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Master's Thesis

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Higher-Spin Holographic Dualities and $\mathcal{W}$-Algebras

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Abstract

In this thesis we will give a review of certain aspects of higher-spin gravity theories on AdS_3 and $\mathcal{W}$-algebras in two-dimensional conformal field theories. Recently the $\mathbb{C}P^N$ Kazama-Suzuki models with the non-linear chiral algebra $\mathcal{SW}_\infty[\lambda]$ have been conjectured to be dual to the fully supersymmetric Prokushkin-Vasiliev theory of higher spin gauge fields coupled to two massive $\mathcal{N}=2$ multiplets on AdS_3. We perform a non-trivial check of this duality by computing three-point functions containing one higher spin gauge field for arbitrary spin s and deformation parameter λ from the bulk theory. We also consider this problem from the CFT where we show that the three-point functions can be calculated using a free ghost system based on the linear $sw_\infty[\lambda]$ algebra. This is the same ghost system known from BRST quantization of perturbative superstring theories. We find an exact match between the two computations. In the 't Hooft limit, the three-point functions only depend on the wedge subalgebra shs$[\lambda]$ and the results are equivalent for any theory with such a subalgebra. In the process we also find the emergence of $\mathcal{N}=2$ superconformal symmetry near the AdS_3 boundary by computing holographic OPE's, consistently with a recent analysis of asymptotic symmetries of higher-spin supergravity.

Contents

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I will also thank Christopher Mudry for inviting me to Zürich to give a talk on higher-spin holography, and to Matthias Gaberdiel for explaining some aspects of his conjecture to me at ETH. I would also like to thank Jørgen Rasmussen and Peter Rønne for discussions during their visits to NBI.

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I will also thank Jacob Christian Lorenzen and Morten Holm Christensen for reading parts of the manuscripts, which is why this thesis contain a few readable spots.

Finally I would like to thank my office mates and neighbors Mizio Spatafora Andersen, Song Chen, Jesper Ipsen and Jacob Christian Lorenzen for their friendship and all the fun we had during this year. I am very glad that our social traditions and the pleasant atmosphere has been adapted by the newcomers Morten Hold Christensen, Christian Marboe, Therkel Zøllner Olesen and Anders Edsberg Møllgaard. I wish you all the best in the future.

Sadly I did not manage to write about most of the topics I studied during the first two-third of this year, nor the many ideas I have had and have made small progress in. But this is just a sign of how much I have learned this year, and for this I really have to thank Konstantinos Zoubos once again. I apologize for the fact that this thesis is not even close to be finished, too many topics are omitted or treated in a shallow fashion. There is without any doubts full of strange sentences, inaccurate (and possibly wrong) claims and tons of typos. Nevertheless I hope the reader won't find it too much of a waste of time.

Chapter 1

Introduction

The holographic principle [2, 3] is one of the core concepts of quantum gravity. It was originally inspired by black hole thermodynamics and was proposed to solve the information loss paradox of Hawking [4]. The number of degrees of freedom in a region in a local theory must scale as the volume of the region. According to the holographic principle however, gravity is non-local in the sense that the number of degrees of freedom inside a volume scale as the surrounding area.[1] This very bizarre feature of gravity rely on general arguments and thought experiments based on combining the laws of quantum mechanics, such as unitary time evolutions, and classical black hole thermodynamics, and must therefore be realized in any consistent theory of quantum gravity. Holography was however not taken seriously until Maldacena proposed a concrete realization in the context of string theory, called AdS/CFT correspondence [6]. Loosely speaking, Maldacena conjectured that type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, with N five-form fluxes on S^5, is dual to $\mathcal{N} = 4$ super(conformal) $SU(N)$ Yang-Mills theory on the asymptotic AdS_5 boundary. Many highly non-trivial tests of this duality have been performed and the conjecture has so far passed in an astonishingly impressive way. Furthermore, many generalizations have been proposed during the years, and recently it has become popular to apply holography to QCD or condensed matter systems as a tool to understand non-perturbative aspects of strongly coupled systems.

There are however many conceptual and technical problems left which are not well-understood, not to mention that a proof of this conjecture remains elusive. A better understanding of these problems are important since holography seems to be a fundamental concept of gravity, but also because of its potential applicability in other areas of physics. It is therefore of great interest to search for simpler realizations of AdS/CFT correspondence, which at the same time is complex enough to capture important features of more realistic theories.

The simplest class of theories are without any doubt free field theories, and it is natural to ask what kind of theories are dual to a CFT of free fields. It is however clear that the dual theory cannot be anything conventional. Free field CFT's are integrable and clearly posses an infinite number of higher-spin conserved currents, the dual bulk theory must therefore contains an infinite number of higher-spin gauge fields.

A particular interesting class of models are the higher-spin theories of Vasiliev on

[1] For an interesting review of the holographic principle and covariant entropy bounds, see [5].

anti-de Sitter space. These theories can evade the usual no-go theorems by containing an infinite tower of massless higher-spin fields. It is widely believed that these theories are a certain tensionless limit of superstring theory, but however not exactly understood how in detail. Vasiliev theory, despite being highly non-linear, is much simpler than full string theory and thereby perfect candidate for toy models of AdS/CFT correspondence.

Based on such reasoning, Klebanov and Polyakov [7], inspired by earlier work of for example Sezgin and Sundell [8], considered the 3D $O(N)$ model of N massless scalars ϕ^a with interactions of the form $(\phi^a\phi^a)^2$. Besides the trivial fixed point, being the free theory, this theory has a non-trivial fixed point. It was conjectured that the two critical points of the 3D $O(N)$ model are dual to Vasiliev theory on AdS_4 in the large N limit, depending on boundary conditions. Note that supersymmetry is not necessary for this duality. Recently three-point functions functions were calculated in these theories and highly non-trivial agreements were found [9], this sparked a renewed interest in this duality (see a recent review in [10]).

It is however possible to find even simpler dualities along these lines. Pure gravity on AdS_3 do not contain dynamical degrees of freedom, in the absence of a boundary, but contain very interesting black hole solutions similar to Kerr black holes in four dimensions. Vasiliev theory on AdS_3 is similarly much simpler than its higher dimensional counterparts, where the massless sector is only dynamical through its coupling to massive matter fields. Furthermore, consistent interacting theories of finite number of massless higher-spin fields in AdS_3 exist and gives a platform of analyzing the massless sector of Vasiliev theory in a much simpler form. When coupling to matter fields, one is however forced to include an infinite tower of higher-spin fields. It was recently shown that higher-spin gravity theories on AdS_3, generically lead to asymptotic higher-spin symmetries known as $\mathcal{W}$-algebras [11, 12].

On the boundary side the situation is even better. Two-dimensional conformal field theories are possibly among the best well-understood non-trivial theories because of their infinite number of symmetries. This power is only enhanced when there are additional higher-spin invariants, not to mention an infinite number of them.

Inspired by the Klebanov-Polyakov conjecture, and the emergence of $\mathcal{W}$-algebras near the AdS_3 boundary of higher-spin theories, Gopakumar and Gaberdiel [13] conjectured that $\mathcal{W}_n$ minimal models are dual to Vasiliev theory on AdS_3. This conjecture has been supported by many non-trivial and detailed checks in the 't Hooft limit, and impressive insight into the finite N regime has already been achieved (which has led to slight refinements of the finite N part of the conjecture). It is hoped that one may eventually be able to prove this duality and thereby gain deep insight into the mechanisms of holography. Subsequently, several variations of the Gaberdiel-Gopakumar conjecture has been proposed an tested.

For example, recently it was conjectured that the $\mathbb{C}P^N$ Kazama-Suzuki model is dual to the $\mathcal{N}=2$ supersymmetric Vasiliev theory on AdS_3. In this case there are an infinite tower of fermionic and bosonic higher-spin fields, coupled to two massive $3d$ $\mathcal{N}=2$ hypermultiplets. To be more precise, Vasiliev theory is a one-parameter family of theories parametrized by λ. There only exist few checks of this conjecture.

In this thesis we will consider this $\mathcal{N}=2$ higher-spin conjecture. We will calculate three-point functions containing two massive scalars and one bosonic higher-spin field from the bulk, for arbitrary spin s and deformation parameter λ. On the boundary side we argue that the full Kazama-Suzuki model is not necessary for this particular class of correlation functions. We will in particular show that all these can be calculated using a simple ghost system known from BRST quantization of perturbative superstring theory. From the CFT side we will also calculate three-point functions containing fermionic matter or higher-spin fields. Let us show two examples of our results

$$\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}(z_1,\bar{z}_1)\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}(z_2,\bar{z}_2)W^{s-}(z_3)\right\rangle = (-1)^{s-1}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda+1)}{\Gamma(2-2\lambda)}\frac{s-1+2\lambda}{2s-1}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \times \left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}(z_1,\bar{z}_1)\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}(z_2,\bar{z}_2)\right\rangle,$$

and

$$\left\langle \mathcal{O}^{\mathcal{F}}_{\Delta_+}(z_1,\bar{z}_1)\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}(z_2,\bar{z}_2)G^{s+}(z_3)\right\rangle = 2(-1)^{s}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda)}{\Gamma(2-2\lambda)}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \times \left\langle \mathcal{O}^{\mathcal{F}}_{\Delta_+}(z_1,\bar{z}_1)\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}(z_2,\bar{z}_2)\right\rangle.$$

Here $\mathcal{O}^{\mathcal{B}}_{\Delta_+}(z_1,\bar{z}_1)$ and $\mathcal{O}^{\mathcal{F}}_{\Delta_+}(z_1,\bar{z}_1)$ are bosonic and fermionic primary fields of the $\mathbb{C}P^N$ Kazama-Suzuki coset, respectively. Furthermore $W^{s-}(z_3)$ and $Q^{s+}(z_3)$ are bosonic and fermionic holomorphic higher-spin fields, respectively. We find non-trivial agreements between correlation functions calculated on both sides of the duality.

We also derive operator product expansions of the boundary CFT currents holographically from the bulk theory. In particular, we show that near the AdS_3 boundary the theory has $\mathcal{N}=2$ $\mathcal{SW}_\infty[\lambda]$ symmetry. This is another consistency check. A preprint of our results was recently published in [1].

The plan of this thesis is as follows. In chapter 2 we will give a basic introduction to higher-spin (super-) gravity theories on AdS_3. It is in particular seen that these theories can be constructed as Chern-Simons theories based on Lie algebras $\mathfrak{g}_k \oplus \mathfrak{g}_{-k}$. Special emphasis is laid upon the infinite dimensional one-parameter family of Lie algebras shs$[\lambda]$, and their associative extensions $\mathcal{SB}[\mu]$, since they play a crucial role in our calculations. Structure constants of these algebras are also derived in a convenient form, not explicitly found in the literature. Hereafter we will show that the calculation of asymptotic symmetries generically lead to classical Drinfeld-Sokolov reduction of Affine Lie algebras, and thereby to $\mathcal{W}$-algebras. Finally we will discuss the coupling of matter fields to higher-spin gauge fields, which leads to Vasiliev theory. This theory on AdS_3 and a linearization needed for our calculation is discussed. Most importantly, we will argue that a slight reformulation of the formalism will lead to tremendous simplifications.

In chapter 3, we will give a brief (and shallow) review of extended symmetries in two-dimensional conformal field theory. In particular systematically introduce superconformal symmetries and $\mathcal{W}$-algebras. In the end we will discuss Kazama-Suzuki models, in particular the subset based on hermitian symmetric spaces. The discussion of many advanced aspects of the topic, such as quantum Drinfeld-Sokolov reduction, is either neglected or very short despite the fact that most of the work on this thesis were based on these CFT topics. This is partly because the advanced technical details of these topics are not relevant for our original results, but mainly due to lack of time.

In chapter 4, we will give a ridiculously short and unjustified review of the conjectures at play, only touching the details necessary (beyond general knowledge about AdS/CFT correspondence) to understand our original results.

Chapter 5 contain the main parts of the original contributions of this thesis. In particular we show how to calculate the relevant three-point functions from the bulk and boundary point of view. On the way, we give a holographic proof of the emergence of $\mathcal{N} = 2$ $\mathcal{SW}_\infty[\lambda]$ algebra near the AdS_3 boundary.

Appendix A contain the solution of a recursion relation. Appendix B contain the structure constants of shs$[\lambda]$ and $\mathcal{SB}_\infty[\lambda]$ which we have found in a particular convenient form, together with certain properties used in our calculations.

Appendix C contains a review of basic aspects of two-dimensional CFT's, including modular invariance, RCFT's, WZW and coset models.

Appendix D contains the structure theory, classification and representation theory of finite-dimensional semi-simple Lie algebras, together with a discussion of regular embeddings and branching rules. Appendix E is about the classification and representation theory of untwisted affine Lie algebras. Finally appendix F contains a list of finite and affine (extended) Dynkin diagrams, together with useful information about these.[2]

[2]These three appendices were written due to my focus on conformal field theory and quantum Drinfeld-Sokolov reduction during most parts of this work. Only in the final few month the focus shifted to three-point functions. We have however chosen to include these anyway.

Chapter 2

2+1D Higher-Spin Gravity

Long before the discovery of the holographic principle [2, 3] and the string theoretical realization by Maldacena [6], Brown and Henneaux analyzed the asymptotic symmetries of 2+1-dimensional Einstein gravity with negative cosmological constant [14]. They found that the asymptotic symmetry algebra was the Virasoro algebra with, quite surprisingly, a central charge even though the bulk theory is purely classical. Back then, it was of course unthinkable to believe that the effective conformal theory on the boundary is equivalent to the bulk theory through a holographic duality.

Three-dimensional gravity is in many ways much simpler than its higher dimensional counterpart, mainly due to the fact that pure Einstein gravity does not have any (local) dynamical degrees of freedom in 2+1 dimensions. One way to see this is by Ricci decomposition

$$R_{\mu\nu\rho\gamma} = S_{\mu\nu\rho\gamma} + E_{\mu\nu\rho\gamma} + C_{\mu\nu\rho\gamma}. \tag{2.1}$$

The first two terms are given by the Ricci scalar and tensor, respectively, and are fixed by the Einstein equations. The last term is the Weyl tensor and contains all dynamical information since it is left undetermined by the equations of motion. But $C_{\mu\nu\rho\gamma} = 0$ for all three dimensional manifolds, thus gravity is non-dynamical.

This implies that the phase space is finite dimensional and that there are only global degrees of freedom present, which makes the theory topological. At the purely classical level, it was shown by Achucarro and Townsend [15] and later Witten [16], that Einstein gravity can be mapped to a Chern-Simons theory with the gauge group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}], \tag{2.2}$$

where the Chern-Simons action is given by

$$S_{\text{CS}}[A] = \frac{k_{\text{CS}}}{4\pi}\int_{\mathcal{M}} \text{tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right). \tag{2.3}$$

This can be mapped to the first order formulation of gravity by

$$e = \frac{l}{2}\left(A - \tilde{A}\right), \qquad \omega = \frac{1}{2}\left(A + \tilde{A}\right), \tag{2.4}$$

with the AdS radius l. Here the coefficients of $e = e^a_\mu J^a \mathrm{d}x^\mu$ are the vielbein, $\omega = \omega^a_\mu J^a \mathrm{d}x^\mu = \frac{1}{2}\epsilon_{abc}\,\omega^{bc}_\mu J^a \mathrm{d}x^\mu$ is the spin connection, and the commutation relations are given by

$$[J_a, J_b] = \epsilon_{abc} J^c.$$

Since $\mathfrak{sl}(2,\mathbb{R})$ is not the compact but rather the normal real form of $\mathfrak{sl}_2 \equiv \mathfrak{sl}(2,\mathbb{C})$, its Killing form is not euclidean but $\mathrm{tr}(J_a J_b) = \frac{1}{2}\eta_{ab}$. Using this, it can be shown that the Chern-Simons coupling constant is related to the Newton constant G as

$$k_{\mathrm{CS}} = \frac{l}{4G}. \tag{2.5}$$

The infinitesimal gauge transformations of the gauge theory translate into transformations of the vielbein and spin connection

$$\begin{aligned} \delta A &= \mathrm{d}\lambda + [A,\lambda], \\ \delta\tilde{A} &= \mathrm{d}\tilde{\lambda} + [\tilde{A},\tilde{\lambda}], \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \delta e &= \mathrm{d}\xi + [\omega,\xi] + [e,\Lambda], \\ \delta\omega &= \mathrm{d}\Lambda + [\omega,\Lambda] + \frac{1}{l^2}[e,\xi], \end{aligned} \tag{2.6}$$

where $\xi = \frac{l}{2}(\lambda - \tilde{\lambda})$ corresponds to diffeomorphisms [16], while $\Lambda = \frac{1}{2}(\lambda + \tilde{\lambda})$ is the local Lorentz transformations associated to change of frame of the tangent bundle. Thus it is necessary to use the first order formalism, in order to have “enough” gauge invariance to map gravity into a gauge theory.

Although irrelevant for this thesis, we cannot resist briefly mentioning the interesting topological features of Chern-Simons theories. In the case of compact gauge groups, the coefficient k_{CS} is quantized due to the fact that $\pi_3(G) = \mathbb{Z}$ and the requirement that the quantum partition function should be invariant under large gauge transformations. The theory was solved by Witten in [17], where he showed that the Hilbert space is isomorphic to the space of conformal blocks of a two-dimensional WZW model (at level k) and expectation values of Wilson loops are given by knot invariants such as the Jones polynomials. Conformal blocks are not monodromy invariant but transform as representations of the Braid group,[1] as is known from the work of Moore and Seiberg [18]. This makes them ideal wave functions of exotic particles in 2+1 dimensions, called non-abelian anyons [19, 20]. It is also possible to axiomatize these topological field theories, similar to the Moore-Seiberg axioms [21, 22], using braided fusion categories which play important roles in mathematics and the field of topological quantum computers. We will however not discuss these extremely interesting topics in detail.

2.1 Higher-Spin Generalizations

Higher-spin theories turn out to be difficult to construct and seem to be forbidden by several no-go theorems. For example, the so-called Weinberg low energy theorem states that higher-spin theories cannot mediate long-range interactions. It is however possible to construct a certain type of theories with an infinite tower of higher-spin fields, such as the class of theories constructed by Vasiliev [23, 24]. These theories can be seen as some sort of tensionless limit of string theory, but in full string theory this higher-spin symmetry is typically dynamically broken. Note that there is no problem with free higher-spin theories, it is due to interactions that inconsistencies arise. See [25] for a detailed review of these

[1]To be more precise, one has to consider the mapping class group of, say, the punctured sphere which contains both the braid group and Dehn twists. In the case of non-abelian statistics, the latter are called topological spins.

no-go theorems and how they can be surpassed.[2] These theorems, however, only apply for dimensions $d > 3$, and therefore it is possible to construct higher-spin theories in three dimensions without resorting to an infinite tower of higher-spins.

Instead of taking the general approach of [25], let us see a glimpse of some of the problems associated to higher-spin theories.

2.1.1 Free Theory and Coupling to Gravity

At the linearized level, Fronsdal [27] constructed equations of motion for massless bosonic higher-spin fields (later generalized to fermions [28]). For example the free propagation of an integer spin s field on Minkowski space (using the notation of [29, 30]) is

$$\mathcal{F}_{\mu_1 \dots \mu_s} \equiv \Box \, \phi_{\mu_1 \dots \mu_s} - \partial_{(\mu_1|} \partial^{\lambda} \, \phi_{|\mu_2 \dots \mu_s)\, \lambda} + \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_s)\lambda}{}^{\lambda} = 0 \,, \tag{2.7}$$

which is invariant under the gauge transformation

$$\delta \, \phi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_s)}, \qquad \xi_{\mu_1 \dots \mu_{s-3} \lambda}{}^{\lambda} = 0 \,. \tag{2.8}$$

The parentheses in (2.8) is the complete symmetrization of the indices, with no normalization. These gauge transformations make sure the fields in $d > 3$ have the correct number of degrees of freedom, while in $d = 3$ they make the theory non-dynamical. Note that these equations and gauge transformations reduce to what we already know for $s = 1$ and $s = 2$, in particular $\mathcal{F}_{\mu\nu}$ is the linearized Ricci tensor. In order to couple these equations to gravity, it is natural to consider minimal coupling, $\partial \to \nabla$ and $\eta \to g$. Consistency requires that the theory must preserve the same gauge symmetries it has on flat space. Taking the spin 3 equation, a calculation shows that $F_{\mu\nu}$ transforms as

$$\begin{aligned} \delta \mathcal{F}_{\mu\nu\rho} = & - 6 \, \xi^{\lambda\sigma} \nabla_{(\mu|} R_{\lambda|\nu\rho)\sigma} - 9 \, R_{\lambda(\mu\nu|\sigma} \nabla_{|\rho)} \, \xi^{\lambda\sigma} + 6 \, R_{\lambda(\mu\nu|\sigma} \nabla^{\lambda} \, \xi_{|\rho)}{}^{\sigma} \\ & - 6 \, \xi^{\lambda}{}_{(\mu|} \nabla_{\lambda} \, R_{|\nu\rho)} + \frac{3}{2} R_{\lambda(\mu|} \nabla^{\lambda} \, \xi_{|\nu\rho)} - 9 \, R_{\lambda(\mu} \nabla_{\nu} \, \xi_{\rho)}{}^{\lambda} \,. \end{aligned} \tag{2.9}$$

This does not vanish on general backgrounds, not even using the vacuum Einstein equations, i.e. vanishing Ricci tensor. Furthermore, it was shown in [31] that for $d > 3$ and spins $s > \frac{3}{2}$, this problem remains, even if one considers non-minimal couplings. The exception $s = \frac{3}{2}$ is crucial for supergravity.

A way out was given by Fradkin and Vasiliev [32], and requires higher-derivative contributions and a negative cosmological constant Λ to be added. It turns out that the interactions are non-analytic functions of Λ, and thus do not have an expansion around flat space [32]. This line of thinking eventually led to the Vasiliev equations, which describe full non-linear interactions and are manifestly invariant under (2.8) [23, 24].

Next, it is natural to consider $d = 3$. As mentioned before, the Weyl tensor vanishes in 2+1 dimensions, so equation (2.9) is proportional to the Ricci tensor. It turns out that these terms can then always be removed by a ξ-dependent gauge transformation, and thus there are no problems with massless higher-spin fields coupled to gravity [31]. Note that the presence of the spin-3 field has extended the diffeomorphisms. We shall use the term "higher-spin diffeomorphisms" to account for all gauge transformations, including all spins.

[2] There is also a very recent review about the no-go theorems in Minkowski space [26].

2.1.1.1 Spin in 2+1 Dimensions

Before proceeding, let us elaborate on the notion of "spin" in 2+1 dimensions. Binegar [33] found the unitary (projective) irreducible representations of the Poincare group $\mathbb{R}^3 \rtimes O(1,3)$ using the usual Mackeys induced representations technique, which essentially reduces the problem to the stabilizer subgroups (little groups). For massive particles, representations with continuous spin are found (which is not surprising).[3] In the massless case, choosing a frame $p = (\frac{1}{2}, \frac{1}{2}, 0)$ for the orbit $\mathcal{O}_0^+ = \{p \in \mathbb{R}^3 \,|\, p^2 = 0, p_0 > 0\}$, one finds the little group [33, eqn. (25)]

$$S_{\mathcal{O}_0^+} \approx \mathbb{Z}_2 \times \mathbb{R}. \tag{2.10}$$

In [33], the continuous representations are discarded, claiming that they are "unphysical" and cannot be used in local field theories. Therefore, there are only two inequivalent representations, $\{1, -1\}$. In $d = 3$ "spin" therefore just reduces to the distinction between fermions and bosons. What do we then mean by "higher-spin"? When constructing field theories of massless fields in four dimensions, it is convenient to use spin rather than helicity and then let gauge invariance kill the unphysical degrees of freedom. Similarly the tensors $\phi_{\mu_1 \ldots \mu_s}$ are non-dynamical, due to the gauge transformations (2.8), as noted before. It is however not all trivial, since tensors of different rank will give rise to different boundary dynamics, as we will shortly see. This distinction motivates us to regard the rank of the different tensors as their "spin" [11].[4]

2.2 Higher-Spin Interactions and Chern-Simons Formulation

We are interested in constructing a full non-linear theory with spins $s \geq 2$, describing their interactions while preserving the higher-spin diffeomorphisms (2.9). It turns out that this is much easier to achieve by first moving the linearized theory into the frame-like formulation (see [34] for some progress using the metric-like formalism). One can introduce generalized vielbeins and spin connections

$$e_\mu^{a_1, \ldots, a_{s-1}}, \qquad \omega_\mu^{a_1, \ldots, a_{s-1}}. \tag{2.11}$$

The generalized spin connections are auxiliary fields, which are introduced due to a generalization of local Lorentz invariance (2.6).[5] Combining these into a gauge connection, it turns out that a higher-spin diffeomorphism invariant interacting theory can be constructed by various types of Chern-Simons theories [11, 35], just like in (2.2).[6]

[3]To see why, recall that we need projective representations according to Wigner and thus must consider the universal covering. Recall that $SU(2)$ is a two-to-one compact universal covering of the d=4 massive little group $SO(3) = SU(2)/\pi_1(SO(3)) = SU(2)/\mathbb{Z}_2$ which gives rise to integer and half-integer representations. The universal covering of $SO(2) = \mathbb{R}/\pi_1(SO(2)) = \mathbb{R}/\mathbb{Z}$ is however ∞-to-one and non-compact, leading to a continuous family of representations. In other words $\mathbb{R}$ is a fiber bundle over $SO(2) \approx S^1$ with infinite discrete fibers $\mathbb{Z}$, and hence $\mathbb{R}$ wraps an infinite number of times around $SO(2)$.

[4]If the reader thinks the concept of "spin" of massless fields in d=3 is a little bit fishy, the author will not disagree. Nonetheless, in this section we made an attempt to motivate the idea.

[5]Doing this in higher dimensions, one is forced to introduce an infinite number of higher-spin vielbeins and more auxiliary fields. One can then define higher-spin generalizations of the (linearized) curvature tensors and formulate Fronsdal equations in terms of them. Attempting to formulate interactions will lead to the Vasiliev theory.

[6]If one allows the Chern-Simons levels to be different, one obtains topologically massive gravity. See [36] for a higher-spin construction along these lines.

Let us however take the opposite route. It is possible to define different types of higher-spin interactions by a quite general Lie algebra $\mathfrak{g} \times \mathfrak{g}$, which can even be infinite-dimensional. This is however not the full story. In order to map back to the metric formulation, we need to identify the gravity sector. This entails choosing an embedding $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$, with different choices corresponding to reorganization of the field content, and thereby different theories, or least to different boundary dynamics.

Given this subalgebra we can identify the physical fields as follows. Decompose $\mathfrak{g}$ into representation spaces of $\mathfrak{sl}(2,\mathbb{R})$ under the adjoint action

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \left(\bigoplus_{s,a} \mathfrak{g}^{(s,a)} \right), \tag{2.12}$$

where $\dim \mathfrak{g}^{(s,a)} = 2s+1$ with $2s \in \mathbb{Z}_+$ and the index a accounts for possible multiplicities. Note there is a subtlety associated with the fact that $\mathfrak{sl}(2,\mathbb{R})$ is the normal real form of $\mathfrak{sl}_2$ and the corresponding group is non-compact, such as it has infinite dimensional irreducible representations. However, for finite dimensional representations, it behaves exactly as the compact real form $\mathfrak{su}(2)$. Thus, we restrict $\mathfrak{g}$ and the choice of embedding, such that the decomposition (2.12) only contains finite dimensional representations. We will further assume only integer $s \in \mathbb{Z}$ representations will arise, in order to avoid certain subtleties (see [35]). This induces the decomposition of the gauge connection

$$A = A^i_\mu L_i \, \mathrm{d}x^\mu + \sum_{s,a} \sum_{m=-s}^{s} A^{[a]\,s,m}_\mu (W^s_m)_{[a]} \, \mathrm{d}x^\mu, \tag{2.13}$$

where L_i generate $\mathfrak{sl}(2,\mathbb{R})$, while $(W^s_m)_{[a]}$ generate $\mathfrak{g}^{(s,a)}$. We have used a basis such that

$$[L_+, L_-] = 2L_0, \qquad [L_\pm, L_0] = \pm L_\pm, \tag{2.14}$$

and

$$\begin{aligned} \left[L_i, (W^s_m)_{[a]}\right] &= (is - m)(W^s_{i+m})_{[a]}, \\ (W^s_m)_{[a]} &= (-1)^{s-m} \frac{(s+m)!}{(2s)!} \, \mathrm{ad}^{s-m}_{L_-} \left((W^s_s)_{[a]} \right). \end{aligned} \tag{2.15}$$

The last equation follows from the (finite) representation theory of $\mathfrak{sl}(2,\mathbb{R})$ with highest weight $(W^s_s)_{[a]}$ and the coefficients are just normalizations. Similar to (2.4), we can now define higher-spin vielbeins and spin connections as

$$e^{[a]\,s}_\mu = \sum_{m=-s}^{s} e^{[a]\,s,m}_\mu (W^s_m)_{[a]}, \qquad \omega^{[a]\,s}_\mu = \sum_{m=-s}^{s} \omega^{[a]\,s,m}_\mu (W^s_m)_{[a]}, \tag{2.16}$$

given by

$$e^{[a]\,s,m}_\mu = \frac{l}{2} \left(A^{[a]\,s,m}_\mu - \bar{A}^{[a]\,s,m}_\mu \right), \qquad \omega^{[a]\,s,m}_\mu = \frac{1}{2} \left(A^{[a]\,s,m}_\mu + \bar{A}^{[a]\,s,m}_\mu \right). \tag{2.17}$$

In the metric formulation, the fields must be invariant under generalized local Lorentz transformations (2.6) generated by Λ. It turns out that $\delta_\Lambda e = [e, \Lambda] \Rightarrow \delta_\Lambda \mathrm{tr}(e^n) = n\,\mathrm{tr}(e^{n-1}[e,\Lambda]) = 0$ by the cyclicity of the trace. Thus this fixes the metric and spin-3 field (up to normalization)

$$g_{\mu\nu} \sim \mathrm{tr}\left(e_\mu \, e_\nu\right), \qquad \phi_{\mu\nu\rho} \sim \mathrm{tr}\left(e_{(\mu} e_\nu e_{\rho)}\right), \tag{2.18}$$

where

$$e = \frac{l}{2}\left(A - \tilde{A}\right) = \sum_{s,a} e_\mu^{[a]\,s}\,\mathrm{d}x^\mu. \tag{2.19}$$

Note that the metric receives in general contributions from the higher-spin vielbeins (2.17). Higher-spin fields require more work. For example for spin-four, there are two possibilities, $\mathrm{tr}(e^4)$ and $\mathrm{tr}(e^2)^2$, and the result turns out to be a linear combination. For further details see [11, 35].

It might seem that the choice of $\mathfrak{sl}(2,\mathbb{R})$ embedding does not matter at all and (2.12) is just a random choice of basis, since the map to the conventional formalism (2.18) uses the whole vielbein and does not make reference to the decompositions (2.12) and (2.17). As will be seen later, the choice of $\mathfrak{sl}(2,\mathbb{R})$ is crucial when specifying asymptotic boundary conditions. Different choices correspond to different embeddings of the gravity sector, leading to inequivalent theories.

2.2.1 Higher-Spin algebras $hs[\lambda]$ and the Lone-Star Product

The most studied example with a finite number of spins is $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{R})$ with the principal embedding, where one has spins $s = 2,\dots,N$ with multiplicity one.

Our current aim is a description of the massless sector of Vasiliev theory. For this we will use the so-called higher-spin algebra $\mathrm{hs}[\lambda]$. This consists of the elements

$$V_n^s, \qquad s \geq 2, \qquad |n| \geq s-1,$$

each of spin $s-1$ under the adjoint action of $\mathfrak{sl}(2,\mathbb{R})$. It can be constructed in various ways; as an analytic continuation of $\mathfrak{sl}(\lambda,\mathbb{R})$ to real λ [37, 38], as an algebra of differential operators [39, 40], or as quotient of a universal enveloping algebra [41, 38]. For now, let us consider the last approach. Let $U(\mathfrak{sl}(2,\mathbb{R}))$ be the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ and $\langle \mathcal{C}_2 - \mu\mathbb{1}\rangle$ be the two-sided ideal generated by elements of the form $\mathcal{C}_2 - \mu\mathbb{1}$ (see appendix D), where the second-order Casimir is given as

$$\mathcal{C}_2 = L_0^2 - \frac{1}{2}\left(L_+L_- + L_-L_+\right), \tag{2.20}$$

and set

$$\mu = \frac{1}{4}(\lambda^2 - 1). \tag{2.21}$$

We can now define an associative algebra by the quotient

$$\mathcal{B}[\mu] = \frac{U\big(\mathfrak{sl}(2,\mathbb{R})\big)}{\langle \mathcal{C}_2 - \mu\mathbb{1}\rangle} = \mathrm{hs}[\lambda] \oplus \mathbb{C}, \tag{2.22}$$

where we have identified the higher-spin algebra $\mathrm{hs}[\lambda]$ as a subspace of $B[\mu]$ by removing the one-dimensional complex space along the identity operator $\mathbb{1}$. This just means that we are allowed to take formal products of $a, b = L_{0,\pm}$, and then identify

$$X \star (a \star b - b \star a) \star Y \approx X \star [a,b] \star Y, \qquad \text{and} \qquad \mathcal{C}_2 \approx \mu,$$

where X and Y are arbitrary products the $\mathfrak{sl}(2,\mathbb{R})$ generators. For the associative product of $B[\mu]$ we have used $\star$, this product is usually called the "lone-star" product. Using the notation $V^2_{0,\pm} \equiv L_{0,\pm}$ and $V^1_0 \equiv \mathbb{1}$, we can construct the rest of the algebra from the adjoint action of the subalgebra $\mathfrak{sl}(2,\mathbb{R})$, and simultaneously decompose it as (2.12) [38]

$$V^s_n = (-1)^{s-1-n}\frac{(s+n-1)!}{(2s-2)!}\,\mathrm{ad}^{s-n-1}_{L_-}\left(L^{s-1}_+\right). \tag{2.23}$$

Compare this to (2.15). In order to use this in a Chern-Simons theory, we need an invariant bilinear form. It turns out one can define a trace as

$$\mathrm{tr}\,(X\star Y) = X\star Y\big|_{L_i=0}, \tag{2.24}$$

or in other words the coefficient proportional to $\mathbb{1}$. Since the trace is symmetric we have that $\mathrm{tr}\,([X,Y]) = 0$. Thus, commutators of elements in hs$[\lambda]$ do not have a term proportional to $\mathbb{1}$ and form a closed Lie algebra. As an example, take

$$V^3_2 = L_+\star L_+, \qquad V^3_0 = \frac{1}{3}\left(L_-\star L_+ + L_0 + 2\,L_0^2\right) \approx L_0^2 - \frac{1}{12}(\lambda^2-1),$$

from which one can calculate the commutator

$$[V^3_2, V^3_0] = 4V^4_2.$$

Luckily, it is possible to write down the full set of commutation relations [41, 38]

$$[V^s_m, V^t_n] = \sum_{u=2\text{ even}}^{s+t-1} g^{st}_u(m,n;\lambda)V^{s+t-u}_{m+n}. \tag{2.25}$$

This can also be done for the whole associative algebra and not just the Lie algebra. It turns out that for integer $\lambda = N \geq 2$, we have that

$$\mathrm{tr}\left(V^s_m V^t_n\right) = 0, \qquad s > N. \tag{2.26}$$

Thus these decouple from the Chern-Simons theory and can consistently be truncated. In other words, an ideal χ_N appears, consisting of generators of spin $s > N$. Factoring over this ideal, one finds

$$\mathfrak{sl}(N,\mathbb{R}) = \mathrm{hs}[N]/\chi_N, \qquad N\geq 2. \tag{2.27}$$

Thus in this sense, hs$[\lambda]$ is an analytic continuation of $\mathfrak{sl}(\lambda,\mathbb{R})$ for $\lambda\in\mathbb{R}$.

2.3 $\mathcal{N}=2$ Higher-Spin Supergravity theory

Before considering the $\mathcal{N}=2$ higher-spin SUGRA of our interest, we will make some general comments about supergravity on AdS$_3$. It turns out that (extended) pure supergravity on AdS$_3$ can also be formulated as a Chern-Simons theory associated to a Lie superalgebra [15, 42]. The classification of finite dimensional Lie superalgebras, in the same spirit as in appendix D, was solved by Kac [43, 44][7]. Not all these algebras will work for us however. Let us denote the Lie superalgebra with its natural $\mathbb{Z}_2$ grading as $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_o$, where $\mathfrak{g}_e$ and $\mathfrak{g}_o$ correspond to its even and odd part respectively.

In order to describe SUGRA on AdS$_3$, there are two basic requirements. (i) the even part of $\mathfrak{g}$ must contain $\mathfrak{sl}(2,\mathbb{R})$, we thus demand the even part to take the form

[7] See [45] for an useful collection of results about Lie superalgebras.

$\mathfrak{g}$	$\mathcal{G}$	ρ	$\dim \mathcal{G}$
$\mathfrak{osp}(N\|2)$	$\mathfrak{so}(N)$	$\boldsymbol{N}$	$N(N-1)/2$
$\mathfrak{su}(1,1\|N)_{N\neq 2}$	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$	$\boldsymbol{N}+\bar{\boldsymbol{N}}$	N^2
$\mathfrak{su}(1,1\|2)/\mathfrak{u}(1)$	$\mathfrak{su}(2)$	$\mathbf{2}+\bar{\mathbf{2}}$	3
$\mathfrak{osp}(4^*\|2M)$	$\mathfrak{su}(2)\oplus\mathfrak{usp}(2M)$	$(\mathbf{2}\boldsymbol{M},\mathbf{2})$	$M(2M+1)+3$
$D(2,1;\alpha)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$(\mathbf{2},\mathbf{2})$	6
$G(3)$	G_2	$\mathbf{7}$	14
$F(4)$	$spin(7)$	$\mathbf{8}_s$	21

Table 2.1: List of Lie superalgebras which can be used to formulate supergravity on AdS_3 spacetimes. Here $\mathfrak{g}$ is the Lie superalgebra, $\mathcal{G}$ is the internal subalgebra and ρ is the representation of $\mathcal{G}$ in which the spinors of $\mathfrak{g}_o$ transform in.

$\mathfrak{g}_e=\mathfrak{sl}(2,\mathbb{R})\oplus\mathcal{G}$. (ii) the fermionic generators of $\mathfrak{g}_o$ must transform in the **2** (spin $\frac{1}{2}$) representation of $\mathfrak{sl}(2,\mathbb{R})$ under the adjoint action. It turns out that only seven classes of algebras in [43, 44] satisfy these requirements [46, 47, 48], see table 2.1. Actually, this is also the list of Lie superalgebras which give rise to two-dimensional superconformal algebras with quadratic non-linearities by Drinfeld-Sokolov reduction of their affinization [49, 50]. As we will see later, this is not a coincidence and Drinfeld-Sokolov reduction comes out naturally from AdS_3, giving a holographic perspective on this two-dimensional CFT problem. The two algebras $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ correspond to $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity [15] and their Drinfeld-Sokolov reduction give rise to the conventional $\mathcal{N}=1$ and $\mathcal{N}=2$ superconformal Virasoro algebras.[8] The other algebras generically give rise to non-linearities after a DS-reduction [49] (similar to $\mathcal{W}$-algebras). We will however not pursue these types of supergravities in this thesis. Note that we are only talking about half of the algebra. The full algebra of the Chern-Simons theory must be of the form

$$\mathfrak{g}_k\oplus\tilde{\mathfrak{g}}_{-k},$$

where the index refers to the CS-level. There might be some restrictions on which $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ one may combine. We will only be concerned with diagonal combinations.

2.3.1 $\mathcal{N}=2$ Higher-Spin SUGRA

In order to find higher-spin generalizations, we must allow higher $\mathfrak{sl}(2,\mathbb{R})$-spin generators in the odd sector $\mathfrak{g}_o$. Since we are interested in $\mathcal{N}=2$ SUGRA, we also modify the requirement of the even part to $\mathfrak{g}_e=\mathfrak{osp}(2|2)\oplus\mathcal{G}$. It turns out that the supersymmetric analogue of $\mathfrak{sl}(N,\mathbb{R})$ (which gives rise to bosonic higher-spin extension of pure AdS_3 gravity), is $\mathfrak{sl}(N|N-1)$. Pure $\mathcal{N}=2$ SUGRA is recovered for $N=2$ since $\mathfrak{sl}(2|1)\approx\mathfrak{osp}(2|2)$ [51, 52]. The $\mathfrak{sl}(2,\mathbb{R})$ decomposition, analogous to (2.12), takes the form

$$\mathfrak{sl}(N|N-1)=\mathfrak{sl}(2,\mathbb{R})\oplus\left(\bigoplus_{s=3}^{N}\mathfrak{g}^{(s)}\right)\oplus\left(\bigoplus_{s=1}^{N-1}\mathfrak{g}^{(s)}\right)\oplus 2\times\left(\bigoplus_{s=1}^{N-1}\mathfrak{g}^{(s+\frac{1}{2})}\right), \tag{2.28}$$

[8]Under the $\mathfrak{sl}(2,\mathbb{R})$ decomposition of $\mathfrak{g}$, $\mathfrak{sl}(2,\mathbb{R})$ transforms as a spin-1 representation (since its the adjoint representation). Drinfeld-Sokolov reduction turns this sector into a spin-2 field, which is nothing but the energy-momentum tensor (and thereby the Virasoro algebra). The generators of $\mathfrak{g}_o$ transform as spin-$\frac{1}{2}$ representations and DS-reduction turns them into spin-$\frac{3}{2}$ fields. Typically $\mathfrak{g}$ ends up as a wedge subalgebra of the resulting algebra of DS-reduction (unless the algebra is non-linear). Thus it is not hard to see why $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ give rise to $\mathcal{N}=1$ and $\mathcal{N}=2$ super Virasoro algebras.

where $\mathfrak{g}^{(s)}$ transforms in the spin-$(s-1)$ representation under the adjoint action of $\mathfrak{sl}(2,\mathbb{R})$ and the last part corresponds to the generators of the odd sector.[9] Note that only for $N=2$ there are only spin-$\frac{1}{2}$ fermionic generators, which is the $\mathfrak{osp}(2|2)$ SUGRA. For $N>2$ there are necessarily higher-spin fields. See the recent paper [52] for some results on this $\mathcal{N}=2$ higher-spin SUGRA.

Analogous to the non-supersymmetric discussion above, the $\mathcal{N}=2$ Vasiliev theory is a one-parameter family of theories with an infinite tower of multiplets containing higher-spin fields. The massless sector can again be formulated as a pair of Chern-Simons theories based on the so-called $\mathcal{N}=2$ higher-spin algebra shs$[\lambda]$.[10] Similar to (2.27), shs$[\lambda]$ can be thought of as an analytic continuation of $\mathfrak{sl}(\lambda|\lambda-1)$ for non-integer λ and it has the following $\mathfrak{sl}(2,\mathbb{R})$ decomposition

$$\text{shs}[\lambda]=\mathfrak{sl}(2,\mathbb{R})\oplus\left(\bigoplus_{s=3}^{\infty}\mathfrak{g}^{(s)}\right)\oplus\left(\bigoplus_{s=1}^{\infty}\mathfrak{g}^{(s)}\right)\oplus 2\times\left(\bigoplus_{s=1}^{\infty}\mathfrak{g}^{(s+\frac{1}{2})}\right). \tag{2.29}$$

In this decomposition, we have the following set of generators

$$\begin{aligned} &L_m^{(s)+} \quad (s\in\mathbb{Z}_{\geq 2},|m|\leq s-1), \qquad && L_m^{(s)-} \quad (s\in\mathbb{Z}_{\geq 1},|m|\leq s-1),\\ &G_r^{(s)+} \quad (s\in\mathbb{Z}_{\geq 2},|r|\leq r-3/2), \qquad && G_r^{(s)-} \quad (s\in\mathbb{Z}_{\geq 2},|r|\leq r-3/2), \end{aligned} \tag{2.30}$$

where $L_m^{(s)\pm}$ generate the even part of shs$[\lambda]$, while $G_r^{(s)\pm}$ generate the odd part. We will spend some time discussing this algebra, since it plays a crucial role in this thesis.

The three generators $L_m^{(2)}$, $m=-1,0,1$, form the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra describing the gravity sector

$$[L_m^{(2)+},L_n^{(2)+}]=(m-n)L_{m+n}^{(2)+}. \tag{2.31}$$

In this basis, $L_0^{(2)+}$ is ad-diagonalized

$$[L_0^{(2)+},L_n^{(s)\pm}]=-n\,L_n^{(s)\pm}, \qquad [L_0^{(2)+},G_r^{(s)\pm}]=-r\,G_r^{(s)\pm}, \tag{2.32}$$

while under the adjoint action of the $m=\pm 1$ $\mathfrak{sl}(2,\mathbb{R})$ generators, the other generators transform as

$$\begin{aligned} &[L_1^{(2)+},L_n^{(s)\pm}]=(-n+s-1)L_{n+1}^{(s)\pm}, \qquad && [L_1^{(2)+},G_r^{(s)\pm}]=(-r+s-3/2)G_{r+1}^{(s)\pm},\\ &[L_{-1}^{(2)+},L_n^{(s)\pm}]=(-n-s+1)L_{n-1}^{(s)\pm}, \qquad && [L_{-1}^{(2)+},G_r^{(s)\pm}]=(-r-s+3/2)G_{r-1}^{(s)\pm}, \end{aligned} \tag{2.33}$$

consistent with the $\mathfrak{sl}(2,\mathbb{R})$ decomposition. There is also a $\mathfrak{osp}(1|2)$ subalgebra spanned by $\{L_0^{(2)+},L_{\pm1}^{(2)+},G_{\pm1/2}^{(2)+}\}$. This can be extended to $\mathfrak{osp}(2|2)$ by adding the generators $\{L_0^{(1)-},G_{\pm1/2}^{(2)-}\}$, where $L_0^{(1)-}$ is the R-charge and spans a $\mathfrak{u}(1)$ internal subalgebra. These

[9]The reason we use s for a spin $s-1$ representation, is that these generators naturally relate to a spin s dual field on the boundary by Drinfeld-Sokolov reduction. Note also that $\mathfrak{sl}(2,\mathbb{R})$ transforms as spin-1 under the adjoint action of itself (adjoint representation), and is labeled by $s=2$.

[10]See [48] for $\mathcal{N}=(N,M)$ extensions of the higher-spin gravity. For extensions above $\mathcal{N}=(2,2)$ it turns out that there does not exist a one-parameter family of theories parametrized by λ.

generators act on the rest of the algebra as follows

$$\begin{aligned}
[G^{(2)\pm}_{1/2}, L^{(s)+}_m] &= -\frac{1}{2}(m-s+1)G^{(s)\pm}_{m+1/2}, & [G^{(2)\pm}_{1/2}, L^{(s)-}_m] &= -G^{(s+1)\mp}_{m+1/2},\\
[G^{(2)\pm}_{-1/2}, L^{(s)+}_m] &= -\frac{1}{2}(m+s-1)G^{(s)\pm}_{m-1/2}, & [G^{(2)\pm}_{-1/2}, L^{(s)-}_m] &= -G^{(s+1)\mp}_{m-1/2},\\
\{G^{(2)+}_{1/2}, G^{(s)+}_r\} &= 2L^{(s)+}_{r+1/2}, & \{G^{(2)+}_{1/2}, G^{(s+1)-}_r\} &= (r-s+1/2)L^{(s)-}_{r+1/2},\\
[L^{(1)-}_0, L^{(s)\pm}_m] &= 0, & [L^{(1)-}_0, G^{(s)\pm}_r] &= G^{(s)\mp}_r.
\end{aligned} \tag{2.34}$$

Note that the $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ supercommutators can be read off these by restricting to $s=2$ and the relevant set of generators. The R-charge maps the G^+ generators to G^-, and vice versa, but one can construct generators with definite $\mathfrak{u}(1)$ R-charge by superpositions $G^{(s)+}_r \pm G^{(s)-}_r$. Please observe that if we truncate away all higher-spin generators ($s>2$) and use the above (anti-)commutators for unrestricted m and r, we find the $\mathcal{N}=0,1,2$ (super)-Virasoro algebras in the Neveu-Schwarz sector with the central charge $c=0$ (the reason for this will become more clear below). $(L^{(s)+}_m, G^{(s)+}_r)$ and $(L^{(s-1)-}_m, G^{(s)-}_r)$ form $\mathcal{N}=1$ multiplets of $\mathfrak{osp}(1|2)$, while combining them we get $\mathcal{N}=2$ multiplets of $\mathfrak{osp}(2|2)$. Note that there is no λ dependence in the commutators involving the $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ subalgebras. This is because by restriction this theory reduces to pure supergravity and as we classified AdS_3 SUGRA, there are no continuous classes of theories (classification is discrete). The commutators written above are obvious properties to be expected from an $\mathcal{N}=2$ higher-spin SUGRA, but commutators between higher-spin generators will generally have complicated λ dependence. Thus, it is not as easy to write these down explicitly.

In this thesis we will mainly use a more compact notation. Allowing the superscript to be half-integer and eliminating the need for the $\pm$ superscript, we define

$$L^{(s)}_m \equiv L^{(s)+}_m, \quad L^{(s+1/2)}_m \equiv L^{(s)-}_m, \quad G^{(s)}_r \equiv G^{(s)+}_r \quad \text{and} \quad G^{(s-1/2)}_r \equiv G^{(s)-}_r. \tag{2.35}$$

Thus, in the following we allow s to be half-integer, but both notations will be used. Together with these definitions, we will use the following notation for the structure constants of shs$[\lambda]$

$$\begin{aligned}
\left[L^{(s)}_m, L^{(t)}_n\right] &= \sum_{u=1}^{s+t-1}{}^{\!\!\frac{1}{2}}\; \hat{g}^{st}_u(m,n;\lambda)\, L^{(s+t-u)}_{m+n}, & \left[L^{(s)}_m, G^{(t)}_q\right] &= \sum_{u=1}^{s+t-1}{}^{\!\!\frac{1}{2}}\; \hat{h}^{st}_u(m,q;\lambda)\, G^{(s+t-u)}_{m+q},\\
\left\{G^{(s)}_p, G^{(t)}_q\right\} &= \sum_{u=1}^{s+t-1}{}^{\!\!\frac{1}{2}}\; \hat{\tilde{g}}^{st}_u(p,q;\lambda)\, L^{(s+t-u)}_{p+q}, & \left[G^{(s)}_p, L^{(t)}_n\right] &= \sum_{u=1}^{s+t-1}{}^{\!\!\frac{1}{2}}\; \hat{\tilde{h}}^{st}_u(p,n;\lambda)\, G^{(s+t-u)}_{p+n}.
\end{aligned}$$

Here the notation $\sum^{\frac{1}{2}}$ means we are summing over half-integer steps. This is the most convenient form of the algebra for the purposes of this thesis, but as far as we are aware there do not exist explicit formulas for the shs$[\lambda]$ structure constants in the literature. It turns out that shs$[\lambda]$ is a subalgebra of a (linear) W-algebra, which we will call $sw_\infty[\lambda]$. This is a higher-spin extension of the super Virasoro algebra. The structure constants of $sw_\infty[\lambda]$ have been explicitly calculated in [39, 40] and from these we can extract the structure constants of shs$[\lambda]$.[11]

[11] Recently, the structure constants of shs$[\lambda]$ were found in [53] in the same way, but explicit expressions of these were not given.

2.3.2 Structure Constants of $\mathcal{SB}[\mu]$ and shs$[\lambda]$

For our calculation of the three-point functions, it will turn out to be crucial to use a slightly more fundamental structure than shs$[\lambda]$: the associative algebra $\mathcal{SB}[\mu]$. Let $U(\mathfrak{osp}(1|2))$ be the universal enveloping algebra of $\mathfrak{osp}(1|2)$, with the second-order Casimir element [39, 45]

$$\mathcal{C}_2 = L_0^2 - \frac{1}{2}\{L_1, L_{-1}\} + \frac{1}{4}[G_{1/2}, G_{-1/2}], \tag{2.36}$$

where $\{L_0, L_{\pm 1}, G_{\pm 1/2}\}$ generate $\mathfrak{osp}(1|2)$. Similar to (2.22), it turns out that the following associative algebra is related to shs$[\lambda]$ [39][12]

$$\mathcal{SB}[\mu] = \frac{U(\mathfrak{osp}(1|2))}{\langle \mathcal{C}_2 - \mu \mathbb{1} \rangle} = \text{shs}[\lambda] \oplus \mathbb{C}, \tag{2.37}$$

where we have defined

$$\mu = \lambda \left(\lambda - \frac{1}{2} \right). \tag{2.38}$$

We will use the notation $\star$ for the product of $\mathcal{SB}[\mu]$, which we will call the super lone-star product. Knowing the structure constants of $\mathcal{SB}[\lambda]$, we can directly recover the structure constants of shs$[\lambda]$, since $X \star Y - Y \star X = [X, Y]$. Using this associative structure, we can as above define the trace as

$$\text{tr}\big(\mathcal{A} \star \mathcal{B}\big) = \left. \frac{\mathcal{A} \star \mathcal{B}}{(2\lambda^2 - \lambda)} \right|_{\mathcal{J}=0}, \qquad \forall \mathcal{J} \neq \mathbb{1}. \tag{2.39}$$

The normalization is chosen for later convenience. We will identify $L_m^{(2)}$ and $G_r^{(2)}$ with $\mathfrak{osp}(1|2)$ and use the notation $L_0^{(1)} \equiv L_0^{(1)+} \equiv \mathbb{1}$ for the identity element of $\mathcal{SB}[\mu]$. The other generators of shs$[\lambda]$ can be constructed as sums and products of the $\mathfrak{osp}(1|2)$ generators, but the analogue of equation (2.23) is not given in the literature as far as we are aware.

We will later show how the $\mathcal{SB}[\mu]$ generators can be expressed as polynomials of $\mathfrak{osp}(1|2)$ generators, which will turn out to be important for us. But first, we will focus on constructing explicit formulas for the structure constants of $\mathcal{SB}[\mu]$ and shs$[\lambda]$.

[12] Note that although we are using $\mathfrak{osp}(1|2)$, the algebra ends up being $\mathcal{N} = 2$ supersymmetric, since another supercharge can be constructed in $\mathcal{SB}[\mu]$ [39]. This seems to imply that if we had used the more natural generalization of (2.22) to $\mathcal{N} = 2$, by using $\mathfrak{osp}(2|2)$ instead of $\mathfrak{osp}(1|2)$, we would find the same algebra $\mathcal{SB}[\mu]$.

2.3.2.1 $sw_\infty[\lambda]$ as an algebra of Super-Operators

In [39, 40], $sw_\infty[\lambda]$ is constructed as an algebra of the following set of super-operators:

$$\begin{aligned}
L_n^{(s)+} &= \sum_{i=0}^{s-1}(n-s+1)_{s-1-i}\, a^i(s,\lambda)\, z^{-n+i}\,\partial^i \\
&\quad + \theta\frac{\partial}{\partial\theta}\sum_{i=0}^{s-1}(n-s+1)_{s-1-i}\left[a^i(s,\lambda+\tfrac{1}{2})-a^i(s,\lambda)\right] z^{-n+i}\,\partial^i, \\
L_n^{(s)-} &= -\frac{s-1+2\lambda}{2s-1}\sum_{i=0}^{s-1}(n-s+1)_{s-1-i}\, a^i(s,\lambda)\, z^{-n+i}\,\partial^i \\
&\quad + \theta\frac{\partial}{\partial\theta}\sum_{i=0}^{s-1}(n-s+1)_{s-1-i}\left[\frac{s-2\lambda}{2s-1}a^i(s,\lambda+\tfrac{1}{2})+\frac{s-1+2\lambda}{2s-1}a^i(s,\lambda)\right] z^{-n+i}\,\partial^i, \\
G_r^{(s)\pm} &= \theta\sum_{i=0}^{s-1}(r-s+\tfrac{3}{2})_{s-1-i}\,\alpha^i(s,\lambda)\, z^{-r+i-1/2}\,\partial^i \\
&\quad \pm\frac{\partial}{\partial\theta}\sum_{i=0}^{s-2}(r-s+\tfrac{3}{2})_{s-2-i}\,\beta^i(s,\lambda)\, z^{-r+i+1/2}\,\partial^i.
\end{aligned} \tag{2.40}$$

Here

$$(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2)\dots(x+n-1), \qquad (x)_0 = 1,$$

is the Pochhammer symbol, $\partial \equiv \frac{\partial}{\partial z}$, and θ is a Grassmann number. See appendix B for the definition of the coefficients $a^i(s,\lambda)$, $\alpha^i(s,\lambda)$ and $\beta^i(s,\lambda)$. One can readily check that the commutators of $L_m^{(2)}$ give rise to the Witt algebra, and the properties discussed earlier are satisfied. It turns out that the products of these operators close as an associative algebra, while the supercommutators give rise to $sw_\infty[\lambda]$. If we restrict n and r to be in the wedge, $|n| \le s-1$ and $|r| \le s-\frac{3}{2}$, we will recover $\mathcal{SB}[\mu]$ and shs$[\lambda]$. In the following we will for the sake of generality, let the modes be $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\frac{1}{2}$ which correspond to $sw_\infty[\lambda]$. However, when we talk about $\mathcal{SB}[\mu]$ and shs$[\lambda]$, we just have to truncate to the Wedge modes.

Using the notation (2.35), we are interested in the following set of structure constants of $\mathcal{SB}[\mu]$

$$\begin{aligned}
L_m^{(s)}\star L_n^{(t)} &= \sum_{u=1}^{s+t-1} g_u^{st}(m,n;\lambda)\, L_{m+n}^{(s+t-u)}, &\qquad L_m^{(s)}\star G_q^{(t)} &= \sum_{u=1}^{s+t-1} h_u^{st}(m,q;\lambda)\, G_{m+q}^{(s+t-u)}, \\
G_p^{(s)}\star G_q^{(t)} &= \sum_{u=1}^{s+t-1} \tilde{g}_u^{st}(p,q;\lambda)\, L_{p+q}^{(s+t-u)}, &\qquad G_p^{(s)}\star L_n^{(t)} &= \sum_{u=1}^{s+t-1} \tilde{h}_u^{st}(p,n;\lambda)\, G_{p+n}^{(s+t-u)}.
\end{aligned} \tag{2.41}$$

The structure constants calculated in [39, 40] are given in a very compact $\mathcal{N}=1$ superspace notation. Since it is very tedious and technical to extract the coefficients above, we will not go through the details. Instead we will just sketch parts of the calculation.

All the operators in equation (2.40) can be expressed in a very compact $\mathcal{N}=1$ supersymmetric language as

$$\mathcal{L}_{\lambda}^{(s)}\big(\Omega^{(s)}\big) = \sum_{i=0}^{2s-2} A^i(s,\lambda)\left(D^{2s-2-i}\Omega^{(s)}\right)D^i, \tag{2.42}$$

where

$$\Omega^{(s)}(z) = \begin{cases} \Lambda^{(s)+}(z) + 2\theta\Theta^{(s)+}(z), & s = \lfloor s \rfloor \in \mathbb{Z}, \\ \Theta^{(\lfloor s \rfloor+1)-}(z) + \theta\Lambda^{(\lfloor s \rfloor)-}(z), & s = \lfloor s \rfloor + \frac{1}{2} \in \mathbb{Z}+\frac{1}{2}, \end{cases} \tag{2.43}$$

and

$$\Lambda^{(s)\pm}(z) = \sum_{n\in\mathbb{Z}} \Lambda_n^{(s)\pm}\, z^{n+s-1}, \qquad \Theta^{(s)\pm}(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Theta_r^{(s)\pm}\, z^{r+s-\frac{3}{2}}. \tag{2.44}$$

Furthermore, $D = \frac{\partial}{\partial\theta} - \theta\partial$ and the relation between $A^i(s,\lambda)$ and $a^i(s,\lambda)$, $\alpha^i(s,\lambda)$ and $\beta^i(s,\lambda)$ is given in appendix B. The coefficients of $\Lambda^{(s)\pm}$ are commuting numbers, while the ones for $\Theta^{(s)\pm}$ are anticommuting. By using relations shown in appendix B, and the form of the operators (2.40), one can show that

$$\mathcal{L}_{\lambda}^{(s)}\big(\Omega^{(s)}\big) = \begin{cases} \sum_{n\in\mathbb{Z}} \Lambda_n^{(s)+} L_{-n}^{(s)+} + \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Theta_r^{(s)+} G_{-r}^{(s)+}, & s = \lfloor s \rfloor \in \mathbb{Z} \\ \sum_{n\in\mathbb{Z}} \Lambda_n^{(\lfloor s \rfloor)-} L_{-n}^{(\lfloor s \rfloor)-} + \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Theta_r^{(\lceil s \rceil)-} G_{-r}^{(\lceil s \rceil)-}, & s = \lfloor s \rfloor + \frac{1}{2} \in \mathbb{Z}+\frac{1}{2} \end{cases}. \tag{2.45}$$

We have used the floor $\lfloor s \rfloor$ and ceiling $\lceil s \rceil$ functions. Note that $\lfloor s \rfloor + 1 = \lceil s \rceil$ if $s \in \mathbb{Z}+\frac{1}{2}$. This means that we can recover $L_n^{(s)\pm}$ and $G_r^{(s)\pm}$ by replacing $\Lambda_{n'}^{(s')\pm} \to \delta_{m',-m}\delta_{s',s}$ and $\Theta_{r'}^{(s')\pm} \to \delta_{r',-r}\delta_{s',s}$, respectively, and putting everything else to zero. For example, we have

$$\begin{aligned} \mathcal{L}_{\lambda}^{(s)}\big(z^{-m+s-1}\big) &= L_m^{(s)} = L_m^{(s)+}, && s = \lfloor s \rfloor \in \mathbb{Z}, \\ \mathcal{L}_{\lambda}^{(s)}\big(\theta z^{-m+\lfloor s \rfloor-1}\big) &= L_m^{(s)} = L_m^{(\lfloor s \rfloor)-}, && s = \lfloor s \rfloor + \frac{1}{2} \in \mathbb{Z}+\frac{1}{2}. \end{aligned} \tag{2.46}$$

The product between the operators has been derived in [39, 40] in the form

$$\mathcal{L}_{\lambda}^{(s)}\big(\Omega^{(s)}\big) \star \mathcal{L}_{\lambda}^{(t)}\big(\Omega^{(t)}\big) = \sum_{u=1}^{s+t-1} \mathcal{L}_{\lambda}^{(s+t-u)}\big(\xi_{(s)(t)}^{(s+t-u)}\big), \tag{2.47}$$

where the function $\xi_{(s)(t)}^{(s+t-u)}(z)$ contains all the structure constants of (2.41). As a function of $\Omega^{(s)}$ and $\Omega^{(t)}$, it is given as

$$\xi_{(s)(t)}^{(s+t-u)}(z) = \sum_{i=0}^{2u-2} F_{st}^u(i,\lambda)\left(D^i\Omega^{(s)}\right)\left(D^{2u-2-i}\Omega^{(t)}\right), \tag{2.48}$$

where $F_{st}^u(i,\lambda)$ is a complicated function given in appendix B. In order to derive explicit formulas for the coefficients in (2.41), we need to choose the appropriate functions $\Omega^{(s)}$ and $\Omega^{(t)}$, and then use (2.48), (2.42) and (2.40) to extract the structure constants. For each of the structure constants, due to the property (B.22), we need to separate the calculation into even/odd i and integer/half-integer s. Having found these four pieces, one then has to "glue" them together to find an explicit formula for each structure constant. The end results of these tedious calculations are given in appendix B.

We have explicitly checked that these structure constants satisfy all possible combinations of $\mathbb{Z}_2$-graded Jacobi identities for many s and t, which is very non-trivial. Furthermore, we have compared various limits and truncations of this algebra to results known in the literature, and we find an exact match. The constants given in appendix B are very complicated, but there are good reasons to believe that they can all be written much simpler in terms of generalized hypergeometric functions. This is at least possible for $\lambda = 0$ [54] and for the non-supersymmetric higher-spin algebra hs[λ] [41].

Before concluding this section, we have to mention that as long as we constrain the modes to be inside the wedge, we can safely restrict the sums of (2.41)

$$1 \leq u \leq s+t-1 \qquad \to \qquad 1 \leq u \leq \mathrm{Min}(2s-1, 2t-1).$$

This is because the structure constants for larger u vanish, as can be seen by a careful analysis of the formulas in appendix B.[13] As we will later see, this is very important when working in our modified formalism of Vasiliev theory, since it will then be manifest that only a finite number of equations couple to each other.

2.3.2.2 Quotient of Universal Enveloping Algebra

As we discussed above, the associative algebra $\mathcal{SB}[\mu]$ can be constructed as a quotient of the universal enveloping algebra of $\mathfrak{osp}(1|2)$ as seen in equation (2.37). This implies that all generators (2.37) can be written as polynomials of $\mathfrak{osp}(1|2)$ generators, which we will denote with $G_\alpha \equiv G_\alpha^{(2)+}$ and $L_m \equiv L_m^{(2)+}$, modulo the equivalence relation $\mathcal{C}_2 \approx \mu = \lambda(\lambda - \frac{1}{2})$. Actually, due to the anticommutator $\{G_\alpha, G_\beta\} = 2L_{\alpha+\beta}$, we only need the fermionic generators G_α. While the anti-commutator is fixed, the commutator is not and corresponds to a new element in the algebra. It is convenient to write it in terms of a new bosonic element Q as follows

$$[G_\alpha, G_\beta] = (Q + \tfrac{1}{2})\epsilon_{\alpha\beta}. \tag{2.49}$$

Due to the $\mathfrak{osp}(1|2)$ commutation relations one has the constraint $\{Q, G_\alpha\} = 0$ and nothing else [39]. Hence the associative algebra generated by G_α and Q modulo $\mathcal{C}_2 \approx \mu$ is isomorphic to $\mathcal{SB}[\mu]$. We can however simplify even more. By direct calculation it turns out that Q is related to the Casimir by[14]

$$\mathcal{C}_2 = L_0^2 - \frac{1}{2}\{L_1, L_{-1}\} + \frac{1}{4}[G_{\frac{1}{2}}, G_{-\frac{1}{2}}] = \frac{1}{4}Q^2 - \frac{1}{16}. \tag{2.50}$$

This is very remarkable, since it implies that we can get rid of the $\mathcal{C}_2 \approx \lambda(\lambda - \frac{1}{2})$ constraint by setting

$$Q = 2(\lambda - \tfrac{1}{4})K, \tag{2.51}$$

where $K^2 = 1$. Thus, we conclude that $\mathcal{SB}[\mu]$ is isomorphic to the associative algebra generated G_α and K with the following relations

$$[G_\alpha, G_\beta] = (cK + \tfrac{1}{2})\epsilon_{\alpha\beta}, \qquad \{K, G_\alpha\} = 0, \qquad K^2 = 1, \tag{2.52}$$

[13] One can actually cut off the sum over u even more, but this will be mode-dependent.

[14] It seems that it has also been noted in math literature [55] that the Casimir posses a "square root" related to the commutator of G_α.

and $c = 2(\lambda - \frac{1}{4})$. By playing around with the (anti-)commutators (2.31),(2.32),(2.33) and (2.34), it becomes obvious that even (odd) numbers of symmetrized products of G_α correspond to the elements $L_m^{(s)+}$ ($G_r^{(s)+}$), while the same objects multiplied with K correspond to $L_m^{(s)-}$ and $G_r^{(s)-}$. We will return to this later, when we discuss Vasiliev theory.

2.4 Asymptotic Conditions and Classical Drinfeld-Sokolov Reduction

We have so far discussed different possibilities for formulating higher-spin (super)gravity theories on AdS_3. Using the Chern-Simons formulation, the input required is a Lie (super)algebra $\mathfrak{g}$ together with an $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$ embedding which corresponds to the gravity sector. The higher-spin content can then be found by a decomposition of $\mathfrak{g}$ under the adjoint action of this $\mathfrak{sl}(2,\mathbb{R})$ embedding.

Having gone through detailed discussions about the algebras relevant for us, we will in this section go back to the Chern-Simons theory, impose boundary conditions and then find the asymptotic symmetries of AdS_3. It is clear that imposing boundary conditions, we must restrict the allowed gauge transformations such that they leave the boundary conditions invariant. This means that an infinite number of previously gauge-equivalent configurations become physically distinct. We will therefore have dynamical (massless) degrees of freedom near the boundary although the bulk is non-propagating. It is well known that for a $\mathfrak{g}_k$ Chern-Simons theory on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, the boundary dynamics is described by a $\hat{\mathfrak{g}}_k$ Wess-Zumino-Witten CFT. This can be seen either by directly rewriting the action in the holomorphic gauge [56, 20], or from the fact that the Poisson structure of the phase space of boundary excitations is an untwisted affine Lie algebra.[15]

However not all solutions, or equivalently all points in phase space, of this Chern-Simons theory are admissible classical (higher-spin) gravity configurations. For this we need to restrict to asymptotically AdS_3 configurations, which in turn impose (first class) constraints on phase space. Turning the first class constraints into second class by gauge-fixing and reducing to the constrained phase space, the Dirac-bracket algebra will generically turn $\hat{\mathfrak{g}}_k$ into a classical $\mathcal{W}$ algebra.

This way of deriving classical $\mathcal{W}$ algebras by constraining affine Lie algebras is known as (classical) Drinfeld-Sokolov reduction [58], and generically associates a centrally extended $\mathcal{W}$ algebra to any semi-simple Lie algebra.[16] This procedure critically depends on how $\mathfrak{sl}(2,\mathbb{R})$ is embedded in $\mathfrak{g}$. Alternatively instead of constraining the phase-space, one can impose these constraints directly on the WZW or Chern-Simons fields, leading to the so-called Hamiltonian reduction. After a reduction, this leads to a Liouville theory[60, 61] for pure gravity, or more generally a Toda Field theory [62], which is known to have higher-spin conserved currents generating $\mathcal{W}$ algebras.

In the following sections we briefly discuss the boundary conditions imposed on the Chern-Simons theory leading to asymptotic AdS_3 solutions, including rotating massive black holes [63]. Then we will see how this induces a classical Drinfeld-Sokolov reduction

[15] This is actually also known from the fractional quantum Hall effect in which the bulk Chern-Simons theory gives rise to gapless edge excitations [57] (known as chiral Luttinger liquid), which has been seen experimentally.

[16] See [59] (PhD thesis) for a very readable account.

of the Chern-Simons gauge connection. The form this field takes in the so-called lowest-weight gauge will be very important for us later in the thesis. Due to lack of time we are sadly forced to be rather shallow and not too detailed, but we will sketch the general features.

2.4.1 Boundary Conditions and Gauge fixing

We will here consider Chern-Simons theory with the gauge group G and Lie algebra $\mathfrak{g}$, on a manifold with topology $\mathcal{M} = \mathbb{R} \times \Sigma$ and boundary $\partial\mathcal{M} = \mathbb{R} \times S^1$. Let t parametrize $\mathbb{R}$, while ρ and θ are the radial and polar coordinate for the disc Σ, respectively. The first thing to note is that the action (2.3) is not well-defined in the presence of a boundary. Following Regge and Teitelboim [64], we need to impose boundary conditions such that the functional derivative $\delta S_{CS}[A]/\delta A$ exists and is well-defined. Using light-cone coordinates $x^{\pm} = t \pm \theta$, it can be shown that a variation of the action contains a boundary contribution

$$\delta S_{CS} = -\frac{k_{CS}}{4\pi} \int_{\partial\mathcal{M}} \mathrm{d}x^+ \mathrm{d}x^- \, \mathrm{tr}(A_+ \delta A_- - A_- \delta A_+), \tag{2.53}$$

where $A_\pm = \frac{1}{2}(A_t \pm A_\theta)$. This boundary contribution to the variation can be set to zero by the boundary condition

$$A_-\big|_{\partial\mathcal{M}} = 0. \tag{2.54}$$

We will now find the basic variables of the physical phase space, which can be thought of as the space of classical solutions modulo gauge transformations. Thus, we need to fix the gauge degrees of freedom. A particularly useful gauge is given by the condition

$$A_\rho = b^{-1}(\rho)\partial_\rho b(\rho), \tag{2.55}$$

which is always possible to obtain.[17] The group-valued function $b(\rho)$ is fixed and depends only on the radial coordinate. We will here choose

$$b(\rho) = e^{\rho L_0}, \tag{2.56}$$

where L_0 and $L_\pm$ are generators of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra. This choice of gauge is particularly useful in the AdS/CFT context, since it naturally leads to a Fefferman-Graham expansion. Decomposing the connection as $A = A_t \mathrm{d}t + A_i \mathrm{d}x^i$, one will find that the action does not contain a time-derivative of A_t and is linear in it. Thus, it can be thought as a Lagrange multiplier. The variation of the action with respect to A_t yields the constraint

$$F_{\rho\theta} = \partial_\rho A_\theta + \left[A_\rho, A_\theta\right] = 0, \tag{2.57}$$

which is solved by

$$A_\theta(t, \rho, \theta) = b^{-1}(\rho)\tilde{a}(t, \theta)b(\rho), \tag{2.58}$$

where $\tilde{a}(t, \theta)$ is an arbitrary $\mathfrak{g}$-valued function of t and θ. The ρ-dependence of the Langrange multiplier A_t is determined from the equations of motion

$$\partial_\rho A_t + [A_\rho, A_t] = 0, \tag{2.59}$$

[17] This is easy to see by starting from an arbitrary A'_ρ and solving the equation $g^{-1}A'_\rho g + g^{-1}\partial_\rho g = b^{-1}\partial_\rho b$. This can be shown to have a solution given by a path-ordered exponential for any group-valued function $b(\rho)$.

which again have a solution of the form (2.58). Due to the boundary condition (2.54) we have that $A_t = A_\theta$ on $\partial\mathcal{M}$, but since the ρ-dependence is completely fixed by $b(\rho)$ in (2.58), this must hold on all $\mathcal{M}$ and not only on the boundary

$$A_- = \frac{1}{2}(A_t - A_\theta) = 0. \tag{2.60}$$

From the final equation $F_{t\theta} = \partial_t A_\theta - \partial_\theta A_t + [A_t, A_\theta] = (\partial_t - \partial_\theta)A_+ = 0$ and (2.60) we find that $\partial_- \tilde{a}(t,\theta) = 0$, and hence a must be a function of x^+ only. The other sector $\bar{A}$ can be treated in a similar way, but it turns out that we must impose the boundary condition $\bar{A}_+ = 0$ instead in order to ensure invertibility of the vielbein [35]. The final results can be summarized as

$$\begin{aligned} A &= b^{-1}(\rho)a(x^+)b(\rho) + b^{-1}(\rho)\,\mathrm{d}b(\rho), \\ \bar{A} &= b(\rho)\bar{a}(x^-)b^{-1}(\rho) + b(\rho)\,\mathrm{d}b^{-1}(\rho), \end{aligned} \tag{2.61}$$

where $a(x^+) = \tilde{a}(x^+)\,\mathrm{d}x^+$ is a $\mathfrak{g}$-valued one-form which can be thought of as the connection for constant ρ-slices. Similar results hold for the other sector.

We have thus found all solutions (2.61) of the equations of motion with the boundary condition (2.54), and $a(x^+)$ and $\bar{a}(x^-)$ parametrize the (reduced) phase-space of the theory. So different choices of $a(x^+)$ and $\bar{a}(x^-)$ correspond to exact gauge inequivalent solutions of the equations of motion. Note that if there were no boundaries present, the general solution would be of the form $A = g^{-1}\mathrm{d}g$. This can be gauge transformed to the trivial solution $A = 0$ and there would be no local degrees of freedom (there might however be global ones measured by holonomies). When boundaries are present, we have the more general solution parametrized by $a(x^+)$. This solution can also be mapped to the trivial solution, but by a transformation which is not generated by a first class constraint. Thus it acts as a global symmetry on the space of solutions and maps inequivalent solutions into each other [65]. In the following section we will, very shallowly, discuss these points and find the Poisson bracket of the basic variables of phase space $a(x^+)$.

2.4.2 Global Symmetries and Poisson Algebra

It turns out that the gauge fixing condition above completely removes all gauge degrees of freedom, i.e. those that are generated by a first class constraint. There are however some residual gauge transformations left, but these do not correspond to first class constraints, but rather to global symmetries of the space of solutions (or reduced phase space of the theory).[18]

The gauge choice (2.55) is preserved by transformations, parametrized by $\Lambda : \mathcal{M} \to \mathfrak{g}$, which satisfy $\partial_\rho \Lambda + [A_\rho, \Lambda] = 0$. This is again of the form (2.57) and the solution is therefore

$$\Lambda(t,\rho,\theta) = b^{-1}(\rho)\lambda(t,\theta)b(\rho). \tag{2.62}$$

The condition that it must preserve the boundary condition $\delta A_- = 0$ forces the gauge parameter to only depend on x^+, $\lambda = \lambda(x^+)$. The Lagrange multiplier A_t gives rise to first-class constraints which can be used to define the smeared generator

$$G(\Lambda) = \frac{K_{CS}}{4\pi} \int_\Sigma \mathrm{d}x^i \wedge \mathrm{d}x^j \; \mathrm{tr}\,(\Lambda F_{ij}) + Q(\Lambda). \tag{2.63}$$

[18] We will only sketch the main ideas here, for more details see [66, 65, 11].

The first term generates gauge transformations, while the second ensures that the variation of $G(\Lambda)$ is well-defined and cancels any surface term. If one assumes that Λ is independent of the fields, one can show that the boundary term is given by

$$Q(\Lambda) = -\frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \mathrm{d}x^i \ \mathrm{tr}\,(\Lambda A_i)\,. \tag{2.64}$$

Using this, a gauge transformation of any phase-space functional is given by the Poisson bracket[19] $\delta_\Lambda F = \{G(\Lambda), F\}$, in particular $\delta_\Lambda A = \{G(\Lambda), A\} = \mathrm{d}\Lambda + [A, \Lambda]$. The boundary term gives rise to a central extension, which can be shown to be

$$\{G(\Lambda), G(\Gamma)\} = G([\Lambda,\Gamma]) + \frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \mathrm{d}x^i \ \mathrm{tr}\,(\Lambda\partial_i\Gamma)\,. \tag{2.65}$$

The crucial point to note is that the charge does not weakly vanish when the constraints $F_{ij} = 0$ are imposed, $G(\Lambda) \approx Q(\Lambda)$. This means that $Q(\Lambda)$ does not correspond to a gauge transformation, but is a global charge, mapping inequivalent configurations into each other. This is the origin of the infinite number of degrees of freedom in the presence of a boundary.

Fixing the gauge as discussed above and going to the reduced (physical) phase-space, the gauge algebra turns into the algebra of global charges

$$\{Q(\Lambda), Q(\Gamma)\}_\star = Q([\Lambda,\Gamma]) + \frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \mathrm{d}x^i \ \mathrm{tr}\,(\Lambda\partial_i\Gamma)\,, \tag{2.66}$$

where $\{\,\cdot\,,\,\cdot\,\}_\star$ is the Dirac bracket. In this reduced phase space the basic variables are not A_i^a anymore, but rather $a(x^+) = \tilde{a}(x^+)\mathrm{d}x^+$ as seen in (2.61). Using the form of A_θ given (2.58) and the allowed transform (2.62) we find the following global charge[20]

$$Q(\Lambda) = -\frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \mathrm{d}\theta \ \mathrm{tr}\,(\Lambda(\theta)A_\theta(\theta)) = -\frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \mathrm{d}\theta \ \mathrm{tr}\,(\lambda(\theta)\tilde{a}(\theta))\,. \tag{2.67}$$

We can now find the canonical Dirac brackets of the dynamical degrees of freedom. These can be found from the transformation

$$\delta_\Lambda \tilde{a}(\theta) = \{Q(\Lambda), \tilde{a}(\theta)\}_\star = -\frac{k_{CS}}{2\pi}\int_{\partial\Sigma} \lambda^a(\theta')\kappa_{ab}\{\tilde{a}^b(\theta'), \tilde{a}(\theta)\}, \tag{2.68}$$

where we have used the Killing form κ_{ab}. We can evaluate the transformation $\delta_\Lambda\tilde{a} = \partial_\theta\lambda + [\tilde{a}, \lambda]$, which is found from $\delta_\Lambda A_\theta = \partial_\theta\Lambda + [A_\theta, \Lambda]$. Expanding in a generic basis of the Lie algebra $\tilde{a}^a = a^a T_a$, this transformation can be shown to be reproduced by the following bracket

$$\{\tilde{a}^a(\theta), \tilde{a}^b(\theta')\}_\star = \frac{2\pi}{k_{CS}}\left[\kappa^{ab}\delta'(\theta-\theta') - f^{ab}_c\tilde{a}^c(\theta)\delta(\theta-\theta')\right], \tag{2.69}$$

[19]The Poisson bracket for two phase-space functionals $F[A_i]$ and $H[A_i]$ is defined as $\{F, H\} = \frac{2\pi}{k_{CS}}\int_\Sigma \mathrm{d}x^i\wedge\mathrm{d}x^j\ \mathrm{tr}\left(\frac{\delta F}{\delta A_i(x)}\frac{\delta H}{\delta A_j(x)}\right) = \frac{2\pi}{K_{CS}}\int_\Sigma \mathrm{d}^2x\,\epsilon_{ij}\frac{\delta F}{\delta A_i^a(x)}\kappa^{ab}\frac{\delta H}{\delta A_j^b(x)}$, where κ_{ab} is the Killing form defined from the trace of $\mathfrak{g}$ (we are ignoring possible subleties of non semi-simple and infinite dimensional algebras). Before gauge-fixing, the basic phase-space variables are $A_i^a(x)$ and they have the Poisson bracket $\{A_i^a(x), A_j^b(y)\} = \frac{2\pi}{k_{CS}}\epsilon_{ij}\kappa^{ab}\delta(x-y)$, which can be derived by calculating the canonical momenta corresponding to $A_j^a(x)$.

[20]We only write the θ-dependence since we are integrating along $\partial\Sigma \approx S^1$ and the t-dependence is completely fixed by θ.

where f_c^{ab} are the structure constants of $\mathfrak{g}$ in the chosen basis. Expanding in terms of modes

$$\tilde{a}^a(\theta) = \frac{1}{k_{CS}} \sum_{n\in\mathbb{Z}} a_m^a e^{-im\theta}, \tag{2.70}$$

we find that this is nothing but a classical untwisted Affine Lie algebra[21]

$$-i\{\tilde{a}_m^a, \tilde{a}_n^b\}_\star = i f_c^{ab} \tilde{a}_{m+n}^c + m k_{CS} \kappa^{ab} \delta_{m,-n}. \tag{2.71}$$

This is the well-known fact discussed earlier, namely that the boundary dynamics of a Chern-Simons theory is given by a Wess-Zumino-Witten model, but derived from a Hamiltonian point-of-view.

2.4.3 Asymptotic AdS_3 Solutions and Asymptotic Symmetries

There is a problem with our analysis so far. Not all solutions (2.61) are admissible since they do not all asymptote to AdS_3. In [11] it was proposed to impose the additional asymptotic fall-off condition[22]

$$(A - A_{\mathrm{AdS}})\big|_{\partial\mathcal{M}} = \mathcal{O}(1), \tag{2.72}$$

which requires the difference between the configuration and AdS_3 to be finite at $\rho \to \infty$. There is a similar condition on $\bar{A}$. Here the gauge configuration corresponding to pure AdS_3 is given by

$$\begin{aligned} A &= b^{-1}\left(L_+ + \frac{1}{4}L_-\right) b\, \mathrm{d}x^+ + L_0\, \mathrm{d}\rho, \\ \bar{A} &= -b\left(L_- + \frac{1}{4}L_+\right) b^{-1}\, \mathrm{d}x^- - L_0\, \mathrm{d}\rho. \end{aligned} \tag{2.73}$$

Note that this crucially depends on the embedding $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$. For inequivalent embeddings, one will therefore obtain different theories. In order to see the consequence of this extra condition on a general Lie (super)algebra $\mathfrak{g}$, it is convenient to use the triangular (Gauss) decomposition (see appendix D)

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+. \tag{2.74}$$

This is just splitting the generators into negative, zero, and positive eigenvalues of $\mathrm{ad}_{L_0} : \mathfrak{g} \to \mathfrak{g}$. Consider expanding A in the basis (2.13), with appropriate $\mathbb{Z}_2$ grading of the coefficients in the case of Lie superalgebras. From the Baker-Campbell-Hausdorff theorem we see that any generator with ad_{L_0} mode m, goes as $e^{m\rho}$, and opposite for the other sector. Using this, the asymptotic fall-off conditions (2.73) imply that $a(x^+) - L_+$ may not contain components of positive ad_{L_0} eigenvalues

$$a(x^+) - L_+ \in \mathfrak{g}_- \oplus \mathfrak{g}_0. \tag{2.75}$$

This constraint essentially corresponds to those of Drinfeld-Sokolov reduction. It turns out that these are first class constraints[23], and they provide enough gauge invariance to

[21] This can also be derived directly from (2.66)

[22] See [67] and [68] for a generalization of these boundary conditions to include Schrödinger, Lifshitz and warped AdS spacetimes, among others.

[23] Except a few cases which are not important for us.

put the action into the so-called lowest weight gauge [58, 35]. In this gauge we have that $a(x^+) = L_+ + a_-(x^+)$ where

$$\mathrm{ad}_{L_-} a_-(x^+) = [L_-, a_-(x^+)] = 0. \tag{2.76}$$

Therefore, only generators with lowest possible mode m for each spin-s generator are kept. It turns out that this completely fixes the gauge. The other sector can again be treated in a similar way. In the case of $\mathfrak{g} = \mathrm{shs}[\lambda]$, we will write the connection for constant ρ-slices as

$$\begin{aligned} a(x^+) &= \left(L_1^{(2)} + \frac{2\pi}{k_{CS}} \sum_{s\geq\frac{3}{2}}^{\infty} \left[\frac{1}{N_s^B}\, \mathcal{L}_s(x^+)\, L^{(s)}_{-\lfloor s\rfloor+1} + \frac{1}{N_s^F}\, \psi_s(x^+)\, G^{(s)}_{-\lceil s\rceil+\frac{3}{2}} \right] \right) \mathrm{d}x^+, \\ \bar{a}(x^-) &= -\left(L_{-1}^{(2)} + \frac{2\pi}{k_{CS}} \sum_{s\geq\frac{3}{2}}^{\infty} \left[\frac{1}{N_s^B}\, \bar{\mathcal{L}}_s(x^-)\, L^{(s)}_{\lfloor s\rfloor-1} + \frac{1}{N_s^F}\, \bar{\psi}_s(x^-)\, G^{(s)}_{\lceil s\rceil-\frac{3}{2}} \right] \right) \mathrm{d}x^-, \end{aligned} \tag{2.77}$$

where one has the freedom of choosing a convenient normalization. It turns out that $\mathcal{L}_s$ and ψ_s can be identified with the bosonic and fermionic currents of the boundary CFT, respectively, and can be thought of as conserved charges of the solutions. In particular, $\mathcal{L}_2$ is related to the energy-momentum tensor.

Since the phase-space has been reduced even more by the additional constraint (2.72), the canonical structure of the phase-space (2.71) is constrained. Drinfeld-Sokolov reduction constrains this affine Lie algebra and turns it into a $\mathcal{W}$-algebra. There are essentially two ways to proceed. Having turned the first-class constraints into second-class ones by fixing the gauge to the lowest weight gauge, we can find the canonical Poisson brackets by computing the Dirac bracket

$$\{f, g\}_\circ = \{f, g\}_\star - \{f, \chi_\alpha\}_\star \left(C^{-1}\right)^{\alpha\beta} \{\chi_\beta, g\}_\star, \tag{2.78}$$

where $C = \{\chi_\alpha, \chi_\beta\}_\star$ and χ_α denotes the constraints.[24] This was for example explicitly done for the case of $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ in [11], where the result was $\mathcal{W}_3$ which is exactly what is expected from Drinfeld-Sokolov reduction.

Alternatively one can consider the most general gauge transformation Λ which leaves the structure (2.77) invariant. For example in the case of $\mathfrak{g} = \mathrm{shs}[\lambda]$,

$$\Lambda = \sum_{s\geq\frac{3}{2}}^{\infty} \left(\sum_{|m|\leq\lfloor s\rfloor-1} \eta_m^{(s)}\, L_m^{(s)} + \sum_{|r|\leq\lceil s\rceil-\frac{3}{2}} \epsilon_r^{(s)}\, G_r^{(s)} \right). \tag{2.79}$$

From the condition of leaving (2.77) invariant, one finds that the highest mode variables of (2.79) $\eta_s \equiv \eta^{(s)}_{\lfloor s\rfloor-1}$ and $\epsilon_s \equiv \epsilon^{(s)}_{\lceil s\rceil-3/2}$, are free, and all other variables can be expressed in terms of these. From this, one finds how the currents transform $\delta\mathcal{L}_s$ and $\delta\psi_s$, which can be used to find the Poisson brackets of the algebra. For example, for the spin 2 current one finds (with appropriate normalization)

$$\delta_2^B \mathcal{L}_2 = 2\mathcal{L}_2\, \partial\eta_2 + \partial\mathcal{L}_2\, \eta_2 + \frac{k_{CS}}{4\pi}\partial^3\eta_2, \tag{2.80}$$

[24] Which are just that all positive and negative modes (except the lowest ones) of $a(x^+)$ have to vanish.

where δ_2^B means that we are only using the gauge transformation w.r.t. $\mathcal{L}_2$. If identifying $T = 2\pi\mathcal{L}_2$ and $k_{CS} = \frac{6}{c}$, this is exactly how the energy-momentum tensor transforms, and this can be used to derive the Virasoro algebra. For this particular algebra, the analysis was done in[25] [53] where the result of this reduction turns out to be the non-linear $\mathcal{SW}_\infty[\lambda]$ algebra. This one-parameter family of algebras, parametrized by λ, are $\mathcal{N} = 2$ supersymmetric extensions of the Virasoro algebra with an infinite tower of higher-spin fields.

We will not go into any of these details since we will give an alternative proof of the emergence of $\mathcal{N} = 2$ $\mathcal{SW}_\infty[\lambda]$ near the AdS$_3$ boundary in section 5.2, using a more holographic approach. We will in particular use aspects of the AdS/CFT dictionary to directly derive the OPE's of the boundary CFT current operators and thereby find higher-spin extensions of $\mathcal{N} = 2$ supersymmetry.

Before closing this section let us briefly comment on non-principal embeddings. Due to the condition (2.72), this procedure is highly dependent on the $\mathfrak{sl}(2,\mathbb{R})$ embedding of $\mathfrak{g}$ (which is of course a feature of Drinfeld-Sokolov reduction). For example, in the case of $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ there are two possible $\mathfrak{sl}(2,\mathbb{R})$ embeddings. The principal embedding gives rise to the usual Zamolodchikov $\mathcal{W}_3$ algebra [69], while for the other embedding we find an algebra usually denoted by $\mathcal{W}_3^{(2)}$. This algebra was found independently by Polyakov [70] and Bershadsky [71]. Different choices of $\mathfrak{sl}(2,\mathbb{R})$ embedding correspond to different AdS$_3$ vacua with different boundary central charges and $\mathcal{W}$-algebras, where the principal embedding gives the highest central charge. In [72] it was shown that there exists a RG flow from the $\mathcal{W}_3$ vacuum to the $\mathcal{W}_3^{(2)}$ one, and suggested that this is readily generalized for more general algebras $\mathfrak{g}$. See also [73].

2.4.4 Higher-Spin Black Holes And Conical Defects

Having all exact solutions of the equations of motions, we will briefly mention a few examples. The most famous solution of AdS$_3$ gravity is definitely that for a rotating massive black hole, called BTZ black hole [63]. We can easily embed this solution in the shs[λ] gravity by putting all charges, $\mathcal{L}_s$, $\bar{\mathcal{L}}_s$, ψ_s and $\bar{\psi}_s$, to zero in (2.77), except that of pure gravity (spin-2). Constant solutions of the form

$$\mathcal{L}_2 = \frac{M-J}{4\pi}, \qquad \bar{\mathcal{L}}_2 = \frac{M+J}{4\pi}, \tag{2.81}$$

correspond to BTZ black holes with angular momentum J and ADM mass M.

It is natural to ask what kind of solutions higher-spin theories have, in particular does there exist a generalization of black holes in these systems? In [74] black holes with spin-3 charges were constructed in $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ gravity, and their thermodynamics analyzed, this has subsequently been extended to $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{R})$ [75, 76] and most importantly $\mathfrak{g} = \mathrm{hs}[\lambda]$ black holes [77]. As we will discuss in next section, only in the case of $\mathfrak{g} = \mathrm{hs}[\lambda]$ (and supersymmetric extension thereof) it is known how to consistently couple massive matter fields to the higher-spin theories, these are the Vasiliev theories. Recently the propagation of scalars on higher-spin black holes in Vasiliev theory were studied in [78].

However, due to the higher-spin extension of diffeomorphisms, many aspects of these black holes are not well understood. For example the usual notion of curvature known from

[25] Here we used the normalization given in (5.19) which is different form [53], but leads to more natural results.

Riemannian curvature, event horizons and singularities are not gauge invariant. It turns out that by using higher-spin gauge transformations, one can transform higher-spin black holes into traversable wormholes and thereby change the causial structure of spacetime [72]. Therefore, there seems to be many conceptual problems to overcome in higher-spin gravity theories.[26]

In the case of $\mathcal{N}=2$ SUGRA, higher-spin black hole solutions have been constructed in the $\mathfrak{g}=\mathfrak{sl}(N+1|N)$ theory [52] [79], but not in the theory we are interested in here based on shs[λ].

In $\mathfrak{sl}(N,\mathbb{R})$ gravity, solutions with conical defects have also been found [80]. These solutions might play an important role in solving certain problems with too many "light-states" in the finite N regime of the higher-spin holographic dualities to be discussed later.

In most parts of this thesis, we will Wick rotate the coordinates $t\to i\tau$, which implies that $x^+\to z$, $x^-\to -\bar{z}$.

2.5 Prokushkin-Vasiliev Theory on AdS$_3$ and the Unfolded Formalism

So far we have studied $\mathcal{N}=2$ higher-spin SUGRA with an infinite tower of fields based on a shs$[\lambda]_k\times$shs$[\lambda]_{-k}$ Chern-Simons theory. This theory corresponds to the massless sector of the $\mathcal{N}=2$ Vasiliev theory. In order to describe the coupling of massive matter multiplets to the higher-spin fields, which makes the theory dynamical, one has to go beyond the Chern-Simons formulation and shs[λ] algebra.[27] In this section, we will give a very brief review of the full non-linear Vasiliev theory on AdS$_3$, as formulated by Prokushkin and Vasiliev in [82, 83]. Then we will suggest a reformulation of the linearized equation which will turn out to vastly simplify our calculation of three-point functions later in the thesis. For more details see [82, 83, 24].[28]

The full non-linear Vasiliev equations are formulated using an associative algebra $\mathcal{A}$, constructed using several auxiliary variables and a Moyal $\star$-product in the following way. Let y_α and z_α ($\alpha=1,2$) be two commuting bosonic twistor variables, where their spinor indices are raised and lowered as

$$y_\alpha = y^\beta\epsilon_{\beta\alpha}, \qquad y^\alpha=\epsilon^{\alpha\beta}y_\beta, \tag{2.82}$$

where $\epsilon_{\alpha\beta}$ is the anti-symmetric tensor satisfying $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma}=-\delta^\alpha_\gamma$. We will use the notation $uv=u_\alpha v^\alpha=-v_\alpha u^\alpha=-vu$ for contracted spinors. Beside these, we have two separate sets of Clifford elements ψ_i ($i=1,2$) and (k,ρ) satisfying the usual relations

$$\{\psi_i,\psi_j\}=2\delta_{ij}, \qquad \{k,\rho\}=0, \qquad k^2=\rho^2=1. \tag{2.83}$$

[26]We have done some interesting attempts on defining killing tensor fields and maximal symmetric spaces in the context of higher-spin geometry. These studies are however not complete and therefore not included in this thesis.

[27]So far it is not known how to formulate the coupling of the massive scalar fields from the Chern-Simons theory point of view. A deeper understanding of this seems to be needed if one wants to derive the holographic duality studied in this thesis. Some progress in this direction has been achieved in a paper published very recently [81].

[28]The so-called unfolded formalism is quite unusual and involves towers of auxiliary fields. We will however not motivate the formalism due to lack of time, but interested readers can look at the reviews and original papers cited.

All auxiliary variables commute with $\psi_{1,2}$. Furthermore ρ and k commute and anti-commute with the twistor variables y_α, z_α, respectively,

$$\{k, y_\alpha\} = 0, \qquad \{k, z_\alpha\} = 0, \qquad [\rho, y_\alpha] = 0, \qquad [\rho, z_\alpha] = 0. \tag{2.84}$$

A generic spacetime function mapping to this algebra has the following form

$$A(z, y; \psi_{12}, k, \rho|x) = \sum_{B,C,D,E=0}^{1} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} A^{BCDE}_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}(x)\, k^B \rho^c \psi_1^D \psi_2^E z^{\alpha_1} \ldots z^{\alpha_m} y^{\beta_1} \ldots y^{\beta_n}. \tag{2.85}$$

For our purposes, we will assume that the space-time functions $A^{BCDE}_{\alpha_1\ldots\alpha_m\,\beta_1\ldots\beta_n}(x)$ are symmetric in the spinor indices. Furthermore, the Grassmann parity of the coefficients $A^{BCDE}_{\alpha_1\ldots\alpha_m\,\beta_1\ldots\beta_n}(x)$ is equal to the number of spinor indices mod 2 and they are defined to commute with all the generating elements y_α, z_α, k, ρ and $\psi_{1,2}$. Thus, commutators of functions of the form (2.85) will automatically turn into supercommutators of polynomials of y_α, z_α, k, ρ and $\psi_{1,2}$.

In order to formulate the theory, we also need the $\star$-product defined on functions of y and z given by

$$(f \star g)(z, y) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2u\, \mathrm{d}^2v\, \exp(iu_\alpha v^\alpha)\, f(z+u, y+u)\, g(z-v, y+v). \tag{2.86}$$

This product turns out to be associative and have a regularity property; the product of two polynomials will also be a polynomial in y and z. Defining the $\star$-commutator $[V, W]_\star = V \star W - W \star V$, it turns out that we have the following commutators

$$[y_\alpha, y_\beta]_\star = -[z_\alpha, z_\beta]_\star = 2i\epsilon_{\alpha\beta}, \qquad [y_\alpha, z_\beta]_\star = 0. \tag{2.87}$$

One can show that the basic variables y_α and z_α behave as derivatives, in particular for a very general class of functions [82] we have $[y_\alpha, f]_\star = 2i\frac{\partial f}{\partial y^\alpha}$ and $[z_\alpha, f]_\star = -2i\frac{\partial f}{\partial z^\alpha}$. Note that, the star product only operates on the twistor components, but the order of all auxiliary variables is important due to the relations (2.83) and (2.84).

Vasiliev theory is formulated in terms of three generating functions depending on spacetime coordinates and the auxiliary variables

$$\begin{aligned} W &= W_\mu(z, y; \psi_{1,2}, k, \rho|x)\, \mathrm{d}x^\mu, \\ B &= B(z, y; \psi_{1,2}, k, \rho|x), \\ S_\alpha &= S_\alpha(z, y; \psi_{1,2}, k, \rho|x). \end{aligned} \tag{2.88}$$

The spacetime 1-form W is the generating function of the higher-spin fields, the 0-form B is the generating function of the massive matter fields, while S_α describes pure gauge degrees of freedom and is necessary for consistent internal symmetries. The full set of non-linear Vasiliev equations are then given by

$$\begin{aligned} \mathrm{d}W - W \star \wedge W &= 0, \\ \mathrm{d}B + [B, W]_\star &= 0, \\ \mathrm{d}S_\alpha + [S_\alpha, W]_\star &= 0, \\ S_\alpha \star S^\alpha + 2i(1 + B \star K) &= 0, \\ [S_\alpha, B]_\star &= 0, \end{aligned} \tag{2.89}$$

where $K = ke^{zy}$ is the Kleinian. The first equation turns out to be the flatness conditions for the massless sector, as we saw in last section. The second equation is the coupling between the matter fields and the higher-spin fields, while the rest are needed due to consistency. For example the last two constraints guarantee that local Lorentz invariance remains unbroken to all orders of interaction. It turns out that there exists an involutive automorphism $\rho \to -\rho$, $S_\alpha \to -S_\alpha$ which can be used to truncate the system such that W and B become ρ-independent, while $S_\alpha(z, y; \psi_{1,2}, k, \rho|x) = \rho\, s_\alpha(z, y; \psi_{1,2}, k|x)$. This is the system studied in this thesis and in [82, 83]. One can readily check that these equations are invariant under the following set of ρ-independent local higher gauge transformations, parametrized by $\epsilon = \epsilon(z, y; \psi_{1,2}, k|x)$

$$\begin{aligned} \delta W &= \mathrm{d}\epsilon + [\epsilon, W]_\star, \\ \delta B &= [\epsilon, B]_\star, \\ \delta S_\alpha &= [\epsilon, S_\alpha]_\star. \end{aligned} \tag{2.90}$$

Note that the equations of motion and gauge transformations for the higher-spin fields look very similar to usual Chern-Simons theory. As mentioned earlier, the commutators in (2.89) and (2.90) are actually supercommutators of polynomials of the generating elements, y_α, z_α, k and $\psi_{1,2}$.

2.5.1 Vacuum Solutions

The full non-linear theory is very difficult to work with. Luckily it turns out that we only need to consider a particular linearization of the theory for our purposes. First we consider vacuum solutions of the Vasiliev equations (2.89), in which the matter fields take a constant value

$$B^{(0)} = \nu = \text{constant}. \tag{2.91}$$

With this ansatz the second and the last equations of (2.89) are trivially satisfied, while the vacuum solutions of W and S_α have to satisfy the three remaining ones

$$\begin{aligned} \mathrm{d}W^{(0)} - W^{(0)} \star \wedge W^{(0)} &= 0, \\ \mathrm{d}S^{(0)}_\alpha + [S^{(0)}_\alpha, W^{(0)}] &= 0, \\ S^{(0)}_\alpha \star S^{(0)\alpha} + 2i(1 + \nu K) &= 0. \end{aligned} \tag{2.92}$$

In [82] three different solutions to the third equation are given, but they are all in the same gauge equivalence class. The simplest is

$$S^{(0)}_\alpha = \rho\, \tilde{z}_\alpha, \qquad \text{where} \qquad \tilde{z}_\alpha = z_\alpha + \nu(z_\alpha + y_\alpha) \int_0^1 \mathrm{d}t\, t\, e^{it\, zy}\, k.$$

Since $\mathrm{d}S^{(0)}_\alpha = 0$, the second equation of (2.92) reduces to $[S^{(0)}_\alpha, W^{(0)}] = 0$. In order to solve this constraint, one can show that the element

$$\tilde{y}_\alpha = y_\alpha + \nu(z_\alpha + y_\alpha) \int_0^1 \mathrm{d}t\, (t-1)\, e^{it\, yz}\, k \tag{2.93}$$

satisfies the following commutation relations

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k), \qquad \{\tilde{y}, k\} = 0, \tag{2.94}$$

and most importantly

$$[\tilde{y}_\alpha, S^{(0)}_\beta] = 0. \tag{2.95}$$

Now the constraint $[S^{(0)}_\alpha, W^{(0)}] = 0$ is solved if $W^{(0)}$ depend only on $\psi_{1,2}$, k and $\tilde{y}_\alpha$ (and not z), since they all commute with $S^{(0)}_\alpha$. We will call the associative algebra generated by $\tilde{y}_\alpha$, k and $\psi_{1,2}$, $\mathcal{A}_S$.

Note the remarkable feature of (2.94), that the vacuum constant ν is deforming the oscillators y_α ($\nu = 0$) into the so-called deformed oscillators $\tilde{y}_\alpha$. This means that ν is parametrizing a continuous family of inequivalent AdS$_3$ vacua (2.91), in which the symmetry algebra is continuously deforming. As we will see later, the higher-spin gauge symmetry is so constraining that even the masses of the matter fields are completely fixed by ν.

2.5.2 Linearized Dynamics of Matter Fields

Next we will consider linearized fluctuations of the matter fields around their vacuum, propagating on the higher-spin background $W^{(0)}$

$$B(z, y; \psi_{1,2}, k) = \nu + \mathcal{C}(z, y; \psi_{1,2}, k). \tag{2.96}$$

In this thesis we will neglect all fluctuations around $W^{(0)}$ and $S^{(0)}_\alpha$. Thus we do not consider higher order effects like backreaction of the matter on the higher-spin fields and interactions among the matter fields. See [82] for more about this. Inserting (2.96) into (2.89) we get two non-trivial equations

$$\begin{aligned} \mathrm{d}\mathcal{C} + [\mathcal{C}, W^{(0)}]_\star &= 0, \\ [S^{(0)}_\alpha, \mathcal{C}]_\star &= 0. \end{aligned} \tag{2.97}$$

The second equation is solved by demanding that $\mathcal{C}$ is a spacetime function mapping into the algebra $\mathcal{A}_S$. In other words, we have now found that both

$$\mathcal{C} = \mathcal{C}(\tilde{y}; k, \psi_{1,2}|x) \qquad \text{and} \qquad W^{(0)} = W^{(0)}(\tilde{y}; k, \psi_{1,2}|x)$$

are elements of $\mathcal{A}_S$ in the linearized approximation. We can now get rid of the $\psi_{1,2}$ Clifford elements and find the equations of motion of the physical fields. For this we need to define the projection operators

$$\mathcal{P}_\pm = \frac{1 \pm \psi_1}{2}, \tag{2.98}$$

with the following properties

$$\mathcal{P}_\pm \mathcal{P}_\mp = 0, \qquad \mathcal{P}_\pm^2 = \mathcal{P}_\pm. \tag{2.99}$$

The usual gauge fields known from AdS$_3$ gravity, A and $\bar{A}$ are extracted as [84]

$$W^{(0)} = -\mathcal{P}_+ A - \mathcal{P}_- \bar{A}.$$

One way to understand this, is that if one finds the pure AdS$_3$ solution, then $W^{(0)} = w_0 + \psi_1 e_0$, where e_0 and w_0 is the vielbein and spin connection, respectively. Thus the above decomposition is related to (2.4). Inserting this into the equations of motion for $W^{(0)}$ (2.92), we find the Chern-Simons flatness conditions

$$\mathrm{d}A + A \star \wedge A = 0, \qquad \mathrm{d}\bar{A} + \bar{A} \star \wedge \bar{A} = 0. \tag{2.100}$$

The matter fields can be decomposed as

$$\mathcal{C}(\tilde{y}; k, \psi_{1,2}|x) = \mathcal{C}_{aux}(\tilde{y}; k, \psi_1|x) + \mathcal{C}_{dyn}(\tilde{y}; k, \psi_1|x)\psi_2. \tag{2.101}$$

It turns out that C_{aux} does not describe any propagating degrees of freedom and can consistently be put to zero. The dynamical part C_{dyn} can be decomposed as

$$\mathcal{C}(\tilde{y}; k, \psi_{1,2}x) = C(\tilde{y}; k|x)\,\mathcal{P}_+\,\psi_2 + \tilde{C}(\tilde{y}; k|x)\,\mathcal{P}_-\,\psi_2. \tag{2.102}$$

By using the identity $\mathcal{P}_\pm\psi_2 = \psi_2\mathcal{P}_\mp$, equation (2.97) finally reduces to

$$\begin{aligned} &\mathrm{d}C + A\star C - C\star\bar{A} = 0,\\ &\mathrm{d}\tilde{C} + \bar{A}\star\tilde{C} - \tilde{C}\star A = 0. \end{aligned} \tag{2.103}$$

These are the equations we will use in this thesis in the calculation of three-point functions from the bulk perspective. The associative algebra generated by $\tilde{y}^\alpha$ and k modulo the relations (2.94) is known as $Aq(2,\nu)$ [85]. The physical fields in this algebra are expanded as

$$\begin{aligned} C(\tilde{y}; k|x) &= \sum_{B=0}^{1}\sum_{n=0}^{\infty}\frac{1}{n!}C^B_{\alpha_1\ldots\alpha_n}(x)\,k^B\,\tilde{y}^{\alpha_1}\star\cdots\star\tilde{y}^{\alpha_n},\\ A(\tilde{y}; k|x) &= \sum_{B=0}^{1}\sum_{n=0}^{\infty}\frac{1}{n!}A^B_{\alpha_1\ldots\alpha_n}(x)\,k^B\,\tilde{y}^{\alpha_1}\star\cdots\star\tilde{y}^{\alpha_n}, \end{aligned} \tag{2.104}$$

and similarly for $\tilde{C}$ and $\bar{A}$. The element k doubles the spectrum. This is needed in order to have $\mathcal{N}=2$ supersymmetry. We can project out two sectors of the generating functions for the matter content as

$$C(\tilde{y}; k|x) = \Pi_+ C^+(\tilde{y}|x) + \Pi_- C^-(\tilde{y}|x), \qquad \text{where} \qquad \Pi_\pm = \frac{1\pm k}{2}. \tag{2.105}$$

There is an analogous decomposition for $\tilde{C}$. The lowest components $\phi_\pm \equiv C_0^\pm$ and $\psi_\pm \equiv C_\alpha^\pm$ correspond to two complex scalars and two fermions, respectively. The are also four corresponding fields from $\tilde{C}_\pm$. All these fields form two sets of $3d$ $\mathcal{N}=2$ hypermultiplets

$$(\phi_+, \psi_+, \psi_-, \phi_-) \qquad \text{and} \qquad (\tilde{\phi}_+, \tilde{\psi}_+, \tilde{\psi}_-, \tilde{\phi}_-). \tag{2.106}$$

These are the matter fields of the Vasiliev theory and key elements of higher-spin holography. The functions $C^B_{\alpha_1,\ldots,\alpha_n}$, for $n > 1$, are auxiliary fields and can all be written as sums of derivatives of the physical fields, using the equations of motion (2.103).

The algebra $Aq(2,\nu)$ contains the bosonic subalgebra of even elements $C(\tilde{y}; k|x) = C(-\tilde{y}; k|x)$, which can be decomposed as $Aq^E(2,\nu) \oplus Aq^E(2,-\nu)$ [85] by the projection operator $\Pi_\pm = \frac{1\pm k}{2}$. Here $Aq^E(2,\nu)$ consists of symmetrized products of even number of $\tilde{y}$ elements, and is isomorphic to the non-supersymmetric higher-spin algebra $Aq(2,\nu) \approx \mathrm{hs}[\frac{1-\nu}{2}]$. This implies that the bosonic fields of $C^\pm$ and $\tilde{C}^\pm$ can be described purely by using $\mathrm{hs}[\lambda]$, instead of $Aq(2,\nu)$. This fact would have made the calculation of three-point functions much simpler, but unfortunately we discovered this important detail toward the final stages of this thesis. We will therefore not use the basis obtained from the projection operators $\Pi_\pm$. The upshot of using our more “unnatural” basis is that we can find the full structure constants of $Aq(2,\nu)$ as described below.

From these facts we can conclude that there is a non-supersymmetric truncation by restriction to even polynomials of $\tilde{y}_\alpha$ and projecting $k = \pm 1$. This was recently used in [84]. There exists also am $\mathcal{N}=1$ truncation [82, 83].

2.5.3 Modified Vasiliev Formalism

The traditional Vasiliev formalism as we have briefly outlined above is very tedious to work with. Mainly due to the fact that it requires us to multiply symmetrized products of the deformed oscillators $\tilde{y}_\alpha$, then by using the relations (2.94) write the result in terms of symmetrized products of $\tilde{y}_\alpha$. Everything would become much simpler if we had explicit expressions for the structure constants of $Aq(2,\nu)$. We will turn to this issue now.

The canonical infinite-dimensional Lie superalgebra corresponding to $Aq(2,\nu)$ is called $hs(2,\nu)$ [83], with the $\mathbb{Z}_2$-grading given by the number of spinor indices modulo 2. The flatness conditions (2.100) involve only (anti-)commutators when written in component form. So turning off the matter content, the theory will only depend on the Lie algebra $hs(2,\nu) \approx \mathrm{shs}[\lambda]$ and not the full $Aq(2,\nu)$. This is nothing but the $\mathrm{shs}[\lambda]_{k_{CS}} \times \mathrm{shs}[\lambda]_{-k_{CS}}$ Chern-Simons higher-spin SUGRA discussed earlier. Since we know that $\mathrm{shs}[\lambda]$ has an associative extension, it is natural to conjecture that $Aq(2,\nu)$ is isomorphic to $\mathcal{SB}[\mu]$. This is actually trivial to see. Using the following following identifications

$$G_\alpha = \left(\frac{-i}{4}\right)^{1/2} \tilde{y}_\alpha \qquad \text{and} \qquad \nu = 2c = 4\lambda - 1, \tag{2.107}$$

the identities (2.94) and (2.52) are equivalent. Thus we have the isomorphism $Aq(2,\nu) \approx \mathcal{SB}[\mu]$ and $sh(2,\nu) \approx \mathrm{shs}[\lambda]$ with $\lambda = \frac{\nu+1}{4}$. By looking at the (anti-)commutators of the $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ together with appendix B of [84], it is clear that the $\mathrm{shs}[\lambda]$ generators are (possibly up to constants) related to the $Aq(2,\nu)$ generators by

$$\left(\frac{-i}{4}\right)^{t-1} S^t_m, \qquad \left(\frac{-i}{4}\right)^{t-1} S^t_m\, k, \tag{2.108}$$

where S^t_m is a symmetric product of $2(t-1)$ $\tilde{y}_\alpha$'s with $N_\pm$ of $y_{\pm\frac{1}{2}}$ and $2m = N_+ - N_-$. The first set of generators are related to $L^{(s)+}_m$ and $G^{(s)+}_m$ for integer and half-integer t, respectively. The second set of generators are similarly related to $L^{(s)-}_m$ and $G^{(s)-}_m$, however for the $U(1)$ R-symmetry generator, we have $L^{(1)-}_0 = k + \nu$. We will however not need the explicit mapping between $\mathcal{SB}[\mu]$ and $Aq(2,\nu)$ in this thesis, only the fact that they are isomorphic. This is a truly marvelous fact, since we explicitly know the structure constants of $\mathcal{SB}[\mu]$.

We will hereby modify the traditional Vasiliev formalism by changing $Aq(2,\nu)$ into $\mathcal{SB}[\mu]$. In this formalism the expansions (2.104) of the generating functions are given as

$$\begin{aligned} A &= \sum_{s=1}^{\infty} \sum_{|m|\leq s-1} A^s_m\, L^{(s)}_m + \sum_{s=\frac{3}{2}}^{\infty} \sum_{|r|\leq s-\frac{3}{2}} A^s_r\, G^{(s)}_r, \\ C &= \sum_{s=1}^{\infty} \sum_{|m|\leq s-1} C^s_m\, L^{(s)}_m + \sum_{s=\frac{3}{2}}^{\infty} \sum_{|r|\leq s-\frac{3}{2}} C^s_r\, G^{(s)}_r, \end{aligned} \tag{2.109}$$

and similarly for $\tilde{C}$ and $\bar{A}$. Note that we can easily distinguish the bosonic components C^s_m from the fermionic ones C^s_r, since m is always an integer while r is half of an odd integer. In this formalism, the physical scalars $\phi_\pm$ and fermions $\psi_\pm$, are given by appropriate superpositions of the lowest components C^1_0, $C^{\frac{3}{2}}_0$, $\left\{C^{\frac{3}{2}}_{+\frac{1}{2}}, C^{\frac{3}{2}}_{-\frac{1}{2}}\right\}$ and $\left\{C^2_{+\frac{1}{2}}, C^2_{-\frac{1}{2}}\right\}$. We will later derive the Klein-Gordon equations for the scalar, find the correct superpositions and show that this modified formalism gives rise to the correct masses.

Chapter 3

Extended Symmetries and $\mathcal{W}$-Algebras

Two dimensional conformal field theories enjoy a vast extension of their symmetry algebra compared to their higher-dimensional counterparts. The energy-momentum tensor splits into holomorphic and anti-holomorphic parts $T(z,\bar{z}) = T(z) + \bar{T}(\bar{z})$, and there are an infinite number of conserved currents in the theory given by $J^{(2)}(z) = \omega(z)\,T(z)$ and $\bar{J}^{(2)}(\bar{z}) = \bar{\omega}(\bar{z})\,\bar{T}(\bar{z})$. These satisfy

$$\bar{\partial} J^{(2)}(z) = 0 \qquad \text{and} \qquad \partial \bar{J}^{(2)}(\bar{z}) = 0, \tag{3.1}$$

for arbitrary holomorphic and anti-holomorphic functions, $\omega(z)$ and $\bar{\omega}(\bar{z})$, respectively. These conserved currents give rise to the infinite-dimensional Virasoro algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12} n(n^2-1)\delta_{m+n,0}, \tag{3.2}$$

where L_n are the Laurent modes of the energy-momentum tensor $T = \sum_n z^{-n-2} L_n$. Here c is a central element and acts as a constant value on irreducible representations due to Schur's lemma, called the central charge. There is also a copy of the Virasoro algebra for the anti-holomorphic sector. Since the holomorphic and anti-holomorphic sectors of a CFT are completely decoupled on the sphere, we will restrict the discussion to one sector only.

As discussed in appendix C, two-dimensional CFT's essentially reduce to representation theory of the Virasoro algebra. One amazing result is that all unitary representations of the Virasoro algebra for $0 < c < 1$ form a discrete sequence, known as the Virasoro minimal models. At these particular values of c, the Verma modules are not irreducible (nor fully reducible) due to null-states forming orthogonal sub-modules which have to be projected out. Due to these null-states, these theories are extremely constrained. One can even, at least in principle, find all correlation functions in these theories purely based on symmetries. The special feature of these minimal models is that the spectrum can be organized into a finite number of irreducible representations, also called Virasoro primary fields or states. All CFT's with $c > 1$ will always contain an infinite number of Virasoro primary fields. For a review of two-dimensional conformal field theories, see appendix C.

There is however, a way to extend the success of Virasoro minimal models. Conformal field theories with primary fields $Q^{(s)}(z)$ of dimension $(h,\bar{h}) = (s,0)$, where s is an integer or half-integer, will have additional conserved currents $J^{(s)}(z) = \omega(z)Q^{(s)}(z)$ satisfying

$$\bar{\partial}J^{(s)}(z) = 0. \tag{3.3}$$

These extra conserved currents will extend the Virasoro algebra with an infinite set of additional generators. CFT's in which the spectrum can be organized into a finite number of families, w.r.t. an extended symmetry algebra, are called rational conformal field theories (RCFT). These theories have a lot of amazing structure which opens up the possibility of axiomatic formulation, see more details in appendix C or in the works of Moore and Seiberg [18, 21, 22].

In this chapter we will very briefly discuss supersymmetric and higher-spin extensions of the Virasoro algebra. Hereafter we will introduce a large class of $\mathcal{N} = 2$ superconformal field theories based on the $\mathcal{N} = 1$ WZW coset construction, called the Kazama-Suzuki models. These models originally played an important role in compactification of superstring theory on Calabi-Yau manifolds which are, surprisingly, related to $\mathcal{N} = 2$ superconformal theories.[1] Unlike another famous construction, the so-called Gepner models,[2] Kazama-Suzuki models do not lead to extra $U(1)$ factors which are generally anomalous and cause problems at string loop level. Our interest in these models is however different. A subclass of these models have higher-spin symmetries and are conjectured to be dual to the Vasiliev theory discussed earlier.

3.1 Higher-Spin Currents and $\mathcal{W}$-algebras

As discussed above, extensions of the Virasoro algebra are possible if the CFT contains holomorphic primary fields of spin s. The most famous and important example is when $s = 1$ currents are present in addition to the Virasoro algebra. In this case the chiral algebra will be a semi-direct product of an untwisted affine Lie algebra $\hat{\mathfrak{g}}_k$ and the Virasoro algebra. These type of currents are generally present in Wess-Zumino-Witten models. The Virasoro algebra is actually contained in the universal enveloping algebra of $\hat{\mathfrak{g}}_k$, where it can be identified with the second order Casimir, and therefore the full CFT can be formulated without the need for an action. This is called the Sugawara construction and is reviewed in appendix C.

The first systematic analysis of higher-spin symmetries was done by Zamolodchikov [69]. Assume that there is one additional holomorphic spin-s field $Q^{(s)}$ in addition to the energy-momentum tensor. In the absence of other fields, the fusion rules must be of the form $Q^{(s)} \times Q^{(s)} = \mathbf{1} + Q^{(s)}$, or written schematically in terms of operator product expansions

$$Q^{(s)}(z)Q^{(s)}(w) \sim a\,[\mathbf{1}] + b\,[Q^{(s)}]. \tag{3.4}$$

The full expression is given in equation (C.46), where the sum over p only include the identity operator and $Q^{(s)}$. The notation $[\phi]$ denotes the contribution of the conformal family corresponding to ϕ, i.e.

$$[\phi] = x^{-2s+h_\phi}\left(\mathbf{1} + x\,\beta_{s,s}^{\phi,\{1\}}L_{-1} + x^2\,\beta_{s,s}^{\phi,\{1,1\}}L_{-1}^2 + x^2\,\beta_{s,s}^{\phi,\{2\}}L_{-2} + \dots\right)\phi, \tag{3.5}$$

[1]For an introduction of Calabi-Yau compactification and their relation to $\mathcal{N} = 2$ superconformal theories see for example [86].

[2]These models are constructed by tensoring several Virasoro minimal models such that $c = 9$. The "reducibility" of these models leads to extra $U(1)$ factors, when used in the context of compactification.

where $x = z - w$. Note that the energy-momentum tensor is contained in $[\mathbf{1}]$ since $L_{-2}\mathbf{1} = T$, it is actually the lowest non-zero term. All the coefficients above are completely fixed by the Virasoro algebra and can be determined as a function of s, h_ϕ and the central charge c, see for example equation (C.48). The value of a depends on the normalization of $Q^{(s)}$ and is conventionally chosen such that $a = c/s$. The OPE (3.4) is therefore completely fixed by conformal invariance except for two free parameters, the central charge c and the coefficient b. By calculating the β coefficients, $L^m_{-n}\mathbf{1}$ and $L^m_{-n}Q^{(s)}$ for low m and n as described in appendix C and references, one will find the following general OPE

$$\begin{aligned} Q^{(s)}(z)Q^{(s)}(w) \sim \frac{c/s}{(z-w)^{2s}} + \frac{2T(w)}{(z-w)^{2s-2}} + \frac{\partial T(w)}{(z-w)^{2s-3}} + \frac{\frac{3}{10}\,\partial^2 T(w) + 2\gamma\,\Lambda(w)}{(z-w)^{2s-4}} \\ + \frac{\frac{1}{15}\,\partial^3 T(w) + \gamma\,\partial\Lambda(w)}{(z-w)^{2s-5}} + \cdots + \frac{b\,Q^{(s)}}{(z-w)^{s}} + \frac{b/2\,\partial Q^{(s)}}{(z-w)^{s-1}} + \ldots, \end{aligned} \tag{3.6}$$

where $\gamma = (5s+1)/(22+5c)$ and $\Lambda(w) = \mathcal{N}(TT)(w) - \frac{3}{10}\partial^3 T(w)$. Note that if $Q^{(s)}$ is a fermion then $b = 0$.

There is one problem remaining. In order for the complete operator algebra TT, $TQ^{(s)}$ and $Q^{(s)}Q^{(s)}$ to be associative, we need to check that the crossing symmetries of the four-point functions are satisfied (see for example equation (C.53)). It turns out that it is not always possible to satisfy the associativity conditions for all values of b and c, we will take a few examples below. Before we proceed, let us mention an alternative route to take. It is possible to write down general commutation relations of Laurent modes of quasi-primary fields [87], which can be used to systematically look for extended symmetry algebras. In this approach the consistency constraints on the four-point functions is replaced by Jacobi identities [88, 89].

3.1.1 Spin-$\frac{1}{2}$ Fermions and $\widehat{\mathfrak{so}}(N)_1$ Current Algebra

We will start by considering N real fermions ψ^i, $1, \ldots, N$. For appropriate normalization of the currents, the currents will have the OPE's

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w}. \tag{3.7}$$

Thus we find the usual free fermion OPE's, but no interesting extension of the Virasoro algebra. However, due to its importance in connection with Kazama-Suzuki coset models consider the following set of currents

$$j^a(z) = \frac{1}{2}\mathcal{N}\left(\psi^i\, t^a_{ij}\, \psi^j\right), \tag{3.8}$$

where t^a, $a = 1, \ldots, \frac{N(N-1)}{2}$, are the generators of $\mathfrak{so}(N)$ in the vector representation. The OPE between spin-1 currents $j^a(z)$, is of the form (C.78) with $k = 1$ and the structure constants are those of $\mathfrak{so}(N)$. This implies that N real spin-$\frac{1}{2}$ fermions generate the $\widehat{\mathfrak{so}}(N)_1$ affine Lie algebra with the central charge $c = \frac{N}{2}$.

3.1.2 Spin-1 Currents and Affine Lie Algebras

For $s = 1$, if $Q^{(1)}$ is a multicomponent field of $(c/k)^{1/2}J^a$, $a = 1, \ldots, d$, then the general OPE will be of the form (C.78). Here κ^{ab} and f^{ab}_c are forced to be symmetric and anti-symmetric in their indices, respectively. Furthermore the crossing symmetries of the four-point functions $\left\langle J^{a_1}(z_1)\ldots J^{a_4}(z_4)\right\rangle$ translate into Jacobi identities for f^{ab}_c. Taking into account the TT and TJ^a OPE's, we conclude that for this case the algebra is a semi-direct product of an affine Lie algebra and the Virasoro algebra. See section C.2 and C.3 for more discussion about this class of theories, and appendix E for more details about affine Lie algebras.

By the addition of a spin-$\frac{1}{2}$ current, one can construct affine Lie superalgebras. This we will return to.

3.1.3 Spin-$\frac{3}{2}$ Currents and Superconformal Algebras

Now assume that $Q^{(\frac{3}{2})} \equiv G$ is a spin $s = \frac{3}{2}$ holomorphic primary field, then from (3.6) it is clear that it has the OPE

$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2\,T(w)}{z-w}. \tag{3.9}$$

Translating the above OPE and $T(z)G(w)$, which just states that G is a primary with $h = \frac{3}{2}$, into (anti-)commutators of their modes we find

$$\Big[L_m, G_r\Big] = \Big(\frac{m}{2} - r\Big)\, G_{m+r}, \qquad \Big\{G_r, G_s\Big\} = 2\, L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{3.10}$$

This, together with the commutators $[L_m, L_n]$, form the $\mathcal{N} = 1$ Virasoro algebra in the Ramond-sector, if $r, s \in \mathbb{Z}$, or the Neveu-Schwarz sector, if $r, s \in \mathbb{Z} + \frac{1}{2}$. We will only consider the Neveu-Schwarz sector in this thesis. The set of generators $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$ form a global $\mathfrak{osp}(1|2)$ subalgebra, extending the definition of quasi-primary fields. We thus conclude that adding one spin $s = \frac{3}{2}$ field, one finds the supersymmetric extension of the Virasoro algebra, and G is the supercharge.

One can now readily define the notion of $\mathcal{N} = 1$ superconformal primary fields in an obvious way. Extending the discussion of Verma modules and null-states given in appendix C, one can show that for $0 < c < \frac{3}{2}$, unitary highest weight representations are only possible at the following discrete values of the central charge

$$c = \frac{3}{2}\left(1 - \frac{8}{(m+2)(m+4)}\right). \tag{3.11}$$

This clearly demonstrate the power of extended symmetries. One can find a new class of minimal models for $c > 1$ by introducing supersymmetry. For $m = 1$ we have $c = \frac{7}{10}$. This is the only CFT in this sequence which is also a Virasoro minimal model. This is nothing but the tri-critical Ising model, and it is quite surprising to find supersymmetry in this model.

3.1.4 $\mathcal{N} = 2$ Superconformal Algebra

In order to construct extended superconformal theories, we need to add several spin-$\frac{3}{2}$ fields G^i, $i = 1, 2, \ldots, \mathcal{N}$, and several spin-1 currents which transform these supercharges into each other. The spin-1 currents correspond to internal R-symmetry and span by themselves an affine Lie algebra. By an obvious generalization of the equation (3.4), and thereby (3.6), one can allow for multiple fields and thereby derive the most general OPE's for this model.

In this thesis we are however only interested in $\mathcal{N} = 2$ superconformal symmetry and will give the results without further details. Here we need two spin-$\frac{3}{2}$ supercharges $G^{\pm}$

and a $SO(2) \approx U(1)$ spin-1 current j. Using a basis such that $G^{\pm}$ have a definite $U(1)$ charge, we find the following set of OPE's [90, 91]

$$\begin{aligned} G^{\pm}(z)G^{\mp} &\sim \frac{2c/3}{(z-w)^3} \pm \frac{2j(w)}{(z-w)^2} + \frac{1}{z-w}\Big(2T(w) \pm \partial j(w)\Big), \\ j(z)G^{\pm}(w) &\sim \pm\frac{G^{\pm}(w)}{z-w}, \\ j(z)j(w) &\sim \frac{c/3}{(z-w)^2}, \\ G^{\pm}(z)G^{\pm}(w) &\sim 0, \end{aligned} \tag{3.12}$$

together with a set of obvious OPE's Tj and $TG^{\pm}$. Most importantly, the $\mathcal{N}=2$ superconformal algebra exists for all values of c. One can again show that the extra symmetry extends the set of RCFT's, in fact all $\mathcal{N}=2$ minimal models exist for a discrete sequence of central charges in the range $1 \leq c < 3$ [92, 93].

One can readily construct algebras with $SO(N)$ or $U(N)$ internal symmetry (leading to $\widehat{\mathfrak{so}}(N)$ affine Lie subalgebras), but for high enough N these turn out not to form Lie algebras due to non-linear terms. These, so-called Knizhnik-Bershadsky algebras, can be vastly extended [50], see [49] for their construction using quantum Drinfeld-Sokolov reduction. We will return to these aspects momentarily.

3.1.5 Spin-$\frac{5}{2}$ Currents

It turns out that adding only a spin-$\frac{5}{2}$ current, does not lead to any interesting conformal field theories. The associativity of the operator algebra (materialized in the crossing symmetry), restricts the central charge to $c = -\frac{13}{14}$. Not only is this just one particular value, its also a non-unitary CFT due to its negativeness.

3.1.6 Spin-3 Currents and the $\mathcal{W}_3$ Algebra

The most interesting thing, for the purposes of this thesis, happens when adding a spin-3 $Q^{(3)} \equiv W^3$ current. Note that a term $(z-w)^{-3}\,W^3$ is not allowed in a $W^3(z)W^3(w)$ OPE, since they have contradicting symmetry properties under the transformation $z \to w$. Thus the constant b in (3.6) is zero, and we can directly write down the OPE

$$\begin{aligned} W^3(z)W^3(w) \sim{}& \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\,\partial^2 T(w) + 2\gamma_3\,\Lambda(w)}{(z-w)^2} \\ &+ \frac{\frac{1}{15}\,\partial^3 T(w) + \gamma_3\,\partial\Lambda(w)}{z-w}, \end{aligned} \tag{3.13}$$

where $\gamma_3 = \frac{16}{22+5c}$. Calculating the four-point function $\langle W^3(z_1)W^3(z_2)W^3(z_3)W^3(z_4)\rangle$, one finds that the crossing symmetry conditions are fulfilled for any c. Using the techniques of appendix C, we find the following commutation relations for the W^3 Laurent modes

$$\begin{aligned} [W^3_m, W^3_n] ={}& \frac{16}{22+5c}(m-n)\Lambda_{m+n} + (n-m)\Big[\frac{1}{15}(m+n+2)(m+n+3) - \frac{1}{6}(m+2)(n+2)\Big]L_{m+n} \\ &+ \frac{c}{360}(m^2-4)(m^2-1)m\,\delta_{m+n,0}, \end{aligned} \tag{3.14}$$

where

$$\Lambda_m = \sum_{n\in\mathbb{Z}} L_{m-n}L_n - \frac{3}{10}(m+3)(m+2)L_m. \tag{3.15}$$

The important thing to notice is that this is not a Lie algebra due to the non-linear terms Λ_m, a Lie algebra is however obtained in the limit $c \to \infty$. This non-linear algebra, is the unique algebra obtained by adding a spin-3 current to the Virasoro sector and is called the $\mathcal{W}_3$ algebra. In the $c \to \infty$ limit, the wedge[3] elements form $\mathfrak{sl}(3)$ and this is a feature that generalizes for more general $\mathcal{W}$ algebras, and will play a crucial role in our arguments in section (5.4).

This enhancement of symmetry again allows for an extension of RCFT's. The $\mathcal{W}_3$ algebra has degenerate representations which lead to minimal models at the following values of the central charge

$$c = 2\left(1 - \frac{12}{m(m+1)}\right), \qquad m = 4, 5, \ldots. \tag{3.16}$$

The only CFT among these which is also a Virasoro minimal model is for $m = 4$, with the central charge $c = \frac{4}{5}$. This is 3-state Potts model (see equation (C.112)), which we analyze from the WZW coset construction in section C.3.1.1.

3.2 $\mathcal{W}$-algebras

The $\mathcal{W}_3$ algebra discussed above can be generalized vastly, by adding various combinations of higher-spin currents. There are in general two classes of $\mathcal{W}$-algebras; (i) "generic" $\mathcal{W}$ algebras which exist for any central charge c, or (ii) "exotic" $\mathcal{W}$-algebras which only exist for special values of c. We will only be interested in generic $\mathcal{W}$ algebras here.

There are several different ways to construct $\mathcal{W}$ algebras. Above we used the direct construction, which entails adding a higher spin current, finding the general OPE's by conformal symmetry, closing the algebra and checking that the associativity conditions are fulfilled. The last step is the most difficult one. The direct construction has been systematized in various ways and many new algebras have been found by adapting the algorithms to a computer.

Beside this, we will highlight two different methods. The first entails constructing higher-spin currents in a WZW coset model by a natural generalization of the Sugawara construction. The second is called quantum Drinfeld-Sokolov reduction and is a quantum version of the classical construction discussed in the previous chapter. Quantum DS-reduction is the most systematic and powerful technique available to construct $\mathcal{W}$ algebras, and even gives rise to a functor between the representation categories of Affine Lie algebras and $\mathcal{W}$ algebras.

Due to lack of time, we will (to the regret of the author) review these beautiful constructions in an unjustifiable, short and crude way.[4]

[3] Here by wedge elements we mean the subset of generators of the chiral algebra, such that the modes of a spin-s element W^s_m is restricted to $|m| \leq s-1$. For example the wedge elements of the Virasoro algebra are L_0 and $L_{\pm 1}$, and they form the wedge subalgebra $\mathfrak{sl}(2)$.

[4] For a general review of $\mathcal{W}$-algebras see [58]. For a particularly readable account of classical and quantum Drinfeld-Sokolov reduction see the PhD thesis of Tjin Tjark [59]. In this thesis the concept of finite $\mathcal{W}$-algebras is also defined. This has grown into an interesting topic in mathematics, however any physical applications are not known to the author.

3.2.1 Generalized Sugawara Construction and Casimir Algebras

In appendix C, we saw that for any untwisted Affine Lie algebra $\hat{\mathfrak{g}}_k$ (see figure F.2), one can define a CFT by the Sugawara Construction. In this approach the Energy-Momentum tensor is given by the second-order Casimir[5] of $\hat{\mathfrak{g}}_k$ with an appropriate normalization constant.

A natural extension to higher-spin symmetries involves Casimir algebras. For any simple finite dimensional Lie algebra $\mathfrak{g}$ of rank r, the center of the universal enveloping algebra $U(\mathfrak{g})$ is r dimensional and spanned by the set of (higher-order) Casimirs of $\mathfrak{g}$. To any of the Casimirs of $\mathfrak{g}$, we can associate an operator belonging to the universal eneloping algebra $U(\hat{\mathfrak{g}}_k)$, corresponding to the affinization of $\mathfrak{g}$. In [94] a generalization of the Sugawara construction was proposed using these operators

$$Q^s(z) = \sum_{a_1 a_2 \dots a_s} d_{a_1 a_2 \dots a_s} \mathcal{N}\left(J^{a_1} J^{a_2} \dots J^{a_s}\right)(z), \tag{3.17}$$

where $d_{a_1 a_2 \dots a_s}$ is some completely symmetric traceless tensor. It turns out that $Q^s(z)$ is a primary field with conformal weight $h = s$, except for $s = 2$ which is the Sugawara energy-momentum tensor.

Except for specific values of c, the Casimir algebras do not close. Let us consider the simplest example $\hat{\mathfrak{g}} = A_2^{(1)} = \widehat{\mathfrak{su}}(3)$, in which there are a second-order and a third-order Casimir operator. The OPE between the spin-3 operator can be shown to be of the form [94]

$$\begin{aligned} Q^3(z)Q^3(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\,\partial^2 T(w) + 2\gamma_3\,\Lambda(w) + R^4(w)}{(z-w)^2} \\ + \frac{\frac{1}{15}\,\partial^3 T(w) + \gamma_3\,\partial\Lambda(w) + \frac{1}{2}\partial R^4(w)}{z-w}, \end{aligned} \tag{3.18}$$

where $R^4(z)$ is a new spin-4 primary field which cannot be written purely in terms of the Casimirs $Q^2 \equiv T$ and Q^3. Since $\dim \mathfrak{su}(3) = 8$ and the dual Coxeter number is $g^\vee = 3$, from the equation (C.82) we find the central charge $c = \frac{8k}{k+3}$. A careful analysis show that the spin-4 field become a null-field for $k = 1$ and decouple from the algebra [94]. Thus the Casimir algebra only close for $c = 2$, in which it is equivalent to the Zamolodchikov $\mathcal{W}_3$ algebra (3.13).

A clue of how to close the Casimir algebra comes from equation (3.16), which is actually the central charge of the WZW coset $\frac{\widehat{\mathfrak{su}}(3)_k \oplus \widehat{\mathfrak{su}}(3)_1}{\widehat{\mathfrak{su}}(3)_{k+1}}$ with $m = k+3$. This actually turn out to be right idea to pursue. In [95] a spin-3 primary was constructed in the universal enveloping algebra $U(\widehat{\mathfrak{su}}(3)_k \oplus \widehat{\mathfrak{su}}(3)_1)$ which commutes with the diagonal subalgebra $\widehat{\mathfrak{su}}(3)_{k+1}$, and hence is a primary of the coset. This Casimir algebra indeed close for all c in (3.16), and give a realization of $\mathcal{W}_3$ minimal models.[6]

[5] Note that these "Casimirs" are not part of the center of $U(\hat{\mathfrak{g}}_k)$, since they do not commute with all the affine Lie algebra elements. Their zero-modes are however Casimirs of the finite Lie algebra's $\mathfrak{g}$ and they naturally extend to elements of $U(\hat{\mathfrak{g}}_k)$. We will therefore talk about Casimirs of $\hat{\mathfrak{g}}_k$ by abuse of language.

[6] By straightforward calculations, one can show that the Casimir algebra does not close for cosets $\frac{\widehat{\mathfrak{su}}(N)_k \oplus \widehat{\mathfrak{su}}(N)_l}{\widehat{\mathfrak{su}}(N)_{k+l}}$ where $l \geq 2$ and other fields are needed. This is actually not surprising at all. For example in the simplest case when $N = 2$ and $l = 2$, we find the coset (C.103), which is a realization the $\mathcal{N} = 1$ minimal models. The chiral algebra thus contain a spin-$\frac{3}{2}$ field, which cannot be realized by a Casimir.

These conclusions can actually be generalized. The cosets

$$\frac{\widehat{\mathfrak{su}}(N)_k \oplus \widehat{\mathfrak{su}}(N)_1}{\widehat{\mathfrak{su}}(N)_{k+1}}, \tag{3.19}$$

give rise to $\mathcal{W}_N$ minimal models with the central charges (using equation (C.100))

$$c = (N-1)\left[1 - \frac{N(N+1)}{p(p+1)}\right] \leq N-1, \tag{3.20}$$

with the parameter p given by $p = k + N \geq N+1$. The $\mathcal{W}_N$ algebra is generated by Casimir elements of $\widehat{\mathfrak{su}}(N)_k \oplus \widehat{\mathfrak{su}}(N)_1$ which commute with the diagonal subalgebra $\widehat{\mathfrak{su}}(N)_{k+1}$. Note that for $N = 2$ we just recover the Virasoro minimal models. This is a beautiful illustration of the powers of the coset construction, the generalization is extremely natural.

In the limit $k \to \infty$ we have that $c = N-1$ and the symmetry algebra is equivalent to the Casimir algebra of $\widehat{\mathfrak{su}}(N)_1$ (since the other factors can be "divided out"). This is in agreement with the $N = 3$ discussion above. This algebra at level $k = 1$ can actually be constructed purely from free bosons.

The above results play a central role in non-supersymmetric higher-spin holography.

3.2.2 Quantum Drinfeld-Sokolov Reduction

Quantum Drinfeld-Sokolov reduction is the most systematic and general approach to construct $\mathcal{W}$-algebras and their representations. The basic idea is as follows. Starting from an affine Lie algebra $\hat{\mathfrak{g}}$ at level k, we will impose a set of constraints by using a BRST operator approach. The reduced algebra, $\mathcal{W}[\mathfrak{g}, k]$, is a $\mathcal{W}$-algebra associated to $\hat{\mathfrak{g}}_k$ and given from the zeroth BRST cohomology class.

It pains the author that due to lack of time we have to skip the details of this beautiful topic. The reader might want to start from the review [58]. Drinfeld-Sokolov reduction of $\widehat{\mathfrak{sl}}(N)_k$ gives rise to the $\mathcal{W}_N$ algebra and was first done by Feigin and Frenkel [96]. In cohomology calculation (using spectral sequences) is done in a much smarter and simpler way by switching the role of the double complex in the thesis [59].

3.3 $\mathcal{N}=2$ Kazama-Suzuki models and Super $\mathcal{W}$-algebras

We have so far briefly discussed some aspects of $\mathcal{W}$-algebras and $\mathcal{W}_N$ minimal models, which play a central role in non-supersymmetric higher-spin holography. In this thesis we are mainly interested in $\mathcal{N} = 2$ higher-spin holography which is based on the so-called $\mathbb{C}P^N$ Kazama-Suzuki model. The chiral algebra of this CFT is related to Drinfeld-Sokolov reduction of the affine Lie superalgera $A(N, N-1)^{(1)} = \widehat{\mathfrak{sl}}(N+1, N)$, which after BRST gauge fixing leads to a $\mathcal{N} = 2$ $\mathcal{SW}_n$ algebra.

The starting point of Kazama-Suzuki models [97, 98] are $\mathcal{N} = 1$ WZW cosets. One then investigates under which conditions the coset actually has $\mathcal{N} = 2$ superconformal symmetry. Let us for the supercoset $\hat{\mathfrak{g}}^1_k/\hat{\mathfrak{h}}^1_{k'}$ associate the coset G/H, where G and H are the Lie groups corresponding to the finite and bosonic subalgebras of the affine Lie superalgebras $\hat{\mathfrak{g}}^1_k$ and $\hat{\mathfrak{h}}^1_{k'}$, respectively.[7] It turns out that the coset $\hat{\mathfrak{g}}^1_k/\hat{\mathfrak{h}}^1_{k'}$ is $\mathcal{N} = 2$ if

[7] In most parts of the literature known to the author, for example the original Kazama-Suzuki papers [97, 98], the WZW supercosets are exclusively written in terms of the Lie group cosets G/H. This is very confusing, so here we will be slightly more precise.

G/H is a Hermitian symmetric space. This means it has to be a both a Kähler manifold and a Riemannian symmetric space. This implies that this class of unitary $\mathcal{N}=2$ superconformal models has a rich geometrical structure.

3.3.1 The $\mathcal{N}=1$ Supersymmetric Coset Models

The $\mathcal{N}=1$ supersymmetric extension of the WZW was systematically considered in superspace formalism in [99]. Amazingly it turns out that it takes a very simple form in component formalism. One simply has to add two free Weyl fermions $\psi_\pm$ in (a complexification of) the adjoint representation of the group in consideration G, and add the following term to the bosonic WZW action (C.73)

$$S^{\mathrm{WZW}}_{\mathcal{N}=1}[g,\psi] = S^{\mathrm{WZW}}[g] + \frac{i}{4\pi}\int_{S^2} \mathrm{d}^2x\,\left(\psi_+\bar{\partial}\psi_+ + \psi_-\partial\psi_-\right). \tag{3.21}$$

One can obtain cosets and thereby Kazama-Suzuki models by gauging the above action appropriately, see the discussion by Witten [100]. Our starting point, however, will be at the level of current operators.

Let J^A be spin-1 current generating the affine Lie algebra $\hat{\mathfrak{g}}_k$ in an orthogonal hermitian basis[8]

$$J^A(z)J^B(w) \sim \frac{k/2\,\delta_{AB}}{(z-w)^2} + \frac{if_{ABC}}{z-w}J^c(w). \tag{3.22}$$

Now add spin-$\frac{1}{2}$ fermic operators j^a transforming in the adjoint representation of $\mathfrak{g}$

$$j^A(z)j^B(w) \sim \frac{k/2\,\delta_{AB}}{z-w}, \tag{3.23}$$

$$J^A(z)j^B(w) \sim j^A(z)J^B(w) \sim \frac{if_{ABC}}{z-w}j^C(w). \tag{3.24}$$

Note that in this basis f_{ABC} is completely anti-symmetric. Together these form an $\mathcal{N}=1$ affine Lie superalgebra which we will denote by $\hat{\mathfrak{g}}^1_k$. An $\mathcal{N}=1$ superconformal algebra can be constructed in the universal enveloping algebra of $\hat{\mathfrak{g}}^1_k$ by an extension of the Sugawara construction. This is most convenient to write down if we decouple J^A and j^A. This can be done by the redefinitions

$$\hat{J}^A(z) = J^A(z) - J^A_f(z), \qquad \text{where} \qquad J^A_f(z) = -\frac{i}{k}f_{ABC}\mathcal{N}\big(j^Bj^C\big)(z). \tag{3.25}$$

One can directly show that $\hat{J}^A$ and j^A are independent since $\hat{J}^a(z)j^b(w)\sim 0$. Furthermore $\hat{J}^A$ and J^A_f generate two separate affine Lie algebras with the levels $\hat{k}=k-g^\vee$ and $k_f=g^\vee$,respectively. Here $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and is equal to the second-order Casimir of the adjoint representation $f_{ACD}f_{BCD}=\mathcal{C}_2(\theta)\delta_{AB}$.[9] The supersymmetric Sugawara currents can be written as

$$\begin{aligned} T_{\mathfrak{g}}(z) &= \frac{1}{k}\left[\mathcal{N}\left(\hat{J}^A\hat{J}^A\right)(z) - \mathcal{N}\left(j^A\partial j^A\right)(z)\right], \\ G_{\mathfrak{g}}(z) &= \frac{2}{k}\left[\mathcal{N}\left(j^A\hat{J}^A\right)(z) - \frac{i}{3k}f_{ABC}\,\mathcal{N}\left(j^Aj^Bj^C\right)(z)\right]. \end{aligned} \tag{3.26}$$

[8] We are using the conventions of [97].

[9] Note that there is a factor of two difference from the discussion in section D.2.3, due to differing conventions.

It can be shown that these two operators satisfy the OPE's (C.17) and (3.9), with the central charge

$$c_{\mathfrak{g}} = \frac{1}{2}\dim\mathfrak{g} + \frac{\hat{k}\dim\mathfrak{g}}{\hat{k}+g^\vee} = \frac{1}{2}\dim\mathfrak{g} + \frac{(k-g^\vee)\dim\mathfrak{g}}{k} = \dim\mathfrak{g}\left(\frac{1}{2}+\frac{k-g^\vee}{k}\right). \tag{3.27}$$

Having covered the $\mathcal{N}=1$ WZW model, we now turn to the coset construction. Let $\mathfrak{h}$ be a semi-simple Lie subalgbra of $\mathfrak{g}$ with the corresponding Lie subgroup H of G. We will use the indices $(a,b,\dots)$ and $(\bar{a},\bar{b},\dots)$ for the generators of $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$, respectively. The capital letters $(A,B,\dots)$ will still denote generators of $\mathfrak{g}$.

Restricting J^A to the subalgbra $\mathfrak{h}$, the decomposition (3.25) becomes

$$J^a = \hat{J}^a - \frac{i}{k} f_{aBC}\mathcal{N}\big(j^Bj^C\big)(z) = \tilde{J}^a - \frac{i}{k} f_{abc}\mathcal{N}\big(j^bj^c\big)(z), \tag{3.28}$$

where the last equation yields the appropriate decomposition for $\mathfrak{g}$ and $\tilde{J}^a = \hat{J}^a - \frac{i}{k} f_{a\bar{b}\bar{c}}\mathcal{N}\big(j^{\bar{b}}j^{\bar{c}}\big)(z)$. We can now, from equation (3.26), find $T_{\mathfrak{h}}$ and $G_{\mathfrak{h}}$ by replacing $\hat{J}^A \to \tilde{J}^a$ and $j^A \to j^a$. The current $\tilde{J}^a$ generates an affine Lie algebra with level $\tilde{k} = k - h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{h}$. We can now define the $\mathcal{N}=1$ superalgebra of the coset in the usual way

$$G_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}} = G_{\mathfrak{g}} - G_{\mathfrak{h}}, \qquad \text{and} \qquad T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}} = T_{\mathfrak{g}} - T_{\mathfrak{h}}. \tag{3.29}$$

One can by direct calculation show that $T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}}$ and $G_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}}$ decouple from j^a, J^a, $\tilde{J}^a$, $T_{\mathfrak{h}}$ and $G_{\mathfrak{h}}$. The central charge of the cosets are

$$c_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}} = c_{\mathfrak{g}} - c_{\mathfrak{h}}. \tag{3.30}$$

3.3.2 $\mathcal{N}=2$ Superconformal Symmetry and Kazama-Suzuki Models

Having given a lightning review of the $\mathcal{N}=1$ supersymmetric coset models, we turn to the Kazama-Suzuki models. As we discussed in section 3.1.4, in order to obtain $\mathcal{N}=2$ superconformal symmetry, we need to add another spin-$\frac{3}{2}$ generator together with a $U(1)$ spin-1 R-symmetry current. The idea of Kazama and Suzuki was to write down the most general spin-1 and -$\frac{3}{2}$ generators and demand that the full operator algebra has $\mathcal{N}=2$ superconformal symmetry.

It turns out that the basis (3.12) is not the most convenient for this purpose. We will instead use the following set of superconformal generators

$$G^0(z) \equiv \frac{1}{\sqrt{2}}\Big[G_+(z)+G_-(z)\Big], \qquad G^1(z) \equiv \frac{1}{\sqrt{2}\,i}\Big[G_+(z)-G_-(z)\Big], \tag{3.31}$$

which have the following OPE

$$G^i(z)G^j(w) \sim \frac{2c/3\,\delta^{ij}}{(z-w)^3} + \frac{2\,J^{ij}(w)}{(z-w)^2} + \frac{2T(w)\,\delta^{ij} + \partial J^{ij}}{z-w}, \tag{3.32}$$

where $J^{ij} = i\,j(z)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ is the R-symmetry current in $SO(2)$ form. We will set $G^0 \equiv G_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{k'}}$. The most general spin-$\frac{3}{2}$ generator one can construct out of the coset fields $j^{\bar{a}}$ and $\hat{J}^{\bar{a}}$ is

$$G^1(z) = \frac{2}{k}\left[h_{\bar{a}\bar{b}}\mathcal{N}\left(j^{\bar{a}}\hat{J}^{\bar{b}}\right)(z) - \frac{i}{3k}S_{\bar{a}\bar{b}\bar{c}}\mathcal{N}\left(j^{\bar{a}}j^{\bar{b}}j^{\bar{c}}\right)(z)\right], \tag{3.33}$$

where $h_{\bar{a}\bar{b}}$ is symmetric while $S_{\bar{a}\bar{b}\bar{c}}$ is completely anti-symmetric. Note that it reduces to G^0 for $h_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}}$ and $S_{\bar{a}\bar{b}\bar{c}} = f_{\bar{a}\bar{b}\bar{c}}$. Demanding that the these operators satisfy the OPE's of $\mathcal{N}=2$ superconformal symmetry, one finds the following constraints

$$\begin{aligned} h_{\bar{a}\bar{b}} &= -h_{\bar{b}\bar{a}}, \qquad h_{\bar{a}\bar{p}}h_{\bar{p}\bar{b}} = -\delta_{\bar{a}\bar{b}}, \\ h_{\bar{a}\bar{d}}f_{\bar{d}\bar{b}e} &= f_{\bar{a}\bar{d}e}h_{\bar{d}\bar{b}}, \\ f_{\bar{a}\bar{b}\bar{c}} &= h_{\bar{a}\bar{p}}h_{\bar{b}\bar{q}}f_{\bar{p}\bar{q}\bar{c}} + h_{\bar{b}\bar{p}}h_{\bar{c}\bar{q}}f_{\bar{p}\bar{q}\bar{a}} + h_{\bar{c}\bar{p}}h_{\bar{a}\bar{q}}f_{\bar{p}\bar{q}\bar{b}}, \\ S_{\bar{a}\bar{b}\bar{c}} &= h_{\bar{a}\bar{p}}h_{\bar{b}\bar{q}}h_{\bar{c}\bar{r}}f_{\bar{p}\bar{q}\bar{r}}. \end{aligned} \tag{3.34}$$

These equations simply constrains the geometry of the coset space G/H of the corresponding Lie groups. For example the equation $h^2 = -\mathbb{1}$, where $(h)_{\bar{a}\bar{b}} = h_{\bar{a}\bar{b}}$, simply states that G/H must have an almost complex structure and hence there is a notion of holomorphic/anti-holomorphic vector fields. The second line implies that the almost complex structure is H invariant. The third line is a consistency condition, while the last equation fixes $S_{\bar{a}\bar{b}\bar{c}}$. For a detailed geometric analysis of these constraints see [98]. Among other things, it is found that when $\operatorname{rank} G = \operatorname{rank} H$ then these spaces are precisely Kähler manifolds.

Schweigert [101] has shown that the above Kazama-Suzuki models completely classify all $\mathcal{N}=2$ superconformal coset models.

3.3.3 Classification of Hermitian Symmetric Spaces

For our purposes, it is enough to restrict attention to the subset of $\mathcal{N}=2$ Kazama-Suzuki models solved by setting

$$f_{\bar{a}\bar{b}\bar{c}} = S_{\bar{a}\bar{b}\bar{c}} = 0, \tag{3.35}$$

which implies that the cosets G/H are a special kind of Kähler manifolds, called Hermitian symmetric spaces. Being a symmetric space, locally, means that we have the following decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with the properties

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}. \tag{3.36}$$

Here $\mathfrak{m}$ is the part corresponding to the indices $(\bar{a},\bar{b},\dots)$. These symmetric spaces can be classified by classifying involutive automorphisms $s : \mathfrak{g} \to \mathfrak{g}$, $s^2 = 1$, which was done by Cartan.[10] One can show that due to the almost complex structure $h_{\bar{a}\bar{b}}$, then one can decompose $m = m_+ \oplus m_-$ such that $m_\pm$ are closed subalgebras individually [98].

The Hermitian symmetric spaces are just a subset of symmetric spaces and there classification can thus be obtain from it. Due to lack of time we will only mention the fact that complex Grassmann manifolds

$$\mathbb{C}G(m,n) = \frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}, \tag{3.37}$$

are among these manifolds.

[10] Symmetric spaces have important applications in random matrix theory, where they are random matrix ensembles corresponding to discrete symmetries s.

3.3.4 Complex Grassmannians, Primary States and Level-Rank Duality

The supersymmetric cosets can be written in terms of ordinary cosets. For example the coset corresponding to the complex Grassmann manifold (3.37) is given by

$$G(m,n,k) = \frac{\mathfrak{su}(m+n)_k \oplus \mathfrak{so}(2mn)_1}{\mathfrak{su}(m)_{n+k} \oplus \mathfrak{su}(n)_{m+k} \oplus \mathfrak{u}(1)_{mn(m+n)(m+n+k)}}, \tag{3.38}$$

with the central charge

$$c = \frac{3mnk}{m+n+k}. \tag{3.39}$$

The $\mathfrak{so}(2mn)_1$ factor arises due to the adjoint fermions of the affine Lie superalgebra, see section 3.1.1. The upshot of using ordinary cosets is that we can use the techniques discussed in appendices C, D and E to analyze the details of the model. For example the spectrum of primary fields (selection rules, field identifications, fix-point resolution), modular properties, fusion rules and so on. Most of the relevant data is quite straightforward to extract, similar to the examples given in the appendices, but due to time constraints the reader is referred to [102] and [103] for details. See also [51] (published version) for more precise details on how the different factors are embedded in the coset.

We will just make a quick comment. Note that besides the trivial permutation $m \leftrightarrow n$, the central charge is also invariant under $m, n \leftrightarrow k$. It turns out that this can be extended to a full level-rank duality

$$G(m,n,k) \approx G(m,k,n) \approx G(k,m,n). \tag{3.40}$$

For more details about this see [102]. The model used in the supersymmetric higher-spin duality is based on the $\mathbb{C}P^N = \mathbb{C}G(N,1)$ manifold , which is given by the coset $G(N,1,k)$.

3.3.5 Drinfeld-Sokolov Reduction of $A(N,N-1)$ and the $\mathbb{C}P^N$ Models

In a series of beautiful papers [104, 105, 106], Ito has applied the Drinfeld-Sokolov reduction to the affine Lie superalgebra $A(N,N-1)^{(1)} = \widehat{\mathfrak{sl}}(N+1,N)$ and found that after the BRST gauge-fixing, the chiral algebra of these models is the $\mathcal{SW}_N$ algebra. The $\widehat{\mathfrak{sl}}(N+1,N)_{k_{DS}}$ WZW theory turns out to be a topological CFT with central charge $c=0$, but after DS-reduction the central charge is non-zero

$$c = \frac{3kN}{N+k+1}. \tag{3.41}$$

Here the coset level k and the level stemming form DS-reduction k_{DS} are related to each other by

$$k_{DS} = -1 + \frac{1}{1+k+N}. \tag{3.42}$$

This is nothing but the central charge of the $\mathbb{C}P^N$ Kazama-Suzuki models. This means that the $\mathbb{C}P^N$ Kazama-Suzuki models are CFT's with $\mathcal{N}=2$ $\mathcal{W}$-algebras. This is exactly why they will play an important role in $\mathcal{N}=2$ higher-spin holography. Again due to time constraints we cannot go through the details involved, the reader is referred to [104, 105, 106] for more details.

Chapter **4**

Higher-Spin $\mathrm{AdS}_3/\mathrm{CFT}_2$ Conjectures

In chapter 2 we discussed in generality how to construct interacting theories on AdS_3 of massless fields with spin $s \geq 2$, respecting the enhancement of diffeomorphism by higher-spin gauge symmetries. This can be done by a $\mathfrak{g}_{k_{CS}} \times \mathfrak{g}_{-k_{CS}}$ Chern-Simons theory together with a choice of embedding $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$ which specifies the gravitational sector. Different embeddings give rise to different boundary dynamics due to the asymptotic AdS_3 fall-off conditions (2.72) and to a different spectrum of massless fields given by the $\mathfrak{sl}(2,\mathbb{R})$ adjoint-action decomposition (2.12). The asymptotic symmetries of such a theory translate into classical Drinfeld-Sokolov reduction of $\mathfrak{g}$ wrt. to the given $\mathfrak{sl}(2,\mathbb{R})$ embedding (and similarly for the other chiral sector). These generically lead to $\mathcal{W}$-algebras for higher-spin theories. For example when $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ the boundary chiral algebra is $\mathcal{W}_3$ for the principal embedding (see quantum version in equation (3.14)), or the Polyakov-Bershadsky algebra $\mathcal{W}_3^{(2)}$ for a non-principal embedding.[1]

Although in 2+1-dimensions it is possible to construct interacting higher-spin theories on AdS_3 with a finite number of higher-spin fields, it is not known how to consistently couple these theories to massive matter fields. As we discussed, there is a one-parameter family of (Vasiliev) theories which is able to achieve this at the cost of having an infinite tower of higher-spin fields. This in turn completely constraints the theory where even the masses of the matter fields are fixed by higher-spin symmetries.

[1]There might be a potential confusion with this notation. In section 3.1.6 we showed that $\mathcal{W}_3$ is the unique algebra containing only the energy-momentum tensor and a spin-3 field, so what do we mean by $\mathcal{W}_3^{(2)}$? This notation comes from the Drinfeld-Sokolov reduction approach to $\mathcal{W}$-algebras. The DS-reduction of $\mathfrak{sl}(N,\mathbb{R})$ is called $\mathcal{W}_N^{(n)}$, where n labels the different types of embeddings $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$. For the principal embedding (where the notation $\mathcal{W}_N$ is used), the algebra contains all integer spin currents $s = 2, \ldots N$ which can be seen from the $\mathfrak{sl}(2,\mathbb{R})$ decomposition. The $\mathcal{W}_3^{(2)}$ turn out to contain a spin-2, two (bosonic) spin-$\frac{3}{2}$ and a $U(1)$ spin-1 currents, but no spin-3 fields. So $\mathcal{W}_3^{(2)}$ is actually more like a non-linear bosonic version of the $\mathcal{N} = 2$ superconformal algebra, than like the $\mathcal{W}_3$ algebra. See for example [73].

4.1 The Gaberdiel-Gopakumar Conjecture

In [107] quadratic fluctuations of higher-spin fields around the thermal AdS_3 vacuum[2] was calculated[3] and it was shown that the partition function can be written in terms of the modular parameter of thermal AdS_3 boundary.

$$Z_{\text{bulk}} = (q\bar{q})^{-c/24} Z_{\text{hs}} Z_{\text{scal}}(h_+)^2 Z_{\text{scal}}(h_-)^2, \tag{4.1}$$

where

$$Z_{\text{hs}} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \qquad \text{and} \qquad Z_{\text{scal}} = \prod_{j,j'=0}^{\infty} \frac{1}{1-q^{h+j}\bar{q}^{h+j'}}. \tag{4.2}$$

Here $(q\bar{q})^{-c/24}$ is the contribution of the AdS_3 background. $Z_{\text{scal}}(h_\pm)$ is the contribution of the two scalars with $h_\pm = \frac{1}{2}(1\pm\lambda)$ (see below), its squared since the scalars are complex. Remarkably, the contributions of different spin-s fields organize themselves into vacuum characters of $\mathcal{W}_N$. This implies that the $\mathcal{W}$-symmetry is (at least perturbatively) realized in the quantum theory of the bulk.

Inspired by these facts, Gaberdiel and Gopakumar proposed the following conjecture. The bosonic truncation of Vasiliev theory, in which the massless sector is based on hs[λ] Chern-Simons theory, coupled to two complex scalars is dual to the following coset CFT

$$\frac{\widehat{\mathfrak{su}}(N)_k \oplus \widehat{\mathfrak{su}}(N)_1}{\widehat{\mathfrak{su}}(N)_{k+1}}, \tag{4.3}$$

in the 't Hooft limit defined as

$$N, k \to \infty : \qquad 0 \le \lambda = \frac{N}{k+N} \le 1 \qquad \text{fixed.} \tag{4.4}$$

As discussed earlier, this coset corresponds to the $\mathcal{W}_N$ minimal models. From the formula (3.20) we see that in the 't Hooft limit there is a continuous one-parameter family of CFT's with central charge $c = N(1-\lambda^2)$. Note that this behavior is different than gauge theories where the number of degrees of freedom go as N^2 [109], this is one indication of why this duality is simpler than the original Maldacena conjecture.

The scalars of Vasiliev theory both have the mass[4]

$$M^2 = -(1-\lambda^2). \tag{4.5}$$

The limit on the range of λ (4.4) implies that the scalar masses squared lie in the window $-1 \le M^2 \le 0$. It is well-known that in this range there are two possible ways of quantizing scalars [110], from $M^2 = \Delta(\Delta-2)$ we find the two possibilities

$$\Delta = 1 \pm \lambda. \tag{4.6}$$

[2]Thermal AdS_3 has a compactified time dimension and therefore the topology of a solid torus. Its boundary is just the torus surface and its complex structure is parametrized by the modular parameter τ, as discussed in section C.1.5.

[3]See also the famous paper by Gibbons and Hawking [108].

[4]Note that these scalars are not tachyonic even though they have negative mass-squared. On AdS_3 particles must transform under irreducible representations of AdS_3 isometry group, rather than the Poincare group. It turns out that due to the negative curvature of AdS_3, there is a (negative) lower bound in which scalars are stable.

In order to match the spectrum in the 't Hooft limit, Gaberdiel and Gopakumar chose opposite quantizations for the two scalars.

Although we, by far, spend the majority of our time studying this duality, we eventually ended up working on the newer $\mathcal{N}=2$ version discussed below. For this reason we will not go into details, since even the most basic aspects would require too much space and time, but largely be irrelevant to understand our contribution [1]. There are also already a huge amount of very interesting results and subtle refinements of the conjecture, which by itself would require a thesis to review appropriately. For this reason we will directly go to the $\mathcal{N}=2$ conjecture.

4.2 The Creutzig-Hikida-Rønne Conjecture

The next natural step is to consider the untruncated $\mathcal{N}=2$ Vasiliev theory of section 2.5, of which the asymptotic symmetry algebra is the Drinfeld-Sokolov reduction of shs$[\lambda]$, also called $\mathcal{SW}_\infty[\lambda]$ [53]. The question is what should the dual theory be.

Recall that the $\mathcal{W}_N$ algebra follows from DS-reduction of $\widehat{\mathfrak{sl}}(N)$, the minimal models of which is the key element of Gaberdiel-Gopakumar conjecture. The natural $\mathcal{N}=2$ supersymmetric extension is to consider DS-reduction of $\widehat{sl}(N+1,N)$. As discussed in section 3.3.5, Ito has done this analysis and found the $\mathcal{SW}_N$ algebra and that the minimal models of this algebra is just given by the $\mathbb{C}P^N$ Kazama-Suzuki model.

Inspired by the non-supersymmetric case and the results above, Creutzig, Hikida and Rønne conjectured the following. The full Vasiliev theory of section 2.5 is dual to the $\mathbb{C}P^N$ Kazama-Suzuki models given by the coset

$$\frac{\widehat{\mathfrak{su}}(N+1)_k \times \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{su}}(N)_{k+1} \times \widehat{\mathfrak{u}}(1)_{N(N+1)(k+N+1)}}, \tag{4.7}$$

with the identification $\lambda = \frac{N}{2(N+k)}$, in the 't Hooft limit

$$0 \leq \lim_{N,k\to\infty} \lambda \leq \frac{1}{2} \quad \text{fixed.} \tag{4.8}$$

The restriction of the range of the parameter $0 \leq \lambda \leq \frac{1}{2}$ again leads to scalar masses with $-1 \leq (M^B)^2 \leq 0$ and for this mass range one can choose two different boundary conditions, with the "usual" quantization being the one with the largest value of the conformal dimension. From the usual AdS/CFT dictionary we have the following relation between masses and conformal weights of dual fields

$$(M^B)^2 = \Delta(\Delta-2), \qquad (M^F)^2 = (\Delta-1)^2, \tag{4.9}$$

for massive scalars and spin 1/2 fermions, respectively. The dual conformal weights are then given by [51]

$$\left(\Delta^B_+,\, \Delta^F_\pm,\, \Delta^B_-\right) = \left(2-2\lambda,\, \tfrac{3}{2}-2\lambda,\, 1-2\lambda\right), \qquad \left(\tilde{\Delta}^B_+,\, \tilde{\Delta}^F_\pm,\, \tilde{\Delta}^B_-\right) = \left(2\lambda,\, \tfrac{1}{2}+2\lambda,\, 1+2\lambda\right). \tag{4.10}$$

The bosonic operators in the first multiplet correspond to the ϕ_+ scalar with the usual quantization and the ϕ_- scalar with the alternative quantization, while the quantizations are opposite in the second multiplet.

Let $(\rho, s; \nu, m)$ label the states of the coset (4.7) up to field identifications due to outer automorphisms of the different factors in the coset. Here ρ and ν are highest weights of $\mathfrak{su}(N+1)$ and $\mathfrak{su}(N)$, respectively, while $m \in \mathbb{Z}_{N(N+1)(k+N+1)}$. In the NS sector we have $s = 0, 2$. In [51], it was proposed that the following holomorphic coset primary fields with chiral conformal weights

$$\begin{aligned} h(f,0;0,N) &= \lambda, & h(0,2;f,-N-1) &= \frac{1}{2} - \lambda, \\ h(f,2;0,N) &= \lambda + \frac{1}{2}, & h(0,0;f,-N-1) &= 1 - \lambda, \end{aligned} \tag{4.11}$$

where f is the fundamental representation, can be used to construct the dual fields (4.10) by gluing holomorphic and anti-holomorphic states as follows

$$\begin{aligned} \mathcal{O}^{\mathcal{B}}_{\Delta_+} &= (0,0;f,-N-1) \otimes (0,0;f,-N-1), & \mathcal{O}^{\mathcal{F}}_{\Delta_+} &= (0,2;f,-N-1) \otimes (0,0;f,-N-1), \\ \mathcal{O}^{\mathcal{B}}_{\Delta_-} &= (0,2;f,-N-1) \otimes (0,2;f,-N-1), & \mathcal{O}^{\mathcal{F}}_{\Delta_-} &= (0,0;f,-N-1) \otimes (0,2;f,-N-1), \end{aligned} \tag{4.12}$$

and for the other multiplet

$$\begin{aligned} \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} &= (f,0;0,N) \otimes (f,0;0,N), & \tilde{\mathcal{O}}^{\mathcal{F}}_{\Delta_+}(z,\bar{z}) &= (f,0;0,N) \otimes (f,2;0,N), \\ \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-} &= (f,2;0,N) \otimes (f,2;0,N), & \tilde{\mathcal{O}}^{\mathcal{F}}_{\Delta_-}(z,\bar{z}) &= (f,2;0,N) \otimes (f,0;0,N). \end{aligned} \tag{4.13}$$

In the 't Hooft limit, the correlation functions we will be considering only depend on the higher-spin algebra shs[λ]. Thus, in section 5.4 we will generate the corresponding highest-weight representations using a free-field CFT having shs[λ] as a subalgebra. Our highest-weight representations will then be constructed in terms of free fields such that they match the above coset primary fields.

Chapter **5**

Three-Point Functions

In this chapter our main task is to calculate three-point functions containing two bulk scalars and one bosonic higher-spin current, for any spin s, both from the bulk and boundary. There are however many difficulties which make the calculation quite difficult. For example in the original formulation of Vasiliev theory one is constantly forced to rewrite products of deformed oscillators in terms of their symmetrizations, which is a very tedious task especially since we are interested in doing the calculations for arbitrary spin. Even if we were able to derive the Klein-Gordon equations in the background of higher-spin fields, we would have to derive bulk-to-boundary propagators for arbitrary higher-spin deformation of AdS_3. It is however possible to simplify the calculations considerably by making use of a few tricks. Let us sketch our strategy.

The first problem is the manipulations of deformed oscillators. We have already discussed this issue in section 2.5.3. The idea was to take advantage of the isomorphism between the infinite dimensional associative superalgebras $Aq(2,\nu)$ and $\mathcal{SB}[\mu]$ by using the latter instead of the former to formulate the theory. Since we have explicit expressions for the structure constants of $\mathcal{SB}[\mu]$, see appendix B, this will prove to be an enormous simplification of the original Vasiliev formalism. In the following section we will show how the Klein-Gordon equations on AdS_3 with the correct masses are derived in our formalism.

Next step is to generalize the boundary conditions of the higher-spin fields and establish precisely the holographic dictionary. Recall that near boundary expansions of fields (suppressing internal indices) are of the form [111]

$$\mathcal{F}(x,r) = r^m \Big(f_{(0)}(x) + f_{(1)}(x)\, r + \cdots + r^n \left[f_{(n)}(x) + \log r\, \tilde{f}_{(n)}(x) \right] + \dots \Big), \qquad (5.1)$$

where the values of m and n are determined by the equations of motion and in these coordinates $r \to 0$ is the boundary. This correspond to two linearly independent solutions, one with a near boundary behavior as r^m and the other as r^{m+n}. The most dominant term near the boundary $f_{(0)}(x)$ can be thought of as the Dirichlet boundary condition and correspond to the source term of the dual operator on the boundary. The equations of motion can be used to iteratively solve $f_{(k)}(x)$, $k < n$, as local functions of $f_{(0)}(x)$.

The function $f_{(n)}(x)$ can be thought of as the Dirichlet boundary condition for the linearly independent solution [111] and is a non-local function of $f_{(0)}(x)$, it is actually proportional to the one-point function of the dual field in the presence of source terms.[1]

[1] The term $\tilde{f}_{(n)}(x)$ is related to conformal anomalies but will not be relevant for our discussion.

It turns out that the bulk constraint equations give rise to Ward identities [112, 113] of these one-point functions. Our strategy is to use these Ward identities to identify which terms in the Chern-Simons gauge connection correspond to the source term of which higher-spin field, with the correct normalization. This will fix the holographic dictionary needed for our calculation.

Finally in order to calculate the three-point functions from the bulk, our strategy is to look at one-point functions of the dual operator in the presence of higher-spin source terms. For this an insight first discussed in [84] will be crucial, starting from solutions on AdS_3 one can generate new solutions with higher-spin deformation by gauge transformations. Thus we will calculate one-point functions of the scalars on AdS_3 using the usual bulk-to-boundary propagator, then use a gauge transformation to include higher-spin sources. This will prove to be an efficient way to derive general formulas for three-point functions.

5.1 Scalars From Modified Vasiliev Formalism

We will start with considering matter coupled to higher-spin fields using our modified Vasiliev formalism. The Vasiliev equations for the higher-spin fields reduce to

$$\begin{aligned} \mathrm{d}A + A \wedge \star A &= 0, \\ \mathrm{d}\tilde{A} + \tilde{A} \wedge \star \tilde{A} &= 0, \end{aligned} \tag{5.2}$$

while for the matter fields, linearized around its vacuum, we have

$$\begin{aligned} \mathrm{d}C + A \star C - C \star \bar{A} &= 0, \\ \mathrm{d}\tilde{C} + \bar{A} \star \tilde{C} - \tilde{C} \star A &= 0. \end{aligned} \tag{5.3}$$

Using our formalism, the gauge and matter fields are given by

$$\begin{aligned} A &= \sum_{s=1}^{\infty}\sum_{|m|\le s-1} A^s_m\, L^{(s)}_m + \sum_{s=\frac{3}{2}}^{\infty}\sum_{|r|\le s-\frac{3}{2}} A^s_r\, G^{(s)}_r, \\ C &= \sum_{s=1}^{\infty}\sum_{|m|\le s-1} C^s_m\, L^{(s)}_m + \sum_{s=\frac{3}{2}}^{\infty}\sum_{|r|\le s-\frac{3}{2}} C^s_r\, G^{(s)}_r, \end{aligned} \tag{5.4}$$

and similarly for $\tilde{C}$ and $\bar{A}$. Note that we can easily distinguish the bosonic components C^s_m from the fermionic ones C^s_r, since m is always an integer while r is half of an odd integer. Recall that we use the following notation

$$L^{(s)}_m = L^{(s)+}_m, \quad L^{(s+1/2)}_m = L^{(s)-}_m, \quad G^{(s)}_r = G^{(s)+}_r \quad \text{and} \quad G^{(s-1/2)}_r = G^{(s)-}_r$$

for $s \in \mathbb{Z}$. Furthermore $L^{(1)}_0 = \mathbb{1}$ is the identity element of the associative algebra $\mathcal{SB}[\mu] = \mathrm{shs}[\lambda] \oplus \mathbb{C}$, while $L^{\frac{3}{2}}_0$ corresponds to the $U(1)$ R-symmetry of the higher-spin (Lie) algebra $\mathrm{shs}[\lambda]$. The true matter fields correspond to superpositions of the lowest components C^1_0, $C^{\frac{3}{2}}_0$, $\{C^{\frac{3}{2}}_{+\frac{1}{2}}, C^{\frac{3}{2}}_{-\frac{1}{2}}\}$ and $\{C^2_{+\frac{1}{2}}, C^2_{-\frac{1}{2}}\}$. The rest of the tower of fields in C are auxiliary fields, which can be written as sums and derivatives of the physical fields.

Let us consider AdS_3, which is given by the connection

$$\begin{aligned} A &= e^\rho\, L^{(2)}_1\, \mathrm{d}z + L^{(2)}_0\, \mathrm{d}\rho \\ \bar{A} &= e^\rho\, L^{(2)}_{-1}\, \mathrm{d}\bar{z} - L^{(2)}_0\, \mathrm{d}\rho \end{aligned} \qquad \Rightarrow \qquad \mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2\rho}\, \mathrm{d}z\mathrm{d}\bar{z}, \tag{5.5}$$

where we have used $g_{\mu\nu} = \frac{1}{2}\mathrm{tr}(e_\mu e_\nu)$, $e = \frac{1}{2}\left(A - \bar{A}\right)$. Turning on other modes, such that (5.2) and appropriate boundary conditions are satisfied, correspond to higher-spin deformations of AdS$_3$. We will for now only consider the scalar fields propagating on AdS$_3$, so we will set $C^s_r = 0$. Plugging (5.4) into Vasiliev equation (5.3) we find,

$$\sum_{s=1}^{\infty}{}^{\circ}\sum_{|m|\leq s-1}\Bigg(\mathrm{d}C^s_m\, L^{(2)}_m + e^\rho\, C^s_m\, L^{(2)}_1 \star L^{(s)}_m\,\mathrm{d}z - e^\rho\, C^s_m\, L^{(s)}_m \star L^{(2)}_{-1}\,\mathrm{d}\bar{z} + C^s_m\left\{L^{(2)}_0 \star L^{(s)}_m + L^{(s)}_m \star L^{(2)}_0\right\}\mathrm{d}\rho\Bigg) = 0. \tag{5.6}$$

The coefficients of linearly independent terms should be set to zero individually, which leads to the following set of equations

$$\partial_\rho C^s_m + \sum_{u=1}^{3}{}^{\circ}\chi_{[-(s+u-3),s+u-3]}(m)C^{s+u-2}_m\left[g^{2(s+u-2)}_u(0,m) + g^{(s+u-2)2}_u(m,0)\right] = 0, \tag{5.7}$$

$$\partial C^s_m + e^\rho\sum_{u=1}^{3}{}^{\circ}\chi_{[-(s+u-4),s+u-2]}(m)C^{s+u-2}_{m-1}g^{2(s+u-2)}_u(1,m-1) = 0, \tag{5.8}$$

$$\bar{\partial}\, C^s_m - e^\rho\sum_{u=1}^{3}{}^{\circ}\chi_{[-(s+u-2),s+u-4]}(m)C^{s+u-2}_{m+1}g^{(s+u-2)2}_u(m+1,-1) = 0, \tag{5.9}$$

which are the coefficients of $L^{(s)}_m\,\mathrm{d}\rho$, $L^{(s)}_m\,\mathrm{d}z$ and $L^{(s)}_m\,\mathrm{d}\bar{z}$ respectively, and the step function is given by

$$\chi_{\mathcal{A}}(m) = \begin{cases} 1, & m \in \mathcal{A}, \\ 0, & m \notin \mathcal{A}. \end{cases} \tag{5.10}$$

The functions $\chi_{\mathcal{A}}(m)$ make sure only generators inside the wedge $|m| \leq s-1$ contribute. For later convenience we will use certain properties of the structure constants given in appendix B, to write these equations as

$$\begin{aligned}
&\partial_\rho C^s_m + 2\left[C^{s-1}_m + C^{s+1}_m\, g^{s+1,2}_3(m,0) + C^{s-\frac{1}{2}}_m\, g^{s-\frac{1}{2},2}_{\frac{3}{2}}(m,0) + C^{s+\frac{1}{2}}_m\, g^{s+\frac{1}{2},2}_{\frac{5}{2}}(m,0)\right] = 0,\\
&\partial C^s_m + e^\rho\Big[C^{s-1}_{m-1} + g^{2,s}_2(1,m-1)C^s_{m-1} + g^{2,s+1}_3(1,m-1)C^{s+1}_{m-1}\\
&\qquad\qquad + g^{2,s-\frac{1}{2}}_{\frac{3}{2}}(1,m-1)C^{s-\frac{1}{2}}_{m-1} + g^{2,s+\frac{1}{2}}_{\frac{5}{2}}(1,m-1)C^{s+\frac{1}{2}}_{m-1}\Big] = 0,\\
&\bar{\partial} C^s_m - e^\rho\Big[C^{s-1}_{m+1} + g^{s,2}_2(m+1,-1)C^s_{m+1} + g^{s+1,2}_3(m+1,-1)C^{s+1}_{m+1}\\
&\qquad\qquad + g^{s-\frac{1}{2},2}_{\frac{3}{2}}(m+1,-1)C^{s-\frac{1}{2}}_{m+1} + g^{s+\frac{1}{2},2}_{\frac{5}{3}}(m+1,-1)C^{s+\frac{1}{2}}_{m+1}\Big] = 0.
\end{aligned} \tag{5.11}$$

We have removed the step functions $\chi_{\mathcal{A}}(m)$ since one can show that they do not play any role as long as we define $C^s_m = 0$ for modes outside of the wedge $|m| > s-1$. These equations can be solved recursively in order to express the auxiliary fields in terms of C^1_0 and $C^{\frac{3}{2}}_0$, and find the equations of motion of these scalars. By a careful analysis, we find the following minimal set of equations needed

$$
\begin{aligned}
L^{(1)}_{0,\rho}: &\quad \partial_\rho C^1_0 + \lambda(2\lambda-1)C^2_0 = 0,\\
L^{(\frac{3}{2})}_{0,\rho}: &\quad \partial_\rho C^{\frac{3}{2}}_0 + \frac{1}{9}(2\lambda^2-\lambda-1)C^{\frac{5}{2}}_0 + \frac{1}{6}(4\lambda-1)C^2_0 = 0,\\
L^{(2)}_{0,\rho}: &\quad \partial_\rho C^2_0 + 2C^1_0 + \frac{2}{3}(1-4\lambda)C^{\frac{3}{2}}_0 + \frac{4}{9}(2\lambda^2-\lambda-1)C^3_0 = 0,\\
L^{(\frac{5}{2})}_{0,\rho}: &\quad \partial_\rho C^{\frac{5}{2}}_0 + 2C^{\frac{3}{2}}_0 + \frac{2}{15}(4\lambda-1)C^3_0 + \frac{4}{25}(2\lambda^2-\lambda-3)C^{\frac{7}{2}}_0 = 0,\\
L^{(1)}_{0,\bar{z}}: &\quad \bar{\partial}\, C^1_0 - e^\rho(1-2\lambda)\lambda C^2_1 = 0,\\
L^{(\frac{3}{2})}_{0,\bar{z}}: &\quad \bar{\partial}\, C^{\frac{3}{2}}_0 - e^\rho\left[\frac{1}{6}(1-4\lambda)C^2_1 - \frac{1}{9}(1+\lambda-2\lambda^2)C^{\frac{5}{2}}_0\right] = 0,\\
L^{(2)}_{1,z}: &\quad \partial C^2_1 + e^\rho\left[C^1_0 + \frac{1}{2}C^2_0 + \frac{1}{9}(1+\lambda-2\lambda^2)C^3_0 + \frac{1}{3}(1-4\lambda)C^{\frac{3}{2}}_0\right] = 0,\\
L^{(\frac{5}{2})}_{1,z}: &\quad \partial C^{\frac{5}{2}}_1 + e^\rho\left[C^{\frac{3}{2}}_0 + \frac{1}{2}C^{\frac{5}{2}}_0 + \frac{1}{25}(3+\lambda-2\lambda^2)C^{\frac{7}{2}}_0 + \frac{1}{30}(1-4\lambda)C^3_0\right] = 0.
\end{aligned}
$$

Solving these recursively we can eliminate all the auxiliary fields and reduce to two coupled equations

$$
\begin{aligned}
\Box C^1_0 + 6\lambda\,(1-2\lambda)\,C^1_0 + 2\lambda\,(1-6\lambda+8\lambda^2)\,C^{3/2}_0 &= 0,\\
\Box C^{3/2}_0 - \frac{1-4\lambda}{6\lambda(1-2\lambda)}\Box C^1_0 + \frac{2}{3}(1+\lambda-2\lambda^2)\,C^{3/2}_0 &= 0,
\end{aligned}
\tag{5.12}
$$

with the Laplacian of AdS_3 in the coordinates (5.5) given by

$$
\Box = \partial_\rho^2 + 2\,\partial_\rho + 4\,e^{-2\rho}\,\partial\bar{\partial}. \tag{5.13}
$$

This is not the standard form of these equations, we can remove the $\Box C^1_0$ term of the second equation by subtracting these two equations with an appropriate weight. This leads to the coupled Klein-Gordon equations

$$
\Box \boldsymbol{C} + \begin{bmatrix} 6\lambda(1-2\lambda) & 2\lambda(1-6\lambda+8\lambda^2) \\ 1-4\lambda & 1-2\lambda+4\lambda^2 \end{bmatrix} \boldsymbol{C} = 0, \tag{5.14}
$$

where

$$
\boldsymbol{C} = \begin{pmatrix} C^1_0 \\ C^{\frac{3}{2}}_0 \end{pmatrix}.
$$

The fields C^1_0 and $C^{\frac{3}{2}}_0$ are clearly not “mass-eigenstates”, but their superpositions must be. Diagonalizing the mass matrix we find

$$
\left[\Box - 4\,\left(\lambda^2-\lambda\right)\right]\phi_+ = 0, \qquad \left[\Box - \left(4\,\lambda^2-1\right)\right]\phi_- = 0. \tag{5.15}
$$

Thus the masses of the two scalars are given by

$$
(M^B_+)^2 = 4(\lambda^2-\lambda) \qquad \text{and} \qquad (M^B_-)^2 = 4\lambda^2-1, \tag{5.16}
$$

and from the eigenvectors of the mass matrix we read off the correct superpositions

$$
C^1_0 = (2\lambda-1)\,\phi_+ + 2\,\lambda\,\phi_-, \qquad C^{\frac{3}{2}}_0 = \phi_+ + \phi_-. \tag{5.17}
$$

By rescaling $\lambda = \frac{1}{2}\tilde{\lambda}$, the masses $(M_-^B)^2 = \tilde{\lambda}^2 - 1$ and $(M_+^B)^2 = \tilde{\lambda}^2 - 2\tilde{\lambda}$, exactly match the results known from Vasiliev theory [51, 82, 83].

This confirms that our formulation works as it should, it reproduces the correct masses of the scalars without very tedious manipulations of deformed oscillators. But it has the disadvantage that the physical fields come out in a little unnatural fashion (5.17) which will complicate our calculations slightly, however the advantages are still enormous compared to the formalism of section 2.5, which would make the calculation of the three-point functions extremely tedious.

If one deforms the AdS$_3$ background by introducing higher-spin deformations, one can show that the Klein-Gordon equation get higher derivative corrections and thus make life more difficult. We will however not need any of these in the calculation of the three-point functions.

5.2 Holographic Ward Identities and the AdS/CFT dictionary

From the classical Drinfeld-Sokolov reduction in the lowest weight gauge, we know that the gauge connection of a constant ρ-slice must be of the form (see section 2.4)

$$\begin{aligned} a(z) &= \left(L_1^{(2)} + \frac{2\pi}{k} \sum_{s\geq\frac{3}{2}}^{\infty} \left[\frac{1}{N_s^B} \mathcal{L}_s L^{(s)}_{-\lfloor s\rfloor+1} + \frac{1}{N_s^F} \psi_s G^{(s)}_{-\lceil s\rceil+\frac{3}{2}} \right] \right) \mathrm{d}z, \\ \bar{a}(\bar{z}) &= \left(L_{-1}^{(2)} + \frac{2\pi}{k} \sum_{s\geq\frac{3}{2}}^{\infty} \left[\frac{1}{N_s^B} \bar{\mathcal{L}}_s L^{(s)}_{\lfloor s\rfloor-1} + \frac{1}{N_s^F} \bar{\psi}_s G^{(s)}_{\lceil s\rceil-\frac{3}{2}} \right] \right) \mathrm{d}\bar{z}, \end{aligned} \tag{5.18}$$

where we have used the following normalizations which will be very useful later on

$$N_s^B = -\mathrm{tr}\left(L^{(s)}_{-\lfloor s\rfloor+1} L^{(s)}_{\lfloor s\rfloor-1} \right), \qquad N_s^F = \mathrm{tr}\left(G^{(s)}_{\lceil s\rceil-\frac{3}{2}} G^{(s)}_{-\lceil s\rceil+\frac{3}{2}} \right). \tag{5.19}$$

According to the rules of AdS/CFT correspondence, in order to calculate correlation functions we have to modify the boundary conditions [114, 115] such that the different higher-spin fields have a boundary value. Using the first-order formalism, all higher-spin fields are packed into the gauge fields $A(z)$ and $\bar{A}(\bar{z})$, but we need to identify which terms correspond to their boundary values. Inspired by [74, 72] for the pure spin 3 non-supersymmetric case, we will consider the more general ansatz

$$\begin{aligned} a = &\left(L_1^{(2)} + \frac{2\pi}{k} \sum_{s\geq\frac{3}{2}}^{\infty} \left[\frac{1}{N_s^B} \mathcal{L}_s L^{(s)}_{-\lfloor s\rfloor+1} + \frac{1}{N_s^F} \psi_s G^{(s)}_{-\lceil s\rceil+\frac{3}{2}} \right] \right) \mathrm{d}z \\ &+ \left(\sum_{s\leq\frac{3}{2}}^{\infty} \sum_{|m|\leq\lfloor s\rfloor-1} \mu_m^s L_m^{(s)} + \sum_{s\leq\frac{3}{2}}^{\infty} \sum_{|r|\leq\lceil s\rceil-\frac{3}{2}} \nu_r^s G_r^{(s)} \right) \mathrm{d}\bar{z}, \end{aligned} \tag{5.20}$$

where the functions $\mu_m^s = \mu_m^s(z,\bar{z})$ and $\nu_r^s = \nu_r^s(z,\bar{z})$ are non-chiral functions. We will show that evaluating the bulk equations of motion to this ansatz, will yield the Ward identities of the dual CFT in the presence of higher-spin sources. We can in particular show the emergence of $\mathcal{N}=2$ $\mathcal{S}W_\infty[\lambda]$ symmetry near the boundary using holographic ideas in contrast to the asymptotic symmetry analysis of [53], by deriving the OPE's of the conserved currents in the dual CFT holographically using the bulk theory. Being able to directly identify source terms of the Lagrangian of the dual theory with components of the bulk gauge connection, will be crucial to us when calculating three-point functions.

In order to get an idea of which of these extra terms could correspond to the source terms, recall that the full gauge field is given as

$$\begin{aligned} A &= b^{-1} a b + b^{-1} \mathrm{d} b, \\ \bar{A} &= b \bar{a} b^{-1} + b \mathrm{d} b^{-1}, \end{aligned} \qquad \text{where} \qquad b = e^{\rho\, L_0^{(2)}}. \tag{5.21}$$

Using the Baker-Campbell-Hausdorff formula (Hadamard lemma)

$$e^X Y e^{-X} = e^{\mathrm{ad} X} Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots ,$$

and the commutation relations

$$[L_0^{(2)}, L_m^{(s)}] = -m\, L_m^{(s)}, \qquad [L_0^{(2)}, G_r^{(s)}] = -r\, G_r^{(s)},$$

we find that

$$e^{-\rho L_0^{(2)}}\, L_m^{(s)}\, e^{\rho L_0^{(2)}} = L_m^{(s)}\, e^{m\rho}, \qquad e^{-\rho L_0^{(2)}}\, G_r^{(s)}\, e^{\rho L_0^{(2)}} = G_r^{(s)}\, e^{r\rho}. \tag{5.22}$$

This implies that terms with highest possible modes, $\mu^s_{\lfloor s \rfloor - 1}$ and $\nu^s_{\lceil s \rceil - \frac{3}{2}}$, are the most dominating near the boundary and can thus be regarded as source terms. Note that this is nothing but a Fefferman-Graham expansion of A, which happens to be finite.

We want to identify these terms with source terms on the boundary of the dual fields $\mathcal{O}_s$

$$S_\partial \to S_\partial - \int \mathrm{d}^2 z\, \mu_s\, \mathcal{O}_s. \tag{5.23}$$

Note that the spin s field $\mathcal{O}_s$ is irrelevant in the renormalization group sense and will therefore change the UV-structure of the dual CFT, which from the bulk perspective corresponds to that the geometry will no longer asymptote to the same AdS_3 geometry [74].

In order to establish the holographic dictionary we need to check that the normalization chosen in (5.19) is the correct normalization that makes sure that $\mu^s_{\lfloor s \rfloor - 1}$ and $\nu^s_{\lceil s \rceil - \frac{3}{2}}$ can be directly identified with the sources (5.23). One way to do this, is to calculate the holographic Ward identities as first discussed in [112, 116, 117] and in particular make use of ideas developed in [113]. This will enable us to develop a powerful way of deriving OPE's of dual fields from the bulk. Not only will this help us fix the holographic dictionary, it will also provide us with an alternative insight into which symmetries emerge near the AdS_3 boundary given our higher-spin fields and boundary conditions.

Using the ansatz (5.20) and the equations of motion we can collect all the terms into coefficients of the Lie algebra generators

$$\partial a_{\bar{z}} - \bar{\partial} a_z + [a_z, a_{\bar{z}}] = \sum_{s \geq \frac{3}{2}}^{\infty} \Bigg[\sum_{|m| \leq \lfloor s \rfloor - 1} c^B_{s,m}\, L^s_m + \sum_{|r| \leq \lceil s \rceil - \frac{3}{2}} c^F_{s,r}\, G^s_r \Bigg] = 0 \tag{5.24}$$

which give rise to the equations

$$\begin{aligned} c^B_{s,m} &= 0, \\ c^F_{s,r} &= 0. \end{aligned} \tag{5.25}$$

The coefficients for the bosonic generators are found to be

$$\begin{aligned} c^B_{s,m} = \partial\mu^s_m - \frac{2\pi}{k}\frac{1}{N^B_s}\bar{\partial}\mathcal{L}_s\,\delta_{m,\text{-}\lfloor s\rfloor+1} + \big(\lfloor s\rfloor - m\big)\mu^s_{m-1}\left(1-\delta_{m,\text{-}\lfloor s\rfloor+1}\right) \\ + \frac{2\pi}{k}\sum_{t\geq\frac{3}{2}}\Bigg\{\frac{1}{N^B_t}\mathcal{L}_t\sum_{\tilde{s}\geq\frac{3}{2}}\chi_{\left[-\lfloor\tilde{s}\rfloor-\lfloor t\rfloor+2,\lfloor\tilde{s}\rfloor-\lfloor t\rfloor\right]}(m)\,\mu^{\tilde{s}}_{m+\lfloor t\rfloor-1}\sum_{u=1}^{\tilde{s}+t-|\tilde{s}-t|-1}\delta_{\tilde{s}+t-u,s} \\ \times g^{t,\tilde{s}}_u\Big(\text{-}\lfloor t\rfloor+1, m+\lfloor t\rfloor-1;\lambda\Big) \\ + \frac{1}{N^F_t}\psi_t\sum_{\tilde{s}\geq\frac{3}{2}}\chi_{\left[-\lceil\tilde{s}\rceil-\lceil t\rceil+3,\lceil\tilde{s}\rceil-\lceil t\rceil\right]}(m)\,\nu^{\tilde{s}}_{m+\lceil t\rceil-\frac{3}{2}}\sum_{u=1}^{\tilde{s}+t-|\tilde{s}-t|-1}\delta_{\tilde{s}+t-u,s} \\ \times \tilde{g}^{t,\tilde{s}}_u\left(\text{-}\lceil t\rceil+\frac{3}{2}, m+\lceil t\rceil-\frac{3}{2};\lambda\right)\Bigg\}, \end{aligned} \tag{5.26}$$

and for the fermionic generators we have

$$\begin{aligned} c^F_{s,r} = \partial\nu^s_r - \frac{2\pi}{k}\frac{1}{N^F_s}\bar{\partial}\psi_s\,\delta_{r,\text{-}\lceil s\rceil+\frac{3}{2}} + \left(\lceil s\rceil - \frac{1}{2} - r\right)\nu^s_{r-1}\left(1-\delta_{r,\text{-}\lceil s\rceil+\frac{3}{2}}\right) \\ + \frac{2\pi}{k}\sum_{t\geq\frac{3}{2}}\Bigg\{\frac{1}{N^B_t}\mathcal{L}_t\sum_{\tilde{s}\geq\frac{3}{2}}\chi_{\left[-\lceil\tilde{s}\rceil-\lfloor t\rfloor+\frac{5}{2},\lceil\tilde{s}\rceil-\lfloor t\rfloor-\frac{1}{2}\right]}(r)\,\nu^{\tilde{s}}_{r+\lfloor t\rfloor-1}\sum_{u=1}^{\tilde{s}+t-|\tilde{s}-t|-1}\delta_{\tilde{s}+t-u,s} \\ \times h^{t,\tilde{s}}_u\Big(\text{-}\lfloor t\rfloor+1, r+\lfloor t\rfloor-1;\lambda\Big) \\ + \frac{1}{N^F_t}\psi_t\sum_{\tilde{s}\geq\frac{3}{2}}\chi_{\left[-\lfloor\tilde{s}\rfloor-\lceil t\rceil+\frac{5}{2},\lfloor\tilde{s}\rfloor-\lceil t\rceil+\frac{1}{2}\right]}(r)\,\nu^{\tilde{s}}_{r+\lceil t\rceil-\frac{3}{2}}\sum_{u=1}^{\tilde{s}+t-|\tilde{s}-t|-1}\delta_{\tilde{s}+t-u,s} \\ \times \tilde{h}^{t,\tilde{s}}_u\left(\text{-}\lceil t\rceil+\frac{3}{2}, r+\lceil t\rceil-\frac{3}{2};\lambda\right)\Bigg\}, \end{aligned} \tag{5.27}$$

where we have used the following

$$\begin{aligned} g^{2s}_u(1,m;\lambda) &= \begin{cases}\lfloor s\rfloor - 1 - m, & u=2\\ 0, & u=1,\frac{3}{2},\frac{5}{2},3\end{cases}, \\ h^{2s}_u(1,r;\lambda) &= \begin{cases}\lceil s\rceil - \frac{3}{2} - r, & u=2\\ 0, & u=1,\frac{3}{2},\frac{5}{2},3\end{cases}. \end{aligned} \tag{5.28}$$

Looking at the form of the equations given by $c^B_{s,m}$ and $c^F_{s,r}$ one can see that by starting from the highest modes, $m = \lfloor s\rfloor - 1$ and $r = \lceil s\rceil - \frac{3}{2}$, we can recursively solve μ^s_m and ν^s_r in terms of the the highest modes $\mu^s_{\lfloor s\rfloor-1}$ and $\nu^s_{\lceil s\rceil-\frac{3}{2}}$, respectively. Finally at the lowest modes, $m = -\lfloor s \rfloor + 1$ and $r = -\lceil s \rceil + \frac{3}{2}$, the equations of motion are reduced to relations containing only $\mathcal{L}_s$, ψ_s, $\mu^s_{\lfloor s \rfloor - 1}$ and $\nu^s_{\lceil s \rceil - \frac{3}{2}}$. These equations can be regarded as Ward identities in the presence of sources, and from these we can identify the correct normalization for the sources by deriving the corresponding OPE's of the dual CFT.

Before we proceed, we will present a general result which will be very useful for us later.

5.2.1 General Formula for Ward Identities from CFT

We will here derive a general formula for the Ward identities in the presence of source terms. Consider two chiral quasi-primary fields $W(z)$ and $X(z)$ of conformal weights h_W and h_X, respectively, and the following general OPE

$$W(z)X(w) \sim \sum_{i=1}^{\infty} \frac{\sigma_i}{(z-w)^i} Z_i(w) = \sum_{i=1}^{\infty} \frac{\sigma_i}{(i-1)!}\, \partial_w^{i-1} \left(\frac{1}{z-w} \right) Z_i(w), \tag{5.29}$$

where $Z_i(w)$ is are chiral quasi-primary fields of weight $h_i = h_W + h_X - i$ and we have used the compact notation $\sigma_i Z_i = \sum_j (\sigma_i)_j (Z_i)_j$ in case there are several fields with the same conformal weight. We have chosen the form of the second equation out of later convenience. We are interested in expectation values of $W(z)$, but with insertions of $X(z)$ source terms

$$\big\langle W \big\rangle_\mu = \big\langle W\, e^{-\int \mu X} \big\rangle, \tag{5.30}$$

where $\mu(w, \bar{w})$ is a non-chiral source. Due to the insertion of $\mu(w, \bar{w})$, the vacuum expectation value $\big\langle W \big\rangle_\mu$ will gain $\bar{z}$ dependence. We can directly derive the following result

$$\begin{aligned}
\bar{\partial} \big\langle W(z) \big\rangle_\mu &= -\bar{\partial} \left\langle \int \mathrm{d}^2 w\, \mu(w, \bar{w})\, \overline{W(z) X}(w) \right\rangle_\mu , \\
&= -\bar{\partial} \left\langle \int \mathrm{d}^2 w\, \mu(w, \bar{w}) \sum_{i=1}^{\infty} \frac{\sigma_i}{(i-1)!}\, \partial_w^{i-1} \left(\frac{1}{z-w} \right) Z_i(w) \right\rangle_\mu , \\
&= \bar{\partial} \left\langle \int \mathrm{d}^2 w\, \frac{1}{z-w} \left[\sum_{i=2}^{\infty} \frac{(-1)^i \sigma_i}{(i-1)!}\, \partial_w^{i-1} \Big\{ Z_i(w) \mu(w, \bar{w}) \Big\} - \sigma_1\, Z_1(w)\, \mu(w, \bar{w}) \right] \right\rangle_\mu , \\
&= 2\pi \left\langle \sum_{i=2}^{\infty} \frac{(-1)^i \sigma_i}{(i-1)!}\, \partial_z^{i-1} \Big\{ Z_i(z) \mu(z, \bar{z}) \Big\} - \sigma_1\, Z_1(z)\, \mu(z, \bar{z}) \right\rangle_\mu , \\
&= 2\pi \left\langle \Big(\sigma_2 \left[\partial Z_2\, \mu + Z_2\, \partial \mu \right] - \sigma_1\, Z_1\, \mu \Big) + \sum_{i=3}^{\infty} \frac{(-1)^i \sigma_i}{(i-1)!} \sum_{q=0}^{i-1} \binom{i-1}{q} \partial^{i-1-q} Z_i\, \partial^q \mu \right\rangle_\mu ,
\end{aligned} \tag{5.31}$$

where we have used the identity $\bar{\partial} \left(\frac{1}{z-w} \right) = 2\pi\, \delta^{(2)}(z-w)$ between the third and fourth line and in the last step used

$$\partial^n (Z\, \mu) = \sum_{q=0}^{n} \binom{n}{q} \partial^{n-q} Z\, \partial^q \mu. \tag{5.32}$$

For illustrative reasons, let us take two simple examples. Let $W = T$ be the energy-momentum tensor and X a primary field, we then have the following coefficients and fields from their OPE

$$\begin{aligned} \sigma_1 &= 1, & \sigma_2 &= h_X, \\ Z_1 &= \partial X, & Z_2 &= X, \end{aligned} \tag{5.33}$$

and all other coefficients are zero. This leads to the identity

$$\frac{1}{2\pi}\bar{\partial}\big\langle T(z)\big\rangle_{\mu_X} = \big\langle h_X\, X\,\partial\mu_X + (h_X - 1)\,\partial X \mu_X\big\rangle_{\mu_X}. \tag{5.34}$$

As our second example let us choose both fields to be the energy-momentum tensor $W = X = T$. For this case we have the following OPE coefficients

$$\begin{aligned} \sigma_1 &= 1, & \sigma_2 &= 2, & \sigma_4 &= \frac{c}{2}, \\ Z_1 &= \partial T, & Z_2 &= T, & Z_4 &= \mathbb{1}, \end{aligned} \tag{5.35}$$

giving us the identity

$$\frac{1}{2\pi}\bar{\partial}\big\langle T(z)\big\rangle_{\mu_T} = \big\langle 2T\,\partial\mu_T + \,\partial T\mu_T + \frac{c}{12}\,\partial^3\mu_T\big\rangle_{\mu_T}. \tag{5.36}$$

As expected, this is just like the above result up to the central charge term. In the following we shall mainly use our result (5.31) in the other way around, we will from the bulk derive the Ward identities then use (5.31) to find the OPE coefficients.

5.2.2 Holographic Operator Product Expansions and Superconformal Symmetries

As discussed above, the terms with highest mode $\mu^s_{\lfloor s\rfloor-1}$ and $\nu^s_{\lceil s\rceil-\frac{3}{2}}$ are the most dominating near the boundary and can thus be identified with sources of the dual field up to normalization. The conserved currents on the boundary can be organized into (holomorphic) $\mathcal{N}=2$ multiplets

$$\Big(W^{s-}, G^{(s+\frac{1}{2})-}, G^{(s+\frac{1}{2})+}, W^{(s+1)+}\Big), \qquad s \in \mathbb{Z}_{\geq 1}, \tag{5.37}$$

where $W^{s\pm}$ are bosonic fields of spin s and and $G^{(s+\frac{1}{2})\pm}$ are fermionic fields of spin $s+\frac{1}{2}$. The modes of these fields should form the $\mathcal{N}=2$ $\mathcal{SW}_\infty[\lambda]$ algebra, which generates the spectrum of the dual CFT. To begin with we will focus on the lowest multiplet $s=1$,

$$\Big(j, G^{\frac{3}{2}-}, G^{\frac{3}{2}+}, T,\Big), \tag{5.38}$$

where we for this special multiplet use the notation $j \equiv W^{1-}$ and $T \equiv W^{2+}$ as is standard in the literature. This multiplet is the most important one and generates the $\mathcal{N}=2$ superconformal algebra in two-dimensions.

In order to find the Ward identities of this multiplet we only need to turn on boundary terms corresponding to these fields $\big(\mu^1_0, \nu^{\frac{3}{2}}_{\pm\frac{1}{2}}, \nu^2_{\pm\frac{1}{2}}, \mu^2_{\pm 1}\big)$, thus all other source terms are turned off. For reasons which will become more clear momentarily, we will rename $\mathcal{L}_2 \to \tilde{\mathcal{L}}_2$. We can now recursively solve the equations (5.25) in order to express all near boundary terms in terms of the highest modes. We will not show the details of these slightly tedious calculations, but the final equations for the lowest modes $c^B_{\frac{3}{2},0} = 0$, $c^B_{2,-1} = 0$, $c^F_{s,-\frac{1}{2}} = 0$ (where $s = \frac{3}{2}, 2$) can be expressed in the following compact form

$$\begin{aligned}
\bar{\partial}\mathcal{L}_{\frac{3}{2}} &= -\psi_2\,\nu^{\frac{3}{2}}_{\frac{1}{2}} - \psi_{\frac{3}{2}}\,\nu^2_{\frac{1}{2}} + \frac{k}{2\pi}\,2\,\partial\mu^{\frac{3}{2}}_0,\\
\bar{\partial}\tilde{\mathcal{L}}_2 &= 2\,\tilde{\mathcal{L}}_2\,\partial\mu^2_1 + \partial\tilde{\mathcal{L}}_2\,\mu^2_1 + \frac{k}{2\pi}\frac{1}{2}\,\partial^3\mu^2_1 + \sum_{s=\frac{3}{2}}^{2}\left(\frac{3}{2}\psi_s\,\partial\nu^s_{\frac{1}{2}} + \frac{1}{2}\partial\psi_s\,\nu^s_{\frac{1}{2}} + \frac{2\pi}{k}\frac{1}{2}\,\mathcal{L}_{\frac{3}{2}}\,\psi_{\bar{s}}\,\nu^s_{\frac{1}{2}}\right),\\
\bar{\partial}\psi_s &= \left(\frac{3}{2}\psi_s\,\partial\mu^2_1 + \partial\psi_s\,\mu^2_1 - \frac{2\pi}{k}\frac{1}{2}\,\mathcal{L}_{\frac{3}{2}}\,\psi_{\bar{s}}\,\mu^2_1\right) + \left(\psi_{\bar{s}}\,\mu^{\frac{3}{2}}_0\right) + (-1)^{2s}\left(\partial\mathcal{L}_{\frac{3}{2}}\,\nu^{\bar{s}}_{\frac{1}{2}} + 2\,\mathcal{L}_{\frac{3}{2}}\,\partial\nu^{\bar{s}}_{\frac{1}{2}}\right),\\
&\quad + (-1)^{2s}\left(\frac{k}{2\pi}\,2\,\partial^2\nu^s_{\frac{1}{2}} + 2\,\tilde{\mathcal{L}}_2\,\nu^s_{\frac{1}{2}} + \frac{2\pi}{k}\frac{1}{2}\left[\mathcal{L}_{\frac{3}{2}}\right]^2\nu^s_{\frac{1}{2}}\right),
\end{aligned} \tag{5.39}$$

where $\bar{s} = \frac{3}{2}$ if $s = 2$ and $\bar{s} = 2$ if $s = \frac{3}{2}$. If we make the following identifications of the currents

$$2\pi\,\tilde{\mathcal{L}}_2 \to \tilde{T}, \qquad 2\pi\,\mathcal{L}_{\frac{3}{2}} \to j, \qquad 2\pi\,\psi_{\frac{3}{2}} \to G^{\frac{3}{2}-}, \qquad 2\pi\,\psi_2 \to G^{\frac{3}{2}+}, \tag{5.40}$$

and of the sources

$$\mu^2_{-1} \to 2\pi\,\mu_{\tilde{T}}, \qquad \mu^{\frac{3}{2}}_0 \to 2\pi\,\mu_j, \qquad \nu^{\frac{3}{2}}_{-\frac{1}{2}} \to 2\pi\,\nu_{G^{\frac{3}{2}-}}, \qquad \nu^2_{-\frac{1}{2}} \to 2\pi\,\nu_{G^{\frac{3}{2}+}}, \tag{5.41}$$

we can use equation (5.31) to derive the following OPE coefficients of the dual currents. The OPE's are given by

$$\begin{aligned}
j(z)j(w) &\sim \frac{2k}{(z-w)^2}, \qquad j(z)G^{\frac{3}{2}\pm}(w) \sim \frac{1}{z-w}\,G^{\frac{3}{2}\mp}(w),\\
\tilde{T}(z)\tilde{T}(w) &\sim \frac{3k}{(z-w)^4} + \frac{2}{(z-w)^2}\,\tilde{T}(w) + \frac{1}{z-w}\,\partial\tilde{T}(w),\\
\tilde{T}(z)G^{\frac{3}{2}\pm}(w) &\sim \frac{3/2}{(z-w)^2}\,G^{\frac{3}{2}\pm}(w) + \frac{1}{z-w}\left(\partial G^{\frac{3}{2}\pm}(w) - \frac{1}{2k}\,[jG^{\frac{3}{2}\mp}](w)\right),\\
G^{\frac{3}{2}\pm}(z)G^{\frac{3}{2}\pm}(w) &\sim \frac{\mp 4k}{(z-w)^3} + \frac{\mp 2}{z-w}\left(\tilde{T}(w) + \frac{1}{4k}[jj](w)\right),\\
G^{\frac{3}{2}\pm}(z)G^{\frac{3}{2}\mp}(w) &\sim \frac{\pm 2}{(z-w)^2}\,j(w) + \frac{\pm 1}{z-w}\,\partial j(w),\\
\tilde{T}(z)j(w) &\sim 0.
\end{aligned} \tag{5.42}$$

These OPE's look somewhat similar to the $\mathcal{N} = 2$ superconformal CFT, however $j(z)$ does not look like a primary field since it decouples completely from $\tilde{T}(z)$. The same problem we encounter for $G^{\frac{3}{2}\pm}$ due to non-linear terms. This seems to indicate that there is something wrong with our identifications between bulk/boundary terms, but which field needs to be modified? One clue comes from the fact that the $j(z)j(w)$ OPE scales as if $j(z)$ is an primary field of conformal weight $h = 1$, while the singular part of $\tilde{T}(z)j(w)$ vanish. This indicates that $\tilde{T}(z)$ might not be the correct energy-momentum tensor. Furthermore note that $[jj](z)$ form an energy-momentum tensor by the usual Sugawara construction, giving rise to a $U(1)$ affine Lie algebra. Given these facts and the form of the OPE's, it is natural to consider the following field

$$T(z) = \tilde{T}(z) + \frac{1}{4k}[jj](z). \tag{5.43}$$

There is an subtle but important thing to note. We are currently looking at the large N limit of the duality, which means that the central charge is very big $c \to \infty$. This is the "classical" limit in which we do not have any information about normal ordering of the products of fields, this means that the OPE's we are working with are "classical" OPE's. We will therefore in the following ignore double (and higher order) contractions when calculating OPE's since we do not have any notion of normal ordering [118], there are however $\mathcal{O}(\frac{1}{c})$ corrections when moving to finite N due to quantum effects. See [119, 120] for some interesting analysis of the $\mathcal{O}(\frac{1}{c})$ corrections.

Now by the following classical OPE's

$$\begin{aligned} \frac{1}{(4k)^2}[jj](z)[jj](w) &\sim \frac{1}{4k}\Big(\frac{2}{(z-w)^2}[jj](w) + \frac{1}{z-w}\partial[jj](w)\Big), \\ \frac{1}{4k}[jj](z)G^{\frac{3}{2}\pm}(w) &\sim \frac{1}{2k}\frac{1}{z-w}[jG^{\frac{3}{2}\mp}](w), \end{aligned} \tag{5.44}$$

and setting the Chern-Simons level to

$$k = \frac{c}{6}, \tag{5.45}$$

we find the OPE's of the $\mathcal{N}=2$ superconformal algebra

$$\begin{aligned} j(z)j(w) &\sim \frac{c/3}{(z-w)^2}, \qquad j(z)G^{\frac{3}{2}\pm}(w) \sim \frac{1}{z-w}G^{\frac{3}{2}\mp}(w), \\ T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w), \\ T(z)G^{\frac{3}{2}\pm}(w) &\sim \frac{3/2}{(z-w)^2}G^{\frac{3}{2}\pm}(w) + \frac{1}{z-w}\partial G^{\frac{3}{2}\pm}(w), \\ G^{\frac{3}{2}\pm}(z)G^{\frac{3}{2}\pm}(w) &\sim \frac{\mp 2c/3}{(z-w)^3} + \frac{\mp 2}{z-w}T(w), \\ G^{\frac{3}{2}\pm}(z)G^{\frac{3}{2}\mp}(w) &\sim \frac{\pm 2}{(z-w)^2}j(w) + \frac{\pm 1}{z-w}\partial j(w), \\ T(z)j(w) &\sim \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}\partial j(w). \end{aligned} \tag{5.46}$$

Here $T(z)$ in the energy-momentum tensor and generate the Virasoro algebra, $j(z)$ is the $U(1)$ R-symmetry and generates an affine Lie algebra while $G^{\frac{3}{2}\pm}$ are the two conformal supercharges. In the literature the supercharges are chosen such that they have definite $U(1)$ charge under R-symmetry, this can be recovered from the superpositions $\tilde{G}^{\pm} = \frac{i}{\sqrt{2}}(G^{\frac{3}{2}+} \pm G^{\frac{3}{2}-})$

$$\begin{aligned} j(z)\tilde{G}^{\pm}(w) &\sim \frac{\pm 1}{z-w}\tilde{G}^{\pm}(w), \\ \tilde{G}^{\pm}(z)\tilde{G}^{\pm}(w) &\sim 0 \\ \tilde{G}^{\pm}(z)\tilde{G}^{\mp}(w) &\sim \frac{2/3c}{(z-w)^3} \pm \frac{2}{(z-w)^2}j(w) + \frac{1}{z-w}\Big(2T(w) \pm \partial j(w)\Big). \end{aligned} \tag{5.47}$$

Note that combining equation (5.45) with (2.5), we find the celebrated Brown-Henneaux [14] central charge

$$c = \frac{3l}{2G}. \tag{5.48}$$

This is in agreement with results obtained using different techniques [11, 35, 53, 12]. Finally we note that even though the modification of the energy-momentum tensor (5.43) looks strange, it seems to have appeared in the literature from a different point of view [46, 48, 53]. In [46] all possible AdS_3 extended supergravity theories (without higher-spin fields) have been systematically investigated in the Chern-Simons formulation and asymptotic symmetry algebras calculated. It is here seen that the energy-momentum tensor is generically shifted by the affine Lie algebra generated by the internal R-symmetries in agreement with our $\mathcal{N} = 2$ higher-spin case.

Let us now consider the second multiplet consisting of the fields

$$\left(W^{2-}, G^{\frac{5}{2}-}, G^{\frac{5}{2}+}, W^{3+}\right). \tag{5.49}$$

Just as above, we can turn off all source terms except the ones corresponding to this multiplet and then proceed recursively. Shifting the energy momentum-tensor as discussed above

$$\mathcal{L}_2 = \tilde{\mathcal{L}}_2 + \frac{\pi}{2k}[\mathcal{L}_{\frac{3}{2}}]^2, \tag{5.50}$$

we find the following two Ward identities corresponding to the energy-momentum tensor and R-symmetry current

$$\begin{aligned} \bar{\partial}\mathcal{L}_2 &= \bar{\partial}\tilde{\mathcal{L}}_2 + \frac{\pi}{k}\bar{\partial}\mathcal{L}_{\frac{3}{2}}\mathcal{L}_{\frac{3}{2}}, \\ &= 3\,\mathcal{L}_3\,\partial\mu_2^3 + 2\,\partial\mathcal{L}_3\,\mu_2^3 + \sum_{s=\frac{5}{2}}^{3}\left(\frac{5}{2}\,\psi_s\,\partial\nu_{\frac{3}{2}}^s + \frac{3}{2}\,\partial\psi_s\,\nu_{\frac{3}{2}}^s\right) + 2\,\mathcal{L}_{\frac{5}{2}}\,\partial\mu_1^{\frac{5}{2}} + \partial\mathcal{L}_{\frac{5}{2}}\,\mu_1^{\frac{5}{2}}. \\ \bar{\partial}\mathcal{L}_{\frac{3}{2}} &= -\psi_3\,\nu_{\frac{3}{2}}^{\frac{5}{2}} - \psi_{\frac{5}{2}}\,\nu_{\frac{3}{2}}^3. \end{aligned} \tag{5.51}$$

Again identifying the currents and sources as

$$\begin{aligned} &2\pi\,\mathcal{L}_{\frac{5}{2}} \to W^{2-}, \quad && 2\pi\,\mathcal{L}_3 \to W^{3+}, \quad && 2\pi\,\psi_{\frac{5}{2}} \to G^{\frac{5}{2}-}, \quad && 2\pi\,\psi_3 \to G^{\frac{5}{2}+}, \\ &\mu_1^{\frac{5}{2}} \to 2\pi\,\mu_{W^{2-}}, && \mu_2^3 \to 2\pi\,\mu_{W^{3+}}, && \nu_{\frac{3}{2}}^{\frac{5}{2}} \to 2\pi\,\nu_{G^{\frac{5}{2}-}}, && \nu_{\frac{3}{2}}^3 \to 2\pi\,\mu_{G^{\frac{5}{2}+}}, \end{aligned} \tag{5.52}$$

we find the following OPE's

$$\begin{aligned} T(z)W^{2-}(w) &\sim \frac{2}{(z-w)^2}\,W^{2-}(w) + \frac{1}{z-w}\,\partial W^{2-}(w), \\ T(z)W^{3+}(w) &\sim \frac{3}{(z-w)^2}\,W^{3+}(w) + \frac{1}{z-w}\,\partial W^{3+}(w), \\ T(z)G^{\frac{5}{2}\pm}(w) &\sim \frac{5/2}{(z-w)^2}\,G^{\frac{5}{2}\pm}(w) + \frac{1}{z-w}\,\partial G^{\frac{5}{2}\pm}(w), \\ j(z)G^{\frac{5}{2}\pm} &\sim \frac{1}{z-w}G^{\frac{5}{2}\mp}(w). \end{aligned} \tag{5.53}$$

We see that the higher-spin fields in the second multiplet are primary fields as expected, and this is also the case for higher multiplets. Thus it seems that the holographic dictionary works consistently given the identifications we have made. It is possible to derive OPE's between higher-spin fields and thereby derive the structure constants of the (classical) $\mathcal{N} = 2\ \mathcal{SW}_\infty[\lambda]$. These interesting results will not be presented here in detail since they are not completely finished and they are peripheral to our main objective, which is to establish the holographic dictionary in order to calculate three-point functions.

Let us however show one last general result which is useful for us later, the leading order singularity of the OPE of higher-spin bosonic currents is found from the term

$$\bar{\partial}\mathcal{L}_s = -\frac{\frac{k}{2\pi}\,N_s^B}{\left(2\lfloor s\rfloor - 2\right)!}\,(-\partial)^{2\lfloor s\rfloor-1}\mu^s_{\lfloor s\rfloor-1} + \dots, \tag{5.54}$$

which leads to the following leading order term

$$W_s(z)W_s(w) \sim \frac{-k\,N_s^B\left(2\lfloor s\rfloor - 1\right)}{(z-w)^{2\lfloor s\rfloor}} + \dots, \tag{5.55}$$

where for simplicity we use the notation that for integer $s \in \mathbb{Z}$ we have the fields $W_s = W^{s+}$, while for half-integers $s = \lfloor s\rfloor + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ we have $W_{\lfloor s\rfloor+\frac{1}{2}} = W^{\lfloor s\rfloor-}$. As a quick check of our results, we can use this to calculate the leading order term of $W^{2-}W^{2-}$. Up to a sign due to different normalizations, this exactly matches the results of [53].

In this section we have established the precise AdS/CFT dictionary for the higher-spin fields. We have in particular shown that using the normalizations given in (5.19), we can identify the bulk terms $\frac{1}{2\pi}\mu^s_{\lfloor s\rfloor-1}$ and $\frac{1}{2\pi}\nu^s_{\lceil s\rceil-\frac{3}{2}}$ with source terms of the boundary CFT (5.23).

5.3 Three-Point Functions From Bulk

We have so far found that our formalism reproduces the correct masses of the scalars in Vasiliev theory and established which terms in the bulk gauge connection correspond to source terms of which dual higher-spin current, and along the way given an alternative proof of the emergence of superconformal $\mathcal{N} = 2$ $\mathcal{SW}_\infty[\lambda]$ symmetry near the AdS_3 boundary. In this section we will use this information to calculate certain classes of three-point functions containing two scalars and one (holomorphic) bosonic higher-spin current.

For our needs we can turn off all higher-spin fields in the bulk, except one of fixed spin s. The gauge connection will take the form

$$A = \left(e^{\rho}\,L_1^{(2)} + \frac{1}{B_s}\,e^{-(\lfloor s\rfloor-1)\rho}\mathcal{L}_s\,L^{(s)}_{-\lfloor s\rfloor+1}\right)\mathrm{d}z + \sum_{|m|\le\lfloor s\rfloor-1} e^{m\rho}\,\mu^s_m\,L^{(s)}_m\,\mathrm{d}\bar{z} + L_0\,\mathrm{d}\rho, \tag{5.56}$$

where out of convenience we will in the following use the notation

$$\frac{1}{B_s} \equiv \frac{2\pi}{k}\frac{1}{N_s^B}, \qquad \frac{1}{F_s} \equiv \frac{2\pi}{k}\frac{1}{N_s^F}. \tag{5.57}$$

Using the standard methods of AdS/CFT correspondence to calculate correlation functions is too cumbersome and does not take full advantage of the higher-spin gauge symmetries. Our strategy for calculating three-point functions of the form $\left\langle\mathcal{O}(z_1,\bar{z}_1)\bar{\mathcal{O}}(z_2,\bar{z}_2)J^s(z_3)\right\rangle$ is based on the observation made in [84]. Starting from the solution of a free scalar field on AdS_3 we can generate new solutions by performing higher-spin gauge transformations.[2]

[2]The gauge transformations we are using are non-vanishing at the boundary and therefore are not real gauge transformations. In other words, they act like global symmetries since they map a configuration to a physically distinct one.

We can therefore start from scalars on AdS_3, then by a gauge transformation introduce higher-spin source terms. From the near boundary expansion of the scalars we can then find the corresponding three-point functions. This means we can reduce the whole calculation into studying how the scalars transform under higher-spin gauge symmetries. We will now find the relevant gauge transformation.

As discussed in the previous section, we can express all functions μ^s_m in eq. (5.56) in terms of the boundary source $\mu^s_{\lfloor s\rfloor-1}$ using the equations of motion (5.2), which for (5.56) reads

$$\begin{aligned}
F_{\bar{z}z} &= \partial A_{\bar{z}} - \bar{\partial} A_z + [A_z, A_{\bar{z}}] \\
&= \sum_{|m|\leq\lfloor s\rfloor-1} e^{m\rho}\,\partial\mu^s_m\,L^{(s)}_m - \frac{1}{B_s}\,\bar{\partial}\mathcal{L}_s\,e^{-(\lfloor s\rfloor-1)\rho}\,L^{(s)}_{-\lfloor s\rfloor+1} + \sum_{|m|\leq\lfloor s\rfloor-1} e^{(m+1)\rho}\,\mu^s_m\,\left[L^{(2)}_1, L^{(s)}_m\right] \\
&\qquad + \frac{1}{B_s}\,\mathcal{L}_s \sum_{|m|\leq\lfloor s\rfloor-1} e^{(m-\lfloor s\rfloor+1)\rho}\,\mu^s_m\,\left[L^{(s)}_{-\lfloor s\rfloor+1}, L^{(s)}_m\right], \\
&= \sum_{|m|\leq\lfloor s\rfloor-1} e^{m\rho}\,\partial\mu^s_m\,L^{(s)}_m - \frac{1}{B_s}\,\bar{\partial}\mathcal{L}_s\,e^{-(\lfloor s\rfloor-1)\rho}\,L^{(s)}_{-\lfloor s\rfloor+1} + \sum_{m'=-\lfloor s\rfloor+2}^{\lfloor s\rfloor} e^{m'\rho}\,\mu^s_{m'-1}\,(\lfloor s\rfloor - m')L^{(s)}_{m'} \\
&\qquad + \frac{1}{B_s}\,\mathcal{L}_s \sum_{|m|\leq\lfloor s\rfloor-1} e^{(m-\lfloor s\rfloor+1)\rho}\,\mu^s_m \sum_{u=1}^{2s-1}{}^{\!\partial}_{\!\!9}\; g^{ss}_u\big(\text{-}\lfloor s\rfloor+1, m;\lambda\big)\,L^{(2s-u)}_{m-\lfloor s\rfloor+1}.
\end{aligned} \tag{5.58}$$

We need to set the coefficients of linearly independent terms equal to zero separately. It is clear that for $u = s$ in the last term we get all terms proportional to L^s, the coefficients are

$$\begin{aligned}
&e^{m\rho}\,\partial\mu^s_m - \frac{1}{B_s}\,\bar{\partial}\mathcal{L}_s\,e^{-(\lfloor s\rfloor-1)\rho}\,\delta_{m,\text{-}\lfloor s\rfloor+1} + e^{m\rho}\,\mu^s_{m-1}\,\big(\lfloor s\rfloor - m\big)\big(1-\delta_{m,\text{-}\lfloor s\rfloor+1}\big) \\
&\quad + \frac{1}{B_s}\,\mathcal{L}_s\,e^{m\rho}\,\mu^s_{m+\lfloor s\rfloor-1}\,g^{ss}_s\big(\text{-}\lfloor s\rfloor+1, m+\lfloor s\rfloor-1;\lambda\big)\,\chi_{[\text{-}\lfloor s\rfloor+1,0]}(m) = 0.
\end{aligned} \tag{5.59}$$

There are also other independent equations for $u \neq s$

$$\frac{1}{B_s}\,\mathcal{L}_s\,e^{(m-\lfloor s\rfloor+1)\rho}\,\mu^s_m\,g^{ss}_u\big(\text{-}\lfloor s\rfloor+1, m;\lambda\big) = 0, \qquad u = 1,\ldots,2s-1, \quad u\neq s,$$

we will however ignore these since these equations will have corrections due to other higher-spin fields (which we have put to zero out of convenience).[3]

For $m > 0$, equation (5.59) reduce to the following recursion relation and solution

$$\partial\mu^s_m = -\big(\lfloor s\rfloor - m\big)\,\mu^s_{m-1} \qquad \Rightarrow \qquad \mu^s_m = \frac{(\text{-}\partial)^{\lfloor s\rfloor-m-1}}{\big(\lfloor s\rfloor - m - 1\big)!}\,\mu^s_{\lfloor s\rfloor-1}, \quad m\geq 0. \tag{5.60}$$

The solution for $m < 0$ is slightly more complicated, the general solution is of the form

$$\mu^s_m = \frac{(\text{-}\partial)^{\lfloor s\rfloor-m-1}}{\big(\lfloor s\rfloor - m - 1\big)!}\,\mu^s_{\lfloor s\rfloor-1} + g^s_m\big(\mathcal{L}_s, \mu^s_{\lfloor s\rfloor-1}\big),$$

[3] These equations were necessary when we derived the holomorphic OPE's in previous section.

where

$$g_m^s\big(\mathcal{L}_s, \mu^s_{\lfloor s\rfloor-1}\big) = \sum_{a,b} \alpha^{s,m}_{a,b}\, \partial^a \mu^s_{\lfloor s\rfloor-1}\, \partial^b \mathcal{L}_s,$$

with the condition $\alpha^{s,m}_{a,b} = 0$ for $m \geq 0$. This implies that the equation for the lowest mode is given by

$$\frac{1}{B_s}\bar{\partial}\mathcal{L}_s = \frac{-1}{(2\lfloor s\rfloor - 2)!}(\text{-}\partial)^{2\lfloor s\rfloor-1}\mu^s_{\lfloor s\rfloor-1} + g(\mathcal{L}_s, \mu^s_{\lfloor s\rfloor-1}), \tag{5.61}$$

where

$$g(\mathcal{L}_s, \mu^s_{\lfloor s\rfloor-1}) = \sum_{a,b} \alpha_{a,b}\, \partial^a \mu^s_{\lfloor s\rfloor-1}\, \partial^b \mathcal{L}_s. \tag{5.62}$$

Note that this is nothing but equation (5.54) with only the spin s field turned on, which is the reason the non-linear terms of $\mathcal{SW}_\infty[\lambda]$ are not present.

We can write any gauge transformation as $\Lambda(\rho, z, \bar{z}) = \sum_{|m|\geq\lfloor s\rfloor-1} \tilde{F}^s_m\, e^{\rho m}\, L^{(s)}_m$. The gauge transformation which maps the AdS_3 connection into chiral higher-spin background with spin s and its boundary source term (5.56) is of the following form

$$\Lambda(\rho, z, \bar{z}) = \sum_{m=0}^{\lfloor s\rfloor-1} \frac{1}{\big(\lfloor s\rfloor - m - 1\big)!}\big(\text{-}\partial\big)^{\lfloor s\rfloor-m-1}\Lambda^s\, e^{m\rho}\, L^{(s)}_m + \sum_{m=0}^{\lfloor s\rfloor-1} \tilde{F}^s_{-m}\, e^{-m\rho}\, L^{(s)}_{-m}, \tag{5.63}$$

and $\bar{\Lambda}(\rho, z, \bar{z}) = 0$, with the following identifications $\mu^s_{\lfloor s\rfloor-1} = \bar{\partial}\Lambda$ and $\mathcal{L}_s = -\frac{B_s}{(2\lfloor s\rfloor-2)!}(\text{-}\partial)^{2\lfloor s\rfloor-1}\Lambda$ which is imposed by the equations of motion (5.61). Note that $\tilde{F}^s_{-m}$ can be explicitly found by using the equations of motion. But as we will briefly see, the negative mode contributions to the connection do not contribute to the three-point functions.

Under infinitesimal gauge transformations, the matter fields transform as

$$\widehat{C} = C + \delta_s C, \qquad \delta_s C = C \star \bar{\Lambda} - \Lambda \star C = -\Lambda \star C. \tag{5.64}$$

Putting the fermions C^s_r to zero in (5.4) we find that the generating function transforms as

$$\begin{aligned}\delta_s C &= -\sum_{t=1}^{\infty}{}^{\!\circ} \sum_{|n|\leq\lfloor t\rfloor-1} \sum_{m=0}^{\lfloor s\rfloor-1} \frac{\big(\text{-}\partial\big)^{\lfloor s\rfloor-m-1}\Lambda^s}{\big(\lfloor s\rfloor - m - 1\big)!} C^t_n\, e^{m\rho}\, L^{(s)}_m \star L^{(t)}_n + \underbrace{\ldots}_{m<0} \\ &= \delta_s C^1_0\, L^{(1)}_0 + \delta_s C^{\frac{3}{2}}_0\, L^{(\frac{3}{2})}_0 + \ldots\end{aligned} \tag{5.65}$$

In order to isolate how the scalars transform, recall that

$$L^{(s)}_m \star L^{(t)}_n = \sum_{u=1}^{\text{Min}(2s-1,2t-1)}{}^{\!\!\circ}\; g^{st}_u(m, n; \lambda)\, L^{(s+t-u)}_{m+n}.$$

In order to isolate the lowest two scalars, we have the following conditions

$$\begin{aligned} m + n = 0 \quad &\Rightarrow \quad m = -n, \\ s + t - u_q = q \quad &\Rightarrow \quad u_q = s + t - q, \end{aligned}$$

where $q = 1, \frac{3}{2}$. Now for $q = 1$, if $t > s$ or $s > t$ we have that $u_1 > \text{Min}(2s-1, 2t-1)$ which implies that $g^{st}_{u_q}(\ldots) = 0$. This implies that only the term with $s = t$ contributes.

For $q = \frac{3}{2}$, besides the $t = s$ terms also the $t = s \pm \frac{1}{2}$ terms contribute. Thus the scalars transform as

$$\delta_s C_0^1 = -\sum_{m=0}^{\lfloor s\rfloor -1} \frac{(\text{-}\partial)^{\lfloor s\rfloor - m-1}\Lambda^s}{(\lfloor s\rfloor - m - 1)!} C^s_{-m}\, g^{ss}_{2s-1}(m,\text{-}m;\lambda)\, e^{m\rho} + \begin{matrix}\text{terms which}\\ \text{vanish as } \rho\to\infty\end{matrix}, \tag{5.66}$$

and

$$\begin{aligned}\delta_s C_0^{\frac{3}{2}} = -\sum_{m=0}^{\lfloor s\rfloor -1} \frac{(\text{-}\partial)^{\lfloor s\rfloor - m-1}\Lambda^s}{(\lfloor s\rfloor - m - 1)!} \Big[& C^s_{-m}\, g^{ss}_{2s-\frac{3}{2}}(m,\text{-}m;\lambda) \\ & + C^{s-1/2}_{-m}\, g^{ss-1/2}_{2s-2}(m,\text{-}m;\lambda)\,\chi_{[0,\lfloor s-1/2\rfloor-1]}(m) + C^{s+1/2}_{-m}\, g^{ss+1/2}_{2s-1}(m,\text{-}m;\lambda)\Big]\, e^{m\rho}.\end{aligned} \tag{5.67}$$

The step function in the second term is put in to ensure we do not go beyond the wedge algebra, which is shs[λ]. Using this we can readily find the transformation of the mass-eigenstates $\widehat{\phi}_i = \phi_i + \delta\phi_i$

$$\begin{aligned}\delta_s\phi_i &= \tilde{a}_i\,\delta_s C_0^1 + \tilde{b}_i\,\delta_s C_0^{\frac{3}{2}}, \\ &= -\sum_{m=0}^{\lfloor s\rfloor -1} \frac{(\text{-}\partial)^{\lfloor s\rfloor - m-1}\Lambda^s}{(\lfloor s\rfloor - m - 1)!} e^{m\rho} \Bigg(\tilde{a}_i\, C^s_{-m}\, g^{ss}_{2s-1}(m,\text{-}m;\lambda) + \tilde{b}_i \Big[C^s_{-m}\, g^{ss}_{2s-\frac{3}{2}}(m,\text{-}m;\lambda) \\ &\quad + C^{s-1/2}_{-m}\, g^{ss-1/2}_{2s-2}(m,\text{-}m;\lambda)\chi_{[0,\lfloor s-1/2\rfloor-1]}(m) + C^{s+1/2}_{-m}\, g^{ss+1/2}_{2s-1}(m,\text{-}m;\lambda)\Big]\Bigg), \\ &\equiv \sum_{m=0}^{\lfloor s\rfloor -1} \Big[f^{s,i}_m(\lambda,\partial_\rho)\,\partial^m\phi_i\Big]\,\partial^{\lfloor s\rfloor - m-1}\Lambda^s, \\ &\equiv D^{(s,i)}(z)\phi_i.\end{aligned} \tag{5.68}$$

This expression requires solving the recursion relations (5.11) in order to express the auxiliary fields C^s_{-m} as sums and derivatives of C_0^1 and $C_0^{\frac{3}{2}}$, which in turn can be expressed as functions of $\phi_\pm$. As will be seen later, it turns out that these will have the form $C^s_{-m} \sim e^{-|m|\rho}\, A(\lambda,\partial_\rho)\,\partial^m\phi_i$,[4] which means that $e^{m\rho}$ is canceled for $m > 0$ and enhanced for $m < 0$. For this reason the terms with $m < 0$ have been neglected in (5.68), since they are vanishing near the AdS$_3$ boundary. The coefficients are given as

$$\tilde{a}_i = \begin{cases} -1, & i = +, \\ 1, & i = -, \end{cases} \qquad \tilde{b}_i = \begin{cases} 2\lambda, & i = + \\ -2\lambda + 1 & i = - \end{cases}, \tag{5.69}$$

which are found by inverting the equations (5.17).The function in the third line of (5.68) contains all the information about the higher-spin deformation and is given as

$$\begin{aligned} f^{s,i}_m(\lambda,\partial_\rho) = \frac{(-1)^{\lfloor s\rfloor - m}}{(\lfloor s\rfloor - m - 1)!} \Bigg(& \tilde{a}_i\,\mathcal{G}^{s,i}_m\, g^{ss}_{2s-1}(m,\text{-}m;\lambda) + \tilde{b}_i \Big[\mathcal{G}^{s,i}_m\, g^{ss}_{2s-\frac{3}{2}}(m,\text{-}m;\lambda) \\ & + \mathcal{G}^{s-1/2,i}_m\, g^{ss-1/2}_{2s-2}(m,\text{-}m;\lambda)\chi_{[0,\lfloor s-1/2\rfloor-1]}(m) + \mathcal{G}^{s+1/2,i}_m\, g^{ss+1/2}_{2s-1}(m,\text{-}m;\lambda)\Big]\Bigg),\end{aligned} \tag{5.70}$$

[4]Note that for our calculation of three-point functions we only need to turn on the boundary source of the relevant scalar. Thus in calculating $\delta_s\phi_+$ we set $\phi_- = 0$ and vice versa.

where $\mathcal{G}_m^{s,i}$ is defined as

$$e^{-|m|\rho}\,\mathcal{G}_m^{s,i}\big(\lambda,\partial_\rho\big)\,\partial^m\phi_i = C^s_{-m}(\lambda,\partial_\rho)\big|_{\phi_{\bar{i}}=0},$$

where $i=\pm$ and the index $\bar{i}$ refers to the opposite sign. Thus we find $\mathcal{G}_m^{s,i}$ by removing a factor of $e^{-|m|\rho}\,\partial^m\phi_i$ from C^s_{-m} and set the other scalar to zero.

5.3.1 Three-Point Functions

Recall that putting a scalar source on the boundary of AdS_3 at z', we can express the bulk solution using the bulk-to-boundary propagator

$$\phi_i(\rho,z) = \int \mathrm{d}^2z'\,G_{b\partial}(\rho,z;z')\,\phi_i^\partial(z'), \tag{5.71}$$

which in our coordinates is given as [114, 121]

$$G_{b\partial}(\rho,z;z') = c_\pm\left(\frac{e^{-\rho}}{e^{-2\rho}+|z-z'|^2}\right)^{\Delta_\pm}. \tag{5.72}$$

Here the conformal weights are determined from the scalar mass $m^2=\Delta_\pm(\Delta_\pm-d)$, where $\Delta_+\geq\Delta_-$, $\Delta_\pm=2-\Delta_\mp$ and $d=2$ here. In this section we will also use the conventional coordinates $r=e^{-\rho}$, in which the metric takes the form $\mathrm{d}s^2=\frac{\mathrm{d}r^2+\mathrm{d}z\mathrm{d}\bar{z}}{r^2}$ and the boundary is at $r\to 0$. The constant in (5.72) is determined by the requirement that near the boundary we have the behavior $\phi_i(\rho,z)\sim r^{d-\Delta_\pm}\,\phi_i^\partial(z)$, which implies that $G_{b\partial}(\rho,z,z')=c_\pm\,r^{d-\Delta_\pm}\,\frac{r^{2\Delta_\pm-d}}{(r^2+|z-z'|^2)^{\Delta_\pm}}\to r^{d-\Delta_\pm}\,\delta^{(2)}(z-z')$. Using a change of coordinates $y=\frac{(z-z')}{r}$ [122], the constant is given by [110, 114]

$$c_\pm = \left[\int \mathrm{d}^2y\,\frac{1}{(1+y^2)^{\Delta_\pm}}\right]^{-1} = \frac{\Gamma(\Delta_\pm)}{\pi\,\Gamma(\Delta_\pm-1)} = \frac{\Delta_\pm-1}{\pi}. \tag{5.73}$$

The near-boundary expansion of the bulk field is of the form [110, 123]

$$\phi_i(\rho,z)\longrightarrow r^{d-\Delta_\pm}\left(\phi_i^\partial(z)+o(r)\right)+r^{\Delta_\pm}\left(\frac{1}{B_\phi^\pm}\left\langle\mathcal{O}_{\Delta_\pm}(z)\right\rangle+o(r)\right), \tag{5.74}$$

where $\mathcal{O}_{\Delta_\pm}$ is the dual field with conformal weight $\Delta_\pm$ and $B_\phi^\pm=2\Delta_\pm-d$ is necessary for a consistent dictionary [121, 110]. The idea is to generate the solution on a background containing a spin s source by a gauge transformation

$$\begin{aligned}\phi_i(\rho,z)\longrightarrow\widehat{\phi}_i(\rho,z) &= \phi_i(\rho,z)+\delta_s\phi_i(\rho,z),\\ &= \left(1+D^{(s,i)}\right)\phi_i(\rho,z),\end{aligned} \tag{5.75}$$

which gives the near boundary expansion

$$\widehat{\phi}_i(\rho,z)\longrightarrow r^{d-\Delta_\pm}\left(\widehat{\phi}_i^\partial(z)+o(r)\right)+r^{\Delta_\pm}\left(\frac{1}{B_\phi^\pm}\left\langle\mathcal{O}_{\Delta_\pm}(z)\right\rangle_\mu+o(r)\right). \tag{5.76}$$

By the notation $\langle \ldots \rangle_\mu$, we mean the vacuum expectation value with a higher-spin source insertion. We will put a scalar point-source at z_2 and a chiral spin s source at z_3 on the AdS_3 boundary

$$\widehat{\phi}_i^\partial(z,\bar{z}) = \mu_\phi\, \delta^{(2)}(z-z_2), \qquad \mu^s_{\lfloor s\rfloor -1}(z,\bar{z}) = \mu_s\, \delta^{(2)}(z-z_3). \tag{5.77}$$

The two and three-point functions can then be read off from the one-point function near the boundary

$$\begin{aligned}\big\langle \mathcal{O}_{\Delta_\pm}(z_1,\bar{z}_1)\big\rangle_\mu =& \mu_\phi \left\langle \mathcal{O}_{\Delta_\pm}(z_1,\bar{z}_1)\, \bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar{z}_2)\right\rangle \\ &+ \mu_\phi\, \mu_s \left\langle \mathcal{O}_{\Delta_\pm}(z_1,\bar{z}_1)\, \bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar{z}_2)\, J^s(z_3)\right\rangle + \ldots.\end{aligned} \tag{5.78}$$

We will now find a general expression for the three-point functions as a function of $f^{s,i}_m$ given in equation (5.70), which characterize the higher-spin deformation the scalars experience. The steps are clear; we need to write down how the scalars transform (5.75) and use (5.71), which requires knowing ϕ_i^∂ as a function of $\widehat{\phi}_i^\partial$. Next we need to find the vacuum expectation value of the dual field from the asymptotics of $\hat{\phi}_i$ (5.76), then isolate the $\mu_\phi\, \mu_s$ order contribution, which gives us the three-point functions as seen in (5.78).

The boundary sources of ϕ_i and $\widehat{\phi}_i$ are related by a gauge transformation

$$\widehat{\phi}_i^\partial(z)\, e^{-\Delta_\mp \rho} = \big(1 + D^{(s,i)}\big) e^{-\Delta_\mp\rho}\, \phi_i^\partial(z) = e^{-\Delta_\mp\rho}\, \big(1 + D^{(s,i)}_\mp\big)\, \phi_i^\partial(z),$$

where we have defined

$$D^{(s,i)}_\pm = D^{(s,i)}\big(\partial_\rho \to -\Delta_\pm\big). \tag{5.79}$$

Inverting this up to first order and using the boundary condition (5.77) we find

$$\phi_i^\partial(z,\bar{z}) = \mu_\phi\, \big(1 - D^{(s,i)}_\mp\big)\, \delta^{(2)}(z-z_2). \tag{5.80}$$

Using this, the gauge transformed scalar field is

$$\hat{\phi}(\rho,z) = \mu_\phi\, \big(1 + D^{(s,i)}(z)\big) \int \mathrm{d}^2 z'\, G_{b\partial}(\rho,z;z')\, \big(1 - D^{(s,i)}_\mp(z')\big)\, \delta^{(2)}(z'-z). \tag{5.81}$$

Going near the boundary $\rho \to \infty$ and keeping only the $e^{-\Delta_\pm\rho}$ contribution we have

$$\begin{aligned}\hat{\phi}_i(\rho,z) &\approx \mu_\phi \Big(1 + D^{(s,i)}(z)\Big) \int \mathrm{d}^2 z'\, \frac{c_\pm\, e^{-\Delta_\pm\rho}}{|z-z'|^{2\Delta_\pm}} \Big(1 - D^{(s,i)}_\mp(z')\Big)\, \delta^{(2)}(z'-z), \\ &= e^{-\Delta_\pm\rho}\, \mu_\phi\, c_\pm \int \mathrm{d}^2 z' \Big(1 + D^{(i,s)}_\pm(z)\Big) \frac{1}{|z-z'|^{2\Delta_\pm}} \Big(1 - D^{(s,i)}_\mp(z')\Big)\, \delta^{(2)}(z'-z), \\ &= e^{-\Delta_\pm\rho}\, \frac{\big\langle \mathcal{O}(z)\big\rangle_\mu}{B^\pm_\phi}, \qquad \rho\to\infty.\end{aligned} \tag{5.82}$$

The two-point function is readily given as

$$\big\langle \mathcal{O}_{\Delta_\pm}(z_1,\bar{z}_1)\, \bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar{z}_2)\big\rangle = \frac{B^\pm_\phi\, c_\pm}{|z_1-z_2|^{2\Delta_\pm}}. \tag{5.83}$$

Next we will look at the $\mu_\phi\, D^{(s,i)}$ contribution of the one-point function given in (5.82), since $D^{(s,i)}$ is proportional to μ_s. Neglecting the other terms, we have

$$\left\langle \mathcal{O}_{\Delta_\pm}(z_1)\right\rangle_\mu = \mu_\phi\, B_\phi^\pm\, c_\pm \left[D_\pm^{(s,i)}(z_1)\, \frac{1}{|z_1-z_2|^{2\Delta_\pm}} - \int \mathrm{d}^2 z' \frac{D_\mp^{(s,i)}(z')\,\delta^{(2)}(z'-z_2)}{|z_1-z'|^{2\Delta_\pm}} \right]. \tag{5.84}$$

Recall that the differential operators describing the infinitesimal gauge transformations take the form

$$D_\pm^{(s,i)}(z) = \sum_{m=0}^{\lfloor s\rfloor -1} \left[f_m^{s,i}(\lambda, -\Delta_\pm)\, \partial^{\lfloor s\rfloor - m-1}\Lambda^s \right] \partial^m + \text{terms vanishing as } \rho\to\infty. \tag{5.85}$$

Using this and the following identity

$$\partial_2^n \frac{1}{|z_1-z_2|^{2\Delta_\pm}} = (-1)^n \partial_1^n \frac{1}{|z_1-z_2|^{2\Delta_\pm}} = \frac{\Gamma(\Delta_\pm+n)}{\Gamma(\Delta_\pm)} \frac{1}{(z_1-z_2)^n} \frac{1}{|z_1-z_2|^{2\Delta_\pm}}, \tag{5.86}$$

we can write the first term of (5.84) as

$$D_\pm^{(s,i)}(z_1) \frac{1}{|z_{12}|^{2\Delta_\pm}} = \sum_{m=0}^{\lfloor s\rfloor -1} (-1)^m \frac{\Gamma(\Delta_\pm+m)}{\Gamma(\Delta_\pm)}\, f_m^{s,i}(\lambda,-\Delta_\pm) \left[\partial_1^{\lfloor s\rfloor -m-1}\Lambda^{(s)}(z_1)\right] \frac{1}{z_{12}^m\, |z_{12}|^{2\Delta_\pm}}. \tag{5.87}$$

For the second term we need to integrate by parts, until there are no derivatives on the delta function

$$\begin{aligned}
\int \mathrm{d}^2 z' \frac{D_\mp^{(s,i)}(z')\,\delta^{(2)}(z'-z_2)}{|z_1-z'|^{2\Delta_\pm}} &= \sum_{m=0}^{\lfloor s\rfloor -1} f_m^{s,i}(\lambda,-\Delta_\mp) \int \mathrm{d}^2 z' \, \frac{\partial_{z'}^{\lfloor s\rfloor -m-1}\Lambda^{(s)}(z')\, \partial_{z'}^m \delta(z'-z_2)}{|z_1-z'|^{2\Delta_\pm}}, \\
&= \sum_{m=0}^{\lfloor s\rfloor -1} f_m^{s,i}(\lambda,-\Delta_\mp) \int \mathrm{d}^2 z'\, (-1)^m\, \partial_{z'}^m \left[\frac{\partial_{z'}^{\lfloor s\rfloor -m-1}\Lambda^{(s)}(z')}{|z_1-z'|^{2\Delta_\pm}}\right] \delta(z'-z_2), \\
&= \sum_{m=0}^{\lfloor s\rfloor -1} (-1)^m\, f_m^{s,i}(\lambda,-\Delta_\mp)\, \partial_2^m \left[\frac{\partial_2^{\lfloor s\rfloor -m-1}\Lambda^{(s)}(z_2)}{|z_{12}|^{2\Delta_\pm}}\right], \\
&= \sum_{m=0}^{\lfloor s\rfloor -1} (-1)^m\, f_m^{s,i}(\lambda,-\Delta_\mp) \sum_{j=0}^{m} \binom{m}{j} \left[\partial_2^{\lfloor s\rfloor -m-1+j}\Lambda^{(s)}(z_2)\right] \partial_2^{m-j} \left[\frac{1}{|z_{12}|^{2\Delta_\pm}}\right], \\
&= \sum_{m=0}^{\lfloor s\rfloor -1} \sum_{j=0}^{m} (-1)^m \frac{\Gamma(\Delta_\pm+m-j)}{\Gamma(\Delta_\pm)}\, f_m^{s,i}(\lambda,-\Delta_\mp) \binom{m}{j} \left[\partial_2^{\lfloor s\rfloor -m-1+j}\Lambda^{(s)}(z_2)\right] \frac{1}{z_{12}^{m-j}\, |z_{12}|^{2\Delta_\pm}},
\end{aligned} \tag{5.88}$$

where in the last line we have used the formula (5.86). In order to get the correct boundary condition for the higher-spin field (5.77), we have to set the gauge parameter to

$$\Lambda^{(s)}(z) = \frac{\mu_s}{2\pi} \frac{1}{z-z_3}. \tag{5.89}$$

We can now make use of the identities

$$\begin{aligned}\partial_1^{\lfloor s\rfloor-m-1}\Lambda^{(s)}(z_1) &= \frac{\mu_s}{2\pi}\,\frac{\big(\lfloor s\rfloor-m-1\big)!}{z_{13}^{\lfloor s\rfloor-m}}(-1)^{\lfloor s\rfloor-m-1},\\ \partial_2^{\lfloor s\rfloor-m-1+j}\Lambda^{(s)}(z_2) &= \frac{\mu_s}{2\pi}\,\frac{\big(\lfloor s\rfloor-m-1+j\big)!}{z_{13}^{\lfloor s\rfloor-m+j}}(-1)^{\lfloor s\rfloor-m-1+j},\end{aligned} \tag{5.90}$$

and write the one-point function as

$$\begin{aligned}\big\langle\mathcal{O}_{\Delta_\pm}(z_1)\big\rangle_\mu &= \frac{\mu_\phi\,\mu_s\,B_m^\pm\,c_\pm\,(-1)^{\lfloor s\rfloor-1}}{2\pi\,|z_{12}|^{2\Delta_\pm}}\sum_{m=0}^{\lfloor s\rfloor-1}\frac{1}{z_{12}^m}\Bigg\{f_s^{s,i}\big(\lambda,-\Delta_\pm\big)\,\frac{\Gamma(\Delta_\pm+m)}{\Gamma(\Delta_\pm)}\,\frac{\big(\lfloor s\rfloor-m-1\big)!}{z_{13}^{\lfloor s\rfloor-m}}\\ &\quad - f_m^{s,i}\big(\lambda,-\Delta_\mp\big)\frac{1}{z_{23}^{\lfloor s\rfloor-m}}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{\Gamma(\Delta_\pm+m-j)}{\Gamma(\Delta_\pm)}\big(\lfloor s\rfloor-m-1+j\big)!\left(\frac{z_{12}}{z_{23}}\right)^j\Bigg\}\end{aligned} \tag{5.91}$$

We have now shown that the three-point functions are known as soon as we know the functions $f_m^{s,i}(\lambda,\Delta_\pm)$. This expression, however, looks very complicated and it is not manifestly conformal invariant. Conformal symmetry constrains the three-point functions to take the form [5]

$$\begin{aligned}\big\langle\mathcal{O}_{\Delta_\pm}(z_1,\bar z_1)\bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar z_2)J^{(s)}(z_3)\big\rangle &= A_\pm(s)\,d_\pm\left(\frac{z_{12}}{z_{13}z_{23}}\right)^{\lfloor s\rfloor}\frac{1}{z_{12}^{2h_\pm}\bar z_{12}^{2\bar h_\pm}},\\ &= A_\pm(s)\left(\frac{z_{12}}{z_{13}z_{23}}\right)^{\lfloor s\rfloor}\big\langle\mathcal{O}_{\Delta_\pm}(z_1,\bar z_1)\bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar z_2)\big\rangle.\end{aligned} \tag{5.92}$$

Note that this, among other things, demands the following relation

$$\big\langle\mathcal{O}_{\Delta_\pm}(z_1,\bar z_1)\bar{\mathcal{O}}_{\Delta_\pm}(z_2,\bar z_2)J^{(s)}(z_3)\big\rangle = (-1)^{\lfloor s\rfloor}\big\langle\mathcal{O}_{\Delta_\pm}(z_2,\bar z_2)\bar{\mathcal{O}}_{\Delta_\pm}(z_1,\bar z_1)J^{(s)}(z_3)\big\rangle. \tag{5.93}$$

Although the full conformal invariance is not manifest in (5.91), we can make the above symmetry manifest in order to simplify (5.91). This implies that the three-point function must be of the form

$$\begin{aligned}\big\langle\mathcal{O}_{\Delta_\pm}(z)\big\rangle_\mu &= \mu_\phi\,B_\phi^\pm\,C_\pm\Big[D_\pm^{(s,i)}(z_1)+(-1)^{\lfloor s\rfloor}\,D_\pm^{(s,i)}(z_2)\Big]\frac{1}{|z_{12}|^{2\Delta_\pm}},\\ &= \frac{\mu_\phi\,\mu_s\,B_\phi^\pm\,C_\pm\,(-1)^{\lfloor s\rfloor-1}}{2\pi\,|z_{12}|^{2\,\Delta_\pm}}\sum_{m=0}^{\lfloor s\rfloor-1}\frac{f_m^{s,i}(\lambda,-\Delta_\pm)}{z_{12}^m}\frac{\Gamma(\Delta_\pm+m)}{\Gamma(\Delta_\pm)}\big(\lfloor s\rfloor-m-1\big)!\\ &\quad\times\left(\frac{1}{z_{13}^{\lfloor s\rfloor-m}}+\frac{(-1)^{\lfloor s\rfloor-m}}{z_{23}^{\lfloor s\rfloor-m}}\right).\end{aligned} \tag{5.94}$$

Note that the second term is acting on z_2, thus the factor $(-1)^{\lfloor s\rfloor-m}$ comes from using the formula (5.86). Furthermore note that making the symmetry (5.93) manifest imposes

[5] Note that in general $z^{2h}\bar z^{2\bar h}=|z|^{2\Delta}e^{i(h-\bar h)\theta}$. For scalars we have that $h-\bar h=0$, while for spin $\frac{1}{2}$ fermions we have $h-\bar h=\pm\frac{1}{2}$.

a constraint on $f_m^{s,i}(\lambda,\Delta_\pm)$, which comes from equating (5.91) with (5.94) and isolating terms of equal powers of z_{12}

$$f^{s,i}_{\lfloor s\rfloor-\tilde{j}}(\lambda,-\Delta_\pm) = -\sum_{m=0}^{\lfloor s\rfloor-1}(-1)^{\lfloor s\rfloor-m}\, f_m^{s,i}(\lambda,-\Delta_\mp)\binom{m}{\tilde{j}-\lfloor s\rfloor+m}, \tag{5.95}$$

where $\tilde{j}=\lfloor s\rfloor-m,\ldots,\lfloor s\rfloor$.

This is quite a non-trivial and non-obvious constraint on $f_m^{s,i}(\lambda,-\Delta_\pm)$ which will be useful as a check of our calculations. Equation (5.94) is one of our main results and gives us the three-point functions when removing[6] $\frac{1}{2\pi}\mu_\phi\mu_s$

5.3.2 Solution of the Vasilev Recursion Relations

According to equation (5.94), the calculation of the three-point functions is reduced to solving the Vasiliev equations (5.11) recursively in order to express the auxiliary fields C^s_{-m} in terms of $\phi_\pm$. This task is most easily solved by splitting it into two steps. We will first express the minimal components C^{m+1}_{-m} and $C^{m+\frac{3}{2}}_{-m}$ in terms of C^1_0 and $C^{\frac{3}{2}}_0$, afterwards express the non-minimal components $C^{s\neq m+1,m+\frac{3}{2}}_{-m}$ in terms of C^{m+1}_{-m} and $C^{m+\frac{3}{2}}_{-m}$. Combining these two solutions, we can express C^s_{-m} in terms of the physical scalars $\phi_\pm$ which is what we need in equation (5.94).

For the first step we need to use the z-equations of (5.11) for the negative mode minimal components

$$\begin{aligned} L^{(m+1)}_{-m,z}:&\quad \partial C^{m+1}_{-m}+e^\rho\, g_3^{2,m+2}(1,-m-1)\, C^{m+2}_{-m-1}=0,\\ L^{(m+\frac{3}{2})}_{-m,z}:&\quad \partial C^{m+\frac{3}{2}}_{-m}+e^\rho\, g_3^{2,m+\frac{5}{2}}(1,-m-1)\, C^{m+\frac{5}{2}}_{-m-1}+e^\rho\, g_{\frac{5}{2}}^{2,m+2}(1,-m-1)\, C^{m+2}_{-m-1}=0.\end{aligned}$$

The first of these equations are readily solved

$$C^{m+1}_{-m}=\left(\prod_{i=1}^m g_3^{2,i+1}(1,-i)\right)^{-1}\left(-e^{-\rho}\partial\right)^m C^1_0. \tag{5.96}$$

The second equation is easier to solve if one considers the more general recursion relation

$$\alpha_m C^{m+\frac{3}{2}}+C^{m+\frac{5}{2}}+\beta_m C^{m+2}=0, \tag{5.97}$$

which have the solution

$$C^{m+\frac{5}{2}}=\prod_{i=0}^m(-\alpha_i)C^{\frac{3}{2}}+\sum_{p=1}^{m+1}\left(\prod_{j=p}^m(-\alpha_j)\right)(-\beta_{p-1})C^{p+1}. \tag{5.98}$$

Putting the coefficients to

$$\alpha_m=e^{-\rho}\left(g_3^{2,m+\frac{5}{2}}(1,-m-1)\right)^{-1}\partial,\qquad \beta_m=\frac{g_{\frac{5}{2}}^{2,m+2}(1,-m-1)}{g_3^{2,m+\frac{5}{2}}(1,-m-1)}, \tag{5.99}$$

[6] Recall our analysis of holographic Ward identities, where we found out that $\frac{\mu_s}{2\pi}$ correspond to the correct normalized source of the dual field operator and not μ_s.

and using the other solution (5.96), one can write down the solution of the second equation as

$$
\begin{aligned}
C_{-m}^{m+\frac{3}{2}} = {} & \left(\prod_{i=1}^{m} g_3^{2,i+\frac{3}{2}}(1,-i)\right)^{-1} \left(-e^{\rho}\partial\right)^{m} C_0^{\frac{3}{2}} + \sum_{p=1}^{m}\left(\prod_{j=p+1}^{m} g_3^{2,j+\frac{3}{2}}(1,-j)\right)^{-1} \\
& \times \left(\prod_{k=1}^{p} g_3^{2,k+1}(1,-k)\right)^{-1}\left(\frac{-g_{\frac{5}{2}}^{2,p+1}(1,-p)}{g_3^{2,p+\frac{3}{2}}(1,-p)}\right)\left(-e^{-\rho}\partial\right)^{m} C_0^{1}.
\end{aligned}
\tag{5.100}
$$

One can find very similar expressions for the auxiliary fields with positive mode using the $\bar{z}$ equations of (5.11), these are given by

$$
C_m^{m+1} = \left(\prod_{i=1}^{m} g_3^{i+1,2}(i,-1)\right)^{-1}\left(e^{-\rho}\bar{\partial}\right)^{m} C_0^{1}, \tag{5.101}
$$

and

$$
\begin{aligned}
C_m^{m+\frac{3}{2}} = {} & \left(\prod_{i=1}^{m} g_3^{i+\frac{3}{2},2}(i,-1)\right)^{-1}\left(e^{-\rho}\bar{\partial}\right)^{m} C_0^{\frac{3}{2}} + \sum_{p=1}^{m}\left(\prod_{j=p+1}^{m} g_3^{j+\frac{3}{2},2}(j,-1)\right)^{-1} \\
& \times \left(\frac{-g_{\frac{5}{2}}^{p+1,2}(p,-1)}{g_3^{p+\frac{3}{2},2}(p,-1)}\right)\left(e^{-\rho}\bar{\partial}\right)^{m-p} C_p^{p+1}.
\end{aligned}
\tag{5.102}
$$

Now for the second step we need to use the ρ-equations of (5.11) given by

$$
\partial_\rho C_m^s + 2C_m^{s-1} + \kappa_s\, C_m^{s+1} + \omega_{s-1/2}\, C_m^{s-\frac{1}{2}} + \sigma_{s+\frac{1}{2}}\, C_m^{s+\frac{1}{2}} = 0, \tag{5.103}
$$

where out of convenience we have defined the quantities

$$
\begin{aligned}
\kappa_s &\equiv 2g_3^{s+1,2}(m,0), \\
\omega_{s-\frac{1}{2}} &\equiv 2g_{\frac{3}{2}}^{s-\frac{1}{2},2}(m,0), \\
\sigma_{s+\frac{1}{2}} &\equiv 2g_{\frac{5}{2}}^{s+\frac{1}{2},2}(m,0).
\end{aligned}
\tag{5.104}
$$

Note that we have suppressed the m dependence since we need to solve the above equation for fixed m. According to the properties of the structure constants listed in appendix B, $\omega_{s-\frac{1}{2}} = 0$ for $s \in \mathbb{Z} + \frac{1}{2}$ and $\sigma_{s+\frac{1}{2}} = 0$ for $s \in \mathbb{Z}$, thus we can split (5.103) into two slightly simpler equations[7]

$$
\begin{aligned}
\partial_\rho C_m^s + 2C_m^{s-1} + \kappa_s\, C_m^{s+1} + \omega_{s-\frac{1}{2}}\, C_m^{s-\frac{1}{2}} &= 0, \\
\partial_\rho C_m^{s+\frac{1}{2}} + 2C_m^{s-\frac{1}{2}} + \kappa_{s+\frac{1}{2}}\, C_m^{s+\frac{3}{2}} + \sigma_{s+1}\, C_m^{s+1} &= 0,
\end{aligned}
\qquad s \in \mathbb{Z}_{\geq 1}.
\tag{5.105}
$$

[7]Note the exceptions $\omega_{\frac{3}{2}-\frac{1}{2}} \propto m$ and $\sigma_{1+\frac{1}{2}} \propto m$, which lead to terms of the form $m\, C_m^1$ and $m\, C_m^{\frac{3}{2}}$. Only for $m = 0$ are these terms inside the wedge and thus they vanish (for $m > 0$, $C_m^1 = C_m^{\frac{3}{2}} = 0$).

Due to the σ and ω terms these two recursion relations are coupled to each other and this makes the equations difficult to solve. In appendix A we show how to solve these equations in the case of $\sigma_s = 0$, the general solution can possibly be obtained by similar techniques. For our needs we can simply solve these equations recursively on a computer, for example using `Mathematica`, to any arbitrary order we would like and then evaluate the expression (5.94). Let us however make a few general and important comments. Note that the general solution will be of the form

$$\begin{aligned} C_m^s &= \mathcal{O}_s\left(\partial_\rho\right) C_m^{m+1} + \mathcal{P}_s\left(\partial_\rho\right) C_m^{m+\frac{3}{2}}, \\ C_m^{s+\frac{1}{2}} &= \tilde{\mathcal{O}}_s\left(\partial_\rho\right) C_m^{m+1} + \tilde{\mathcal{P}}_s\left(\partial_\rho\right) C_m^{m+\frac{3}{2}}, \end{aligned} \tag{5.106}$$

where the differential operators clearly do not explicitly depend on ρ but only on ∂_ρ. In order to find the functions $\mathcal{G}_m^{s,i}(\lambda, \partial_\rho)$ of equation (5.70), we need to move the exponential factors of (5.96), (5.100), (5.101), (5.102), outside in equation (5.106). Since the operators $\mathcal{O}_s$, $\mathcal{P}_s$, $\tilde{\mathcal{O}}_s$ and $\tilde{\mathcal{P}}_s$ are polynomials of ∂_ρ (see appendix A), consider the following short calculation

$$\begin{aligned} \partial_\rho^n \left(e^{-m\rho}\,\phi\right) &= \sum_{q=0}^{n} \binom{n}{q} \partial_\rho^{n-q}\left(e^{-m}\right) \partial_\rho^q\,\phi, \\ &= \sum_{q=0}^{n} \binom{n}{q} (-m)^{n-q}\,\partial_\rho^q \phi\, e^{-m\rho}, \\ &= \Big[\left(\partial_\rho - m\right)^n \phi \Big]\, e^{-m\rho}, \end{aligned} \tag{5.107}$$

where we have used the binomial theorem for the differential operator in the last line. Thus if we remove by hand the exponential factors of (5.96), (5.100), (5.101), (5.102), and then shift the operators of equation (5.106) by

$$\partial_\rho \to \partial_\rho - m,$$

we will find the functions $\mathcal{G}_m^{s,i}(\lambda, \partial_\rho)$. This is an important detail to remember when implementing these recursion relations (5.105) in a mathematical software.

As a final remark, let us note that the ρ and ∂ dependence of the auxiliary fields are of the form[8] $C_{-m}^s \sim e^{-|m|\rho}\, A\big(\lambda, \partial_\rho\big)\, \partial^m \phi_i$ as claimed and used earlier.

5.3.3 Final Results for Three-Point Functions

We can finally calculate the three-point functions by using equation (5.94), removing the $\frac{1}{2\pi}\mu_\phi\mu_s$ factor, together with the solution of the above recursion relations. It is however difficult to proceed analytically partly because we do not have a general closed formula for the recursion relations, but mainly because the structure constants (B.2) are very complicated expressions and it is hard to rewrite the whole thing as simple functions of λ. We will therefore proceed by explicitly calculating the different three-point functions for low spin s, then extrapolate the result to arbitrary s. These closed expressions can then be checked on a computer for many spins s.

Let us briefly comment on some consistency checks. We have checked that the constraints (5.95) are satisfied for wide range of values s. Remarkably if we modify the

[8] Recall that we always put one of the scalars $\phi_\pm$ to zero.

expression (5.70), even slightly, then the constraint (5.95) will no longer be satisfied. Furthermore it turns out that the expression (5.94) for the three-point functions exactly end up having the correct (z_1, z_2, z_3)-dependence which is required by conformal symmetry (5.92), but not manifest from (5.94) at all. Here we also observe that even the smallest changes of the equations (5.70) or (5.94), will result in "three-point functions" with complicated (z_1, z_2, z_3)-dependence and the result will not respect conformal symmetry (5.92)! There are many other similar checks which seem quit remarkable that things work out. These tests are highly non-trivial and its very encouraging that our results seem to be quit consistent and "robust".

Since all three-point functions we are considering are of the form

$$\left\langle \mathcal{O}_\Delta(z_1,\bar{z}_1)\bar{\mathcal{O}}_\Delta(z_2,\bar{z}_2)J^{(s)}(z_3)\right\rangle = \left\langle \mathcal{O}_\Delta\bar{\mathcal{O}}_\Delta J^{(s)}\right\rangle \left(\frac{z_{12}}{z_{13}z_{23}}\right)^s \left\langle \mathcal{O}_\Delta(z_1,\bar{z}_1)\bar{\mathcal{O}}_\Delta(z_2,\bar{z}_2)\right\rangle,$$

we will use the notation $\left\langle \mathcal{O}_\Delta\bar{\mathcal{O}}_\Delta J^{(s)}\right\rangle$ to denote the coefficients. Let us take the dual operator of ϕ_+ with conformal weight $\Delta_+ = 2(1-\lambda)$. By solving the recursion relations above and following the detailed procedure developed in this chapter, equation (5.94) give us the following coefficients for low spin

$$\begin{aligned}
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{2+}\right\rangle &= -(\lambda-1),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{3+}\right\rangle &= -\frac{1}{3}(\lambda-1)(2\lambda-3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{4+}\right\rangle &= -\frac{1}{5}(\lambda-2)(\lambda-1)(2\lambda-3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{5+}\right\rangle &= -\frac{2}{35}(\lambda-2)(\lambda-1)(2\lambda-5)(2\lambda-3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{6+}\right\rangle &= -\frac{2}{63}(\lambda-3)(\lambda-2)(\lambda-1)(2\lambda-5)(2\lambda-3).
\end{aligned} \tag{5.108}$$

Note that $W^{2+}(z)$ is the holographic part of the energy-momentum tensor and therefore the coefficient of the three-point function must be the holomorphic conformal weight of $\mathcal{O}^{\mathcal{B}}_{\Delta_+}$ which is $h_+ = 1-\lambda$ (see equations (4.11) and (4.12)). Encouragingly this is exactly what we find. Let us also show a few low-spin results of three-point functions with the same scalar but with the other bosonic higher-spin current

$$\begin{aligned}
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{2-}\right\rangle &= -\frac{1}{3}(\lambda-1)(2\lambda+1),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{3-}\right\rangle &= -\frac{2}{15}(\lambda-1)(\lambda+1)(2\lambda-3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{4-}\right\rangle &= -\frac{1}{35}(\lambda-2)(\lambda-1)(2\lambda-3)(2\lambda+3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{5-}\right\rangle &= -\frac{4}{315}(\lambda-2)(\lambda-1)(\lambda+2)(2\lambda-5)(2\lambda-3),\\
\left\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}W^{6-}\right\rangle &= -\frac{2}{693}(\lambda-3)(\lambda-2)(\lambda-1)(2\lambda-5)(2\lambda-3)(2\lambda+5).
\end{aligned} \tag{5.109}$$

Amazingly it turns out that all three-point functions factorize as the above examples and thus make it easy for us to guess the correct closed form expression for all spin. For the CFT dual fields corresponding to $\tilde{\phi}_\pm$, we need to multiply by a factor of $(-1)^s$ due to the different coupling to the higher-spin fields (5.3). The general expressions are given by

$$\begin{aligned}
\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+} \bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} W^{s+} \rangle &= (-1)^s \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s-2\lambda+1)}{\Gamma(2-2\lambda)}, \\
\langle \mathcal{O}^{\mathcal{B}}_{\Delta_-} \bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_-} W^{s+} \rangle &= (-1)^s \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s-2\lambda)}{\Gamma(1-2\lambda)}, \\
\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} \bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_+} W^{s+} \rangle &= (-1)^{s-1} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)}, \\
\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-} \bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_-} W^{s+} \rangle &= (-1)^{s-1} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(-2\lambda)}{\Gamma(-2\lambda-s+1)},
\end{aligned} \tag{5.110}$$

We have checked these closed-form expressions with our actual calculation for many spins and find perfect match. It is possible to combine these results into more unified formulas which depend only on s, the holomorphic conformal weights and the type of the fields involved, as

$$\begin{aligned}
\langle \mathcal{O}^{\mathcal{B}}_{h} \bar{\mathcal{O}}^{\mathcal{B}}_{h} W^{s+} \rangle &= (-1)^s \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s+2h-1)}{\Gamma(2h)}, \\
\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{h} \bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{h} W^{s+} \rangle &= \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s+2h-1)}{\Gamma(2h)}.
\end{aligned} \tag{5.111}$$

Comparing these general formulas with the non-supersymmetric results of [84], and accounting for the different conformal weights in that case ($h_\pm = \frac{1}{2}(1 \pm \lambda)$), we find perfect agreement (up to a normalization-dependent factor of $-1/(2\pi)$).

We can follow the same procedure to find the three-point functions containing the other bosonic higher spin fields, which are not present in the non-supersymmetric case:

$$\begin{aligned}
\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+} \bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} W^{s-} \rangle &= (-1)^{s-1} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s-2\lambda+1)}{\Gamma(2-2\lambda)} \frac{s-1+2\lambda}{2s-1}, \\
\langle \mathcal{O}^{\mathcal{B}}_{\Delta_-} \bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_-} W^{s-} \rangle &= (-1)^{s} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(s-2\lambda)}{\Gamma(1-2\lambda)} \frac{s-2\lambda}{2s-1}, \\
\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} \bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_+} W^{s-} \rangle &= (-1)^{s} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)} \frac{s-1+2\lambda}{2s-1}, \\
\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-} \bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_-} W^{s-} \rangle &= (-1)^{s-1} \frac{\Gamma^2(s)}{\Gamma(2s-1)} \frac{\Gamma(-2\lambda)}{\Gamma(-2\lambda-s+1)} \frac{s-2\lambda}{2s-1}.
\end{aligned} \tag{5.112}$$

The coefficients (5.110) and (5.112) are our main results from the bulk calculation.

We notice that the coefficients of the same primaries with the W^{s+} and W^{s-} currents are very closely related. Although the reason for this similarity is not very clear from the bulk calculation, it is obvious from the boundary theory perspective, as we will see in the following section.

It is straightforward to generalize the above in order to obtain correlation functions containing fermions. For fermionic matter, one would need to set the scalar fields to zero in 5.4 while keeping the fermionic ones. In order to include a fermionic higher-spin current, one would need to keep only a particular fermionic higher-spin generator in 5.20. Then the procedure in this section can be repeated with minor modifications.

5.4 Three-Point Functions from Boundary

Now we want to switch gears and consider the problem from the point of view of the dual CFT. Initially this problem seems to be quite difficult. Recall that the conjectured dual theory is the $\mathcal{N}=2$ $\mathbb{CP}^N$ Kazama-Suzuki model given by the following bosonic coset

$$\frac{\widehat{\mathfrak{su}}(N+1)_k \times \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{su}}(N)_{k+1} \times \widehat{\mathfrak{u}}(1)_{N(N+1)(k+N+1)}}, \tag{5.113}$$

where it should be dual to the classical Vasiliev theory with the parameter identification $\lambda = \frac{1}{2}\frac{N}{N+k}$ in the 't Hooft limit

$$\lim_{N,k\to\infty} \lambda = \text{fixed}. \tag{5.114}$$

Since the dual CFT is defined by a double scaling limit, one has to calculate the three-point functions for arbitrary N and k then take the 't Hooft limit. One way to do this is to consider a Feigin-Fuchs type free-field realization, as first done for the $c<1$ minimal models [124, 125].[9] This involves calculating conformal blocks of screened Vertex operators and then solve the monodromy problem to find the four-point function, from which one can extract the coefficient of the three-point function. This was done for the $\mathcal{N}=0$ case in [127]. In order to construct such a free-field realization of the Kazama-Suzuki models, one can with benefit start from the constructions given in [105, 128, 129]. It is however not the best way to proceed for several reasons. Although the calculation is definitely doable, it is not so simple to perform for arbitrary N and k and it involves working with complicated contour integrals. Furthermore in this approach the solution of a harder problem has to be calculated in order to extract the results of the simpler 't Hooft limit, it would be much smarter to go directly to the 't Hooft limit.[10]

We will here take a much simpler and smarter route to solve this problem, our calculation will be based purely on the symmetry $\mathcal{SW}_\infty[\lambda]$.[11] First note that all three-point functions take the form (5.92), where the coefficient is just given by the leading order pole of the OPE[12]

$$J^{(s)}(z)\mathcal{O}_\Delta(w,\bar{w}) \sim \frac{A(s)}{(z-w)^s}\,\mathcal{O}_\Delta(w,\bar{w}) + \ldots . \tag{5.115}$$

If we use a standard Laurent expansion $J^{(s)}(z) = \sum_n J^{(s)}_n\, z^{-n-s}$, we can turn this into

$$J^{(s)}_0(z)|\mathcal{O}_\Delta\rangle = A(s)|\mathcal{O}_\Delta\rangle. \tag{5.116}$$

Thus the three-point functions can found by calculating the higher spin charges of $\mathcal{O}_\Delta$, which is a problem in representation theory of $\mathcal{SW}_\infty[\lambda]$. This is in general not such a simple problem due to non-linearities of the algebra, especially for arbitrary central charge c. There are however certain simplifications which makes this much more straightforward.

As we discussed earlier, the super higher-spin algebra shs$[\lambda]$ give rise to $\mathcal{SW}_\infty[\lambda]$ by a quantum Drinfeld-Sokolov reduction but shs$[\lambda]$ is not a subalgebra due to non-linear

[9] This is a BRST construction [126] in which the relevant model is constructed by constraining a certain model of free-fields and primary fields are identified with (screened) Vertex operators.

[10] But on the other hand knowing the full finite N and k results will be useful in the future when one calculates $\mathcal{O}(1/N)$ quantum corrections of the Vasiliev theory.

[11] Note that this relies on the assumption that the Kazama-Suzuki model has $\mathcal{SW}_\infty[\lambda]$ as symmetry algebra, which is not a priory clear at all. For certain strong arguments in favor of this, see the recent paper [120].

[12] Here $J^{(s)}(z)$ represents a general higher spin current, both the bosonic and fermionic ones.

term. In [38] the non-supersymmetric hs[λ] and $\mathcal{W}_\infty[\lambda]$ were analyzed. It was argued that all the non-linear terms appearing in the commutator of elements in the wedge

$$[W^s_m, W^t_n], \qquad |m| < s, \quad |t| < t,$$

vanish in the limit of large central charge $c \to \infty$.[13] This has been shown recently to also hold in the $\mathcal{N} = 2$ case in [120], so we have that[14]

$$\mathrm{shs}[\lambda] \xrightarrow{\text{Drinfeld-Sokolov}} \mathcal{SW}_\infty[\lambda] \xrightarrow{c\to\infty,\, |n|<s} \mathrm{shs}[\lambda].$$

Since in the 't Hooft limit, and the bulk calculation we want to compare to, we have $c \to \infty$, we can assume that shs[λ] is a closed subalgebra of $\mathcal{SW}_\infty[\lambda]$.

Now assume that $\mathcal{O}_\Delta$ is a highest weight representation of $\mathcal{SW}_\infty[\lambda]$ with conformal weight $\Delta = h + \bar{h}$, then this must also be a representation of shs[λ] by restriction since it is a closed subalgebra in the limit we are interested in. Conversely any representation of shs[λ], with the highest weight state $\mathcal{O}_\Delta$, gives rise to a representation of $\mathcal{SW}_\infty[\lambda]$ by the usual procedure know from pure Virasoso algebra; we just have to assume $\mathcal{O}_\Delta$ is annihilated by all positive modes and construct a Verma module by the action of negative modes. This is in general a irreducible representation of $\mathcal{SW}_\infty[\lambda]$ except for very specific values of c, in which there are null-states generating submodules making the full module degenerate (this is what happens for the minimal models). To summarize, in the $c \to \infty$ limit we expect the spectrum of primary fields to fall into representations of shs[λ].

We have thus reduced the calculation of three-point functions of the Kazama-Suzuki models in the 'Hooft limit, to studying representations of shs[λ] which is a much simpler task even though it is an infinite dimensional Lie algebra.

As we discussed earlier, shs[λ] can be constructed as a the quotient of $U(\mathfrak{osp}(1|2))$ with some ideal and all generators of the higher spin algebra can be expressed as products of $\mathfrak{osp}(1|2)$ generators. In particular if we specify the spin-two zero mode we can calculate the eigenvalues the all higher-spin zero modes

$$L^{(2)}_0 \mathcal{O}_\Delta = h\, \mathcal{O}_\Delta, \qquad \Rightarrow \qquad L^{(s)}_0 \mathcal{O}_\Delta = A(s)\, \mathcal{O}_\Delta, \tag{5.117}$$

thus the $\mathfrak{osp}(1|2)$ representation, specified by h, gives rise to a shs[λ] representation. This is not such a difficult problem, but it requires us to express the generators of shs[λ] in terms of those of $\mathfrak{osp}(1|2)$, similar to (2.23), with the correct normalizations. We will however take another route.

5.4.1 Field Theoretic Approach

Instead of working directly with representation theory, we will use a more field theoretic approach to generate the necessary representation theory data we need. We have argued that the calculation of the three-point functions reduce to symmetry and therefore we are allowed to pick any CFT, simpler than the $\mathbb{CP}^N$ Kazama-Suzuki model, with the correct symmetry algebra. Our arguments are actually much stronger than that, we can use any CFT we like as long as it contains hs[λ] as a closed subalgebra. The simplest class of

[13] This was shown in the case of finite dimensional Lie algebras $\mathfrak{g}$ and the Drinfeld-Sokolov reduction of their affinization $\hat{\mathfrak{g}}$ in [130].

[14] This was a conjecture we were assuming during this work, until [120] recently appeared.

CFT's of them all are the free CFT's, in which there are an infinite number of (higher spin) conserved currents.[15]

The possibly simplest CFT realization of the shs[λ] algebra is given by the ghost CFT as known from superstring theory [131]

$$S = \frac{1}{\pi}\int \mathrm{d}^2 z \left(b\bar{\partial}c + \beta\bar{\partial}\gamma + \tilde{b}\partial\tilde{c} + \tilde{\beta}\partial\tilde{\gamma}\right). \tag{5.118}$$

which has the free field OPE's:

$$\gamma(z)\beta(w) \sim \frac{1}{z-w} \quad , \quad \text{and} \quad c(z)b(w) \sim \frac{1}{z-w} \tag{5.119}$$

and similarly for the tilded fields. Here b, c, $\tilde{b}$ and $\tilde{c}$ are anti-commuting fermions while β, γ, $\tilde{\beta}$ and $\tilde{\gamma}$ are bosons. It was shown in [39] that this free CFT has an infinite number of conserved currents which together form the $\mathcal{N}=2$ linear $sw_\infty[\lambda] \oplus sw_\infty[\lambda]$ algebra. Although this is not equivalent to the $\mathbb{C}P^N$ Kazama-Suzuki model or even have the non-linear $\mathcal{SW}_\infty[\lambda] \oplus \mathcal{SW}_\infty[\lambda]$ algebra in common, they both have an shs[λ] ⊕ shs[λ] closed subalgebra. This implies that if we can construct primary fields with the correct conformal weights in this free theory, then the coefficients in the leading order pole (5.115) would exactly correspond to the higher-spin zero mode and thereby the coefficients of three-point function of the Kazama-Suzuki CFT in the 't Hooft limit.

The conformal weights of the fields are given by

	b	c	β	γ	$\tilde{b}$	$\tilde{c}$	$\tilde{\beta}$	$\tilde{\gamma}$
h	$\lambda+\frac{1}{2}$	$\frac{1}{2}-\lambda$	λ	$1-\lambda$	0	0	0	0
$\bar{h}$	0	0	0	0	$\lambda+\frac{1}{2}$	$\frac{1}{2}-\lambda$	λ	$1-\lambda$

Remarkably, this is exactly the same as the coset primaries discussed in section 4.2.

We will use these fields to construct CFT operators that are dual to the bulk fields $\phi_\pm, \tilde{\phi}_\pm, \psi_\pm, \tilde{\psi}_\pm$. Recall that [82, 83, 51] the bulk fields are arranged in multiplets of $\mathcal{N}=2$ supersymmetry:

$$\left(\phi_+, \psi_\pm, \phi_-\right) \qquad \text{and} \qquad \left(\tilde{\phi}_+, \tilde{\psi}_\pm, \tilde{\phi}_-\right), \tag{5.120}$$

where the scalars appearing in each multiplet have different masses, $(M_+^B)^2 = (\tilde{M}_+^B)^2 = -4\lambda(1-\lambda)$ and $(M_-^B)^2 = (\tilde{M}_-^B)^2 = -1+4\lambda^2$, but are oppositely quantized (ϕ_+ and $\tilde{\phi}_-$ have the normal quantization, ϕ_- and $\tilde{\phi}_+$ the alternative one).

Identifying these fields with the coset fields, we can construct the dual fields as discussed in section 4.2 (see equations (4.12) and (4.13))

$$\begin{aligned} \mathcal{O}^{\mathcal{B}}_{\Delta_+}(z,\bar{z}) &= \gamma(z)\otimes\tilde{\gamma}(\bar{z}), \qquad & \mathcal{O}^{\mathcal{F}}_{\Delta_+}(z,\bar{z}) &= c(z)\otimes\tilde{\gamma}(\bar{z}), \\ \mathcal{O}^{\mathcal{B}}_{\Delta_-}(z,\bar{z}) &= c(z)\otimes\tilde{c}(\bar{z}), \qquad & \mathcal{O}^{\mathcal{F}}_{\Delta_-}(z,\bar{z}) &= \gamma(z)\otimes\tilde{c}(\bar{z}), \end{aligned} \tag{5.121}$$

and

$$\begin{aligned} \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}(z,\bar{z}) &= \beta(z)\otimes\tilde{\beta}(\bar{z}), \qquad & \tilde{\mathcal{O}}^{\mathcal{F}}_{\Delta_+}(z,\bar{z}) &= b(z)\otimes\tilde{\beta}(\bar{z}), \\ \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-}(z,\bar{z}) &= b(z)\otimes\tilde{b}(\bar{z}), \qquad & \tilde{\mathcal{O}}^{\mathcal{F}}_{\Delta_-}(z,\bar{z}) &= \beta(z)\otimes\tilde{b}(\bar{z}). \end{aligned} \tag{5.122}$$

[15] Note that this is not a free CFT realization of the full Kamaza-Suzuki models, like in the Feigin-Fuchs type constructions. In that case the CFT is not really free and a BRST procedure has to be used to project out unwanted states. We are looking for truly free models since we are only interested in shs[λ] and not the full CFT.

The scaling dimensions of these fields $\Delta = h + \bar{h}$ precisely match the dimensions corresponding to the bulk fields with the appropriate quantization, as discussed earlier.

The higher spin currents corresponding to the linear $sw_\infty[\lambda] \oplus sw_\infty[\lambda]$ algebra are given by [39]:

$$V_\lambda^{(s)+}(z) = \sum_{i=0}^{s-1} a^i(s,\lambda)\partial^{s-1-i}\left\{(\partial^i\beta)\gamma\right\} + \sum_{i=0}^{s-1} a^i(s,\lambda+\tfrac{1}{2})\partial^{s-1-i}\left\{(\partial^i b)c\right\}, \tag{5.123}$$

$$V_\lambda^{(s)-}(z) = -\frac{s-1+2\lambda}{2s-1}\sum_{i=0}^{s-1} a^i(s,\lambda)\partial^{s-1-i}\left\{(\partial^i\beta)\gamma\right\} + \frac{s-2\lambda}{2s-1}\sum_{i=0}^{s-1} a^i(s,\lambda+\tfrac{1}{2})\partial^{s-1-i}\left\{(\partial^i b)c\right\}, \tag{5.124}$$

and

$$Q_\lambda^{(s)\pm}(z) = \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\partial^{s-1-i}\left\{(\partial^i\beta)c\right\} \mp \sum_{i=0}^{s-2} \beta^i(s,\lambda)\partial^{s-2-i}\left\{(\partial^i b)\right\}\gamma, \tag{5.125}$$

and similarly for the anti-holomorphic sector. The coefficients are given in equation (B.18).

These currents are normalized such that their Laurent modes (when restricting to the wedge) correspond to the shs[λ] generators (2.30) in the exactly same basis [39]. Thus the higher-spin zero modes of the dual fields (5.116), and thereby three-point functions should be directly comparable to the bulk calculation.

5.4.2 Operator Product Expansions

In order to compute three-point funtions involving the higher spin currents we need to compute the coefficient of the leading order pole of the OPE between higher spin currents and the primaries (5.121) and (5.122). It is straightforward to do this using (5.119) and the form of the higher spin currents given in (5.123), (5.124) and (5.125), we will list the result here. For $V_\lambda^{(s)+}$ we have

$$\begin{aligned}
V_\lambda^{(s)+}(z)\beta(w) &\sim a^0(s,\lambda)\frac{(-1)^{s-1}(s-1)!}{(z-w)^s}\beta(w) + \cdots, \\
V_\lambda^{(s)+}(z)b(w) &\sim a^0(s,\lambda+\tfrac{1}{2})\frac{(-1)^{s-1}(s-1)!}{(z-w)^s}b(w) + \cdots, \\
V_\lambda^{(s)+}(z)\gamma(w) &\sim \left(\sum_{i=0}^{s-1} a^i(s,\lambda)\right)\frac{(-1)^{s}(s-1)!}{(z-w)^s}\gamma(w) + \cdots, \\
V_\lambda^{(s)+}(z)c(w) &\sim \left(\sum_{i=0}^{s-1} a^i(s,\lambda+\tfrac{1}{2})\right)\frac{(-1)^{s}(s-1)!}{(z-w)^s}c(w) + \cdots.
\end{aligned} \tag{5.126}$$

In a similar manner we find the OPE's involving the $V_\lambda^{(s)-}$ currents are given by

$$\begin{aligned}
V_\lambda^{(s)-}(z)\beta(w) &\sim \frac{s-1+2\lambda}{2s-1} a^0(s,\lambda)\frac{(-1)^s(s-1)!}{(z-w)^s}\beta(w)+\cdots\,,\\
V_\lambda^{(s)-}(z)b(w) &\sim \frac{s-2\lambda}{2s-1} a^0(s,\lambda+\tfrac{1}{2})\frac{(-1)^{s-1}(s-1)!}{(z-w)^s}b(w)+\cdots\,,\\
V_\lambda^{(s)-}(z)\gamma(w) &\sim \frac{s-1+2\lambda}{2s-1}\left(\sum_{i=0}^{s-1} a^i(s,\lambda)\right)\frac{(-1)^{s-1}(s-1)!}{(z-w)^s}\gamma(w)+\cdots\,,\\
V_\lambda^{(s)-}(z)c(w) &\sim \frac{s-2\lambda}{2s-1}\left(\sum_{i=0}^{s-1} a^i(s,\lambda+\tfrac{1}{2})\right)\frac{(-1)^{s}(s-1)!}{(z-w)^s}c(w)+\cdots\,.
\end{aligned} \tag{5.127}$$

Finally for the fermionic higher-spin currents $Q_\lambda^{(s)\pm}$ we find

$$\begin{aligned}
Q_\lambda^{(s)\pm}(z)\beta(w) &\sim \mp\beta^0(s,\lambda)\frac{(-1)^s(s-2)!}{(z-w)^{s-1}}b(w)+\cdots\,,\\
Q_\lambda^{(s)\pm}(z)b(w) &\sim \alpha^0(s,\lambda)\frac{(-1)^{s-1}(s-1)!}{(z-w)^s}\beta(w)+\cdots\,,\\
Q_\lambda^{(s)\pm}(z)\gamma(w) &\sim \left(\sum_{i=0}^{s-1}\alpha^s(s,\lambda)\right)\frac{(-1)^s(s-1)!}{(z-w)^s}c(w)+\cdots\,,\\
Q_\lambda^{(s)\pm}(z)c(w) &\sim \mp\left(\sum_{i=0}^{s-2}\beta^i(s,\lambda)\right)\frac{(-1)^s(s-2)!}{(z-w)^{s-1}}\gamma(w)+\cdots\,.
\end{aligned} \tag{5.128}$$

In order to be able to compare the CFT three-point functions with the bulk results (5.110) and (5.112), we will write the coefficients in the following form

$$\begin{aligned}
a^0(s,\lambda)(s-1)! &= \frac{\Gamma(s)^2}{\Gamma(2s-1)}\frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)},\\
\beta^0(s,\lambda)(s-2)! &= \frac{\Gamma(s-1)\Gamma(s)}{\Gamma(2s-2)}\frac{\Gamma(-2\lambda)}{\Gamma(-2\lambda-s+2)},\\
\alpha^0(s,\lambda)(s-1)! &= \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(2s-2)}\frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)}.
\end{aligned} \tag{5.129}$$

Furthermore it is straightforward to perform the necessary sums over the coefficients, which results in

$$\begin{aligned}
\sum_{i=0}^{s-1} a^i(s,\lambda) &= \frac{4^{1-s}\sqrt{\pi}\,\Gamma(1+s-2\lambda)}{\Gamma(s-\frac{1}{2})\Gamma(2-2\lambda)} = \frac{\Gamma(s)}{\Gamma(2s-1)}\frac{\Gamma(1+s-2\lambda)}{\Gamma(2-2\lambda)},\\
\sum_{i=0}^{s-2} \beta^i(s,\lambda) &= \frac{2^{3-2s}\sqrt{\pi}\,(s-1)\,\Gamma(s-2\lambda)}{\Gamma(s-\frac{1}{2})\Gamma(2-2\lambda)} = 2\frac{\Gamma(s)(s-1)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda)}{\Gamma(2-2\lambda)},\\
\sum_{i=0}^{s-1} \alpha^i(s,\lambda) &= \frac{(-1)^{s-1}2^{3-2s}\sqrt{\pi}\,\Gamma(2\lambda)}{\Gamma(s-\frac{1}{2})\Gamma(1-s+2\lambda)} = (-1)^{s-1}2\frac{\Gamma(s)}{\Gamma(2s-1)}\frac{\Gamma(2\lambda)}{\Gamma(1-s+2\lambda)},
\end{aligned} \tag{5.130}$$

for $s>1$.

5.4.3 Bosonic Three-Point Correlators from the CFT

We now have all the necessary ingredients to compute all three-point correlators of two bosonic or fermionic operators with a spin-s current. In this section we list all the bosonic three-point functions, using the notation used in section 5.3.3.

First Multiplet with $V_\lambda^{(s)+}$

$$\begin{aligned}\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}V_\lambda^{(s)+}\rangle &= (-1)^s\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda+1)}{\Gamma(2-2\lambda)},\\ \langle \mathcal{O}^{\mathcal{B}}_{\Delta_-}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_-}V_\lambda^{(s)+}\rangle &= (-1)^s\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda)}{\Gamma(1-2\lambda)}.\end{aligned} \tag{5.131}$$

Second multiplet with $V_\lambda^{(s)+}$

$$\begin{aligned}\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}\bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_+}V_\lambda^{(s)+}\rangle &= (-1)^{s-1}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)},\\ \langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-}\bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_-}V_\lambda^{(s)+}\rangle &= (-1)^{s-1}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(-2\lambda)}{\Gamma(-2\lambda-s+1)}.\end{aligned} \tag{5.132}$$

First multiplet with $V_\lambda^{(s)-}$

$$\begin{aligned}\langle \mathcal{O}^{\mathcal{B}}_{\Delta_+}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}V_\lambda^{(s)-}\rangle &= (-1)^{s-1}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda+1)}{\Gamma(2-2\lambda)}\frac{s-1+2\lambda}{2s-1},\\ \langle \mathcal{O}^{\mathcal{B}}_{\Delta_-}\bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_-}V_\lambda^{(s)-}\rangle &= (-1)^s\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(s-2\lambda)}{\Gamma(1-2\lambda)}\frac{s-2\lambda}{2s-1}.\end{aligned} \tag{5.133}$$

Second multiplet with $V_\lambda^{(s)-}$

$$\begin{aligned}\langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_+}\bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_+}V_\lambda^{(s)-}\rangle &= (-1)^s\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(-2\lambda+1)}{\Gamma(-2\lambda-s+2)}\frac{s-1+2\lambda}{2s-1},\\ \langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_-}\bar{\tilde{\mathcal{O}}}^{\mathcal{B}}_{\Delta_-}V_\lambda^{(s)-}\rangle &= (-1)^{s-1}\frac{\Gamma^2(s)}{\Gamma(2s-1)}\frac{\Gamma(-2\lambda)}{\Gamma(-2\lambda-s+1)}\frac{s-2\lambda}{2s-1}.\end{aligned} \tag{5.134}$$

Comparing with the bulk computation of the same quantities we find precise agreement. This provides a non-trivial check of the $\mathcal{N}=2$ proposal of [51].

5.4.4 Fermionic Three-Point Correlators from the CFT

The above methods can also be used to compute boundary three-point functions involving fermions. It is immediately clear that the coefficients of correlators involving two fermionic operators and one holomorphic bosonic higher-spin current will be the same as those of the bosonic correlators of operators that share the same chiral part. More precisely this class of fermionic three-point functions are given by

$$\langle \mathcal{O}^{\mathcal{F}}_{\Delta_\pm}\bar{\mathcal{O}}^{\mathcal{F}}_{\Delta_\pm}V_\lambda^{(s)p}\rangle = \langle \mathcal{O}^{\mathcal{B}}_{\Delta_\mp}\bar{\mathcal{O}}^{\mathcal{F}}_{\Delta_\mp}V_\lambda^{(s)p}\rangle, \qquad \langle \tilde{\mathcal{O}}^{\mathcal{F}}_{\Delta_\pm}\bar{\tilde{\mathcal{O}}}^{\mathcal{F}}_{\Delta_\pm}V_\lambda^{(s)p}\rangle = \langle \tilde{\mathcal{O}}^{\mathcal{B}}_{\Delta_\mp}\bar{\tilde{\mathcal{O}}}^{\mathcal{F}}_{\Delta_\mp}V_\lambda^{(s)p}\rangle, \tag{5.135}$$

for $p = \pm$. On the other hand, the coefficients of the three-point functions involving one bosonic primary, one fermionic primary and a fermionic higher-spin current will be different. As as example, we find

$$\langle \mathcal{O}^{\mathcal{F}}_{\Delta_+} \bar{\mathcal{O}}^{\mathcal{B}}_{\Delta_+} Q^{(s)\pm}_{\lambda} \rangle = \pm 2(-1)^s \frac{\Gamma(s)^2}{\Gamma(2s-1)} \frac{\Gamma(s-2\lambda)}{\Gamma(2-2\lambda)}. \tag{5.136}$$

It would clearly be interesting to compute the above fermionic coefficients from the bulk side of the duality. This computation will require a straighforward generalisation of the discussion in section 5.

Chapter **6**

Conclusion

In this thesis we considered the resent conjectures about holographic duality between Prokushkin-Vasiliev theory on AdS_3 and $\mathcal{W}$-algebraic minimal models. We gave an introduction to some aspects of higher-spin gravity on AdS_3 and to $\mathcal{W}$-algebras. Unfortunately we did not have time to discuss the details of the conjectures and the interesting results there are known about them.

Our main focus was on the proposal of [51] that the $\mathcal{N} = 2$ Prokushkin-Vasiliev theory on AdS_3 is dual to the $\mathbb{C}P^N$ Kazama-Suzuki model with the non-linear chiral algebra $\mathcal{SW}_\infty[\lambda]$. In the 't Hooft limit, we showed exact matching between three-point functions involving two bulk scalars and one bosonic higher-spin field as computed from the bulk and the same quantities computed in the dual CFT. Since the correlation functions in this class only depend on the linear shs[λ] algebra, they can be computed in any CFT that shares this symmetry. We chose to compute them in a free-field ghost CFT. This greatly simplified the boundary side of the computation. These results were recently published in [1].

In [51], a specific gluing of coset chiral states was proposed as dual to the bulk fields (see (4.12) and (4.13)). Our bulk calculation only has information about the full conformal weight $\Delta = h + \bar{h}$ of the coset primaries, but the results correctly capture the dependence on the chiral conformal weights separately. This provides further evidence for the identification of states in [51].

Using the CFT, we have also obtained results for three-point functions involving fermionic operators, and it would clearly be of interest to compare those with the corresponding bulk quantities. This will require a slight generalization of our bulk techniques, in particular in order to isolate the physical fermionic fields from the Vasiliev equations.

Of course, our approach of using a surrogate free-field CFT instead of the full-fledged Kazama-Suzuki model has severe limitations. It would be interesting to check whether other types of three-point functions (for instance, those involving three scalar fields) match between the bulk and the boundary theory. But it is unlikely that the free-field CFT can correctly capture those correlation functions, so any mismatch would be likely to be an artifact of this. Even if one could reproduce all three-point functions, the simple fact that the spectrum of the free theory is not the same as that of the $\mathbb{C}P^N$ model indicates that four-point functions will differ and matching those would require a more intricate boundary computation. Such checks would be essential in order to better establish the $\mathcal{N} = 2$ correspondence beyond the level of symmetries.

The correspondence is currently formulated by taking a double scaling limit of the $\mathbb{C}P^N$ Kazama-Suzuki models. In order to go beyond the quantities captured by the free CFT one would have to perform a computation at finite N and k, then take the 't Hooft limit at the end. One might instead imagine a procedure by which one could obtain the nonlinear $\mathcal{S}W_\infty[\lambda]$ symmetry and the Kazama-Suzuki models directly in the 't Hooft limit. A natural idea is to impose certain constraints on the current algebra of the linear $sw_\infty[\lambda]$ or the free ghost CFT, by a BRST procedure and thereby deform the theory to become non-linear. Or just directly perform quantum Drinfeld-Sokolov reduction on shs[λ], but there might be many subtleties since shs[λ] is infinite dimensional. If the dual CFT could be obtained directly in the 't Hooft limit, it would probably provide a much more efficient way to check the duality at large N, k.

Recently, an $\mathcal{N} = 1$ version of the higher-spin/minimal model correspondence was proposed [132]. We expect the techniques used in this paper to transfer to that case with minor modifications, allowing the comparison of three-point functions in that model as well.

Appendix A

Solution to a Recursion Relation

In this appendix we will sketch how to solve the two coupled recursion relations

$$\begin{aligned} \partial_\rho C_m^s + 2C_m^{s-1} + \kappa_s\, C_m^{s+1} + \omega_{s-\frac{1}{2}}\, C_m^{s-\frac{1}{2}} &= 0, \\ \partial_\rho C_m^{s+\frac{1}{2}} + 2C_m^{s-\frac{1}{2}} + \kappa_{s+\frac{1}{2}}\, C_m^{s+\frac{3}{2}} &= 0, \end{aligned} \qquad s \in \mathbb{Z}_{\geq 1}, \tag{A.1}$$

which are nothing but (5.103) for $\sigma_s = 0$. The second equation is now not coupled to the first one, and can thus be solved separately. It is actually just a slight generalization of a recursion relation in [84]. The solution can be expressed as

$$C^{s+\frac{1}{2}} = \mathcal{O}_{s+\frac{1}{2}} C^{m+\frac{3}{2}}, \tag{A.2}$$

where the differential operator is of the form

$$\mathcal{O}_s = (-1)^{\lfloor s \rfloor - 1 - m} \left(\prod_{p=2+m}^{\lfloor s \rfloor} \kappa_{p+s-\lfloor s \rfloor - 1} \right)^{-1} \left[\sum_{\alpha=0}^{\left\lfloor \frac{\lfloor s \rfloor - 1 - m}{2} \right\rfloor} A_\alpha(s,m)\, \partial_\rho^{\lfloor s \rfloor - 2\alpha - m - 1} \right], \tag{A.3}$$

with

$$A_\alpha(s,m) = (-2)^\alpha \sum_{i_1,\ldots,i_\alpha} \prod_{k=1}^{\alpha} \kappa_{i_k + s - \lfloor s \rfloor - 1}, \tag{A.4}$$

and the limits of the sums are given by

$$\begin{aligned} 2k + m \leq i_k \leq 2k + \lfloor s \rfloor - 1 - 2\alpha, \\ i_k \geq i_{k-1} + 2, \quad \forall k \geq 2. \end{aligned} \tag{A.5}$$

The trick is now to exploit what we know to find a simpler recursion relations that the first equation of (A.1). Looking at the form of (A.1), it is clear that an ansatz of the following form will work

$$C^s = \mathcal{O}_s C^{m+1} + \mathcal{P}_s C^{m+\frac{3}{2}}, \tag{A.6}$$

for some differential operator $\mathcal{P}_s$. Since $\mathcal{O}_s$ satisfy the operator equation and boundary condition

$$\mathcal{O}_s\, \partial_\rho + 2\mathcal{O}_{s-1} + \kappa_s\, \mathcal{O}_{s+1} = 0, \qquad \mathcal{O}_{m+1} = \mathbb{1},$$

we find the following recursion relation for $\mathcal{P}_s$

$$\mathcal{P}_s\,\partial_\rho + 2\mathcal{P}_{s-1} + \kappa_s\,\mathcal{P}_{s+1} + \omega_{s-\frac{1}{2}}\,\mathcal{O}_{s-\frac{1}{2}} = 0, \tag{A.7}$$

with the important boundary condition

$$\mathcal{P}_{m+1} = 0. \tag{A.8}$$

This means that we can express $\mathcal{P}_s$ in terms of $\mathcal{O}_s$ which is known, this greatly simplifies the original problem which otherwise would have been more difficult so solve. Using the notation $\tilde{\mathcal{O}}_s \equiv \omega_s\,\mathcal{O}_s$, one can show that the solution is given by

$$\begin{aligned}
\mathcal{P}_s &= \kappa_{s-1}^{-1} \sum_{i=0}^{s-m-1} (-1)^{s-m-i}\tilde{A}_i\,\partial_\rho^{s-m-1-i}, \\
\tilde{A}_i &= \sum_{\alpha=1-\tilde{h}\left(\frac{i}{2}\right)}^{\lfloor\frac{i}{2}\rfloor} (-2)^{\lfloor\frac{i}{2}\rfloor-\alpha} a_{i,\alpha}\,\tilde{\mathcal{O}}_{m-\frac{1}{2}+2\alpha+\tilde{h}\left(\frac{i}{2}\right)}, \\
a_{i,\alpha} &= \left(\prod_{\beta=2\alpha+\tilde{h}\left(\frac{i}{2}\right)}^{s-m-2} \kappa_{\beta+m}^{-1}\right) \sum_{\gamma_1,\dots\gamma_{\lfloor\frac{i}{2}\rfloor-\alpha}} \prod_{\omega=1}^{\lfloor\frac{i}{2}\rfloor-\alpha} \kappa_{\gamma_\omega+m} \prod_{\bar{\omega}=1}^{\lfloor\frac{i}{2}\rfloor-\alpha-1} \left(1-\delta_{\gamma_{\bar{\omega}+1}-\gamma_{\bar{\omega}},1}\right),
\end{aligned} \tag{A.9}$$

with the following intervals

$$2\alpha + \tilde{h}\left(\frac{i}{2}\right) + 1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{\lfloor\frac{i}{2}\rfloor-\alpha-1} < \gamma_{\lfloor\frac{i}{2}\rfloor-\alpha} \leq s-m-2. \tag{A.10}$$

Notice that these expressions are quite complicated since one has to insert (A.3), we will however refrain from showing the full expression containing the structure constants.

Appendix **B**

The $\mathcal{SB}[\mu]$ and $\mathrm{shs}[\lambda]$ Algebras

This appendix contains information and definitions of functions related to the algebras $\mathcal{SB}[\mu]$ and $\mathrm{shs}[\lambda]$, together with several properties used in the thesis.

B.1 Structure Constants of $\mathcal{SB}[\mu]$

In this section we will list explicit formulas for the structure constants of the infinite dimensional associative algebra, $\mathcal{SB}[\mu]$. See section 2.3.2 for a sketch of how these are derived from the results of [39, 40]. We will use the following notation for the $\mathcal{SB}[\mu]$ products

$$\begin{aligned}
L_m^{(s)} \star L_n^{(t)} &= \sum_{u=1}^{s+t-1} g_u^{st}(m,n;\lambda)\, L_{m+n}^{(s+t-u)}, &\qquad L_m^{(s)} \star G_q^{(t)} &= \sum_{u=1}^{s+t-1} h_u^{st}(m,q;\lambda)\, G_{m+q}^{(s+t-u)}, \\
G_p^{(s)} \star G_q^{(t)} &= \sum_{u=1}^{s+t-1} \tilde{g}_u^{st}(p,q;\lambda)\, L_{p+q}^{(s+t-u)}, &\qquad G_p^{(s)} \star L_n^{(t)} &= \sum_{u=1}^{s+t-1} \tilde{h}_u^{st}(p,n;\lambda)\, G_{p+n}^{(s+t-u)}.
\end{aligned} \tag{B.1}$$

If one does not put any constraints on the modes, this then corresponds to an associative algebra related to $sw_\infty[\lambda]$. If one restricts to the wedge subalgebra, one can show that it is safe to restrict the sums to $1 \leq u \leq \mathrm{Min}(2s-1, 2t-1)$ since the structure constants for higher u vanish (this is not the case for modes outside the wedge).

The $L \star L$ structure constant is given as

$$\begin{aligned}
g_u^{st}(m,n;\lambda) = \sum_i F_{st}^u &\left[h\Big(u + \tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big)i + \tilde{h}(s)\tilde{h}\Big(u + \tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big); \lambda\right] \\
&\times (m - \lfloor s \rfloor + 1)_{\lceil i,u,s,t \rceil_1} \, (n - \lfloor t \rfloor + 1)_{\lfloor u \rfloor - 1 + \tilde{h}\left(s+\frac{1}{2}\right)\tilde{h}\left(t+\frac{1}{2}\right) - \tilde{h}\left(u+\frac{1}{2}\right)\tilde{h}\left(s+t+\frac{1}{2}\right) - \lceil i,u,s,t \rceil_1},
\end{aligned} \tag{B.2}$$

where the range of the sum is

$$0 \leq i \leq h\Big(u + \tfrac{1}{2}\tilde{h}(s+t)\Big)\,(\lfloor u \rfloor - 1) + \tilde{h}(u)\tilde{h}\big(s+t+\tfrac{1}{2}\big) - \tilde{h}(s)\tilde{h}\Big(u + \tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big)\tilde{h}\big(u+\tfrac{1}{2}\big).$$

Similarly we have for the $G \star G$

$$\begin{aligned}\tilde{g}_u^{st}(p,q;\lambda) = -h\big(s+\tfrac{1}{2}\big)h\big(t+\tfrac{1}{2}\big)\sum_i (-1)^{[i+\tilde{h}(s)]\tilde{h}\left(u+\frac{1}{2}\tilde{h}(s+t)\right)} \\ \times F_{st}^u\Big[h\Big(u+\tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big)i+\tilde{h}\big(s+\tfrac{1}{2}\big)\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big);\lambda\Big] \\ \times \big(p-\lceil s\rceil+\tfrac{3}{2}\big)_{\lceil i,u,s,t\rceil_2}\big(q-\lceil t\rceil+\tfrac{3}{2}\big)_{\lfloor u\rfloor-\tilde{h}\left(s+\frac{1}{2}\right)-\tilde{h}(s+t)\tilde{h}(s)-\tilde{h}\left(s+t+\frac{1}{2}\right)\tilde{h}\left(u+\frac{1}{2}\right)-\lceil i,u,s,t\rceil_2},\end{aligned} \tag{B.3}$$

where,

$$\begin{aligned}0\le i\le h\Big(u+\tfrac{1}{2}\tilde{h}(s+t)\Big)(u-1)-\Big[\tilde{h}\big(s+\tfrac{1}{2}\big)+\tilde{h}(s+t)\tilde{h}(s)\Big]\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}\big(s+t+\tfrac{1}{2}\big)\Big) \\ \times\Big(\tilde{h}\big(u+\tfrac{1}{2}\big)+\tfrac{1}{2}\tilde{h}(s+t)\Big).\end{aligned} \tag{B.4}$$

And for $L \star G$

$$\begin{aligned}h_u^{st}(m,q;\lambda) = h^{(-1)^{\tilde{h}(t)}}\Big(u+\tfrac{1}{2}\tilde{h}(s)\Big)\sum_i F_{st}^u\Big[h\Big(u+\tfrac{1}{2}\tilde{h}\Big(s+\tfrac{1}{2}\tilde{h}(t)\Big)\Big)i+\tilde{h}(s) \\ \times\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}\big(t+\tfrac{1}{2}\big)\Big);\lambda\Big] \\ \times(m-\lfloor s\rfloor+1)_{\lceil i,u,s,t\rceil_3}\big(q-\lceil t\rceil+\tfrac{3}{2}\big)_{\lfloor u\rfloor-\tilde{h}\left(t+\frac{1}{2}\right)-\tilde{h}(t)\tilde{h}(s)-\tilde{h}\left(s+\frac{1}{2}\tilde{h}(u)\right)\tilde{h}\left(u+\frac{1}{2}\right)-\lceil i,u,s,t\rceil_3},\end{aligned} \tag{B.5}$$

where,

$$\begin{aligned}0\le i\le h\Big(u+\tfrac{1}{2}\tilde{h}\Big(s+\tfrac{1}{2}\tilde{h}\big(t+\tfrac{1}{2}\big)\Big)\Big)(u-1)-\tilde{h}(s)\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}\big(t+\tfrac{1}{2}\big)\Big)\tilde{h}\big(u+\tfrac{1}{2}\big) \\ -\tfrac{1}{2}\tilde{h}\Big(s+\tfrac{1}{2}\tilde{h}\big(t+\tfrac{1}{2}\big)\Big)\tilde{h}(u).\end{aligned} \tag{B.6}$$

And finally for the $G \star L$ product

$$\begin{aligned}\tilde{h}_u^{st}(p,n;\lambda) = h^{(-1)^{\tilde{h}(s)}}\Big(u+\tfrac{1}{2}\tilde{h}(t)\Big)\sum_i(-1)^{[i+\tilde{h}(s)]\tilde{h}\left(u+\frac{1}{2}\tilde{h}\left(t+\frac{1}{2}\tilde{h}\left(s+\frac{1}{2}\right)\right)\right)} \\ F_{st}^u\Big[h\Big(u+\tfrac{1}{2}\tilde{h}\Big(t+\tfrac{1}{2}\tilde{h}(s)\Big)\Big)i+\tilde{h}\big(s+\tfrac{1}{2}\big)\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}(t)\Big);\lambda\Big] \\ \times\big(p-\lceil s\rceil+\tfrac{3}{2}\big)_{\lceil i,u,s,t\rceil_4}(n-\lfloor t\rfloor+1)_{\lfloor u\rfloor-\tilde{h}\left(s+\frac{1}{2}\right)-\tilde{h}(s)\tilde{h}(t)-\tilde{h}\left(t+\frac{1}{2}\tilde{h}(s)\right)\tilde{h}\left(u+\frac{1}{2}\right)-\lceil i,u,s,t\rceil_4},\end{aligned} \tag{B.7}$$

where,

$$\begin{aligned}0\le i\le h\Big(u+\tfrac{1}{2}\tilde{h}\Big(t+\tfrac{1}{2}\tilde{h}\big(s+\tfrac{1}{2}\big)\Big)\Big)(u-1)-\tilde{h}\big(s+\tfrac{1}{2}\big)\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}(t)\Big)\tilde{h}\big(u+\tfrac{1}{2}\big) \\ -\tfrac{1}{2}\tilde{h}\Big(t+\tfrac{1}{2}\tilde{h}\big(s+\tfrac{1}{2}\big)\Big)\tilde{h}\Big(u+\tfrac{1}{2}\tilde{h}\big(s+\tfrac{1}{2}\big)\tilde{h}(t)\Big).\end{aligned} \tag{B.8}$$

The functions used in the above structure constants are

$$\begin{aligned}
\lceil i,u,s\rceil &= \left\lceil h(u)\frac{\left[i+\tilde h(u+\frac{1}{2})\tilde h(s)\right]}{2}\right\rceil \\
\lceil i,u,s,t\rceil_1 &= \lceil i, u+\tfrac{1}{2}\tilde h(s+t+\tfrac{1}{2}), s\rceil \\
\lceil i,u,s,t\rceil_2 &= \left\lceil i, u+\tfrac{1}{2}\tilde h(s+t+\tfrac{1}{2}), s+\tfrac{1}{2}\tilde h(s+t+\tfrac{1}{2})+\tfrac{1}{2}\tilde h(s+t)\left\{\tilde h(s+\tfrac{1}{2})+\tilde h(s)\tilde h(u+\tfrac{1}{2})\right\}\right\rceil \\
\lceil i,u,s,t\rceil_3 &= \left\lceil i, u+\tfrac{1}{2}\tilde h\Big(s+\tfrac{1}{2}\tilde h(t)\Big), s+\tfrac{1}{2}\tilde h(t)\tilde h(s)\tilde h(u)\right\rceil \\
\lceil i,u,s,t\rceil_4 &= \left\lceil i, u+\tfrac{1}{2}\tilde h\Big(t+\tfrac{1}{2}\tilde h(s)\Big), s+\tfrac{1}{2}\right\rceil
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
h(u) &= \lceil u-\lfloor u\rfloor+1\rceil \\
\tilde h(u) &= \lceil u-\lfloor u\rfloor\rceil
\end{aligned} \tag{B.10}$$

$$|n|_2 = n-2\lfloor n/2\rfloor, \tag{B.11}$$

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \qquad (a)_0=1, \tag{B.12}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\lfloor a\rfloor!}{\lfloor b\rfloor!\lfloor a-b\rfloor!} \tag{B.13}$$

$$F^u_{st}(\lambda) = (-1)^{\lfloor s+t-u-1\rfloor}\frac{(2s+2t-2u-2)!}{(2s+2t-\lfloor u\rfloor-3)!}\sum_{i=0}^{2s-2}\sum_{j=0}^{2t-2}\delta(i+j-2s-2t+2u+2) \tag{B.14}$$

$$\times A^i(s,\tfrac{1}{2}-\lambda)A^j(t,\lambda)(-1)^{2s+2i(s+t-u)},$$

$$A^i(s,\lambda) = (-1)^{\lfloor s\rfloor+1+2s(i+1)}\begin{bmatrix} s-1 \\ i/2 \end{bmatrix}\frac{([(i+1)/2]+2\lambda))_{\lfloor s-1/2\rfloor-\lfloor(i+1)/2\rfloor}}{(\lfloor s+i/2\rfloor)_{2s-1-\lfloor s+i/2\rfloor}}. \tag{B.15}$$

$$F^u_{st}(i,\lambda) = F^u_{st}(\lambda)(-1)^{\lfloor i/2\rfloor+2i(s+u)}\begin{bmatrix} u-1 \\ i/2 \end{bmatrix}(\lfloor 2s-u\rfloor)_{\lfloor u-1-i/2\rfloor+|2u|_2|2u-2-i|_2} \tag{B.16}$$

$$\times(\lfloor 2t-u\rfloor)_{\lfloor i/2\rfloor+|2u|_2|i|_2}$$

B.2 Structure Constants of shs[λ]

Similar to above, we can write the commutation relations of the infinite dimensional Lie algebra shs[λ] as

$$\left[L^{(s)}_m, L^{(t)}_n\right] = \sum_{u=1}^{s+t-1}{}^{\circ}\, \hat g^{st}_u(m,n;\lambda)\, L^{(s+t-u)}_{m+n}, \qquad \left[L^{(s)}_m, G^{(t)}_q\right] = \sum_{u=1}^{s+t-1}{}^{\circ}\, \hat h^{st}_u(m,q;\lambda)\, G^{(s+t-u)}_{m+q},$$

$$\left\{G^{(s)}_p, G^{(t)}_q\right\} = \sum_{u=1}^{s+t-1}{}^{\circ}\, \hat{\hat g}^{st}_u(p,q;\lambda)\, L^{(s+t-u)}_{p+q}, \qquad \left[G^{(s)}_p, L^{(t)}_n\right] = \sum_{u=1}^{s+t-1}{}^{\circ}\, \hat{\hat h}^{st}_u(p,n;\lambda)\, G^{(s+t-u)}_{p+n}.$$

These structure constants are directly given by the formulas for the $\mathcal{SB}[\mu]$ structure constants, but the constants $F^u_{st}(\lambda)$ has to be replaced by

$$f^u_{st}(\lambda) = F^u_{st}(\lambda) + (-1)^{\lfloor -u\rfloor+4(s+u)(t+u)}\, F^u_{st}(\frac{1}{2}-\lambda). \tag{B.17}$$

B.3 Some Definitions and Useful Relations

In this section we will give the definition of few other functions and some of their properties which is used in the thesis. The functions used in the definition of the operators (2.40) are

$$\begin{aligned} a^i(s,\lambda) &= \binom{s-1}{i} \frac{(-2\lambda - s + 2)_{s-1-i}}{(s+i)_{s-1-i}}, && 0 \le i \le s-1, \\ \alpha^i(s,\lambda) &= \binom{s-1}{i} \frac{(-2\lambda - s + 2)_{s-1-i}}{(s+i-1)_{s-1-i}}, && 0 \le i \le s-1, \\ \beta^i(s,\lambda) &= \binom{s-2}{i} \frac{(-2\lambda - s + 2)_{s-2-i}}{(s+i)_{s-2-i}}, && 0 \le i \le s-2. \end{aligned} \tag{B.18}$$

For showing the relation (2.45), one has to use the following identities

$$\begin{aligned} -\frac{\beta^i(s,\lambda)}{2} &= a^i(s,\lambda+\tfrac{1}{2}) - a^i(s,\lambda), \\ \alpha^i(s,\lambda) &= 2a^i(s,\lambda) - \beta^{i-1}(s,\lambda). \end{aligned} \tag{B.19}$$

Furthermore one has to know the relation between $A^i(s,\lambda)$ (see (B.15) and (2.42)) and the $a^i(s,\lambda)$, $\alpha^i(s,\lambda)$ and $\beta^i(s,\lambda)$ (see (B.18) and (2.40)). This is given by

$$\begin{aligned} A^{2i}(s,\lambda) &= (-1)^i a(s,\lambda), & A^{2i+1}(s,\lambda) &= -\frac{1}{2}(-1)^i \beta^i(s,\lambda), & s &= \lfloor s \rfloor \in \mathbb{Z}, \\ A^{2i}(s,\lambda) &= \frac{\lfloor s \rfloor - 1 + 2\lambda}{2\lfloor s \rfloor - 1} a^i(\lfloor s \rfloor,\lambda) & A^{2i+1}(s,\lambda) &= (-1)^i \beta^i(\lfloor s \rfloor + 1,\lambda), & s &= \lfloor s \rfloor + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}. \end{aligned} \tag{B.20}$$

Remember that for $s \in \mathbb{Z} + \frac{1}{2}$ we have that $\lfloor s \rfloor + 1 = \lceil s \rceil$. Another useful fact to know is that the operator

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}, \tag{B.21}$$

satisfies the following relations

$$D^{2q} = (-\partial)^q, \qquad D^{2q+1} = (-\partial)^q\, D, \qquad q \in \mathbb{Z}. \tag{B.22}$$

Another very useful relation is

$$\begin{aligned} (-\partial)^m z^k &= (-1)^m\,(k-m+1)_m\, z^{k-m}, \\ &= (-k)_m\, z^{k-m}, \end{aligned} \tag{B.23}$$

where in the second line we have used the property of the Pochhammer symbol $(-x)_n = (x-n+1)_n\,(-1)^n$.

B.4 Properties of The Structure Constants

For reference, we will in this section list a few properties of some of the $\mathcal{SB}[\mu]$ structure constants which are quite useful for our calculations.

$$g_u^{st}(m,n;\lambda) = \begin{cases} (-1)^{\lfloor u \rfloor +1}\, g_u^{ts}(n,m;\lambda) & \begin{cases} u \in \mathbb{Z}, & \left(s,t \in \mathbb{Z} \quad \text{or} \quad s+t \in \mathbb{Z}+\frac{1}{2}\right) \\ u \in \mathbb{Z}+\frac{1}{2}, & \left(s,t \in \mathbb{Z}+\frac{1}{2} \quad \text{or} \quad s+t \in \mathbb{Z}+\frac{1}{2}\right) \end{cases} \\ (-1)^{\lfloor u \rfloor}\, g_u^{ts}(n,m;\lambda) & \begin{cases} u \in \mathbb{Z}, & s,t \in \mathbb{Z}+\frac{1}{2} \\ u \in \mathbb{Z}+\frac{1}{2}, & s,t \in \mathbb{Z} \end{cases} \end{cases} \tag{B.24}$$

$$g_1^{st}(m,n;\lambda) = \begin{cases} 1 & \left(s,t \in \mathbb{Z} \quad \text{or} \quad s+t \in \mathbb{Z}+\frac{1}{2}\right) \\ 0 & s,t \in \mathbb{Z}+\frac{1}{2} \end{cases}$$
$$g_{\frac{3}{2}}^{st}(m,n;\lambda) = \begin{cases} m/2 \text{ or } n/2 & \left(s=1,\ t \in \mathbb{Z}\right) \quad \text{or} \quad \left(s \in \mathbb{Z},\ t=1\right) \\ 0 & s,t \in \mathbb{Z} \quad \text{and} \quad s,t \neq 1 \\ g_{\frac{3}{2}}^{st}(0,0;\lambda) & \left(s,t \in \mathbb{Z}+\frac{1}{2}\right) \quad \text{or} \quad \left(s+t \in \mathbb{Z}+\frac{1}{2}\right) \end{cases} \tag{B.25}$$

$$\hat{g}_u^{2s}(1,m;\lambda) = \begin{cases} \lfloor s \rfloor - 1 - m, & u=2 \\ 0, & u=1,\frac{3}{2},\frac{5}{2},3 \end{cases},$$
$$\hat{h}_u^{2s}(1,r;\lambda) = \begin{cases} \lceil s \rceil - \frac{3}{2} - r, & u=2 \\ 0, & u=1,\frac{3}{2},\frac{5}{2},3 \end{cases}. \tag{B.26}$$

Appendix **C**

Introduction to Conformal Field theory

In this appendix we will give a quick introduction to certain basic aspects of two-dimensional conformal field theory which are necessary to understand the more advanced topics used in the thesis. For more details see [133, 134, 135, 136, 137].

C.1 Basic Concepts

Given a (pseudo-)Riemannian Manifold (M, g), a conformal transformation is a diffeomorphism $f : M \to M$ (possibly only defined on a open set $U \subset M$) which preserves the metric up to a local scaling

$$f^* g_{f(x)} = \omega(x)\, g_x, \qquad x \in M, \tag{C.1}$$

where $\omega \in \mathcal{F}(M)$ is a smooth map and f^* is the pull-back. Acting on a set of tangent vectors $X, Y \in T_x M$ we can write the definition as $g_{f(x)}(f_* X, f_* Y) = \omega(x) g_x(X, Y)$ where f_* is the push forward. For this thesis we shall mainly choose local charts and consider the components, in which the definition takes the following form

$$g_{\alpha\beta}(y) \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} = \Omega(x) g_{\mu\nu}(x), \tag{C.2}$$

with $y = f(x)$. Note that while scales are not preserved, local angles between tangent vectors $\cos^2\theta_x = \frac{g_x^2(X,Y)}{g_x(X,X) g_x(Y,Y)}$ are invariant. In this thesis we shall only consider conformal field theories on flat (Minkowskian or Euclidean) spaces with the topology $M = \mathbb{R} \times S^1$ or (Euclidean) $M = S^1 \times S^1 = T^2$. Consider the metric $g = dx^0 \otimes dx^0 + dx^1 \otimes dx^1 = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$. An infinitesimal diffeomorphism $y^\alpha(x) = x^\alpha + \epsilon^\alpha(x) + \mathcal{O}(\epsilon^2)$ preserving the metric up to a local scaling

$$\begin{aligned} \delta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} &= \delta_{\mu\nu} + \left(\frac{\partial \epsilon_\mu}{\partial x^\nu} + \frac{\partial \epsilon_\nu}{\partial x^\mu} + \mathcal{O}(\epsilon^2) \right) \\ &\overset{!}{=} \delta_{\mu\nu} + \omega(x) \delta_{\mu\nu} + \mathcal{O}(\epsilon^2) \end{aligned}$$

must satisfy the constraint

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega(x)\delta_{\mu\nu} = \partial_\rho \epsilon^\rho \delta_{\mu\nu}, \tag{C.3}$$

where $\omega(x) = \partial_\rho \epsilon^\rho$ is found by tracing. We can see that $\epsilon^\mu \partial_\mu$ defines a conformal Killing vector. Amazingly this is nothing but the Cauchy-Riemann equations

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0,$$

which implies that any holomorphic[1] transformation is allowed and thus the symmetry is infinite dimensional. It is convenient to switch to complex coordinates

$$\begin{aligned} z &= x^0 + ix^1, & \epsilon &= \epsilon^0 + i\epsilon^1, & \partial &= \frac{1}{2}(\partial_0 - i\partial_1), \\ \bar{z} &= x^0 - ix^1, & \bar{\epsilon} &= \epsilon^0 - i\epsilon^1, & \bar{\partial} &= \frac{1}{2}(\partial_0 + i\partial_1), \end{aligned}$$

in which the metric tensor takes the simple form[2]

$$g = \frac{1}{2}\mathrm{d}z \otimes \mathrm{d}\bar{z} + \frac{1}{2}\mathrm{d}\bar{z} \otimes \mathrm{d}z \equiv \mathrm{d}z\mathrm{d}\bar{z},$$

with the inverse $g^{z\bar{z}} = g^{\bar{z}z} = 2$, $g^{zz} = g^{\bar{z}\bar{z}} = 0$. We have used the notation convenient notation $\partial_z = \partial$ and $\partial_{\bar{z}} = \bar{\partial}$. Thus under a holomorphic transformation $f(z) = 1 + \epsilon(z)$, the metric transforms as

$$g = \mathrm{d}z\mathrm{d}\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}\mathrm{d}z\mathrm{d}\bar{z} = \left|\frac{\partial f}{\partial z}\right|^2 \mathrm{d}z\mathrm{d}\bar{z}.$$

In the following we shall extend x^0 and x^1 to $\mathbb{C}$ so that z and $\bar{z}$ become independent, but in the end we can restrict to the physics surface $\bar{z} = (z)^*$. Using a Laurent expansion around $z = 0$, $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$ and similarly for $\bar{\epsilon}$, we see that to linear order

$$\begin{aligned} \delta\phi(z,\bar{z}) &= \phi(z - \epsilon(z), \bar{z} - \bar{\epsilon}) - \phi(z,\bar{z}) \\ &= -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi \\ &= \sum_{n\in\mathbb{Z}} \left(\epsilon_n l_n + \bar{\epsilon}_n \bar{l}_n\right) \phi(z,\bar{z}), \end{aligned}$$

$l = -z^{n+1}\partial$ and $\bar{l} = -\bar{z}^{n+1}\bar{\partial}$ generate the transformations. These generators satisfy the Witt algebra

$$[l_m, l_n] = (m-n)l_{m+n}, \tag{C.4}$$

and similarly for $\bar{l}_n$. The separation of holomorphic and anti-holomorphic degrees of freedom can be regarded as the essence of Conformal Field Theories. It turns out that it is necessary to work with the one-point compactification of $\mathbb{C}$, the Riemann Sphere $S^2 \simeq \mathbb{C} \cup \infty$. However only the subset $\{l_{\pm 1}, l_0\}$ is globally well-defined on the Riemann sphere (due to ambiguities at $z = \infty$) and forms a Lie subalgebra. Here l_{-1} generates translations $z \to z + a$ while l_1 generates special conformal transformations $z \to \frac{z}{cz+1}$.

[1] We will be rather sloppy and use the term holomorphic also for meromorphic functions.

[2] Here we, as is conventional, abuse the notation due to the symmetry property of the metric.

The geometric meaning of l_0 is most clearly seen if we use polar coordinates $z = re^{i\theta}$, $l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\theta$, and combine with the anti-holomorphic part

$$l_0 + \bar{l}_0 = -r\partial_r, \quad \text{and} \quad i(l_0 - \bar{l}_0) = -\partial_\theta, \tag{C.5}$$

or in other words $l_0 + \bar{l}_0$ generates dilations while $i(l_0 - \bar{l}_0)$ generates rotations. So this global part, called the conformal group, generates the Möbius group $SL(2,\mathbb{C})/\mathbb{Z}_2$ on the Riemann Sphere S^2

$$z \to \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}, \quad ad - bc = 1. \tag{C.6}$$

Quantum mechanics however introduces an extremely important subtlety. According to Wigner's theorem, symmetries are realized projectively on the Hilbert space so we can either consider projective representations of the Witt algebra or linear representations of its central extension. A central extension of $\mathfrak{g}$ is a short exact sequence

$$0 \longrightarrow \mathfrak{C} \xrightarrow{\ i\ } \tilde{\mathfrak{g}} \xrightarrow{\ \pi\ } \mathfrak{g} \longrightarrow 0,$$

such that $\mathfrak{C}$ is in the center of $\tilde{\mathfrak{g}}$. By the properties of exact sequences, i is injective, π is surjective and thereby $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{C}$. More concretely we can start from the vector space $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}\, c$ and give it a Lie bracket by finding a anti-symmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ that satisfies

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0. \tag{C.7}$$

This will give a central extension with the Lie bracket

$$[x + \alpha c, y + \beta c] = [x,y] + \omega(x,y)c, \quad x,y \in \mathfrak{g}, \quad \alpha,\beta \in \mathbb{C}.$$

There is some ambiguity however, two bilinear forms give rise to isomorphic Lie algebra structures on $\bar{\mathfrak{g}}$ if there exists a linear map $\mu : \mathfrak{g} \to \mathbb{C}$, such that $\omega(x,y) = \omega'(x,y) + \mu([x,y])$. It turns out that isomorphism classes of central extensions are in one-to-one correspondence with elements of the second Lie algebra cohomology group of $\mathfrak{g}$, $H^2(\mathfrak{g},\mathbb{C})$. Actually (C.7) is the statement that ω is a 2-cycle, ie. a two-chain with zero boundary $d\omega = 0$. See more details in [138, 139] and especially chapter 6 of [140]. It turns out that there is a unique, up to isomorphism, central extension of the Witt algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}, \tag{C.8}$$

known as the Virasoro algebra. There is an analog algebra with $\bar{L}_n$ but the same central element. According to Schur's lemma, the central element c acts as a constant on irreducible representations. This number is known as the central charge and plays an important role in the representation theory of the Virasoro algebra, and thereby conformal field theory.

If a field transforms under conformal transformations $z \to f(z)$ according to

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})), \tag{C.9}$$

it is called a primary field with conformal weight $(h, \bar{h})$. If it only transforms like this under $SL(2,\mathbb{C})/\mathbb{Z}_2$, it is called a quasi-primary field. We shall also define the scaling dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$ since under a rotation and scaling $f(z) = z\lambda e^{i\theta}$ and $\bar{f}(\bar{z}) = \bar{z}\lambda e^{-i\theta}$ we have that

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \lambda^{\Delta} e^{is\theta}\, \phi(z\lambda e^{i\theta}, \bar{z}\lambda e^{-i\theta}), \tag{C.10}$$

Associated to a conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, there is a conserved Noether current $j_\mu = T_{\mu\nu}\epsilon^\nu$. From translation symmetry it gives a conserved energy-momentum tensor, for rotations it implies that it is symmetric $T^{\mu\nu} = T^{\nu\mu}$ and most importantly conformal symmetry implies

$$T^\mu_\mu = 0.$$

Using this information, in complex coordinates the energy-momentum tensor has the structure $T_{z\bar{z}} = T_{\bar{z}z} = 0$, $T_{zz}(z,\bar{z}) \equiv T(z)$ and $T_{\bar{z}\bar{z}}(z,\bar{z}) = \bar{T}(\bar{z})$ with the infinite number of conserved currents

$$\bar{\partial}(\epsilon(z)T(z)) = 0, \qquad \partial(\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})) = 0. \tag{C.11}$$

Later we shall see that conserved currents of this type with higher spin will lead to extensions of the Virasoro algebra.

C.1.1 Ward Identities and Operator Product Expansions

We can translate much of these statements about symmetries and conserved currents into quantum mechanics. Consider the expectation value of local fields

$$\big\langle \mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\big\rangle = \int \mathcal{D}\phi\, e^{-S[\phi]}\mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)$$

and assume that under a holomorphic transformation $z \to z + \epsilon(z)$, the fields transform as $\mathcal{O}_i \to \mathcal{O}_i + \delta_\epsilon\mathcal{O}_i$. Then one can derive the identity

$$\sum_{i=1}^{n}\big\langle \mathcal{O}_1(z_1,\bar{z}_1)\ldots\delta_\epsilon\mathcal{O}_i(z_i,\bar{z}_i)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\big\rangle = \int_{\mathcal{C}}\frac{\mathrm{d}z}{2\pi i}\big\langle T(z)\epsilon(z)\mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\big\rangle, \tag{C.12}$$

which can be regarded as alternative definition of a primary field. Here $\mathcal{C}$ is a contour enclosing the points z_i. There is a similar expression for anti-holomorphic transformations. Using the transformation properties of a primary field in eq. (C.9), we find the operator product expansions

$$\begin{aligned} T(z)\mathcal{O}_i(w,\bar{w}) &\sim \frac{h_i}{(z-w)^2}\,\mathcal{O}_i(w,\bar{w}) + \frac{1}{z-w}\,\partial\mathcal{O}_i(w,\bar{w}),\\ \bar{T}(\bar{z})\mathcal{O}_i(w,\bar{w}) &\sim \frac{\bar{h}_i}{(\bar{z}-\bar{w})^2}\,\mathcal{O}_i(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\bar{\partial}\mathcal{O}_i(w,\bar{w}), \end{aligned} \tag{C.13}$$

where $\sim$ means equal up to regular terms. Notice that this expression is valid under correlation functions and the time-ordering in the RHS in implicit. These OPE's are convergent up to the nearest insertion, see [141] for a detailed account on convergence issues. We shall mainly be working in radial quantization, mapping coordinates from the cylinder w to the complex plane z by $z = e^{-iw}$. Time-ordering is then changed into radial ordering

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & \text{for } |z| > |w|,\\ B(w)A(z) & \text{for } |w| > |z|. \end{cases} \tag{C.14}$$

Operator product expansions can be equivalently expressed through commutation relations between their modes using

$$\begin{aligned}\oint [A(z), B(w)] &= \oint_{|z|>|w|} \mathrm{d}z\, A(z)B(w) - \oint_{|z|<|w|} \mathrm{d}z\, B(w)A(z) \\ &= \oint_{\mathcal{C}(w)} \mathrm{d}z\, R(A(z)B(w)).\end{aligned} \tag{C.15}$$

A primary field will generally have the Laurent expansion

$$\phi(z, \bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{m}-\bar{h}} \phi_{n,\bar{m}}. \tag{C.16}$$

$T(z)$ is a quasi-primary field with the conformal weight $(2,0)$ with the OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w), \tag{C.17}$$

the extra singular term is the reason the energy-momentum tensor is not primary. Using the expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \qquad L_n = \oint_{\mathcal{C}(0)} \frac{\mathrm{d}z}{2\pi i}\, z^{n+1} T(z),$$

one can show that $[L_n, L_m]$ gives rise to the Virasoro algebra in eq. (C.8).

Two dimensional conformal field theories are essentially given by representation theory of the Virasoro algebra, and we have just seen that the energy-momentum tensor encodes this information. Thus one can actually define conformal field theories just by specifying the $T(z)$ and $\bar{T}(\bar{z})$, without thinking about the action. This fact will be very important for us. Also note that the Virasoro algebra is best considered as a spectrum generating algebra, since not all elements commute with the Hamiltonian as usual quantum symmetries.

C.1.1.1 Normal Ordering and Generalized Wick Contractions

In CFT's the spectrum of local operators plays an extremely important role due to the one-to-one correspondence between local operators and states in the Verma module. Local operators will correspond to derivatives and products of operators at the same space-time point and thus we need a way to regularize these products. In this section we will define a more general form of normal ordering of quantum operators and develop a generalized version of a weak form of Wick's theorem. This will enable us to work with interacting conformal field theories and their OPE's.

In general we can decompose an operator product into singular and regular parts,

$$A(z)B(w) = \overbrace{A(z)B}(w) + \mathcal{N}(A(z)B(w)) \tag{C.18}$$

where the regular part is the normal ordering

$$\mathcal{N}(A(z)B(w)) = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \mathcal{N}\left(\partial^n AB\right)(w), \tag{C.19}$$

while the singular part

$$\overbrace{A(z)B}(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}, \tag{C.20}$$

is called the (Wick) contraction. Here we have assumed that the largest singular pole is of order N and used a Taylor expansion in (C.19) since it is regular. The z independent term of (C.19) is exactly what we need to define a regular product of local operators, this is given as

$$\mathcal{N}(AB)(w) = \oint_{\mathcal{C}(w)} \frac{\mathrm{d}z}{2\pi i} \frac{A(z)B(w)}{z-w} = \lim_{z\to w} \left[A(z)B(w) - \overbrace{A(z)B}(w) \right]. \tag{C.21}$$

In the case of free field theories, it is customary to use the notation $:AB:(z)$ for normal ordering. An alternative definition of normal ordering for free field theories is to require that annihilation operators are always put to the right of creation operators. In order to connect to this definition, let us look at a Laurent expansion of the normal ordered product

$$\mathcal{N}(AB)(w) = \sum_{m\in\mathbb{Z}} w^{-m-h_A-h_B} \mathcal{N}(AB)_m, \tag{C.22}$$

where

$$\mathcal{N}(AB)_m = \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi i}\, w^{m+h_A+h_B-1} \mathcal{N}(AB)(w), \tag{C.23}$$

are the expansion coefficients. What we need to do is to find the relation between $\mathcal{N}(AB)_m$ and the coefficients of the expansions $A(w) = \sum_{m\in\mathbb{Z}} w^{-m-h_A} A_m$ and $B(w) = \sum_{m\in\mathbb{Z}} w^{-m-h_B} B_m$. Using (C.22), (C.21), (C.15) together with the standard deformation of contours one can show that

$$\mathcal{N}(AB)_m = \sum_{n\leq -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n} A_n, \tag{C.24}$$

which is analog to the usual normal ordering of modes, with the difference that normal ordering is not commutative $\mathcal{N}(AB)(z) \neq \mathcal{N}(BA)(z)$. Next we want a simple calculus for contracting products of normal ordered fields. This would suggest we need a generalized version of a weak version of Wick's theorem that even works for interacting field theories. [3] A generalization sufficient for our needs is

$$\overbrace{A(z)\mathcal{N}(BC)}(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} \frac{\mathrm{d}w}{x-w} \left\{ \overbrace{A(z)B}(x)C(w) + B(x)\overbrace{A(z)C}(w) \right\}.$$

The integral essentially works as a point splitting regularization of $\mathcal{N}(BC)(z)$ in order to extract the singular terms, which can only come from contracting A with B and C, respectively. See more details in [133, 58].

[3] A full version of Wick's theorem, however, does not exist for interacting field theories.

C.1.2 Verma Modules and Descendant States

In radial quantization dilations and rotations correspond to time and space translations, respectively (see (C.10)). This implies that we can identity the Hamiltonian and Momentum operators with

$$H = L_0 + \bar{L}_0, \qquad P = i(L_0 - \bar{L}_0). \tag{C.25}$$

Conformal invariance implies that we can collect the states in our Hilbert space into representations of the Virasoro algebra[4] $\texttt{Vir} \oplus \overline{\texttt{Vir}}$ or some extension thereof $\mathcal{A} \oplus \bar{\mathcal{A}}$. A physical spectrum must be bounded from below, which implies that Highest Weight modules [5] is what we should study.

In Radial quantization, it is natural to define an asymptotic in-state of the form

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle = \phi_{-h,-\bar{h}}|0\rangle, \tag{C.26}$$

where in order to keep (C.26) regular, we have required (see (C.16))

$$\phi_{n,\bar{m}}|0\rangle = 0, \qquad \text{for} \qquad n > -h, \quad \bar{m} > -\bar{h}. \tag{C.27}$$

The hermitian conjugate of ϕ will be defined as

$$\phi^\dagger(z,\bar{z}) = \bar{z}^{-2h}\, z^{-2\bar{h}}\, \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right), \tag{C.28}$$

this strange form is related to radial quantization. The mode expansion of $\phi^\dagger$ as obtained from (C.16) and (C.28)

$$\phi^\dagger = \sum_{n,\bar{m}\in\mathbb{Z}} \bar{z}^{h-h} z^{\bar{m}-\bar{h}} \phi_{n,\bar{m}},$$

reveals that the hermitian conjugate of the Laurent modes are given as

$$(\phi_{n,\bar{m}})^\dagger = \phi_{-n,-\bar{m}}.$$

In particular, the modes of the energy-momentum tensor satisfy $(L_n)^\dagger = L_{-n}$ and we can start talking about unitary representations. For completeness, let us mention that these definitions lead to the asymptotic out-states

$$\langle\phi| = \lim_{z,\bar{z}\to 0} \langle 0|\phi^\dagger(z,\bar{z}) = \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}}\langle 0|\phi(w,\bar{w}) = \langle 0|\phi_{h,\bar{h}}, \tag{C.29}$$

and

$$\langle 0|\phi_{n,\bar{m}} = 0 \qquad \text{for} \qquad n < h, \quad \bar{m} < \bar{h}.$$

Now using the energy momentum tensor in equation (C.27) implies that

$$\begin{matrix} L_n|0\rangle = 0 \\ \bar{L}_n|0\rangle = 0 \end{matrix}, \qquad n \geq -1, \tag{C.30}$$

meaning that the vacuum is invariant under the global conformal group. A highest weight module of the Virasoro algebra is characterized by a central charge c and highest weights

[4] We shall use a sloppy language and call both $\texttt{Vir} \oplus \overline{\texttt{Vir}}$ or its chiral parts the Virasoro algebra.

[5] Although it would be better terminology to call it lowest weight modules.

$(h, \bar{h})$. Under the operator-state correspondence of eq. (C.26), any primary field ϕ gives rise to a highest weight state $|\phi\rangle = |h, \bar{h}\rangle = \phi_{-h,-\bar{h}}|0\rangle$ satisfying

$$\begin{aligned} L_0\,|h,\bar{h}\rangle &= h\,|h,\bar{h}\rangle, & L_{-n}\,|h,\bar{h}\rangle &= 0 \\ \bar{L}_0\,|h,\bar{h}\rangle &= \bar{h}\,|h,\bar{h}\rangle, & \bar{L}_{-n}\,|h,\bar{h}\rangle &= 0 \end{aligned}, \qquad n > 0, \tag{C.31}$$

which is seen from the commutator relations

$$[L_m, \phi_n] = ((h-1)m - n)\,\phi_{m+n}, \tag{C.32}$$

and similarly for the anti-holomorphic part. Since everything works in parallel, we shall mainly be concerned with the holomorphic part in the following. The module consisting of finite linear combinations of the states

$$V_{h,c} = \mathrm{span}_{\mathbb{C}}\left\{ L_{-k_1}L_{-k_2}\dots L_{-k_n}|h,c\rangle \,\middle|\, k_1,\dots,k_n > 0 \right\},$$

is called a Verma module. From the Virasoro algebra we know that $[L_0, L_{-m}] = mL_{-m}$, which on combination with eq. (C.31) means that that L_{-m} increases the eigenvalue of L_0. The Verma module thus admits a L_0-eigenspace decomposition of the form

$$V_{h,c} = \bigoplus_{m\geq 0} V_{h,c}^{(m)}, \qquad V_{h,c}^{(m)} = \left\{ |v\rangle \in V_{h,c} \,\middle|\, L_0|v\rangle = (h+m)|v\rangle \right\}, \tag{C.33}$$

where $V_{h,c}^{(m)}$ is spanned by

$$L_{k_1}\dots L_{k_r}|h,c\rangle, \qquad \sum_{i=1}^{r} k_i = m, \quad k_1 \geq \dots \geq k_r > 0.$$

The number m is called the level. The number of states at level m is the number of positive integer partition of m and is given by the Euler partition function $p(m)$. A well-known generating function of $p(m)$ is given by

$$\frac{1}{\phi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n.$$

States in the Verma module for $m \neq 0$ are called descendant states of ϕ. Using the 1-1 correspondence of operators and states in the Verma module, we can find operator representations for the descendant states by $L_{-k_1}\dots L_{-k_n}\phi(z)$. For example by using eq. (C.18)

$$\begin{aligned} L_{-n}\phi(0) &= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}z}{2\pi i}\, z^{-n+1} T(z)\phi(0) \\ &= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}z}{2\pi i}\, z^{-n+1} \left(\overline{A(z)B}(w) + \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{N}(\partial^n T\phi)(0) \right), \qquad n \geq 2 \\ &= \frac{1}{(n-2)!}\,\mathcal{N}(\partial^{n-2}T\phi)(0), \end{aligned}$$

and by the same way one can also show that

$$\underbrace{L_{-1}\dots L_{-1}}_{n}\phi(0) = \partial^n\phi(0), \qquad n \geq 0.$$

This motivates the concept of a conformal family of a primary field ϕ

$$\begin{aligned}[\phi] &= \left\{ L_{k_1} \dots L_{k_n} \phi \,\middle|\, k_1, \dots, k_n \leq -1 \right\}, \\ &= \left\{ \phi \,, \partial\phi \,, \partial^2\phi \,, \dots, \mathcal{N}(T\phi) \,, \mathcal{N}(T\partial\phi) \,, \dots, \mathcal{N}(\partial T\phi) \,, \dots \right\}. \end{aligned} \tag{C.34}$$

One important consequence is

$$L_{-2}\mathbf{1}(w) = \oint_{\mathcal{C}(w)} \frac{1}{z-w} T(z)\mathbf{1} = T(w), \tag{C.35}$$

so the energy-momentum tensor is a descendant field of the identity operator, which explains why it does not have the canonical OPE of primary fields.

C.1.3 Virasoro Minimal Models

The Verma module V_{hc} is generally not irreducible, nor even fully reducible, i.e. cannot be written as a direct sum of irreducible modules. This is due to invariant subspaces generated by null-states[6], which are annihilated by all L_n $(n > 0)$ and therefore generate their own Verma submodules. It can be shown that null-states $|\chi\rangle$ are orthogonal to the whole Verma module and in particular have zero norms

$$\langle\chi|\chi\rangle = 0,$$

and this is also true for all of its descendants. A irreducible representation can be found by modding out the null submodules

$$L_{h,c} = V_{h,c}/V_\chi.$$

There are however other problems. We are interested in unitary representation of the Virasoro algebra and we therefore have to avoid negative norm states. This condition will put certain constrains on the values of h and c. For example take the following inner-products

$$\langle h, c|L_1 L_{-1}|h, c\rangle = 2h, \qquad \langle 0|L_2 L_{-2}|0\rangle = \frac{c}{2},$$

implying that for unitary representations it is necessary to require $c \geq 0$ and $h \geq 0$. For a more systematic approach, it is convenient to introduce the unitary Gram matrix $M_{ab} = \langle a|b\rangle$, for all states $|a\rangle$ in the Verma module $V_{h,c}$. Since the decomposition (C.33) is orthogonal, the Gram matrix decomposes into a block diagonal form with the blocks $M_{ab}^{(m)}$ for each level m. The condition for $|v\rangle = \sum_a \Lambda_a|a\rangle$ to have vanishing norm

$$\|v\|^2 = \sum_{a,b} \Lambda_a \langle a|b\rangle \Lambda_b = \Lambda^T M \Lambda = 0,$$

is that Λ is a eigenvector with eigenvalue $\lambda = 0$ of M. Thus following Friedan, Qiu and Shenker [142], we will consider the determinant $\det(M - \lambda I) = \det M$. There is a general formula for $\det M^{(m)}(h, c)$ called the Kac-determinant. A careful analysis gives the following conclusions about unitary irreducible representations of the Virasoro algebra [133]:

[6] Null-states are states which are both primary and secondary.

- For $c > 1$ and $h \geq 0$ there are no zeros and all eigenvalues of $M^{(m)}(h,c)$ are positive, thus there can exist unitary representations but with necessarily infinite number of primary fields.
- For $c = 1$, $\det M^{(m)} = 0$ for $h = \frac{n^2}{4}$ where $n \in \mathbb{Z}$.
- For $c < 1$ and $h \geq 0$, there are a discrete set of unitary irreducible modules $L_{h,c}$ for

$$c(m) = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \ldots, \tag{C.36}$$

with only $\binom{m}{2}$ allowed primary states with the conformal weights

$$h_{p,q}(m) = \frac{\left((m+1)p - mq\right)^2 - 1}{4m(m+1)}, \qquad 1 \leq p \leq m-1, 1 \leq q \leq m. \tag{C.37}$$

The modules given by (C.36) and (C.37) are called Virasoro unitary minimal models and were first discussed in [143]. It turns out that they cover all unitary irreducible representations with finite number of primary states and they are much easier to control since all fields can be ordered into finite number of families. It is however possible to have conformal field theories with a larger symmetry algebra $\mathcal{A} \oplus \bar{\mathcal{A}}$, where fields can be organized into finite families of modules of this larger symmetry algebra. Conformal field theories of this type are usually called Rational Conformal Field Theories (RCFT). It turns out that RCFT's have many very interesting properties and admit a useful axiomatic formulation [18, 144, 21, 22]. Also note that the CFT's in (C.36) and (C.37) do not give rise to unique theories, since the holomorphic and anti-holomorphic parts can be combined in various ways. We will return to this point when we discuss modular invariance.

C.1.4 Correlation Functions, Null States and the Fusion Algebra

In this section we will see one the main powers of conformal invariance in two-dimensions. In particular, we will investigate how null states in the Virasoro unitary minimal models for $c < 1$ put very strong constrains on correlation functions. Many of these methods can be generalized to more general RCFT's.

First we note that global conformal invariance $SL(2,\mathbb{C})/\mathbb{Z}_2$ restricts the form of two- and three-point functions of quasi-primary fields

$$\left\langle \phi_i(z)\, \phi_j(w) \right\rangle = \frac{d_{ij}\, \delta_{h_i,h_j}}{(z-w)^{2h_i}}, \tag{C.38}$$

$$\left\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3) \right\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{13}^{h_1+h_3-h_2}}, \tag{C.39}$$

where $z_{ij} = z_i - z_j$. The coefficients in the two-point function d_{ij} can be fixed by normalization of the fields, but the three-point coefficients have to be calculated by different means and play an important role in CFT's. We will later talk about how the full Virasoro algebra puts strong constrains on these. Another important feature of the Virasoro algebra is that fields can be organized into conformal families (C.34), which in turn implies that correlation functions contaning descendant fields can be calculated using those with the primary ones. More concretely consider the descendant field

$$L_{-n}\phi(w) = \oint_{\mathcal{C}(w)} (z-w)^{-n+1}\, T(z)\phi(w).$$

Now insert this into a correlation function with primary fields $\phi_1(w_1), \ldots, \phi_N(w_N)$, choose the contour $\mathcal{C}(w)$ such that no other $w_1, \ldots, w_N$ are enclosed and then wrap the contour around the Riemann sphere such that it decomposes into contours $\mathcal{C}(w_i)$. Being careful about the orientation of the contours and using (C.13) one ends up with the result

$$\big\langle L_{-n}\,\phi(w)\,\phi_1(w_1)\,\ldots\,\phi_N(w_N)\big\rangle = \mathcal{L}_{-n}\big\langle\phi(w)\,\phi_1(w_1)\,\ldots\,\phi_N(w_N)\big\rangle, \tag{C.40}$$

where

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right). \tag{C.41}$$

This expression generalizes naturally to more general descendants, such as $L_{k_1}\ldots L_{k_n}\phi(w)$. Using these relations with null-fields will provide us with extremely powerful constraints on correlation functions. For example the null field $L_{-2}\phi(z) - \frac{3}{2(2h+1)}L_{-1}^2\phi(z)$ will give the constraint

$$\left[\sum_{i=1}^{N}\left(\frac{h_i}{(w_i-w)^2} - \frac{1}{w_i-w}\partial_{w_i}\right) - \frac{3}{2(2h+1)}\partial_w^2\right]\langle\phi(w)\,\phi_1(w_1)\,\ldots\,\phi_N(w_N)\rangle = 0.$$

The two-point function (C.38) will trivially satisfy this constraint but for the three-point function (C.39) we find that $C_{123} = 0$ unless

$$h_2 = \frac{1}{6} + \frac{h}{3} + h_1 \pm \frac{2}{3}\sqrt{h^2 + 3hh_1 - \frac{1}{2}h + \frac{3}{2}h_1 + \frac{1}{16}}. \tag{C.42}$$

In the context of Virasoro unitary minimal models (C.37), using $h = h_{2,1}(m)$ and $h_1 = h_{p,q}(m)$ the two solutions (C.42) are just $\mathcal{Z} = \{h_{p-1,q}(m), h_{p+1,q}(m)\}$. This implies that the three-point function $\langle\phi_{2,1}\phi_{p,q}\phi_{p',q'}\rangle$ vanishes unless $h_{p',q'} \in \mathcal{Z}$, and this obviously extends to descendant fields using (C.40). This motivates the concept of fusion rules, which for this case can be written as

$$[\phi_{2,1}] \times [\phi_{p,q}] = [\phi_{p+1,q}] + [\phi_{p-1,q}]\,.$$

This can be readily generalized to higher level null states in the case of minimal models, the general result is [133, 135]

$$[\phi_{p_1,q_1}] \times [\phi_{p_2,q_2}] = \sum_{p_3=|p_1-p_2|+1}^{\substack{\min(p_1+p_2-1,\\ 2m-1-(p_1+p_2))}} \; \sum_{q_3=|q_1-q_2|+1}^{\substack{\min(q_1+q_2-1,\\ 2m+1-(q_1+q_2))}} [\phi_{p_3,q_3}]\,. \tag{C.43}$$

The simplest, and probably most famous minimal model is for $m = 3$ which gives the central charge $c(3) = \frac{1}{2}$. This CFT (when combined with the anti-holomorphic part) describes the critical point of the 2D Ising model [133] and so-called Ising anyons due to their relation to topological field theories [19, 20, 17] among many other applications. It is customary to use the notation

$$\begin{aligned} \mathbf{1} &= [\phi_{1,1}] \quad &\text{or} \quad &[\phi_{2,3}] \\ \sigma &= [\phi_{2,2}] \quad &\text{or} \quad &[\phi_{1,2}] \\ \varepsilon &= [\phi_{2,1}] \quad &\text{or} \quad &[\phi_{1,3}]\,, \end{aligned}$$

with the fusion rules

$$\sigma \times \sigma = \mathbf{1} + \varepsilon, \qquad \sigma \times \varepsilon = \sigma, \qquad \varepsilon \times \varepsilon = \mathbf{1}. \tag{C.44}$$

Notice the similarity with decomposition of tensor products of $SU(2)$ representations, if one identifies $\mathbf{1}$ with a spin 0, σ with spin $\frac{1}{2}$ and ε with spin 1, and cut-off spins larger than $\frac{m-1}{2} = 1$. This is not a coincident, as we will later discuss this is actually related to the representation theory of the affine Lie algebra $\widehat{\mathfrak{su}}(2)_k$.

The concept of fusion rules is very useful even for more general RCFT's, which motivates the definition of a fusion algebra [145] (omitting brackets)

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k, \tag{C.45}$$

where the sum runs over all primary fields in the theory and is by definition finite for RCFT's. The numbers $N_{ij}^k \in \mathbb{N}_0$ can be interpret as the number of independent fusion paths from ϕ_i and ϕ_j to ϕ_k, and is naturally $N_{ij}^k = 0$ whenever $C_{ijk} = 0$. The fusion algebra is commutative and associative. And a final important fact to mention is the neutrality condition: a correlator is zero unless there exists a fusion channel such that all fields can fuse together to get the identity

$$\phi \times \phi^* = \mathbf{1} + \ldots .$$

As our notation silently imply, there always exist a unique "dual" field ϕ^* associated to any other field ϕ such that they fuse to the identity operator and possibly some more, this is the unique field in which the two-function is non-zero. As is evident from the Ising model fusion rules, all fields in that theory are self-dual. The fusion algebra is an important step towards an axiomatic formulation of RCFT's, but we first need to consider certain other important details.

C.1.4.1 Conformal Blocks, Duality and the Bootstrap Approach

In this section we will reintroduce the anti-holomorphic part of the CFT, so for example the correlators (C.38) and (C.39) have to be multiplied by the $\bar{z}$ and $\bar{h}$ dependent part. We will also assume that the coefficient of the two point function (C.38) is $d_{ij} = \delta_{ij}$, which can also be done by normalization [133]. The OPE of two primary fields can be expressed as a sum over other primary fields and their descendants, due to the decoupling of holomorphic and anti-holomorphic parts it will take the following general form

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_p \sum_{\{k,\bar{k}\}} C_{ij}^p \frac{\beta_{ij}^{p,\{k\}}\, \bar{\beta}_{ij}^{p,\{\bar{k}\}}\, \phi_p^{\{k,\bar{k}\}}(w,\bar{w})}{(z-w)^{h_i+h_j-h_p-K}(\bar{z}-\bar{w})^{\bar{h}_i+\bar{h}_j-\bar{h}_p-\bar{K}}}, \tag{C.46}$$

where p run over all primary fields in the theory, $K = \sum_i k_i$ and $\bar{K} = \sum_i \bar{k}_i$ and the multi-indexed field $\phi_p^{\{k,\bar{k}\}}$ label the descendants

$$L_{-k_1} \dots L_{-k_n} \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_m} \phi_p, \tag{C.47}$$

of ϕ_p. In particular $\phi_p^{\{0,0\}} = \phi_p$. The $z - w$ and $\bar{z} - \bar{w}$ dependence is fixed by conformal invariance, coefficient C_{ij}^p determine whether the conformal family of ϕ_p participates, while the $\beta_{ij}^{p,\{k\}}$ and $\bar{\beta}_{ij}^{p,\{\bar{k}\}}$ are the coefficients of the descendants. If we use the convention $\beta_{ij}^{p,\{0\}} = \bar{\beta}_{ij}^{p,\{0\}} = 1$, then C_{ij}^p will be equal to the coefficient of the three-point function. It turns out that the β's are fixed by conformal invariance and depend on the conformal weights and the central charge. For example a straightforward calculation for the case $h = h_i = h_j$ gives [133]

$$\beta_{ij}^{p,\{1,1\}} = \frac{c - 12h - 4h_p + c\,h_p + 8h_p^2}{4(c - 10h_p + 2c\,h_p + 16h_p^2)}. \tag{C.48}$$

The fact that these can be calculated in such a general setting is related to the fact that correlation functions of descendants can be obtained from the primaries (C.40).

These observations are quite striking. They imply that given the set of primary fields, their conformal weights, the central charge and C_{ij}^p, one has fully specified the operator algebra and possibly the whole CFT. This again hints at a possible route to axiomatically formulate RCFT's. Before turning to that, let us investigate how we can constrain C_{ij}^p and calculate them. For this, let us consider the four point-function

$$\big\langle \phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2)\phi_l(z_3, \bar{z}_3)\phi_m(z_4, \bar{z}_4)\big\rangle.$$

It turns out that by the same reasoning leading to (C.38) and (C.39), the four point function is completely fixed up to an overall function depending only on the so-called crossing ratios

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}. \tag{C.49}$$

Although conformal invariance cannot fix this overall function, associativity can constrain it a lot. It is convenient to use global $SL(2,\mathbb{C})/\mathbb{Z}_2$ invariance to map the four points to, say, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$. Consider the four point-function

$$\begin{aligned} G_{lm}^{ji}(x, \bar{x}) &= \lim_{z_1, \bar{z} \to \infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} \big\langle \phi_i(z_1, \bar{z}_1)\phi_j(1,1)\phi_l(x, \bar{x})\phi_m(0,0)\big\rangle, \\ &= \langle i|\phi_j(1,1)\phi_l(x, \bar{x})|m\rangle, \end{aligned} \tag{C.50}$$

where we have used (C.26) and (C.29). If we now take the OPE $\phi_l(x, \bar{x})\phi_m(0,0)$ using (C.46), we find the following expression

$$G_{lm}^{ji}(x, \bar{x}) = \sum_p C_{lm}^p C_{ij}^p \mathcal{F}_{lm}^{ji}(p|x) \bar{\mathcal{F}}_{lm}^{ji}(p|\bar{x}), \tag{C.51}$$

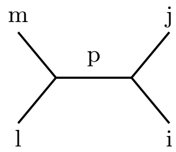
where $\mathcal{F}_{lm}^{ji}(p|x)$ and $\bar{\mathcal{F}}_{lm}^{ji}(p|\bar{x})$ are called conformal blocks and express the contribution of the conformal family $[\phi_p]$ to the four-point function. Actually, conformal blocks are important building blocks where even higher-point functions can be build out of them. They are given by

$$\mathcal{F}_{lm}^{ji}(p|x) = x^{h_p - h_l - h_m} \sum_{\{k\}} \beta_{lm}^{p,\{k\}} x^K \frac{\langle i|\phi_j(1,1)L_{-k_1} \dots L_{-k_N}|p\rangle}{\langle i|\phi_j(1,1)|p\rangle}, \tag{C.52}$$

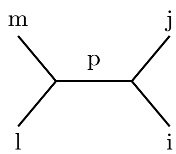
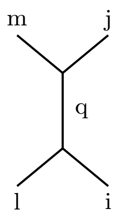
and similarly for $\bar{\mathcal{F}}^{ji}_{lm}(p|\bar{x})$. The denominator of (C.52) is put in, so that there is a factor of C^p_{ij} in (C.51) where we have identified this with the constant in (C.39). In order to have a consistent theory we will require that we will get the same result if we change the order (C.50) and use another OPE. Let us then perform the conformal transformation $z \to 1 - z$, which entails $z_2 \to 0$, $z_4 \to 1$ and $z_3 \to 1 - x$, thus we require the crossing symmetry (sometimes called duality)

$$G^{ji}_{lm}(x, \bar{x}) = G^{mi}_{lj}(1 - x, 1 - \bar{x}). \tag{C.53}$$

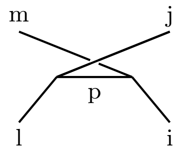
BPZ [143] introduced a useful graphical notation inspired by Feynman diagrams in which conformal blocks take the form

$$\mathcal{F}^{ji}_{lm}(p|x)\bar{\mathcal{F}}^{ji}_{lm}(p|\bar{x}) =$$  . (C.54)

With this, we can express the condition (C.53) as

$$\sum_p C^p_{lm} C^p_{ij}$$  $$= \sum_q C^p_{im} C^p_{lj}$$  . (C.55)

One can derive another condition by $z \to \frac{1}{z}$ which gives rise to the conformal block

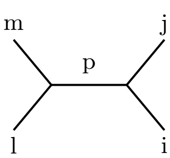
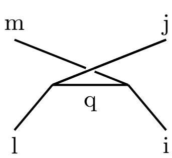


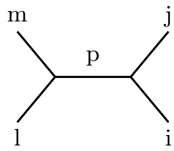
.

Since conformal blocks can, at least in principle, be completely determined by conformal invariance alone, these conditions can be thought of as constraints on the coefficients C^p_{ij}.

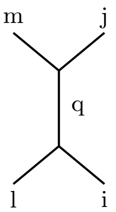
C.1.4.2 Rational Conformal Field Theories and Modular Tensor Categories

There are simplifications when considering RCFT's, since there are only finite number of conformal families that can propagate as intermediate states. It turns out that the conformal blocks form a finite-dimensional vector space and crossing symmetries can be thought of as linear maps relating different choices of basis

 $$= \sum_q B \begin{bmatrix} m & j \\ l & i \end{bmatrix}_{p,q}$$  , (C.56)



$$= \sum_q F \begin{bmatrix} m & j \\ l & i \end{bmatrix}_{p,q}$$



. (C.57)

The matrices B and F are usually called the braid and fusing matrices, respectively. By considering five-point functions one can show that these matrices have to satisfy two very important constraints called pentagon and hexagon equations [18, 22, 21]. It turns out that RCFT's can be formulated as modular tensor categories and have very deep connections to three-dimensional topological field theories, knot invariants and exotic particle statistics in 2+1 dimensions. There are a lot more to say about this extremely interesting topic, but we will move on due to constraints on time.

C.1.5 Moduli of Algebraic Curves, Modular Invariance and Partition functions

As we have seen, the essence of 2D conformal field theory is separation of holomorphic and anti-holomorphic degrees of freedom and so far these have been completely independent. For example one could in principle construct different variations of minimal models (C.37) by different combinations of holomorphic and anti-holomorphic sectors. There are however two arguments for why we cannot keep these completely independent.

One argument relies on the fact that in 2D, scaling invariance implies conformal invariance [146] and thus CFT's describe fixed points of quantum field theories. The separation of right and left modes is only a feature of this fixed point and small perturbations away from it necessarily couple them back again. But not all combinations of right and left modes necessarily give rise to consistent couplings. Another arguments relies on that a CFT should be consistent on the torus either because one is interested in string perturbation theory or thermodynamic properties of the CFT. Let us see how this imposes further constraints on the CFT.

Let us recall that compact Riemann surfaces are one-dimensional complex manifolds, or complex-algebraic curves in the language of algebraic geometry. Even though these curves can be classified topologically by their genus g, they can still be inequivalent due to differing complex structures. Given two non-vanishing complex numbers $\omega_1, \omega_2 \in \mathbb{C}$, we can construct a lattice $L(\omega_1, \omega_2) = \{n\omega_1 + m\omega_2 | n, m \in \mathbb{Z}\}$. We can now construct a torus by identifying points of the complex plane

$$\Sigma_1 \approx \mathbb{C}/L(\omega_1, \omega_2). \qquad \text{(C.58)}$$

The upshot of this approach is that the torus automatically inherits a complex structure from the complex plane,[7] so we have reduced our classification problem to studying different choices of the lattice. We are interested in lattices up to multiplication, so it is convenient to normalize and define the modular parameter

$$\tau = \frac{\omega_2}{\omega_1} = \tau_1 + i\tau_2 \in \mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\} = \mathtt{Teich}(\Sigma_1), \qquad \text{(C.59)}$$

[7] It turns out that all complex structures on the torus can be induced in this way.

where we have assumed that $\tau \in \mathbb{H}$ without loss of generality. In algebraic geometry, in order to solve a classification problem (isomorphism classes of Riemann surfaces in our case) it is conventional to introduce extra structure, classify that problem and then investigate consequences of letting the structure go away. The Teichmüller space $\texttt{Teich}(\Sigma_1) = \mathbb{H}$ is the isomorphism classes of elliptic curves, which are genus one Riemann surfaces with a marked point. If we let this extra structure go, then any two set of complex numbers $\omega_1, \omega_2 \in \texttt{Teich}(\Sigma_1)$ related by

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})/\mathbb{Z}_2, \tag{C.60}$$

define equivalent complex structures. For the modular parameter this entails a transformation of the form

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{C.61}$$

The moduli space of Riemann surfaces of genus 1 is then given as

$$\texttt{Moduli}(\Sigma_1) = \frac{\texttt{Teich}(\Sigma_1)}{\texttt{MCG}(\Sigma_1)} = \frac{\mathbb{H}}{SL(2,\mathbb{Z})/\mathbb{Z}_2}. \tag{C.62}$$

It turn out that $\texttt{Moduli}(\Sigma_1) = \{\tau \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}\tau < \frac{1}{2} \text{ and } |\tau| \geq 1\}$, see [147, page 388], we will however work with $\mathbb{H}$ and keep track of equivalent complex structures. The mapping class group of the torus is $\texttt{MCG}(\Sigma_1) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ and can be generated by the following two transformations

$$\begin{aligned} \mathcal{T}: &\quad \tau \to \tau + 1, &\quad \text{or} &\quad \mathcal{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ \mathcal{S}: &\quad \tau \to -\frac{1}{\tau}, &\quad \text{or} &\quad \mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{C.63}$$

These transformations satisfy the important relations

$$\mathcal{S}^2 = \mathbf{1}, \qquad (\mathcal{S}\mathcal{T})^3 = \mathbf{1}. \tag{C.64}$$

The transformation $\tau \to \tau + 1$ generates a Dehn twist along the meridian[8], while $\tau \to -\frac{1}{\tau}$ switches the roles of the meridian and longitude.[9] Similarly, the mapping class group of higher Riemann surfaces are generated by a series of 2π Dehn twists, see figure C.1.

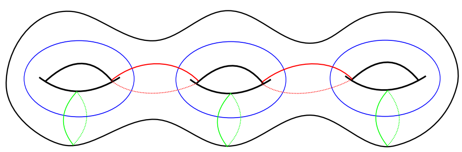


Figure C.1: The mapping class group of genus g Riemann surface is generated by $\dim \texttt{MCG}(\Sigma_g) = 3g - 1$ Dehn twists. The figure illustrates this for Σ_3. (Courtesy of Wikimedia Commons).

Let us now consider the CFT partition function, which is usually defined by compactifying the time direction and tracing the Boltzmann factor $e^{-\beta H}$. Mapping back

[8] The meridian is the small circle along the torus, while longitude is the other.

[9] The transformation $\mathcal{U} : \tau \to \frac{\tau}{\tau+1}$ is the other Dehn twist. It is however customary to instead use $\mathcal{S} = \mathcal{U}\mathcal{T}^{-1}\mathcal{U}$.

on the cylinder the Schwarzian derivative will modify the energy-momentum zero mode $L_0 \to L_0 - \frac{c}{24}$. Next observe that for a non-trivial modular parameter $\tau = \tau_1 + i\tau_2$, we don't get a closed loop in time[10] by translation along τ_2, so we need a translation in space $i\tau_1$. Equation (C.25) implies that we may write

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}_{\mathcal{H}} \left(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} \right), \\ &= \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \qquad q = e^{2\pi\tau}, \\ &= \sum_{\hat{\mu},\hat{\nu}} \chi_{\hat{\mu}}(\tau) \, \mathcal{M}_{\hat{\mu}\hat{\nu}} \, \chi_{\hat{\nu}}(\bar{\tau}), \end{aligned} \tag{C.65}$$

where we trace over the Hilbert space $\mathcal{H} = \bigoplus_{\hat{\mu}\hat{\nu}} \mathcal{M}_{\hat{\mu}\hat{\nu}} L_{\hat{\mu}} \otimes \bar{L}_{\hat{\nu}}$ decomposed into products of irreducible representations of some (possibly extended) symmetry algebra and we have defined the character

$$\chi_{\hat{\mu}}(\tau) = \mathrm{Tr}_{L_{\hat{\mu}}} \left(q^{L_0 - \frac{c}{24}} \right). \tag{C.66}$$

The mass matrix $\mathcal{M}_{\hat{\mu}\hat{\nu}}$ contains non-negative integers and specifies how the holomorphic and anti-holomorphic sectors are combined, note that $\mathcal{M}_{00} = 1$. In order to have a consistent theory on the torus, we need to require that the partition function is modular invariant

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau). \tag{C.67}$$

This poses strong constraints on the matrix $\mathcal{M}_{\hat{\mu}\hat{\nu}}$. In a RCFT there are a finite number of conformal families and it turns out that the characters transform into each other under modular transformations

$$\begin{aligned} \chi_{\hat{\mu}}(\tau + 1) &= \sum_{\hat{\nu}} \mathcal{T}_{\hat{\mu}\hat{\nu}} \chi_{\hat{\nu}}(\tau), \\ \chi_{\hat{\mu}}(-1/\tau) &= \sum_{\hat{\nu}} \mathcal{S}_{\hat{\mu}\hat{\nu}} \chi_{\hat{\nu}}(\tau). \end{aligned} \tag{C.68}$$

The space of characters actually form a unitary representation of the modular group $SL(2,\mathbb{Z})/\mathbb{Z}_2$. Since the transformation $\tau \to \tau + 1$ is a 2π Dehn twist, it is natural to speculate that it is represented by a pure phase transformation. This is actually seen to be correct from the definition (C.66)

$$\mathcal{T}_{\hat{\mu}\hat{\nu}} = \delta_{\hat{\mu}\hat{\nu}} e^{2\pi i (h_{\hat{\mu}} - c/24)}. \tag{C.69}$$

The modular $\mathcal{S}_{\hat{\mu}\hat{\nu}}$ turns out to be much more interesting and much harder to calculate, since it changes the two cycles of the torus it actually transforms into other characters. See the result for affine Lie algebras in appendix E and references. It is clear that for a diagonal mass matrix $\mathcal{M}_{\hat{\mu}\hat{\nu}} = \delta_{\hat{\mu}\hat{\nu}}$, the partition function (C.65) is modular invariant. However, more general mass matrices lead to modular invariance if they satisfy the following conditions

$$\mathcal{T}^\dagger \mathcal{M} \mathcal{T} = \mathcal{S}^\dagger \mathcal{M} \mathcal{S} = \mathcal{M}. \tag{C.70}$$

In a beautiful paper [148], Cappelli, Itzykson and Zuber found a complete classification of modular invariant mass matrices in the case of $\widehat{\mathfrak{su}}(2)_k$ WZW models called the A-D-E

[10] We have chosen time to be along Imaginary axes, but this doesn't matter too much since a modular $\mathcal{S}$ transformation changes it into the other axes.

classification, since all solutions can be labeled using simply-laced Lie algebras.[11] As will be discussed later, the Virasoro unitary minimal models can be realized by a coset WZW model $\frac{\widehat{su}(2)_k\times\widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}$ and it turns out that all modular invariant partition functions can be realized by combining modular invariants of $\widehat{su}(2)_k$ and $\widehat{su}(2)_{k+1}$. Thus the A-D-E classification also covers all $c<1$ unitary CFT's.

C.1.6 The Verlinde Formula

An extremely powerful and unexpected feature of RCFT's is the relation between fusion rules and modular invariance. A priori one would not expect any such relation since modular invariance is related to non-chiral features of a CFT, while fusion rules are very holomorphic in nature. However in a beautiful paper [145], Verlinde, then a graduate student, defined the fusion algebra (C.45) and conjectured that the coefficients are given by the modular $\mathcal{S}$-matrix as

$$N_{\hat\lambda\hat\mu}^{\hat\nu}=\sum_{\hat\sigma}\frac{\mathcal{S}_{\hat\lambda\hat\sigma}\mathcal{S}_{\hat\mu\hat\sigma}\mathcal{S}_{\hat\nu\hat\sigma}}{\mathcal{S}_{0\hat\sigma}}. \tag{C.71}$$

This was later proved by Moore and Seiberg [18], which in the process described the main general features of RCFT's. There is another way to state the formula which is in the spirit of the original paper by Verlinde. Define the matrix $(N_{\hat\lambda})_{\hat\mu\hat\nu}=N_{\hat\lambda\hat\mu}^{\hat\nu}$, then it turns out that the modular $\mathcal{S}$-matrix diagonalizes the fusion rules

$$\left(\mathcal{S}^\dagger N_{\hat\lambda}\mathcal{S}\right)_{\hat\mu\hat\nu}=\left(\frac{\mathcal{S}_{\hat\lambda\hat\mu}}{\mathcal{S}_{0\hat\mu}}\right)\delta_{\hat\mu\hat\nu}. \tag{C.72}$$

These eigenvalues for $\hat\mu=\hat\nu=0$ are called quantum dimensions and play an interesting role in CFT's and topological field theories.

C.2 Wess-Zumino-Witten Models and Affine Lie Algebras

In this section we will consider one of the most important constructions in 2D CFT, the Wess-Zumino-Witten model. This will be the first example of a CFT with enhanced symmetry algebra, which originates from conserved spin one currents besides the spin two ones which started everything (C.11). This will also allow us to construct unitary RCFT's with $c>1$, which is not possible without extra symmetry.

Take a compact connected Lie group G with a semi-simple Lie algebra $\mathfrak{g}$, a (unitary) highest weight representation Λ and a group valued function $g:S^2\to G$. The Wess-Zumino-Witten action is given by

$$S^{\mathrm{WZW}}[g]=-\frac{k}{8\pi}\int_{S^2}\mathrm{d}^2x\,K_\Lambda\left(g^{-1}\partial^\mu g,g^{-1}\partial_\mu g\right)+k\,\Gamma[g] \tag{C.73}$$

here $K_\Lambda(X,Y)=\frac{1}{2x_\Lambda}\mathrm{Tr}_\Lambda\left(\mathcal{R}(X),\mathcal{R}(Y)\right)$ is the Killing form in the Highest weight representation Λ. For the adjoint representation $\Lambda=\theta$, the Dynkin index is just the dual Coxeter number $x_\theta=g^\vee$ and we get the usual Killing form (see appendix D). In the following we will not distinguish between fields valued in the group g or in some representation $\mathcal{R}(g)$.

[11] To my knowledge, there is no deep understanding of why there is this relation to simply-laced Lie algebras.

The non-linear σ model is asymptotically free and not conformally invariant. Conformal symmetry can however be restored using arguments of Witten [149]. Since $\pi_2(G) = 0$ for any compact connected Lie group, we can extend the map g to the interior of the sphere with no obstruction $\tilde{g} : B \to G$, with $\partial B = S^2$.[12] Any compact non-abelian Lie group has a non-trivial harmonic form $\omega \sim \mathrm{Tr} g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g$ called the Cartan 3-form, this implies that we have a non-trivial de Rahm cohomology group $H^3(G) \neq 0$. The Wess-Zumino term is defined as the pull-back of this form, with an appropriate normalization

$$\Gamma[\tilde{g}] = i \int_B \tilde{g}^* \omega. \tag{C.74}$$

There is however a possible ambiguity, since $\pi_3(G) = \mathbb{Z}$ the extension $\tilde{g}$ can belong to any homotopy class and it turns out that the Wess-Zumino term gets shifted by $\Gamma + 2\pi i N$, $n \in \mathbb{Z}$, when changing the homotopy class of $\tilde{g}$. This is not a problem since the Euclidean functional integral $e^{-k\Gamma[\tilde{g}]}$ depends only on g, not the extension and is therefore well-defined. Expressing the Cartan 3-form using the Killing form, the term in given as

$$\Gamma[g] = \frac{-i}{24\pi} \int_B \mathrm{d}^3 y \, \epsilon_{\alpha\beta\gamma} \, K\left(\tilde{g}^{-1}\partial^\alpha \tilde{g}, \left[\tilde{g}^{-1}\partial^\beta \tilde{g}, \tilde{g}^{-1}\partial^\gamma \tilde{g}\right]\right). \tag{C.75}$$

This can be put in a more conventional form by using $\partial(g^{-1}g) = \partial I = 0$,

$$S^{\mathrm{WZW}}[g] = \frac{k}{16\pi} \int \mathrm{d}^2 x \, \mathrm{Tr'}\left(\partial^\mu g^{-1} \partial_\mu g\right) - \frac{ik}{24\pi} \int_B \mathrm{d}^3 y \, \epsilon_{\alpha\beta\gamma} \mathrm{Tr'}\left(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}\right), \tag{C.76}$$

where $\mathrm{Tr'}(\dots) = \frac{1}{x_\lambda}\mathrm{Tr}(\dots)$. Witten showed that with this choice of relative coupling constants, the theory is conformally invariant even quantum mechanically (it describes an infrared fixed point of the model with more general coupling constants). Turning to complex coordinates, it turns out that the theory is invariant under

$$g(z, \bar{z}) \to \Omega(z) g(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}),$$

with the conserved currents

$$\begin{aligned} \bar{\partial} J(z) &= 0, \qquad & J(z) &= -k \partial g g^{-1}, \\ \partial \bar{J}(\bar{z}) &= 0. \qquad & \bar{J}(\bar{z}) &= k g^{-1} \bar{\partial} g. \end{aligned} \tag{C.77}$$

A similar analysis with Ward identities to what we discussed before, leads to the OPE[13]

$$J^a(z) J^b(w) \sim \frac{k\,\kappa^{ab}}{(z-w)^2} + i f^{ab}_c \frac{J^c(w)}{z-w}, \tag{C.78}$$

where we have used the matrix representation of the Killing form $\kappa^{ab} = K(T^a, T^b)$ in a basis $\{T^a\}_{a=1}^{\dim \mathfrak{g}}$. This indicates that $J^a(z)$ has conformal weights $(h, \bar{h}) = (1, 0)$. A mode expansion $J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n$ leads to the commutator relations

$$\left[J^a_n, J^b_m\right] = i f^{ab}_c J^c_{n+m} + k n \kappa^{ab} \delta_{n+m,0}. \tag{C.79}$$

This is nothing but the affine Lie algebra $\hat{\mathfrak{g}}_k$, see appendix E. We have also the OPE $J^a(z)\bar{J}^b(\bar{z}) \sim 0$, which means that the two sectors decouple as expected.

[12] It is easy to construct examples with a manifold $\pi_2(M) \neq 0$, where is it clear that maps in a non-trivial homotopy class cannot be extended in such a way.

[13] Note that we could have found this result purely by assuming $h = 1$ and using dimensional arguments together with requiring the Jacobi identity for the commutator of their modes.

C.2.1 The Sugawara Construction and the WZW Primary Fields

The natural next step is to figure out where the Virasoro algebras is, which means we have to find the energy-momentum tensor. The energy-momentum tensor turns out to have the form $T(z) = \gamma \sum_a \kappa_{ab}\mathcal{N}(J^a J^b)(z)$, where the coefficient γ can be fixed by demanding that $J^a(z)$ has conformal weight 1, either by calculating the OPE[14] $T(z)J^a(z)$ or the commutator of their modes. Using the relation[15] $\mathrm{Tr}(t^a_{\mathrm{ad}}t^b_{\mathrm{ad}}) = -\sum_{dc} f^{ac}_{\ \ d} f^{bd}_{\ \ c} = C_2(\theta)\delta^{ab} = 2g^\vee\delta^{ab}$ from appendix D we find the energy-momentum tensor[16]

$$T(z) = \frac{1}{2(k+g^\vee)}\sum_{a,b=1}^{\dim\mathfrak{g}} \kappa_{ab}\,\mathcal{N}(J^a J^b)(z). \tag{C.80}$$

Here κ_{ab} is the inverse of κ^{ab}. Note the similarity to the second order Casimir element, this point will turn out to be important when we talk about $\mathcal{W}$-algebras. The central charge is calculated similarly by using (C.1.1.1)

$$\overbrace{T(z)T}(w) = \frac{\frac{1}{2}k\dim\mathfrak{g}/(k+g^\vee)}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w), \tag{C.81}$$

where we have used $\kappa_{ab}\kappa^{ab} = \dim\mathfrak{g}$. The central charge is thus given by

$$c = \frac{k\dim\mathfrak{g}}{k+g^\vee}. \tag{C.82}$$

Note that we could have started the whole story from the energy-momentum tensor (C.80), without ever talking about the WZW Lagrangian. This approach is called the Sugawara construction and in certain situations allow the construction of more general CFT's than the Langragian approach [133]. We can also translate (C.80) into modes using eq. (C.24) which give us

$$\begin{aligned} L_n &= \frac{1}{2(k+g^\vee)}\sum_{a,b=1}^{\dim\mathfrak{g}} \kappa_{ab}\left\{\sum_{m\le 1} J^a_m J^b_{n-m} + \sum_{m\ge 0} J^b_{n-m}J^a_m\right\}, \\ &= \frac{1}{2(k+g^\vee)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m\in\mathbb{Z}} :J^a_m J^a_{n-m}: \qquad \text{if} \quad \kappa_{ab} = \delta_{ab}. \end{aligned} \tag{C.83}$$

In the second line by $:\cdots:$ we mean that the lowest mode has to be put to the left, this only affects the $n = 0$ mode since in a orthonormal basis $\kappa_{ab} = \delta_{ab}$ we only have product with the same a index and for these the commutator (C.79) reduces to $[J^a_m, J^a_{n-m}] = km\delta_{n,0}$. Note that this means the Virasoro algebra is contained in the universal enveloping algebra of our affine Lie algebra, $\mathtt{Vir} \subset U(\hat{\mathfrak{g}}_k)$, which is not surprising given that the Sugawara energy-momentum tensor is some sort of Casimir operator.

[14] Note that we have to use the definition (C.1.1.1) since the WZW model is not a free field theory.

[15] Recall that $(t^a_{\mathrm{ad}})^b_{\ c} = -if^{ab}_{\ \ c}$.

[16] It is possible to generalize this energy-momentum tensor. A usual extension used in the literature is to add a term $p\cdot\partial H$, where H^i are the generators of the Cartan subalgebra.

C.2.1.1 WZW Primary Fields

There is an important point to note about the central charge (C.82), it satisfies the inequality $r \leq c \leq \dim \mathfrak{g}$, where r is the rank of $\mathfrak{g}$. This means that these WZW models generically have central charge larger than one and according to our earlier discussions, they will always contain an infinite number of Virasoro primary fields. However, as we also discussed earlier, when we have some extended symmetry algebra available we can organize fields into larger conformal families with respect to this larger symmetry algebra. Just as Virasoro primary fields gave rise to highest weight modules, we can define WZW primary fields which will give rise to affine Lie algebra highest weight modules.

A non-chiral field $\Phi_{\Lambda,\Omega}$ labeled with the highest weights Λ and Ω of $\mathfrak{g}$ (for each chirality) is said to be a WZW primary field if it satisfies the following OPE's

$$\begin{aligned} J^a(z)\Phi_{\Lambda,\Omega}(w,\bar{w}) &\sim \frac{-t^a_\Lambda\,\Phi_{\Lambda,\Omega}(w,\bar{w})}{z-w}, \\ \bar{J}^a(z)\Phi_{\Lambda,\Omega}(w,\bar{w}) &\sim \frac{\Phi_{\Lambda,\Omega}(w,\bar{w})\,t^a_\Omega}{z-w}, \end{aligned} \tag{C.84}$$

where $R_{\lambda/\mu}(T^a) = t^a_{\lambda/\mu}$ are the representations of the generators of $\mathfrak{g}$. We will denote the holomorphic and anti-holomorphic parts with small letters $\Phi_{\Lambda,\Omega}(z,\bar{z}) = \phi_\Lambda(z)\bar{\phi}_\Omega(\bar{z})$. On the level of the Hilbert space, the WZW primary field corresponds to a state satisfying

$$\begin{aligned} J^a_0|\Lambda,\Omega\rangle &= -t^a_\Lambda|\Lambda,\Omega\rangle, \qquad & \bar{J}^a_0|\Lambda,\Omega\rangle &= t^a_\Omega|\Lambda,\Omega\rangle, \\ J^a_n|\Lambda,\Omega\rangle &= 0, & \bar{J}^a_n|\Lambda,\Omega\rangle &= 0, \qquad n>0, \end{aligned} \tag{C.85}$$

with the definition $|\Lambda,\Omega\rangle = \lim_{z,\bar{z}\to 0}\Phi_{\Lambda,\Omega}(z,\bar{z})|0\rangle$. The next natural question is how the Virasoro algebra acts on these states. Using eq. (C.83) and (C.85) it is clear that

$$L_n|\Lambda,\Omega\rangle = \bar{L}_n|\Lambda,\Omega\rangle = 0, \qquad n>0, \tag{C.86}$$

and

$$L_0|\Lambda,\Omega\rangle = \frac{1}{2(k+g^\vee)}\sum_{a,b=1}^{\dim\mathfrak{g}}\kappa_{ab}J^a_0J^b_0|\Lambda,\Omega\rangle = \frac{1}{2(k+g^\vee)}C_2(\Lambda)|\Lambda,\Omega\rangle, \tag{C.87}$$

and similarly for $\bar{L}_0$. This implies that $|\Lambda,\Omega\rangle$ is a primary field of $\mathtt{Vir}\oplus\overline{\mathtt{Vir}}$ with the conformal weights

$$h_\Lambda = \frac{(\Lambda,\Lambda+2\rho)}{2(k+g^\vee)}, \qquad \bar{h}_\Omega = \frac{(\Omega,\Omega+2\rho)}{2(k+g^\vee)}. \tag{C.88}$$

There is a small issue we need to resolve. The set of states (C.85) form a multiplet transforming irreducibly under the horizontal subalgebra $\mathfrak{g}\subset\hat{\mathfrak{g}}_k$ but have the same L_0 eigenvalue, and so do not constitute a unique "vacuum". Using Cartan-Weyl basis, eq. (C.85) says that these states are annihilated by all positive mode generators H^i_n and $E^{\pm\alpha}_n$ with $n>0$. In order to have a true highest weight representation of $\hat{\mathfrak{g}}_k$ we also need to require these states are annihilated by the positive roots of the zero modes E^α_0 for $\alpha>0$. In other words, the true highest weight state, labeled with highest weights $\hat{\Lambda}$ and $\hat{\Omega}$, has to satisfy the relations (C.85) together with these

$$E^\alpha_0|\hat{\Lambda},\hat{\Omega}\rangle = \bar{E}^\alpha_0|\hat{\Lambda},\hat{\Omega}\rangle = 0, \qquad \forall\alpha>0. \tag{C.89}$$

The rest of the "vacuum multiplet" $|\Lambda,\Omega\rangle$, can be constructed by acting with $E^{-\alpha}_0$ on $|\hat{\Lambda},\hat{\Omega}\rangle$. We will now label everything with respect to highest weights of the affine Lie algebra, so for example $h_{\hat{\Lambda}} = h_\Lambda$. Descendant fields in the WZW conformal family is found by successive action of the Virasoro negative modes $\{L_n, \bar{L}_n\}$ and WZW negative modes and $\{J^a_{-n}, \bar{J}^a_{-n}\}$. An argument similar to (C.35) shows that the WZW currents $J^a(z)$ are descendants of the identity operator, thus not WZW primaries although they are Virasoro primaries.

C.2.2 Knizhnik-Zamolodchikov and Gepner-Witten Equations

All this symmetry puts many constraints of correlation functions on WZW primary fields, two immediate ones come from the Ward identities

$$\begin{aligned} \sum_{i=1}^{N} t^a_{\Lambda_i} \big\langle \phi_{\Lambda_i}(z_1) \dots \phi_{\Lambda_N}(z_N) \big\rangle = 0, \\ \sum_{i=1}^{N} \Big\{ z_i^m \Big(z_i \partial_i + (m+1) h_{\hat{\Lambda}_i} \Big) \Big\} \big\langle \phi_{\Lambda_i}(z_1) \dots \phi_{\Lambda_N}(z_N) \big\rangle = 0, \end{aligned} \tag{C.90}$$

where the first constraint comes from global G invariance and the second from global $SL(2,\mathbb{C})/\mathbb{Z}_2$ invariance ($m = 0, \pm 1$). There are however at times much stronger constraints stemming from the fact that states in the WZW Verma module generated by the action of L_{-n} and J^a_{-n}'s, are not all linearly independent although they formally appear so. The existence of null-vectors, which generate their own Verma module that need to be modded out, give rise to such constraints. Following Gepner and Witten [150], let us summarize the three types of null-states

1. From purely Virasoro algebra.
2. Combined Virasoro and current algebra.
3. Purely current algebra.

We have already discussed case 1., where null-vectors exist for certain values of central charge $c < 1$ and give rise to the Virasoro unitary minimal models. An important example of case 2. was discussed by Knizhnik and Zamolodchikov [151]. Due to form of the Virasoro algebra generators (C.83) we can see that (choosing an orthonormal basis $\kappa_{ab} = \delta_{ab}$)

$$L_{-1}|\Lambda_i\rangle = \frac{1}{k+g^\vee} \sum_{a=1}^{\dim \mathfrak{g}} J^a_{-1} J^a_0 |\Lambda_i\rangle = \frac{-1}{k+g^\vee} \sum_{a=1}^{\dim \mathfrak{g}} J^a_{-1} t^a_{\Lambda_i} |\Lambda_i\rangle, \tag{C.91}$$

which implies that we have the zero null-state

$$|\chi\rangle = \Big[L_{-1} + \frac{1}{k+g^\vee} \sum_{a=1}^{\dim \mathfrak{g}} J^a_{-1} t^a_{\Lambda_i} \Big] |\Lambda_i\rangle = 0. \tag{C.92}$$

Putting the corresponding field into correlators $\langle \phi_{\Lambda_1}(z_1) \dots \chi(z_i) \dots \phi_{\Lambda_N}(z_N) \rangle$ and requiring this has to vanish, leads to the so-called Knizhnik-Zamolodchikov equation

$$\left[\partial_i + \frac{1}{k+g^\vee} \sum_{i \neq j} \frac{\sum_{a=1}^{\dim \mathfrak{g}} t^a_{\Lambda_i} \otimes t^a_{\Lambda_j}}{z_i - z_j} \right] \big\langle \phi_{\Lambda_1}(z_1) \dots \phi_{\Lambda_N}(z_N) \big\rangle = 0. \tag{C.93}$$

Knizhnik and Zamolodchikov were able to solve this equation in the case of $\hat{\mathfrak{g}}_k = \widehat{su}(N)_k$ four-point functions with ϕ_{Λ_i} all in the fundamental representation [151, 133].

One can now turn case 3., as analyzed by Gepner and Witten [150]. Due to lack of time, we shall not give the details. If one concentrates on the class of integrable highest weight representations $\hat{\Lambda} \in P_+^k$, one needs to require that these states generate finite representations with respect to any $\mathfrak{su}(2)$ subalgebra of $\widehat{\mathfrak{su}}(2)$, which implies that they should be annihilated if one applies the negative roots of these subalgebras enough times. This implies these states are null-states. Putting these states into correlation functions, one can derive an equation called the Gepner-Witten equation. One important consequence is that one can see from these equations that all non-integrable representations decouple from the theory since their correlators vanish with arbitrary fields. This means we only have to consider integrable highest weight modules which makes WZW models RCFT's since the condition

$$k \geq (\Lambda, \theta), \tag{C.94}$$

states that there are only finite number of these representations for finite k. See more details in appendix E and [150, 133, 136].

C.2.3 Fusion Rules of WZW Models

We will not have enough time and space to go into all these glory details of fusion rules of WZW models. There is however a few important things to note. Let us write the fusion rules in the following way

$$\hat{\Lambda} \times \hat{\Omega} = \bigoplus_{\hat{\Xi} \in P_+^k} N_{\hat{\Lambda}\hat{\Omega}}^{(k)\hat{\Xi}}\, \hat{\Xi}. \tag{C.95}$$

The actions of outer automorphisms of fusion rules turns out to put a constraint on the fusion rules, the fusion coefficient is zero unless

$$\Lambda + \Omega - \Xi \in Q, \tag{C.96}$$

where Q is the root lattice. Another important fact is that in the $k \to \infty$ limit, the fusion rules becomes decomposition of tensor products of the finite Lie algebra $\mathfrak{g}$.

C.3 The WZW Coset Construction

So far we have discussed two classes of CFT's, the minimal models which cover all Unitary representations of the Virasoro algebra for $c < 1$ and the WZW models which for each simple Lie algebra gives a class of RCFT's for $c > 1$ with respect to the larger algebra. The latter can easily be generalized to any semi-simple Lie algebra. In this section we will discuss a vast generalization of the WZW models called the coset construction and was introduced by Goddard, Kent and Olive [152, 153]. This class of CFT's are so general that it is believed that all RCFT's can be constructed in this way, we will in particular discuss the $c < 1$ minimal models (see also [144] where it is conjectured that all RCFT's can be classified by 2+1D Chern-Simons theory).

Recall that any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can be embedded in several ways $\mathfrak{h} \hookrightarrow \mathfrak{g}$ characterized by an embedding index x_e. This can be lifted to an embedding of affine Lie algebras $\hat{\mathfrak{h}}_{\bar{k}} \hookrightarrow \hat{\mathfrak{g}}_k$ with the level given by $\bar{k} = x_e k$. Assume that the currents $J^a_{\hat{\mathfrak{g}}_k}$ and $J^b_{\hat{\mathfrak{h}}_{\bar{k}}}$ generate

$\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_{\bar{k}}$ respectively, then in their universal enveloping algebras there are the following Sugawara energy-momentum tensors

$$\begin{aligned} T_{\hat{\mathfrak{g}}_k}(z) &= \frac{1}{2(k+g^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} \mathcal{N}\left(J^a_{\hat{\mathfrak{g}}_k} J^a_{\hat{\mathfrak{g}}_k}\right)(z), \\ T_{\hat{\mathfrak{h}}_{\bar{k}}}(z) &= \frac{1}{2(k+h^\vee)} \sum_{b=1}^{\dim \mathfrak{h}} \mathcal{N}\left(J^b_{\hat{\mathfrak{h}}_{\bar{k}}} J^b_{\hat{\mathfrak{h}}_{\bar{k}}}\right)(z). \end{aligned} \tag{C.97}$$

Note that the currents $J^b_{\hat{\mathfrak{h}}_{\bar{k}}}$ are $h=1$ primary fields of both energy-momentum tensors

$$T_{\hat{\mathfrak{g}}_k}(z) J^b_{\hat{\mathfrak{h}}_{\bar{k}}}(w) \sim T_{\hat{\mathfrak{h}}_{\bar{k}}}(z) J^b_{\hat{\mathfrak{h}}_{\bar{k}}}(w) \sim \frac{1}{(z-w)^2} J^b_{\hat{\mathfrak{h}}_{\bar{k}}}(w) + \frac{1}{z-w} \partial J^b_{\hat{\mathfrak{h}}_{\bar{k}}}(w). \tag{C.98}$$

We are interested in constructing a theory in this we decouple the sector corresponding to the subalgebra $\hat{\mathfrak{h}}_{\bar{k}}$. This can be achieved by the decomposition

$$T_{\hat{\mathfrak{g}}_k} = T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} + T_{\hat{\mathfrak{h}}_{\bar{k}}} \qquad \Rightarrow \qquad T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} = T_{\hat{\mathfrak{g}}_k} - T_{\hat{\mathfrak{h}}_{\bar{k}}}.$$

This decomposes the Virasoro algebra into two commuting sectors since we have the regular OPE's

$$T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} J^b_{\hat{\mathfrak{h}}_{\bar{k}}} \sim T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} T_{\hat{\mathfrak{h}}_{\bar{k}}} \sim 0,$$

which on the level of modes means that $L_m^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} = L_m^{\hat{\mathfrak{g}}_k} - L_m^{\hat{\mathfrak{h}}_{\bar{k}}}$ satisfy the commutator $[L_m^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}}, L_n^{\hat{\mathfrak{h}}_{\bar{k}}}] = 0$. Either by calculating $[L_m^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}}, L_n^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}}]$ or observing that $T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} T_{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}} \sim T_{\hat{\mathfrak{g}}_k} T_{\hat{\mathfrak{g}}_k} - T_{\hat{\mathfrak{h}}_{\bar{k}}} T_{\hat{\mathfrak{h}}_{\bar{k}}}$ we find the central charge

$$c(\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}) = c(\hat{\mathfrak{g}}_k) - c(\hat{\mathfrak{h}}_{\bar{k}}) = \frac{k \dim \mathfrak{g}}{k+g^\vee} - \frac{x_e k \dim \mathfrak{h}}{x_e k + h^\vee}. \tag{C.99}$$

As is evident from our notation, these CFT's are labeled by the coset $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}$.

A very important example are the diagonal cosets $(\hat{\mathfrak{g}}_{k_1} \oplus \hat{\mathfrak{g}}_{k_2})/\hat{\mathfrak{g}}_k$, where the algebras $\hat{\mathfrak{g}}_{k_i}$ are generated by $J^a_{(i)}$ and $\hat{\mathfrak{g}}_k$ is generated by $J^a = J^a_{(1)} + J^a_{(2)}$. Since $[J^a_{(1)}, J^b_{(2)}] = 0$ it follows that that the level and structure constants of $\hat{\mathfrak{g}}_k$ are just the sum the two others $k = k_1 + k_2$. The central charge is thus given by

$$c = \dim \mathfrak{g} \left(\frac{k_1}{k_1+g^\vee} + \frac{k_2}{k_2+g^\vee} - \frac{k_1+k_2}{k_1+k_2+g^\vee} \right). \tag{C.100}$$

The fact that we subtract the central charge of the subalgebra indicates that one might be able to construct cosets with central charge $c < 1$, these must necessarily be identified with the minimal models if the representations are unitary (which they are for integer levels). There are thus not a unique way of constructing different minimal models CFT's using the coset construction. It was however shows in [153] that all Virasoro minimal models can be constructed using the diagonal coset

$$\frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{k+1}}, \tag{C.101}$$

which gives rise to the central charge

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}, \qquad k \geq 1, \tag{C.102}$$

which has to be compared to (C.36) with $m = k+2$. A possibly even more surprising fact is that the coset construction (of ordinary affine Lie algebras) can even given rise to representations of the super-Virasoro algebra, which was also notices in [153]. In particular the coset

$$\frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_2}{\widehat{\mathfrak{su}}(2)_{k+2}}, \tag{C.103}$$

gives rise to the $\mathcal{N} = 1$ super-Virasoro minimal models with the central charge

$$c = \frac{3}{2}\left(1 - \frac{8}{(k+2)(k+4)}\right). \tag{C.104}$$

C.3.1 Primary Fields, Fix Points and Field Identifications

Next we need to find the spectrum of primary fields of the coset $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}$. We will not go into many details since we did not have time to write too much about the relevant math in appendix E, but only mention the main aspects. The branching rules gives rise to a corresponding character identity

$$\hat{\Lambda} \to \bigoplus_{\hat{\Omega} \in P_+^{\bar{k}}(\hat{\mathfrak{h}})} b_{\hat{\Lambda}\hat{\Omega}}\, \hat{\Omega} \qquad \Rightarrow \qquad \mathrm{ch}_{\mathcal{P}\hat{\Lambda}} = \sum_{\hat{\Omega} \in P_+^{\bar{k}}(\hat{\mathfrak{h}})} b_{\hat{\Lambda}\hat{\Omega}}\, \mathrm{ch}_{\hat{\Omega}}, \tag{C.105}$$

where $\mathcal{P}$ is the projection matrix of the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Evaluating this on an affine weight and multiplying with the relevant exponential we can identify the normalized character of the coset with the branching rules

$$\chi_{\{\hat{\Lambda};\hat{\Omega}\}}(\tau) = e^{2\pi i \tau (m_{\hat{\Lambda}} - m_{\hat{\Omega}})} b_{\hat{\Lambda}\hat{\Omega}}(\tau), \tag{C.106}$$

where $m_{\hat{\Lambda}}$ is the modular anomaly. One immediate consequence is that in order for the characters to be non-zero the branching rules must be non-zero as in equation (D.37). This imposes the requirement

$$\mathcal{P}\Lambda - \Omega \in \mathcal{P}Q(\mathfrak{g}). \tag{C.107}$$

This selection rule requires $\mathcal{P}\Lambda$ and Ω to be in the same congruence class. If there is nontrivial branching of outer automorphisms $A \to \tilde{A}$ there is some over counting we need to take care of. In the case of no fixed points $\mathcal{P}\hat{\Lambda} = \hat{\Lambda}$ and $\mathcal{P}\hat{\Omega} = \hat{\Omega}$, it turns out that we must make the following the identification

$$\{\hat{\Lambda}; \hat{\Omega}\} \sim \{A\hat{\Lambda}; \tilde{A}\hat{\Omega}\}. \tag{C.108}$$

These two conditions are actually related. In the case there are fixed points subtleties arise, we will however not discuss the resolution of fixed points since it is not important for us and it is not well understood in general.

In order to summarize, primary fields of $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}_{\bar{k}}$ can be labeled by integral highest weights $\hat{\Lambda} \in P_+^k(\hat{\mathfrak{g}})$ and $\hat{\Omega} \in P_+^{\bar{k}}(\hat{\mathfrak{h}})$, written $\{\hat{\Lambda}; \hat{\Omega}\}$, satisfying the constraints and identifications discussed above. Finally the conformal weights of the Virasoro primary fields are given as

$$h_{\{\hat{\Lambda};\hat{\Omega}\}} = h_{\hat{\Lambda}} - h_{\hat{\Omega}} + n, \tag{C.109}$$

where n is an integer as can be calculated by knowing the details of the branching rules, which is hard in general.

In the case of diagonal cosets we have that $\mathcal{P}(\Lambda, \Xi) = \Lambda + \Xi$, $\mathcal{P}(Q \oplus Q) = Q$ and $A \otimes A \to A$. Thus we can label the primary fields by three $\hat{\mathfrak{g}}$ integrable highest weights $\{\hat{\Lambda}, \hat{\Xi}; \hat{\Omega}\}$ at levels k_1, k_2 and $k_1 + k_2$, respectively, satisfying the selection rule

$$\Lambda + \Xi - \Omega \in Q, \tag{C.110}$$

and the field identifications

$$\{\hat{\Lambda}, \hat{\Xi}; \hat{\Omega}\} = \{A\hat{\Lambda}, A\hat{\Xi}; A\hat{\Omega}\}, \qquad \forall A \in \mathcal{O}(\hat{\mathfrak{g}}). \tag{C.111}$$

C.3.1.1 Three-State Potts Model using the Coset Construction

As a very simple and concrete example let us consider the following diagonal coset, which is studied in more details in this thesis,

$$\frac{\widehat{\mathfrak{su}}(3)_1 \oplus \widehat{\mathfrak{su}}(3)_1}{\widehat{\mathfrak{su}}(3)_2}. \tag{C.112}$$

This coset can be shown to have another conserved current in the vacuum sector of spin 3, extending the Virasoro algebra to the Zamolodchikov W_3 algebra [69]. The central charge is $c = \frac{4}{5}$. Since this is less than one and the CFT is unitary it must correspond to a Virasoro minimal model, it is actually the 3-state Potts model. The coset can be characterized by three integrable highest weights[17] $\{\hat{\rho}, \hat{\mu}; \hat{\nu}\}$. Using the highest root $\theta = \omega_1 + \omega_2 = (1, 1)$, the condition (C.94) gives the constraints

$$1 \geq \rho_1 + \rho_2, \qquad 1 \geq \mu_1 + \mu_2, \qquad 2 \geq \nu_1 + \nu_2.$$

The selection rule (C.110) requires that the three weights lie in the same congruence class. As discussed in appendix D, for A_2 we have three congruence classes $P/Q = \mathbb{Z}_3$ which can also be seen in figure D.1. Let us recast the condition $\rho + \mu - \nu \in Q$ into

$$(\rho_1 + \mu_1 - \nu_1) + 2(\rho_2 + \mu_2 - \nu_2) = 0 \mod 3.$$

With this we can pick $\rho = (\rho_1, \rho_2)$, $\nu = (\nu_1, \nu_2)$ and then calculate what $\mu = (\mu_1, \mu_2)$ should be. It turns out there are 18 possibilities, 6 of them are listed here:

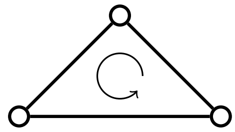
$[\rho_0, \rho]$	$[\nu_0, \nu]$		$[\mu_0, \mu]$
$[1, 0, 0]$	$[2, 0, 0]$	$\Rightarrow$	$[1, 0, 0]$
$[1, 0, 0]$	$[1, 1, 0]$	$\Rightarrow$	$[0, 1, 0]$
$[1, 0, 0]$	$[1, 0, 1]$	$\Rightarrow$	$[0, 0, 1]$
$[1, 0, 0]$	$[0, 1, 1]$	$\Rightarrow$	$[1, 0, 0]$
$[1, 0, 0]$	$[0, 2, 0]$	$\Rightarrow$	$[0, 0, 1]$
$[1, 0, 0]$	$[0, 0, 2]$	$\Rightarrow$	$[0, 1, 0]$

Finally we must remember the identifications using the outer automorphisms

$$\{\hat{\rho}, \hat{\mu}, \hat{\nu}\} \sim \{A\hat{\rho}, A\hat{\mu}, A\hat{\nu}\}.$$

[17] We will for now use small Greek letters to describe highest weights.

The structure of the outer automorphism group $\mathcal{O}(\widehat{\mathfrak{su}}(3))$ is clear from the affine Dynkin diagram



One can easily show that the other 12 primary fields are the ones one get by using these outer automorphisms on the primaries written in the table, thus there are only 6 distinct coset primary fields. This is in agreement with the analysis of the three-state Potts Model in [133] section 7.4.4, using different techniques.

C.3.2 Fusion Rules and Modular Properties

The fusion coefficients and the modular $\mathcal{S}$ and $\mathcal{T}$ on the coset can be shown to be products of the ones from $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_{\bar{k}}$.

Appendix **D**

Semi-simple Lie Algebras

In this appendix we will sketch the relevant aspects of the structure and representation theory of semi-simple Lie algebras, for more details see [154, 155, 133, 156]. Unless explicitly stated, we will only consider finite-dimensional Lie algebras over $\mathbb{C}$.

Given a basis $\{J^a | a = 1, \ldots, d\}$ for a d-dimensional Lie Algebra $\mathfrak{g}$, the commutator relations are characterized by the *structure constant* f_c^{ab},

$$[J^a, J^b] = \sum_c i f_c^{ab} J^c.$$

A *simple* Lie algebra is a Lie algebra with no proper ideal, meaning there is no subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \in \mathfrak{h}$, other than the trivial ideals 0 and $\mathfrak{g}$. A Lie algebra is *semi-simple* if it is a direct sum of simple Lie algebras.

D.1 Structure Theory and Classification

D.1.1 Cartan-Weyl basis

Many aspects of semi-simple Lie algebras are best considered after choosing a special basis, e.g. we would like to write down the structure constants in a canonical way. The Cartan-Weyl basis will be convenient.

Let $\mathfrak{g}_0 := \text{span}_{\mathbb{C}} \left\{ H^i | i = 1, 2, \ldots, r \right\}$ be a maximal set of linearly independent elements H^i among the ad-diagonalizable elements of $\mathfrak{g}$, such that

$$[H^i, H^j] = 0, \qquad \text{for} \quad i, j = 1, 2, \ldots, r. \tag{D.1}$$

The *rank* of $\mathfrak{g}$ is defined as rank $\mathfrak{g} = \dim \mathfrak{g}_0 = r$ and $\mathfrak{g}_0$ is called the *Cartan subalgebra.* We can find simultaneous eigenvectors for the generators of $\mathfrak{g}_0$ in the adjoint representation

$$\text{ad}_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha = \alpha(H^i) E^\alpha. \tag{D.2}$$

The vector $\alpha = (\alpha^1, \ldots, \alpha^r)$ is called a *root* (if non-zero) and E^α is the corresponding *ladder operators.* The set of roots are called the *root system* Δ. Note that $\alpha : \mathfrak{g}_0 \to \mathbb{C}$ can be extended to a linear functional on $\mathfrak{g}_0$ and roots can therefore be considered as elements of the dual space $\alpha \in \mathfrak{g}_0^*$. This leads to the *root space decomposition* relative to $\mathfrak{g}_0$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \,\middle|\, [h, x] = \alpha(h) x \text{ for all } h \in g_0 \right\}. \tag{D.3}$$

Some fundamental properties are

- The roots span $\mathfrak{g}_0^*$: $\text{span}_{\mathbb{C}}(\Delta) = \mathfrak{g}_0^*$.
- Roots are non-degenerate, thus root spaces $\mathfrak{g}_\alpha$ are one-dimensional.
- The only multiplets of $\alpha \in \Delta$ which are roots are $\pm\alpha$.
- One can choose a basis $\{H^i\}$ of the Cartan subalgebra such that $\alpha(H^i)$ are real (even integers), for all i and each root $\alpha \in \Delta$.

Therefore we have $|\Delta| = d - r \in 2\mathbb{N}$ number of roots and E^α is uniquely specified up to normalization. The basis

$$\mathfrak{B} = \{H^i | r = 1, \ldots, r\} \cup \{E^\alpha | \alpha \in \Delta\},$$

is called the Cartan-Weyl basis. The commutation relations are given by

$$\begin{aligned}
[H^i, H^j] &= 0 \\
[H^i, E^\alpha] &= \alpha^i E^\alpha \\
[E^\alpha, E^\beta] &= \mathcal{N}_{\alpha,\beta} E^{\alpha+\beta} && \text{if } \alpha + \beta \in \Delta \\
&= \tilde{\alpha} \cdot H && \text{if } \alpha = -\beta \\
&= 0 && \text{otherwise,}
\end{aligned} \tag{D.4}$$

where $\tilde{\alpha} \cdot H = \sum_{i=1}^r \tilde{\alpha}^i H^i$ and $\tilde{\alpha}$ are some expansion coefficients to be determined.

D.1.2 The Killing Form

Using the adjoint representation, we can define an inner product $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on the Lie algebra $\mathfrak{g}$ by

$$K(X, Y) \equiv \frac{1}{I_{ad}} \text{Tr}(\text{ad}_X \circ \text{ad}_Y) = \frac{1}{2g^\vee} \text{Tr}(\text{ad}_X \circ \text{ad}_Y), \tag{D.5}$$

where the normalization in the adjoint representation $I_{ad} = 2g^\vee$ is given in terms of the *dual Coxeter number* $g^\vee$ of $\mathfrak{g}$, to be defined below. It is obvious that the Killing form is symmetric and bilinear, furthermore the cyclic property of the trace yield the identity[1]

$$K([Z, X], Y) + K(X, [Z, Y]) = 0. \tag{D.6}$$

Actually, it turn out that the Killing form is uniquely characterized by this property. The standard basis J^a is assumed to be orthonormal $K(J^a, J^b) = \delta^{a,b}$, and the same will we assume for the Cartan subalgebra

$$K(H^i, H^j) = \delta^{i,j}. \tag{D.7}$$

[1] Using that $\text{ad}_{[X,Y]} = \text{ad}_x \circ \text{ad}_y - \text{ad}_y \circ \text{ad}_x$.

For semi-simple Lie algebras the Killing form is nondegenerate, this can in fact be used as an alternative definition of semi-simplicity and one of the reasons why our following construction works for this class of Lie algebras.

Actually, the restriction of K to the Cartan subalgebra is nondegenerate as well. Now any nondegenerate bilinear form on a vector space can be used to identify the vector space and its dual space. Hence we are led to associate to any root α an element $H^\alpha \in \mathfrak{g}_0$, which up to normalization is unique, such that

$$\alpha(h) = c_\alpha K(H^\alpha, h), \quad \text{for all } h \in \mathfrak{g}_0,$$

where c_α are normalization constants. As an important consequence, we can define a nondegenerate inner product on $\mathfrak{g}^*$ by

$$(\alpha, \beta) \equiv c_\alpha c_\beta K(H^\alpha, H^\beta) = c_\beta \alpha(H^\beta),$$

for all root $\alpha, \beta \in \Delta$, and extend by bilinearity to all $\mathfrak{g}^* \times \mathfrak{g}^*$.

Now we need to fix the normalization constants $\tilde{\alpha}$ and c_α. Choosing $c_\alpha = 1$, one can easily show that using (D.7)

$$\gamma \in \mathfrak{g}_0^* \quad \Leftrightarrow \quad H^\gamma = \sum_{i=1}^{r} \gamma^i H^i \in \mathfrak{g}_0, \tag{D.8}$$

are the corresponding duals. Furthermore using (D.6) with $X = H^\gamma \in \mathfrak{g}_0$, $Y = E^\alpha$ and $Z = E^{-\alpha}$ we find

$$\begin{aligned} K(H^\gamma, [E^\alpha, E^{-\alpha}]) &= K([E^{-\alpha}, H^\gamma], E^\alpha) \\ \gamma([E^\alpha, E^{-\alpha}]) &= \alpha(H^\gamma) K(E^{-\alpha}, E^\alpha) \\ &= K(H^\alpha, H^\gamma) K(E^{-\alpha}, E^\alpha) \\ &= \gamma(K(E^{-\alpha}, E^\alpha) H^\alpha), \end{aligned}$$

which implies that

$$[E^\alpha, E^{-\alpha}] = K(E^\alpha, E^{-\alpha})\, \alpha \cdot H.$$

We can now fix the normalization by choosing $K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2}$. Defining the coroots $\alpha^\vee = 2\frac{\alpha}{|\alpha|^2}$, we get the commutator[2]

$$[E^\alpha, E^{-\alpha}] = \frac{2}{|\alpha^2|} \alpha \cdot H = \alpha^\vee \cdot H. \tag{D.9}$$

D.1.3 Weights and $\mathfrak{sl}_2$ Subalgebras

So far, we have been dealing with a specific representation, the adjoint representation of $\mathfrak{g}$ onto itself. For a general finite-dimensional representation, $R : \mathfrak{g} \to V$, we can find a basis $\{|\lambda\rangle\}$ for the representation space such that

$$R(H^i)|\lambda\rangle = \lambda^i |\lambda\rangle.$$

[2]Note that we are following the conventions of [133]. In [154, 155], $c_\alpha = \frac{1}{2}|\alpha|^2$ so $H^\beta = \sum_{i=1}^r \beta^{i\vee} H^i$ and $[E^\alpha, E^{-\alpha}] = \alpha \cdot H$.

The collection $\lambda = (\lambda^1, \ldots, \lambda^r)$ is called a *weight* and clearly live in the dual space of the Cartan subalgebra $\lambda \in \mathfrak{g}_0^*$ with $\lambda(H^i) = \lambda^i$. Roots are nothing but weights, for the adjoint representation. From the commutators (D.4) we see that E^α changes the eigenvalue with of a state by α

$$R(H^i)R(E^\alpha)|\lambda\rangle = (\lambda^i + \alpha^i)R(E^\alpha)|\lambda\rangle,$$

so if $R(E^\alpha)|\lambda\rangle$ is nonzero, it must be proportional to $|\lambda + \alpha\rangle$.

Now, for any state $|\lambda\rangle$ in a finite-dimensional representation, there are necessarily two positive integers p and q, such that

$$R(E^\alpha)^{p+1}|\lambda\rangle \propto R(E^\alpha)|\lambda + p\alpha\rangle = 0,$$
$$R(E^{-\alpha})^{q+1}|\lambda\rangle \propto R(E^{-\alpha})|\lambda - q\alpha\rangle = 0,$$

for any root $\alpha \in \Delta$. Actually the generators $J_\alpha^+ = E^\alpha$, $J_\alpha^- = E^{-\alpha}$ and $J_\alpha^3 = \alpha \cdot H/|\alpha|^2$, form a $\mathfrak{sl}_2$ subalgebra with the commutation relations

$$[J_\alpha^+, J_\alpha^-] = 2J_\alpha^3, \quad [J_\alpha^3, J_\alpha^\pm] = \pm J_\alpha^\pm.$$

The projection of a finite-dimensional $\mathfrak{g}$-module to the $\mathfrak{s}l_2$ subalgebra associated with the root α must also be finite-dimensional. Let the dimension of the latter be $2j + 1$, then from the state $|\lambda\rangle$, the state with highest $J_\alpha^3 = \alpha \cdot H/|\alpha|^2$ projection (m=j) can be reached by $p_{\alpha,\lambda}$ applications of $J_\alpha^+ = E^\alpha$, whereas $q_{\alpha,\lambda}$ applications of $J^- = E^{-\alpha}$ leads to the state with $m = -j$:

$$j_{\alpha,\lambda} = \frac{(\alpha, \lambda)}{|\alpha|^2} + p_{\alpha,\lambda}, \quad -j_{\alpha,\lambda} = \frac{(\alpha, \lambda)}{|\alpha|^2} - q_{\alpha,\lambda}.$$

This leads to the important result

$$2\frac{(\alpha, \lambda)}{|\alpha|^2} = (\alpha^\vee, \lambda) = -(p_{\alpha,\lambda} - q_{\alpha,\lambda}) \in \mathbb{Z}. \tag{D.10}$$

D.1.4 Simple Roots and the Cartan Matrix

We noted above that for a root $\alpha \in \Delta$, we have that $n\alpha \in \Delta$ iff $n \in \{\pm 1\}$. This implies that we can separate the root system into positive/negative subsets. Fix a basis $\{\beta_1, \beta_2, \ldots, \beta_r\}$ in $\mathfrak{g}_0^*$ such that any root can be expanded as

$$\alpha = \sum_{i=1}^{r} n_i \beta_i, \qquad n_i \in \mathbb{Z}. \tag{D.11}$$

In this basis we call $\alpha \in \Delta$ a *positive root* iff the first non-zero component in the sequence $(n_1, \ldots, n_r)$ is positive, otherwise we call α a *negative root*. If α is positive (negative), we write $\alpha > 0$ ($\alpha < 0$). Furthermore, for roots α, β we use the notation $\alpha > \beta$ iff $\alpha - \beta > 0$, this defines a partial order of the root system. Denote the set of positive, respectively negative roots, by

$$\Delta_\pm \equiv \{\alpha \in \Delta \mid \pm\alpha > 0\}, \tag{D.12}$$

clearly $\Delta_- = \Delta \backslash \Delta_+$. As a consequence, one has $\Delta_+ = -\Delta_-$, i.e. $\alpha \in \Delta_+ \Leftrightarrow (-\alpha) \in \Delta_-$. The step operators E^α and $E^{-\alpha}$ for $\alpha \in \Delta_+$ are called *raising* and *lowering* operators, respectively. This shows that

$$\{E^\alpha | \alpha \in \Delta\} = \{E^\alpha | \alpha > 0\} \cup \{E^{-\alpha} | \alpha > 0\},$$

and the number of elements $|\Delta_+| = |\Delta_-| = \frac{1}{2}(d-r) \in \mathbb{N}$. The raising and lowering operators span each a subspace of $\mathfrak{g}$ denoted by

$$\mathfrak{g}_\pm = \mathrm{span}_\mathbb{C}\{E^{\pm\alpha}|\alpha > 0\},$$

thus $\mathfrak{g}$ can be decomposed as[3]

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+. \tag{D.13}$$

This is the *triangular* or *Gauss decomposition* of $\mathfrak{g}$ and will be important in constructing the highest weight modules.

A *simple root* α_i is defined to be a positive root that cannot be written as a linear combination of other positive roots with positive coefficients.[4] Two consequences are (i) $\alpha_i - \alpha_j \notin \Delta$ and (ii) any positive root is a linear combination of simple roots with positive integral coefficients. It turns out, independently of the chosen basis, there are exactly $r = \mathrm{rank}\ \mathfrak{g}$ simple roots. Hence the set of simple roots is

$$\Delta_s \equiv \{\alpha_i | r = 1, \ldots, r\}.$$

It can be shown that $\mathrm{span}_\mathbb{C}\Delta_s = \mathrm{span}_\mathbb{C}\Delta = \mathfrak{g}_0^*$ (so simple roots are linearly independent). Generically this basis is not orthonormal and the non-orthonormality is encoded in the *Cartan matrix*

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_j|^2} = (\alpha_i, \alpha_j^\vee) \in \mathbb{Z},$$

where it is seen from (D.10) that all entries are integers. Diagonal entries are all equal to 2 and its not generally symmetric. Using the Schwarz inequality for the inner product, we find the condition $A_{ij}A_{ji} = 4\cos^2\phi_{\alpha_i\alpha_j} < 4$ for $i \neq j$. Since $\alpha_i - \alpha_j$ is not a root, $R(E^{-\alpha_j})|\alpha_i\rangle = 0$, thus $q_{\alpha_j\alpha_i} = 0$ in (D.10), hence

$$(\alpha_i, \alpha_j^\vee) = -p_{\alpha_j\alpha_i} \leq 0, \quad i \neq j.$$

Thus all off-diagonal elements of A_{ij} are nonnegative integers. In the view of the above inequality we find $A_{ij} \in \{0, -1, -2, -3\}$ for $i \neq j$. One can also easily see that

$$\frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)} = \frac{A_{ij}}{A_{ji}}.$$

It can be shown that in the root system Δ, at most two different lengths (long and short) are possible. When all the roots have the same length, the algebra is said to be simply laced. These results are summarized in the table below.

A_{ij}	A_{ji}	$\phi_{\alpha_i\alpha_j}$	$\frac{(\alpha_i,\alpha_i)}{(\alpha_j,\alpha_j)}$
-1	-1	120°	1
-2	-1	135°	2
-3	-1	150°	3
0	0	90°	arbitrary

[3]We are being imprecise here. The direct sum refers to direct sum as vector spaces, not Lie algebras since none of the subalgebras are ideals of $\mathfrak{g}$.

[4]Note that the subindex is a labeling index and does not refer to a root component.

Note that we can expand any weight $\lambda \in \mathfrak{g}_0^*$ as

$$\lambda = \sum_{i=1}^{r} b_i \alpha_i = \sum_{i=1}^{r} b_i^\vee \alpha_i^\vee,$$

we call the coefficients b_i and $b_i^\vee$ for *Kac* and *dual Kac labes*, respectively. The *hight* is defined as the sum of the Kac labels $\mathrm{ht}(\lambda) = \sum_{i=1}^{r} b_i$. Since root are just special examples of weights, this introduce a natural $\mathbb{Z}$-grading of $\mathfrak{g}$, the so-called *root space gradation* (for each $\mathrm{ht}(\alpha) = j$). It turns out that there exist a unique *highest root* characterized by $\mathrm{ht}(\theta) > \mathrm{ht}(\alpha)$ for all $\alpha \in \Delta \backslash \{\theta\}$

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee, \quad a_i, a_i^\vee \in \mathbb{N},$$

where we use the special names *mark* and *comark* for the Kac and dual Kac labels of θ. Any other element of Δ can be obtained by repeated subtraction of simple root from θ. Marks and comarks are related by $a_i = a_i^\vee \frac{2}{|\alpha_i|^2}$. Another important property of the highest root is

$$(\theta, \theta) \geq (\alpha, \alpha) \quad \text{for all } \alpha \in \Delta.$$

The *Coxeter* and *dual Coxeter number* are defined by

$$g = \sum_{i=1}^{r} a_i + 1 \quad \text{and} \quad g^\vee = \sum_{i=1}^{r} a_i^\vee + 1. \tag{D.14}$$

D.1.5 The Chevalley Basis and Dynkin Diagrams

Given a Cartan matrix we can reconstruct the set of simple roots, which then provides us with all roots and thereby the whole algebra. The point that the Cartan matrix is sufficient to characterize the algebra is fully manifest in the *Chevalley basis* where to each simple root α_i there corresponds the three generator

$$e^i = E^{\alpha_i} \quad f^i = E^{-\alpha_i} \quad h^i = \frac{2\alpha_i \cdot H}{|\alpha_i|^2} = \alpha_i^\vee \cdot H,$$

with the commutator relations

$$\begin{aligned}
[h^i, h^j] &= 0, \\
[h^i, e^j] &= A_{ji} e^j, \\
[h^i, f^j] &= -A_{ji} f^j, \\
[e^i, f^j] &= \delta_{ij} h^j.
\end{aligned}$$

One can show that the remaining step operators are obtained by the so-called *Serre relations*

$$\begin{aligned}
(\mathrm{ad}_{e^i})^{1-A_{ji}} e^j &= 0, \\
(\mathrm{ad}_{f^i})^{1-A_{ji}} f^j &= 0,
\end{aligned}$$

for $i \neq j$. So clearly the Lie algebra can be reconstructed from the Cartan matrix alone. An important fact is that Cartan matrices related by relabeling of the rows and columns will give rise to isomorphic Lie algebras, thus semi-simple Lie algebras and Cartan matrices (up to this ambiguity) are one-to-one. This means that the classification of semi-simple Lie algebras over $\mathbb{C}$ can be solved by classifying matrices with the following properties

1. $A_{ii} = 2$,
2. $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$,
3. $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
4. $\det A > 0$,
5. A is not equivalent to a block diagonal matrix.

We have shown the first three points above. Number five means that we are restricting to simple Lie algebras, since this automatically classifies semi-simple Lie algebras. To understand number 4, recall that we can choose a basis $\{H^i\}$ such that $\alpha(H^i) \in \mathbb{R}$. Therefore it makes sense to consider the vector space of the real $\mathrm{span}_{\mathbb{R}}(H^i)$, and the dual space $\mathrm{span}_{\mathbb{R}}(\Delta)$ contains all roots (called the *root space*). The inner product induced by the Killing form is also real; it follows that for any real linear combination λ of the roots one has $(\lambda, \lambda) \geq 0$, zero iff λ is zero. Thus the root space is euclidean and isomorphic to $\mathbb{R}^r$, this leads to the very important condition $\det A > 0$.

We shall not go through the lengthy procedure of finding all solutions to the above problem, just state the result. The information in the Cartan matrix can be encapsulated in a simple diagram: the *Dynkin diagram.* To every simple root α_i, we associate a node and join the node i and j with $A_{ij}A_{ji}$ lines. Two disconnected simple roots means $A_{ij} = A_{ji} = 0$ and hence they are orthogonal. Those with relative angles of 120, 135 and 150 degrees are linked by one, two or three lines, respectively. Finally an arrowhead ´$>$´ is added to the lines from the ith and jth node if $|A_{ij}| > |A_{ji}|$. Instead of the arrows, some books use open and full dots to denote long and short roots respectively, since the analysis shows that only two different length (and thereby angles) will be present.

The result can be seen in figure F.1. There are four infinite families

$$A_r \simeq \mathfrak{sl}_{r+1} = \mathfrak{su}(r+1, \mathbb{R})^{\mathbb{C}}, \qquad B_r \simeq \mathfrak{so}_{2r+1} = \mathfrak{so}(2r+1, \mathbb{R})^{\mathbb{C}},$$
$$C_r \simeq \mathfrak{sp}_r = \mathfrak{sp}(r, \mathbb{R})^{\mathbb{C}}, \qquad D_r \simeq \mathfrak{so}_{2r} = \mathfrak{so}(2r, \mathbb{R})^{\mathbb{C}},$$

where we have also written the corresponding classical Lie algebra and their compact real forms. In addition there are five isolated cases E_6, E_7, E_8, F_4 and G_2, called exceptional Lie algebras. A, D and E are simply-laced since all roots have the same length. In the case of B_r C_r and F_4 the long root are $\sqrt{2}$ time longer than short roots, while $\sqrt{3}$ for G_2. The dual root system $\Delta^\vee(\mathfrak{g}) = \{\alpha^\vee | \alpha \in \Delta(\mathfrak{g})\}$ is isomorphic to the root system of another simple Lie algebra, which is called the *dual Lie algebra* $\mathfrak{g}^\vee$ of $\mathfrak{g}$, $\Delta^\vee(\mathfrak{g}) \simeq \Delta(\mathfrak{g}^\vee)$. Simply laced algebras are self-dual $\mathfrak{g} = \mathfrak{g}^\vee$, but this is also true for C_2, G_2 and F_4, while $(B_r)^\vee = C_r$ and vice versa.

D.1.6 Fundamental Weights

As pointed out above, weights and roots live in the same real r-dimensional euclidean space and weights can therefore be expanded in the basis of simple root. There exists, however, a more convenient basis for the weight space for which the coefficient are always integers for weights. The *fundamental weights* $\{\omega_i\}$ are defined to be dual to the simple coroot basis

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}.$$

We can expand any weight in some representation as

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \quad \Leftrightarrow \quad \lambda_i = (\lambda, \alpha_i^\vee) \in \mathbb{Z}.$$

The expansion coefficients λ_i of a weight λ, in the fundamental weight basis are called *Dynkin labels*, and these are always integers as seen from equation (D.10). Any weight written in component form $\lambda = (\lambda_1, \ldots, \lambda_r)$ in understood to refer to Dynkin labels. In other words Dynkin labels (lower index) are eigenvalues of the Chevalley generators of the Cartan subalgebra

$$h^i|\lambda\rangle = \lambda(h^i)|\lambda\rangle = (\lambda, \alpha_i^\vee)|\lambda\rangle = \lambda_i|\lambda\rangle,$$

while the upper index notation λ^i refers to $\lambda(H^i)$, the eigenvalue of H^i. Note that the elements of the Cartan matrix are the Dynkin labels of the simple roots

$$\alpha_i = \sum_{j=1}^{r} A_{ij} \omega_j, \tag{D.15}$$

as seen by the definition of A_{ij}.

A weight of special importance is the one for which all Dynkin label are unity

$$\rho = \sum_i \omega_i = (1, 1, \ldots, 1), \tag{D.16}$$

called the *Weyl vector* and has also the alternative expression (which one can prove using the Weyl group)

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{D.17}$$

The inner product of weights can be expressed in terms of a symmetric *quadratic form matrix* F_{ij}

$$(\omega_i, \omega_j) = F_{ij}.$$

Clearly this can be used to change basis from $\{\omega_i\}$ to $\{\alpha_i^\vee\}$

$$\omega_i = \sum_j F_{ij} \alpha_j^\vee,$$

conversely using (D.15) we have that

$$\alpha_i^\vee = \sum_j \frac{2}{|\alpha_i|^2} A_{ij} \omega_j.$$

This leads to the following relation

$$F_{ij} = (A^{-1})_{ij} \frac{\alpha_j^2}{2}.$$

This provides us with a metric in the fundamental weight basis; the scalar product of two weights $\lambda = \sum_i \lambda_i \omega_i$ and $\mu = \sum_i \mu_i \omega_i$ reads

$$(\lambda, \mu) = \sum_{ij} \lambda_i \mu_j (\omega_i, \omega_j) = \sum_{ij} \lambda_i \mu_j F_{ij}.$$

D.1.7 Lattices

Given a basis of the d-dimensional Euclidean space $\mathbb{R}^d$, a lattice is the $\mathbb{Z}$-span of this basis (so its a basis dependent notion). There are three important lattices for Lie Algebras, the *weight lattice*

$$P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r,$$

the *root lattice*

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r,$$

and the *coroot lattice*

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_r^\vee.$$

We will show that states in finite dimensional representations of $\mathfrak{g}$ are labeled by points in the weight lattice, thus $Q \subset P$ since Q since roots are weights in the adjoint representation. The integers specifying the position in P are eigenvalues of the Chevalley generator h^i and the step operators shifts the eigenvalues by a root lattice element Q. Define *dominant weights* as elements in the set

$$P_+ = \mathbb{Z}_+\omega_1 + \cdots + \mathbb{Z}_+\omega_r.$$

The coset P/Q is a finite group and the elements are called *congruence classes* (the identity element is the root lattice Q), and its order $|P/Q|$ is equal to the determinant of the Cartan matrix. For G_2, F_4 and E_8 it turns out that $Q = P$ (and thus $P/Q = \{1\}$), while in all other cases Q is a proper subset of P. Any weight λ lie in exactly one congruence class (since starting from the highest weight, we move with elements of Q and thus from one class to another class).

For $\mathfrak{s}l_3$ we have three classes $[(0,0)]$, $[(1,0)]$, $[(0,1)]$ as can be seen in figure D.1. Here we use the notation

$$[(\lambda_1, \ldots, \lambda_r)] = (\lambda_1, \ldots, \lambda_r) \star Q = \big\{(\lambda_1, \ldots, \lambda_2) + \sum_{i=1}^{r} n_i\alpha_i \mid n_i \in \mathbb{Z}\big\}.$$

Alternatively one can characterize the three classes as: $\lambda_1 + 2\lambda_2 \bmod 3$. This generalizes to $\mathfrak{s}l_N$ as

$$\lambda_1 + 2\lambda_2 + \cdots + (N-1)\lambda_{N-1} \bmod N.$$

For any Lie algebra $\mathfrak{g}$, the congruence classes take the form

$$\lambda \cdot \nu = \sum_{i=1}^{r} \lambda_i\nu_i \mod |P/Q| \ \ (\bmod\ \mathbb{Z}_2 \quad \text{for} \quad \mathfrak{g} = D_{2l}),$$

where the vector $(\nu_1, \ldots, \nu_r) = (1, 2, \ldots, N-1)$ for $\mathfrak{sl}_N$ and called the *congruence vector*.

D.1.8 The Weyl Group

Consider the $\mathfrak{sl}_2$-subalgebra corresponding to the root α and the $J_\alpha^3 = \alpha \cdot H/|\alpha|^2$ eigenvalue in the adjoint representation of $\mathfrak{g}$

$$\begin{aligned} \mathrm{ad}_{J_\alpha^3}(E^\beta) &= mE^\beta \\ &= \frac{1}{2}\alpha^\vee \cdot [H, E^\beta] = \frac{1}{2}(\alpha^\vee, \beta)E^\beta, \end{aligned}$$

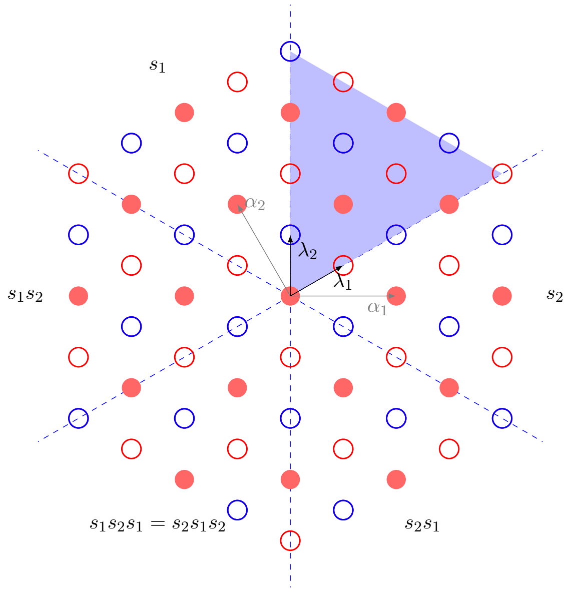


Figure D.1: Root system of $\mathfrak{sl}_3$ and the corresponding Weyl chambers. The circles correspond to the weight lattice P, restricting to the filled red circles we have the root lattice Q, while the different circles represent the three congruence classes P/Q.

thus

$$2m = (\alpha^\vee, \beta) \in \mathbb{Z}.$$

If m is non-zero, there should be another vector with the eigenvalue $-m$. This is given by

$$s_\alpha \beta = \beta - (\alpha^\vee, \beta)\alpha,$$

as seen by

$$\begin{aligned} \mathrm{ad}_{J^3_\alpha}(E^{s_\alpha \beta}) &= \frac{1}{2}(\alpha^\vee, s_\alpha\beta)E^{s_\alpha\beta} \\ &= \frac{1}{2}\Big((\alpha^\vee, \beta) - (\alpha^\vee, \beta)(\alpha^\vee, \alpha)\Big)E^{s_\alpha\beta} \\ &= -mE^{s_\alpha\beta}, \end{aligned}$$

since $m = \frac{1}{2}(\alpha^\vee, \beta)$ and $(\alpha^\vee, \alpha) = 2$. The operation s_α is a reflection with respect to the hyperplane perpendicular to α. The set of all such reflections with respect to roots forms the *Weyl group* of the algebra, denoted W. It is generated by the r elements $s_i \equiv s_{\alpha_i}$, the *simple Weyl reflections*, in the sense that any $w \in W$ can be decomposed as

$$w = s_i s_j \dots s_k. \tag{D.18}$$

Its can be presented as a *Coxeter group*, meaning that it is freely generated by the simple Weyl reflection modulo the relations

$$(s_i s_j)^{m_{ij}} = 1. \tag{D.19}$$

Clearly $m_{ii} = 1$ for all i, and it turns out that all $m_{ij} \in \{2, 3, 4, 6\}$, when $i \neq j$. This Coxeter presentation can be encoded in a *Coxeter diagram*: nodes are drawn for each primitive reflection, and $\{0, 1, 2, 3\}$ lines between nodes for $m_{ij} = \{2, 3, 4, 6\}$, respectively. For simple $\mathfrak{g}$, we find that the Coxeter diagrams are just the corresponding Dynkin diagrams (with the arrows omitted). Alternatively we can write $m_{ij} = \frac{\pi}{\pi-\theta_{ij}}$ for $i \neq j$, where θ_{ij} is the angle between the simple roots α_i and α_j. On the simple roots we have that $s_i\alpha_j = \alpha_j - A_{ij}\alpha_i$, in particular $s_i\alpha_i = -\alpha_i$. Note that W maps Δ into itself, i fact it provides a way to generate the complete set Δ from the simple roots

$$\Delta = \{w\alpha_1, \ldots, w\alpha_r | w \in W\}.$$

Recall that in order to define simple roots we needed to choose a basis, this construction shows that any set $\{w'\alpha_i\}$ for fixed $w' \in W$ could serve as basis of simple roots.

The action of the Weyl group can be readily extended to weights

$$s_\alpha\lambda = \lambda - (\alpha^\vee, \lambda)\alpha, \tag{D.20}$$

and one can easily show it leaves the inner product invariant

$$(s_\alpha\lambda, s_\alpha\mu) = (\lambda, \mu). \tag{D.21}$$

So the Weyl group is the isometry group on the weight space. Thinking of the weight Lattice P as a infinite crystal, its point group is isomorphic to the Weyl group (which explains the restriction $m_{ij} \in \{2, 3, 4, 6\}$, familiar from crystallography).

The Weyl group induces a natural splitting of the r-dimensional weight space into *Weyl Chambers*, whose number equal the order of W. These are simplicial cones defined as

$$C_w = \left\{\lambda \in \mathfrak{g}_0^* \,\middle|\, (w\lambda, \alpha_i) \geq 0, i = 1, \ldots, r\right\}, \quad w \in W, \tag{D.22}$$

where $\mathfrak{g}_0^*$ is the weight space (considered as a real space, as discussed earlier). The chamber corresponding to the identity element of W is called the *fundamental chamber*, and it will be denoted by C_0. Note that the orbit of any weight λ, $\{w\lambda|\, w \in W\}$, has exactly one point in C_0. In other words, for any $\lambda \notin C_0$ there exist a $w \in W$ such that $w\lambda \in C_0$.

We will define some notations used later on. The *shifted Weyl reflection* is

$$w \cdot \lambda \equiv w(\lambda + \rho) - \rho.$$

As a consequence one can show that

$$w \cdot (w' \cdot \lambda) = (ww') \cdot \lambda.$$

The *length* of w, denoted $l(w)$, in the minimum number of s_i among all possible decompositions of $w = \prod_i s_i$. The *signature* of w is defined as

$$\epsilon(w) = (-1)^{l(w)}.$$

One can show that in a linear representation of the Weyl group, the signature is simply given by $\det(w)$. Finally the longest element of W will be denoted as w_0 and is the unique element mapping Δ_+ to Δ_-.

In the following we will use the following normalization of long roots

$$|\theta|^2 = 2.$$

D.2 Representation Theory

D.2.1 Highest-Weight Representations

It turns out that all irreducible representations of finite-dimensional semi-simple are so-called *highest weight representations*. The highest weight state $|\Lambda\rangle$ is unique and thus completely specified by its eigenvalues $\Lambda(h^i) = \Lambda_i$ (so $\mathrm{ht}(\Lambda)$ is maximal). We can choose a basis such that the Cartan subalgebra acts diagonally and this naturally introduces the decomposition

$$V_\Lambda = \bigoplus_{\lambda \in \Omega_\Lambda} V_{(\lambda)}, \qquad V_{(\lambda)} = \left\{ \, |\lambda\rangle \, \middle| \, R(h^i)|\lambda\rangle = \lambda_i |\lambda\rangle \, \right\}, \tag{D.23}$$

where Ω_Λ is the *weight system*, the set of all weights in the representation. Since $\mathrm{ht}(\Lambda) > \mathrm{ht}(\lambda)$ for any $\lambda \in \Omega_\Lambda \backslash \{\Lambda\}$, for any root $\alpha > 0$, $\Lambda + \alpha$ cannot be a weight in Ω_Λ, so we require

$$R(E^\alpha)|\lambda\rangle = 0, \qquad \forall \alpha \in \Delta_+. \tag{D.24}$$

For the highest weight $\Lambda = \sum_{i=1}^r \Lambda_i \omega_i$ it is clear from equation (D.10) that (since $p = 0$)

$$\Lambda_i = (\alpha_i^\vee, \Lambda) \in \mathbb{Z}_+, \qquad \text{for} \quad i = 1, \ldots, r, \tag{D.25}$$

thus Λ is a dominant weight. Conversely for any dominant Weight, we have a irreducible representation. Note that is the Highest weight is not unique, the representation is necessarily reducible. For the adjoint representation, θ is the highest weight.

The elements of V_Λ can be obtained by applying step operators for negative roots to $|\Lambda\rangle$, i.e. any $|\lambda\rangle \in V_\Lambda$ is of the form $R(x)|\Lambda\rangle$ for some x in the universal enveloping algebra of $\mathfrak{g}_-$,

$$R(E^{-\beta_1} E^{-\beta_2} \ldots E^{-\beta_m})|\Lambda\rangle \qquad \text{for} \quad \beta_1, \ldots, \beta_m \in \Delta_+.$$

Making use of commutator relations between step operators, we may assume without loss of generality that these roots obey $\beta_p > \beta_q$ if $p > q$ and if $\beta_p - \beta_q$ is a root.

We can find all elements in the weight system using the $\mathfrak{sl}_2$ subalgebras for simple roots. Equation (D.10) gives us

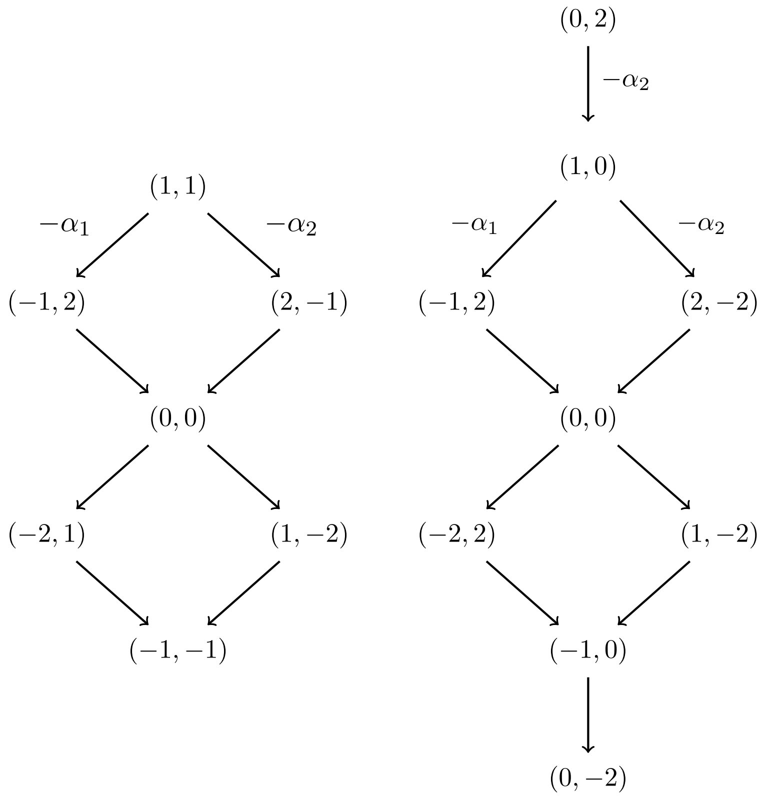
$$(\alpha_i^\vee, \lambda) = \lambda_i = -(p_i - q_i),$$

in particular $\Lambda_i = q_i$. All weights are of the form $\lambda = \Lambda - \sum_i n_i \alpha_i = \Lambda - \mu$, with $n_i \in \mathbb{Z}$. The *depth* of Λ is defined as $\mathrm{dp}(\lambda) = \mathrm{ht}(\mu) = \sum_i n_i$, clearly Λ is the unique weight with zero depth. Starting from the highest weight Λ, for each positive Dynkin label Λ_i we construct the following sequence of weights $\Lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \Lambda - \Lambda_i \alpha_i$, which all belong to Ω_Λ. The process is then repeated with all the other weights until there are no more weights with positive Dynkin label. Figure D.2 shows two examples with highest weights $\Lambda = \theta = (1, 1)$ and $\Lambda = (0, 2)$. However, this procedure does not keep track of multiplicities

$$\mathrm{mult}_\Lambda(\lambda) \equiv \dim V_{(\lambda)}.$$

For this we can use the *Freudenthal recursion formula*

$$\mathrm{mult}_\Lambda(\lambda) = 2\big[|\Lambda + \rho|^2 - |\lambda + \rho|^2\big]^{-1} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\infty} (\lambda + k\alpha, \alpha) \mathrm{mult}_\Lambda(\lambda + k\alpha) \tag{D.26}$$



(a) **(b)**

Figure D.2: Weight systems for $A_2 = \mathfrak{sl}_3$.

Using this we can show that all states in the adjoint representation has multiplicity 1 except $\text{mult}_\theta(0,0) = 2$.

Every irreducible module can be made unitary with $(H^i)^\dagger = H^i$ and $(E^\alpha)^\dagger = E^{-\alpha}$, the norm of any state is positive definite

$$|\lambda\rangle = E^{-\beta} \dots E^{-\gamma} |\Lambda\rangle \quad \to \quad \langle \lambda | \lambda \rangle > 0,$$

and also for any linear combination.

The lowest state (highest depth) is unique and can be used to define *conjugate representations*. It lies in the W orbit, in the exactly opposite chamber to the fundamental one. Thus we can find the lowest state by applying the longest element $w_0 \in W$, $\lambda_{\text{min}} = w_0 \Lambda$. Turning the representation upside down we find the conjugate representation with the highest weight

$$\Lambda^* = -(w_0 \Lambda) = -\lambda_{\text{min}}. \tag{D.27}$$

Representations that satisfy $\Lambda^* = \Lambda$ are called self-conjugate. All the weights in Ω_{λ^*} are the negatives of Ω_λ. For $\mathfrak{sl}_N$ we have the longest element

$$w_0 = s_1 s_2 \dots s_{N-1} s_1 s_2 \dots s_{N-2} \dots s_1 s_2 s_1,$$

in particular for $N = 3$

$$\Lambda^* = (-w_0)(\Lambda_1, \Lambda_2) = (\Lambda_2, \Lambda_1).$$

These properties can be actually be extracted from the symmetries of the Dynkin diagrams.

D.2.2 Universal Enveloping Algebra

Let $T(\mathfrak{g})$ denote the tensor algebra generated by the Lie algebra $\mathfrak{g}$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \left(\bigotimes_{i=0}^{n} \mathfrak{g} \right).$$

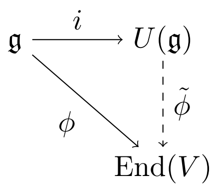
Now let J be the two-sided ideal generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y],$$

for $X, Y \in \mathfrak{g}$. Then the *universal enveloping algebra* of $\mathfrak{g}$ is defined as

$$U(\mathfrak{g}) = T(\mathfrak{g})/J.$$

According to the Poincaré-Birkhoff-Witt theorem, one can regard elements of $U(\mathfrak{g})$ as formal products of elements in $\mathfrak{g}$ modulo the commutator relations. In other words, this construction gives an associative algebra $U(\mathfrak{g})$ to any Lie algebra, with usual commutation relations $X \otimes Y - Y \otimes X = [X, Y]$. This construction has a universal property, which can be used as the definition: for a finite-dimensional irreducible representation $\phi : \mathfrak{g} \to \mathrm{End}(V)$, there exists a unique map $\tilde{\phi} : U(\mathfrak{g}) \to \mathrm{End}(V)$ such that the following diagram commutes



where $i : \mathfrak{g} \to U(\mathfrak{g})$ is the natural embedding of $\mathfrak{g}$ in $U(\mathfrak{g})$ (injective Lie algebra homomorphism).

D.2.3 Quadratic Casimir Element and Index of a Representation

So far we have seen how to construct highest-weight representations of semi-simple Lie algebras by diagonalizing the Cartan subalgebra. It is however sometimes useful to label representations according to certain central elements, as is well-known from non semi-simple algebras like the Poincare algebra in which central elements have direct physical interpretation. Semi-simple Lie algebras do not have such central elements and by calculating their second Lie algebra cohomology group $H^2(\mathfrak{g}, \mathbb{C})$, one can see that they do not even admit central extensions. What we are looking for actually exist in the center of the universal enveloping algebra $U(\mathfrak{g})$ and is called the quadratic, or second-order, Casimir element

$$\mathcal{C}_2 = \sum_{a,b=1}^{\dim \mathfrak{g}} K(J^a, J^b)^{-1} J^a J^b.$$

Given the universal property of $U(\mathfrak{g})$, this is what we need for representation theory. One can by show $\mathcal{C}_2$ commutes with all generators and is thus part of the center. Using the Cartan-Weyl basis

$$\mathcal{C}_2 = \sum_{i=1}^{r} H^i H^i + \sum_{\alpha\in\Delta_+} \frac{|\alpha|^2}{2}\left(E^\alpha E^{-\alpha} + E^{-\alpha}E^\alpha\right), \tag{D.28}$$

and acting on a vector with highest weight Λ, we find

$$\mathcal{C}_2|\Lambda\rangle = \left[|\Lambda|^2 + 2(\Lambda,\rho)\right]|\Lambda\rangle. \tag{D.29}$$

The first part comes from $\sum_i H^i H^i|\Lambda\rangle = (\Lambda,\Lambda)|\Lambda\rangle$, while the second part comes from

$$\left[E^\alpha, E^{-\alpha}\right]|\Lambda\rangle = \frac{2}{|\alpha|^2}\alpha\cdot H\,|\Lambda\rangle = \frac{2}{|\alpha|^2}(\alpha,\Lambda)\,|\Lambda\rangle,$$

then using (D.17). Since the Casimir element commutes with the other generators, the number

$$\mathcal{C}_2(\Lambda) = (\Lambda, \Lambda + 2\rho), \tag{D.30}$$

is the same for the whole irreducible module and can be used to characterized these representations. It is however not unique, for example it does not distinguish between a representation and its conjugate $\mathcal{C}_2(\Lambda) = \mathcal{C}_2(\Lambda^*)$. There can also exist higher order Casimir elements, their degrees minus one is usually called the exponents of the algebra. As one application of the Casimir, let us mention that one can show that the invariant bilinear form is given by

$$\mathrm{Tr}_\Lambda\left(R(J^a)R(J^b)\right) = |\theta|^2 x_\Lambda K(J^a,J^b) = 2x_\Lambda K(J^a,J^b), \tag{D.31}$$

with the *index for the representation* Λ given by

$$x_\Lambda = \frac{\dim V_\Lambda}{2\dim\mathfrak{g}}(\Lambda,\Lambda+2\rho). \tag{D.32}$$

For the adjoint representation $\Lambda = \theta$ one can show that $\mathcal{C}_2(\theta) = 2g^\vee \Rightarrow x_\theta = g^\vee$, which matches with (D.5).

D.2.4 Characters

A character of the representation with highest weight Λ is formally defined as

$$\chi_\Lambda = \sum_{\lambda\in\Omega_\Lambda} \mathrm{mult}_\Lambda(\lambda)e^\lambda \tag{D.33}$$

where e^λ denotes a formal exponential satisfying

$$e^\lambda e^\mu = e^{\lambda+\mu},$$
$$e^\lambda(\xi) = e^{(\lambda,\xi)},$$

here ξ is an arbitrary weight and the r.h.s. of the second equation is a genuine exponential. For example for $\mathfrak{sl}_2$ we have $\text{mult}_\Lambda(\lambda) = 1$ and $\Omega_\Lambda = \{-\Lambda, -\Lambda - 2, \ldots, \Lambda - 2, \Lambda\}$, so using the formula for geometric series we find

$$\begin{aligned}\chi_\Lambda(\xi) &= e^{-\Lambda\xi}\left(1 + e^{2\xi} + e^{4\xi} + \cdots + e^{2\Lambda\xi}\right)\\ &= e^{-\Lambda\xi}\frac{1 - e^{2(\Lambda+1)\xi}}{1 - e^{2\xi}} = \frac{\sinh\left([\Lambda+1]\xi\right)}{\sinh\left(\xi\right)}.\end{aligned}$$

For $\xi = 0$ we should find the dimension of the irreducible module, but the expression is ill-defined. A Taylor expansion shows that $\chi_\Lambda(\xi) = \Lambda + 1 + O(\xi)$ and as expected $\text{dim}V_\Lambda = \Lambda + 1$.

One can show that the character can also be expressed as

$$\chi_\Lambda = \frac{\sum_{w\in W}\epsilon(w)e^{w(\Lambda+\rho)}}{\sum_{w\in W}\epsilon(w)w^{w\rho}},$$

called the *Weyl's Character Formula*. Evaluated at the weight ξ we find

$$\chi_\Lambda(\xi) = \frac{\sum_{w\in W}\epsilon(w)e^{(w(\lambda+\rho),\xi)}}{\sum_{w\in W}\epsilon(w)e^{(w\rho,\xi)}}. \tag{D.34}$$

One can easily verify that this formula agrees with the former calculation in the case of $\mathfrak{sl}_2$. Note that we can alternatively define the character in the highest weight module V_Λ by the map

$$\chi_\Lambda : g_0^* \to \mathbb{C}, \quad \xi \mapsto \chi_\Lambda(\xi) = \text{tr}\exp\left(R_\lambda(H^\xi)\right),$$

where $R_\lambda(H^\xi)$ is the representation of the Cartan subalgebra element dual to ξ. Since the generator H^ξ acts as $R_\lambda(H^\xi)|\lambda\rangle = (\lambda, \xi)|\lambda\rangle$, its clearly equivalent to the former definition.

Evaluating the character at $\xi = 0$, the find the dimension of the module $\text{dim}V_\Lambda$. But $\sum\epsilon(w) = 0$ since the number of even and odd elements are the same, thus setting $\xi = 0$ in (D.34) gives zero divided by zero which isn't well-defined. Therefore we need a limiting procedure to evaluate this, first we set $\xi = t\rho$ and then consider the limit $t \to 0$. One can show that

$$\chi_\Lambda(t\rho) = \prod_{\alpha\in\Delta_+}\frac{\sinh\left(\alpha, (\lambda+\rho)t/2\right)}{\sinh\left(\alpha, \rho t/2\right)},$$

and the zeroth-order term in a Taylor expansion gives the *Weyl dimension formula*

$$\text{dim}V_\Lambda = \lim_{t\to 0}\chi_\Lambda(t\rho) = \prod_{\alpha>0}\frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}. \tag{D.35}$$

Again, this is in agreement for $\mathfrak{sl}_2$. For $\mathfrak{sl}_3$ we find $\text{dim}V_\Lambda = \frac{1}{2}(\Lambda_1+1)(\Lambda_2+1)(\Lambda_1+\Lambda_2+2)$ and for $\mathfrak{sp}_4$, $\text{dim}V_\Lambda = \frac{1}{6}(\Lambda_1 + 1)(\Lambda_2 + 1)(\Lambda_1 + 2\Lambda_2 + 3)(\Lambda_1 + \Lambda_2 + 2)$. By using Taylor expansions and few simple tricks, the *Freudenthal-de Vries strange formula* can be derived

$$|\rho|^2 = \frac{g^\vee}{12}\,\text{dim}\mathfrak{g}.$$

Other useful relations for characters are

$$\chi_{\oplus_i\Lambda_i} = \sum_i \chi_{\Lambda_i} \qquad \chi_{\Lambda\otimes\Lambda'} = \chi_V\chi_{\Lambda'}.$$

D.3 Branching Rules and Embeddings

It is often important to consider subalgebras $\mathfrak{p} \subset \mathfrak{g}$ of semi-simple Lie algebras, but the same algebra $\mathfrak{p}$ can be embedded in several different ways in $\mathfrak{g}$. Thus we will consider embeddings $i : \mathfrak{p} \to \mathfrak{g}$ where i is an injective Lie algebra homomorphism and write $\mathfrak{p} \hookrightarrow \mathfrak{g}$ instead of $\mathfrak{p} \subset \mathfrak{g}$. In this section we will be concerned with the classification of these embeddings.

D.3.1 Embedding Index

There are several ways of characterizing an embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$, some important for representation theory and some to distinguish inequivalent embedding of the same subalgebra.

(i) Branching rules:
Restricting a irreducible highest weight module V_Λ of $\mathfrak{g}$, to $\mathfrak{p}$ the representation decomposes in general into several irreducible representation of $\mathfrak{p}$. Such decompositions are called *branching rules* and are denoted as

$$\Lambda \mapsto \bigoplus_{\Lambda' \in P_+(\mathfrak{p})} b_{\Lambda\Lambda'} \Lambda',$$

where $\Lambda \in P_+(\mathfrak{g})$. Note that we use the highest weight to denote the corresponding module. The branching coefficients $b_{\Lambda\Lambda'}$ gives the multiplicity of $\Lambda' \in P_+(\mathfrak{p})$ in the decomposition of $\Lambda \in P_+(\mathfrak{g})$, when restricted to $\mathfrak{p}$. It turns out that the decomposition of the lowest-dimensional nontrivial module is sufficient to characterize an embedding. To each of its inequivalent branching rules correspond a distinct embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$.
(ii) Projection matrix:
It can be shown that the embedding respects the triangular decomposition for simple $\mathfrak{g}$, in the sense that $i(\mathfrak{p}_0) \subseteq \mathfrak{g}_0$ and $i(\mathfrak{p}_\pm) \subseteq \mathfrak{g}_\pm$. Restricting the embedding to the Cartan subalgebra $i : \mathfrak{p}_0 \hookrightarrow \mathfrak{g}_0$, gives rise to a dual map $i^* : \mathfrak{g}_0^* \to \mathfrak{p}_0^*$ of the weight spaces. The map i^* is surjective and a projection of the weight space of $\mathfrak{g}$ to the weight space of $\mathfrak{p}$. Thus the weights of $\mathfrak{p}$ can be regarded as projections of weights of $\mathfrak{g}$, i.e. there is a *projection matrix* $\mathcal{P}$ of size $\operatorname{rank}\mathfrak{p} \times \operatorname{rank}\mathfrak{g}$, such that for any $\mathfrak{g}$-weight λ, the associated $\mathfrak{p}$-weight is given by

$$i^*(\lambda) = \mathcal{P}\lambda \in P(\mathfrak{p}).$$

Note that we need to choose a basis in the weight space in order to get a matrix representation of i^*, thus the projection matrices are not unique: a Weyl reflection of the root diagram modifies them without affecting the embedding.
(iii) Embedding index: The *embedding index* x_e is defined as the ratio of the square length of the projection of θ, the highest root of $\mathfrak{g}$, to the square length of the highest root of $\mathfrak{p}$, which is denoted by ϑ:

$$x_e = \frac{|\mathcal{P}\theta|^2}{|\vartheta|^2}.$$

Given the branching rule, the embedding index can also be calculated from

$$x_e = \sum_{\Lambda' \in P_+(\mathfrak{p})} b_{\Lambda\Lambda'} \frac{x_{\Lambda'}}{x_\Lambda},$$

where x_Λ index of the representation with highest weight Λ.

D.3.2 Classification of Regular Embeddings

Clearly there are usually many subalgebras embedded into a simple Lie algebra $\mathfrak{g}$. We will only consider proper maximal subalgebras $\mathfrak{p}$, i.e. subalgebras such that there does not exist any intermediate algebra $\mathfrak{h}$ obeying $\mathfrak{p} \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{g}$. Non-maximal subalgebras can then be treated in a step-wise procedure, first considering maximal subalgebras $\mathfrak{p}$, then in turn the maximal subalgebras of $\mathfrak{p}$, and so on.

Regular embeddings are those for which there exists a basis of $\mathfrak{g}$ in which a subset of generators form generators of $\mathfrak{p}$. Thus if the we have that $\{\tilde{E}^\alpha\} \subset \{E^\alpha\}$ and $\{\tilde{H}^i\} \subset \{H^i\}$, where tilde denotes generators of $\mathfrak{p}$. More generally, subalgebras which are contained in some regular subalgebra are called *R-subalgebras*.

The root system and simple root system of the embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}$ will be denoted by $\tilde{\Delta}$ and $\tilde{\Delta}_s$. According to one of the main properties of simple roots, $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Delta}_s$ implies $\tilde{\alpha} - \tilde{\beta} \notin \tilde{\Delta}_s$. If $\mathfrak{p} \hookrightarrow \mathfrak{g}$ is a regular embedding, this also means that $\tilde{\alpha} - \tilde{\beta} \notin \Delta_s$, because otherwise $E^{\tilde{\alpha}-\tilde{\beta}} \propto [E^{\tilde{\alpha}}, E^{-\tilde{\beta}}]$ would lie in $\mathfrak{p}$, in contraction to $\tilde{\alpha} - \tilde{\beta} \notin \tilde{\Delta}_s$. Thus regular embeddings are in one to one correspondence to the subsets $\tilde{\Delta}_s \subset \Delta$, which obey

$$\tilde{\alpha}, \tilde{\beta} \in \tilde{\Delta}_s \subset \Delta \;\Rightarrow\; \tilde{\alpha} - \tilde{\beta} \notin \Delta.$$

There exist a simple algorithm due to Dynkin to find all such sets. All maximal regular embeddings can be obtained by choosing

$$\tilde{\Delta}_s \in \Delta_s \cup \{-\theta\}.$$

Note that promoting a $-\theta$ to a "simple root" preserves the characteristic property that the difference between two simple roots is not a root (i.e., $\alpha_i + \theta$ cannot be a root since θ is the highest root). However the roots are not linearly independent and we must remove at least one α_i in order to restore linear independence. This is most easily done using Dynkin diagrams; construct *extended Dynkin diagrams* by adding the node $-\theta$. For example for A_r the Dynkin labels of the highest root is $\theta = (1, 0, \ldots, 0, 1)$, and the node should therefore be connected to α_1 and α_r. Figure F.2 contains all extended Dynkin diagrams.

For $\mathfrak{g} \neq A_r$, there are no other maximal regular semi-simple subalgebras besides the ones with simple root systems

$$\tilde{\Delta}_s = \Delta_s \cup \{-\theta\} \backslash \{\alpha_i\}, \quad \text{for some} \quad i = 1, \ldots, r,$$

and conversely, with very few exceptions each such choice does yield such a subalgebra. In contrast for $\mathfrak{g} = A_r$, such prescription will just return A_r itself. As a consequence, for A_r the relevant semi-simple subalgebras are precisely the ones which have a simple root system

$$\tilde{\Delta}_s = \Delta_s \cup \{-\theta\} \backslash \{\alpha_i, \alpha_j\}, \quad \text{with} \quad i, j = 1, \ldots, r, \quad i \neq j.$$

Maximal subalgebras that are not semi-simple are constructed from removal of two nodes with mark $a_i = 1$ and the addition of a $u(1)$ factor (thus the maximal non-semi-simple subalgebras of a semi-simple algebra are reductive).

The exceptions just mentioned are only encountered when removing a simple root with a non-prime number mark, and thus only for exceptional algebras. The chains of embeddings are

$$
\begin{array}{llrcl}
F_4, & \Delta_s \cup \{-\theta\}\backslash\{\alpha_3\}: & A_3 \oplus A_1 \hookrightarrow & B_4 & \hookrightarrow F_4, \\
E_7, & \Delta_s \cup \{-\theta\}\backslash\{\alpha_3\}: & A_3 \oplus A_3 \oplus A_1 \hookrightarrow & D_6 \oplus A_1 & \hookrightarrow E_7, \\
E_8, & \Delta_s \cup \{-\theta\}\backslash\{\alpha_3\}: & A_3 \oplus D_5 \hookrightarrow & D_8 & \hookrightarrow E_8, \\
E_8, & \Delta_s \cup \{-\theta\}\backslash\{\alpha_5\}: & A_5 \oplus A_2 \oplus A_1 \hookrightarrow & E_6 \oplus A_2 & \hookrightarrow E_8, \\
E_8, & \Delta_s \cup \{-\theta\}\backslash\{\alpha_6\}: & A_7 \oplus A_1 \hookrightarrow & E_7 \oplus A_1 & \hookrightarrow E_8.
\end{array}
$$

Note that E_8 is the only algebra without any nodes with $a_i = 1$, and thus all maximal subalgebras are semi-simple.

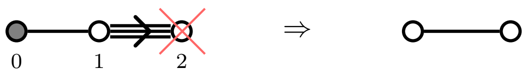
D.3.3 Branching Rules

One can also go further and calculate the branching rules. We first add to all the weights in the representation V_Λ an extra Dynkin label, associated with the extra simple root $-\theta$. Since the decomposition if θ in terms of the simple coroots is known, this extra Dynkin label is simply

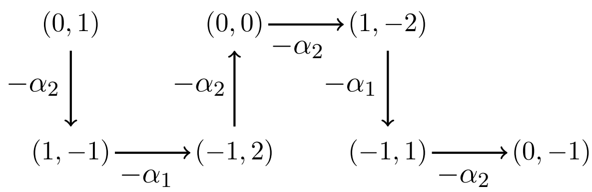
$$\lambda_{-\theta} = -\sum_{i=1}^{r} a_i^\vee \lambda_i. \tag{D.36}$$

If the regular subalgebra $\mathfrak{p}$ is obtained by deleting the simple root α_i, we simply delete the Dynkin label λ_i from all weights. The resulting weights are exactly the projected weights, and they can be reorganized into irreducible representations of $\mathfrak{p}$. The same procedure works for the semi-simple algebra obtained from the removal of two nodes.

As an example, using these techniques one can easily calculate the branching rules for the embedding $A_2 \hookrightarrow G_2$



From (D.15) and the Dynkin diagram of G_2, figure F.1, we find the simple roots $\alpha_1 = (2,-3)$ and $\alpha_2 = (-1,2)$. The weight system of $(0,1)_{G_2}$ contains the weights



where one can check there is no degeneracy, so $\dim V_{(0,1)} = 7$. Using (D.36) we can map $(\lambda_1, \lambda_2)_{G_2}$ into weights of A_2, $(\lambda_{-\theta}, \lambda_1)_{A_2}$:

$$\{\, (-1,0)_{A_2},\ (-1,1)_{A_2},\ (0,-1)_{A_2},\ (0,0)_{A_2},\ (0,1)_{A_2},\ (1,-1)_{A_2},\ (1,0)_{A_2} \,\}.$$

The last step is to reorganize these into A_2 irreducible representations. It is clear that only three of the weights can be highest weights, thus we find the three sectors $\{\,(1,0),\ (-1,1),\ (0,-1)\,\}$,

$\{(0,1), (1,-1), (-1,0)\}$ and $\{(0,0)\}$. Thus we find the following branching rules for the embedding $A_2 \hookrightarrow G_2$

$$\begin{aligned} (0,1) &\quad\to\quad (1,0) \oplus (0,1) \oplus (0,0) \\ \mathbf{7} &\quad\to\quad \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}, \end{aligned}$$

with the projection matrix

$$\mathcal{P} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that there is an necessary condition for the branching coefficient $b_{\Lambda\Lambda'}$ to be non-zero

$$\mathcal{P}\Lambda - \Lambda' \in \mathcal{P}Q(\mathfrak{g}) \tag{D.37}$$

which just says that the integrable wight Λ' must lie somewhere in the integrable representation of Λ after projection. There also exist other types of embeddings called special embedding, which we will not discuss too much here. For more information see [133, 154, 155].

Appendix **E**

Kac-Moody Algebras

Recall that there is a one-to-one correspondence between finite-dimensional simple Lie algebras and $(r+1)\times(r+1)$ matrices satisfying

$$\begin{aligned}
&\widehat{A}_{ii} = 2,\\
&\widehat{A}_{ij} = 0 \quad \Leftrightarrow \quad \widehat{A}_{ji} = 0,\\
&\widehat{A}_{ij} \in \mathbb{Z}_{\leq 0} \quad \text{for} \quad i \neq j,\\
&\widehat{A} \text{ is not equivalent to a block diagonal matrix,}
\end{aligned} \tag{E.1}$$

together with the important condition

$$\det \widehat{A} > 0. \tag{E.2}$$

In particular the rank of $\widehat{A}$ is $r+1$. It turns out that one can obtain a particular class of infinite dimensional Lie algebras by removing the condition (E.2), this lead to the general class of *Kac-Moody algebras.* We will, however, only consider the most important subclass of Kac-Moody algebras obtained by relaxing (E.2) to

$$\det \widehat{A}_{\{i\}} > 0 \quad \text{for all} \quad i = 0, \ldots, r, \tag{E.3}$$

where $\widehat{A}_{\{i\}}$ are the matrices obtained from $\widehat{A}$ by deleting the ith row and column ($\det \widehat{A}_{\{i\}}$ are called principal minors of $\widehat{A}$). Thus for general Kac-Moody algebras the rank of $\widehat{A}$ is arbitrary, but the sub-class satisfying (E.3) has at least rank r. Matrices satisfying only (E.1) are called *generalized Cartan matrices.* If they also satisfy (E.3) then they are called *affine Cartan matrices* and the corresponding algebra generated by the Cartan-Serre relations, *affine Lie algebras.* Note that for rank $\widehat{A} = r+1$ we will recover the usual finite-dimensional Lie algebras, thus in the following we will only have rank $\widehat{A} = r$ in mind when discussing affine Lie algebras.

We will not go through this classification in detail, but note that the condition (E.3) implies that when we remove any node from a affine Dynkin diagram we must recover diagrams for finite-dimensional semi-simple Lie algebras, and that one of the simple roots must the linearly dependent of the rest r simple roots. It turns out that the corresponding affine Dynkin diagrams are just the extended Dynkin diagrams in figure F.2, together with a few others not shown. The diagrams shown in figure F.2 correspond to so-called *untwisted affine Lie algebras* and the ones not shown are *twisted affine Lie algebras*.[1]. We will only consider the former in what follows.

E.1 Loop Algebras and Central Extensions

Rather than pursuing the approach discussed in the last section, we will turn to a more explicit construction of untwisted affine Lie algebras from a finite-dimensional semi-simple Lie algebra.

Let $\mathbb{C}[t, t^{-1}]$ correspond to the set of Laurent polynomials in t, then the *loop algebra* $\tilde{\mathfrak{g}}$ is defined as

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

with generators $J^a \otimes t^n$, where $\mathfrak{g}$ is a simple Lie algebra. The Lie bracket can be extended from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$ in a natural way

$$[J^a \otimes t^n, J^b \otimes t^m] = \sum_c i f^{ab}_c J^c \otimes t^{n+m}.$$

We will use the notation $J^a_n \equiv J^a \otimes t^n$. There turns out to be a unique central extension of the loop algebra of the form

$$[J^a_n, J^b_m] = \sum_c i f^{ab}_c J^c_{n+m} + \hat{k} n K(J^a, J^b) \delta_{n+m,0}, \tag{E.4}$$

augmented with the commutation relation

$$[J^a_n, \hat{k}] = 0.$$

For a orthonormal basis we of course have $K(J^a, J^b) = \delta_{ab}$. By applying this procedure to the Cartan-Weyl basis and recalling that $K(H^i, H^i) = \delta^{ij}$ and $K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2}$, we end up with the *affine Cartan-Weyl basis*

$$
\begin{aligned}
[H^i_n, H^j_m] &= \hat{k} n \delta^{ij} \delta_{n+m,0} \\
[H^i_n, E^\alpha_m] &= \alpha^i E^\alpha_{n+m} \\
[E^\alpha_n, E^\beta_m] &= \frac{2}{|\alpha|^2} \left(\alpha \cdot H_{n+m} + \hat{k} n \delta_{n+m,0} \right) && \text{if} \quad \alpha = -\beta \\
&= \mathcal{N}_{\alpha,\beta} E^{\alpha+\beta}_{n+m} && \text{if} \quad \alpha + \beta \in \Delta \\
&= 0 && \text{otherwise}
\end{aligned}
$$

The set of generators $\{H^1_0, \ldots, H^r_0, \hat{k}\}$ is manifestly abelian, but their eigenvalues in the adjoint representation, $\{\alpha^1, \ldots, \alpha^r, 0\}$, are infinitely degenerate (since they are the same for all the E^α_m). Hence $\{H^1_0, \ldots, H^r_0, \hat{k}\}$ is not a maximal abelian subalgebra and must be augmented by the addition of the grading operator L_0, whose eigenvalues in the adjoint representation depend on n; it is defined as

$$L_0 = -t \frac{\mathrm{d}}{\mathrm{d}t},$$

[1] The removal of the zeroth root of both untwisted and twisted Dynkin diagrams produces the correct finite-dimensional Lie algebra, but only in the former case will the (dual) Coxeter labels coincide with the finite-dimensional case.

with

$$[L_0, J_n^a] = -nJ_n^a.$$

Thus the maximal Cartan subalgebra is generated by $\{H_0^1, \ldots, H_0^r, \hat{k}, L_0\}$ and operators E_m^α for any n and H_n^i for $n \neq 0$ play the role of ladder operators. The algebra

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0$$

turns out to be an affine Lie algebra. Actually, one can formulate this algebra in terms of its affine Cartan matrix $\widehat{A}$ (by adding the affine simple root) in the Chevalley basis, but then the infinite-dimensionality will be hidden in the Serre-relations.

E.2 Affine Roots

Now we need to extend several structures from semi-simple Lie algebras to their corresponding (untwisted) affine Lie algebra. First we need a affine Killing form. Recall that the Killing form is uniquely characterized by the identity

$$K([Z, X], Y) + K(X, [Z, Y]) = 0,$$

for $X, Y, Z \in \hat{\mathfrak{g}}$. Choosing different combinations of $X, Y, Z \in \{J_n^a, \hat{k}, L_0\}$ one finds the following relations

$$\begin{aligned} &K(J_n^a, J_m^b) = \delta^{ab}\delta_{n+m,0}, \qquad &&K(J_n^a, \hat{k}) = 0, \qquad &&K(\hat{k}, \hat{k}) = 0, \\ &K(J_n^a, L_0) = 0, &&K(L_0, \hat{k}) = -1, &&K(L_0, L_0) = 0. \end{aligned}$$

The last relation is actually not fixed by invariance of the Killing form, but is fixed by convention to yield zero. The arbitrariness stems from that any redefinition $L_0 \to L_0' = L_0 + a\hat{k}$, where a is some constant, doesn't affect the Lie algebra and it shift the Killing form by $-2a$.

Just like before, the Killing form leads to an isomorphism between the elements in the Cartan subalgebra and its dual. Let the components of the (dual) vector $\hat{\lambda}$ be the eigenvalues of a state that is simultaneous eigenvector of all the generators of the Cartan subalgebra

$$\hat{\lambda} = \left(\hat{\lambda}(h_0^1), \ldots, \hat{\lambda}(h_0^r); \hat{\lambda}(\hat{k}); \hat{\lambda}(-L_0)\right), \tag{E.5}$$

where the first r components characterize the finite part λ, we will use the short-hand notation

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda).$$

We will call $\hat{\lambda}$ an *affine weight*. The extended Killing form induces a scalar product on the dual space

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda.$$

Note the similarity with the inner product in light-cone gauge. Now let us concentrate on the weights in the adjoint representation, the roots. Since $\hat{k}$ commutes with all other elements in the algebra, its ad eigenvalue is zero. Thus the affine roots are of the form

$$\hat{\beta} = (\beta; 0; n).$$

Clearly the inner product on affine roots are the same as their finite part

$$(\hat{\beta}, \hat{\alpha}) = (\beta, \alpha).$$

Thus for $\alpha \in \Delta$ all affine weights are of the form $\hat{\alpha} = (\alpha; 0; n)$ for $n \in \mathbb{Z}$. Using the notation $\delta = (0; 0; 1)$ and $\alpha \equiv (\alpha; 0; 0)$, we can express the affine roots as

$$\hat{\alpha} = \alpha + n\delta.$$

Is this notation, $n\delta$ is the root associated to H_n^i. Finally, the full set of affine roots are given by

$$\hat{\Delta} = \{\alpha + n\delta | n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\delta | n \in \mathbb{Z}, n \neq 0\}.$$

Note that δ has zero length, $(\delta, \delta) = 0$. Therefore all roots of the form $\{n\delta\}$ are called *imaginary* and have multiplicity r, while all others are called *real* and have multiplicity 1.

E.3 Simple Affine Roots and the Cartan Matrix

Now we need to identify a basis simple roots for $\hat{\mathfrak{g}}$. The basis must contain $r + 1$ elements, there r of then correspond to the simple roots of the finite part α_i, whereas the remaining simple root must be linear combination involving δ. The proper choice is

$$\alpha_0 \equiv (-\theta; 0; 1) = -\theta + \delta,$$

where as always θ is the highest root of $\mathfrak{g}$. The basis of simple roots are then $\{\alpha_i\}$, $i = 0 \ldots, r$ and the set of positive affine roots is

$$\hat{\Delta}_+ = \{\alpha + n\delta | n > 0, \alpha \in \Delta\} \cup \{\alpha | \alpha \in \Delta_+\}. \tag{E.6}$$

One can indeed see this, let $n > 0$ and $\alpha \in \Delta$, then

$$\alpha + n\delta = \alpha + n\alpha_0 + n\theta = n\alpha_0 + (n-1)\theta + (\theta + \alpha)$$

where the last term is a positive root. Thus any positive root can be expanded in the basis of simple roots with nonnegative coefficients. One important difference between the finite and affine case is that, there are no highest affine root (and thus the adjoint representation is not a highest weight representation).

We can now define the extended Cartan matrix as

$$\widehat{A}_{ij} = (\alpha_i, \alpha_j^\vee) \qquad 0 \leq i, j \leq r,$$

where we have defined the affine coroots by

$$\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2}(\alpha; 0; n) = \frac{2}{|\alpha|^2}(\alpha; 0; n) = (\alpha^\vee; 0; \frac{2}{|\alpha|^2} n).$$

The extended Cartan matrix $\widehat{A}$ has an extra row and column compared with its finite counterpart A. The extra diagonal element is $(\alpha_0, \alpha_0^\vee) = |\theta|^2 = 2$, while using $\theta = \sum_{i=1}^r a_i \alpha_i$ we find the other components

$$(\alpha_0, \alpha_j^\vee) = -(\theta, \alpha_j^\vee) = -\sum_{i=1}^r a_i (\alpha_i, \alpha_j^\vee).$$

It is convenient to define the zeroth mark as $a_0 = 1$. Since the finite part of α_0 is θ we find $a_0^\vee = a_0 \frac{|\alpha_0|^2}{2} = 1$. Using this, the last equation can be rewritten as

$$\sum_{i=0}^{r} a_i \widehat{A}_{ij} = \sum_{j=0}^{r} \widehat{A}_{ij} a_j^\vee = 0,$$

which means that the mark and comark are right, respectively left, eigenvectors with zero eigenvalue. This is of course due to the linear dependence of the rows and columns of the affine Cartan matrix. A useful relation is obtained by observing that $\sum_{i=0}^{r} a_i \alpha_i = -\theta + \delta + \sum_{i^a}^{r} a_i \alpha_i$ leading to

$$\delta = \sum_{i=0}^{r} a_i \alpha_i = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee. \tag{E.7}$$

Also note that we can now express the dual Coxeter number as $g^\vee = \sum_{i=1}^{r} a_i^\vee + 1 = \sum_{i=0}^{r} a_i^\vee$, and similarly for the Coxeter number.

E.4 Fundamental Weights

Just like the finite case, we define the affine fundamental weights $\{\hat{\omega}_i\}_{i=0}^{r}$ as the basis dual to the simple coroots; $(\hat{\omega}_i, \hat{\alpha}_j^\vee) = \delta_{ij}$. Its easy to verify that these are given by

$$\hat{\omega}_i = \begin{cases} (0;1;0) & \text{for } i = 0 \\ (\omega_i; a_i^\vee; 0) & \text{for } i \neq 0 \end{cases}$$

Again its convenient to use the notation $\omega_i \equiv (\omega_i; 0; 0)$ and $\omega_0 = (0;0;0)$ to express the fundamental weights as

$$\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i.$$

The affine quadratic form matrix is given by

$$\begin{aligned} (\hat{\omega}_i, \hat{\omega}_j) &= (\omega_i, \omega_j) = F_{ij} && \text{for} \quad i, j \neq 0, \\ (\hat{\omega}_0, \hat{\omega}_i) &= (\omega_0, \omega_0) = 0 && \text{for} \quad i \neq 0, \end{aligned}$$

and is essentially the equal to the finite case. Affine weights with zero L_0 eigenvalue can thus the expanded as

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + l\delta, \quad l \in \mathbb{R}.$$

The $\hat{k}$ eigenvalue is called the *level*. Using (E.5) and (E.7) we find

$$k \equiv \hat{\lambda}(\hat{k}) = (\hat{\lambda}, \delta) = \sum_{i=0}^{r} a_i^\vee (\hat{\lambda}, \alpha_i^\vee) = \sum_{i=0}^{r} a_i^\vee \lambda_i.$$

The zeroth Dynkin label depends on the rest, the relation is given by

$$\lambda_0 = k - \sum_{i=1}^{r} a_i^\vee \lambda_i = k - (\lambda, \theta). \tag{E.8}$$

This will turn out to be an important relation for the representation theory of affine Lie algebras. We will use the following notation

$$\hat{\lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_r]$$

for Dynkin labels (modulo L_0 eigenvalues, which sometimes are written as an index). As in the finite case the Cartan matrix contain Dynkin labels of simple roots $\alpha_i = [\widehat{A}_{i0}, \widehat{A}_{i1}, \ldots, \widehat{A}_{ir}]$. Finally the affine Weyl vector is defined as

$$\hat{\rho} = \sum_{i=0}^{r} \hat{\omega}_i = [1, 1, \ldots 1],$$

but it cannot be written as half the sum of positive affine roots. Note that $\rho(\hat{k}) = \sum_{i=0}^{r} a_i^\vee = g^\vee$. As last thing to note is that the concept of dominant weight is k-dependent through the zeroth Dynkin label (E.8). We let P_+^k denote the set of dominant weights at level k.

E.5 Outer Automorphisms

A notion that will be important for us is the group of outer automorphisms of $\hat{\mathfrak{g}}$ which is defined as

$$\mathcal{O}(\hat{\mathfrak{g}}) = D(\hat{\mathfrak{g}})/D(\mathfrak{g}). \tag{E.9}$$

Here $D(\hat{\mathfrak{g}})$ and $D(\mathfrak{g})$ are the groups of transformations of simple roots that leave the inner product, and therefore Cartan matrix, invariant. It is most easily though of as the group of symmetries of the Dynkin diagrams, figure F.1 and F.2. This means that $\mathcal{O}(\hat{\mathfrak{g}})$ contains the set of transformations that does not leave the zeroth root invariant. For example, it is clear from the diagrams that $D(A_r) = \mathbb{Z}_2$ while $\mathcal{O}(A_r^{(1)}) = \mathbb{Z}_{r+1}$. We will not list these groups since they all can easily be seen from figure F.2. There are actually one slightly surprising isomorphism

$$\mathcal{O}(\hat{\mathfrak{g}}) \simeq B(G),$$

where $B(G)$ is the center of the universal covering of all groups which has $\mathfrak{g}$ as Lie algebra. Take for example (using the compact real forms) $G = SU(r+1)$ which has the center $B(SU(r+1)) = \mathbb{Z}_{r+1}$ composed by $(r+1)$'th root of unity multiplied by the identity matrix. These automorphisms have an action on affine weights which have important applications, for example in the case of $A_r^{(1)}$ the automorphism group is generated by $a \in \mathcal{O}(A_r^{(1)})$

$$a[\lambda_0, \lambda_1, \ldots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_0, \ldots, \lambda_{r-2}, \lambda_{r-1}],$$

and similarly for other affine Lie algebras.

E.6 Integrable Highest Weight Representations

As we discussed before, all irreducible representations of the semi-simple Lie algebras are highest weight representations. This is not the case for affine Lie algebras, for example there does not exit any highest root and therefore the adjoint representation is not a highest weight representation. For physical applications however, it turns out that highest weight representations are the most relevant ones to study.

As in the finite case, a highest weight representation is characterized by a unique highest weight $|\hat{\Lambda}\rangle$ which is annihilated by all positive roots (E.6)

$$E_0^{\alpha}|\hat{\Lambda}\rangle = E_n^{\pm\alpha}|\hat{\Lambda}\rangle = H_n^i|\hat{\Lambda}\rangle = 0, \qquad n > 0,\ \alpha > 0,\ i = 1, \ldots, r, \tag{E.10}$$

with the eigenvalues

$$H_0^i|\hat{\Lambda}\rangle = \lambda^i|\hat{\Lambda}\rangle, \qquad \hat{k}|\hat{\Lambda}\rangle = k|\hat{\Lambda}\rangle, \qquad L_0|\hat{\Lambda}\rangle = 0. \tag{E.11}$$

Note that it is purely a matter of convention to put the L_0 eigenvalue to zero, in applications this is fixed for other reasons and is extremely important. In the following we will mostly use eigenvalues of h^i, the Dynkin labels Λ_i, since they turn out to be integers for the type of representations we are interested in.

The class of highest weight representations that are most important to us, are the ones that are analogous to the finite case. We require that the projections onto $\mathfrak{sl}_2$ subalgebras associated to any positive real root, are finite. Using the same argument as in the finite case leading to (D.10), we find that any affine weight in the weight system $\hat{\lambda} \in \Omega_{\hat{\Lambda}}$ must satisfy

$$\left(\hat{\lambda}, \alpha_i^{\vee}\right) = -\left(p_{\hat{\alpha},\hat{\lambda}} - q_{\hat{\alpha},\hat{\lambda}}\right) \in \mathbb{Z}, \tag{E.12}$$

which implies that

$$\lambda_i \in \mathbb{Z}, \qquad \Lambda_i \in \mathbb{Z}_+, \qquad i = 0, \ldots, r. \tag{E.13}$$

The last condition follows from the fact that all p's are zero for the highest weight and all the q's are positive. Since $\Lambda_0, (\Lambda, \theta) \in \mathbb{Z}_+$, using (E.8) we find the bound

$$k \in \mathbb{Z}_+, \quad k \geq (\Lambda, \theta). \tag{E.14}$$

These two conditions are among the most important ones we have found, the level k is a positive integer and makes sure that there are only a finite number of irreducible modules. In other words, the highest weight must be a dominant weight $\hat{\Lambda} \in P_+^k$. It turns out that these representations satisfy a so-called integrability condition which is why we will call them *integrable highest-weight representations*. Due to lack of time and space, we are forced to neglect many beautiful topics.

In order to obtain the weight system $\Omega_{\hat{\Lambda}}$ of a integrable highest weight module $V_{\hat{\Lambda}}$, we can use a very similar algorithm to the one in the finite case. Starting from $|\hat{\Lambda}\rangle$, one can grade-by-grade subtract the relevant affine roots but whenever we apply the zeroth root, the grade is increased by one. Projecting so a fixed grade, one will then see that the representation is organized into direct sums of irreducible representations of $\mathfrak{g}$. Note however, that this procedure will never end, unlike the finite case. It is easy to construct the weight system of the the highest affine weight $[1, 0]$ of $\widehat{\mathfrak{su}}(2)_1$. For $\widehat{\mathfrak{su}}(2)_2$ the simplest highest weight is $[2, 0]$, but already at level one it becomes messy to write down the weights by hand.

E.7 Missing Topics

Due to lack of time we cannot write about several very important topics such as the affine Weyl group, characters and modular transformations, affine embeddings (important for the WZW coset construction) and many others. For more details see the references given.

Appendix **F**

Finite and Affine Dynkin Diagrams

In this appendix we have collected the set of finite and (extended) Dynkin diagram, together with useful information.

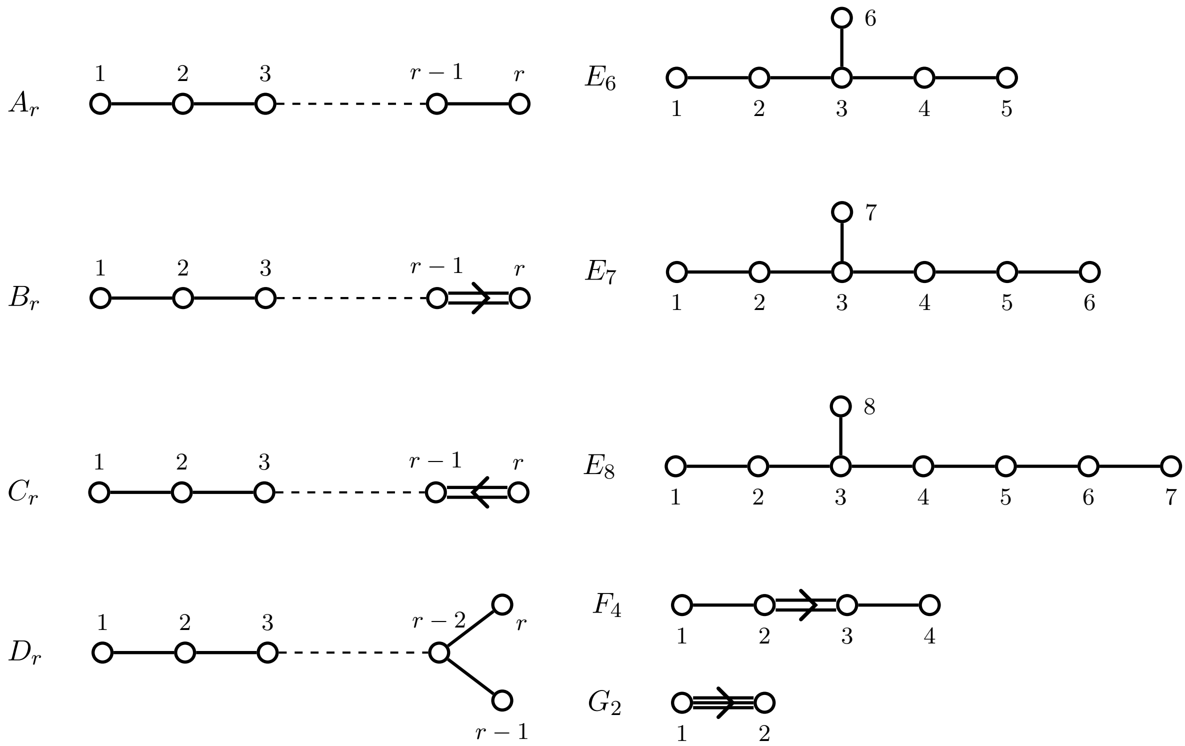


Figure F.1: The Coxeter-Dynkin diagrams of finite-dimensional simple Lie algebras.

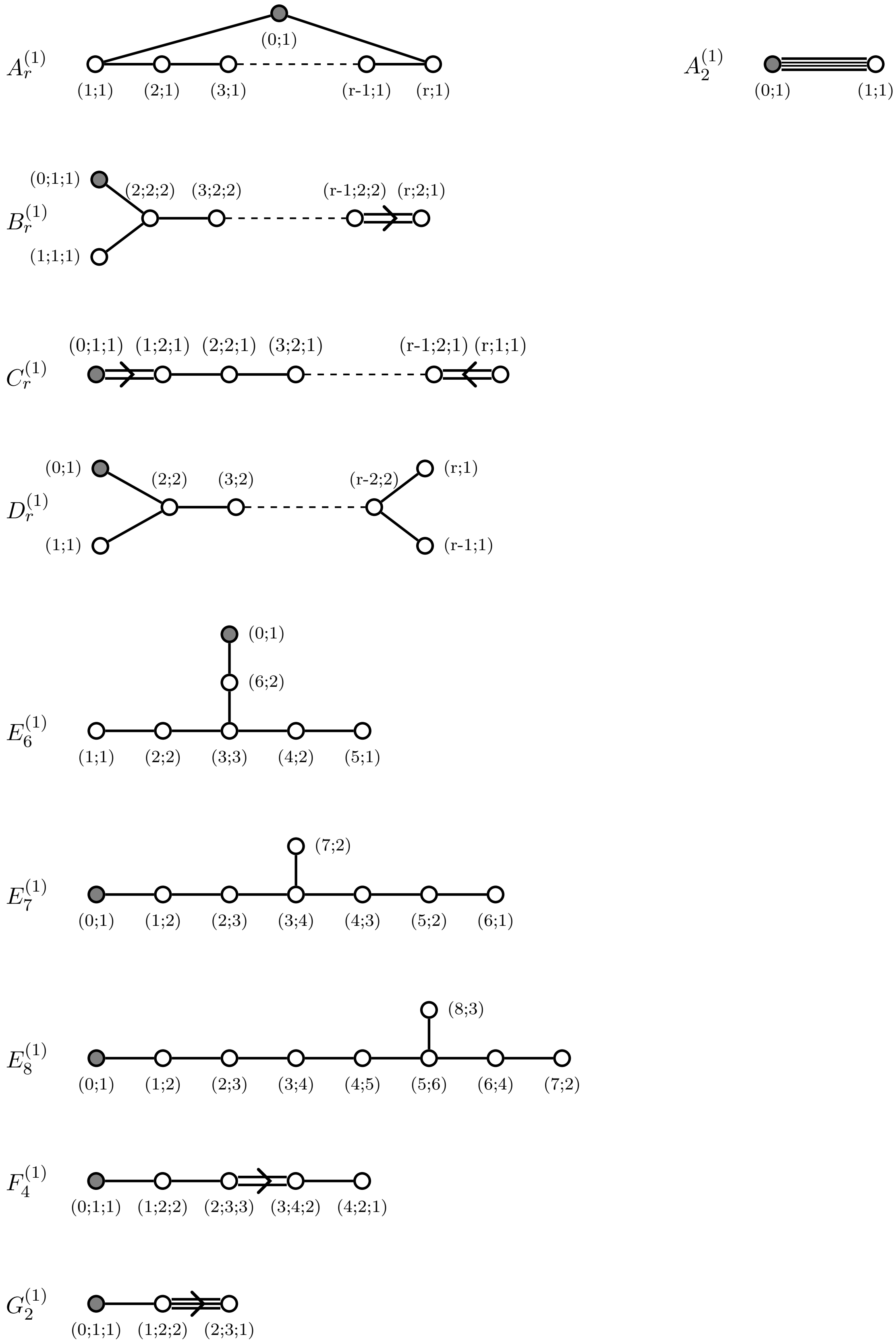


Figure F.2: The Extended Dynkin diagrams corresponding to untwisted affine Lie algebras. Labels $(i, a, a^\vee)$ stands for the simple root label, mark and comark. If only two labels are present, then $a = a^\vee$.

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