologies, and with two charges of different kind on it. The main focus is on D0-Dp and F1-Dp solutions of the ten dimensional supergravity action. Two different alignments of the fundamental string on the probe brane are considered. Finally, in ch. 5, we will analyze the thermodynamics of the six dimensional thin black ring with D1-D5-P charges on it.

Notation and terminology:

We summarize here some of our notation.

For a blackfold with p spatial dimensions in D-dimension we introduce

$$n = D - p - 3\,.$$

Background and worldvolume quantities are denoted in the following way:

- Background coordinates: X^μ, $\mu, \nu \ldots = 0, \ldots, D-1$.
  Background metric: $g_{\mu\nu}$.
  Background covariant derivative: ∇_μ.
  Background tensors indices: $\mu, \nu, \ldots$.

- Worldvolume coordinates: σ^a, $a, b \ldots = 0, \ldots, p$.
  Worldvolume metric: γ_{ab}.
  Worldvolume covariant derivative: D_a.
  Worldvolume tensors indices: $a, b, \ldots$.

Chapter 1

Black holes in supergravity and string theory

The physics of the 20th century has been founded on two pillars: general relativity and quantum mechanics. The first describes successfully the physics at cosmological scales, while the latter describes with an enormous precision the physics at short scales. Even though these two theories work so successfully at their own scales, one has enormous troubles when one needs to use both, a theory of quantum gravity, to describe physical systems or interactions. Attempts in constructing a quantum theory of gravity face the problem of non-renormalizability of the theory.

There are several reasons for physicists to believe that a quantum theory of gravity is necessary. Indeed at scales of order of the Planck length black holes arises simply as consequence of the uncertainty principle. Therefore, it seems necessary to develop a unified theory to describe the physics at these scales. Nowadays the leading candidate for a unified theory is string theory. Apart from reconciling general relativity and quantum mechanics on small scales, this theory gives new deep insights on how physics might work at these scales, but more generally it gives a new very powerful organizing framework to deal with physical problems.

Since in this thesis work we are mainly interested in black hole physics, in the next sections we will briefly review the main aspects of black holes in string theory, and how this theory answers to some open questions arising from classical and semi-classical black holes.

1.1 Black holes in string theory

Classically, black holes are regions of spacetime in which photons (and any massive particle) are confined by the gravitational force. This defines an event horizon as a surface in which future directed null paths become tangent to the boundaries of the surface. In addition, there is a singularity hidden behind the horizon.

In the early seventies, a number of laws that govern the physics of black holes were established [24, 25, 26], and it has been argued there is a very close analogy between these laws and the four laws of thermodynamics[25], [26].

It is natural to wonder whether this formal similarity is more than just an analogy. At the classical level one immediately runs into problems: classically black holes only absorbs, so their temperature is strictly zero. In [27], however, Hawking showed that quantum mechanically, black holes emit thermal radiation corresponding to a certain temperature. The temperature was found to be $T = \kappa/2\pi$, where κ is the surface of gravity. Then from the first law follows the "Bekenstein-Hawking entropy formula" [26],

$$S = \frac{A}{4G_N}\,. \tag{1.1}$$

There are a lot of subtleties in black hole thermodynamics. Firstly the entropy of black holes scales as the area and not as the volume like in classical systems. Secondly from statistical mechanics we know that temperature and entropy arise from an averaging over the microstates of the systems, but at the classical level black holes (at least four-dimensional ones) possess at most three parameters (mass, angular momenta, charge) which completely characterize the solutions (" No-Hair" theorem). Last but not least, since black holes radiate, they lose mass and they may eventually evaporate, and, even if this phenomenon does not occur for stellar masses black holes (where the time for a full evaporation may be longer than the lifetime of the universe), it is supposed to occur for microscopic black holes. This fact leads to an important paradox. The matter that falls into a black hole carries information. The outgoing radiation, however, is purely thermal, so structure-less. What happens to the information stored in the black hole when it completely evaporates? This is still an open question, for which there are no answers if one continues to deal with these systems at a classical or semiclassical level. Let us note, however, that the solutions of these problems may be related to the question of understanding the microscopic description of black holes.

Any consistent theory of quantum gravity should address and provide answers to these open problems. In the following we will see, with a specific example, how string theory provide an answer to the microscopic nature of black holes entropy, while, since the information paradox is outside the scope of this thesis, we just briefly mention now that the string theory interpretation is the holography principle.

We need firstly to identify what are good candidates in string theory to understand black hole physics. Black holes arise in string theory as solutions of the corresponding low-energy supergravity theory. String theory lives in 10 dimensions (or 11 from the M-theory perspective). Suppose now one compactifies the theory on a compact manifold down to $p + 1$ spacetime dimensions. Branes wrapped in the compact dimensions will look like point-like objects in the $p+1$-dimensional spacetime. So, the idea is to construct a configuration of intersecting wrapped branes which upon dimensional reduction yields a black hole spacetime.

1.1.1 BPS branes

Let us consider type II A/B supergravities, which are the low energy limit of type II A/B string theories. There are two sectors of massless bosonic modes of Type-II strings: NS-NS and R-R.

In the NS-NS sector we have the string metric $G_{\mu\nu}$, the two-form potential B_2, and the scalar dilaton Φ. In the R-R sector we have the n-form potential C_n, where n is even for IIB and it is odd for IIA. Thus these are all the fields involved in our construction.

Now recall that in $d=4$ electromagnetism, an electrically charged particle naturally couples to A_1 (or its field strength F_2), while the dual field strength ${}^*F_2$ gives rise to a magnetic coupling to point particles. This is a " miracle" happening only in four dimensions, where both electrical and magnetic charges are carried by point particles. Indeed let's uplift to $d=10$ dimensions, here a p-brane couples to $C_{n=p+1}$ "electrically", or C_{7-p} "magnetically". As a result, we find 1-branes 'F1' and 5-branes 'NS5' coupling respectively electrically and magnetically to the NSNS potential B_2, and p-branes 'Dp' coupling to the R-R potentials C_{p+1} (or their Hodge duals). Reviews of p-branes in string theory can be found in [28], [29], while review on couplings of branes can be found in [30], [31].

In the following we will focus mostly on BPS systems, since rules and basic ideas are much simpler when the BPS condition is satisfied. Basically this is a condition involving equality of mass and charge that leads to unbroken supersymmetry in the spacetime near the horizon, and has the consequence that gravitational attractions are balanced by electro-magnetic repulsions. This results in the cancellation of quantum corrections to the effective action for string theory, so that we can work to lowest order in perturbation theory.

Parallel BPS branes are then in static equilibrium with each other. In addition, BPS p-branes and q-branes, for some choice of p and q, can be in equilibrium under certain conditions, but before trying to describe complicated configurations of branes, it is useful to consider the description of parallel branes of the same kind in supergravity and string theory.

Let us now consider eleven dimensional supergravity. Then BPS Mp-branes are solitonic objects charged with respect to a $(p+2)$-form U(1) field strength $F_{(p+2)}$. If the brane carries no other charges then it is a solution of a subsector of the supergravity theory with eleven dimensional action, in the Einstein frame,

$$S=\frac{1}{16\pi G_{11}}\int \mathrm{d}^{11}x\sqrt{-g}\left(R-\frac{1}{2(p+2)!}F^2_{(p+2)}\right)\,, \tag{1.2}$$

where G_{11} is the 11-dimensional Newton's constant,

$$G_{11}=\frac{(2\pi)^8 l_P^9}{16\pi}\,. \tag{1.3}$$

It can easily be checked that a solution to the equations of motion is

$$\mathrm{d}s^2 = H^{-\frac{8-p}{9}}\mathrm{d}x^2_{(1,p)}+H^{\frac{p+1}{9}}\mathrm{d}x^2_{(10-p)} \tag{1.4}$$

$$F_{(p+2)} = \pm\mathrm{d}(H^{-1})\wedge\epsilon_{1,p}\,, \tag{1.5}$$

where $\mathrm{d}x^2_{(1,p)}$ is the $(p+1)$-dimensional Minkowski metric with volume form $\epsilon_{1,p}$ and

$$\mathrm{d}x^2_{(10-p)}=\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{9-p}$$

is the $(10-p)$-dimensional Euclidean metric. Observe that the same function appears both in front of the metric and the gauge field. This is not surprising since it is a consequence of supersymmetry. Anyway supersymmetry alone is not enough to derive the equations of motion the function H must satisfy. One, instead, needs to solve the supergravity equations of motion. Then one finds that H must be harmonic as it satisfies the Laplace equations $\partial_\perp^2 H_p(r) = 0$, in the transverse space $x_\perp \equiv x_{(10-p)}$. Thus the solution is

$$H_p = 1 + \frac{c_p N_p}{r^{8-p}}\,, \tag{1.6}$$

where N_p is the number of coincident branes and c_p a constant revealing the harmonic nature of H_p (it is the eleven dimensional generalization of d_p defined below (1.12)).

Let us now move down to ten dimensions. In the string frame, the relevant part of the supergravity action is[1] [32], [33],

$$S = \frac{1}{16\pi^7 G_{10}} \int d^{10}x \sqrt{-g}\big[e^{-2\phi}\big(R + 4(\partial\phi)^2 - \frac{1}{12}|H_3|^2\big) - \frac{1}{2(p+2)!}\,|F_{p+2}|^2\big]\,, \tag{1.7}$$

the solutions are given by [33],

$$ds^2_{st} = H_i^\alpha[H_i^{-1}\left(-dt^2 + dx^2_\parallel\right) + dx^2_\perp]\,, \tag{1.8}$$

$$e^\phi = H_i^\beta\,, \tag{1.9}$$

$$A^{(p+1)}_{01\cdots p} = H_i^{-1} - 1,\ \text{“electric”},\quad \text{or}\quad F_{8-p} = \star dH_i,\ \text{“magnetic”}\,, \tag{1.10}$$

where $A^{(p+1)}$ is either the RR potential $C^{(p+1)}$, or the NSNS 2-form B, depending on the solution. The subscript $i = \{Dp, F1, NS5\}$ denotes which solution (Dirichelet p-brane, fundamental string or solitonic fivebrane, respectively) we are describing.

The function $H_p(r)$ is harmonic; it satisfies $\partial^2_\perp H_p(r) = 0$, and we can write it as

$$H_i = 1 + \frac{\mathcal{Q}_i}{r^{(7-p)}}, \qquad p < 7\,. \tag{1.11}$$

The constant part was chosen to be non-zero in order the solution to be asymptotically flat, and then equal to one as a normalization condition. The values of the parameters α and β for each solution are given in tab. 1.1.1. In the same table there are also the values of the charges $\mathcal{Q}_i$, as function of the constant

$$d_p = (2\sqrt{\pi})^{5-p}\Gamma(\frac{7-p}{2})\,, \tag{1.12}$$

required for H_i to be harmonic.

Dp-branes	$\alpha = 1/2$	$\beta = (3-p)/4$	$\mathcal{Q}_{Dp} = d_p N_p g_s l_s^{7-p}$
F1	$\alpha = 0$	$\beta = -1/2$	$\mathcal{Q}_{F1} = d_1 N_1 g_s^2 l_s^6$
NS5	$\alpha = 1$	$\beta = 1/2$	$\mathcal{Q}_{NS5} = N_5 l_s^2$

Table 1.1: *p-brane solutions of Type II theories.*

These solutions are interpreted as N_p coincident branes with a $(p+1)$-dimensional Minkowski worldvolume located at $r = 0$.

[1] this is only the bosonic sector of the action

1.1.2 Smearing of branes

Consider now only the BPS Dp-branes solutions in (1.8). Since the equation for H_p, $\nabla_\perp^2 H_p = 0$, is linear, we can now construct BPS multi-centre solutions of the form

$$H_{\overline{p}} = 1 + d_p g_s N_p l_s^{7-p} \sum_i \frac{1}{|x_\perp - x_{\perp\, i}|^{7-p}} \,. \tag{1.13}$$

This works because, as we have seen before, BPS branes of the same kind are in static equilibrium as a result of the balance between repulsive (gauge) and attractive (gravitational and dilatonic) forces.

Let us now make an infinite array of Dp-branes along the x^{p+1} direction with periodicity $2\pi R$,

$$H_{\overline{p}} = 1 + d_p g_s N_p l_s^{7-p} \sum_{n=-\infty}^{\infty} \frac{1}{[\hat{r}^2 + (\, x^{p+1} - 2\pi R n\,)^2]^{\frac{1}{2}(7-p)}} \,, \tag{1.14}$$

where

$$r^2 \equiv \hat{r}^2 + (x^{p+1})^2 \,. \tag{1.15}$$

Assuming R to be very small, or alternatively $x_\perp \gg R$, then the sum varies slowly with n and we can approximate it by an integral. Changing variables to u,

$$x^{p+1} \equiv 2\pi R n - \hat{r} u \,, \tag{1.16}$$

we obtain

$$\begin{aligned} H_{\overline{p}} &\simeq 1 + d_p g_s N_p l_s^{7-p} \frac{1}{2\pi R \hat{r}^{7-[p+1]}} \int du \frac{1}{\sqrt{1+u^2}^{(7-p)}} \\ &= 1 + d_p g_s N_p l_s^{7-p} \frac{1}{2\pi R \hat{r}^{7-[p+1]}} \sqrt{\pi} \frac{\Gamma\left[\frac{1}{2}\,(7-(p+1))\right]}{\Gamma\left[\frac{1}{2}(7-p)\right]} \,. \end{aligned} \tag{1.17}$$

If we identify, in analogy with (1.12), $d_{p+1} = (2\sqrt{\pi})^{5-(p+1)}\Gamma(\frac{7-(p+1)}{2})$,, we can write (1.17) as

$$H_{\overline{p}} \simeq 1 + \left[\frac{N_p}{(R/l_s)}\right] g_s d_{p+1} \left(\frac{l_s}{\hat{r}}\right)^{7-[p+1]} \,. \tag{1.18}$$

Let us take the limit where the arrayed objects make a linear density of branes. Then matching the smeared harmonic function with the $(p+1)$-brane harmonic function H_{p+1} gives

$$N_{p+1} = \frac{N_p}{(R/l_s)} \,. \tag{1.19}$$

We see that the linear density of p-branes per unit length in string units becomes the number of $(p+1)$-branes. This procedure of arraying branes and then take the limit is known as "smearing" and it results in a larger brane, as depicted in fig. 1.1.

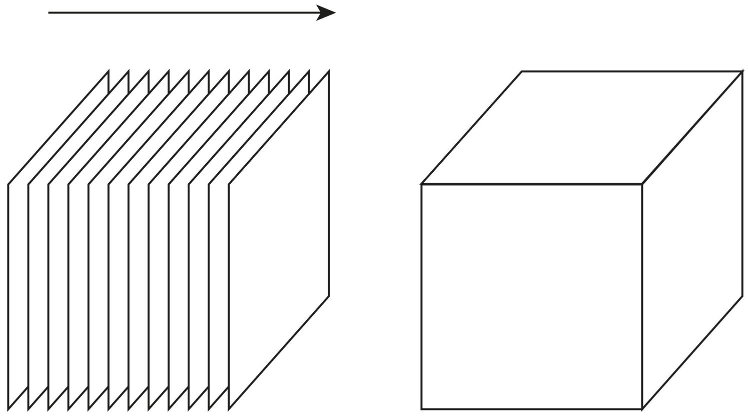

Figure 1.1: *Smearing of a Dp-brane. In the limit where the arrayed objects make a linear density of branes the brane results in a Dp+1-brane.*

1.1.3 Intersection rules

We have just seen that it is possible to array BPS branes of the same kind, and we have anticipated that under certain conditions BPS p-branes and q-branes can be in equilibrium to each other. Furthermore it is possible to have different (or equal) branes intersecting each other. These configurations are of major importance in constructing lower dimensional solutions which have the structure of black holes with finite horizon area at extremality, and in the next we will try to describe which configurations are allowed and which are not.

As example consider the M5-brane solution of the $D = 11$ supergravity, action (1.2). Moreover consider, in the action only the part for the gauge potential A_3, but allowing the possibility of a Chern-Simons term, thus

$$S\left(A_3\right) = \frac{1}{16\pi G_{11}} \int \left\{ -\left[d^{11}x\sqrt{-g}\frac{|F_4|^2}{2(4!)} \right] + \left[\# F_4 \wedge F_4 \wedge A_3\right] \right\} . \tag{1.20}$$

The constant # can be changed by a field redefinition. The field strength F_4 is defined as

$$F_4 = dA_3 \, , \tag{1.21}$$

and it obeys a Bianchi identity

$$dF_4 = 0 \, . \tag{1.22}$$

This implies that the charge

$$Q_5 = \int_{S^4} F_4 \, , \tag{1.23}$$

is conserved, where the integral is over a transverse four-sphere. This is the M5-brane charge. The Bianchi identity also implies that the M5-brane cannot end on anything else. Anyway we can write the field equation for A_3, with a convenient normalisation of #, as

$$\begin{aligned} d^*F_4 &= -F_4 \wedge F_4 \\ &= -\left(dA_3\right) \wedge F_4 = -d\left(A_3 \wedge F_4\right) \, , \end{aligned} \tag{1.24}$$

This tells us that the conserved charge is

$$\hat{Q}_2 = \int_{S^7} \left[{}^*F_4 + A_3 \wedge F_4 \right] . \tag{1.25}$$

This has to be interpreted as the M2-brane charge when the M2-brane ends on the M5-brane, because it is the only object which carries A_3.

To picture this observe that far from the intersection the gauge field A_3 vanishes, thus the charge will be with a good approximation

$$\hat{Q}_2 = \int_{S^7} {}^*F_4 \, , \tag{1.26}$$

which shows it is indeed an M2 brane charge.

As the integration sphere moves towards the intersection, the field due to the presence of the other brane becomes stronger, and the above equality will be no longer true. Anyway, if we reduce the radius of the seven sphere as it approaches the intersection, then the value of the integral remains constant, and we can continue to identify it as the brane charge (1.25). At the intersection the seven sphere will have almost zero radius, but here, and only here, we can deform $S^7 \to S^4 \times S^3$. Furthermore, the components of the field strength F_4 parallel to the M5-brane are approximately zero there, because the flux threads the S^4 in a spherically symmetric way. As a consequence, on the M5, one can write the approximate relation $A_3 \simeq dV_2$, for some two-form V. Then the charge splits into

$$\hat{Q}_2 \simeq \underbrace{\int_{S^3} dV_2}_{\text{string charge}} \ \underbrace{\int_{S^4} F_4}_{Q_5} \, . \tag{1.27}$$

The first factor is the (magnetic) charge of the string which is the boundary of the M2-brane in the M5-brane worldvolume. This means the M2 brane can end on a M5 brane, and the intersection will be a string-like object, and this string is just the boundary of an open brane. We will denote this configuration as M2⊥M5(1).

This procedure can be generalised to find other branes intersections, and we can summarize the main steps of these intersections rules valid also in other supergravity theories in various dimensions (see [34] for more details). One firstly considers the worldvolume field content of the q-brane. If this includes a p-form gauge field V_p, and a $(p+1)$-form gauge potential A_{p+1}, then one can assume a coupling of the form $\int |dV_p - A_{p+1}|^2$ in the q-brane's effective worldvolume action. This leads the q-brane to be as a source for A_{p+1} such that

$$Q_p = \int_{S^{q-p}} *dV_p \, , \tag{1.28}$$

where the integral in the q-brane is over a $(q-p)$-sphere surrounding the $(p-1)$-brane boundary of the p-brane. Thus, the p-brane charge can be transferred to the electric charge of the $(p-1)$-brane boundary living in the q-brane.One can read off from the Chern-Simons terms in the supergravity action if any given p-brane posses a boundary and, if so, in what q-brane the boundary must lie.

1.1.4 Making black holes with branes

Once we know how to construct multiple BPS-branes configurations, we can now address our analysis to the study of how to get black holes solutions from these intersecting configurations.

In order to do that, we need to look at how the metric looks like when we have multiple intersecting branes. A systematic ansatz [35] is available for constructing supergravity solutions corresponding to intersections of BPS branes, which is known as the "harmonic function rule". The ansatz states the metric factories into a product structure and one simply "superposes" the harmonic functions. This ansatz works for both parallel and perpendicular intersections, with the restriction that the harmonic functions can depend only on the overall transverse coordinates. In this way we get only smeared intersecting brane solutions.

Let us discuss it with an example. The convention is as the one used in [30] where $-$ indicates that the brane is extended in that direction, $\cdot$ indicates that it is point-like, and $\sim$ indicates that, although the brane is not extended in that direction, it could be smeared along it. As an example, consider a D5 with a (smeared) D1:

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathrm{D1} & - & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \\ \mathrm{D5} & - & - & - & - & - & - & \cdot & \cdot & \cdot & \cdot \end{array} \tag{1.29}$$

Let us further define $r^2 = x_\perp^2 \equiv \sum_{i=6}^{9}(x^i)^2$ to be the overall transverse coordinate. Then the string frame metric is, using the harmonic function rule,

$$\begin{aligned} ds_{10}^2 = {} & H_1(r)^{-1/2}H_5(r)^{-1/2}\left(-dt^2+dx_1^2\right)+H_1(r)^{1/2}H_5(r)^{-1/2}dx_{2\cdots5}^2 \\ & +H_1(r)^{1/2}H_5(r)^{1/2}\left(dr^2+r^2d\Omega_3^2\right) \,, \end{aligned} \tag{1.30}$$

and dilaton is

$$e^{\Phi} = H_1(r)^{1/2}H_5(r)^{-1/2}\,, \tag{1.31}$$

while the gauge fields are as before,

$$C_{01} = g_s^{-1}\left[1-H_1(r)^{-1}\right]\,, \qquad C_{01\dots5} = g_s^{-1}\left[1-H_5(r)^{-1}\right]\,. \tag{1.32}$$

The independent harmonic functions both go like r^{-2}, which is natural for a D5-brane and also for a D1-brane smeared over four coordinates:

$$H_5(r) = 1+\frac{\mathcal{Q}_5}{r^2}\,, \qquad H_1(r) = 1+\frac{\mathcal{Q}_1}{r^2}\,. \tag{1.33}$$

Observe that all the BPS black holes are extremal: they do not radiate and thus they have zero temperature.

1.1.5 Non-extremal black holes

There is a simple algorithm which leads to the non-extremal version of a given supersymmetric solution [36](constructed according to standard intersection rules). We will give these rules for

M-branes intersections. This is sufficient since thanks to dimensional reduction and T-duality one can get all the other standard intersections of type II branes. It consists of the following steps:

(1) Make the following replacements in the d-dimensional transverse spacetime part of the metric:

$$dt^2 \to f(r)dt^2 \ , \qquad dx_1^2 + \cdots + dx_{d-1}^2 \to f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2 \ , \qquad f(r) = 1 - \frac{r_0^{d-3}}{r^{d-3}} \ , \quad (1.34)$$

and use the following harmonic functions,

$$H_2 = 1 + \frac{\mathcal{Q}_2}{r^{d-3}} \ , \qquad \mathcal{Q}_2 = r_0^{d-3} \sinh^2 \alpha_2 \ ,$$

$$H_5 = 1 + \frac{\mathcal{Q}_5}{r^{d-3}} \ , \qquad \mathcal{Q}_5 = r_0^{d-3} \sinh^2 \alpha_5 \ , \qquad (1.35)$$

for the two-branes and five-branes, respectively.

(2) In the expression for the field strength F_4 of the three-form field make the following replacements:

$$H_2'^{-1} \to H_2'^{-1} = 1 - \frac{\mathcal{Q}_2}{r^{d-3}} H_2^{-1}, \qquad \mathcal{Q}_2 = r_0^{d-3} \sinh \alpha_2 \cosh \alpha_2 \ ,$$

$$H_5 \to H_5' = 1 + \frac{\mathcal{Q}_5}{r^{d-3}}, \qquad \mathcal{Q}_5 = r_0^{d-3} \sinh \alpha_5 \cosh \alpha_5 \ , \qquad (1.36)$$

in the "electric" (two-brane) part, and in the "magnetic" (five-brane) part, respectively. In the extreme limit $r_0 \to 0, \ \alpha_5 \to \infty$, and $\alpha_2 \to \infty$, while the charges Q_F and Q_T are kept fixed. In this case $\mathcal{Q}_5 = Q_5$ and $\mathcal{Q}_2 = Q_2$, so that $H_2' = H_2$, and all the previous analysis holds.

(3) In the case there is a common string along some direction x, one can add momentum along x. Then

$$-f(r)dt^2 + dx^2 \to -K^{-1}(r)f(r)dt^2 + K(r)\left(dx - [K'^{-1}(r) - 1]dt\right)^2 \ , \qquad (1.37)$$

where

$$K = 1 + \frac{\mathcal{Q}_K}{r^{d-3}}, \qquad \mathcal{Q}_K = r_0^{d-3} \sinh^2 \alpha_K$$

$$K'^{-1} = 1 - \frac{\mathcal{Q}_K}{r^{d-3}} K^{-1} \ , \qquad \mathcal{Q}_K = r_0^{d-3} \sinh \alpha_K \cosh \alpha_K \ . \qquad (1.38)$$

In the extreme limit $r_0 \to 0, \ \alpha_K \to \infty$, the charge Q_K is held fixed, $K = K'$ and thus the metric (1.37) becomes $dudv + (K-1)du^2$, where $u, v = x \pm t$.

1.1.6 Entropy counting

Although charged black holes can be constructed in supergravity theories, the important point in constructing them from intersecting brane solutions in ten or eleven dimensions is that we automatically have a string (or M-)theory interpretation. In particular the interpretation of a black hole as a particular configuration of branes allows us to calculate the entropy by counting the number of massless degrees of freedom in string theory. This can be then compared to the area of the lower dimensional black hole horizon, providing a microscopic derivation of the black hole entropy [37]. This assumes one can get the results from a weak coupling computation, which is the case due to the supersymmetry, so that it may be possible that one can follow the black hole from strong to weak coupling.

As an example, consider the extremal black D1-D5-P system which is obtained from (1.30) by boosting it along the x_1 direction [2]. At extremality we can write the harmonic functions as

$$H_5(r) = 1 + \frac{q_5}{r^{d-3}}\,, \qquad H_1(r) = 1 + \frac{q_1}{r^{d-3}}\,, \qquad H_1(r) = 1 + \frac{q_p}{r^{d-3}}\,. \tag{1.39}$$

Then the entropy can be calculated by counting the states [30] of this system,

$$s = 2\pi\sqrt{q_1 q_5 q_p}\,. \tag{1.40}$$

This agrees with the geometrical entropy calculated from the horizon area (1.1). This agreement will be shown explicitly for this system in ch. 5.1.5, and we skip this proof here. Even though this is a very special class of supersymmetric solutions which show this agreement, it gives hope that string theory is on the right track to provides an answer to the statistical entropy of black holes, even if there is still lack of a complete understanding on it.

[2] P is indeed a momentum charge

Chapter 2

Blackfolds

In [7] black rings were introduced as the result of bending a black string into the shape of a circle and spinning it up to balance forces. Even if, as we have mentioned in the introduction, we cannot extend to $D \geq 6$ the techniques that allow us to construct exact black hole solutions in four and five dimensions, we want to hold on the intuition that a long circular black string, or more generally a smooth black brane, could be obtained as a perturbation of a straight one, also in $D \geq 6$. Thus if we could bend the worldvolume of a black p-brane into the shape of a compact hypersurface (e.g. T^p or S^p), we would obtain new geometries and topologies of black hole horizons.

Examples of this kind of perturbation method applied to brane-like objects are often used nowadays. Consider for instance D-branes in string theory. The bending and vibrations of D-branes are generally intractable in an exact manner in string theory, but they are efficiently captured by the Dirac-Born-Infield worldvolume field theory, which is applicable as long as the scale of the perturbations is sufficiently large that locally the brane continues to look like a flat D-brane. Thus the full dynamic of the brane is replaced and approximated by an effective worldvolume theory for a set of "collective coordinates".

With this picture in mind we are looking for an effective theory for describing black branes whose worldvolume is not exactly flat, but where the deviations from it occur on scales much larger than the brane thickness. In [18] this long-distance effective theory has been developed, and the term *blackfold* was introduced to define *black* brane (possibly boosted locally) whose worldvolume spans a curved sub-mani*fold* of the embedding spacetime. Thus in the limit in which we have two widely separate scales, the black hole can be regarded as a blackfold. This approach reveals black objects with novel horizon geometries and topologies more complex than a black ring, but more generally it provides a new organizing framework for the dynamics of higher-dimensional black holes.

One could now wonder when this separation of scales occurs and how to quantify it in order to verify the applicability of the blackfold approach. One should firstly observe that this is a feature of higher dimensional ($D \geq 5$) neutral black holes, where in some regimes their horizons are characterized by at least two separate scales, $r_0 \ll R$, and this feature has no counterpart in

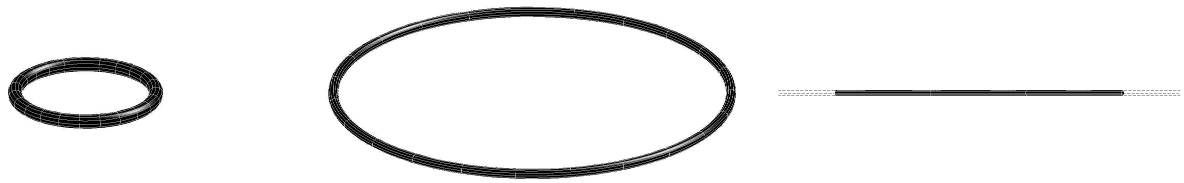

Figure 2.1: *Black ring regimes in $D \geq 5$: 1. small angular momenta; 2.ultraspinnig regime; 3.the black ring becomes very thin and looks locally like a black string.*

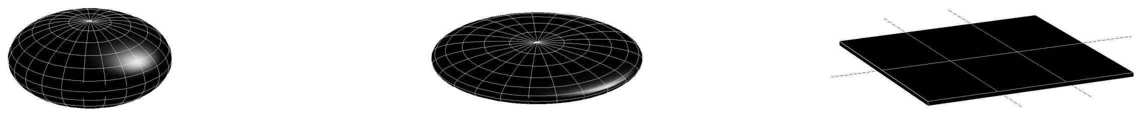

Figure 2.2: *MP black holes regimes in $D \geq 5$: 1. small angular momenta, the black holes behave qualitatively similarly to the four-dimensional Kerr black hole; 2.ultraspinnig regime; 3.the MP black holes pancake and result, locally, in a flat infinitely extended black brane.*

four dimensional ones, since the shape of a Kerr black hole is always approximately round with radius $r_0 \sim GM$ due to a bound in the angular momentum ($J \leq GM^2$) which is required in order to avoid a naked singularity. The effect of this bound is to forbid large distortion of the horizon, so that widely separate scales do not occur. In $D \geq 5$ that bound does not generally hold and, as far as we know from explicit solutions, we can widely separate the two classical length scales J/M and $(GM)^{1/(d-3)}$. Indeed one can see from the exact solution that five-dimensional black rings can have arbitrary large angular momentum for a given mass, and at large J the ring's radius R is much bigger than its thickness r_0 [6]. There is nothing that forbids this regime to hold in $D \geq 6$, even if explicit solutions are not known yet. Thus such a black ring looks locally like, and can be approximated by, a boosted black string slightly curved into a submanifold of a background spacetime as illustrated in fig. 2.1. In a similar manner $D \geq 6$ Myers-Perry black holes have ultra-spinning regimes, in which their horizons pancake, and they look locally like black membranes of small thickness r_0 and large extent R along the plane of rotation [1] [38], as illustrated in fig. 2.2.

When one has to face up with a problem with two widely separate length scales, usually the natural approach is to integrate out the short distance physics and deal with a long distance effective theory. In general relativity there are two different techniques to do this: matched asymptotic expansion [40] and the classical effective field theory [41]. To leading order there is no difference between them, and we will refer to this leading order theory for the long-distance dynamics of higher dimensional black holes as the blackfold approach.

2.1 Effective theory for black 0-branes

In order to understand the general-relativistic aspects of the effective theory of black p-brane dynamics it is instructive to first consider firstly the simpler case of $p = 0$. Thus, our aim is now to study the effective dynamics of a black hole that moves in a background whose curvature radii R are much larger than the black hole horizon radius r_0,

$$r_0 \ll R. \tag{2.1}$$

Thus we have two distinct regions in this geometry. In the region close to the horizon the black hole geometry is well approximated by the Schwarzschild-Tangherlini solution. We will refer to this region as the "near-zone" and we choose a coordinate r centred at the black hole, thus the Schwarzschild geometry is a good approximation as long as $r \ll R$,

$$ds^2_{(near)} = ds^2_{(Schwarzschild)} + \mathcal{O}(r/R)\,. \tag{2.2}$$

The other region is the one far enough from the black hole that its effects on the background geometries are very small, and we can consider this effect as small perturbations. We will refer to this region as the "far-zone" ($r \gg r_0$) and we can write

$$ds^2_{(far)} = ds^2_{(background)} + \mathcal{O}(r_0/r). \tag{2.3}$$

An observer sitting in this "far-zone" is too far away to resolve the size the black hole, and indeed the black hole looks like a point from that point of view. In order to describe its dynamics and its effects on the background geometry we need to describe its motion in spacetime. Thus we label its effective trajectory with $X^\mu(\tau)$, with proper time τ and with velocity $\dot{X}^\mu(\tau) = \partial_\tau X^\mu$ so that $g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = -1$.

We also need to find an effective-stress-energy tensor which encodes how this point-like source affects the gravitational field in the far-zone. In the following we will assume the acceleration and other higher-order derivatives of the particle's velocity to be small. Since these variations are of the order $\sim R$, these higher derivatives must be suppressed by power of r_0/R. Thus to leading order in this expansion, the effective stress-energy tensor is fixed by symmetries and worldline reparametrization invariance,

$$T^{\mu\nu} = m\dot{X}^\mu\dot{X}^\nu\,. \tag{2.4}$$

Thus we have just "integrated-out" the short-wavelength degrees of freedom of the near-zone and we have just replaced them with an effective worldline theory of a point particle. Anyway we need somehow to get rid again that we are dealing with a black object, with a small but finite size, and not with a point-particle. To this end we need to relate the only parameter of the solution, m, to one of the parameters of the "microscopic" configuration, which in this case can only be the horizon thickness r_0, which in turn is related to the temperature. This can be done through a matching calculation (see [42] for more details). Due to the separation of scales (2.1), the near and far zone overlap in

$$r_0 \ll r \ll R \tag{2.5}$$

so that we can match our fields here. In this region the near-zone Schwarzschild solution is in a weak-field regime and can be linearized around Minkowski spacetime. In this zone the background curvature of the far-zone geometry can be neglected, so the far field is the linear perturbation of Minkowski spacetime sourced by (2.4). Thus this two fields match if m is nothing but the ADM mass of the Schwarzschild solution with horizon radius r_0

$$m = \frac{(D-2)\,\Omega_{(D-2)}}{16\pi G} r_o^{D-3} \tag{2.6}$$

This is the first step in the method of "matching asymptotic expansion" [42]. Then one can continue and solve the Einstein equations in a perturbative manner, before in the far-zone, then in the near-zone, and at each step the solutions in one zone provide the boundary terms for the equations one need to solve in the other zone, through the matching of fields in the overlap region. This construction is well understood now but it is a quite complicated procedure. Thus if we are interested only to leading order equations we can restrict to (2.6) and use symmetries and conservation principles as our guideline towards finding the full dynamic. Thus we impose general covariance

$$\overline{\nabla}_\mu T^{\mu\nu} = 0\,, \tag{2.7}$$

where

$$\overline{\nabla}_\mu = -\dot{X}_\mu \dot{X}^\nu \nabla_\nu\,. \tag{2.8}$$

The equation (2.7) has components both in directions orthogonal and parallel to the particle's velocity, so that we can split them up by multiplying it by the following factors

$$(g_{\rho\nu} + \dot{X}_\rho \dot{X}_\nu)\overline{\nabla}_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad m a^\mu = 0\,, \tag{2.9}$$

$$\dot{X}_\nu \overline{\nabla}_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\tau m(\tau) = 0\,, \tag{2.10}$$

with $a^\mu = D_\tau \dot{X}^\mu = \dot{X}^\nu \nabla_\nu \dot{X}^\mu$ being the effective particle's acceleration.

The first equation is nothing but the geodesic equation which determines the trajectory of a test particle [43]. The second equation just states that along the trajectory m is conserved.

Thus the method of the matched asymptotic expansions provides the conceptual backdrop to the blackfold approach, but at the practical level we will remain at the leading order, considering the black brane as a "test-brane" in a background spacetime.

2.2 Effective theory for black p-branes

Our aim is now to extend the previous formalism to a worldvolume theory that describes the collective dynamic of a black p-brane. Here all the complexity of extended objects comes into play. It is then necessary to review in more details the basics of an effective theory. Basically when we are dealing with a system in which there are two widely separate scales, we can split up the degrees of freedom of general relativity into long and short wavelength components,

$$g_{\mu\nu} = \{g_{\mu\nu}^{(\text{long})}, g_{\mu\nu}^{(\text{short})}\} \,. \tag{2.11}$$

So that we can write the Einstein-Hilbert action as

$$I_{EH} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}R \approx \frac{1}{16\pi G}\int d^D x\sqrt{-g^{(\text{long})}}R^{(\text{long})} + I_{\text{eff}}[g_{\mu\nu}^{(\text{long})}, \phi] \,, \tag{2.12}$$

where $I_{\text{eff}}[g_{\mu\nu}^{(\text{long})}, \phi]$ is an effective action obtained after integrating-out the short-wavelength gravitational degrees of freedom, and ϕ is a set of "collective coordinates" which encodes the coupling of the short-wavelength components to the long-wavelength ones. We need now to quantify the length scales that allow us to split these degrees of freedom, and then identify these "collective coordinates". To this end we need to introduce dimensionless quantities in order to be able to compare them. Thus for the case of neutral, vacuum black holes, we can associate to the mass and the angular momentum two length scales,

$$\ell_M \sim (GM)^{\frac{1}{D-3}} \,, \qquad \ell_J \sim \frac{J}{M} \,. \tag{2.13}$$

Thus for higher dimensional black holes we will have three different regimes:

1. $\ell_J \lesssim \ell_M$: black holes behave qualitatively similarly to the four-dimensional Kerr black hole.

2. $\ell_J \approx \ell_M$: threshold of new black hole dynamics.

3. $\ell_J \gg \ell_M$: the separation of scales suggests an effective description of long-wavelength dynamics.

The effective theory sits, and it's applicable, in the last of the above regimes. Indeed in the limit $l_M/l_J \to 0$ higher-dimensional black holes result in a flat, infinitely extended black brane, possibly boosted along some of its worldvolume directions. This has been observed in all the possible exact higher dimensional solutions known so far: in that limit indeed the horizon of Myers-Perry black holes pancakes along the plane of rotation and can be regarded as a black brane [38], fig. 2.2, while black rings become thin and locally look like boosted black string [7] [39], fig. 2.1. The effective theory will then describe the collective dynamics of a black p-brane, which we are going now to identify.

We start with the metric of a straight, flat, static black p-brane in D-dimensions which can be obtained by 'dragging' a n+1-dimensional Schwarzschild-Tangherlini black hole along p straight spatial directions (thus for $p = 0$ one recovers the Schwarzschild-Tangherlini black hole of the previous section, for $p = 1$ a black string, for $p = 2$ a black membrane, and so on)

$$ds^2 = -\left(1 - \frac{r_0^n}{r^n}\right)dt^2 + \sum_{i=1}^{p}(dz^i)^2 + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \,. \tag{2.14}$$

It worth to stress some comments on this solution. The horizon topology is $S^{n+1} \times R^p$, and the brane worldvolume is spanned by p+1 coordinates $\sigma^a = (t, z^i)$. Thus in our description the parameters of this solution are the horizon thickness r_0 and the $D-p-1$ coordinates which parametrize the position of the brane in the directions transverse to the worldvolume, which we denote collectively by $X^\perp$. Let us now boost it along the worldvolume to get the metric of a boosted black brane,

$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \tag{2.15}$$

where $\sigma^a = (t, z^i)$ as before, η_{ab} is the brane worldvolume Minkowski metric, and u^a is a velocity (or boost parameters) such that $u^a u^b \eta_{ab} = -1$.

One should notice we are in the set of the exact solutions since constant shifts of r_0, u^i, and $X^\perp$ still give solutions to the Einstein equations.

In the next we will assume $\partial X^\perp$, $\ln r_0$ and u_i to change slowly along the worldvolume $\mathcal{W}_{p+1}$ under perturbations over a length scale $R \gg r_0$. Under these perturbations we require our metric to preserve manifest diffeomorphism invariance, thus we introduce a gauge redundancy in our system enlarging the set of embedding coordinates of the black brane worldvolume to include all the spacetime coordinates $X^\mu(\sigma)$.

We are now ready to perturb the metric (2.15) with long-wavelength perturbation. The metric we are looking for is a direct consequence of few considerations. In the far-zone $r \gg r_0$ this metric must match the geometry of the far-zone background with metric $g_{\mu\nu}$, in which the thin brane lives. The short-wavelength degrees of freedom live in the 'near-zone' $r \ll R$, thus in the limit where $R \to \infty$, the near-zone solution is (2.15), but when R is large but finite, the collective coordinates take a dependence on the worlvolume ones $x^\mu = X^\mu(\sigma)$. The two metrics $g_{\mu\nu}^{long}$ and $g_{\mu\nu}^{short}$ must then match in the 'overlap-zone' $r_0 \ll r \ll R$.

With this requirements we can now write the near-zone geometry after long-wavelength perturbations as,

$$ds^2 = \left(\gamma_{ab}\left(X^\mu(\sigma)\right) + \frac{r_0^n(\sigma)}{r^n} u_a(\sigma) u_b(\sigma)\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n(\sigma)}{r^n}} + r^2 d\Omega_{n+1}^2 + \dots \tag{2.16}$$

where the dots stand for additional terms, of order $O(r_0/R)$, we have neglected but are required for this to be a solution to Einstein's equations, and where γ_{ab} depends explicitly on the transverse coordinates, and it is naturally interpreted as the geometry induced on the worldvolume

$$\gamma_{ab} = g_{\mu\nu}^{(long)} \partial_a X^\mu \partial_b X^\nu \,. \tag{2.17}$$

Thus the D collective coordinates of the black brane can now be identified as

$$\phi(\sigma^a) = \{X^\perp(\sigma^a), r_0(\sigma^a), u^i(\sigma^a)\} \,. \tag{2.18}$$

2.2.1 Effective stress-energy tensor

The stress energy tensor of the black brane plays the same role as the mass m of the black hole in our previous example, thus, as before, it has to be computed in the overlap region $r_0 \ll r \ll R$. This stress tensor plays a fundamental role in the blackfold construction, since the effects of the solution at distances $r \gg r_0$ are encoded in it and it depends only on the collective coordinates. In other words this stress-energy tensor is such that its effect on the long-wavelength field $g_{\mu\nu}^{(long)}$ is the same as that of the black brane at distances $r \gg r_0$. Thus we can write

$$R_{\mu\nu}^{(long)} - \frac{1}{2} R^{(long)} g_{\mu\nu}^{(long)} = 8\pi G T_{\mu\nu}^{eff} \tag{2.19}$$

where the effective worldvolume stress tensor is

$$T_{\mu\nu}^{eff} = -\frac{2}{\sqrt{-g^{(long)}}} \frac{\delta I_{eff}}{\delta g_{(long)}^{\mu\nu}} \tag{2.20}$$

This is nothing but a generalization of the ADM stress tensor, or as we will see in a while it is equivalent to the Brown-York quasilocal stress-energy tensor [44]. The latter is defined by considering a timelike hypersurface, away from the black brane but which encloses it by extending the worldvolume directions and the angular directions $\Omega_{(n+1)}$, so that it acts as a boundary. Since in our description the angular directions are in the set of the collective coordinates we are integrating out, and do not play any role to leading order, we can focus only on the worldvolume directions of the boundary. Thus we can write

$$T_{ab}^{(quasilocal)} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta I_{cl}}{\delta \gamma^{ab}} \tag{2.21}$$

where γ_{ab} is the one defined in (2.17), and where I_{cl} is the classical on-shell action where the short-distance degrees of freedom are integrated, so it must be the same function of the collective variables as I_{eff}. Thus we can identify (2.20) with (2.21).

In [44] it has been shown the quasi local stress tensor obeys the equation of conservation

$$D_a T^{ab}_{(quasilocal)} = 0 \tag{2.22}$$

where D_a is the covariant derivative arising from the boundary metric γ_{ab}.

The effective stress tensor is computed in the overlap zone ($r_0 \ll r \ll R$) where the gravitational field is weak, and where the quasilocal stress tensor is, to leading order in r_0/R, the same as the ADM stress tensor. The stress tensor of the black p-brane with components along the worldvolume direction is then given by integrating the quasilocal stress-energy tensor over the sphere

$$T_{ab} = \int_{s^{n+1}} T_{ab}^{(quasilocal)} \, . \tag{2.23}$$

Thus we can now proceed further and we can compute this stress tensor for the boosted black p-brane (2.15),

$$T^{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n u^a u^b - \eta^{ab} \right) \tag{2.24}$$

and after introducing slow variations of the collective coordinates

$$T^{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n(\sigma) \Big(n u^a(\sigma) u^b(\sigma) - \gamma^{ab}(\sigma) \Big) + ... \tag{2.25}$$

where the dots stand again for terms with gradients of $\ln r_0$, u^a and γ_{ab} which we are taking to be small and thus are neglected.

As pointed out in [19], the long-wavelength effective theory of any kind of brane will take the form of a derivative expansion for an effective stress-energy tensor that satisfies the conservation equation (2.22), as one can easily see in (2.25). This can be viewed as the dynamics of an effective fluid that lives on the worldvolume spanned by the brane. With this picture in mind we are considering the leading order effective tensor (2.25) as the stress tensor of an isotropic perfect fluid, where the terms we have neglected are the one responsible for dissipative effects. Thus, it is then natural to require the stress-tensor (2.23) to be the one of an isotropic perfect fluid

$$T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab} \,. \tag{2.26}$$

For the black p-brane (2.25) this means to identify

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G}(n+1) r_0^n \,, \qquad P = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \,. \tag{2.27}$$

If we are going into the rest frame of the fluid, at any given point on the worldvolume, the Bekenstein-Hawking identification between horizon area and entropy still applies. Thus we will get an entropy density which is a density over the worldvolume. Thus we calculate it from (2.14), and we get

$$s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \,. \tag{2.28}$$

At any given point over the worldvolume also the identification between surface of gravity and temperature still holds, thus

$$\mathcal{T} = \frac{n}{4\pi r_0} \,, \tag{2.29}$$

which again is a density over the worldvolume.

As one can easily check all these local quantities obey locally to the first law of black hole thermodynamics

$$d\varepsilon = \mathcal{T} ds \tag{2.30}$$

and the Euler-Gibbs-Duhem relation

$$\varepsilon + P = \mathcal{T} s \,. \tag{2.31}$$

2.2.2 Blackfold dynamics

We have seen that the effective theory of brane dynamics can be formulated as a theory of a fluid living on a dynamical worldvolume. The fluid must satisfy a conservation equation (2.22), but we have not told anything yet about the dynamics of its worldvolume geometry.

Worldvolume geometry

The worldvolume $\mathcal{W}_{p+1}$ is embedded in a background whose metric is $g_{\mu\nu}$, and its induced metric is (2.17). Indices $\mu\,,\nu$ are raised and lowered by $g_{\mu\nu}$, and a, b by γ_{ab}. We can then define the first fundamental form of the submanifold to be

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab}\,, \tag{2.32}$$

(for a worldline, $h^{\mu\nu} = -\dot{X}^\mu \dot{X}^\nu$) and we can define the tensor

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu}\,, \tag{2.33}$$

one can see that $h^\mu{}_\nu$ acts as a projector onto $\mathcal{W}_{p+1}$, while $\perp^\mu{}_\nu$ projects along directions orthogonal to it.

Background tensors $t^{\mu\cdots}{}_{\nu\ldots}$ can be converted into worldvolume tensors $t^{a\cdots}{}_{b\ldots}$ and vice-versa using $\partial_a X^\mu$, for instance

$$u^\mu = \partial_a X^\mu u^a \tag{2.34}$$

or

$$T^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \, T^{ab}\,. \tag{2.35}$$

It has been shown in [19] that the last satisfies

$$h^\rho{}_\nu \overline{\nabla}_\mu T^{\mu\nu} = \partial_b X^\rho D_a T^{ab}\,. \tag{2.36}$$

The covariant differentation of tensors living in the worldvolume is well defined only along tangential directions, which we will denote with a bar,

$$\overline{\nabla}_\mu = h_\mu{}^\nu \nabla_\nu\,, \tag{2.37}$$

which is the natural generalization of (2.8).

Then we define the *extrinsic curvature tensor*, or second fundamental form,

$$K_{\mu\nu}{}^\rho = h_\mu{}^\sigma \overline{\nabla}_\nu h_\sigma{}^\rho = -h_\mu{}^\sigma \overline{\nabla}_\nu \perp_\sigma{}^\rho \tag{2.38}$$

This tensor has lower indices μ, ν along the directions tangent to $\mathcal{W}_{p+1}$, and upper index ρ along directions orthogonal to $\mathcal{W}_{p+1}$. Its trace is the *mean curvature vector*

$$K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho = \overline{\nabla}_\mu h^{\mu\rho}\,. \tag{2.39}$$

All of these definitions and relevant proofs can be found in [19] [21].

Blackfold equations

The extrinsic dynamics of a brane has been analysed by Carter in [45]. The equations are formulated in terms of a stress-energy tensor with supports on the p+1-dimensional worldvolume $\mathcal{W}_{p+1}$ satisfying the tangentiality condition

$$\perp^{\rho}{}_{\mu} T^{\mu\nu} = 0\,. \tag{2.40}$$

There are some assumptions in order to apply all of this: we assume this stress tensor must obey the conservation equations

$$\overline{\nabla}_{\mu} T^{\mu\rho} = 0\,. \tag{2.41}$$

which is a direct consequence of the underlying conservative dynamics (General Relativity in this case), even if the effective worldvolume theory may be dissipative. Thus we can decompose (2.41) along direction parallel and orthogonal to $\mathcal{W}_{p+1}$

$$\begin{aligned}
\overline{\nabla}_{\mu} T^{\mu\rho} &= \overline{\nabla}_{\mu}(T^{\mu\nu} h_{\nu}{}^{\rho}) = T^{\mu\nu}\overline{\nabla}_{\mu} h_{\nu}{}^{\rho} + h_{\nu}{}^{\rho}\overline{\nabla}_{\mu} T^{\mu\nu} \\
&= T^{\mu\nu} h_{\nu}{}^{\sigma}\overline{\nabla}_{\mu} h_{\sigma}{}^{\rho} + h_{\nu}{}^{\rho}\overline{\nabla}_{\mu} T^{\mu\nu} \\
&= T^{\mu\nu} K_{\mu\nu}{}^{\rho} + \partial_b X^{\rho} D_a T^{ab}\,,
\end{aligned} \tag{2.42}$$

where in the last line we have used (2.36) and (2.38). These are the most important equations in the blackfold approach since the D equations in (2.41) now have been naturally decoupled into $D-p-1$ equations in direction orthogonal to $\mathcal{W}_{p+1}$ and $p+1$ equations parallel to $\mathcal{W}_{p+1}$,

$$T^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0 \qquad \text{(extrinsic equations)}\,, \tag{2.43}$$

$$D_a T^{ab} = 0 \qquad \text{(intrinsic equations)}\,. \tag{2.44}$$

The extrinsic equations (2.43), if written in terms of the embedding $X^{\mu}(\sigma^a)$ become

$$T^{ab} \perp_{\sigma}{}^{\rho} \left(\partial_a \partial_b X^{\sigma} + \Gamma^{\sigma}_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu}\right) = 0\,, \tag{2.45}$$

or alternatively

$$T^{ab} \left(D_a \partial_b X^{\rho} + \Gamma^{\rho}_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu}\right) = 0\,. \tag{2.46}$$

These can be regarded as a generalization of the geodesic equation to p-branes (where the acceleration is the extrinsic curvature of the worldline, $a^{\rho} = -K^{\rho}$). In other words, this is the generalization of Newton's equation "mass×acceleration= 0" (2.9) to relativistic extended objects.

The second set of equations, (2.44), express energy-momentum conservation on the worldvolume, and generalize (2.10). For a black hole this was a rather trivial equation, but for a p-brane all the complexity of the hydrodynamics of a perfect fluid arises.

If we insert the stress-energy tensor of the black brane (2.24) into the blackfold equations, after some manipulation the extrinsic equations (2.43) become

$$K^\rho = n\perp^\rho{}_\mu \dot{u}^\mu\,, \tag{2.47}$$

and the intrinsic equations (2.44),

$$\dot{u}_a + \frac{1}{n+1} u_a D_b u^b = \partial_a \ln r_0 \tag{2.48}$$

where $\dot{u}^\mu = u^\nu \nabla_\nu u^\mu$ and $\dot{u}^b = u^c D_c u^b$.

Eqs. (2.47) and (2.48) are the *blackfold equations* of a neutral black p-brane. It is a set of D equations that describe the dynamics of the D collective variables of a neutral black brane, in the 'test brane' approximation (to leading order we neglect the backreaction of the blackfold on the background and the dissipative effects on its worldvolume).

It should be stressed that blackfolds are not like other branes. They possess an event horizon, even if in the long-distance effective theory we lose sight of it, since its thickness r_0 is one of the variables we integrate out. Thus in the far-zone they indeed look like other brane objects, but the presence of the horizon is reflected in the existence of an entropy and in the local thermodynamics of the effective fluid. Indeed, we shall assume, the regularity of the event horizon under long-wavelength perturbations is satisfied when the blackfold equations are satisfied. Indeed, as argued in [19], there is a significant evidence that horizon regularity is preserved in the balckfold solutions, moreover [17] and [46] provide evidences that this should be true both for the extrinsic and the intrinsic equations, and finally in [47] this has been proved in details.

2.3 Stationary blackfolds

Equilibrium configurations for blackfolds that remain stationary in time are of particular interest since they represent stationary black holes. Furthermore in the blackfold approach if our system is in stationary equilibrium it is then possible to solve explicitly the intrinsic equations for the worldvolume variables (r_0 and u), and one is left only with the extrinsic equations for the worldvolume embedding $X^\mu(\sigma)$.

For every kind of fluid configuration the requirement of stationarity implies there are no dissipative effects in the fluid, then the stationary fluid (intrinsic) equations translate into the requirement the velocity field to be proportional to a worldvolume Killing field $\mathbf{k} = k^a \partial_a$. That is,

$$u = \frac{\mathbf{k}}{|\mathbf{k}|}\,, \qquad |\mathbf{k}| = \sqrt{-\gamma_{ab} k^a k^b} \tag{2.49}$$

where $\mathbf{k}$ satisfies the worldvolume Killing equation $D_{(a} k_{b)} = 0$.

It should be emphasized that this Killing field is not necessarily a Killing field of the background since the worldvolume could be at a locus of enhanced symmetry. Anyway, this is not generic, thus till the blackfold thickness r_0 is small but not zero, $\mathbf{k}$ should satisfy the killing equation on a finite region around the worldvolume. We will assume it to be the pull-back of a timelike Killing vector $k^\mu \partial_\mu$ in the background,

$$\nabla_{(\mu} k_{\nu)} = 0\,, \tag{2.50}$$

so that $k_a = \partial_a X^\mu k_\mu$, in analogy with (2.34) and (2.35). The existence of this timelike Killing vector is a necessary assumption if we want to describe stationary black holes.

The Killing equation (2.50) implies

$$\nabla_{(\mu} u_{\nu)} = -u_{(\mu} \nabla_{\nu)} \ln |\mathbf{k}|\,, \tag{2.51}$$

so the acceleration is

$$\dot{u}^\mu = \partial^\mu \ln |\mathbf{k}|\,. \tag{2.52}$$

Which plugged into the the intrinsic equation (2.48), and considering the expansion of u to vanish,

$$\partial_a \ln |\mathbf{k}| = \partial_a \ln r_0 \tag{2.53}$$

so we can write

$$\frac{r_0}{|\mathbf{k}|} = \text{constant}\,. \tag{2.54}$$

In [19] it has been argued, due to the matching considerations in the overlap-zone, that it is natural to extend $\mathbf{k}$ to all the geometry (including the near-zone), and it has been shown this Killing vector become null as we approach the horizon $r \to r_0$, so that it is a null generator of the horizon and we can use it to obtain the surface of gravity,

$$\kappa = \frac{n|\mathbf{k}|}{2r_0}\,. \tag{2.55}$$

Thus we have fixed the constant above (2.54) to be

$$\text{constant} = \frac{n}{2\kappa} = \frac{n}{4\pi T} \tag{2.56}$$

where T is the global temperature of the black hole. Plugging (2.56) into (2.54) we can read out the thickness

$$r_0(\sigma) = \frac{n}{4\pi T}|\mathbf{k}| \tag{2.57}$$

adjusts its value along the worldvolume and that T is a constant.

Equations (2.49) and (2.54) allow us to completely solve the intrinsic equations for stationary blackfolds. To proceed, we need to give them a more explicit expression choosing a preferred set of orthogonal commuting Killing vectors of the background geometry. Thus we introduce a timelike ξ and spacelike χ_i Killing vectors on the background, such that $\mathbf{k}$ is a linear combination of them

$$\mathbf{k} = \xi + \sum_i \Omega_i \chi_i\,, \tag{2.58}$$

where Ω_i is a constant we are going to define later, and the index i running at most up to p.

We assume that the background Killing vectors ξ and χ_i are canonically normalized to generate respectively unit time translations and unit space translations at asymptotic infinity, thus their norms on the worldvolume are

$$R_0 = \sqrt{-\xi^2}\Big|_{\mathcal{W}_{p+1}}, \qquad R_i = \sqrt{\chi_i^2}\Big|_{\mathcal{W}_{p+1}}. \tag{2.59}$$

where $R_a(\sigma)$ are a set of worldvolume functions that we must regard to the embedding coordinates $X^\mu(\sigma)$. Thus if ξ is the canonically-normalized generator of asymptotic time translations then we can regard R_0 as a redshift factor that measures the local gravitational redshift between worldvolume and asymptotic time. Furthermore, since χ_i are the generators of the orbits along the worldvolume, we can identify R_i as the proper radii of the orbits generated by them, and the Ω_i as the horizon angular velocities relative to observers that follow the orbits of ξ.

We can thus relate them to vectors that are orthonormal to the metric γ_{ab} on the worldvolume

$$\frac{\partial}{\partial t} = \frac{1}{R_0}\xi, \qquad \frac{\partial}{\partial z^i} = \frac{1}{R_i}\chi_i. \tag{2.60}$$

We also further assume that ξ is an hypersurface orthogonal to the spacelike slices $\mathcal{B}_p$ of the worldvolume $\mathcal{W}_{p+1}$, with unit norm

$$n^a = \frac{1}{R_0}\xi^a. \tag{2.61}$$

It is now convenient to introduce the rapidity η of the fluid velocity relative to the worldvolume time generated by n^a

$$-n^a u_a = \cosh\eta. \tag{2.62}$$

This rapidity has to be seen as the relativistic rapidity associated to worldvolume spatial velocity field

$$\cosh\eta = 1/\sqrt{1-\sum_i V_i^2(\sigma)} = 1/\sqrt{1-V^2(\sigma)}. \tag{2.63}$$

where

$$V_i = \Omega_i R_i. \tag{2.64}$$

It's now trivial to see that we can write

$$\tanh^2\eta = \frac{\sum_i \Omega_i^2 R_i^2}{R_0^2}, \tag{2.65}$$

so that

$$\mathbf{k} = R_0\left(\frac{\partial}{\partial t} + \tanh\eta \sum_i \frac{\partial}{\partial z^i}\right) \tag{2.66}$$

and

$$\begin{aligned} |\mathbf{k}| &= \left(-\xi^2 - \sum_i \Omega_i^2 \chi_i^2\right)^{1/2} \\ &= \frac{R_0}{\cosh\eta}. \end{aligned} \tag{2.67}$$

Thus $|\mathbf{k}|$ can be regarded as the relativistic Lorentz factor at a point in $\mathcal{W}_{p+1}$, with a local redshift, relative to a frame of static observers sitted along ξ. If now we plug (2.67) into (2.55) we get

$$r_0(\sigma) = \frac{nR_0(\sigma)}{2\kappa \cosh\eta} \,. \tag{2.68}$$

2.4 Blackfold thermodynamics

With the blackfold approach we are considering on any point of the spatial section $\mathcal{B}_p$ of $\mathcal{W}_{p+1}$ a small transverse sphere s^{n+1} with radius $r_0(\sigma)$. Thus we can regard the blackfold to a black hole whose horizon geometry is the warped product of $\mathcal{B}_p$ with s^{n+1} (the radius of s^{n+1} is not necessarily constant over $\mathcal{B}_p$).

If r_0 is non-zero everywhere on $\mathcal{B}_p$, then the horizon geometry is a fibration of s^{n+1} on $\mathcal{B}_p$, and the full horizon topology will be

$$(topology\ of\ \mathcal{B}_p) \times s^{n+1} \,. \tag{2.69}$$

If $\mathcal{B}_p$ has boundary then r_0 will shrink to zero size there and this results in a different fibration and horizon topology, but we will see this in details in sec. 2.6.1.

Since in our construction we have regarded the blackfold as a black hole, is now natural to wonder what are the global physical parameters of such black hole. We can thus apply all the formalism of the previous sections.

From (2.29) plugged into (2.57) we can read out

$$\mathcal{T}(\sigma) = \frac{T}{|\mathbf{k}|} = \frac{T}{R_0}\cosh\eta \tag{2.70}$$

the local temperature $\mathcal{T}$ (2.29) along the worldvolume is dictated by the local redshift.

To analyse the horizon geometry we need to look at the metric (2.15). As pointed out in [19] the local area density of the horizon a_H at any given point of $\mathcal{B}_p$ and is, to lowest order in r_0/R,

$$a_{\mathrm{H}} = \Omega_{(n+1)} \left(\frac{n}{2\kappa}\right)^{n+1} \frac{R_0^{n+1}(\sigma^a)}{\cosh^n\eta} \,. \tag{2.71}$$

The total area of the horizon is then

$$A_{\mathrm{H}} = \int_{\mathcal{B}_p} dV_{(p)}\; a_{\mathrm{H}}(\sigma^a) \,, \tag{2.72}$$

where $dV_{(p)}$ denotes the volume form in $\mathcal{B}_p$.

The total entropy is

$$S = \int_{\mathcal{B}_p} dV_{(p)} s u_a n^a \,. \tag{2.73}$$

The entropy density for an effective fluid was given in (2.28), and taking the relativistic Lorentz factor into account,

$$S = \frac{\Omega_{(n+1)}}{4G} \left(\frac{n}{2\kappa}\right)^{n+1} \int_{\mathcal{B}_p} dV_{(p)}\; \frac{R_0^{n+1}(\sigma^a)}{\cosh^n\eta} = \frac{A_H}{4G} \,, \tag{2.74}$$

which agrees with the geometric area computed from (2.72).

The mass and angular momenta are conjugated to the generators of asymptotic time translations and rotations ξ and χ_i. Thus

$$M = \int_{\mathcal{B}_p} dV_{(p)}\; T_{\mu\nu} n^\mu \xi^\nu\,, \qquad J_i = -\int_{\mathcal{B}_p} dV_{(p)}\; T_{\mu\nu} n^\mu \chi_i^\nu\,. \tag{2.75}$$

And using the results of the previous section, they reduce to

$$M = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)}\, \frac{R_0^{n+1}(\sigma^a)}{\cosh^{n-2}\eta}\left(n + \frac{1}{\cosh^2\eta}\right)\,, \tag{2.76}$$

and

$$J_i = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n n\Omega_i \int_{\mathcal{B}_p} dV_{(p)}\, \frac{R_0^{n-1}(\sigma^a)}{\cosh^{n-2}\eta} R_i^2\,. \tag{2.77}$$

2.5 Action principle and first law

In the previous section we have seen a general solution to the intrinsic equations for a stationary blackfold. Using the intrinsic solutions (2.52) and (2.54), the extrinsic equations (2.47) that determine the embedding reduce to

$$K^\rho = n \perp^{\rho\mu} \partial_\mu \ln |\mathbf{k}|^n\,.$$

It has been shown in [19] (see also [21]) that this equation can be obtained by varying the action

$$I = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\, |\mathbf{k}|^n \tag{2.78}$$

in respect to X^μ in direction transverse to $\mathcal{W}_{p+1}$.

We can rewrite it in a more useful form by making some considerations. In (2.60) we have seen the asymptotic time ξ is related to the proper time on the worldvolume t by the redshift factor R_0. Integrations over $\mathcal{W}_{p+1}$ reduce, over an interval $\Delta t = \beta$ of the asymptotic time, to integrals over $\mathcal{B}_p$ with measure $dV_{(p)}$, so

$$\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\, \sqrt{-\gamma}\, |\mathbf{k}|^n = \beta \int_{\mathcal{B}_p} dV_{(p)}\; R_0\, |\mathbf{k}|^n\,. \tag{2.79}$$

Using now (2.67) we find

$$\begin{aligned} I[X^\mu(\sigma)] &= \beta \int_{\mathcal{B}_p} dV_{(p)}\, \frac{R_0^{n+1}(\sigma^a)}{\cosh^n\eta} \\ &= \beta \int_{\mathcal{B}_p} dV_{(p)}\; R_0(\sigma)\left(R_0^2(\sigma) - \sum_i \Omega_i^2 R_i^2(\sigma)\right)^{\frac{n}{2}}\,. \end{aligned} \tag{2.80}$$

Using eqs. (2.72), (2.76), (2.77), it is straightforward to check that the action (2.80) can be written as

$$I = \beta\left(M - \sum_i \Omega_i J_i - \frac{\kappa}{8\pi G} A_H\right)\,. \tag{2.81}$$

which is valid for any embedding.

Thus if we now vary it in respect to $X^\mu(\sigma)$, considering M, J_i and A_H as functional of the embedding, we have for constants κ and Ω_i

$$\frac{\delta I}{\delta X^\mu} = 0 \quad \Leftrightarrow \quad \frac{\delta M}{\delta X^\mu} - \sum_i \Omega_i \frac{\delta J_i}{\delta X^\mu} - \frac{\kappa}{8\pi G}\frac{\delta A_H}{\delta X^\mu} = 0\,. \tag{2.82}$$

Hence, the first law of black hole mechanics is satisfied by solutions of the blackfold equations.

If now we Wick rotate $t = i\tau$, it is natural to take β to be the period of Euclidean time, and associate it to the temperature $\beta = 1/T$ of the system. Now, due to $\kappa A_H/8\pi G = TS$ and eq. (2.81) we see that I is equal, up to a factor β, to the Gibbs free energy $\mathcal{G}$,

$$\beta^{-1} I = \mathcal{G} = M - \sum_i \Omega_i J_i - TS\,. \tag{2.83}$$

Therefore (2.78) can be identified as the effective action that approximates the gravitational Euclidean action of black holes [48].

2.6 Blackfold solutions

With the blackfold formalism in our hands we can try to recover in certain limits some known exact solutions (ultraspinning Myers-Perry black holes and five dimensional thin black ring) in order to provide non-trivial checks to this method. We can further extend this analysis in order to explore new types of vacuum stationary solutions, with new possible horizon topologies, and nevertheless novel spatial isometry group. This work has been developed in details in [18], [19], [20] and summarized in [21], so in the following we will refer to those references for further details and proofs.

The essence of this tool is to take a black p-fold and regard it as, or wrap it around, a topological object. One then uses the extrinsic equations (2.47) in order to find a configuration of equilibrium for the resulting geometry. This will provide a set of solutions for the parameters of the embedding. Then the intrinsic equations, (2.44), will relate the local physical parameters of the blackfold to the global physical parameters of the resulting black object.

2.6.1 Ultraspinning Myers-Perry black holes as even-ball blackfolds

Myers-Perry black holes have ultraspinning regimes in which the horizon pancakes and they exhibit locally a black-brane behaviour. This suggests we can reproduce this regimes through the blackfold construction.

In Minkowski background the blackfold equations do not admit solutions where $\mathcal{B}_p$ is a topological even-sphere. The tension at fixed points of the rotation group cannot be counterbalanced by centrifugal forces, so regular solutions of this kind do not exist. Instead they admit solutions where $\mathcal{B}_p$ is an ellipsoidal even-ball, with thickness r_0 vanishing smoothly at the boundary of the ball, so the resulting horizon topology is S^{D-2}. These configurations are possible for black p-branes.

We will see this spherical solution will reproduce *precisely* the physical parameters of an ultra-spinning Myers-Perry black hole with $p/2$ ultra-spins.

Blackfolds with boundaries

Let us start by analyzing what means for a stationary blackfold to have boundaries, and show why its thickness r_0 should vanish there. If the worldvolume of the black p-brane have some boundaries, we can specify a function $f(\sigma^a)$ such that $f|_{\partial\mathcal{W}_{p+1}} = 0$. If the effective fluid remains within these boundaries, the velocity must remain parallel to them,

$$u^a \partial_a f|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{2.84}$$

At the boundaries the intrinsic equations (2.44) become

$$\left((\varepsilon + P)u_a u_b + P\gamma_{ab}\right) \partial^a f\Big|_{\partial\mathcal{W}_{p+1}} = 0\,, \tag{2.85}$$

and imposing (2.84) we find the pressure must approach zero there,

$$P\Big|_{\partial\mathcal{W}_{p+1}} = 0\,. \tag{2.86}$$

For a neutral blackfold, vanishing pressure (2.27) means

$$r_0\Big|_{\partial\mathcal{W}_{p+1}} = 0\,, \tag{2.87}$$

the horizon closes off at the edge of the blackfold.

If the blackfold is stationary, the condition (2.87) translates, thanks to (2.54), into $|\mathbf{k}| \to 0$. This, by virtue of (2.67), means the fluid velocity approaches the speed of light at the boundary

$$V^2\Big|_{\partial\mathcal{W}_{p+1}} = 1 \tag{2.88}$$

or the boundary is an infinite-redshift surface, $R_0 \to 0$, which will not be taken into consideration in this present case.

Physical parameters

We are now ready to construct the solutions. Let us take a D-dimensional Minkowski spacetime as a background and a planar 2k-fold that spans a (2k+1)-dimensional subspace within it. Since the brane is taken to be flat, the extrinsic equations are trivially satisfied. We assign to this blackfold plane polar coordinates (r_i, ϕ_i). Thus the subspace is

$$ds^2 = -dt^2 + \sum_{i=1}^{k} \left(dr_i^2 + r_i^2 d\phi_i^2\right). \tag{2.89}$$

The blackfold is setted to rotate rigidly (as required by stationarity) along ϕ_i directions, with angular velocities Ω_i. Eqs. (2.49) and (2.66) lead now to

$$u = \frac{1}{\sqrt{1 - \sum_{i=1}^{k}\left(\Omega_i^2 r_i^2\right)}} \left(\frac{\partial}{\partial t} + \sum_{i=1}^{k} \Omega_i \frac{\partial_i}{\partial \phi_i}\right). \tag{2.90}$$

Then (2.68) gives r_0 as function of the radial coordinates r_i

$$r_0\left(r_i\right) = \frac{n}{2\kappa}\sqrt{1 - \sum_{i=1}^{k}\left(\Omega_i r_i\right)^2}\ . \tag{2.91}$$

real values of r_0 imply the extent of the blackfold is limited by

$$\sum_{i=1}^{k}\Omega_i^2 r_i^2 \leq 1\ , \tag{2.92}$$

which defines an ellipsoidal even-ball.

Thus (2.91) agrees with (2.87) and (2.88). Let us now introduce the constant

$$r_+ = \frac{n}{2\kappa}\,, \tag{2.93}$$

and inserting it and the worldvolume velocity field $V^2 = \sum_{i=1}^{k}\Omega_i^2 r_i^2$ into (2.76), (2.77) and (2.74) one gets the physical properties of these solutions to be (for details we refer to [20])

$$M = \frac{r_+^n}{8G}\frac{(1+2k+n)\pi^{k+\frac{n}{2}}}{\Gamma(1+k+\frac{n}{2})}\prod_{\ell=1}^{k}\Omega_\ell^{-2}\ , \tag{2.94a}$$

$$J_i = \frac{r_+^n}{4G}\frac{\pi^{k+\frac{n}{2}}}{\Gamma\left(1+k+\frac{n}{2}\right)}\Omega_i^{-1}\prod_{\ell=1}^{k}\Omega_\ell^{-2}\ , \tag{2.94b}$$

$$S = \frac{\pi}{2G}r_+^{n+1}\frac{\pi^{k+\frac{n}{2}}}{\Gamma(1+k+\frac{n}{2})}\prod_{\ell=1}^{k}\Omega_\ell^{-2}\ . \tag{2.94c}$$

Let us now compare them to the ultraspinning Myers-Perry solutions [1]. The mass, angular momenta and angular velocities are

$$M = \frac{D-2}{16\pi G}\Omega_{(D-2)}\mu\,, \tag{2.95a}$$

$$J_i = \frac{2}{D-2}Ma_i\,, \tag{2.95b}$$

$$\Omega_i = \frac{a_i}{r_+^2 + a_i^2}\ . \tag{2.95c}$$

In the ultraspinning limit in which

$$a_i \gg \mu^{1/(D-3)} \tag{2.96}$$

we have

$$r_+ \to \left(\frac{\mu}{\prod_{i=1}^{k}a_i^2}\right)^{1/(D-2k-3)}\ , \tag{2.97}$$

and

$$\kappa \to \frac{D-2k-3}{2r_+}\,, \tag{2.98}$$

$$S \to \frac{\Omega_{(D-2)}}{4G} r_+^{D-2k-2} \prod_{i=1}^{k} a_i^2 \,. \tag{2.99}$$

Then, substituting

$$\Omega_{(D-2)} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \tag{2.100}$$

into (2.95a) we obtain

$$M \to \frac{r_+^{D-2k-3}}{8G} \frac{(D-2)\pi^{\frac{D-3}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \prod_{i=1}^{k} a_i^2 \,. \tag{2.101}$$

Also in this limit,

$$\Omega_i \to \frac{1}{a_i} \,. \tag{2.102}$$

Thus setting $n = D - 2k - 3$ we can easily see the expressions for the mass, angular momenta, and entropy in (2.94a)-(2.94c), reproduce exactly the ones of the ultraspinning Myers-Perry black hole.

2.6.2 Odd-spheres and product of odd-spheres geometries

In this section, we aim to recover a large class of solutions for black holes in D-dimensional flat space with horizon topology

$$\Big(\prod_{p_a = \text{odd}} S^{p_a} \Big) \times s^{n+1} \,, \qquad \sum_{a=1}^{\ell} p_a = p \,. \tag{2.103}$$

so that the spatial section of the blackfold worldvolume $\mathcal{B}_p$ is now a product of odd-spheres. Our first example will be the case of a single odd-sphere S^{2k+1}, and then we will extend this analysis to the case of a product of odd-spheres. The simplest of the former is the thin black ring with horizon topology $S^1 \times s^{n+1}$.

Black S^{2k+1}-folds

We start as usual taking the Minkowski space $\mathbb{R}^D$ as our background, and we embed the S^{2k+1} sphere into its $(2k+2)$-dimensional spatial flat subspace with metric (in polar coordinates)

$$dr^2 + r^2 \sum_{i=1}^{k+1} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) \,, \qquad \sum_{i=1}^{k+1} \mu_i^2 = 1 \tag{2.104}$$

where the S^{2k+1} is parametrized by $k+1$ Cartan angles ϕ_i and k independent director cosines μ_i. In terms of these, the metric of a k-dimensional sphere can be written as

$$d\Omega_k^2 = \sum_{i=1}^{k+1} d\mu_i^2 = \sum_{i,j=1}^{k} \left(\delta_{ij} + \frac{\mu_i \mu_j}{\mu_{k+1}^2} \right) d\mu_i d\mu_j \ . \tag{2.105}$$

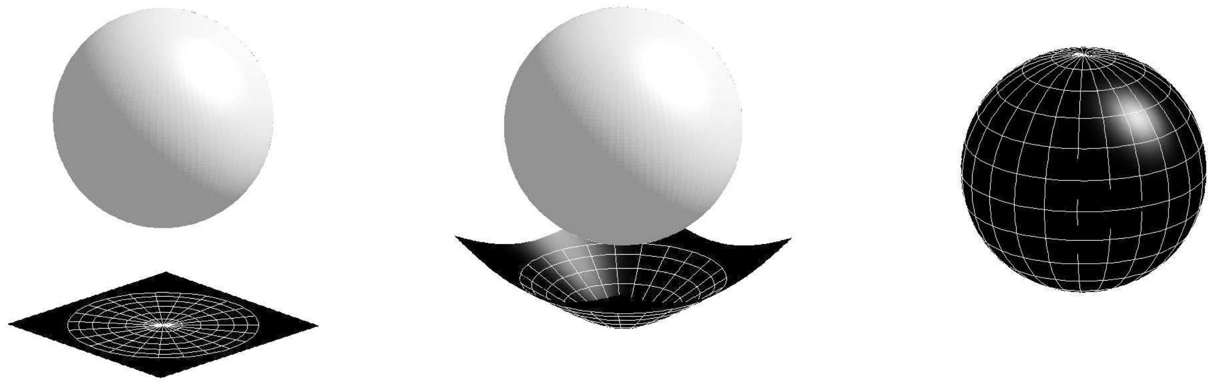

Figure 2.3: *A black p-fold wrapping a single odd-sphere, embedded in a sub-space of Minkowski background. The extrinsic equations give the configuration of equilibrium.*

The $D-2k-3 = n+1$ spatial dimensions of the background which are orthogonal to the blackfold worldvolume play only a spectator role in the blackfold equations, since in these directions the horizon of the black brane is 'thickened' into a transverse s^{n+1} of radius r_0, which is integrated out in our construction.

We now choose a gauge in which the embedding of the blackfold worldvolume $\mathcal{B}_{2k+1}$ is described by a single scalar $r = R(\mu_1, \ldots, \mu_k)$. Then the extrinsic equations give a set of differential equations for $R(\mu_i)$ involving k angular velocities Ω_i. Thus focusing on stationary solutions, we align the velocity field u along the Killing vector field

$$\mathbf{k} = \frac{\partial}{\partial t} + \sum_{i=1}^{k+1} \Omega_i \frac{\partial}{\partial \phi_i} \; . \tag{2.106}$$

In this case the blackfold equations are rather complicated to solve, and details are given in [20]. We now restrict ourselves to the simpler but instructive case of a geometrically round sphere with constant radius $r = R$ that rotates with the same angular velocity Ω in all $k+1$ directions ϕ_i. The embedding is depicted in fig. 2.3. Then (2.106) reduces to

$$\mathbf{k} = \frac{\partial}{\partial t} + \Omega \sum_{i=1}^{k+1} \frac{\partial}{\partial \phi_i} \, . \tag{2.107}$$

The blackfold is now homogeneous so the thickness r_0 is constant over the worldvolume. The extrinsic equations, assuming R to be constant, are now easily obtained from the stationary blackfold action (2.80)

$$I[R] = \beta \Omega_{(p)} R^p \left(1 - R^2 \Omega^2\right)^{\frac{n}{2}} \; , \qquad p = 2k+1 \, . \tag{2.108}$$

Varying it with respect to R we obtain the equilibrium condition

$$R = \sqrt{\frac{p}{n+p}} \frac{1}{\Omega} \tag{2.109}$$

for a spherical stationary blackfold.

We can now easily compute the physical parameters for round black S^{2k+1}-folds, from the general formulae of (2.74), (2.76) and (2.77). Setting $p = 2k+1$ and $V_{(p)} = R^p \Omega_{(p)}$ for the volume of a round p-sphere with radius R we find

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\, r_0^n (n+p+1)\,, \tag{2.110a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1} \sqrt{\frac{n+p}{n}}\ , \quad T = \frac{n}{4\pi}\sqrt{\frac{n}{n+p}}\frac{1}{r_0}\,, \tag{2.110b}$$

$$J_i = \frac{1}{k+1}\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\, R\, r_0^n \sqrt{p(n+p)}\,, \qquad \Omega_i = \sqrt{\frac{p}{n+p}}\frac{1}{R}\ , \quad i = 1\ldots k+1\,. \tag{2.110c}$$

When $p = 1$ they agree with the results in [17] for black rings.

General product of odd-spheres

It is now easy to extend the previous analysis to obtain solutions where $\mathcal{B}_p$ is now a product of round odd-spheres, which we will label with $a = 1, \ldots, \ell$. We will further denote the radius of the S^{p_a} sphere (p_a =odd) by R_a, and as before we assume the angular momenta of the a-th sphere to be all equal, *i.e.,*

$$\Omega_i^{(a)} = \Omega^{(a)}\,, \qquad \forall i = 1, \ldots, \frac{p_a+1}{2}\,. \tag{2.111}$$

We embed the product of ℓ odd-spheres in a flat $(p+\ell)$-dimensional subspace of $\mathbb{R}^{D-1}$ with metric

$$\sum_{a=1}^{\ell}\left(dr_a^2 + r_a^2 d\Omega_{(p_a)}^2\right)\,, \qquad \sum_{a=1}^{\ell} p_a = p \tag{2.112}$$

The embedding now is governed by the set of l scalars given by $r_a = R_a$. The number l of spheres cannot be arbitrary, but it is bounded by $\ell \leq n+2$ (this is basically a consequence that the blackfold worldvolume must be a subspace of the background spacetime). As before we assume R_a to be constant functions on the worldvolume, thus the stationary blackfold action (2.80) reduces in this case to

$$I[\{R\}] = \beta \prod_{b=1}^{\ell} \Omega_{(p_b)} R_b^{p_b} \left(1 - \sum_{a=1}^{\ell}\left(R_a \Omega^{(a)}\right)^2\right)^{n/2} \tag{2.113}$$

which is the straightforward generalization of eq. (2.108).

Varying it with respect to each of the R_a's we get a set of ℓ equations, the solution of which gives the equilibrium conditions on each sphere

$$R_a = \sqrt{\frac{p_a}{n+p}}\frac{1}{\Omega^{(a)}}\ . \tag{2.114}$$

A subset of these solution is the torus obtained by setting all the $p_a = 1$, thus the global horizon topology of these solution will now take the form

$$\mathbb{T}^p \times s^{n+1}\,, \qquad p \leq n+2\;.$$

The thermodynamics of a product of odd-spheres can be now easily computed. The expressions for M, S and T coincide with the ones in (2.110a), (2.110b), provided we set $V_{(p)} = \prod_a V_{(p_a)}$. The angular momenta and velocities are given by the expressions

$$J_i^{(a)} = \frac{2}{p_a+1}\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\,R_a\,r_0^n\sqrt{p_a(n+p)}\,, \qquad \Omega_i^{(a)} = \sqrt{\frac{p_a}{n+p}}\frac{1}{R_a} \tag{2.115}$$

where for each label a, the index i runs from 1 to $(p_a+1)/2$.

2.6.3 Black 1-fold

We mentioned before the blackfold approach allows us to explore novel solutions, with non trivial horizon topology and isometry groups. We will focus now on the first simpler case of these solutions: black one-folds constructed out of black strings.

General results

As usual we take a D-dimensional Minkowski spacetime as our background

$$ds^2 = -dt^2 + d\mathbf{x}^2_{(D-1)}\,. \tag{2.116}$$

and as usual we further assume the effective fluid to be a perfect one, where dissipation effects are absent.

Let us start by analyzing the effective dynamic of a one-brane. We can always choose two orthonormal vectors, u and v, tangent to the string worldsheet such that $u^2 = -1$, $v^2 = 1$ and $u^\mu v_\mu = 0$. We can then write the first fundamental form as

$$h_{\mu\nu} = -u_\mu u_\nu + v_\mu v_\nu\,. \tag{2.117}$$

We take u as the timelike unit vector that defines the velocity field of the effective fluid. Thus the stress tensor is

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu + P v_\mu v_\nu\,. \tag{2.118}$$

The extrinsic equations (2.43) is now

$$\perp^\rho{}_\mu(\varepsilon \nabla_u u^\mu + P\nabla_v v^\mu) = 0\,. \tag{2.119}$$

All of that is valid for any kind of perfect effective fluid, not necessary stationary. The vectors u and v that we have introduced above define an orthonormal local rest frame for the effective fluid.

We now restrict ourselves to the case of stationary brane, and we identify the timelike unit vector as the one defined in (2.49). Then using the notation of (2.60),

$$\xi = \frac{\partial}{\partial t}\,, \qquad \zeta = \frac{\partial}{\partial z}\,, \tag{2.120}$$

we can write,

$$u = \cosh\alpha\xi + \sinh\alpha\zeta\,, \qquad v = \sinh\alpha\xi + \cosh\alpha\zeta\,, \tag{2.121}$$

where α is a boost parameter and ξ and ζ are respectively the unit timelike and unit spacelike translations on the worldsheet. The boosted stress tensor is thus

$$\begin{aligned} T_{\mu\nu} \;=\; & (\varepsilon\cosh^2\alpha + P\sinh^2\alpha)\,\xi_\mu\xi_\nu + (\varepsilon\sinh^2\alpha + P\cosh^2\alpha)\,\zeta_\mu\zeta_\nu \\ & +(\varepsilon + P)\sinh\alpha\cosh\alpha\,\xi_{(\mu}\zeta_{\nu)}\,. \end{aligned} \tag{2.122}$$

If now we consider that ξ is a unit-normalized Killing vector, so that its orbits are geodesics, $\nabla_\xi\xi = 0$, we can find how the extrinsic equations (2.119) reduce for a stationary black one-fold

$$(\varepsilon\sinh^2\alpha + P\cosh^2\alpha)\nabla_\zeta\zeta = 0\,. \tag{2.123}$$

Thus, as long as the orbits of ζ are not themselves geodesics, these equations require that

$$\tanh^2\alpha = -\frac{P}{\varepsilon} = c_T^2\,, \tag{2.124}$$

and therefore the local fluid velocity $\tanh\alpha$ must be the same as the propagation speed c_T of elastic, transverse oscillations of the string. For the specific black string fluid (2.27), the condition (2.124) can be written as

$$\sinh^2\alpha = \frac{1}{n}\,. \tag{2.125}$$

When $n = 1$, i.e., in $D = 5$, this is precisely the value for the boost that was found in [7] from the exact black ring solution in the limit where the ring is very thin and long, $r_0/R \to 0$.

It is straightforward now to calculate the physical parameters (2.74), (2.76) and (2.77) once we identify the above boost parameter α with the rapidity η, and we take ζ on the worldsheet to be proportional to a background Killing vector χ, yielding to a complete recovering of all the physical magnitudes of the exact black ring solution to leading order in r_0/R.

All stationary one-folds in Minkowski background: helical black strings and rings

In this section we extend the previous analysis in order to describe all the possible solutions of stationary one-folds in Minkowski background.

In the previous analysis we did not recovered yet all the possible geometries of the curve along which the string lies on. We have considered only the case where the orthogonal commuting vectors χ_i were generators of Minkowski spatial translations. Let us now further consider the case where the χ_i are the generators either of spatial translations or of rotations.

We can always choose a frame so that the curve where the string lies on has at most only one translational symmetry ∂_x. We assume it has compact orbits of length $2\pi R_x$. We then introduce a coordinate $\phi_x = x/R_x$ with periodicity 2π. We denote the generators of rotations with ∂_{ϕ_i}, so we can write the spatial subspace of Minkowski spacetime in which the string is embedded as

$$ds^2 = R_x^2 d\phi_x^2 + \sum_{i=1}^{m} \left(dr_i^2 + r_i^2 d\phi_i^2 \right) \tag{2.126}$$

The string lies along the curve parametrized by:

$$\phi_x = n_x \sigma\,, \qquad r_i = R_i\,, \qquad \phi_i = n_i \sigma\,, \qquad 0 \le \sigma < 2\pi\,, \tag{2.127}$$

with tangent unit vector

$$\zeta = \frac{1}{\sqrt{\sum_a n_a^2 R_a^2}} \sum_a n_a \frac{\partial}{\partial \phi_a}\,. \tag{2.128}$$

Here the vectors ∂_{ϕ_a} are evaluated on the worldsheet, $r_i = R_i$. The indices run in

$$\begin{aligned} i &= 1, \ldots, m \le \left\lfloor \frac{D-1}{2} \right\rfloor = \left\lfloor \frac{n+3}{2} \right\rfloor\,, \\ a &= x, 1 \ldots, m\,. \end{aligned} \tag{2.129}$$

We can now treat translations and rotations jointly, even if in our construction the direction x does not need to be always present. The upper limit on m is setted by the rank of spatial rotation group in $D = n - 4$ spacetime dimensions. As we can see in (2.126), $D - 2m - 2$ dimensions of space are ignored. They are directions totally orthogonal to the string, thus they only play a spectator role in providing, together with the m directions r_i, the $n + 2$ dimensions orthogonal to the worldsheet in which the horizon of the black string is "thickened" into a transverse s^{n+1} of radius r_0.

We shall assume the n_a to be integers (and without loss of generality $n_a \ge 0$) in order to close the curve in on itself, and to avoid multiple recovering we further assume the smallest of the n_a, which we will call n_{min}, to be coprime with all the n_a. Thus we can specify the whole set of n_a by m positive rational numbers $n_a/n_{min} \ge 1$. Observe that n_x now plays the role of a winding number in the x direction. If we take $n_x \neq 0$ and all $n_i = 0$ we recover a straight string. If $n_x = 0$ and all $n_i = 1$ we obtain a circular planar ring along an orbit of $\sum_i \partial_{\phi_i}$ with radius $\sqrt{\sum_i R_i^2}$. What is not considered in the previous examples falls into a novel set of horizon topologies, analysed for the first time in [20].

It should also be stressed the black objects these blackfolds give rise to, are not globally asymptotically flat. These new topologies rise from black 1-folds with $n_x \neq 0$, and at least one $n_i \neq 0$, and we will refer to them as *helical black strings*, and from black 1-folds with $n_x = 0$, and at least two $n_i > n_j > 0$, and we will refer to them as *helical black rings*. The latter, as all the black 1-folds with $n_x = 0$, give rise to asymptotically flat black holes. These solutions provide the first instance of asymptotically flat black holes with a single spatial $U(1)$ isometry.

This exhausts all the possible stationary black 1-folds in Minkowski background with a spatially compact worldsheet. We can classify all these possibilities into:

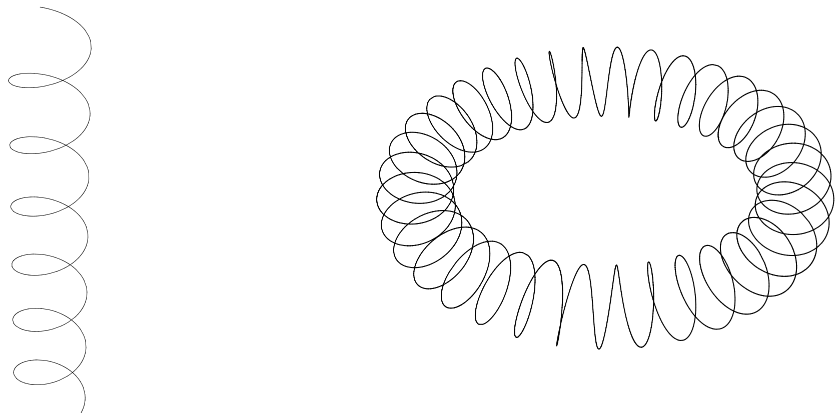

Figure 2.4: *Helical black strings and helical black rings.*

- Straight strings: $n_x \neq 0$, $n_i = 0$.
- Helical strings: $n_x \neq 0$, $n_i \neq 0$.
- Planar rings: $n_x = 0$, $n_i = 1$.
- Helical rings: $n_x = 0$, and at least two $n_i > n_j > 0$.

Where the helical ring and string are depicted in fig. 2.4

Without loss of generality we can now decompose the velocity V, along the string, into a linear velocity $V_x = \Omega R_x$ and into components along the Killing directions that are the generators of the rotations, giving the angular velocities Ω_i. We can then write the Killing generator of the worldsheet velocity field as

$$\mathbf{k} = \frac{\partial}{\partial t} + \sum_a \Omega_a \frac{\partial}{\partial \phi_a} \,. \tag{2.130}$$

The equilibrium condition (2.124) then fixes

$$V^2 = \sum_a \Omega_a^2 R_a^2 = -\frac{P}{\varepsilon} = \frac{1}{n+1} \,. \tag{2.131}$$

ζ can now be rewritten in terms of the velocities

$$\zeta = \frac{1}{V} \sum_a |\Omega_a| \frac{\partial}{\partial \phi_a} \,. \tag{2.132}$$

Observe that the ratios between angular velocities must be rational

$$\left| \frac{\Omega_a}{\Omega_b} \right| = \frac{n_a}{n_b} \quad \forall a, b \,. \tag{2.133}$$

In order to these configurations to be in equilibrium we need to assume the geometry along the curve to be homogeneous, so that gravitational self-attraction is counterbalanced by an increment in the velocity.

Physical properties

The total length of the string along the curve (2.127) is

$$L = 2\pi\sqrt{\sum n_a^2 R_a^2}\,. \tag{2.134}$$

and using (2.131) and (2.133)

$$L = \frac{2\pi}{\sqrt{n+1}}\frac{n_a}{|\Omega_a|} \tag{2.135}$$

for any a. The entropy, the mass and the angular momenta can be computed from (2.74) and (2.75) using (2.131),

$$M = \frac{\Omega_{(n+1)}}{8G}(n+2)r_0^n\sqrt{\sum n_a^2 R_a^2}\,, \tag{2.136}$$

$$J_a = \pm\frac{\Omega_{(n+1)}}{8G}\sqrt{n+1}\,r_0^n\,n_a R_a^2\,, \tag{2.137}$$

$$S = \frac{\pi\Omega_{(n+1)}}{2G}\sqrt{\frac{n+1}{n}}\,r_0^{n+1}\sqrt{\sum n_a^2 R_a^2}\,. \tag{2.138}$$

In [20] it has been shown the entropy behaves as

$$S(M,J) \propto \left(\sum_a n_a|J_a|\right)^{-1/n} M^{\frac{n+2}{n}}\,. \tag{2.139}$$

Thus, according to this formula, the helical black ring has the largest entropy among the above solutions. And in the subset of the helical black ring solutions, the ones with the minimum value of n_a are those with the highest entropy. This is reasonable since decreasing n_a makes the ring shorter (2.135) and hence thinner for fixed mass (2.136), thus the entropy (2.138) decreases. Meanwhile, within the class of the black objects, with a single large angular momentum, mentioned above, the planar black ring is the one which maximize the entropy.

Chapter 3

Blackfolds with q-brane currents

Higher dimensional charged black holes play a fundamental role in supergravity and string theory. A large class of solutions are explicitly known, but if we turn on an angular momentum, the number of exact solutions known with both charge and rotation is really small. A lot of different techniques have been employed in order to get solutions with a dilaton coupling or in the presence of Chern-Simons terms both in five dimensions [49, 50, 51, 39, 52] or in $D > 5$ [53, 54], but it seems a big challenge to get exact solutions for rotating charged black hole in the higher dimensional Einstein-Maxwell theory without a dilaton or Chern-Simons terms to be turned on. Some progress has been made perturbatively or numerically [55, 56, 57], but none of them recovers ultraspinning regimes, where the horizon is distorted significantly far from spherical symmetry. Moreover, apart from few exceptions [58], dipole black holes remain a large class of black holes not very well understood.

Within this contest the blackfold approach has been employed in order to recover the previous solutions. In [22] the approach has been extended to black p-folds with $q = 0, 1$ charges dissolved on the worldvolume, and it has been employed to study the thermodynamics of black holes with different horizon topologies (as in the spirit of the previous chapter). In [23] the approach has been naturally generalized to black p-folds with p-brane currents, and to black Dp-folds with $q = 0, 1$ charges dissolved in their worldvolume. In the following we will give a brief summary of these results with particular focus only on blackfold wrapping single and product of odd-spheres, referring the interested reader to [22] or to the results of the previous chapter in order to extend this analysis to other non-trivial topologies.

3.1 Black p-folds with p-brane currents

We want to describe the dynamics of a perfect fluid that lives in a $(p + 1)$-dimensional worldvolume $\mathcal{W}_{p+1}$ and carries a p-brane current

$$J = Q_p \hat{V}_{(p+1)} \,. \tag{3.1}$$

Here $\hat{V}_{(p+1)}$ is the volume form on $\mathcal{W}_{p+1}$. We assume that this current is conserved, so the charge must be constant along the worldvolume,

$$\partial_a Q_p = 0 \,. \tag{3.2}$$

The main difference now is that the charge Q_p is *not* a collective variable of the fluid: there are no modes in the worldvolume that describe local fluctuations of this charge. Thus the collective coordinates of the fluid are the same than for neutral blackfolds. The charge will play a role only as parameter in the equation of state of the fluid.

3.1.1 Effective stress-energy tensor

The effective, perfect Dp-fold fluid is uniformly charged along all its p-directions, thus it is characterized by an isotropic stress energy-tensor analogous to the one for neutral blackfolds (2.26),

$$T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab} \,. \tag{3.3}$$

Locally, (3.3) satisfies the thermodynamic relations,

$$d\varepsilon = \mathcal{T} ds , \qquad \varepsilon + P = \mathcal{T} s \,. \tag{3.4}$$

We now need to specify to the kind of brane we are dealing with and find the equation of state of an effective fluid living in its worldvolume. Thus we consider the the charged dilatonic black p-branes solutions of the action

$$I = \frac{1}{16\pi G} \int d^D x \, \sqrt{-g} \left(R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2(p+2)!} e^{a\phi} F^2_{(p+2)} \right) \,. \tag{3.5}$$

Defining

$$n = D - p - 3 \,, \qquad a^2 = \frac{4}{N} - \frac{2(p+1)n}{D-2} \,, \tag{3.6}$$

the flat black p-brane solution reads

$$ds^2 = H^{-\frac{Nn}{D-2}} \left(-f dt^2 + \sum_{i=1}^{p} dx_i^2 \right) + H^{\frac{N(p+1)}{D-2}} \left(f^{-1} dr^2 + r^2 d\Omega^2_{n+1} \right) \,, \tag{3.7}$$

$$e^{2\phi} = H^{aN} \;, \quad A_{(p+1)} = \sqrt{N} \coth \alpha (H^{-1} - 1) dt \wedge dx_1 \wedge \cdots \wedge dx_p \,, \tag{3.8}$$

with

$$H = 1 + \frac{r_0^n \sinh^2 \alpha}{r^n} \,, \qquad f = 1 - \frac{r_0^n}{r^n} \,. \tag{3.9}$$

Since it must be that $a^2 \geq 0$, the parameter N is bounded by

$$N \leq 2 \left(\frac{1}{n} + \frac{1}{p+1} \right) \,. \tag{3.10}$$

In string/M-theory, N is typically an integer up to 3 (when $p \geq 1$) that corresponds to the number of different types of branes in an intersection. The D-branes in type II string theory are obtained for $D = 10$, $N = 1$, $p = 0, \ldots, 6$.

The properties of the effective stress tensor of the fluid are [1],

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + nN \sinh^2 \alpha\right), \qquad P = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1 + nN \sinh^2 \alpha\right), \tag{3.11}$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \left(\cosh \alpha\right)^N}, \qquad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \left(\cosh \alpha\right)^N, \tag{3.12}$$

$$Q_p = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \sinh\alpha \cosh\alpha\,, \qquad \Phi_p = \sqrt{N} \tanh\alpha\,, \tag{3.13}$$

where Q_p couples to the field strength $F_{[p+2]}$ of the background, and Φ_p corresponds to the potential of $F_{[p+2]}$ on the worldvolume.

In the case of a Minkowski background the black hole will be asymptotically flat and it must satisfy a Smarr relation,

$$(D-3)M - (D-2)(TS + \Omega J) - n\Phi_H^{(p)} Q_p = 0 \tag{3.14}$$

In term of the above physical quantities (3.11), (3.12), (3.13), the stress-energy tensor (3.3) can be written as

$$T_{ab} = \mathcal{T} s \left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \Phi_p Q_p\, \gamma_{ab} \tag{3.15}$$

This suggests that the stress tensor has a brane-tension component $\propto -\Phi_p Q_p$, and a thermal component $\propto \mathcal{T} s$.

3.1.2 Blackfold dynamics

Intrinsic dynamics

The general analysis is very similar to that of perfect neutral fluids, and as before the fluid equations $D_a T^{ab} = 0$ decompose into components parallel (timelike) and transverse (spacelike) to u^a. The former gives the energy continuity equation, which for a perfect fluid, and using (3.4) is equivalent to the conservation of entropy

$$D_a \left(s u^a\right) . \tag{3.16}$$

The latter, instead, gives rise to the (spacelike) Euler force equations,

$$(\gamma^{ab} + u^a u^b)(\dot{u}_b + \partial_b \ln \mathcal{T}) = 0\,. \tag{3.17}$$

Thus (3.16) and (3.17), complete the set of intrinsic equations for this charged blackfold effective fluid.

[1] we overcome the issue to show a detailed compute of these quantities here since in this thesis work it will be done in details for a specific class of solutions, see appendix C. We refer to [23] for more details

Extrinsic dynamics

The extrinsic dynamic is governed by the Carter's equations

$$T^{\mu\nu} K_{\mu\nu}{}^{\rho} = \frac{1}{(p+1)!} \perp^{\rho}{}_{\sigma} J_{\mu 0 \ldots \mu_p} H^{\mu_0 \ldots \mu_p \sigma} \,, \tag{3.18}$$

where $H_{[p+2]}$ is the background field strength that couples electrically to the p-brane charge. In the following we will focus only on systems where that background field is absent, so that (3.18) reduces to (2.43),

$$T^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0\,. \tag{3.19}$$

Using now (3.3), the extrinsic equations become

$$-PK^{\rho} = \perp^{\rho}{}_{\mu} s\mathcal{T}\dot{u}^{\mu}\,. \tag{3.20}$$

If we restrict ourselves to stationary solutions (2.49), then the extrinsic equation (3.20) becomes

$$K^{\rho} = \perp^{\rho\mu} \partial_{\mu} \ln(-P)\,. \tag{3.21}$$

This equation can be obtained by extremizing the action

$$\tilde{I} = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\; P \tag{3.22}$$

for variations of the brane embedding among stationary fluid configurations with the same Q_p.

Integrations on the worldvolume $\mathcal{W}_{p+1}$ reduce, over an interval Δt of the Killing time, to integrals over $\mathcal{B}_p$ with measure $dV_{(p)}$ as

$$\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\,(P) = \Delta t \int_{\mathcal{B}_p} R_0 dV_{(p)}\,(P)\,. \tag{3.23}$$

Thus keeping fixed the charges Q_p, we can write (3.22) in terms of (3.11), (3.12), and after integration,

$$\tilde{I} = -\Delta t(M - \Omega J - TS)\,, \tag{3.24}$$

and rotating to Euclidean time, $it \to \tau$, with periodicity $\Delta\tau = \beta = 1/T$, the Euclidean action, $i\tilde{I} \to -\tilde{I}_E$ is

$$\tilde{I}_E = \beta(M - \Omega J - TS)\,, \tag{3.25}$$

i.e., the thermodynamic potential at constant T and Ω, with Q_p fixed.

Since T and Ω are integration constants, the extrinsic equations (3.21) are equivalent to

$$dM = TdS + \Omega dJ \qquad (\text{fixed } Q_p)\,. \tag{3.26}$$

Note that Q_p is not a worldvolume density but rather a global quantity. It is indeed one of the conserved charges, together with M and J. This can easily be seen as a consequence of the conservation of the charge along the worldvolume (3.2). Indeed since the p-brane charge spans all the worldvolume directions of the black p-brane, the latter results in a uniformly charged brane, so that the p-charge density is constant over $\mathcal{W}_{p+1}$, and we can identify it as the global charge carried by the brane, so that

$$Q_p = \mathcal{Q}_p\,. \tag{3.27}$$

Furthermore, in this analysis, we should be able to consider variations among stationary configurations with different charges. To this end we need to require the existence of a global potential Φ_H^p conjugated to Q_p, which will be the integral over spatial directions of the worldvolume of the potential density $\Phi_p(\sigma^a)$,

$$\Phi_H^{(p)} = \int_{\mathcal{B}_p} dV_{(p)} R_0 \Phi_p(\sigma^a)\,. \tag{3.28}$$

We can now reformulate the variational principle for stationary solutions. We then introduce locally the Gibbs free energy density,

$$\mathcal{G} = \varepsilon - \mathcal{T}s - \Phi_p \mathcal{Q}_p = -P - \Phi_p \mathcal{Q}_p\,, \tag{3.29}$$

and considering only variations of the worldvolume embedding where this time is the potential $\Phi_H^{(p)}$ in (3.28) which is kept constant. Then the extrinsic equations (3.21) take the form

$$K^\rho = \perp^{\rho\mu} \partial_\mu \ln \mathcal{G} \tag{3.30}$$

which we can derive by extremizing the action

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\, \mathcal{G} \tag{3.31}$$

for variations of the embedding among stationary fluid configurations where now T, Ω and $\Phi_H^{(p)}$ are kept fixed. By integrating it, using (3.29), and going to Euclidean time it is straightforward to see that the extrinsic equations are now equivalent to the complete form of the global first law,

$$dM = TdS + \Omega dJ + \Phi_H^{(p)} dQ_p\,. \tag{3.32}$$

Thus if we restrict to stationary solutions (2.49), then the blackfold equations are completely determined and (2.73), (2.75) and (2.70), hold again, while the global charge and the global potential are determined by (3.27) and (3.28) respectively.

3.1.3 Odd-spheres and product of odd-spheres geometries

We can now proceed further and wrap the black Dp-fold around an odd-sphere S^p ($p = 2n + 1$), or a product of them. These recover all the possible horizon topologies for stationary non-extremal black holes in type IIA/B string theory constructed using blackfolds with p-brane charges. Indeed, in contrast to neutral blackfold, or blackfolds with lower-form currents, they do not admit open boundaries, since the charge Q_p would not be conserved at them. Thus blackfold disk and ball solutions constructed out these kind of blackfolds are forbidden now. A list of allowed horizon topologies is shown in tab. 3.1.

Brane (IIA)	Worldvolume	$\perp$ Sphere
F1	S^1	s^7
D2	$\mathbb{T}^2$	s^6
D4	$S^3 \times S^1$, $\mathbb{T}^4$	s^4
NS5	S^5 , $S^3 \times \mathbb{T}^2$	s^3
D6	$S^3 \times S^3$, $S^5 \times S^1$	s^2

Brane (IIB)	Worldvolume	$\perp$ Sphere
D1	S^1	s^7
F1	S^1	s^7
D3	S^3, $\mathbb{T}^3$	s^5
D5	S^5 , $S^3 \times \mathbb{T}^2$	s^3
NS5	S^5 , $S^3 \times \mathbb{T}^2$	s^3

Table 3.1: *Allowed horizon topologies for stationary non-extremal black holes in type IIA/IIB string theory constructed out by p-brane charged blackfold. The s^{n+1} denotes the 'small' sphere in horizon directions orthogonal to the worldvolume.*

For odd-spheres solutions the analysis is very similar than before: we embed the sphere S^p in a $\mathbb{R}^{p+1}$ subspace of Minkowski background

$$ds^2 = dr^2 + r^2 d\Omega^2_{(p)} \tag{3.33}$$

as surfaces at $r = R$. Then one reduces to the simpler case where all the angular velocities along the Cartan generators $\partial/\partial\phi_i$ are all equal to each other and aligned along the Killing vector χ

$$\chi = \frac{\partial}{\partial \phi} = \sum_{i=1}^{m+1} \frac{\partial}{\partial \phi_i} \,. \tag{3.34}$$

Then

$$u = \frac{1}{\sqrt{1-\Omega^2 R^2}} \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) . \tag{3.35}$$

The extrinsic equation is then easily solved, since all the fields are constant on S^p. The solution in terms of rapidity, so that $\tanh\eta = \Omega R$, is

$$\sinh^2 \eta = \frac{p}{n}(1 + nN \sinh^2 \alpha_p) \,. \tag{3.36}$$

This fixes the radius of equilibrium to be

$$R = \frac{1}{\Omega} \sqrt{\frac{p(1+nN\sinh^2\alpha_p)}{n + p(1+nN\sinh^2\alpha_p)}} \,. \tag{3.37}$$

Given the homogeneity of the blackfold solution, (2.70) and (3.27) that give T and Q_p, are easily solved, and the integrals (2.75),(2.73), (3.28) that give M, J, S and $\Phi_H^{(p)}$, are obtained by simply multiplying the corresponding energy densities by the volume $V_{(p)} = R^p \Omega_{(p)}$ of a round p-sphere with radius R. Then,

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\, r_0^n (n + p + 1 + Nn(1+p)\sinh^2\alpha_p), \tag{3.38a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1} \sqrt{\frac{(n+p+np\sinh^2\alpha_p)(1+\sinh^2\alpha_p)^N}{n}} \,, \tag{3.38b}$$

$$T = \frac{n}{4\pi}\sqrt{\frac{n}{(n+p+Nnp\sinh^2\alpha_p)(1+\sinh^2\alpha_p)^N}}\frac{1}{r_0}\,, \tag{3.38c}$$

$$J = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\,R\,r_0^n\sqrt{p(1+Nn\sinh^2\alpha_p)(n+p+Nnp\sinh^2\alpha_p)}\,, \tag{3.38d}$$

$$\Omega = \sqrt{\frac{p(1+Nn\sinh^2\alpha_p)}{n+p+Nnp\sinh^2\alpha_p}}\frac{1}{R}\,, \tag{3.38e}$$

$$Q_p = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n\sqrt{N}n\sinh\alpha_p\sqrt{1+\sinh^2\alpha_p}\,, \qquad \Phi_H^{(p)} = V_{(p)}\frac{\sqrt{N}\sinh\alpha_p}{\sqrt{1+\sinh^2\alpha_p}}\,. \tag{3.38f}$$

Observe that if we set $\alpha_p = 0$ we precisely recover the physical parameter, (2.110), of the neutral blackfold.

We can now generalize the analysis to products of round odd-spheres, where the worldvolume spatial subspace $\mathcal{B}_p = \prod_{I=1}^{\ell} S^{p_I}$ is now embedded as the surfaces $r_I = R_I$ in

$$ds^2 = \sum_{I=1}^{\ell}\left(dr_I^2 + r_I^2 d\Omega^2_{(p_I)}\right)\,. \tag{3.39}$$

Taking the velocity along $\chi = \sum_{I=1}^{\ell}\Omega_I\partial_{\phi_I}$,

$$u = \cosh\eta\left(\frac{\partial}{\partial t} + \sum_{I=1}^{\ell}\Omega_I\frac{\partial}{\partial\phi_I}\right) \tag{3.40}$$

where η is the total rapidity so that

$$\tanh^2\eta = \sum_{I=1}^{\ell}\Omega_I^2R_I^2\,. \tag{3.41}$$

The extrinsic equations are now solved by

$$K^{r_I} = -\frac{p_I}{R_I}\,, \qquad \dot{u}^{r_I} = -\Omega_I^2R_I\cosh^2\eta\,, \tag{3.42}$$

whose solutions, given in terms of the rapidity, are

$$\Omega_IR_I = \sqrt{\frac{p_I}{p}}\tanh\eta \tag{3.43}$$

and $\sinh^2\eta$ as (3.36). These solutions fix then the radius of equilibrium on each sphere to be

$$R_I = \frac{1}{\Omega_I}\sqrt{\frac{p_I(1+nN\sinh^2\alpha_p)}{n+p(1+nN\sinh^2\alpha_p)}}\,. \tag{3.44}$$

The physical parameter $M, S, T, Q_p, \Phi_H^{(p)}$ are the same as (3.38a), (3.38b), (3.38f), identifying $V_{(p)} = \prod_I\Omega_{(p_I)}R_I^{p_I}$. The angular momenta and angular velocities on each sphere are now

$$J_I = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}\,R_I\,r_0^n\sqrt{p_I(1+Nn\sinh^2\alpha_p)(n+p+Nnp\sinh^2\alpha_p)}\,, \tag{3.45}$$

$$\Omega_I = \sqrt{\frac{p_I(1+Nn\sinh^2\alpha_p)}{n+p+Nnp\sinh^2\alpha_p}}\frac{1}{R_I}\,. \tag{3.46}$$

3.2 Black p-folds with electric charge and with string dipoles

When we turn on charges dissolved in the worldvolume of the brane the long wavelength effective theory of the brane is not described anymore only by an effective stress tensor $T_{\mu\nu}$, but there are additional collective variables that combine into an effective charge current J flowing along the worldvolume $\mathcal{W}_{p+1}$. This will be in general a $q+1$-form describing a conserved q-brane current in $\mathcal{W}_{p+1}$. These new collective variables are the q-brane charge density $\mathcal{Q}(\sigma)$ and a set of worldvolume vectors for the polarization of the q-brane inside the worldvolume.

3.2.1 Effective stress-energy tensor

We want to analyze the formalism for perfect fluids with a conserved q-brane current living in the worldvolume $\mathcal{W}_{p+1}$.

q=0

As we have previously seen the stress tensor for an isotropic perfect fluid has a unit-normalized timelike eigenvector u, (2.26). This still holds even if the blackfold carries a $q=0$ brane charge, since in absence of dissipative effects the particle current must be proportional to u,

$$J_a = \mathcal{Q}_0\, u_a\,, \tag{3.47}$$

where $\mathcal{Q}_0$ is the charge density on the fluid.

q=1

When $q=1$ the string introduces an anisotropy on the worldvolume, and we need to find a spacelike vector orthogonal to u, which characterizes the direction along which the string lies. We can thus contract the timelike vector u with the string two-form current J_{ab} to find the spacelike vector $v^b = u_a J^{ab}$. We normalize it to one, thus

$$-u^2 = v^2 = 1\,, \qquad u\cdot v = 0\,. \tag{3.48}$$

Observe the two-form current is now

$$J_{ab} = \mathcal{Q}_1(u_a v_b - v_a u_b)\,, \tag{3.49}$$

where now $\mathcal{Q}_1$ is the string charge density. If dissipative effects are absent also v must be an eigenvector of the fluid stress tensor T_{ab}. The fluid continues to be isotropic in the spatial directions transverse to it, thus we can write the stress tensor as

$$T_{ab} = \varepsilon u_a u_b + P_{\parallel} v_a v_b + P_{\perp}\left(\gamma_{ab} + u_a u_b - v_a v_b\right)\,. \tag{3.50}$$

Unified description

We can unify the above description introducing a projector onto the space parallel to the particle/string worldline/worldsheet,

$$h^{(q)}_{ab} = -u_a u_b + q\, v_a v_b \qquad (q = 0, 1)\,, \tag{3.51}$$

and onto worldvolume directions orthogonal to it,

$$\perp_{ab} = \gamma_{ab} - h^{(q)}_{ab}\,. \tag{3.52}$$

Thus we can re-write (3.50) as

$$T_{ab} = (\varepsilon + P_{\|}) u_a u_b + (P_{\|} - P_{\perp}) h^{(q)}_{ab} + P_{\perp}\gamma_{ab}\,. \tag{3.53}$$

The (q+1)-form current can be then written as

$$J_{(q+1)} = \mathcal{Q}_q \hat{V}_{(q+1)} \qquad (q = 0, 1)\,, \tag{3.54}$$

where the volume form $\hat{V}_{(q+1)}$ on the worldline/sheet is

$$\hat{V}_{(q+1)} = \begin{cases} u & \text{for } q = 0 \\ u \wedge v & \text{for } q = 1\,, \end{cases} \tag{3.55}$$

The conservation equation $d^{*}J_{(q+1)} = 0$ requires ${}^{*}\hat{V}_{(q+1)} \wedge d^{*}\hat{V}_{(q+1)} = 0$, where * is the Hodge dual on the worldvolume $\mathcal{W}_{p+1}$. Observe that in the $q = 1$ case the difference between the pressure in the directions orthogonal and parallel to them is due to the tension of the string, given by $\Phi_q \mathcal{Q}$,

$$P_{\perp} - P_{\|} = \Phi_q \mathcal{Q}_q\,, \tag{3.56}$$

but we will be back to this issue later, since it will appear more apparent.

The thermodynamic equilibrium is still locally satisfied, and the first law is now

$$d\varepsilon = \mathcal{T} ds + \Phi_q d\mathcal{Q}_q\,, \tag{3.57}$$

and the Gibbs-Duhem relations

$$\varepsilon + P_{\perp} = \mathcal{T} s + \Phi_q \mathcal{Q}_q\,, \tag{3.58}$$

$$dP_{\perp} = s d\mathcal{T} + \mathcal{Q}_q d\Phi_q\,, \qquad dP_{\|} = s d\mathcal{T} - \Phi_q d\mathcal{Q}_q\,. \tag{3.59}$$

We can now compute the effective stress tensor for a black p-brane with q-brane charges in its worldvolume. As we have previously seen the effective stress tensor of the blackfold is defined as the quasi-local Brown-York stress-tensor, thus at leading order we need to compute the latter for the corresponding planar brane. This has been done, for any $q < p$, in [22], and since the main steps are the same of appendix C, we will now show the main results, skipping their derivations. Thus the local thermodynamics parameters are,

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + nN \sinh^2 \alpha_q\right)\,, \tag{3.60}$$

$$P_{\|} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1 + nN \sinh^2 \alpha_q\right), \qquad P_{\perp} = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n. \tag{3.61}$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \left(\cosh \alpha_q\right)^N}, \qquad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \left(\cosh \alpha_q\right)^N. \tag{3.62}$$

$$\Phi_q = \sqrt{N} \tanh \alpha_q\,, \tag{3.63}$$

$$\mathcal{Q}_q = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \sinh \alpha_q \cosh \alpha_q\,, \tag{3.64}$$

where α_q is the charge parameter of the solution. Observe that all these densities depend on n and N, but not on p nor q. The reason is that these densities, in the dilatonic black hole arising under the compactification of the p- and q-brane directions, become the conserved charges, which depend only on the number $n+3$ of dimensions that the black hole lives in, and on the dilaton coupling through the parameter N. Furthermore it is now clear the identification previously asserted (3.56).

The effective stress energy tensor, (3.53), can now be written in terms of the local physical parameter as

$$T_{ab} = \mathcal{T} s \left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \Phi_q \mathcal{Q}_q h_{ab}^{(q)}\,. \tag{3.65}$$

3.2.2 Intrinsic dynamics

The general analysis is very similar to the one of a perfect neutral fluid. The main difference is that adding charges makes the intrinsic equations $D_a T^{ab}$ to decompose into components parallel and transverse to u^a. The former gives the current continuity equations, which can be written for $q=0$, as [22],

$$D_a(\mathcal{Q}_0 u^a) = 0\,, \tag{3.66}$$

and for $q=1$,

$$D_a(\mathcal{Q}_1 u^a) + \mathcal{Q}_1 u^a D_v v_a = 0\,, \tag{3.67}$$

$$D_a(\mathcal{Q}_1 v^a) - \mathcal{Q}_1 v^a \dot{u}_a = 0\,. \tag{3.68}$$

Now using these and (3.58) and (3.59) the intrinsic equation

$$D_a(s u^a) = 0 \tag{3.69}$$

states that the entropy density is conserved onto directions parallel to u^a.

For directions perpendicular to u^a is the Euler force equations to govern the physics of the fluid,

$$\perp^{ab} s\mathcal{T}\left(\dot{u}_b + \partial_b \ln \mathcal{T}\right) - \mathcal{Q}_q \Phi_q \left(K^a - \perp^{ab} \partial_b \ln \Phi_q\right) = 0\,, \tag{3.70}$$

and

$$\left(h^{ab}_{(q)} + u^a u^b\right)\left(\dot{u}_b + \partial_b \ln \mathcal{T}\right) = 0\,, \tag{3.71}$$

where $K^a_{(q)}$ is the mean curvature vector of the worldlines/sheets embedded in $\mathcal{W}_{p+1}$,

$$K^a_{(q)} = h^{bc}_{(q)} D_b h_c{}^a{}_{(q)} = -\perp^a{}_b \left(\dot{u}^b - q\, D_v v^b\right) \qquad (q = 0, 1)\,. \tag{3.72}$$

Here the notation is a bit sloppy, (q) is not an index of the tensor but rather it is a label (3.51).

3.2.3 Extrinsic dynamics

The extrinsic dynamic of a charged blackfold fluid is governed by the Carter's equations

$$T^{\mu\nu} K_{\mu\nu}{}^\rho = \frac{1}{(q+1)!} \perp^\rho{}_\sigma J_{\mu_0 \ldots \mu_q} H^{\mu_0 \ldots \mu_q \sigma}\,, \tag{3.73}$$

where $H_{[q+2]}$ is the background field strength that couples electrically to the q-brane charge. In the following we will focus only on situations in which that field is absent, thus that reduce, as before, to (2.43). The main difference is that now the charges enter into the extrinsic equations towards the stress tensor $T^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu T^{ab}$ which now is is the push-forward of the q-brane-charged fluid stress tensor (3.53). We can then write the extrinsic equations (**??**) as

$$P_\perp K^\rho = -\perp^\rho{}_\mu \left(s\mathcal{T}\dot{u}^\mu - \Phi_q \mathcal{Q}_q K^\mu_{(q)}\right)\,, \tag{3.74}$$

where

$$K^\mu_{(q)} = h^{\nu\sigma}_{(q)} \nabla_\nu h_\sigma{}^\mu{}_{(q)} = h^{\nu\sigma}_{(q)} K_{\nu\sigma}{}^\mu \tag{3.75}$$

is the mean curvature of the worldlines/sheets of the particles/strings embedded in the D-dimensional background spacetime, with the first fundamental form $h^{\mu\nu}_{(q)}$ obtained by pushing forward (3.51).

3.2.4 Stationary charged blackfolds and their thermodynamics

The study of charged stationary blackfold is very similar to the study of the neutral one, done in the previous chapter. Requiring stationarity solves the extrinsic equation completely, and the extrinsic equations can be encoded in a variational principle. Stationarity implies the fluid velocity is aligned with a Killing vector k along the worldvolume, (2.49).

In general we can write the the extrinsic curvature vector of the worldlines/sheets inside $\mathcal{W}_{p+1}$ as,

$$K^a = -\perp^{ab} \partial_b \ln\left(|h_{(q)}|^{1/2}\right)\,, \tag{3.76}$$

where $|h_{(q)}|^{1/2}$ is the area element of the worldlines/sheets. Then (3.70) and (3.71) become

$$s\mathcal{T} \perp^{ab} \partial_b \ln\left(\mathcal{T}|k|\right) + \mathcal{Q}_q \Phi_q \perp^{ab} \partial_b \ln\left(|h_{(q)}|^{1/2} \Phi_q\right) = 0 \tag{3.77}$$

and

$$\left(h^{ab}_{(q)} + u^a u^b\right) \partial_b \ln\left(\mathcal{T}|k|\right) = 0\,. \tag{3.78}$$

Stationarity then requires $\mathcal{T}|k|$ to be constant over the worldvolume, which was true also for stationary neutral blackfold (2.70), where the integration constant was interpreted as the global temperature of the blackfold. Furthermore now we have the condition the potential $\Phi_q(\sigma)$ must not depend on the Killing time coordinate, and one can show that it neither depends on the directions transverse to the current. Thus we can find this constant by integrating it over the worldvolume subsurface $\mathcal{C}_q$ spanned by the q-brane,

$$\Phi_H = \int_{\mathcal{C}_q} d^q\sigma |h_{(q)}|^{1/2} \Phi_q(\sigma) \tag{3.79}$$

which should be interpreted as the global q-brane potential for stationary charged black fold.

As before, we assume that k is of the form (2.58), where ξ foliates $\mathcal{W}_{p+1}$ into spatial sections $\mathcal{B}_p$ with unit timelike norm n^a, (2.61). The entropy, the total mass and the angular momenta are obtained as in (2.73), (2.75).

In order to obtain the q-brane charge for stationary configurations we need to consider the spatial sections of the q-brane worldvolume $\mathcal{C}_{q+1}$ that are orthogonal to n^a. On these, the unit q-form $\omega_{(q)}$ orthogonal to n^a is

$$\omega_{(q)} = \frac{-\hat{V}_{(q+1)} \cdot n}{\sqrt{-h^{(q)}_{ab} n^a n^b}}\,. \tag{3.80}$$

The total q-brane charge Q_q is obtained by integrating its density over the directions transverse to the q-brane current,

$$Q_q = -\int_{\mathcal{B}^{\perp}_{p-q}} dV_{(p-q)} J_{(q+1)} \cdot \left(n \wedge \omega_{(q)}\right) = \int_{\mathcal{B}^{\perp}_{p-q}} dV_{(p-q)} \sqrt{-h^{(q)}_{ab} n^a n^b}\, \mathcal{Q}_q(\sigma)\,. \tag{3.81}$$

The global potential (3.79) is

$$\Phi^{(q)}_H = \int dV_{(q)} \frac{R_0}{\sqrt{-h^{(q)}_{ab} n^a n^b}} \Phi_q(\sigma)\,. \tag{3.82}$$

Note that (3.27) and (3.28) are a subset of these solutions and are recovered for $p = q$. Furthermore for $q = 0$ we have $\sqrt{-h^{(q)}_{ab} n^a n^b} = 1/|k|$, and the potential redshifts like the temperature

$$\Phi^{(0)}_H = \Phi_0(\sigma)|k| \tag{3.83}$$

Then all the collective variables for the fluid are determined. Thus stationarity completely solve the intrinsic equations for the fluid with 0-brane charge.

For $q = 1$ the solution is not completed yet. We need to specify the geometry along the currents. We specify our analysis to configurations in which the strings lie on a spacelike Killing vector ψ that commutes with k,

$$[\psi, k] = 0\,. \tag{3.84}$$

Then we construct its component orthogonal to k by projecting it along the fluid velocity

$$\zeta^a = \psi^a + \left(\psi^b u_b\right) u^a \tag{3.85}$$

and assuming that ζ is spacelike over the blackfold worldvolume. Indeed if ζ were found to become timelike on a region of the worldvolume, then that region should be excluded and the blackfold would have a boundary. Observe that in general ζ is not a Killing vector, but nevertheless it satisfies

$$\zeta^a D_{(a}\zeta_{b)} = 0\,, \qquad k^a D_{(a}\zeta_{b)} = 0\,, \qquad D_a\zeta^a = 0\,. \tag{3.86}$$

If we take

$$v^a = \frac{\zeta^a}{|\zeta|}\,, \tag{3.87}$$

then

$$[u, v] = 0 \tag{3.88}$$

so eq. (3.84) is satisfied.

Furthermore we have

$$h^{(1)}_{ab} = -u_a u_b + \frac{\zeta_a\zeta_b}{|\zeta|^2}\,, \qquad |h^{(1)}|^{1/2} = |k||\zeta|\,, \qquad K^{\rho}_{(1)} = -\left(g^{\rho\mu} - h^{\rho\mu}_{(1)}\right)\partial_\mu \ln(|k||\zeta|)\,. \tag{3.89}$$

Additionally eq. (3.86) and eq. (2.51) implies that $v^a D_v u_a = 0$, or equivalently, $u^a D_v v_a = 0$. With these (3.68) becomes

$$v^a \partial_a \mathcal{Q}_q = 0\,, \tag{3.90}$$

which implies together with (2.70) that Φ_q cannot vary along v, so in (3.79) we set Φ_q to be a constant. If we assume that the orbits of ψ (and hence ζ) are closed we can normalize it so that the periodicity is 2π. The area element on $\mathcal{C}_{q+1}$ is

$$|h^{(q)}(\sigma^a)|^{1/2} = \begin{cases} |k| & \text{for } q = 0 \\ |\zeta||k| & \text{for } q = 1\,, \end{cases} \tag{3.91}$$

and then we can write the solution for the potential, both for $q = 0, 1$, as

$$\Phi(\sigma) = \frac{1}{(2\pi)^q}\frac{\Phi^{(q)}_H}{|h^{(q)}(\sigma^a)|^{1/2}}\,. \tag{3.92}$$

3.2.5 Odd-spheres and product of odd-spheres solutions

As we have mentioned in the previous chapter, there is a large class of black holes where the horizon geometry along the worldvolume directions is an odd-dimensional sphere S^{2k+1} or a product of them. In general with non-equal rotations Ω_i on the Cartan planes of the sphere one has to deal with a complicate set of equations, thus as before we restrict our analysis to the simpler case where the rotations $\Omega_i \equiv \Omega$ are equal in all the planes. The extrinsic curvature vector of the sphere is easily seen to be $K^r = -p/R$, thus plugging (3.65) into (**??**), and using (3.62)(3.63)(3.64) one can find the configuration of equilibrium for a charged blackfold wrapping a single odd-sphere.

Solutions for q=0 currents

The configuration of equilibrium for a blackfold with $q = 0$ currents dissolved in its worldvolume, wrapping a single p-sphere ($p = 2k + 1$) is thus given by

$$\sinh^2\eta = \frac{p}{n(1 + N\sinh^2\alpha_0)}, \qquad R = \frac{\tanh\eta}{\Omega}\,, \tag{3.93}$$

Since all the fields are constant over the worldvolume $\mathcal{W}_{p+1}$ it is straightforward to compute all the global physical parameters ((2.70),(2.73), (2.75), (3.81), (3.82))for these blackfolds,

$$M = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \left(n + p + 1 + nN\sinh^2\alpha_0\right), \tag{3.94a}$$

$$J = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} R r_0^n \sqrt{p\left(n + p + nN\sinh^2\alpha_0\right)}, \tag{3.94b}$$

$$\Omega = \frac{1}{R}\sqrt{\frac{p}{n + p + nN\sinh^2\alpha_0}}. \tag{3.94c}$$

$$S = \frac{\Omega_{(n+1)}V_{(p)}}{4G} r_0^{n+1} \frac{1}{n^{1/2}} \sqrt{\frac{n + p + nN\sinh^2\alpha_0}{1 + N\sinh^2\alpha_0}} \left(\cosh\alpha_0\right)^N, \tag{3.94d}$$

$$T = \frac{n^{3/2}}{4\pi r_0} \sqrt{\frac{1 + N\sinh^2\alpha}{n + p + nN\sinh^2\alpha}} \frac{1}{\left(\cosh\alpha_0\right)^N}\,, \tag{3.94e}$$

$$Q = \frac{\Omega_{(n+1)}V_{(p)}}{16\pi G} r_0^n \sqrt{nN}\sinh\alpha_0\cosh\alpha_0 \sqrt{\frac{n + p + nN\sinh^2\alpha_0}{1 + N\sinh^2\alpha_0}}\,, \tag{3.94f}$$

$$\Phi_H = \sqrt{nN}\sqrt{\frac{1 + N\sinh^2\alpha_0}{n + p + nN\sinh^2\alpha_0}} \tanh\alpha_0\,. \tag{3.94g}$$

Again, setting $\alpha_0 = 0$ we recover the physical parameter, (2.110), of the neutral blackfold.

In the previous chapter we have shown that the blackfold worldvolume can wrap any product of odd-spheres, resulting in black holes in D-dimensional flat space with horizon topology (2.103).

We now obtain their charged generalization. We embed the blackfold in Minkowski spacetime with metric

$$dr_a^2 + r_a^2 \sum_{a,i} \left(d\mu_{a,i}^2 + \mu_{a,i}^2 d\phi_{a,i}^2\right), \qquad \sum_{i=1}^{k_a+1} \mu_{a,i}^2 = 1, \tag{3.95}$$

where the ath sphere is parametrized by the director cosines and Cartan angles $(\mu_{a,i}, \phi_{a,i})$ The analysis is the same as the neutral case, and as before this is a set of rather complicated equations to solve, and since it is outside the scope of this thesis we refer to [22] for detailed calculations. In here we focus in the simpler case in which all the scalars R_a defining the radii of the l spheres wrapped by the blackfold are constants, and the angular momenta of the ath sphere are all equal ($\Omega_{a,i} = \Omega_i$). The configuration of equilibrium is now

$$\Omega_a R_a = \sqrt{\frac{p_a}{n + p + nN \sinh^2 \alpha}} = \sqrt{\frac{p_a}{p}}\, \Omega R\,. \tag{3.96}$$

In this simpler case all the thermodynamic quantities (3.94a), (3.94d),(3.94e),(3.94f),(3.94g) are the same as before, identifying $V_{(p)} = \prod_a V_{(p_a)}$. The angular momenta and angular velocities on each sphere are

$$J_a = \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} R_a r_0^n \sqrt{p_a \left(n + p + nN \sinh^2 \alpha_0\right)}, \tag{3.97}$$

$$\Omega_a = \frac{1}{R_a} \sqrt{\frac{p_a}{n + p + nN \sinh^2 \alpha_0}}. \tag{3.98}$$

Solutions for q=1 currents

Let us analyze the thermodynamics of blackfolds with q=1 currents wrapping a round odd-sphere S^p, $p = 2k + 1$, of radius R in Minkowski background. In this case the string current introduces an anisotropy on the worldvolume, and there are different configurations depending on how it is aligned relatively to the velocity field. In order to apply the previous formalism we require the string to lie along any spatial Killing vector. Then the string could be aligned along different directions in respect to the velocity field u. Following [22], we align the string parallel to the velocity field (2.49), and hence we choose $\psi = \sum_I \Omega_I \partial_{\phi_I}$ in (3.85), thus

$$\zeta = \sinh^2 \eta \frac{\partial}{\partial t} + \cosh^2 \eta \sum_I \Omega_I \frac{\partial}{\partial \phi_I} \tag{3.99}$$

with $|\zeta| = \sinh \eta$ and $|h^{(1)}|^{1/2} = \tanh \eta$. The extrinsic equations (3.74) are now solved by

$$R = \frac{1}{\Omega} \sqrt{\frac{p + nN \sinh^2 \alpha}{n + p + nN \sinh^2 \alpha}}\,, \tag{3.100}$$

or in terms of the rapidity η such that $\tanh \eta = \Omega R$,

$$\sinh^2 \eta = \frac{p + nN \sinh^2 \alpha}{n}\,. \tag{3.101}$$

Since the worldvolume fields are again constant it is easy to perform the integrals and obtain the thermodynamics variables for these blackfolds. Then

$$M = \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} r_0^n \left(n + p + 1 + 2nN \sinh^2 \alpha_1\right), \tag{3.102a}$$

$$J = \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} R r_0^n \sqrt{\frac{n + p + nN \sinh^2 \alpha_1}{p + nN \sinh^2 \alpha_1}}, \tag{3.102b}$$

$$\Omega = \frac{1}{R} \sqrt{\frac{p + nN \sinh^2 \alpha_1}{n + p + nN \sinh^2 \alpha_1}}. \tag{3.102c}$$

$$S = \frac{\Omega_{(n+1)} V_{(p)}}{4G} r_0^{n+1} \frac{1}{n^{1/2}} \sqrt{n + p + nN \sinh^2 \alpha_1} \, (\cosh \alpha)^N \,, \tag{3.102d}$$

$$T = \frac{n^{3/2}}{4\pi r_0} \sqrt{\frac{1}{n + p + nN \sinh^2 \alpha_1}} \frac{1}{(\cosh \alpha_1)^N} \,, \tag{3.102e}$$

$$Q = \frac{\Omega_{(n+1)} V_{(p-1)}}{16\pi G} r_0^n n \sqrt{N} \sinh \alpha_1 \cosh \alpha_1 \,, \tag{3.102f}$$

$$\Phi_H = \tanh \alpha_1 \sqrt{N} 2\pi \frac{n + p + nN \sinh^2 \alpha_1}{n} \,, \tag{3.102g}$$

where $dV_{(p-1)} = dV_{(p)}/(2\pi |h^{(1)}|^{1/2})$ is the volume element of (3.81).

Again for $\alpha_1 = 0$ we recover the thermodynamics of a neutral blackfold.

Chapter 4

Black D0-Dp and F1-Dp -folds

4.1 Black Dp-folds with electric charge and with string dipole

On the worldvolume the local gauge symmetry is not present, but nevertheless the black brane can support a global conserved current that sources the background spacetime gauge field. Furthermore D-branes in string theory can support worldvolume currents that correspond to charges of strings or other D-branes 'dissolved' in their worldvolume. The effective theory then allows us to describe thermal excitations of the worldvolume of the D-brane with these currents. Thus these brane currents greately expand the possible dynamic of the new class of black holes that can be constructed out of them. As in the previous chapter, we will now focus only on $q = 0$-brane (particle) and $q = 1$-brane (string) currents, dissolved in the worldvolume of a Dp-brane. Then, through the blackfold approach, one can construct different classes of black holes with two charges.

4.1.1 Effective stress energy tensor

The analysis is very similar to the one of the first section in chapter 3. Each q-brane current foliates $\mathcal{W}_{p+1}$ into sub-worldvolumes $\mathcal{C}_{q+1} \subset \mathcal{W}_{p+1}$, so that the current is (3.54). Now the thermodynamic relation (3.57) can be generalized

$$d\varepsilon = \mathcal{T} ds + \sum_q{}' \Phi_q d\mathcal{Q}_q \,, \tag{4.1}$$

where the prime in $\sum_q{}'$ indicates that the $q = p$ case is excluded from the sum, whereas it is considered included in $\sum_q$. Thus (4.1) is in this case the same as (3.57), but make the equation suitable for adding more currents on the worldvolume. The effective stress tensor can now be written as

$$T_{ab} = \mathcal{T} s\, u_a u_b - \mathcal{G}\, \gamma_{ab} - \sum_q \Phi_q \mathcal{Q}_q h^{(q)}_{ab} \,, \tag{4.2}$$

where

$$\mathcal{G} = \varepsilon - \mathcal{T} s - \sum_q \Phi_q \mathcal{Q}_q \,, \tag{4.3}$$

is the local Gibbs free energy density (which generalizes (3.29)) and $h^{(q)}_{ab}$ is the projector onto $\mathcal{C}_{q+1}$ (3.51).

Observe that the q-brane charge densities $\mathcal{Q}_q(\sigma)$ are only "quasi-local" since they are constant along $\mathcal{C}_{q+1}$ and can vary in the $p-q$ directions transverse to the current.

The energy density now reads

$$\varepsilon = \frac{n+1}{n}\mathcal{T}s + \sum_q \Phi_q \mathcal{Q}_q \,, \tag{4.4}$$

which plugged in (4.3) gives

$$\mathcal{G} = \frac{1}{n}\mathcal{T}s \,. \tag{4.5}$$

Thus the general form of the stress-energy tensor for blackfold fluids which support more charges on their worldvolume is

$$T_{ab} = \mathcal{T}s\left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \sum_q \Phi_q \mathcal{Q}_q h^{(q)}_{ab} \,, \tag{4.6}$$

which generalize (3.65).

4.1.2 Extrinsic dynamics

The extrinsic equations, $K_{ab}{}^{\rho}T^{ab} = 0$, for an effective fluid whose energy momentum tensor is (4.6) reduce to

$$\mathcal{T}s \perp^{\rho}{}_{\mu}\dot{u}^{\mu} = \frac{1}{n}\mathcal{T}sK^{\rho} + \perp^{\rho}{}_{\mu}\sum_q \Phi_q \mathcal{Q}_q K^{\mu}_{(q)} \tag{4.7}$$

where

$$K^{\mu}_{(q)} = h^{ab}_{(q)} K_{ab}{}^{\mu} \tag{4.8}$$

is the mean curvature vector of the embedding of $\mathcal{C}_{q+1}$ in the background spacetime.

Stationary configurations

We restrict ourselves, as usual, to stationary configurations, in which the local velocity field u is aligned with a Killing vector field (2.49),

$$u = \frac{k}{|k|} = \cosh\eta\left(\frac{\partial}{\partial t} + \sum_{I=1}^{l}\Omega_I \frac{\partial}{\partial\phi_I}\right) \,, \tag{4.9}$$

where for $l = 1$ we recover the single odd-sphere case and $\partial/\partial\phi = \sum_{i=1}^{m+1}\partial/\partial\phi_i$ are the Cartan generators of the I'th sphere.

It is now possible to extend the previous results and obtain the extrinsic equations for stationary fluid configurations on the brane from the variation of the action

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\,\mathcal{G} = -\Delta t\left(M - TS - \Omega J - \sum_q \Phi^{(q)}_H Q_q\right)$$

keeping T, Ω, $\Phi_H^{(q)}$ constant. Then the global first law

$$dM = TdS + \Omega dJ + \sum_q \Phi_H^{(q)} dQ_q \tag{4.10}$$

for variations of the parameter of the embedding is equivalent to these extrinsic equations.

The Smarr relation is now generalized to

$$(D-3)M - (D-2)(TS + \Omega J) - \sum_q (D-3-q)\Phi_H^{(q)} Q_q = 0\,. \tag{4.11}$$

For stationary configurations the analysis for charges and potentials made in chapter 3 still hold, and the total charges and the global potentials are as (3.81) and (3.82),

$$Q_q = \int_{\mathcal{B}^{\perp}_{p-q}} dV_{(p-q)} \sqrt{-h_{ab}^{(q)} n^a n^b}\; \mathcal{Q}_q(\sigma)\,, \tag{4.12}$$

$$\Phi_H^{(q)} = \int_{\mathcal{C}_q} dV_{(q)} \frac{R_0}{\sqrt{-h_{ab}^{(q)} n^a n^b}} \Phi_q(\sigma)\,, \tag{4.13}$$

where the Dp-brane charge $\mathcal{Q}_p = Q_p$ is given by $q = p$.

Observe that $\mathcal{B}^{\perp}_{p-q}$ is a $(p-q)$-surface transverse to the q-directions of the $\mathcal{C}_q$ surface, and that $\mathcal{C}_{q+1}$ is a subset of the worldvolume $\mathcal{W}_{p+1}$. Thus

$$\int_{\mathcal{C}_q} d\hat{V}_{(q)} \int_{\mathcal{B}^{\perp}_{p-q}} dV_{(p-q)} = \int_{\mathcal{B}^{\perp}_p} dV_{(p)}\,, \tag{4.14}$$

is a relation which holds for any q and p, and it will be useful in computing the first law of thermodynamic and the Smarr relation (4.10), (4.11).

q = 0 current

The global D0 charge (4.12) is now

$$Q_{D0} = \int_{\mathcal{B}_{p-0}} dV_{(p-0)} \sqrt{-h_{ab}^{(0)} n^a n^b}\; \mathcal{Q}_{D0}(\sigma)\,, \tag{4.15}$$

and the global potential conjugated to Q_D0 (4.13),

$$\Phi_H^{(D0)} = \sqrt{-h_{ab}^{(0)} n^a n^b} \Phi_{D0}(\sigma) \tag{4.16}$$

For stationary configurations (4.9) we have

$$\sqrt{-h_{ab}^{(0)} n^a n^b} = \sqrt{-u^a n_a} = \frac{1}{|k|} = \cosh\eta\,, \tag{4.17}$$

so the global D0 charge, and its global chemical potential supported by a stationary blackfold are

$$Q_{D0} = \int_{\mathcal{B}_p} dV_{(p)} \frac{\mathcal{Q}_{D0}(\sigma)}{|k|} = \int_{\mathcal{B}_p} dV_{(p)} \cosh\eta \mathcal{Q}_{D0}(\sigma)\,, \tag{4.18}$$

$$\Phi_H^{(D0)} = \Phi_{D0}(\sigma)|k| = \frac{\Phi_{D0}(\sigma)}{\cosh\eta} \tag{4.19}$$

q = 1 current

The strings introduce an anisotropy on the worldvolume, and there are different configurations depending on how the string currents are aligned in the worldvolume $\mathcal{W}_{p+1}$ geometry. The analysis is the same as chapter 3: we restrict ourselves to the case in which the strings lie on a spatial Killing vector ψ that commutes with k. So $d\hat{V}_{(q)}$ is the integration measure over $\mathcal{C}_q$ along the direction given by $\psi/|\psi|$. Then introducing

$$\zeta^a = \psi^a + \left(\psi^b u_b\right) u^a \,, \tag{4.20}$$

we have

$$h^{(1)}_{ab} = -u_a u_b + \frac{\zeta_a \zeta_b}{|\zeta|^2} = -u_a u_b + v_a v_b \,, \qquad |h^{(1)}|^{1/2} = |k||\zeta| \,. \tag{4.21}$$

Thus the global F1 charge (4.12) can be written as

$$Q_{F1} = \int_{\mathcal{B}^{\perp}_{p-1}} dV_{(p-1)} \sqrt{-\left(-u_a u_b + v_a v_b\right) n^a n^b}\; \mathcal{Q}_{F1}(\sigma) \,, \tag{4.22}$$

and the global potential (4.13)

$$\Phi^{(F1)}_H = \int_{\mathcal{C}_1} d\hat{V}_{(1)} \frac{1}{\sqrt{-\left(-u_a u_b + v_a v_b\right) n^a n^b}} \Phi_{F1}(\sigma) \,. \tag{4.23}$$

Everything now depends on how the strings are aligned relative to the velocity field. In this thesis work we will analyse two relevant cases: configurations in which the string current is parallel to the velocity field (as the one considered in chapter 3, as well as [22] and [23]), and configurations in which the string current is perpendicular to the velocity field. In this chapter we will focus only on configurations where the string dipole current is sourced by the gauge potential of the fundamental string $F1$, and thus in the next we will restrict to this relevant case, referring to ch. 5 for currents arising from the Dirichlet string D1.

q=1 current parallel to the boost

Taking the configuration in which the string current is parallel to the velocity field u^a, we then choose $\psi = \sum_I \Omega_I \frac{\partial}{\partial \phi_I}$, and requiring stationarity (4.9) in (4.20),

$$\zeta = \sinh^2 \eta \frac{\partial}{\partial t} + \cosh^2 \eta \sum_I \Omega_I \frac{\partial}{\partial \phi_I} \,, \tag{4.24}$$

we find $|\zeta| = \sinh \eta$, $|h^{(1)}|^{1/2} = \tanh \eta$, and $\sqrt{-h^{(1)}_{ab} n^a n^b} = 1$.

It is now straightforward to calculate the global F1 charge (4.22)

$$Q_{F1} = \int_{\mathcal{B}^{\perp}_{p-1}} dV_{(p-1)}\; \mathcal{Q}_{F1}(\sigma) \,, \tag{4.25}$$

and the global potential conjugated to Q_{F1} (4.23) for this configuration,

$$\Phi^{(F1)}_H = \int_{\mathcal{C}_1} d\hat{V}_{(1)} \Phi_{F1}(\sigma) \,, \tag{4.26}$$

where

$$d\hat{V}_{(1)} = d\sigma |h^{(1)}(\sigma)|^{1/2} = d\sigma \tanh\eta\,, \tag{4.27}$$

thus assuming ψ to have compact orbits of periodicity 2π, the potential (4.23) now reads,

$$\Phi_H^{(F1)} = 2\pi \Phi_{F1} \tanh\eta\,. \tag{4.28}$$

q=1 current perpendicular to the boost

The configurations where the string current is aligned into directions perpendicular to the boost is studied in appendix A.

There are many configurations in which the string current is perpendicular to the velocity field u^a (so that $\psi_a u^a = 0$), but each of them ends up with the same stringy boost v^a (A.16). With this value of the stringy boost, the "redshift element" is now

$$\sqrt{-h_{ab}^{(1)} n^a n^b} = \sqrt{-u_a u_b n^a n^b} = \cosh\eta\,. \tag{4.29}$$

Thus the global F1 charge (4.22) carried by the blackfold with this configuration is now

$$Q_{F1} = \int_{\mathcal{B}_{p-1}^{\perp}} dV_{(p-1)}\ \mathcal{Q}_{F1}(\sigma)\cosh\eta\,, \tag{4.30}$$

and the global potential (4.23) conjugated to it,

$$\Phi_H^{(F1)} = \int_{\mathcal{C}_1} d\hat{V}_{(1)} \frac{\Phi_{F1}(\sigma)}{\cosh\eta}\,, \tag{4.31}$$

where the volume element, using (4.21), is now

$$d\hat{V}_{(1)} = d\sigma |h^{(1)}(\sigma)|^{1/2} = |k||\zeta| d\sigma = d\sigma\,, \tag{4.32}$$

4.1.3 Odd-spheres and product of odd-spheres geometries

In [23] it has been proposed a list of the only possible horizon topologies for D0-Dp and F1-Dp blackfolds wrapping a single, or a product of odd-spheres. This list is showed in table 4.1. We analyze now the global thermodynamics of the corresponding black holes.

Observe that in the D0-Dp systems p is taken to be even since they are solutions only of type IIA string theory. This implies there is no possibility for such a fluid to wrap a single odd-sphere. Thus in the next we will focus only on the product of odd-spheres for the D0-Dp fluid configuration.

The F1-Dp system differs in the possibility to be a solution of both type IIA/B string theory. Thus, depending on which theory the solution is derived, p can be even or odd. These configurations can then wrap also a single odd-sphere, when solutions of type IIB string theory.

		Worldvolume	$\perp$ Sphere
	F1-D1	(helical) S^1	s^7
D0-D2	F1-D2	$\mathbb{T}^2$	s^6
	F1-D3	S^3 , $\mathbb{T}^3$	s^5
D0-D4	F1-D4	$S^3 \times S^1$, $\mathbb{T}^4$	s^4
	F1-D5	S^5 , $S^3 \times \mathbb{T}^2$	s^3
D0-D6	F1-D6	$S^3 \times S^3$, $S^5 \times S^1$	s^2

Table 4.1: *A list of the allowed possibilities for D0-Dp and F1-Dp blackfolds wrapping products of odd-spheres.*

Product of odd-spheres

D0-Dp fluid

The effective stress-energy tensor (4.6) for a D0-Dp blackfold fluid reduces to

$$T_{ab} = (\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})\, u_a u_b - \left(\frac{\mathcal{T}s}{n} + Q_{\rm Dp}\Phi_{\rm Dp}\right)\gamma_{ab}\,. \tag{4.33}$$

Assuming stationary configurations, the solutions of the extrinsic equations (4.7) on each sphere are easily found to be

$$\Omega_I R_I = \sqrt{\frac{p_I}{p}}\tanh\eta\,, \tag{4.34}$$

$$\sinh^2\eta = \frac{p}{n}\frac{\mathcal{T}s + n\Phi_{Dp}Q_{Dp}}{\mathcal{T}s + \Phi_{D0}\mathcal{Q}_{D0}}\,. \tag{4.35}$$

Thus focusing on the simpler case where all the fields are constant over the blackfold, and identifying $V_{(p)} = \prod_{I=1}^{l}\Omega_{p_I}R_I^{p_I}$, the global physical parameters (2.70),(2.73), (2.75), (3.81), (3.82) now reduce to

$$T = \mathcal{T}\frac{1}{\cosh\eta} = \mathcal{T}\sqrt{\frac{n(\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}{(n+p)\mathcal{T}s + n(pQ_{\rm Dp}\Phi_{\rm Dp} + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}} \tag{4.36a}$$

$$S = V_{(p)}s\cosh\eta = V_{(p)}s\sqrt{\frac{(n+p)\mathcal{T}s + n(pQ_{\rm Dp}\Phi_{\rm Dp} + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}{n(\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}} \tag{4.36b}$$

$$\begin{aligned} M &= V_{(p)}\left((\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})\cosh^2\eta + \left(\frac{\mathcal{T}s}{n} + Q_{\rm Dp}\Phi_{\rm Dp}\right)\right) \\ &= V_{(p)}\frac{((n+p+1)\mathcal{T}s + n\mathcal{Q}_{\rm D0}\Phi_{\rm D0} + n(p+1)Q_{\rm Dp}\Phi_{\rm Dp})}{n}\,, \end{aligned} \tag{4.36c}$$

$$J_I = V_{(p)}R_I\,(\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})\left(\sqrt{\frac{p_I}{p}}\sinh\eta\cosh\eta\right)$$

$$= V_{(p)} R_I \frac{\sqrt{p_I\,(\mathcal{T}s + nQ_{\rm Dp}\Phi_{\rm Dp})\,((n+p)\mathcal{T}s + n\mathcal{Q}_{\rm D0}\Phi_{\rm D0} + npQ_{\rm Dp}\Phi_{\rm Dp})}}{n}\,, \tag{4.36d}$$

$$\Omega_I = \frac{\tanh\eta}{R_I} = \frac{1}{R_I}\sqrt{\frac{p_I\,(\mathcal{T}s + nQ_{\rm Dp}\Phi_{\rm Dp})}{(n+p)\mathcal{T}s + n\mathcal{Q}_{\rm D0}\Phi_{\rm D0} + npQ_{\rm Dp}\,,\Phi_{\rm Dp}}} \tag{4.36e}$$

$$Q_{Dp} = \mathcal{Q}_{Dp}\,, \qquad \Phi_H^{(p)} = V_{(p)}\Phi_{\rm Dp}\,, \tag{4.36f}$$

$$Q_{D0} = V_{(p)}\cosh\eta\,\mathcal{Q}_{\rm D0} = V_{(p)}\mathcal{Q}_{\rm D0}\sqrt{\frac{(n+p)\mathcal{T}s + n(pQ_{\rm Dp}\Phi_{\rm Dp} + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}{n(\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}}\,, \tag{4.36g}$$

$$\Phi_H^{(D0)} = \frac{\Phi}{\cosh\eta}_{\rm D0} = \Phi_{\rm D0}\sqrt{\frac{n(\mathcal{T}s + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}{(n+p)\mathcal{T}s + n(pQ_{\rm Dp}\Phi_{\rm Dp} + \mathcal{Q}_{\rm D0}\Phi_{\rm D0})}}\,. \tag{4.36h}$$

We have thus found the global physical parameter for a D0-Dp blackfold fluid wrapping a product of odd-spheres, in terms of the local thermodynamics. These relations are general for any kind of black D0-Dp brane solutions of type IIA string theory. We just need to plug into those relations the local thermodynamic quantities as found in [23] for each different system.

D0-D2 brane

The supergravity solutions of the D0-D2 brane configuration in type IIA string theory are:

Metric in string frame:

$$ds^2 = H^{-\frac{1}{2}}\left[-f dt^2 + D\sum_{i=1}^{2} dx_i^2 + H(f^{-1}dr^2 + r^2 d\Omega_6^2)\right]\,, \tag{4.37}$$

with dilaton

$$e^{2\phi} = DH^{\frac{1}{2}}\,, \tag{4.38}$$

NSNS and RR potentials:

$$B_{12} = \tan\theta[DH^{-1} - 1]\,,\quad A_0 = \coth\alpha\sin\theta[H^{-1} - 1]\,,\quad A_{012} = \coth\alpha\sec\theta[DH^{-1} - 1]\,, \tag{4.39}$$

where

$$f = 1 - \frac{r_0^5}{r^5}\,,\quad H = 1 + \frac{r_0^5\sinh^2\alpha}{r^5}\,,\quad D = (H^{-1}\sin^2\theta + \cos^2\theta)^{-1}\,. \tag{4.40}$$

Using the prescription of chapter 1 we can visualize this configuration as

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{D0} & - & \sim & \sim & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \text{D2} & - & - & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{4.41}$$

a D0-brane sited in the worldvolume of the D2-brane, and smeared along its worldvolume directions.

The local thermodynamic quantities of the D0-D2 blackfold effective fluid are then,

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n (n + 1 + n\sinh^2\alpha)\,, \tag{4.42a}$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \cosh\alpha}\,, \qquad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1}\cosh\alpha\,, \tag{4.42b}$$

$$Q_{\rm D2} = \frac{n\Omega_{(n+1)}}{16\pi G} r_0^n \sinh\alpha\cosh\alpha\cos\theta\,, \qquad \mathcal{Q}_{\rm D0} = Q_{\rm D2}\tan\theta\,, \tag{4.42c}$$

$$\Phi_{\rm D2} = \tanh\alpha\cos\theta\,, \qquad \Phi_{\rm D0} = \Phi_{\rm D2}\tan\theta\,, \tag{4.42d}$$

with $n = 5$ for the present case. These are the solutions as given in [23], but in order to parallel the notation for all the systems, we introduce α_0 and α_2 such that,

$$\sin^2\theta\sinh^2\alpha = \sinh^2\alpha_0\,, \qquad \cos^2\theta\sinh^2\alpha = \sinh^2\alpha_2\,. \tag{4.43}$$

Thus (4.36) are now

$$T = \frac{n}{4\pi}\sqrt{\frac{n(1+\sinh^2\alpha_0)}{\left(1+\sinh^2\alpha_0+\sinh^2\alpha_2\right)\left(n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)\right)}}\frac{1}{r_0}\,, \tag{4.44a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1}\sqrt{\frac{\left(1+\sinh^2\alpha_0+\sinh^2\alpha_2\right)\left(n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)\right)}{n(1+\sinh^2\alpha_0)}}\,, \tag{4.44b}$$

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n\left(n+p+1+n\left(\sinh^2\alpha_0+(p+1)\sinh^2\alpha_2\right)\right)\,, \tag{4.44c}$$

$$J_I = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n R_I\sqrt{p_I(1+n\sinh^2\alpha_2)\left(n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)\right)}\,, \tag{4.44d}$$

$$\Omega_I = \frac{1}{R_I}\sqrt{\frac{p_I(1+n\sinh^2\alpha_2)}{\left(n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)\right)}}\,, \tag{4.44e}$$

$$Q_{D2} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n\sinh\alpha_2\sqrt{1+\sinh^2\alpha_0+\sinh^2\alpha_2}\,, \qquad \Phi_H^{(D2)} = V_{(p)}\frac{\sinh\alpha_2}{\sqrt{1+\sinh^2\alpha_0+\sinh^2\alpha_2}}\,, \tag{4.44f}$$

$$Q_{D0} = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n n\sinh\alpha_0\sqrt{1+\sinh^2\alpha_0+\sinh^2\alpha_2}\sqrt{\frac{\left(n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)\right)}{n(1+\sinh^2\alpha_0)}}\,, \tag{4.44g}$$

$$\Phi_H^{(D0)} = \frac{\sinh\alpha_0}{\sqrt{1+\sinh^2\alpha_0+\sinh^2\alpha_2}\sqrt{\frac{n+p+n\left(p\sinh^2\alpha_2+\sinh^2\alpha_0\right)}{n(1+\sinh^2\alpha_0)}}}\,, \tag{4.44h}$$

where $p = \sum_I p_I$. As usual the physical parameter of a blackfold wrapping a torus, are obtained setting $p_I = 1$.

Smarr relation and 1st law

The Smarr relation for this system is (4.11) which can be rewritten as

$$\begin{aligned}(D-3)M =&(D-3)\left(TS+\sum_I \Omega_I J_I + \Phi_H^{(D2)} Q_{D2} + \Phi_H^{(D0)} Q_{D0}\right)\\ &+TS+\sum_I \Omega_I J_I - 2\Phi_H^{(D2)} Q_{D2}\,,\end{aligned} \tag{4.45}$$

in units where $\frac{r_0^3 \Omega_{(3)} V(S^1)}{16\pi G}=1$, we have

$$M = n+p+1+n\left(\sinh^2\alpha_0 + (p+1)\sinh^2\alpha_2\right)\,, \tag{4.46a}$$

$$TS = n \tag{4.46b}$$

$$\sum_I \Omega_I J_I = p(1+n\sinh^2\alpha_2) \tag{4.46c}$$

$$\Phi_H^{(D2)} Q_{D2} = n\sinh^2\alpha_2 \tag{4.46d}$$

$$\Phi_H^{(D0)} Q_{D0} = n\sinh^2\alpha_0 \tag{4.46e}$$

It is straightforward to check the Smarr relation is indeed satisfied.

It has also been checked numerically that the above quantities satisfy the first law of thermodynamics (4.10) evaluated for variation of each of the parameters r_0, α_0, α_2 of the embedding.

D0-D4 brane

The supergravity solution for the D0-D4 brane is:

Metric in string frame:

$$ds^2 = H_0^{-\frac{1}{2}} H_4^{-\frac{1}{2}}\left[-f dt^2 + H_0 \sum_{i=1}^{4} dx_i^2 + H_0 H_4 (f^{-1} dr^2 + r^2 d\Omega_4^2)\right]\,. \tag{4.47}$$

with dilaton

$$e^{2\phi} = H_0^{\frac{3}{2}} H_4^{-\frac{1}{2}}\,. \tag{4.48}$$

RR potentials:

$$A_0 = \coth\alpha_0 [H_0^{-1}-1]\,,\quad A_{01234} = \coth\alpha_6 [H_4^{-1}-1]\,, \tag{4.49}$$

where

$$f = 1-\frac{r_0^3}{r^3}\,,\quad H_i = 1+\frac{r_0^3\sinh^2\alpha_i}{r^3} \qquad i=0,4\,. \tag{4.50}$$

As before we can depict this configuration as

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{D0} & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot & \cdot \\ \text{D4} & - & - & - & - & - & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{4.51}$$

The local thermodynamic quantities of the D0-D4 blackfold effective fluid are then

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n (n + 1 + n\sinh^2\alpha_0 + n\sinh^2\alpha_4) \tag{4.52a}$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \cosh\alpha_0 \cosh\alpha_4}\ , \quad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \cosh\alpha_0 \cosh\alpha_4 \tag{4.52b}$$

$$\mathcal{Q}_{\mathrm{D}i} = \frac{\Omega_{(n+1)}}{16\pi G} n r_0^n \cosh\alpha_i \sinh\alpha_i\ , \quad \Phi_{\mathrm{D}i} = \tanh\alpha_i \qquad i = 0, 4\,, \tag{4.52c}$$

where $n = 3$. Then the global physical quantities (4.36) are now,

$$T = \frac{n}{4\pi}\sqrt{\frac{n}{\left(1+\sinh^2\alpha_4\right)\left(n+p+n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)\right)}}\frac{1}{r_0}\,, \tag{4.53a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1}\sqrt{\frac{\left(1+\sinh^2\alpha_4\right)\left(n+p+n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)\right)}{n}}\,, \tag{4.53b}$$

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + p + 1 + n\sinh^2\alpha_0 + n(1+p)\sinh^2\alpha_4\right)\ , \tag{4.53c}$$

$$J_I = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n R_I \sqrt{p_I(1+n\sinh^2\alpha_4)(n+p+n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)}\,, \tag{4.53d}$$

$$\Omega_I = \frac{1}{R_I}\sqrt{\frac{p_I\left(1+n\sinh^2\alpha_4\right)}{p+n\left(1+\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)}}\,, \tag{4.53e}$$

$$Q_{D4} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n \sinh\alpha_4\sqrt{1+\sinh^2\alpha_4}\,, \qquad \Phi_H^{(D4)} = V_{(p)}\frac{\sinh\alpha_4}{\sqrt{1+\sinh^2\alpha_4}}\,, \tag{4.53f}$$

$$Q_{D0} = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \sinh\alpha_0\sqrt{n\left(n+p+n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)\right)}\,, \tag{4.53g}$$

$$\Phi_H^{(D0)} = \sinh\alpha_0\sqrt{\frac{n}{n+p+n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_4\right)}}\,, \tag{4.53h}$$

Smarr relation and 1st law

The Smarr relation for this system is (4.11) which can be rewritten as

$$\begin{aligned}(D-3)M =&(D-3)\left(TS+\sum_I \Omega_I J_I + \Phi_H^{(D4)}Q_{D4}+\Phi_H^{(D0)}Q_{D0}\right)\\ &+TS+\sum_I \Omega_I J_I - 4\Phi_H^{(D4)}Q_{D4}\,,\end{aligned} \tag{4.54}$$

setting $\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}r_0^n = 1$, we have

$$M = \left(n+p+1+n\sinh^2\alpha_0 + n(1+p)\sinh^2\alpha_4\right)\,, \tag{4.55a}$$

$$TS = n\,, \tag{4.55b}$$

$$\sum_I \Omega_I J_I = p(1+n\sinh^2\alpha_4)\,, \tag{4.55c}$$

$$\Phi_H^{(D4)}Q_{D4} = n\sinh^2\alpha_4\,, \tag{4.55d}$$

$$\Phi_H^{(D0)}Q_{D0} = n\sinh^2\alpha_0\,. \tag{4.55e}$$

Thus the Smarr relation is again satisfied. It has also been checked numerically that the above quantities satisfy the first law of thermodynamics (4.10) evaluated for variation of each of the parameters r_0, α_1, α_4.

D0-D6 brane

The supergravity solution for the D0-D6 brane is:

Metric in string frame:

$$ds^2 = H_0^{-\frac{1}{2}}H_6^{-\frac{1}{2}}\left[-f dt^2 + H_0\sum_{i=1}^{6}dx_i^2 + H_0H_6(f^{-1}dr^2 + r^2 d\Omega_2^2)\right]\,, \tag{4.56}$$

with dilaton

$$e^{2\phi} = H_0^{\frac{3}{2}}H_6^{-\frac{3}{2}}\,, \tag{4.57}$$

RR potentials:

$$A_0 = -\frac{q_0}{r}\left[1+\frac{r_0}{2r}\sinh^2\alpha_6\right]H_0^{-1}\,,\quad A_\phi = -q_6\cos\theta\,, \tag{4.58}$$

where

$$f = 1-\frac{r_0}{r}\,,\quad H_i = 1+\frac{r_0\sinh^2\alpha_i}{r}+\frac{r_0^2\cosh^2\alpha_i}{2r^2}\frac{\sinh^2\alpha_0\sinh^2\alpha_6}{\cosh^2\alpha_0+\cosh^2\alpha_6}\,,\qquad i=0,6 \tag{4.59}$$

and

$$q_0 \equiv r_0 \cosh\alpha_0 \sinh\alpha_0 \sqrt{\frac{\cosh^2\alpha_0 + 1}{\cosh^2\alpha_0 + \cosh^2\alpha_6}}\,, \tag{4.60}$$

$$q_6 \equiv r_0 \cosh\alpha_6 \sinh\alpha_6 \sqrt{\frac{\cosh^2\alpha_6 + 1}{\cosh^2\alpha_0 + \cosh^2\alpha_6}}\,, \tag{4.61}$$

as introduced in [23].

As before we can visualize this configuration as

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathrm{D0} & - & \sim & \sim & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot \\ \mathrm{D6} & - & - & - & - & - & - & - & \cdot & \cdot & \cdot \end{array} \tag{4.62}$$

The local thermodynamic quantities of the D0-D6 blackfold effective fluid are

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0 (n + 1 + n\sinh^2\alpha_0 + n\sinh^2\alpha_6) \tag{4.63a}$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \cosh\alpha_0 \cosh\alpha_6} \frac{(n+1)(\cosh^2\alpha_0 + \cosh^2\alpha_6)}{(\cosh^2\alpha_0 + 1)(\cosh^2\alpha_6 + 1)} \tag{4.63b}$$

$$s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \cosh\alpha_0 \cosh\alpha_6 \frac{(\cosh^2\alpha_0 + 1)(\cosh^2\alpha_6 + 1)}{(n+1)(\cosh^2\alpha_0 + \cosh^2\alpha_6)} \tag{4.63c}$$

$$\mathcal{Q}_{\mathrm{D}i} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n \cosh\alpha_i \sinh\alpha_i \sqrt{\frac{\cosh^2\alpha_i + 1}{\cosh^2\alpha_0 + \cosh^2\alpha_6}} \tag{4.63d}$$

$$\Phi_{\mathrm{D}i} = \tanh\alpha_i \sqrt{\frac{\cosh^2\alpha_0 + \cosh^2\alpha_6}{\cosh^2\alpha_i + 1}} \tag{4.63e}$$

with $i = 0, 6$, and $n = 1$

Plugging these into (4.36) now give,

$$T = \frac{n}{4\pi} \frac{1}{\left(\sinh^2\alpha_0 \sinh^2\alpha_6\right)} \sqrt{\frac{n}{\left(1 + \sinh^2\alpha_6\right)\left(n + p + n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_6\right)\right)}} \frac{1}{r_0}\,, \tag{4.64a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1} \sinh^2\alpha_0 \sinh^2\alpha_6 \sqrt{\frac{\left(1 + \sinh^2\alpha_6\right)\left(n + p + n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_6\right)\right)}{n}}\,, \tag{4.64b}$$

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + p + 1 + n\sinh^2\alpha_0 + n(1+p)\sinh^2\alpha_6\right)\,, \tag{4.64c}$$

$$J_I = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n R_I \sqrt{p_I (1 + n\sinh^2\alpha_6)(n + p + n\left(\sinh^2\alpha_0 + p\sinh^2\alpha_6\right)}\,, \tag{4.64d}$$

$$\Omega_I = \frac{1}{R_I}\sqrt{\frac{p_I\left(1+n\sinh^2\alpha_6\right)}{p+n\left(1+\sinh^2\alpha_0+p\sinh^2\alpha_6\right)}}\,, \tag{4.64e}$$

$$Q_{D6} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n\sinh\alpha_6\sqrt{1+\sinh^2\alpha_6}\sqrt{\frac{2+\sinh^2\alpha_6}{2+\sinh^2\alpha_0+\sinh^2\alpha_6}}\,, \tag{4.64f}$$

$$\Phi_H^{(D6)} = V_{(p)}\frac{\sinh\alpha_6}{\sqrt{1+\sinh^2\alpha_6}}\sqrt{\frac{2+\sinh^2\alpha_0+\sinh^2\alpha_6}{2+\sinh^2\alpha_6}}\,, \tag{4.64g}$$

$$Q_{D0} = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n\sinh\alpha_0\sqrt{\frac{n\left(2+\sinh^2\alpha_0\right)\left(n+p+n\left(\sinh^2\alpha_0+p\sinh^2\alpha_6\right)\right)}{2+\sinh^2\alpha_0+\sinh^2\alpha_6}}\,, \tag{4.64h}$$

$$\Phi_H^{(D0)} = \sinh\alpha_0\sqrt{\frac{n\left(2+\sinh^2\alpha_0+\sinh^2\alpha_6\right)}{\left(2+\sinh^2\alpha_0\right)\left(n+p+n\left(\sinh^2\alpha_0+p\sinh^2\alpha_6\right)\right)}}\,, \tag{4.64i}$$

where $p=\sum_I p_I$.

Observe that setting $\alpha_p = 0$ in (4.44), (4.53) and (4.64) we recover the thermodynamics, (3.94), of the the neutral p-fold with 0-brane charges dissolved in its worldvolume, while setting $\alpha_0 = 0$ we recover (3.38) of a black p-fold with p-brane charges in its worlvolume. Furthermore if we set $\alpha_p = \alpha_0 = 0$ the neutral black p-fold, (2.110), is recovered as well.

Smarr relation and 1st law

The Smarr relation for this system is (4.11) which can be rewritten as

$$\begin{aligned}(D-3)M =& (D-3)\left(TS+\sum_I \Omega_I J_I + \Phi_H^{(D6)}Q_{D6} + \Phi_H^{(D0)}Q_{D0}\right)\\ &+TS+\sum_I\Omega_I J_I - 6\Phi_H^{(D6)}Q_{D6}\,,\end{aligned} \tag{4.65}$$

thus in units where $\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G}r_0^n = 1$,

$$M = n+p+1+n\sinh^2\alpha_0+n(1+p)\sinh^2\alpha_6\,, \tag{4.66a}$$

$$TS = n\,, \tag{4.66b}$$

$$\sum_I \Omega_I J_I = p(1+n\sinh^2\alpha_6)\,, \tag{4.66c}$$

$$\Phi_H^{(D6)}Q_{D6} = n\sinh^2\alpha_6\,, \tag{4.66d}$$

$$\Phi_H^{(D0)}Q_{D0} = n\sinh^2\alpha_0\,. \tag{4.66e}$$

The Smarr relation is then trivially satisfied, and also the first law of thermodynamics (4.10), evaluated for variation of each of the parameters r_0, α_1, α_6 has been checked numerically to be satisfied.

Single odd-spheres

When the F1-Dp black brane is solution of the ten dimensional supergravity action of type IIB string theory, p is odd, thus the related blackfold can wrap single, as well as a product of, odd-spheres.

F1-Dp brane

As before, as an aid to the reader we will present the solutions for F1-Dp black brane as given in [23].

Metric in string frame:

$$ds^2 = D^{-\frac{1}{2}} H^{-\frac{1}{2}} \left[-f dt^2 + dx_1^2 \right] + D^{\frac{1}{2}} H^{-\frac{1}{2}} \sum_{i=2}^{p} dx_i^2 + D^{-\frac{1}{2}} H^{\frac{1}{2}} \left[f^{-1} dr^2 + r^2 d\Omega_{n+1}^2 \right] , \qquad (4.67)$$

dilaton

$$e^{2\phi} = D^{\frac{p-5}{2}} H^{\frac{3-p}{2}} , \qquad (4.68)$$

NSNS and RR potentials:

$$B_{01} = \sin\theta (H^{-1} - 1) \coth\alpha \ , \quad A_{2\cdots p} = \tan\theta (H^{-1} D - 1) \ , \quad A_{01\cdots p} = \cos\theta D (H^{-1} - 1) \coth\alpha \, , \qquad (4.69)$$

where

$$f = 1 - \frac{r_0^n}{r^n} \ , \quad H = 1 + \frac{r_0^n \sinh^2\alpha}{r^n} \ , \quad D^{-1} = \cos^2\theta + \sin^2\theta H^{-1} \, . \qquad (4.70)$$

We can depict this configuration as

$$\begin{array}{lcccccccc} & 0 & 1 & \ldots & p & p+1 & \ldots & 9 \\ \text{F1} & - & - & \sim & \sim & \cdot & & \cdot & \cdot \\ \text{Dp} & - & - & - & - & \cdot & & \cdot & \cdot \end{array} \qquad (4.71)$$

the fundamental string charged along x_1 and smeared along the other $p-1$ worldvolume directions of the Dp-brane.

The local thermodynamic quantities of the F1-Dp blackfold effective fluid are

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n (1 + n \cosh^2\alpha) \, , \qquad (4.72a)$$

$$\mathcal{T} = \frac{n}{4\pi r_0 \cosh\alpha} \ , \quad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \cosh\alpha \, , \qquad (4.72b)$$

$$Q_{\text{D}p} = \frac{\Omega_{(n+1)}}{16\pi G} n r_0^n \cos\theta \cosh\alpha \sinh\alpha \ , \quad \Phi_{\text{D}p} = \cos\theta \tanh\alpha \, , \qquad (4.72c)$$

$$\mathcal{Q}_{\rm F1} = \frac{\Omega_{(n+1)}}{16\pi G} n r_0^n \sin\theta \cosh\alpha \sinh\alpha \ , \quad \Phi_{\rm F1} = \sin\theta \tanh\alpha \, . \tag{4.72d}$$

With the same spirit as (4.43), we introduce in the next α_1 and α_p such that,

$$\sin^2\theta \sinh^2\alpha = \sinh^2\alpha_1 \, , \qquad \cos^2\theta \sinh^2\alpha = \sinh^2\alpha_p \, . \tag{4.73}$$

Moreover the effective stress-energy tensor (4.6) for a charged F1-Dp blackfold fluid with string current parallel to the velocity field reduces to

$$T_{ab} = (\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}) u_a u_b - \mathcal{Q}_{\rm F1}\Phi_{\rm F1} v_a v_b - \left(\frac{\mathcal{T}s}{n} + Q_{\rm Dp}\Phi_{\rm Dp}\right)\gamma_{ab} \, . \tag{4.74}$$

F1-Dp $\parallel$ fluid

If we require stationarity, we can easily solve the extrinsic equations, (4.7), and the solutions fix the radius of the wrapped sphere in order to achieve an equilibrium configuration. In terms of the rapidity parameter, this is achieved when

$$\sinh^2\eta = \frac{p\mathcal{T}s + n\Phi_{F1}\mathcal{Q}_{F1} + np\Phi_{Dp}Q_{Dp}}{n\mathcal{T}s} \tag{4.75}$$

where $\Omega R = \tanh\eta$.

As usual we require regularity of the event horizon all over $\mathcal{B}_p$, thus the global physical parameters (2.70),(2.73), (2.75), (3.81), (3.82) are easily integrated

$$T = \mathcal{T}(\sigma^a)\frac{1}{\cosh\eta} = \mathcal{T}\sqrt{\frac{n\mathcal{T}s}{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}} \tag{4.76a}$$

$$S = V_{(p)} s \cosh\eta = V_{(p)} s \sqrt{\frac{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{n\mathcal{T}s}} \tag{4.76b}$$

$$\begin{aligned} M &= V_{(p)}\left(\mathcal{T}s\cosh^2\eta + \left(\frac{\mathcal{T}s}{n} + Q_{\rm Dp}\Phi_{\rm Dp} + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)\right) \\ &= V_{(p)}\frac{((n+p+1)\mathcal{T}s + n(1+p)Q_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1})}{n} \end{aligned} \tag{4.76c}$$

$$\begin{aligned} J_I &= V_{(p)} R\,(\mathcal{T}s)\,(\sinh\eta\cosh\eta) \\ &= V_{(p)} R \frac{\sqrt{(p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1})\,((n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1})}}{n} \end{aligned} \tag{4.76d}$$

$$\Omega = \frac{1}{R}\sqrt{\frac{p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}} \tag{4.76e}$$

$$Q_{Dp} = \mathcal{Q}_{Dp} \, , \qquad \Phi_H^{(Dp)} = V_{(p)}\Phi_{\rm Dp} \tag{4.76f}$$

$$Q_{F1} = V_{(p-1)} \mathcal{Q}_{\rm F1}\,, \qquad \Phi_H^{(F1)} = \hat{V}_{(1)} \Phi_{\rm F1} \tag{4.76g}$$

where $\hat{V}_{(1)}$ is given by (4.27). We can write the above in a more useful form, if before integrating we would have used (4.27) in the identity (4.14), thus if we further consider all the fields to be constant over the worldvolume, we get

$$Q_{F1} = \frac{V_{(p)}}{2\pi \tanh \eta} \mathcal{Q}_{\rm F1} = \frac{V_{(p)}}{2\pi} \mathcal{Q}_{\rm F1} \sqrt{\frac{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}}\,, \tag{4.77}$$

$$\Phi_H^{(F1)} = 2\pi \tanh \eta \Phi_{\rm F1} = 2\pi \Phi_{\rm F1} \sqrt{\frac{p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}}\,. \tag{4.78}$$

These complete the thermodynamics of a F1-Dp blackfold fluid wrapping a single odd-sphere, where the string current is parallel to the velocity field, and where the global physical quantities are expressed in terms of the local ones. It's now straightforward to plug into (4.76) the local physical quantities (4.72),

$$T = \frac{n}{4\pi} \sqrt{\frac{n}{\left(1 + \sinh^2\alpha_1 + \sinh^2\alpha_p\right)\left(n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)}} \frac{1}{r_0}\,, \tag{4.79a}$$

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1} \sqrt{\frac{\left(1 + \sinh^2\alpha_1 + \sinh^2\alpha_p\right)\left(n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)}{n}}\,, \tag{4.79b}$$

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + p + 1 + 2n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)\,, \tag{4.79c}$$

$$J = R\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \sqrt{\left(p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)\left(n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)}\,, \tag{4.79d}$$

$$\Omega = \frac{1}{R} \sqrt{\frac{p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}{n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}}\,, \tag{4.79e}$$

$$Q_{Dp} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n \sinh\alpha_p \sqrt{1 + \sinh^2\alpha_p + \sinh^2\alpha_1}\,, \tag{4.79f}$$

$$\Phi_H^{(Dp)} = V_{(p)} \frac{\sinh\alpha_p}{\sqrt{1 + \sinh^2\alpha_p + \sinh^2\alpha_1}}\,, \tag{4.79g}$$

$$Q_{\rm F1} = \frac{V_{(p)}\Omega_{(n+1)}}{32\pi^2 G} r_0^n n \sqrt{\frac{n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}{p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}} \sinh\alpha_1 \sqrt{1 + \sinh^2\alpha_p + \sinh^2\alpha_1}\,, \tag{4.79h}$$

$$\Phi_H^{(F1)} = 2\pi \sqrt{\frac{p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}{n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}} \frac{\sinh\alpha_1}{\sqrt{1 + \sinh^2\alpha_p + \sinh^2\alpha_1}}\,, \tag{4.79i}$$

Again setting $\alpha_p = 0$ in (4.79) we recover the thermodynamics, (3.102), of the the neutral p-fold with string dipoles in its worldvolume. For $\alpha_1 = 0$ we recover (3.38) of a black p-fold with p-brane charges in its worlvolume. Setting both $\alpha_p = \alpha_0 = 0$ we recover again the neutral black p-fold thermodynamic, (2.110).

F1-Dp $\perp$ fluid

Again, requiring stationarity, completely solve the extrinsic equations, whose solutions fix the configuration of equilibrium for the blackfold wrapping a single odd-sphere,

$$\sinh^2\eta = \frac{p\mathcal{T}s + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1} + npQ_{\rm Dp}\Phi_{\rm Dp}}{n\left(\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)}\,, \tag{4.80}$$

with $\Omega R = \tanh\eta$.

Requiring all the fields to be constant all over $\mathcal{B}_p$, the global physical parameter (2.70),(2.73), (2.75), (3.81), (3.82) are easily solved,

$$T = \mathcal{T}\frac{1}{\cosh\eta} = \mathcal{T}\sqrt{\frac{n\left(\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)}{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}} \tag{4.81a}$$

$$S = V_{(p)}s\cosh\eta = V_{(p)}s\sqrt{\frac{(p+n)\,\mathcal{T}s + pnQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{n\left(\mathcal{T}s + Q_{\rm F1}\Phi_{\rm F1}\right)}} \tag{4.81b}$$

$$\begin{aligned} M &= V_{(p)}\left(\left(\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)\cosh^2\eta + \left(\frac{\mathcal{T}s}{n} + Q_{\rm Dp}\Phi_{\rm Dp}\right)\right) \\ &= V_{(p)}\frac{\left((n+p+1)\mathcal{T}s + n(1+p)Q_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)}{n} \end{aligned} \tag{4.81c}$$

$$J = V_{(p)}R\left(\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)\left(\sinh\eta\cosh\eta\right)$$

$$= V_{(p)}R\frac{\sqrt{\left(p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)\left((n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)}}{n} \tag{4.81d}$$

$$\Omega = \sqrt{\frac{p\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{(n+p)\mathcal{T}s + npQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}} \tag{4.81e}$$

$$Q_{Dp} = \mathcal{Q}_{Dp}\,, \qquad \Phi_H^{(Dp)} = V_{(p)}\Phi_{\rm Dp} \tag{4.81f}$$

$$Q_{F1} = \frac{V_{(p)}}{2\pi}\cosh\eta\,\mathcal{Q}_{\rm F1} = \frac{V_{(p)}}{2\pi}\mathcal{Q}_{\rm F1}\sqrt{\frac{(p+n)\,\mathcal{T}s + pnQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}{n\left(\mathcal{T}s + Q_{\rm F1}\Phi_{\rm F1}\right)}} \tag{4.81g}$$

$$\Phi_H^{(F1)} = 2\pi\frac{\Phi_{\rm F1}}{\cosh\eta} = 2\pi\Phi_{\rm F1}\sqrt{\frac{n\left(\mathcal{T}s + \mathcal{Q}_{\rm F1}\Phi_{\rm F1}\right)}{(p+n)\,\mathcal{T}s + pnQ_{\rm Dp}\Phi_{\rm Dp} + 2n\mathcal{Q}_{\rm F1}\Phi_{\rm F1}}} \tag{4.81h}$$

where we have used (4.32) before integration.

It is now straightforward to regard this effective fluid to the thermodynamics of the related black brane, by using the local physical quantities (4.72),

$$S = \frac{V_{(p)}\Omega_{(n+1)}}{4G} r_0^{n+1} \sqrt{\frac{\left(1+\sinh^2\alpha_p+\sinh^2\alpha_1\right)\left(n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)\right)}{n\left(1+\sinh^2\alpha_1\right)}} \tag{4.82a}$$

$$T = \frac{n}{4\pi}\sqrt{\frac{n\left(1+\sinh^2\alpha_1\right)}{\left(1+\sinh^2\alpha_p+\sinh^2\alpha_1\right)\left(n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)\right)}}\frac{1}{r_0} \tag{4.82b}$$

$$M = \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \left(n+p+1+2n\left(p\sinh^2\alpha_p+\sinh^2\alpha_1\right)\right) \,, \tag{4.82c}$$

$$J = R\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \sqrt{\left(p+n\left(p\sinh^2\alpha_p+\sinh^2\alpha_1\right)\right)\left(n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)\right)} \,, \tag{4.82d}$$

$$\Omega = \frac{1}{R}\sqrt{\frac{p+n\left(p\sinh^2\alpha_p+\sinh^2\alpha_1\right)}{n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)}} \,, \tag{4.82e}$$

$$Q_{Dp} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n n \sinh\alpha_p \sqrt{1+\sinh^2\alpha_1+\sinh^2\alpha_p} \,, \tag{4.82f}$$

$$\Phi_H^{(Dp)} = V_{(p)}\frac{\sinh\alpha_p}{\sqrt{1+\sinh^2\alpha_1+\sinh^2\alpha_p}} \,, \tag{4.82g}$$

$$Q_{\mathrm{F1}} = \frac{V_{(p)}\Omega_{(n+1)}}{32\pi^2 G} r_0^n n \sinh\alpha_1 \sqrt{1+\sinh^2\alpha_1+\sinh^2\alpha_p}\sqrt{\frac{\left(n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)\right)}{n\left(1+\sinh^2\alpha_1\right)}} \,, \tag{4.82h}$$

$$\Phi_H^{(F1)} = 2\pi\sqrt{\frac{n\left(1+\sinh^2\alpha_1\right)}{\left(n+p+n\left(p\sinh^2\alpha_p+2\sinh^2\alpha_1\right)\right)}}\frac{\sinh\alpha_1}{\sqrt{1+\sinh^2\alpha_1+\sinh^2\alpha_p}} \,. \tag{4.82i}$$

Observe that even if we can recover the p-brane charged black p-fold, (3.38), by setting $\alpha_1 = 0$ and the neutral black p-fold, (2.110), by setting both $\alpha_p = \alpha_0 = 0$, we have no counterpart of neutral black p-fold with string dipoles boosted perpendicularly to the velocity field u^a, since finding this configuration has been a part of this thesis work. Anyway the thermodynamics of this neutral blackfold with string dipoles can now be obtained by setting $\alpha_p = 0$ in (4.82). It is instructive to note that the configuration where the string is boosted parallel to the velocity field has highest entropy.

Product of odd-spheres

F1-Dp ∥

The configuration of equilibrium on each odd-spheres is given by

$$\Omega_I R_I = \sqrt{\frac{p_I}{p}} \tanh \eta \,, \tag{4.83}$$

and

$$\sinh^2 \eta = \frac{p\mathcal{T}s + n\mathcal{Q}_{\mathrm{F1}}\Phi_{\mathrm{F1}} + np\Phi_{Dp}Q_{Dp}}{n\mathcal{T}s} \,. \tag{4.84}$$

The global thermodynamics are the same of (4.76), and so of (4.79), identifying $V_{(p)} = \prod_I \Omega_{(p_I)} R_I^{p_I}$, but the angular momenta and the angular velocities are on each sphere now

$$J_I = R_I \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \sqrt{\frac{p_I}{p}} \sqrt{\left(p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)\left(n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)} \tag{4.85}$$

$$\Omega_I = \frac{1}{R_I}\sqrt{\frac{p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}{n + p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}} \tag{4.86}$$

F1-Dp ⊥

The configuration of equilibrium on each odd-spheres is given by

$$\Omega_I R_I = \sqrt{\frac{p_I}{p}} \tanh \eta \,, \tag{4.87}$$

and

$$\sinh^2 \eta = \frac{p\mathcal{T}s + n\mathcal{Q}_{\mathrm{F1}}\Phi_{\mathrm{F1}} + np\Phi_{Dp}Q_{Dp}}{n\left(\mathcal{T}s + \mathcal{Q}_{\mathrm{F1}}\Phi_{\mathrm{F1}}\right)} \,. \tag{4.88}$$

The global thermodynamics are the same of (4.81), thus (4.82), identifying $V_{(p)} = \prod_I \Omega_{(p_I)} R_I^{p_I}$, but the angular momenta and the angular velocities are on each sphere now

$$J = R_I \frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n \sqrt{\frac{p_I}{p}} \sqrt{\left(p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)\left(n + p + n\left(p\sinh^2\alpha_p + 2\sinh^2\alpha_1\right)\right)} \tag{4.89}$$

$$\Omega = \frac{1}{R_I}\sqrt{\frac{p_I}{p}}\sqrt{\frac{p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)}{n + p + n\left(p\sinh^2\alpha_p + 2\sinh^2\alpha_1\right)}} \tag{4.90}$$

Smarr relation and 1st law

We now check if the physical parameters of a charged F1-Dp fluid satisfy the Smarr relation. There is no difference between configurations in which the string current is parallel or perpendicular to the boosts, since all the factors of the global physical quantities entering in the Smarr relation are the same for both the configurations.

The Smarr relation (4.11) can be written for this system as

$$\begin{aligned}(D-3)M =&(D-3)\left(TS+\sum_I \Omega_I J_I + \Phi_H^{(Dp)} Q_{Dp} + \Phi_H^{(F1)} Q_{F1}\right)\\&+TS+\sum_I \Omega_I J_I - p\Phi_H^{(Dp)} Q_{Dp} - \Phi_H^{(F1)} Q_{F1}\,,\end{aligned} \tag{4.91}$$

thus in units where $\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n = 1$,

$$M = \left(n+p+1+2n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\right)\,, \tag{4.92a}$$

$$TS = n\,, \tag{4.92b}$$

$$\sum_I \Omega_I J_I = p + n\left(p\sinh^2\alpha_p + \sinh^2\alpha_1\right)\,, \tag{4.92c}$$

$$\Phi_H^{(Dp)} Q_{Dp} = n\sinh^2\alpha_p\,, \tag{4.92d}$$

$$\Phi_H^{(F1)} Q_{F1} = n\sinh^2\alpha_1\,. \tag{4.92e}$$

The Smarr relation is then satisfied again, and also the I law of thermodynamics (4.10) evaluated for variation of each of the parameters r_0, α_1, α_p has been checked numerically to be satisfied.

Chapter 5

Six-dimensional D1-D5-P thin black ring

The aim of this chapter is to study the thermodynamics of a non-extremal six dimensional thin black ring with three charges dissolved on it. To parallel the solution analysed in [23] for the extremal case, we will focus on the D1-D5-P [1] solution of the ten dimensional low-energy action of type IIB string theory.

This solution is of a particular interest since black branes with D1-D5 charges are particularly prominent in string theory, but they show a non-finite horizon area at extremality. Indeed the D1-D5 system is known to have the microscopic entropy[2]

$$S = 2\pi\sqrt{2N_1 N_5}\,. \tag{5.1}$$

but this entropy cannot be reproduced as a thermodynamic Bekenstein-Hawking entropy, because this geometry has a zero horizon area at extremality.

Adding a longitudinal momentum wave make these branes to develop an horizon even at extremality. Moreover if one reduces the non-extremal D1-D5-P black brane solution to five dimensions, the system results in a static D1-D5-P black hole [60], whose BPS state preserves the whole supersymmetry. Anyway that is the only supersymmetric black hole solution one can get from the D1-D5-P black brane.

Nevertheless in $D \geq 5$, at extremality, the resulting black brane is marginally stable to fluctuation of the D1 charge and the Gregory-Laflamme instability seems to be absent. Thus, these solutions could be the first stable, asymptotically flat, extremal, non-supersymmetric black holes with non-spherical horizon topology. In this class the six dimensional black ring appears as the more stable. It should be stressed that when the worldvolume is not flat but it is bent in a curved shape, the travelling waves will emit gravitational radiation. Therefore these D1-D5-P extremal black holes, even if classicaly stable, will decay at quantum-mechanical level. This seems to be a general feature of all the extremal non-supersymmetric rotating black holes.

[1] observe P does not give rise to an electric or magnetic charge, but rather to a momentum charge
[2] See [59] for a derivation

There are several ways to construct a six dimensional D1-D5-P black string solution. One can start from the D1-D5 supergravity solution, boost it along the direction where the D-string lies on, and compactify it to six dimensions An other possibility is to start with the D0-F1-D4 solution of type IIA string theory, T-dualize it to get directly the D1-D5-P solution of type IIB string theory, and compactify it to six dimensions. In this thesis work we will follow the latter, and the building up of the solution has been done in appendices B.1 and B.2.

We can then bend the six dimensional black string into a circle S^1 with large extent radius R, and regard it as a thin black ring. Indeed the horizon topology of the six dimensional thin black ring is $S^1 \times s^3$, and in the blackfold regime, when the radius of the internal sphere is much smaller than the radius of the ring $r_0 \ll R$, this solution looks locally like a boosted black string. Thus we need to study the dynamic of a black 1-fold (which could have eventually a helical shape) with three charges dissolved in its worldvolume. Then through the blackfold equations we can bend the black 1-fold along a circle S^1, to recover the thermodynamics of the thin six dimensional black ring.

5.1 Black D1-D5-P 1-fold

We start our analysis by analyzing the extrinsic and intrinsic dynamic of a black 1-fold with D1-D5-P charges dissolved on its worldvolume. The analysis is a straightforward application of the study made in the previous chapter, where the equations were arranged in such a way to allow the possibility of adding more than two charges.

5.1.1 Effective stress energy tensor

We can write write the stress energy tensor (4.2) of the effective D1-D5-P fluid as,

$$T_{ab} = \mathcal{T} s \, u_a u_b - \mathcal{G} \, \gamma_{ab} - \sum_q \Phi_q \mathcal{Q}_q h^{(q)}_{ab} - \mathcal{P} v_H h^{(P)}_{ab} \,, \tag{5.2}$$

where the Gibbs free energy is now

$$\mathcal{G} = \varepsilon - \mathcal{T} s - \left(\Phi_{D1} \mathcal{Q}_{D1} + \Phi_{D5} \mathcal{Q}_{D5} + \mathcal{P} v_H \right) \,. \tag{5.3}$$

The local thermodynamic parameters for the effective D1-D5-P fluid are computed in (C.10), (C.2), (C.23), (C.25), (C.26), (C.32), (C.7), (C.38) and we now rewrite them here in a compact form as,

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n + 1 + n \sinh^2(\alpha_1) + n \sinh^2(\alpha_5) + n \sinh^2(\alpha_p) \right) \,, \tag{5.4a}$$

$$s = \frac{r_0^{n+1} \Omega_{(n+1)} \cosh \alpha_1 \cosh \alpha_5 \cosh \alpha_p}{4G} \,, \qquad \mathcal{T} = \frac{2}{4\pi r_0 \cosh \alpha_p \cosh \alpha_1 \cosh \alpha_5} \,, \tag{5.4b}$$

$$\mathcal{Q}_{D1} = \frac{\Omega_{(3)}}{16\pi G} n r_0^n \sinh \alpha_1 \cosh \alpha_1 \,, \qquad \Phi_{D1} = \tanh \alpha_1 \,, \tag{5.4c}$$

$$\mathcal{Q}_{D5} = \frac{\Omega_{(n+1)}}{16\pi G} n r_0^n \sinh\alpha_5 \cosh\alpha_5 \,, \qquad \Phi_{D5} = \tanh\alpha_5 \,, \tag{5.4d}$$

$$\mathcal{P} = \frac{\Omega_{(n+1)} r_0^n}{16\pi G} n \cosh\alpha_p \sinh\alpha_p \,, \qquad v_H = \tanh\alpha_p \,, \tag{5.4e}$$

where $n = 2$ for the six dimensional D1-D5-P black string. Moreover the energy density satisfies

$$\varepsilon = \frac{n+1}{n}\mathcal{T}s + \Phi_{D1}\mathcal{Q}_{D1} + \Phi_{D5}\mathcal{Q}_{D5} + \mathcal{P}v_H \,, \tag{5.5}$$

as expected in order to recover (4.4) in this present case.

With this value of the energy density, it's easy to see the Gibbs free energy (5.3) of this three charged blackfold fluid satisfies (4.5).

Thus the effective stress energy tensor for a the D1-D5-P blackfold fluid (5.2) can be written as

$$T_{ab} = \mathcal{T}s\left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \sum_q \Phi_q \mathcal{Q}_q h_{ab}^{(q)} - \mathcal{P}v_H h_{ab}^{(P)} \,, \tag{5.6}$$

which recovers (4.6).

It should be stressed the validity of the above relations in any number of dimensions of the background space time. They indeed represent the intrinsic dynamic of the fluid living on the worldvolume, and the dependence on the geometrical configuration of the charges enters through the projector tensors h_{ab}.

From table B.27 we can read off that our system in the six dimensional background consists of a D1-brane charged along z, a D5-brane which, after compactification, is a string-like object charged along z, and a momentum charge P along z. Thus we need to study the dynamic of a black 1-fold, whose worldsheet spatial direction is identified along z, and we have

$$h_{ab}^{(D1)} = h_{ab}^{(D5)} = \gamma_{ab} \,, \tag{5.7}$$

and

$$h_{ab}^{(P)} = -u_a u_b \,. \tag{5.8}$$

where u_a is the boost along the 1-fold, and a and b runs along the worldsheet directions t and z.

Thus (5.6) reduces now to

$$T_{ab} = \left(\mathcal{T}s + v_H\mathcal{P}\right) u_a u_b - \gamma_{ab}\left(\frac{\mathcal{T}s}{n} + \Phi_{D1}\mathcal{Q}_{D1} + \Phi_{D5}\mathcal{Q}_{D5}\right) \,, \tag{5.9}$$

which is the effective stress energy tensor of the D1-D5-P black 1-fold fluid in a six-dimensional background[3].

[3]for the ten dimensional background the configuration is given by tab. B.13, thus $h_{ab}^{(D5)} = \gamma_{ab}$, $h_{ab}^{(D1)} = -u_a u_b + v_a v_b$, $h_{ab}^{(P)} = -u_a u_b$, so that $T_{ab} = \left(\mathcal{T}s + v_H\mathcal{P} + \Phi_{D1}\mathcal{Q}_{D1}\right) u_a u_b - \left(\Phi_{D1}\mathcal{Q}_{D1}\right) v_a v_b - \gamma_{ab}\left(\frac{\mathcal{T}s}{n} + \Phi_{D5}\mathcal{Q}_{D5}\right)$

5.1.2 Extrinsic dynamics

The extrinsic equation, $K_{ab}{}^{\rho} T^{ab} = 0$, for the stress energy tensor (5.9), reduces to

$$\mathcal{T}s \perp^{\rho}{}_{\mu} \dot{u}^{\mu} = \frac{1}{n}\mathcal{T}sK^{\rho} + \perp^{\rho}{}_{\mu} \sum_{q} \Phi_q \mathcal{Q}_q K^{\mu}_{(q)} + \perp^{\rho}{}_{\mu} v_H \mathcal{P} K^{\mu}_{(P)}\,, \tag{5.10}$$

where $K^{\mu}_{(q')} = h^{ab}_{(q')} K_{ab}{}^{\mu}$ with $q' = D1, D5, P$ is the mean curvature vector of the embedding $\mathcal{C}_{q'+1}$ in the background spacetime.

Stationary configurations

We can now apply the results of the previous chapter to obtain the extrinsic equations for our fluid in a stationary configuration. The action (4.10)

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\, \mathcal{G}$$

keeping T, Ω, $\Phi^{(q)}_H, v_H$ constant, now leads to

$$I = -\Delta t \left(M - TS - \Omega J - \sum_q \Phi^{(q)}_H Q_q - V_H P \right) . \tag{5.11}$$

Then the global first law

$$dM = TdS + \Omega dJ + \sum_q \Phi^{(q)}_H dQ_q + V_H P\,, \tag{5.12}$$

evaluated for variations of the parameter of the embedding is equivalent to the extrinsic equations. The Smarr relation is now

$$(D-3)M - (D-2)(TS + \Omega J + V_H P) - \sum_q (D-3-q)\Phi^{(q)}_H Q_q = 0\,. \tag{5.13}$$

For stationary configurations the analysis for the global charges made in the previous chapters still holds, and eq. (3.81), (3.82) apply also to the momentum charge and the conjugate quantities associated to it, once we identify p=1 and q=0. Thus,

$$P = \int_{\mathcal{B}_1} dV_{(1)} \sqrt{-h^{(P)}_{ab} n^a n^b}\, \mathcal{P}(\sigma) = \int_{\mathcal{B}_1} dV_{(1)} \cosh\eta\, \mathcal{P}(\sigma)\,, \tag{5.14}$$

$$V_H = \frac{v_h(\sigma)}{\sqrt{-h^{(P)}_{ab} n^a n^b}} = \frac{v_h(\sigma)}{\cosh\eta}\,. \tag{5.15}$$

5.1.3 Odd-sphere

The three charges thin black ring can now be constructed out as a result of bending the D1-D5-P black 1-fold around a circle S^1 whose curvature radius, for an equilibrium configuration, is given by the solution of the extrinsic equation (5.10),

$$\sinh^2\eta = \frac{\mathcal{T}s/n + \Phi_{D1}\mathcal{Q}_{D1} + \Phi_{D5}\mathcal{Q}_{D5}}{\mathcal{T}s + v_H\mathcal{P}}, \tag{5.16}$$

with $\Omega R = \tanh\eta$. We restrict, as usual, to the simpler case where all the local physical quantities are constant over the worldvolume, thus the global physical parameters are trivially integrated and for a D1-D5-P black 1-fold fluid wrapping an S^1 sphere (2.70),(2.73), (2.75), (3.81), (3.82), (5.14), (5.15) are now

$$\begin{aligned} M &= V(S^1)\left((\mathcal{T}s + v_H\mathcal{P})\cosh^2\eta + \left(\frac{\mathcal{T}s}{n} + \Phi_{D1}\mathcal{Q}_{D1} + \Phi_{D5}\mathcal{Q}_{D5}\right)\right) \\ &= V(S^1)\left(\left(1+\frac{2}{n}\right)\mathcal{T}s + 2\Phi_{D1}\mathcal{Q}_{D1} + 2\Phi_{D5}\mathcal{Q}_{D5} + v_H\mathcal{P}\right), \end{aligned} \tag{5.17a}$$

$$J = V(S^1)R\,(\mathcal{T}s + v_H\mathcal{P})\cosh\eta\sinh\eta$$

$$= V(S^1)R\frac{\sqrt{((n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + nv_H\mathcal{P})\,(\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5})}}{n}, \tag{5.17b}$$

$$\Omega = \frac{1}{R}\tanh\eta = \frac{1}{R}\sqrt{\frac{\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5}}{(n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + nv_H\mathcal{P}}}, \tag{5.17c}$$

$$S = V(S^1)s\cosh\eta = V(S^1)s\sqrt{\frac{(n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + nv_H\mathcal{P}}{n\mathcal{T}s + nv_H\mathcal{P}}}, \tag{5.17d}$$

$$T = \mathcal{T}\frac{1}{\cosh\eta} = \mathcal{T}\sqrt{\frac{n\mathcal{T}s + nv_H\mathcal{P}}{(n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + nv_H\mathcal{P}}}, \tag{5.17e}$$

$$Q_{D1} = \mathcal{Q}_{D1}\,, \qquad \Phi_H^{(D1)} = V(S^1)\Phi_{D1}\,, \tag{5.17f}$$

$$Q_{D5} = \mathcal{Q}_{D5}\,, \qquad \Phi_H^{(D5)} = V(S^1)\Phi_{D5}\,, \tag{5.17g}$$

$$\begin{aligned} P &= V(S^1)\mathcal{P}\cosh\eta = V(S^1)\mathcal{Q}_P\sqrt{\frac{(n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + n\Phi_P\mathcal{Q}_P}{n\mathcal{T}s + nv_H\mathcal{P}}}, \\ V_H &= v_H\frac{1}{\cosh\eta} = v_H\sqrt{\frac{n\mathcal{T}s + nv_H\mathcal{P}}{(n+1)\mathcal{T}s + n\Phi_{D1}\mathcal{Q}_{D1} + n\Phi_{D5}\mathcal{Q}_{D5} + nv_H\mathcal{P}}}, \end{aligned} \tag{5.17h}$$

where $n = 2$ in the present case.

Using now the local thermodynamic quantities (5.4) of the effective fluid living on the brane, we can now find the global thermodynamics of a three charges six dimensional thin black ring,

$$M = \frac{r_0^3 \Omega_{(3)} V(S^1)}{16\pi G} \left(2 + n + n \left(2\sinh^2(\alpha_1) + 2\sinh^2(\alpha_5) + \sinh^2(\alpha_P)\right)\right) \tag{5.18a}$$

,

$$J = \frac{r_0^3 \Omega_{(3)} V(S^1)}{16\pi G} R \sqrt{\left(1 + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5\right)\right)\left(1 + n + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)\right)}\,, \tag{5.18b}$$

$$\Omega = \frac{1}{R}\sqrt{\frac{1 + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5\right)}{1 + n + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)}}\,, \tag{5.18c}$$

$$S = \frac{r_0^4 \Omega_{(3)} V(S^1)}{4G} \cosh\alpha_1 \cosh\alpha_5 \sqrt{\frac{\left(1 + n + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)\right)}{n}}\,, \tag{5.18d}$$

$$T = \frac{n}{4\pi r_0 \cosh\alpha_1 \cosh\alpha_5 \sqrt{\frac{\left(1+n+n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)\right)}{n}}}\,, \tag{5.18e}$$

$$Q_{D1} = \frac{r_0^3 \Omega_{(3)}}{16\pi G} n \sinh\alpha_1 \cosh\alpha_1\,, \qquad \Phi_H^{(D1)} = V(S^1)\tanh\alpha_1\,, \tag{5.18f}$$

$$Q_{D5} = \frac{r_0^3 \Omega_{(3)}}{16\pi G} n \sinh\alpha_5 \cosh\alpha_5\,, \qquad \Phi_H^{(D5)} = V(S^1)\tanh\alpha_5\,, \tag{5.18g}$$

$$P = \frac{r_0^3 \Omega_{(3)} V(S^1)}{16\pi G} \sinh\alpha_P \sqrt{n\left(1 + n + n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)\right)}\,, \tag{5.18h}$$

$$V_H = \frac{\sinh\alpha_P}{\sqrt{\frac{1+n+n\left(\sinh^2\alpha_1 + \sinh^2\alpha_5 + \sinh^2\alpha_P\right)}{n}}}\,. \tag{5.18i}$$

Observe that if we set all the charges $\alpha_P = \alpha_1 = \alpha_5 = 0$ we recover all the physical parameters of the six-dimensional black ring (2.110) (for $p = 1$). Turning on the momentum charge α_P, the thermodynamics of (3.94) are recovered for $p = 1$ and by identifying α_P with α_0. Indeed it is now well known to exist an equivalence between momentum charge and charge in string theory, [62]. Turning on only one of α_5, or α_1, we can recover the thermodynamics of (3.38) a six-dimensional black 1-fold, with a string charge on it.

5.1.4 Smarr relation and 1st law

The Smarr relation for this system is (5.13) which can be rewritten as

$$\begin{aligned}(D-3)\,M = &(D-3)\left(\Omega J + TS + Q_{D1}\Phi_H^{D1} + Q_{D5}\Phi_H^{D5} + V_H P\right) + \\ &\Omega J + TS - \left(Q_{D1}\Phi_H^{D1} + Q_{D5}\Phi_H^{D5}\right)\,,\end{aligned} \tag{5.19}$$

where all the quanties, in units where $\frac{r_0^3\Omega_{(3)}V(S^1)}{16\pi G}=1$, are,

$$M = 2+n+n\left(2\sinh^2(\alpha_1)+2\sinh^2(\alpha_5)+\sinh^2(\alpha_P)\right)\,, \tag{5.20a}$$

$$TS = n\,, \tag{5.20b}$$

$$\Omega J = 1+n\left(\sinh^2\alpha_1+\sinh^2\alpha_5\right)\,, \tag{5.20c}$$

$$\Phi_H^{D1}Q_{D1} = n\sinh^2\alpha_1\,,\qquad \Phi_H^{D5}Q_{D5} = n\sinh^2\alpha_5\,,\qquad V_HP = n\sinh^2\alpha_P\,. \tag{5.20d}$$

It is straightforward to check the Smarr relation is satisfied. It has also been checked numerically the above quantities satisfy the first law of thermodynamics (5.12) evaluated for variation of each of the parameters $r_0,\alpha_1,\alpha_5,\alpha_P$ of the embedding.

5.1.5 Extremal limit

Extremal D1-D5-P blackfolds have been studied in [23]. We aim now to recover those solutions taking the extremal limit of the results found in the previous sections.

The limit to an extremal solution is obtained by taking

$$r_0\to 0\,,\qquad \alpha\to\infty \tag{5.21}$$

while keeping the charge Q_p fixed. In this limit, $\mathcal{T}s\to 0$ and $v_H,\Phi_q\to\sqrt{N}=1$ so (5.5) is now

$$\varepsilon=\sum_q \mathcal{Q}_q+\mathcal{P}\,. \tag{5.22}$$

The only dynamics now is extrinsic, and it is governed by the minimal-surface equations

$$K^\rho=0\,. \tag{5.23}$$

It is interesting to study if besides the limit (5.21), we moreover scale the velocity field components to infinity in such a way that $r_0^{n/2}u^a$ remains finite in that limit. We then introduce a 'momentum density' $\mathcal{K}$ and a vector l that remain finite in the limit,

$$\left(\frac{\Omega_{(n+1)}}{16\pi G}nr_0^n\right)^{1/2}u^a=\mathcal{K}^{1/2}l^a\,. \tag{5.24}$$

Thus, at extremality, $r_0\to 0$, l becomes a lightlike vector

$$l_al^a=0\,, \tag{5.25}$$

the fluid on the brane is boosted to the speed of light.

We now take the extremal limit (5.21) of the six dimensional metric (B.20). At extremality the functions

$$H_1\to 1+\frac{q_1}{r^2}\,,\,,\quad H_5\to 1+\frac{q_5}{r^2}\,,\,,\quad H_p\to 1+\frac{q_p}{r^2}\,,\,,\quad f\to 1\,, \tag{5.26}$$

thus introducing the lightcone coordinates $u = t + z$ and $v = t - z$, the extremal metric is

$$ds^2 = H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \left[-du dv + \frac{q_p}{r^2} dv^2 + H_5 H_1 (dr^2 + r^2 d\Omega_3^2) \right] . \tag{5.27}$$

The local thermodynamic parameters (5.4), are now

$$\varepsilon = \frac{\pi}{4G}(q_1 + q_5 + q_p) , \tag{5.28a}$$

$$s = \frac{\pi^2}{2G}\sqrt{q_1 q_5 q_p} , \qquad \mathcal{T} = 0 , \tag{5.28b}$$

$$\mathcal{Q}_{1,5} = \frac{\pi}{4G} q_{1,5} , \qquad \Phi_{1,5} = 1 , \tag{5.28c}$$

$$\mathcal{K} = \mathcal{P} = \frac{\pi}{4G} q_p , \qquad v_H = 1 . \tag{5.28d}$$

Observe the agreement between the geometrical entropy and the entropy counting of microstates, as analysed in ch.1. These solutions correctly reproduce the ones find in [23]. Furthermore the stress energy tensor (5.10), in the extremal limit reduces to

$$T_{ab} = \mathcal{P} l_a l_b - \sum_q \mathcal{Q}_q \gamma_{ab} . \tag{5.29}$$

The global physical parameters (5.18) for the extremal six dimensional black ring become

$$M = \frac{\pi}{2G} V_{(1)}(q_1 + q_5) , \tag{5.30a}$$

$$J = \frac{\pi}{4G} R V_{(1)} q_p \qquad \Omega = R^{-1} , \tag{5.30b}$$

$$Q_{D1,D5} = \frac{\pi}{4G} q_{1,5} , \qquad \Phi_H^{D1,D5} = V_{(1)} , \tag{5.30c}$$

$$P = \frac{\pi}{4G} V_{(1)} q_p , \qquad V_H = 1 . \tag{5.30d}$$

At extremality the first law reads

$$dM = \Omega dJ + \Phi_H^{(D1)} dQ_{\rm D1} + \Phi_H^{(D5)} dQ_{\rm D5} . \tag{5.31}$$

All these results reproduce correctly the one found in [23].

The stress tensor becomes

$$T_{ab} = \mathcal{K}\, l_a l_b - Q_p\, \gamma_{ab} . \tag{5.32}$$

Thus, this extremal brane supports a null momentum wave with momentum density $\mathcal{K}$, without breaking locally the translational invariance along the wave.

Outlook

In this thesis work we have considered stationary configurations of $D \geq 5$ black holes, arising from black Dp-branes with D0-brane charges, or F1-string dipole dissolved on it, regarding them as thin black branes that wrap a submanifold of the background spacetime. Furthermore we have built the six-dimensional thin D1-D5-P black ring and studied its thermodynamics. All the thermodynamics found satisfies the Smarr relation and the global first law.

This thesis then provides a non-trivial check of the applicability of the blackfold approach, if one needs further proofs. Indeed this approach has just been tested and developed in different scenarios. In [74] a charged and dipole black rings in string/M theory in Taub-Nut have been constructed; in [75], the approach has been developed to other non trivial backgrounds, like dS and AdS spaces. Furthermore we have worked to leading order in the blackfold expansion, anyway recently higher order corrections has been found and studied, [76]. Moreover, as we have seen, blackfolds are not like other branes, since they posses a thin event horizon in the transverse space, and the blackfold action only at extremality reduces to the Nambu-Goto action. Thus in [77] it has been studied the thermal versions of BIon solutions. Last but not least it has been recently shown that blackfold equations can be derived from the Einstein equations [47]. All of these reveal the approach is self consistent, which means it is a new, very powerful tool to study the dynamics and the thermodynamics of higher dimensional black branes in certain regimes.

The main results of this thesis can be summarized in:

- Thermodynamics of D0-Dp brane charges black holes, (4.44), (4.53), (4.64), arising from solutions of type IIA string theory
- The thermodynamics of F1-Dp brane charges black holes, (4.79), (4.82), arising from solutions of both type IIA/B string theory, and where we have considered two different alignments of the fundamental string in the probe brane.
- The thermodynamics, (5.18), of a six-dimensional black ring with three charges arising from type IIB string theory.

What is novel, and somehow unexpected *a priori*, is that all these results show a very deep symmetry. The thermal $\propto TS$ and charged $\propto Q_i \Phi_H^{(i)}$ factors entering in the Smarr relations, (4.46), (4.55), (4.66), (4.92), (5.20), are every time the same [4]. Moreover the values of the mass

[4] to be honest in each system we wrote these factors in units where $\frac{V_{(p)}\Omega_{(n+1)}}{16\pi G} r_0^n = 1$, where the volume is a

and of the kinetic term ΩJ, is always the same for black holes arising from D0-Dp solutions, as well as for the six-dimensional three charges black ring if we turn off the D1- or the D5-brane charge. This is slightly different for black holes arising from F1-Dp solutions, since the fundamental string creates an anisotropy in the worldvolume of the probe brane increasing the total energy of the system.

A possible explanation of these symmetries could be found in T-duality (or more generally U-duality) relations between these systems. Anyway, these results suggest a very strong and deep possibility that all the black holes with perfectly round spheres horizon topologies (the approximation we used in all these work), have all the same thermal, and charge components, as they enter in the Smarr relation, characterized only by their own parameters: spatial extension of the brane probe, p, the dimension D (through the parameter $n = D - P - 3$) and the microstates α_i, as well as the horizon thickness r_0 and the volume of the odd-spheres $V_{(p)}$. That suggests there could be a new and deep insight to black hole physics in string theory, and deserve further studies and explanations.

function of the radius of equilibrium, which is every time different in the systems we have considered. Anyway, the symmetry of the results seems to be deeper than a merely coincidence.

Appendix A

String current perpendicular to boosts

In this appendix we analyze the geometry and the stringy boost v, for stationary configurations in which the string current is aligned perpendicularly to the fluid velocity u. The current (3.54), is

$$J_{(q+1)} = \mathcal{Q}_q \hat{V}_{(q+1)}\,, \tag{A.1}$$

where the unit 1-form orthogonal to n^a is

$$\omega_a = \frac{-\hat{V}_{ab}n^b}{\sqrt{-h_{ab}n^a n^b}}\,. \tag{A.2}$$

Thus in order to find the configuration in which the string current is perpendicular to the boosts we need

$$\omega_a u^a = 0\,, \tag{A.3}$$

then

$$-\hat{V}_{ab}n^a u^b = -(u_a v_b - v_a u_b)n^a u^b = u_a n^a v_b u^b - v_a n^a u_b u^b = 0\,, \tag{A.4}$$

which means

$$v_a n^a = 0\,, \tag{A.5}$$

and

$$v_a u^a = 0\,. \tag{A.6}$$

The vector along which the string lies on is a Killing vector ψ which commutes with the worldvolume Killing vector k (3.84). In stationary configurations the velocity field is aligned along the Killing field (4.9), thus we can write in the most general way

$$\psi^a = \alpha\partial_t + \beta\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} + \gamma\sum_{J=k+1}^{l}\Omega_J\,,\partial_{\phi_J}\,, \tag{A.7}$$

where $1 < k < l$ is an arbitrary number of odd-spheres [1] and α, β, γ are parameters to be determined.

Then

$$\zeta^a = \alpha\partial_t + \beta\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} + \gamma\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J} +$$

$$\left(-\alpha + \sum_{I=1}^{k}\frac{p_I}{p}\beta\tanh^2\eta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\tanh^2\eta\right)\cosh^2\eta\left(\partial_t + \sum_{I=1}^{k}\Omega_I\partial_{\phi_I} + \sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}\right), \tag{A.8}$$

and rearranging the terms

$$\zeta^a = \left(\left(-\alpha + \sum_{I=1}^{k}\frac{p_I}{p}\beta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\right)\sinh^2\eta\right)\partial_t +$$
$$\left(\beta - \alpha\cosh^2\eta + \left(\sum_{I=1}^{k}\frac{p_I}{p}\beta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\right)\sinh^2\eta\right)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} +$$
$$\left(\gamma - \alpha\cosh^2\eta + \left(\sum_{I=1}^{k}\frac{p_I}{p}\beta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\right)\sinh^2\eta\right)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J} \tag{A.9}$$

If we want the string current to be orthogonal to the boost, we need (A.5) and (A.6) to be satisfied. The first means to require

$$\alpha = \sum_{I=1}^{k}\frac{p_I}{p}\beta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma \tag{A.10}$$

in (A.9).

Thus we can write (A.9) as

$$\zeta^a = (\beta-\alpha)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} + (\gamma-\alpha)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}, \tag{A.11}$$

or, using (A.10),

$$\zeta^a = \left(\beta\left(1 - \sum_{I=1}^{k}\frac{p_I}{p}\right) - \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\right)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} +$$
$$\left(\gamma\left(1 - \sum_{J=k+1}^{l}\frac{p_J}{p}\right) - \sum_{I=1}^{k}\frac{p_I}{p}\beta\right)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}, \tag{A.12}$$

which can be rewritten as

$$\zeta^a = (\beta-\gamma)\left(\sum_{J=k+1}^{l}\frac{p_J}{p}\right)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} + (\gamma-\beta)\left(\sum_{I=1}^{k}\frac{p_I}{p}\right)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}. \tag{A.13}$$

[1] the $l = k = 1$ case is not considered now, but will be analyzed later

We then require v to be canonically normalized $v_a v^a = 1$. This fixes

$$\beta - \gamma = \sqrt{\frac{1}{\left(\sum_{I=1}^{k} p_I\right)\left(\sum_{J=k+1}^{l} p_J\right)}}\,, \tag{A.14}$$

indeed this condition means to have

$$\zeta^a = \left(\sqrt{\frac{\sum_{J=k+1}^{p} p_J}{\sum_{I=1}^{k} p_I}}\right)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} - \left(\sqrt{\frac{\sum_{I=1}^{k} p_I}{\sum_{J=k+1}^{l} p_J}}\right)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}\,, \tag{A.15}$$

then $|\zeta| = \tanh\eta$, thus

$$v^a = \frac{1}{\tanh\eta}\left(\left(\sqrt{\frac{\sum_{J=k+1}^{l} p_J}{\sum_{I=1}^{k} p_I}}\right)\sum_{I=1}^{k}\Omega_I\partial_{\phi_I} - \left(\sqrt{\frac{\sum_{I=1}^{k} p_I}{\sum_{J=k+1}^{l} p_J}}\right)\sum_{J=k+1}^{l}\Omega_J\partial_{\phi_J}\right) \tag{A.16}$$

It is straightforward to see that (A.16) satisfies (A.5) and (A.6), although we have never imposed the latter, which then plays a redundancy role in the analysis. Moreover we have not imposed at all any value to α or γ or β. We have instead required orthogonality, (A.5), and we have normalized the stringy boost v^a. We then ended up with (A.16) which is independent on all those parameters. Thus, two parameters among α or γ or β are determined as function of the other one by

$$\beta - \gamma = \sqrt{\frac{1}{\left(\sum_{I=1}^{k} p_I\right)\left(\sum_{J=k+1}^{l} p_J\right)}}\,, \tag{A.17}$$

$$\alpha = \sum_{I=1}^{k}\frac{p_I}{p}\beta + \sum_{J=k+1}^{l}\frac{p_J}{p}\gamma\,. \tag{A.18}$$

Nevertheless, there is still a free parameter in (A.16), namely the number of odd-spheres k. Anyway, in the blackfold construction this liberty plays no role in the compute of the worldvolume quantities. This occurs every time we contract ζ with a Killing vector, since we get a quantity which is independent on the number k as we have just seen, for example, in $|\zeta| = \sqrt{\zeta^a\zeta_a}$.

In the case of a single odd-sphere everything holds again, but now the sums are performed over Cartan-angles. This implies to consider these kind of configurations

$$\psi^a = \alpha\partial_t + \beta\sum_{i=1}^{k}\Omega_i\partial_{\phi_i} + \gamma\sum_{j=k+1}^{l}\Omega_j\partial_{\phi_j}\,. \tag{A.19}$$

All the analysis is the same, and one finally ends up with

$$\zeta^a = \left(\sqrt{\frac{p-k}{k}}\right)\sum_{i=1}^{k}\Omega_i\partial_{\phi_i} - \left(\sqrt{\frac{k}{p-k}}\right)\sum_{j=k+1}^{l}\Omega_j\partial_{\phi_j}\,, \tag{A.20}$$

thus $|\zeta| = \tanh\eta$ and

$$v^a = \frac{1}{\tanh\eta}\left(\left(\sqrt{\frac{p-k}{k}}\right)\sum_{i=1}^{k}\Omega_i\partial_{\phi_i} - \left(\sqrt{\frac{k}{p-k}}\right)\sum_{j=k+1}^{p}\Omega_j\partial_{\phi_j}\right) \tag{A.21}$$

with the orthogonality and normalizing condition which now reads

$$\beta - \gamma = \sqrt{\frac{1}{k\,(p-k)}}\,, \tag{A.22}$$

$$\alpha = \frac{k}{p}\beta + \frac{p-k}{p}\gamma\,. \tag{A.23}$$

Appendix B

Six-dimensional boosted black D1-D5-P string, building up the solution

B.1 T-duality

The starting point is the D0-F1-D4 black brane solution of type IIA string theory, which is the T-dual of the D1-D5-P black brane solution of type IIB string theory. String frame metric:

$$ds^2 = H_1^{-1} H_4^{-\frac{1}{2}} H_0^{-\frac{1}{2}} \left[-f dt^2 + H_4 H_0 dz^2 + H_1 H_0 \sum_{i=1}^{4} dx_i^2 + H_1 H_4 H_0 (f^{-1} dr^2 + r^2 d\Omega_3^2) \right] . \tag{B.1a}$$

Dilaton:

$$e^{2\phi} = H_1^{-1} H_4^{-\frac{1}{2}} H_0^{\frac{3}{2}} . \tag{B.1b}$$

NSNS and RR potentials:

$$B_{0z} = \coth \alpha_1 (H_1^{-1} - 1) \ , \quad A_{01234} = \coth \alpha_4 (H_4^{-1} - 1) \ , \quad A_0 = \coth \alpha_0 (H_0^{-1} - 1) . \tag{B.1c}$$

Functions:

$$f = 1 - \frac{r_0^2}{r^2} \ , \quad H_a = 1 + \frac{r_0^2 \sinh^2 \alpha_a}{r^2} \tag{B.1d}$$

where $a = 1, 4, 0$.

This configuration consists of a D0-brane smeared along z and x_1, x_2, x_3, x_4, an F-string charged along z and smeared along x_1, x_2, x_3, x_4 and finally a D4-brane charged along x_1, x_2, x_3, x_4 and smeared along z. Using the prescription of ch. 1 we can depict this configuration as

$$\begin{array}{lcccccccccc} & 0 & z & x_1 & x_2 & x_3 & x_4 & 6 & 7 & 8 & 9 \\ \text{D0} & - & \sim & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \\ \text{F1} & - & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \\ \text{D4} & - & \sim & - & - & - & - & \cdot & \cdot & \cdot & \cdot \end{array} \tag{B.2}$$

The operation of T-duality on a circle switches winding and momentum modes and exchanges Type IIA and IIB. The effect on units is:

$$\frac{\tilde{R}}{\tilde{l}_s} = \frac{l_s}{R} \quad , \qquad \frac{\tilde{g}_s}{\sqrt{\tilde{R}/\tilde{l}_s}} = \frac{g_s}{\sqrt{R/l_s}} \quad , \qquad \tilde{l}_s = l_s \, . \tag{B.3}$$

It should be emphasized that T-duality does not leave all the branes invariant. It changes the dimension of a brane depending on whether the transformation is performed on a circle parallel ($\|$) or perpendicular ($\perp$) to the worldvolume

$$Dp \leftrightarrow Dp-1(\|) \; or \, Dp+1(\perp) \, , \tag{B.4}$$

but above all, the brane must be smeared along that direction. For quick reference, we briefly recall here the Buscher rules of T-duality, as given in [30], [61]. In a general background that has an isometry along the direction z, metric of the form $(\mu, \nu, ... \neq z)$

$$ds^2 = G_{zz}\big(dx^0\big)^2 + 2G_{z\mu}dzdx^\mu + 2G_{\mu\nu}dx^\mu dx^\nu \, , \tag{B.5a}$$

and B-field components $B_{z\mu}$, $B_{\mu\nu}$, the T-duality transformation along z acts in the following manner

$$G'_{zz} = \frac{1}{G_{zz}} \; , \; \; G'_{z\mu} = \frac{B_{\mu z}}{G_{zz}} \; , \; \; G'_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu z}G_{z\nu} - B_{z\mu}B_{z\nu}}{G_{zz}} \, , \tag{B.5b}$$

$$B'_{\mu z} = \frac{G_{z\mu}}{G_{zz}} \; , \; \; B'_{\mu\nu} = B_{\mu\nu} - \frac{G_{z\mu}B_{z\nu} - B_{z\mu}G_{z\nu}}{G_{zz}} \, , \tag{B.5c}$$

$$C'^{(n)}_{\mu\ldots\nu\alpha z} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}G_{|\alpha]z}}{G_{zz}} \, , \tag{B.5d}$$

and the dilaton transforms as

$$g^2_{s,\text{new}} = g^2_{s,\text{old}}\frac{\det G_{\text{new}}}{\det G_{\text{old}}} \, . \tag{B.5e}$$

Note that applying T-duality to a supergravity Dp-brane solution in a direction which is not an isometry (i.e. in a direction perpendicular to its worldvolume) we should first "smear" the Dp-brane in that direction to create an isometry and then T-dualize it. Indeed only smeared brane can satisfies (B.4), as depicted in fig. 1.1.

We can now T-dualize our configuration (B.1) using (B.5), so that

$$G'_{zz} = \frac{1}{G_{zz}} = H_p H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \; , \; \; G'_{z\mu} = G'_{z0} = \frac{B_{0z}}{G_{zz}} = \coth\alpha_p(1-H_p)H_5^{-\frac{1}{2}}H_1^{-\frac{1}{2}} \, , \tag{B.6}$$

and

$$\begin{aligned} G'_{00} &= G_{00} - \frac{G_{0z}G_{z0} - B_{z0}B_{z0}}{G_{zz}} = H_5^{-\frac{1}{2}}H_1^{-\frac{1}{2}}\left(-fH_p^{-1} + \coth^2\alpha_p(1-H_p)^2H_p^{-1}\right) = \\ &= H_5^{-\frac{1}{2}}H_1^{-\frac{1}{2}}\left(H_p^{-1}\left(-1 + \frac{r_0^2}{r^2} + \frac{r_0^4}{r^4}\sinh^2\alpha_p\cosh^2\alpha_p\right)\right) = \\ &= H_5^{-\frac{1}{2}}H_1^{-\frac{1}{2}}\left(H_p^{-1}\left(\left(-1 + \frac{r_0^2}{r^2}\cosh^2\alpha_p\right)\left(1 + \frac{r_0^2}{r^2}\sinh^2\alpha_p\right)\right)\right) = \\ &= H_5^{-\frac{1}{2}}H_1^{-\frac{1}{2}}\left(-1 + \frac{r_0^2}{r^2}\cosh^2\alpha_p\right) \, , \end{aligned} \tag{B.7}$$

and for $i, j, ... \neq 0, z$,

$$G'_{ij} = G_{ij} \;, \tag{B.8}$$

the NS-NS field now is

$$B'_{\mu z} = 0 \;, \quad B'_{\mu\nu} = 0 \;. \tag{B.9}$$

The dilaton transforms as

$$\Phi' = \Phi - \frac{1}{2}\log G_{zz} = \Phi + \frac{1}{2}\log G'_{zz} \,. \tag{B.10}$$

Thus

$$g^2_{s,\mathrm{new}} = g^2_{s,\mathrm{old}} G'_{zz} \,. \tag{B.11}$$

We can now write down the supergravity solutions for the non-extremal D1-D5-P system thus obtained.

Dilaton [1]:

$$e^{2\Phi} = H_5^{-1} H_1 \,. \tag{B.12a}$$

Metric in string frame

$$\begin{aligned} ds^2 = H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \Bigg[& -dt^2 + dz^2 + \frac{r_0^2}{r^2}(\cosh\alpha_p dt + \sinh\alpha_p dz)^2 + \\ & + H_1 \sum_{i=1}^{4} dx_i^2 + H_5 H_1 (f^{-1} dr^2 + r^2 d\Omega_3^2) \Bigg] \,. \end{aligned} \tag{B.12b}$$

Metric in Einstein frame

$$\begin{aligned} ds^2 = H_5^{-\frac{1}{4}} H_1^{-\frac{3}{4}} \Bigg[& -dt^2 + dz^2 + \frac{r_0^2}{r^2}(\cosh\alpha_p dt + \sinh\alpha_p dz)^2 + \\ & + H_1 \sum_{i=1}^{4} dx_i^2 + H_5 H_1 (f^{-1} dr^2 + r^2 d\Omega_3^2) \Bigg] \,. \end{aligned} \tag{B.12c}$$

RR-potential

$$A_{01234z} = \coth\alpha_5 (H_5^{-1} - 1) \;, \quad A_{0z} = \coth\alpha_1 (H_1^{-1} - 1) \,. \tag{B.12d}$$

Field strengths:

$$F_{01234zr} = n r_0^n \sinh\alpha_5 \cosh\alpha_5 (r^{-n-1} H_5^{-2}) \;, \quad F_{0zr} = n r_0^n \sinh\alpha_1 \cosh\alpha_1 (r^{-n-1} H_1^{-2}) \,. \tag{B.12e}$$

Functions:

$$f = 1 - \frac{r_0^2}{r^2} \;, \quad H_a = 1 + \frac{r_0^2 \sinh^2\alpha_a}{r^2} \tag{B.12f}$$

where $a = 1, 5, P$.

[1] from now on we omit the prime since we will not deal with the D0-F1-D4 system anymore.

This configuration consists of a D1-brane charged along z and smeared along x_1, x_2, x_3, x_4, a momentum charge P along z and a D5-brane charged along z, x_1, x_2, x_3, x_4. Thus this configuration can be depicted as

$$\begin{array}{lcccccccccc} & 0 & z & x_1 & x_2 & x_3 & x_4 & 6 & 7 & 8 & 9 \\ \text{D1} & - & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \\ \text{P} & - & \rightarrow & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \text{D5} & - & - & - & - & - & - & \cdot & \cdot & \cdot & \cdot \end{array} \tag{B.13}$$

where $\rightarrow$ indicates the direction in which the momentum charge moves. Observe the necessity to have smeared branes along the transverse direction z in (B.2) in order to get higher extended objects charged along that direction after T-duality. Furthermore it's worth to notice how T-duality acts with the F1 string that was charged along z. It transforms the gauge charge of the string (charged under the NSNS potential), into a momentum charge, whose momentum flows along z.

B.2 Compactification on T^4

We now reduce our space to six dimensions, by compactifying the solutions (B.12) on T^4 whose compact directions are taken to be x_1, x_2, x_3, x_4. The resulting object will be a three charges six dimensional black string whose charges are a momentum charge P and two string charges, one from the D1-brane and the other one as a result of compactify the D5-brane along its four space-like directions, as one can easily see from (B.13).

Einstein Frame

The metric (B.12c) is diagonal along x_1, x_2, x_3, x_4, thus under dimensional reduction the gauge vectors arising in the compact space have vanishing components. This allow us to write the KK ansatz [2] directly in the following form,

$$ds_{10}^2 = e^{2\alpha\varphi} ds_6^2 + e^{2\beta\varphi} \sum_{i=1}^{4} dx_i^2 \,, \tag{B.14}$$

and to compactify simultaneously four directions. To get a six dimensional metric out from (B.12c) it is straightforward to see we need

$$e^{2\beta\varphi} = |g_{x_i x_i}^{10}| = e^{2\Phi} \,. \tag{B.15}$$

Where Φ is the ten dimensional dilaton field (B.12a). Using (B.15) in (B.14) we can write,

$$|g_{10}| = \left(e^{2\alpha\varphi}\right)^6 |g_6| e^{2\Phi} \,. \tag{B.16}$$

On the other hand the Ricci scalar under dimensional reduction transforms as

$$R_{10} = g_{10}^{\mu\nu} R_{\mu\nu}^{10} = g_{10}^{\mu\nu} \left(R_{\hat\mu\hat\nu}^6 + ...\right) = e^{-2\alpha\varphi} g_6^{\hat\mu\hat\nu} \left(R_{\hat\mu\hat\nu}^6 + ...\right) = e^{-2\alpha\varphi} R_6 \,, \tag{B.17}$$

[2] we use the convention of [30],[63], [64]

where hatted indices are running on the 6-dimensional non-compact space, and the dots stand for factors which do not contribute in this analysis. Thus,

$$\sqrt{|g_{10}|}R_{10} = e^{\Phi} e^{6\alpha\varphi} e^{-2\alpha\varphi} \sqrt{|g_6|}R_6 \,. \tag{B.18}$$

Requiring the six dimensional Einstein Hilbert action to be in the canonical form fully determine the value of $\alpha\varphi$ in (B.14), to be

$$\alpha\varphi = -\Phi/4 \,. \tag{B.19}$$

Thus the 6-dimensional metric in the Einstein frame is

$$ds^2 = H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \left[-dt^2 + dz^2 + \frac{r_0^2}{r^2} (\cosh \alpha_p dt + \sinh \alpha_p dz)^2 + H_5 H_1 (f^{-1} dr^2 + r^2 d\Omega_3^2) \right] \,. \tag{B.20}$$

String Frame

Let's compactify the ten dimensional string frame metric (B.12b), using the KK-ansatz (B.14).

The analysis is very similar than before, the main difference is that now

$$e^{2\beta\varphi} = |g^{10}_{x_i x_i}| = e^{4\Phi} \,, \tag{B.21}$$

where Φ is the ten dimensional dilaton field. The relation (B.17) between the six and the ten dimensional Ricci scalar holds again since it is a consequence only of the KK-ansatz (B.14), and it does not depend on the frame of the metric.

The gravity part of the Einstein-Hilbert action in string frame thus scales under dimensional reduction as,

$$\sqrt{|g_{10}|} e^{-2\Phi} R_{10} = e^{2\Phi} e^{6\alpha\varphi} e^{-2\alpha\varphi} e^{-2\Phi} \sqrt{|g_6|} R_6 \,. \tag{B.22}$$

Thus the six dimensional dilaton appears to be $e^{-2\Phi_6} = e^{6\alpha\varphi}$, but this is wrong, since under dimensional reduction the dilaton transforms as

$$e^{-2\Phi_6} \equiv e^{-2\Phi_{10}} \sqrt{G_{ab}} \,, \tag{B.23}$$

where the indices a, b are running on the internal compact 4-dimensional space. Thus

$$e^{-2\Phi_6} = e^{-2\Phi_{10}} H_5^{-1} H_1 = 1 \,, \tag{B.24}$$

which implies

$$\alpha\varphi = 0 \,. \tag{B.25}$$

So the 6-dimensional metric in string frame,

$$ds^2 = H_5^{-\frac{1}{2}} H_1^{-\frac{1}{2}} \left[-dt^2 + dz^2 + \frac{r_0^2}{r^2} (\cosh \alpha_p dt + \sinh \alpha_p dz)^2 + H_5 H_1 (f^{-1} dr^2 + r^2 d\Omega_3^2) \right] \,, \tag{B.26}$$

takes the same form as the one in Einstein frame. The 6-dimensional 3-charges black string is then a non-dilatonic black object, and this configuration can be depicted (with the notation of (B.13)) by

$$\begin{array}{ccccccc} & 0 & z & 6 & 7 & 8 & 9 \\ \mathrm{D1} & - & - & \cdot & \cdot & \cdot & \cdot \\ \mathrm{P} & - & \rightarrow & \cdot & \cdot & \cdot & \cdot \\ \mathrm{D5} & - & - & \cdot & \cdot & \cdot & \cdot \end{array} \tag{B.27}$$

RR fields

The next step is to find the six dimensional RR fields, after compactification on T^4. For our purpose it is easier to work with p-forms in order to clean up the notation as much as possible. Since we don't have gauge vectors arising from compactification of the metric, and we have only two non-vanishing gauge fields (B.12d), the ansatz [63], [64] for the RR potential can be written in a short-hand notation as,

$$A^{(10)}_{(n-1)} = A^{(6)}_{(n-1)} + A^{(6)}_{(n-5)} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,, \tag{B.28}$$

where the other 2^3 terms, two for each dimension reduced, vanish, since we do not have gauge vectors arising in the compact space. Thus the ten dimensional field strength can be split into

$$F^{(10)}_{(n)} = dA^{(6)}_{(n-1)} + dA^{(6)}_{(n-5)} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,. \tag{B.29}$$

Usually one identifies the lower dimensional field strength adding and subtracting a term, so that

$$F^{(10)}_{(n)} = F^{(6)}_{(n)} + F^{(6)}_{(n-4)} \left(dx^1 + \mathcal{A}^{x^1}_{(1)}\right) \wedge \left(dx^2 + \mathcal{A}^{x^2}_{(1)}\right) \wedge \left(dx^3 + \mathcal{A}^{x^3}_{(1)}\right) \wedge \left(dx^4 + \mathcal{A}^{x^4}_{(1)}\right) \,, \tag{B.30}$$

where $\mathcal{A}^{x^i}_{(1)}$ is the gauge vector arising from the compactifcation of the metric along the x^i direction. In our present case, each of those vectors has vanishing components, that's enough to write

$$F^{(10)}_{(n)} = F^{(6)}_{(n)} + F^{(6)}_{(n-4)} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,, \tag{B.31}$$

and to identify

$$F^{(6)}_{(n)} = dA^{(6)}_{(n-1)} \,, \qquad F^{(6)}_{(n-4)} = dA^{(6)}_{(n-5)} \,. \tag{B.32}$$

We now need to study how the components of these p-forms change under the procedure of dimensional reductions. This calculation is easier in the vielbein basis, which is possible since the ten dimensional metric is diagonal along x_1, x_2, x_3, x_4. Under the KK ansatz (B.14) the 10-dimensional vielbeins are related to the 6-dimensional one by

$$e^a_{(10)} = e^{\alpha\varphi} e^a_{(6)} \,, \tag{B.33}$$

and

$$e^{x^1}_{(10)} \wedge e^{x^2}_{(10)} \wedge e^{x^3}_{(10)} \wedge e^{x^4}_{(10)} = e^{\beta\varphi} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,. \tag{B.34}$$

Thus

$$F^{(10)}_{(n)} = \frac{1}{n!} F^{(10)}_{M_1 ... M_n} e^{M_1}_{(10)} \wedge ... \wedge e^{M_n}_{(10)} =$$

$$= \frac{e^{\alpha\varphi}}{n!} F^{(10)}_{\mu_1 ... \mu_n} e^{\mu_1} \wedge ... \wedge e^{\mu_n} + \frac{e^{\beta\varphi}}{(n-4)!} F^{(10)}_{\mu_1 ... \mu_{n-4} x^1 x^2 x^3 x^4} e^{\mu_1} \wedge ... \wedge e^{\mu_{n-4}} \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 =$$

$$= \frac{1}{n!} F^{(6)}_{\mu_1 ... \mu_n} e^{\mu_1} \wedge ... \wedge e^{\mu_n} + \frac{1}{(n-4)!} F^{(6)}_{\mu_1 ... \mu_{n-4}} e^{\mu_1} \wedge ... \wedge e^{\mu_{n-4}} \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,, \quad \text{(B.35)}$$

where we have identified

$$F^{(10)}_{\mu_1 ... \mu_n} = e^{-\alpha\varphi} F^{(6)}_{\mu_1 ... \mu_n}, \qquad F^{(10)}_{\mu_1 ... \mu_{n-4} x^1 x^2 x^3 x^4} = e^{-\beta\varphi} F^{(6)}_{\mu_1 ... \mu_{n-4}} \,. \quad \text{(B.36)}$$

Thus, using (B.15) and (B.19) in (B.12a),

$$F^{(10)}_{\mu_1 ... \mu_n} = e^{\Phi/4} F^{(6)}_{\mu_1 ... \mu_n}, \qquad F^{(10)}_{\mu_1 ... \mu_{n-4} x^1 x^2 x^3 x^4} = e^{-\Phi} F^{(6)}_{\mu_1 ... \mu_{n-4}} \,. \quad \text{(B.37)}$$

Appendix C

Local thermodynamics

The local thermodynamics of the six dimensional D1-D5-P system can now be easily carried out from the solutions (B.20) (B.37). In order to apply the effective theory, we focus on the subset of solutions where the black ring is very thin. Thus if r_0 is the radius of the transverse sphere s^{n+1}, and R the radius of the ring S^1 we restrict to the case where $r_0 \ll R$. In this regime the rotating black ring looks locally like a boosted black string, whose worldsheet, in this case, extends along z. The local thermodynamic parameters will be then densities over this worldsheet, and will be obtained by formally omitting to integrate over z. Similar quantities have been computed in [65], [66].

C.1 Entropy

The local entropy is a function of the area density of the event horizon,

$$s = \frac{a_H}{4G} = \frac{1}{4G} \int d\Omega_{(3)} \sqrt{g|_{z,\Omega_{(3)}}|}_{r=r_0} = \frac{1}{4G} \int d\Omega_{(3)} \left(H_1 H_5 H_p\right)^{1/2} |_{r=r_0} \,, \tag{C.1}$$

where $g|_{z,\Omega_{(3)}}|$ is the determinant of the metric (B.20) at constant t and r slices. Thus

$$s = \frac{r_0^3 \Omega_{(3)}}{4G} \cosh\alpha_1 \cosh\alpha_5 \cosh\alpha_p \,. \tag{C.2}$$

C.2 Temperature

The temperature is a function of the surface of gravity. The metric (B.20) has the particularity the time-time component does not vanish where the radial-radial component of the inverse metric does (at the horizon). This is due to the presence of the momentum wave which makes the metric (B.20) to develop an off-diagonal component in t and z, and indeed we do not have this issue by setting $\alpha_p = 0$ in the metric as in the D1-D5 solution.

Thus, to calculate the surface of gravity, we need to find a frame in which $g_{\tilde{t}\tilde{t}} = 0 = g^{rr}$ at the horizon. To this end, we write, as in [67],

$$\tilde{t} = t, \qquad \tilde{z} = z - vt \,. \tag{C.3}$$

In this new frame we have

$$g_{\tilde{t}\tilde{t}} = g_{tt} + 2vg_{tz} + v^2 g_{zz}\,, \tag{C.4}$$

and at horizon $g_{\tilde{t}\tilde{t}}|_{r=r_0} = 0$. The off-diagonal component in this frame has the property to vanish at the horizon,

$$g_{\tilde{t}\tilde{z}}|_{r=r_0} = g_{tz}|_{r=r_0} + vg_{zz}|_{r=r_0} = 0\,, \tag{C.5}$$

which means

$$v = -\frac{g_{tz}}{g_{zz}} = -\frac{\left(\frac{r_0}{r}\right)^2 \sinh\alpha_p \cosh\alpha_p}{1 + \left(\frac{r_0}{r}\right)^2 \sinh^2\alpha_p}\,. \tag{C.6}$$

At the horizon this reduces to

$$v_H \equiv -v|_{r=r_0} = \tanh\alpha_p\,, \tag{C.7}$$

where we have defined $v_H = \frac{g_{tz}}{g_{zz}}|_{r=r_0}$ for later convenience (one can get rid of this minus sign in a redefinition of the coordinate transformation (C.3) and deal with a negative value in front of g_{tz} in (C.4)). In this new frame

$$g_{\tilde{t}\tilde{t}} = \left(H_1^{-1/2} H_5^{-1/2}\right)\left(-1 + \left(\frac{r_0}{r}\right)^2 \cosh^2\alpha_p - \frac{\left(\frac{r_0}{r}\right)^2 \sinh^2\alpha_p \cosh^2\alpha_p}{1 + \left(\frac{r_0}{r}\right)^2 \sinh^2\alpha_p}\right)$$

$$= \left(H_1^{-1/2} H_5^{-1/2}\right)\left(\frac{-1 + \left(\frac{r_0}{r}\right)^2 \sinh^2\alpha_p \cosh^2\alpha_p}{1 + \left(\frac{r_0}{r}\right)^2 \sinh^2\alpha_p}\right) = -f H_p^{-1} H_1^{-1/2} H_5^{-1/2}\,. \tag{C.8}$$

The surface of gravity is then

$$\kappa = \sqrt{-\partial_r g_{\tilde{t}\tilde{t}}|_{r=r0} \partial_r g^{rr}|_{r=r0}} = \frac{2}{r_0 \cosh\alpha_p \cosh\alpha_1 \cosh\alpha_5}\,, \tag{C.9}$$

and the temperature density

$$\mathcal{T} = \frac{2}{4\pi r_0 \cosh\alpha_p \cosh\alpha_1 \cosh\alpha_5} \tag{C.10}$$

.

C.3 Charges

In D dimensions the electric charge is given by [68], [69],

$$\mathcal{Q} = \frac{1}{16\pi G_D} \int_{S_\infty^{n+1}} {}^*F_{n+1} e^{a\Phi_D}\,, \tag{C.11}$$

where the integral is over a (n+1)-sphere evaluated at infinity, Φ_D is the D-dimensional dilaton and *F is the Hodge dual of the field strength,

$${}^*F_{\mu_{n+1}\dots\mu_D} = \frac{1}{n!}\sqrt{|g|}\varepsilon_{\mu_1\dots\mu_D} F^{\mu_1\dots\mu_n}\,, \tag{C.12}$$

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}) \equiv \frac{1}{D-n!}\epsilon_{\nu_1\dots\nu_{D-n}}{}^{\mu_1\dots\mu_n} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-n}}\,, \tag{C.13}$$

where $\epsilon_{\mu_1 \dots \mu_D}$ is the totally antisymmetric Levi-Civita tensor, whose component are $\pm\sqrt{|g|}$ or 0, given by

$$\epsilon_{\mu_1 \dots \mu_D} = \sqrt{|g|}\varepsilon_{\mu_1 \dots \mu_D} \,, \tag{C.14}$$

where $\varepsilon_{\mu_1 \dots \mu_D} \equiv (+1, -1, 0)$ is the totally antisymmetric Levi-Civita tensor density.

Thus under the above definition we have in general [68]

$$*F = \frac{1}{n!} d^{D-n} \Xi_{\mu_1 \dots \mu_n} F^{\mu_1 \dots, \mu_n} \,, \tag{C.15}$$

where

$$d^{D-n}\Xi_{\mu_1 \dots \mu_n} \equiv \frac{1}{(D-n)!} \epsilon_{\nu_1 \dots \nu_{D-n} \mu_1 \dots \mu_n} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-n}} \,. \tag{C.16}$$

Thus plugging all together we have

$$\mathcal{Q} = \frac{1}{16\pi G_D} \int_{S_\infty^{n+1}} \frac{e^{a\Phi_D}}{n!(D-n)!} \sqrt{|g|} \varepsilon_{\nu_1 \dots \nu_{D-n} \mu_1 \dots \mu_n} g^{\mu_1 \mu_1} \dots g^{\mu_n \mu_n} F_{\mu_1 \dots \mu_n} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-n}} \,. \tag{C.17}$$

We can now refer to our present case. We have $D = 6$, $n = 2$, a vanishing six dimensional dilaton field, and six dimensional field strengths related by (B.37) to the ten dimensional ones (B.12e). We can write the integral (C.17) as

$$\mathcal{Q} = \frac{1}{16\pi G} \int_{S_\infty^3} \frac{\sqrt{|g|}}{2!4!} \varepsilon_{\nu_1 \nu_2 \nu_3 tzr} g^{tt} g^{zz} g^{rr} F^{(i)}_{tzr} dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\nu_3} \,, \tag{C.18}$$

for $i = 1, 5$. The RR potentials depend only on the radial component, and for a spherically symmetric system the above integral reduces to

$$\mathcal{Q} = \frac{1}{16\pi G} \int_{S_\infty^{n+1}} d\Omega_{n+1} r^{n+1} \sqrt{|g|} g^{tt} g^{zz} g^{rr} F^{(i)}_{tzr} \,. \tag{C.19}$$

Observe that for the metric (B.20), we can write

$$\sqrt{|g|} g^{tt} g^{zz} g^{rr} = \sqrt{\frac{|g_{(\Omega_3)}|}{g_{tt} g_{zz} g_{rr}}} = \sqrt{\frac{g_\perp}{g_\|}} = H_1 H_5 \,, \tag{C.20}$$

where $|g_{(\Omega_3)}|$ is the determinant of the sub-metric transverse to the worldsheet, at constant r.

We can now calculate the charges by using (B.12e) in (B.37) [1]

$$\mathcal{Q}_1 = \frac{1}{16\pi G} \int_{S_\infty^3} d^2\Omega_{(3)} 2r_0^n e^{-\Phi/4} \sinh\alpha_1 \cosh\alpha_1 (\frac{H_5}{H_1}) \,, \tag{C.21}$$

$$\mathcal{Q}_5 = \frac{1}{16\pi G} \int_{S_\infty^3} d^2\Omega_{(3)} 2r_0^n e^{\Phi} \sinh\alpha_5 \cosh\alpha_5 (\frac{H_1}{H_5}) \,. \tag{C.22}$$

These integrals has to be computed at asymptotic infinity, thus

$$\begin{aligned} \mathcal{Q}_1 &= \frac{\Omega_{(3)}}{16\pi G} 2r_0^2 \sinh\alpha_1 \cosh\alpha_1 \,, \\ \mathcal{Q}_5 &= \frac{\Omega_{(3)}}{16\pi G} 2r_0^2 \sinh\alpha_5 \cosh\alpha_5 \,. \end{aligned} \tag{C.23}$$

[1] observe that the D1-brane gives rise to an electric charge through ${}^*F_{tzr}$, while the D5-brane gives rise to a magnetic charge through $F_{\Omega_{(3)}}$, which is the dual of (B.12e)

It is worth to notice the $\mathcal{Q}_5$ and the $\mathcal{Q}_1$ charge densities as seen from the ten dimensional or six dimensional point of view are the same. They only change in the value of the gravitational constant and thus they are just related by the volume of the compact space, $\mathcal{Q}_i^{(6)} \sim \mathcal{Q}_i^{(10)} V(T^4)$.

C.4 Chemical potentials

The KK ansatz for the RR potential is (B.28) and the field strength forms under dimensional reduction become (B.32). Thus using these results and (B.37) we can identify

$$A_6^{(10)} = \tilde{A}_2^{(6)}, \qquad A_2^{(10)} = A_2^{(6)}, \tag{C.24}$$

where $\tilde{A}$ is the RR potential, after compactification, arising from the D5 brane solutions.

The electric potential is given by the difference between the asymptotic value and the value at the horizon of the RR potential, thus

$$\Phi_5 = \tilde{A}_{tz}^{(6)}(\infty) - \tilde{A}_{tz}^{(6)}(r_0) = -\tilde{A}_{tz}^{(6)}(r_0) = -A_{tzx^1..x^4}^{(10)}(r_0) = \tanh\alpha_5\,, \tag{C.25}$$

and

$$\Phi_1 = A_{tz}^{(6)}(\infty) - A_{tz}^{(6)}(r_0) = -A_{tz}^{(10)}(r_0) = \tanh\alpha_1\,. \tag{C.26}$$

Its remarkable to see the chemical potentials are left invariant by the dimensional reduction procedure. Anyway this does not generally hold, but it is a direct consequence of the particular form of the metric (B.20) and the related lack of gauge vectors arising from compactification.

C.5 Momentum charge

To see quantitatively that the solution (B.20) carries a momentum charge, there are two equivalent methods. We can compute it from the $t-z$ component of the ADM-like stress tensor as in [70], or as a function of the first order correction around flat space, [72],

$$g_{tz} \simeq 1 + \frac{c_{tz}}{r^{d-3}}\,, \tag{C.27}$$

The ADM-like stress tensor is given by the following asymptotic integrals

$$T_{ab} = \frac{1}{16\pi G}\oint d\Omega_{(n+1)}\hat{r}^{n+1}n^i\left(\eta_{ab}\left(\partial_i h_c^c + \partial_i h_j^j - \partial_j h_i^j\right) - \partial_i h_{ab}\right)\,, \tag{C.28}$$

where n^i is a radial unit vector in the transverse subspace, $a,b,c \in \{t,z\}$; i,j run on the transverse coordinates and $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the deviation of the asymptotic metric from flat space. To apply this formula the asymptotic metric should be given in Cartesian coordinate. Thus we change frame setting $r = \hat{r}\left(1 + \left(1 + (\frac{r_0}{\hat{r}})^n\right)/2n\right)$ where $\hat{r}^2 = \sum_{i=1}^{n+1}\left(y^i\right)^2$. Using this coordinates transformation and that $n=2$,

$$ds^2 = \hat{H}_5^{-\frac{1}{2}}\hat{H}_1^{-\frac{1}{2}}\left[-dt^2 + dz^2 + \frac{r_0^2}{\hat{r}^2}\left(\cosh\alpha_p dt + \sinh\alpha_p dz\right)^2 + \hat{H}_5\hat{H}_1\left(1 + \frac{1}{2}\left(\frac{r_0}{\hat{r}}\right)^2\right)dy^i dy_i\right]\,, \tag{C.29}$$

which is the 6-dimensional D1-D5-P black string metric in Cartesian coordinates, and the hats on the harmonic functions means they are now functions of $\hat{r}$. Thus

$$T_{tz} = \frac{1}{16\pi G} \oint d\Omega_{(n+1)} \hat{r}^{n+1} \left(-\partial_{\hat{r}} h_{tz}\right)|_{\hat{r}\to\infty} \, , \tag{C.30}$$

where

$$h_{tz} = \hat{H}_5^{-\frac{1}{2}} \hat{H}_1^{-\frac{1}{2}} \left(\frac{r_0}{\hat{r}}\right)^2 \cosh\alpha_p \sinh\alpha_p \, . \tag{C.31}$$

The contribution of the harmonic function is asymptotically constant, thus the $\mathcal{P}$ momentum charge density is

$$\mathcal{P} = T_{tz} = \frac{\Omega_{(3)} r_0^2}{16\pi G} 2\cosh\alpha_p \sinh\alpha_p \, . \tag{C.32}$$

Observe that the first order correction around flat space (C.27) is

$$c_{tz} = 2r_0^2 \cosh\alpha_p \sinh\alpha_p \tag{C.33}$$

and the momentum charge [66],[71],

$$\mathcal{P} = T_{tz} = \frac{\Omega_{(3)}}{16\pi G} c_{tz} \quad , \tag{C.34}$$

agrees with (C.32).

C.6 Horizon velocity

The remaining thermodynamic density to be evaluated is the one associated to the linear momentum charge density (C.32). As [66] and [73] noticed, the conjugated quantity to the momentum is the horizon velocity (C.7) $v_H = \frac{g_{tz}}{g_{zz}}|_{r=r_0}$. Indeed in parallel as the case of rotating black holes, where one considers zero angular momentum observers and identifies the angular velocity Ω_H of the horizon as the limit of the angular velocity of that observer as it approaches the horizon radius. For a boosted black string, we may analogously consider observers with zero linear momentum along the string. If we confine ourselves into the set of stationary solutions, and we align the linear momentum along the Killing vector field which generates the translational symmetry along z, then the condition $\mathcal{P} = 0$ of vanishing linear momentum is $dz/dt = g_{tz}/g_{zz}$, which evaluated at horizon is equal to (C.7).

C.7 Energy density

The energy density can be evaluated from the time-time component of (C.28), since to leading order it is the same as the Brown-York stress energy tensor. As before we can evaluate it alternatively studying the asymptotic behaviour of the time-time component of the metric tensor. To this end we need to find a frame in which $g_{\tilde{t}\tilde{t}} = 0$ when $g^{rr} = 0$, but that has been done with the coordinate transformation in (C.4). Thus we will use this new frame and we will use (C.8) as time-time component. To find the leading order corrections, in $(r_0/r)^2$, to asymptotic flat space, we expand $g_{\tilde{t}\tilde{t}}$ and g_{zz},

$$\begin{aligned} g_{\tilde{t}\tilde{t}} &= -1 + \frac{r_0^2}{r^2}\left(1 + \frac{1}{2}\sinh^2\alpha_1 + \frac{1}{2}\sinh^2\alpha_5 + \sinh^2\alpha_P\right) , \\ g_{zz} &= 1 + \frac{r_0^2}{r^2}\left(-\frac{1}{2}\sinh^2\alpha_1 - \frac{1}{2}\sinh^2\alpha_5 + \sinh^2\alpha_P\right) . \end{aligned} \tag{C.35}$$

The energy density is [71], [72], [66],

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^2 \left((D-3)\, c_t - c_z \right) , \tag{C.36}$$

where c_t and c_z are the leading order correction of the metric at infinity

$$g_{\tilde{t}\tilde{t}} = -1 + \frac{r_0^2}{r^2} c_t \qquad g_{zz} = 1 + \frac{r_0^2}{r^2} c_z \, . \tag{C.37}$$

Then

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^2 \left(3 + 2\sinh^2(\alpha_1) + 2\sinh^2(\alpha_5) + 2\sinh^2(\alpha_p)\right) . \tag{C.38}$$

This complete the local thermodynamics of the boosted six dimensional D1-D5-P black string.

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