FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Master Thesis

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Conformal Symmetry for Black Holes

Subtracted Geometry & Hidden Conformal Symmetry

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Submitted on October 6, 2014

Abstract

In this thesis we give a brief review of modern black hole physics. We point out that a microscopic derivation of entropy for black holes, is facilitated by the mere presence of some conformal field theory, whose central charges and conformal weights give us the entropy via the Cardy formula. Ultimately we are interested in the continuation of the recently proposed *subtracted geometry* approach, which manifests hidden conformal symmetry for non-extremal asymptotically flat black holes. We review this approach in detail, reproducing several of the results found in the literature, amongst which is the fact that for the four-charged black holes in the $\mathcal{N} = 2$ minimal supergravity model (STU model), the warp factor Δ_0 may be altered to $\Delta = \bar{\Delta} + \Theta\, G$, leaving the thermodynamic potentials unchanged while altering the asymptotics. This change in warp factor maintains separability of the massless Klein-Gordon equation, and gives the radial equation a hypergeometric form; making the conformal symmetry manifest. In the literature, the minimal warp factor $\bar{\Delta}$ (going like r for large r) has been studied extensively, thus we focus our attention on the class of subtracted geometries with warp factors Δ with $\Theta \neq 0$. We identify a set of warp factors $\Delta_{\mathrm{NHEK};A}$ given by $\Theta = Am^2 + (7 - A)a^2 + a^2 \cos^2\theta$, for which the near-horizon extremal Kerr limit coincides with the very same limit on Δ_0. This is significant as the same limit on $\bar{\Delta}$ failed to coincide with the limit on Δ_0. In pursuit of matter supporting the subtracted geometry with $\Delta_{\mathrm{NHEK};A}$, we find matter supporting a large class of warp factors in the simpler case $\Theta = \Theta_0$ (a constant for static geometries). In the rotating case we find matter for the warp factor $\Delta_- \equiv \mathcal{A}_{\mathrm{red}}^2 = 4m^2((\Pi_c - \Pi_s)r + 2m\Pi_s)^2$ via a scaling limit on the original matter. It remains to find matter for $\Delta_{\mathrm{NHEK};A}$, as a scaling limit is not applicable. We also present the perspective of subtracted geometry from the dual CFT point of view, where in an interpolating black hole family, the flow from subtracted to original geometry is understood as the effect of irrelevant deformations. We conclude with an outlook, presenting several interesting future directions.

Acknowledgments

First and foremost I thank Niels Obers and Jelle Hartong for taking me on as a master student. They have guided me through this process, and made my first encounter with this incredibly grand subject manageable. I thank them in particular for their patience during numerable meetings. I also thank Niels Obers in particular for nominating me to the Lørup scholar stipend. The stipend has allowed me to focus entirely on my thesis during its final stage. I am very grateful for this.

I would also like to thank the students at the Niels Bohr Institute for a pleasant study environment. In particular I would like to thank the basement dwellers, Callum Kift and Svavar Gunnarsson for making the last year particularly memorable. I also thank Patryk Kusek and Jochen Heinrich for sharing their enthusiasm for physics and being great company.

Lastly I thank my family for their support, and Meeli for making my life in Copenhagen so much brighter.

One geometry cannot be more true than another;
it can only be more convenient.

∼ Henri Poincaré

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Introduction

The introduction is divided into five sections. The first gives a short historical introduction to black holes in theoretical physics, with focus on general relativity. The second section scratches the surface of the Schwarzschild solution, in a manner that allows for a fluid introduction to concepts central to black hole physics. The third gives a brief review of black hole thermodynamics, with emphasis on the questions that it allows us to ask. The fourth section sets the stage for the remainder of this thesis, briefly introducing novel ideas that may answer the questions of black hole mechanics, and resolve the microscopic structure of black holes. Finally, the last section gives an overview of the remainder of this thesis.

1.1 The Birth of Black Hole Physics

The advent of black hole physics is hard to pinpoint, but perhaps it was the idea put forward by John Michell in a letter to Henry Cavendish in 1783. Michell reasoned that particles of light emitted from distant stars would slow down due to the attractive nature of gravity. He argued that in extreme cases, when the star was sufficiently dense, the escape velocity would be greater than the speed of light effectively making the star invisible to distant observers.

Perhaps not surprisingly, Michell's idea did not catch on at the time, it was too wild! A lucid treatment remained out of reach until Einstein shared his new insight regarding time and space, which lead him to a novel understanding of gravity in the context of the theory of general relativity (GR) published in 1916. GR is a pillar of modern physics, and a short review of its principles is in order.

GR follows from a set of postulates and principles. Primarily, Einstein postulated the constancy of the speed of light, which lead to the theory of *special* relativity. Unfortunately, special relativity did not incorporate gravity, ultimately it treated spacetime as a flat spacetime, Minkowski-space, where coordinates and distances have their usual intuitive global aspects. He extended the theory to explain gravity by an incredibly simple, and ingenious insight, which we refer to as the *equivalence principle*. The equivalence principle asserts that an accelerating frame and a stationary frame in a gravitational field, are *locally* indistinguishable: An observer accelerating in a space ship experiences the same "force" as another observer stationary in a gravitational field[1]. GR was as such born out of an adaptation of special relativity, by discarding the idea of global inertial frames, and replacing it with the notion of local inertial frames. A patchwork of such local inertial frames, translates into the abstract mathematical construction of a differentiable manifold, the curvature of which manifests itself locally as what we experience as a gravitational "force".

As a side note, the mathematics used to deal with the notion of spacetime in GR, is the mathematics of differential geometry. An account of differential geometry and other important mathematical results are largely omitted in this thesis. The literature available is vast as well as brilliant, and the interested reader should not find accessibility a problem.

Now that the main underpinnings of GR have been sketched, we jump to the central pillar of GR, namely Einstein's field equations. These can be derived from fundamental principles,

[1]This is strictly only true in the limit of very tiny regions of space, the equivalence principle is a local statement.

as well as from a Lagrangian point of view. Denoting a general spacetime (M, g), a manifold with an associated metric, Einstein's field equations present themselves as a set of coupled non-linear differential equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where G is the four-dimensional Newton constant and we are using units in which the speed of light $c = 1$, we will stick to units in which $c = 1$ throughout this thesis. These equations comprise the key relation between *matter* and *geometry.* The Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ with its contractions $R_{\mu\nu}$ and R, are invariant measures of the curvature of spacetime, while the energy-momentum tensor $T_{\mu\nu}$ encodes the energy content of the spacetime.

In the Lagrangian formalism of gravity, choosing the metric as the dynamical variable, the field equations can be derived from the Einstein-Hilbert action supplemented with a matter term S_{matter}:

$$S_{\text{EH}} + S_{\text{matter}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_{\text{matter}}. \tag{1.2}$$

By the principle of least action (stationary action $\delta S = 0$), one readily finds that $\delta S_{EH} = 0$ for all metric fluctuations $\delta g^{\mu\nu}$ only if (1.1) is satisfied, where the energy-momentum tensor is identified with

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{1.3}$$

The idea is that $R^{\mu}{}_{\nu\rho\sigma}$ depends uniquely on the metric $g_{\mu\nu}$ (using the Christoffel connection), therefore, the Einstein field equations relate $g_{\mu\nu}$ to the energy and momentum of spacetime. A $g_{\mu\nu}$ that solves these equations for a given $T_{\mu\nu}$ encodes the curvature of a given spacetime manifold. Inserting a suitable energy-momentum tensor, such a solution reveals what kind of spacetimes are supported by massive stars. As we shall see, stars with a density beyond a critical value will support remarkable spacetimes, namely black holes.

1.2 Black Holes in General Relativity

Despite the fact that Einstein himself is said to have rejected black holes, his theory supports their existence; they appear as solutions to his field equations. We thus move on to discuss some of the, by now, well known black hole solutions.

The Einstein equations are highly non-linear, and finding a solution given some arbitrary $T_{\mu\nu}$ is not straight forward at all, if at all possible. To make the search for solutions feasible, it is common practice to a priori look for solutions that posses a certain degree of symmetry. Obviously, empty space, for instance Minkowski-space has the maximum symmetry, and black holes will in general only inherit a subset of these symmetries. The black hole solutions that we will discuss, and the black hole solutions that one in general looks for, are ones with a large degree of symmetry, and this required symmetry simplifies the search for solutions tremendously.

The first discovered *exact* black hole solution to Einstein's vacuum[2] field equations is the Schwarzschild black hole, discovered shortly after the arrival of Einstein's GR. The characteristics of this spacetime are largely attributed to the symmetries employed in its construction, being a spherically symmetric and *stationary* spacetime. These symmetry requirements turn out to be very restrictive, and the corresponding *unique* solution takes on the astonishingly simple form

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.4}$$

[2]The vacuum field equations are acquired by setting $T_{\mu\nu} = 0$, rendering Einstein's field equations $R_{\mu\nu} = 0$.

where t and r denote respectively time and radial coordinate, G is Newtons constant, M is the mass of the black hole and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit two-sphere[3]. The signature of this metric is mostly positive, $(- + + +)$, we will stick with this convention throughout this thesis.

There are several things to note about this metric, firstly, due to the spherical symmetry of the solution, it is to very good approximation the geometry of spacetime in the vicinity of a spherical energy density, such as a non-rotating (i.e *static*) stationary star. Secondly, notice the singularity at $r = 2GM$. This singularity gives rise to an event horizon, let us emphasize that the horizon is only present for objects with mass confined within a sphere of radius $2GM$, since the solution is only valid in vacuum.

In Schwarzschild coordinates $\{t, r, \theta, \phi\}$, t represents the proper time of an observer stationary at infinity, and to such an observer, infalling matter will never be seen to cross the horizon. This is a direct consequence of the singularity present in these coordinates which dictates the closing up of lightcones with decreasing r, closing completely at $r = 2GM$ where for radial null geodesics $dt/dr \to \pm\infty$.

In terms of coordinates adapted to an infalling observer however - one moving along a geodesic parameterized with a proper time τ, there is nothing stopping the observer from crossing the event horizon within a finite proper time. Eddington-Finkelstein coordinates are such adapted coordinates in which light cones gradually tilt toward the center of the black hole. The light cones no longer close up at the event horizon, but are found to be tilted to such an extent that the entire future light cone is confined within the horizon. Consequently, signals of light emitted beyond and at the event horizon will never reach future null infinity ($\mathscr{I}^+$ in figure 1.1), and an outside observer at $r > 2GM$ sees nothing from within the region enclosed by the horizon, the black hole.

In Eddington-Finkelstein coordinates, we can follow a freely falling observer past the horizon, and we see that the coordinate singularity at $r = 2GM$ is only an artifact of Schwarzschild coordinates. However the physical singularity located at $r = 0$ is still present. To convince oneself that the metric diverges independent of coordinates as $r \to 0$, one can compute the Riemann curvature tensor fully contracted with itself

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}. \tag{1.5}$$

Clearly this expression diverges as $r \to 0$, indicating that $r = 0$ represents a true singularity of the spacetime. In figure 1.1, a conformal diagram of the maximally extended Schwarzschild solution is shown. It brings the infinities to a finite distance, and allows for a compact representation of the causal structure of spacetime. This diagram nicely summarizes our discussion of the Schwarzschild black hole, clearly showing the event horizon as the intersecting 45 degree lines. The diagram captures the essential fact that future directed time-like and null geodesics that cross the future event horizon are destined to hit the singularity in the future.

There are a few other exact black hole solutions to Einstein's field equations in four spacetime dimensions, they are the following generalizations of the Schwarzschild solution; the charged, the rotating and the charged-rotating black holes, respectively named Reissner-Nordström, Kerr, and Kerr-Newman black holes. These solutions are unique; they are the only stationary spherically/axially symmetric solutions to Einstein's field equations in vacuum (the charged are solutions to the Einstein-Maxwell field equations where $T_{\mu\nu}$ is specified by the electromagnetic field-strength tensor $F_{\mu\nu}$). These solutions are completely specified by their mass, charge and angular momentum, a fact referred to as a *no hair theorem*. Explicit expressions for their metrics can be found in most modern textbooks treating general relativity, such as [10]. Although the solutions presented in [10] are highly symmetrical and

[3]The tensor product $\otimes$ is omitted, as is usual, strictly speaking $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$.

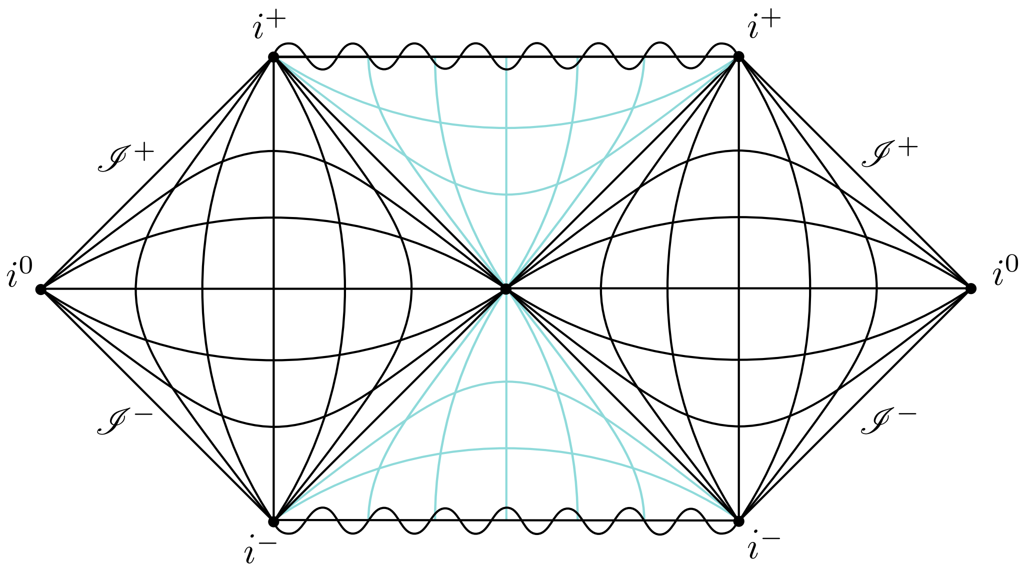

Figure 1.1: Conformal diagram of maximally extended Schwarzschild. The wavy lines indicate the locations of spacetime singularities, the top being a black hole singularity and the bottom a white hole. Horizontal lines are lines of constant time, and vertical constant radius. Lines at 45 degrees are null geodesics.

stationary, they do in fact represent realistic or at least near realistic end products of gravitational collapse, since any deviations from stationarity in the past (upon early formation of the black hole) is likely to decay quickly.

Before we go on to discuss the Kerr-Newman family we will mention a thing or two about uniqueness of solutions to Einstein's equations in general. Most notably, uniqueness is only partially true as it breaks down in higher than 4-dimensional spacetime. In 5-dimensional spacetime, restricting to asymptotically flat solutions with both mass and angular momentum, Emparan and Reall found the existence of a black-ring, i.e a geometry with horizon topology $S^1 \times S^2$ [24]. Furthermore, for specific ranges of mass M and the angular momentum J, there is not a unique solution, in fact, they found that there are several possible solutions; two black-rings (of different horizon area) and a black hole (horizon S^3). However, for special cases, such as the static asymptotically flat black hole solutions to Einstein-Maxwell-Dilaton theory, uniqueness continues to hold in dimensions $D \geq 4$ [33]. In general, uniqueness depends on the matter coupled to Einsteins GR as well as the dimensionality of spacetime. For a detailed review on uniqueness of black hole solutions take a look at [33]. Even though black hole solutions are non-unique in general, they are still characterized by very few parameters, too few to describe the matter that formed the black hole, e.g a massive star.

Let us get back to discussing the family of four-dimensional black holes of Einstein-Maxwell theory, namely the Kerr-Newman family. This family encompasses a variety of solutions corresponding to different regimes of the parameters M, Q and J. As expected, the family contains of course, the Schwarzschild ($Q = J = 0$), the Reissner Nordström ($J = 0$), and the Kerr ($Q = 0$) –black holes. Without going into the details, let us simply mention that one finds solutions lacking an event horizon – *naked singularities*, solutions with an inner and an outer horizon, and *extremal* solutions where the event horizon is degenerate[4]. Apart from the naked singularities which are thought to be unphysical by the *cosmic censorship* conjecture, the different types of horizon configurations are very interesting. For our purpose, the extremal black holes are most noteworthy and will be the focus of later discussion.

In the previous paragraph we left out a feature specific to rotating black holes ($J \neq 0$).

[4]This implies that the surface gravity vanishes on the horizon, and as we will learn in the next section, that the black hole will have zero Hawking temperature.

Such black holes will possess an ergo region, within which observers *can not* remain stationary, they are forced to partially move along orbits generating the axial symmetry. A curious implication of this was discovered by Penrose, namely the *Penrose process* suggesting that it is possible to extract energy from such a Kerr black hole by carefully tossing matter into the ergosphere.

The Penrose process is a nice toy for learning about black hole mechanics. From it, we can motivate an analogy between black hole parameters and the parameters characterizing a thermodynamical system. In all fairness however, the pillars of black hole mechanics are more fundamental and rely on deep geometrical truths as well as general observations in certain GR settings. In the next section we will discuss the analogy between black hole mechanics and thermodynamics, motivated by the Penrose process, and backed up by their various rigorous geometrical/GR proofs.

1.3 Black Hole Thermodynamics

There is a limit to how much energy can be extracted from a thermodynamical system in the form of 'useful' work. In other words, there is only a certain amount of energy available to us. Likewise, there is a limit to how much energy can be extracted from a Kerr black hole in the Penrose process. As we are about to learn, this has to be so as one would otherwise violate Hawking's area theorem, also known as the second law of black hole mechanics. There is also a zeroth and a first law of black hole mechanics, and a more general version of the second law (GSL), all of which are conveniently named after corresponding thermodynamical laws. We shall briefly review these laws in this section, starting with the zeroth law.

The zeroth law of black hole mechanics is a statement about the surface gravity κ of black holes in stationary equilibrium. The zeroth law states that κ is constant on the event horizon. Under certain assumptions this can be proved geometrically as was done by Carter [56]. Another proof alleviates some of the geometric assumptions replacing them with the dominant energy condition and validity of Einstein's equations. This proof is due to Bardeen, Carter and Hawking [4]. This law of black hole mechanics suggests that κ is analogous to temperature; all parts of a system in thermal equilibrium are at the same temperature – the zeroth law of thermodynamics.

The first law is an equation relating variations of the different black hole parameters. It was originally derived in [4]. Their derivation requires the perturbation to be stationary, and makes use of Einstein's equations. More general derivations of this law have been established, for instance it has been shown to be a consequence of an identity holding for the variation of the Noether current when considering general properties of field equations, specifically holding for any field equation derived from a diffeomorphism covariant Lagrangian [36].

A less intensive study; simply calculating the differential mass in the Penrose process for a Kerr black hole with angular momentum J, horizon angular velocity Ω_{H}, surface gravity κ and horizon area A, reveals the following 'first' law

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_{\mathrm{H}}\delta J, \tag{1.6}$$

in units with $\hbar = c = k_{\mathrm{B}} = 1$. One should compare this with the first law of thermodynamics

$$dE = TdS - pdV, \tag{1.7}$$

where E, T, S, p and V denote respectively, internal energy, temperature, entropy, pressure and volume.

Before we jump to conclusions based on these observations, let us also review the second law of black hole mechanics and its generalized version. This law is perhaps the most interesting of all the mechanical laws. Assuming the weak energy condition $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$, where t^μ are tangents to null-geodesics, and that the cosmic censorship hypothesis is valid, it can be proven that the area of a future event horizon in an asymptotically flat spacetime is non-decreasing with time. The original proof by Hawking as found in [32] is rather involved. Roughly speaking it relies on the study of the expansion θ of geodesic congruences generating the null hypersurface (the horizon).

The expansion is a measure of the rate of change of the local area elements defined by nearby geodesics in the congruence. If the area element were to decrease with time, that is, if $\theta < 0$, then, within a *finite* affine parameter nearby geodesics *must* intersect at some caustic which would imply that such null geodesic generators would have a future endpoint. A theorem by Penrose states that generators (geodesics) on the future horizon have *no* future endpoints, which contradicts with $\theta < 0$, implying that θ *must* be greater than or equal to zero, and thus that the rate of change of area elements on the future horizon *must* be positive with increasing time. Reference [14] gives a detailed and referenced account of this proof.

The increase in horizon area with increase in time, is of course analogous to the increase of entropy with increase in time – a way of stating the second law of thermodynamics. In the early 70's, this and the fact that information crossing the horizon is irretrievably lost, lead Bekenstein to propose that the entropy of a black hole was up to an unknown constant given by the area of the horizon in units of the Planck area [5].

Bekenstein's observation went something along the following lines. Working in units where $c = k_{\mathrm{B}} = 1$, $[E] = [m]$,[5] the Planck length is simply $\sqrt{G\hbar}$. Observing furthermore that $\Omega_{\mathrm{H}}\delta J$ is just a work term associated with changing the angular momentum of the Kerr black hole, we find that that the expression for the differential mass for the Penrose process can be expressed as

$$dE = TdS + (\text{work terms}), \tag{1.8}$$

with

$$E \sim M, \qquad S_{\mathrm{BH}} \sim \frac{A}{\hbar G}, \qquad T \sim \hbar\kappa. \tag{1.9}$$

These are not precise relations; we can multiply the entropy by an arbitrary constant α and the temperature by $1/\alpha$, and the first law would be unchanged, due to their appearance as TdS. However, this analogy gives us, with certainty, that $S_{\mathrm{BH}} \sim A$ and $T \sim \kappa$. Furthermore, the laws of black hole mechanics, such as constancy of κ over the horizon (stationary BH) and $\delta A > 0$ coincide respectively with the zeroth and second laws of thermodynamics as we have already pointed out. Bekenstein proposed that in general the following generalized second law should hold

$$\delta S_{\mathrm{tot}} \geq 0, \qquad S_{\mathrm{tot}} \equiv S_{\mathrm{BH}} + S_{\mathrm{matter}}, \tag{1.10}$$

where S_{BH} denotes the entropy of the black hole, and S_{matter} the entropy ascribed to the matter outside the black hole. This ensures that the second law is upheld even when the black hole is Hawking radiating, we will get to that in a moment.

Finally only the third law remains to be discussed. It states that a system cooled to absolute zero also has to have vanishing entropy[6]. For black holes, this translates into $\kappa \to 0$ implies that $A \to 0$. However, evaluating κ for the Schwarzschild solution, and noting that at any moment in time the horizon is the spatial two sphere of radius $r = 2GM$, we have

$$\kappa = \frac{1}{4GM}, \qquad A = 16\pi G^2 M^2. \tag{1.11}$$

Clearly, taking $M \to \infty$ gives us $\kappa \to 0$ and $A \to \infty$, in contradiction with the third law. In general there are many such extremal ($\kappa = 0$) black holes whose entropy remains finite. Although this is in violation with the third law this is not taken very seriously as the third

[5] By $[\cdot]$ we mean the dimension of $\cdot$

[6] For simplicity we assume no residual entropy associated with degenerate ground state

law is known to be violated by ordinary quantum systems as well, it is in other words not truly fundamental [56].

Although the third-law is somewhat brushed aside, we cannot ignore the fact that black holes are ”black”: Recall section 1.2 - nothing can ever escape from the region of spacetime enclosed by the event horizon - including the horizon itself. This implies that black holes cannot emit radiation, which in turn makes it difficult to see how black holes can have a temperature proportional to κ.

The temperature problem is not the only puzzle in store, what about the entropy? The *no hair theorem* states that black holes are characterized by very few parameters. This is a formal way of stating that the complexity of whatever collapsed to form the black hole is not 'registered' in the resulting black hole. As an example the Kerr solutions is completely specified by its angular momentum and mass with no reference to its predecessor. This theorem begs the question. How is the entropy realized, is there underlying 'microscopic' physics that can support $S_{\text{BH}} \sim A$?

The temperature problem was surprisingly alleviated by a semi-classical calculation carried out by Hawking [30]. Treating quantum fields on a fixed GR background, no back reaction, Hawking showed that black holes radiate thermally, with a black body spectrum at a temperature

$$T_{\text{H}} = \frac{\hbar\kappa}{2\pi}. \tag{1.12}$$

Actually, to an asymptotic observer the radiation is not precisely given by the black body spectrum, it deviates by so called greybody factors [29].

We will not dive into the details of the semi-classical calculation. There is a quick and rather sketchy way of deriving the correct result, by analytically continuing the coordinate time t to $-i\tau$ such that the time signature is now 'Euclidean'. The near horizon geometry then turns out to be Euclidean space in spherical coordinates, with the exception of a conical singularity that is only avoided by requiring τ to have period $2\pi/\kappa$, which is identified with the inverse temperature in the Euclidean partition function.

A more instructive and physically pleasant derivation relies on a phenomenon that describes the physics of a situation very similar to being near the horizon of a Schwarzschild black hole, namely the *Unruh effect*. Unruh radiation [52] is the thermal radiation predicted to be recorded by a constantly accelerated observer in Minkowski-space. This is a *quantum* mechanical result, that roughly speaking asserts that there will in general be disagreement on the particle number, and in particular on the vacuum state among different observers.

The relation to black holes is easily seen when studying the local structure of spacetime for stationary observers (observers on timelike orbits). The structure is found to be that of Rindler space – Minkowski-space as observed by a constantly accelerated observer. The temperature that such a constantly accelerating observer measures is

$$T_{\text{U}} = \frac{\hbar a}{2\pi}, \tag{1.13}$$

as predicted by Unruh, where a is the magnitude of the four-acceleration. To be stationary close to the horizon requires a tremendous acceleration. In the limiting case where we consider an observer on a timelike orbit at $r \to 2GM$, $a \to \infty$. However, on the watch of a second observer stationary at infinity where the Schwarzschild time t is the proper time, we measure an acceleration

$$\tilde{a} = \frac{d\tau}{dt} \cdot a, \tag{1.14}$$

where τ is the proper time of the stationary observer close to the horizon. In the limit $r \to 2GM$, $\tilde{a} \to \kappa$, thus we identify the Hawking temperature $T_{\text{H}} = \frac{\hbar\kappa}{2\pi}$, as the temperature measured by a stationary observer at infinite r. With this identification of temperature, it was possible for the first time to identify the previously unknown constant in (1.9) with a quarter, thus fixing the expression for entropy

$$S_{\rm BH} = \frac{A}{4\hbar G}. \tag{1.15}$$

It is clear that Hawking radiation solves the classical problem of vanishing black hole temperature. On the other hand, addressing the question regarding the micro-states comprising the entropy is extremely involved. It is in a very real sense a modern problem, that despite much effort and successes in specific cases, lacks a generally fulfilling answer. As we will discuss in the next sections, string theory gives us a possible scenario.

In light of Hawking's discovery the third law seems to hold for 'normal' black holes: They have finite temperature and entropy, and attempts at cooling (reducing κ) of black holes, becomes increasingly difficult with decreasing temperature. However, extremal black holes, for which κ is zero may very well have non-vanishing horizon area. An example is the extremal Reissner-Nordström black hole with $GM^2 = Q^2 + P^2$, the gravitational charge (M) exactly balancing the electromagnetic charges (Q and P). In this case the degenerate horizon is located at $r = GM$, implying finite entropy in violation of the third law. Even though we do not consider the violation of the third law as a problem, we do acknowledge the fact that cooling a black hole to absolute zero, is in effect physically impossible. It is conventional to think of extremal black holes as finely tuned black holes, that despite being highly unrealistic have simplifying features that help shed light on the mysteries of black hole thermodynamics. We will indeed have more to say about extremal black holes later.

Although Hawking resolved both the issue of classically vanishing black hole temperature and the fixation of Bekenstein's constant, his semi-classical calculation poses a new riddle. Observing that $\kappa \sim 1/M$, we notice that Hawking radiation facilitates a runaway process for the mass of black holes. Large black holes can have lifetimes comparable or even greater than the timespan of the universe, while smaller ones have *very short* lives, and completely evaporate within the lifetime of the universe. At first thought this does not seem too bad, the energy-momentum tensor does not satisfy the dominant energy condition in the semi-classical framework [56], and thus the area theorem is not violated; allowing for the horizon area to decrease with time. Furthermore, despite the fact that the area decreases during the evaporation process, the total entropy $S = S_{\rm BH} + S_{\rm matter}$ does not, since the particles in the Hawking radiation contribute to $S_{\rm matter}$ accordingly [56]. So what exactly is the problem with Hawking radiation?

The problem with the runaway process of Hawking evaporation, lies in the completely thermal nature of the radiation. Consider a region of spacetime containing matter which prior to collapsing to form a black hole bears quantum correlations with matter that will remain far outside the black hole once it forms. Now, as the black hole evaporates, the emitted Hawking radiation is completely uncorrelated with the matter that remained outside the black hole, i.e the matter remaining in the black hole becomes increasingly correlated retaining the original correlations with the matter that remained outside the horizon. However, as the black hole will eventually evaporate completely, it seems we have lost all correlations, that is, the originally pure state has evolved to a mixed state. Thus Hawking radiation facilitates the loss of information [56], and this paradox or puzzle is accordingly named the *information paradox* (puzzle).

It seems very difficult to resolve this paradox. A reasonable approach would be to realize correlations within the Hawking radiation, however this seems difficult, if not impossible, in a local quantum field theory [43]. It is however, thought, that non-local quantum field theories such as string theory, could allow for correlations giving Hawking radiation the ability to encode the original information. String theory has indeed aided in the discovery of certain idealized scenarios, in which information retention is realized. These take the shape of holographic dualities such as the famous AdS/CFT correspondence that literally suggests that a conformal field theory lives at the boundary of a bulk string theory in anti-de Sitter space. Thus is seems quite feasible, at least in certain settings, to realize the information return scenario, and the problem lies in understanding where the semi-classical Hawking derivation goes wrong [43].

In this thesis we will not dwell on the information puzzle, instead we will be addressing the related entropy puzzle. The entropy of black holes has the puzzling feature of scaling as the horizon area. Specifically, we are having a hard time trying to understand how to derive the area law (1.15) from supposed microscopic degrees of freedom. As we shall see, for certain extremal black holes, string theory gives a beautiful account of the entropy when certain Dirichlet branes are used to model the microscopic degrees of freedom. In general, we do not have the luxury of a string theory description. However, it seems that an underlying conformal field theory (CFT) can account for the entropy of general black hole microstates.

1.4 A Theory of Quantum Gravity

To better understand black hole thermodynamics, it became necessary to incorporate quantum mechanical effects as exemplified by Hawking's semi-classical derivation of black hole temperature. It is safe to say that his discovery revolutionized black hole physics; Hawking radiation, gave rise to the information paradox and set the stage for a puzzling entropy. Resolving these issues has been a common goal for many theoretical physicists intrigued by black holes. To address the information paradox and the entropy puzzle we need to look beyond semi-classical results. Indeed the presence of $\hbar$, G and c (which we set to 1) in the Bekenstein Hawking entropy formula, is a clear giveaway to the quantum gravitational nature of black hole entropy. What we really need to get our hands on, is a fully fledged quantum theory of gravity, in which we can treat general relativity and quantum mechanics simultaneously.

We are out of luck, it is not easy to quantize gravity, as a consequence, gravity is not incorporated in the triumphant standard model. This might sound like a big flaw, but the gravitational force is incredibly weak when compared with the electromagnetic, the strong and the weak nuclear forces, to the extent that its presence hardly makes a difference at the scales probable by todays accelerators. However, a clear understanding of quantum gravity becomes necessary in order to understand the physics of black holes, precisely because these are regions of spacetime where gravity dominates.

Roughly stated, the difficulty with quantizing gravity lies in the ultraviolet (UV) divergences that arise (divergences at high energy - short length scale). It is our inability to subtract these divergent terms by a *finite* number of counter terms, known as non-renormalizability, that stops us [22].

Despite the difficulties in quantizing gravity, it looks like string theory may give us a way to avoid UV divergences. In string theory, extended one-dimensional objects, strings, interact with each other at finite distances characterized by the string length scale ℓ_s, providing a natural UV cutoff [22]. In other words, the stringy nature is thought to comprise the ultraviolet degrees of freedom that can regulate the UV divergence, and the string length, usually taken to be of the oder of the Planck length ℓ_p, provides us with a cutoff. We should point out that at the moment we do not seem to know for sure whether string theory is UV finite or not [50]. String theory as we know it, may not be enough.

In addition to 'taming' the UV divergence, one finds that spacetime emerges in the low energy limit. There are several types of string theory to which low energy effective actions correspond: A bosonic string theory in 26 spacetime dimensions and five supersymmetric string theories in 10 spacetime dimensions, labeled by type - I, IIA, IIB, and HO, HE. Their low energy effective actions are reminiscent of the Einstein-Hilbert action, with additional field content. These are so called supergravity (SUGRA) theories, i.e supersymmetric theories of gravity, we elaborate on this in section 2.1. Finally, although probably obvious, string theory abides quantum mechanics by construction, thus string theory is a candidate theory for quantum gravity, with luck it could be *the theory*.

To get a better understanding of black hole physics, it is clear that we need to tackle quantum gravity. In this thesis we choose the beautiful, yet not fully understood framework of string theory. Although conceptually a challenge, string theory brings us closer to a resolution of the information paradox [22]. In particular the high degree of supersymmetry that supergravity theories support, allows for partly supersymmetric objects, p-branes, whose supersymmetry aids in identifying the precise solution in the supergravity theory. Such p-branes (in particular charged p-branes with their Dp-brane counterparts), upon dimensional reduction and compactification, comprise the ingredients of many lower dimensional black holes [43]. In general, procedural steps such as 'lifts', 'reductions' and 'boosts', as well as the general T, S and compacted U-duality, allow us to construct and relate such solutions with ease [43].

In some respect we are testing the applicability of string theory. If string theory should fail to reproduce the Bekenstein-Hawking entropy, we would have to tone down our enthusiasm for string theory. As we will see in what follows, the news is for the better, and string theory somewhat miraculously reproduces the semi-classical results for a special class of extremal solutions, identified with so called BPS states in the supersymmetric theory. These are solitons of the low energy effective action of type IIA/IIB string theory that preserve some fraction of the background supersymmetry. They are characterized by the fact that they saturate a lower mass bound that is believed to be untouched by renormalization, which in turn allows for an exact extrapolation between weak and strong string coupling g_s. The upshot being that stringy degrees of freedom are identified as the microscopic states that constitute the entropy of extremal black holes. These successes are exemplified in *holographic dualities*, of which we most notably point out the AdS/CFT correspondence. The AdS/CFT correspondence is a remarkable cornerstone of modern theoretical physics. It has far reaching applications, not just for black hole physics, but has proven to be an invaluable tool when dealing with conformal field theories at strong coupling, however, we will not have space in this thesis to give it a worthy presentation.

It is the goal of this thesis to extend the search for a string theory description of black hole entropy to more general non-extremal black holes. We are still a far way away from finding out exactly how string theory realizes the entropy for non-extremal black holes. However, we are seeing evidence for an underlying 'hidden' conformal symmetry that derives the entropy, which is a significant step forward. In this thesis the focus is on aspects of revealing hidden conformal symmetry for certain non-extremal black holes, with focus on the recent work by Cvetič, Larsen and Gibbons [16, 19, 20] and also the closely related work by Baggio et al. [2].

1.5 Outline of the Thesis

Before getting to the core of hidden conformal symmetry and subtracted geometry, we cover the necessary background: chapters 2 - 4. Of main interest to the core topic of this thesis is chapter 3, which covers an important example of microscopic entropy "counting", but also the more general means of deriving an entropy without counting, i.e by indirectly identifying the central charges of the relevant conformal field theory and applying the Cardy formula.

In chapter 5 we briefly address the notion of hidden conformal symmetries, but mainly our focus is on subtracted geometry. We will derive important results from the previous work done by Cvetič and Larsen, setting the stage for the subtracted geometries considered. Specifically we consider subtracted geometries for the four-dimensional four-charged black holes that are solutions to the minimal supergravity theory, the STU-model [13]. The focus is thereafter on warp factors $\Delta \sim r^2$ for large r. We also study the effect of the near-horizon extremal Kerr limit on the warp factors, and find a possible Δ_{NHEK} that coincides with the NHEK limit on Δ_0. Furthermore we study the extent to which subtracted geometries $\Delta \sim r^2$ for large r are asymptotically conical.

In chapter 6 we identify matter in the static case. We start by studying the Einstein equations, and find that for special warp factors $\Delta_\pm$ ($\Delta_- = \mathcal{A}^2_{\text{red}}$), the equations of motion simplify greatly. We furthermore find matter for a large class of warp factors by identifying overlap between subtracted geometries and the members of the four-parameter family studied in [2].

In chapter 7 we reproduce several of the calculations in [2]. In particular we study the uplift and dimensional reduction, that allow for the identification of a dual CFT description, where irrelevant deformations are found to be the dual operators that start the flow in the four parameter family from subtracted to original geometry. We also extend part of the analysis to the case $\Delta \sim r^2$.

In chapter 8 we review the use of scaling limits to obtain supporting matter for rotating subtracted geometries. Notably we manage to find supporting matter in the rotating case for $\Delta_- = \mathcal{A}^2_{\text{red}}$ via a scaling limit akin to the one considered in [16]. We also address the trouble with a scaling limit for Δ_{NHEK}.

In chapter 9 we conclude, and give an outlook. We have set aside a discussion of dimensional reduction for appendix A. Further appendices deal with the STU model and conventions.

2

Black Holes in String Theory

We briefly mention some relevant aspects of string theory, and continue with a quick derivation of the bosonic sector of the low-energy effective action of type superstring theories. In these theories supersymmetry is a key ingredient, and states invariant under part of the supersymmetry transformations, so called BPS states will be discussed, along with a derivation of the BPS bound. We then introduce the solitons and use an embedding of a 5-dimensional Reissner Nordström in a supersymmetric background as an example. We finalize with the correspondence principle that matches supergravity solitons with the Dp-branes from perturbative string theory.

2.1 The Low-Energy Effective Action of String Theory

We will merely familiarize ourselves with the low-energy effective actions that arise in string theory. To this end references [50, 57] were very useful. Our focus is on type IIA and IIB string theories as they possess the maximum degree of supersymmetry, and contain a massless spin 2 particle, i.e the graviton.

Type IIA and IIB are set in a $D = 10$ dimensional background. They solely contain closed strings, this periodic nature of the string imposes a strict condition on the world-sheet bosons X^μ, where $\mu = 0, \cdots, D-1$, that is they have to be periodic. On the other hand, the world-sheet fermions Ψ^μ only need to be periodic up to a sign. When Ψ comes back to itself without a change in sign we have the so called Ramond (R) periodicity condition, and if Ψ comes back to $-\Psi$ going once around the string, we have the so called Neveu-Schwarz (NS) anti-periodic condition. Enforcing either R or NS boundary conditions on either the left movers or the right movers, we get the four sectors of superstring theory; R-R, R-NS, NS-R, NS-NS. The R-NS and NS-R sectors support spacetime fermionic degrees of freedom, while the R-R and NS-NS sectors are bosonic. It is the latter two sectors that we are interested in, since we will in general consider a background free of fermions.

The low-energy effective description should only involve massless modes, a reasonable assumption as the excitations are separated by mass gaps $\sim 1/\ell_s$, where $\ell_s \ll 1$ is the string length. The massless field content of the NS-NS sector involves the graviton $G_{\mu\nu}$, the Kalb-Ramond two-form gauge field $B_{\mu\nu}$ and the scalar dilaton Φ. The massless field content of the R-R sector is slightly different for IIA and IIB. In both the massless degrees of freedom are packaged into q-form gauge fields C_q. Where q takes on the values 1 and 3 in IIA, while it has values $0, 2$ and 4 in type IIB.

The low-energy effective actions of type IIA/IIB string theory are actions of IIA/IIB SUGRA which can be thought of as generalized versions of the Einstein-Hilbert action. They produce the equations of motion of the low-energy background which other strings probe (feel), i.e, the emergent 10-dimensional spacetime.

For the free relativistic bosonic quantum string one has the Polyakov action, which is also the form the superstring action takes if one only considers the bosonic fields X^μ:

$$S_{\text{Pol}} = \frac{1}{4\pi\ell_s^2} \int d\sigma^2 \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, \eta_{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the flat target-space metric and $g_{\alpha\beta}$ is the world-sheet metric. This action reduces to the intuitive Nambu-Goto action via the equations of motion for the world-sheet metric. It is important to understand that the world-sheet[1] metric is an unphysical parameter that is simply added for convenience. The reason one considers this action, is that it is readily quantized as opposed to the Nambu-Goto action. The action is also invariant under local rescaling of the world-sheet metric

$$g_{\alpha\beta} \to \Omega^2(x) g_{\alpha\beta}. \tag{2.2}$$

We refer to this as Weyl invariance. The key observation is that Weyl invariance together with general reparameterization invariance amounts to conformal invariance. This allows us to fix the world-sheet metric (by a choice of gauge) which is reassuring since the world-sheet metric is unphysical. Let us emphasize that in order for the world-sheet metric to appear as an unphysical (gaugeable freedom) we require conformal invariance. We will use this required conformal invariance to impose constraints on the low-energy modes in the form of differential equations for which one identifies a corresponding action, namely the low-energy effective action.

For the general interacting string action, one would have to add all kinds of terms associated with the possible interactions. A way to approach the low-energy effective action, is to argue that in this regime, one only has massless modes $G_{\mu\nu}(X), B_{\mu\nu}(x)$ and $\Phi(X)$ (other massless fermionic modes enter for the superstring), ignoring all higher (massive) string excitations. Consequently, there are only three interaction terms in the action that play a role; the ones coupling to each of the massless modes. By means which we will not dwell upon, one calculates the vertex operators associated with the three fields and arrives at the following action, where we see that quite naturally the field $G_{\mu\nu}(X)$ resulting from the graviton mode replaces the Minkowski metric. The low-energy effective action of such an interacting string is thus the string probe action

$$\begin{aligned} S_{\text{probe}} = \frac{1}{4\pi\ell_s^2} \int d\sigma^2 \sqrt{-g} \Big(g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, G_{\mu\nu}(X) + i\epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu\, B_{\mu\nu}(X) + \ell_s^2 \Phi(X) R^{(2)} \Big) \\ + \text{(fermionic part)}. \end{aligned} \tag{2.3}$$

Here $R^{(2)}$ is specifically the Ricci scalar associated with the world-sheet metric.

The dilaton term $\ell_s^2 \Phi(x) R^{(2)}$ is not explicitly Weyl invariant, and hence spoils the explicit Weyl invariance of the action. As we emphasized earlier, Weyl invariance together with general reparameterization invariance gives us the freedom to fix the world-sheet metric. Without it the world-sheet metric would appear as a physical dynamical quantity in the action. We therefore require Weyl invariance by fixing the behavior of the background fields $G_{\mu\nu}(X), B_{\mu\nu}(X)$ and $\Phi(X)$ order by order in ℓ_s^2, the sigma model loop expansion parameter. This ensures that the physics does not care about the world-sheet metric.

In order to restore Weyl invariance at $\mathcal{O}(\ell_s^2)$, the background fields must solve three coupled differential equations. This requirement stems from the corresponding requirement $\langle T^\alpha{}_\alpha \rangle = 0$, and the fact that

$$\langle T^\alpha{}_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{\mu\nu}(G) \partial_\alpha X^\mu \partial^\alpha X^\nu - \frac{i}{2\alpha'} \beta_{\mu\nu}(B) \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \tfrac{1}{2}\beta(\Phi) R^{(2)}. \tag{2.4}$$

In general this implies that each of the beta functions must vanish independently yielding three sets of coupled differential equations, as can be seen from the one-loop expressions of the beta functions displayed in [50]. An action whose equations of motion coincide with this set of differential equations is the action governing the low-energy physics of the background. The action turns out to be

$$S_{\text{S}} = \frac{1}{2\kappa_0^2} \int d^{10}X \sqrt{-G} e^{-2\Phi} \Big(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi \Big), \tag{2.5}$$

[1]When referring to the world-sheet we mean the surface swept out by the string as it propagates in its target spacetime.

where R is the Ricci scalar associated with $G_{\mu\nu}$, and $H = \mathrm{d}B$. From $[R] = -2$ we must have for dimensional reasons $\kappa_0 \sim \ell_s^8$. This action S_S is *part* of the bosonic sector of IIA/IIB SUGRA. We will come back to the complete IIA/IIB SUGRA actions when we discuss p-branes.

The effective action S_S is reminiscent of the Einstein-Hilbert action, however it is not quite the same, the term $e^{-2\Phi}$ spoils it. We are free to absorb this term without changing the physics, by simply making a field redefinition. We refer to the action in its present form S_S as the string frame action, and $G_{\mu\nu}$ as the string metric. While we refer to

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} e^{-4\tilde{\Phi}/(D-2)}, \quad \tilde{\Phi} = \Phi - \Phi_0, \quad \Phi_0 \equiv \langle \Phi \rangle \tag{2.6}$$

as the Einstein metric. The corresponding Ricci scalar is

$$\tilde{R} = e^{4\tilde{\Phi}/(D-2)} R + (\text{terms proportional to } \tilde{\Phi}). \tag{2.7}$$

Where D is the dimension of spacetime. With these field redefinitions and $D = 10$, the action reads

$$S_\mathrm{E} = \frac{1}{2\kappa_0^2 e^{2\Phi_0}} \int d^{10}X \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-\tilde{\Phi}} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{2} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \right). \tag{2.8}$$

In string theory, e^{Φ_0} is recognized as the string coupling g_s. Comparing with the Einstein-Hilbert action we therefore find

$$G_N \sim \ell_s^{D-2} g_s^2, \tag{2.9}$$

where we put a subscript N just to emphasize that it is Newton's constant and not the determinant of some metric. It is convenient to enhance it to an identification by the definition

$$16\pi G_N^{(10)} \equiv 2\kappa_{10}^2 = (2\pi)^7 g_s^2 \ell_s^8. \tag{2.10}$$

In the next section we go into more detail with regard to supersymmetry and the related BPS bound.

2.2 Supersymmetry and the BPS Bound

Since the BPS condition, and BPS states are of such great use in establishing a match between quantum states and the effective gravity at strong coupling, it is almost cheating not to look into the physics that supports this bound.

The 'magic' of supergravity lies in the supersymmetry (SUSY). We will look at one of the central consequences of extended supersymmetry, i.e the BPS bound. We will start off with a brief introduction to supersymmetry, and work our way toward the BPS bound by considering $N = 2$ SUSY in $D = 4$ spacetime. Our discussion bases for the most part on [53, 59].

For a time it was believed that one could not extend the external continuous symmetries of spacetime beyond the Poincaré group, at least not without jeopardizing the axioms dictating properties that a reasonable theory should have, such as locality, positivity of energy, etc. Under these assumptions (locality, positivity ...), Coleman and Mandula showed that it was not possible to extend the group. Among many of their reasonable assumptions, one was that the algebra consisted solely of commutators. This was reasonable as generators of continuous symmetries usually satisfy a Lie algebra (involving only commutators).

Supersymmetry extends the set of known symmetries by supposing the existence of supersymmetry charges. They are packaged into spinors that appear in anti-commutators in the extended super-algebra, i.e the symmetry Lie algebra is extended to a graded Lie algebra.

In this sense the above no-go theorem is avoided as it assumes only commutators while now we are considering anti-commutators as well.

Super-Poincaré is achieved by extending the Poincaré group by adding to the set of commutators, a set of anti-commutators between spinors. Haag, Lopuszanski and Sohnius narrowed down the possible spin of the spinors to spin-$\frac{1}{2}$. As an important example, the super-Poincaré algebra in four dimensions is achieved by the addition of Majorana spinors Q

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}, \tag{2.11}$$

where Q_α is a left handed Weyl spinor, and $\bar{Q}^{\dot{\alpha}}$ a right handed Weyl spinor. Depending on the number of spinors (supersymmetry charges) we add, we have either regular supersymmetry where $N = 1$ spinor is added, or we have extended $N > 1$ supersymmetry.

Consider extended supersymmetry in $D = 4$ spacetime, i.e we extend the Poincaré group by adding

$$Q^I = \begin{pmatrix} Q^I_\alpha \\ \bar{Q}^{\dot{\alpha}I} \end{pmatrix}, \quad I = 1, \cdots, N. \tag{2.12}$$

The graded lie algebra that one obtains is comprised of the usual commutator relations of the Poincaré algebra and the following additional commutators and anti-commutators between Poincaré generators and the spinors Q^I.

$$\begin{aligned} &[P_\mu, Q^I_\alpha] = 0, \quad [Q^I_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta, \\ &[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \quad [\bar{Q}^I_{\dot{\alpha}}, M_{\mu\nu}] = -\bar{Q}^I_{\dot{b}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J, \\ \{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ}, \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{b}J}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ}. \end{aligned} \tag{2.14}$$

Here $M_{\mu\nu}$ are the Lorentz generators, and P_μ are the generators of spacetime translations. The indices α and β are spinor indices, while μ and ν are spacetime indices. We denote the completely antisymmetric symbol $\epsilon_{\alpha\beta}$, and Z^{IJ} is an antisymmetric complex $N \times N$ matrix.

The first two commutators on the left in (2.13) are not actually that straight forward. On the LHS, one gets an object belonging to

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2}, 0\right) = \left(0, \tfrac{1}{2}\right) \oplus \left(1, \tfrac{1}{2}\right). \tag{2.15}$$

The RHS clearly needs to transform accordingly, but generators transforming in $(1, \frac{1}{2})$ under the Lorentz group would posses spin $\frac{3}{2}$ as well as $\frac{1}{2}$. We only allow for spinors with spin $\frac{1}{2}$, and for consistency the RHSs must vanish as shown by Haag et al.

The two commutators on the right in (2.13) follow from the fact that the Q's are spinors, and $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ generate Lorentz transformations for respectively the left handed and right handed Weyl spinors that are carried in the Majorana spinors Q^I.

The first of the last three clearly transforms as $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$, that is, as a vector. The RHS must therefore transform as a vector, and the only generator in the algebra that transforms as a vector is P_μ. To contract away the spacetime index and get an expression with two lower spinor indices, the four-vector of Pauli matrices $\sigma^\mu_{\alpha\dot{\alpha}}$ is introduced. The ϵ symbol and the matrix Z are introduced to ensure that both sides of the equality have the same symmetry with respect to interchanging indices. Since the Z^{IJ} commute with all other generators in the algebra (to ensure closure and consistency), they only appear on the RHS and are therefore called central charges of the algebra.

Although we are currently only considering the four-dimensional case, the following calculation generalizes to higher dimensions. We will now show that for the present case, and focusing explicitly on $N = 2$ supersymmetry, we will get a lower bound on the mass M, the so called Bogomolnyi bound. It is an inequality between the mass and the central charges. In the present case we will only have a single complex central charge Z, since Z^{IJ} is anti-symmetric, and in the case $N = 2$ it is specified by the number $Z^{12} = -Z^{21} = Z \in \mathbb{C}$.

Boosting to the rest frame $P^\mu = M\delta_0^\mu$ and choosing a basis[2] where Z^{IJ} is replaced with a *real* antisymmetric matrix $\tilde{Z}^{IJ}$, the anti-commutators read

$$\begin{aligned}\{Q_\alpha^I, \bar{Q}_{\dot{\beta}J}\} &= 2\sigma^0_{\alpha\dot{\beta}} M \delta^I_J, \\ \{Q_\alpha^I, Q_\beta^J\} &= 2\epsilon_{\alpha\beta}\epsilon^{IJ}\tilde{Z}, \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}\tilde{Z},\end{aligned} \tag{2.16}$$

where $\tilde{Z} = \tilde{Z}^{12}$. This set of commutators is readily diagonalized by introducing suitable creation and annihilation operators

$$\begin{aligned} a_\alpha &= \tfrac{1}{2}(Q^1_\alpha + \epsilon_{\alpha\beta}\bar{Q}^2_{\dot{\gamma}}\bar{\sigma}^{0\dot{\gamma}\beta}), \qquad & b_\alpha &= \tfrac{1}{2}(Q^1_\alpha - \epsilon_{\alpha\beta}\bar{Q}^2_{\dot{\gamma}}\bar{\sigma}^{0\dot{\gamma}\beta}), \\ a^\dagger_{\dot{\alpha}} &= \tfrac{1}{2}(\bar{Q}^1_{\dot{\alpha}} + \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{0\dot{\beta}\gamma}Q^2_\gamma), \qquad & b^\dagger_{\dot{\alpha}} &= \tfrac{1}{2}(\bar{Q}^1_{\dot{\alpha}} - \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{0\dot{\beta}\gamma}Q^2_\gamma).\end{aligned} \tag{2.17}$$

The anti-commutators read

$$\begin{aligned}\{a_\alpha, a^\dagger_{\dot{\beta}}\} &= \sigma^0_{\alpha\dot{\beta}}(M + \tilde{Z}), \\ \{b_\alpha, b^\dagger_{\dot{\beta}}\} &= \sigma^0_{\alpha\dot{\beta}}(M - \tilde{Z}), \\ \{a_\alpha, b_\beta\} &= \cdots = 0.\end{aligned} \tag{2.18}$$

Here $\sigma^0_{\alpha\dot{\beta}} = \mathbb{1}_{\alpha\dot{\beta}}$, and $\bar{\sigma}^{\mu\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\delta}}\epsilon^{\beta\gamma}\sigma^\mu_{\gamma\dot{\delta}}$. To ensure a unitary representation such that one obtains only positive norm states, the following inequality must be satisfied

$$M \geq |\tilde{Z}|. \tag{2.19}$$

This inequality is referred to as the Bogomolnyi bound. It is clear that if $M < |\tilde{Z}|$, the RHS of one of the anti-commutators in (2.18) will be negative. From now on we take $\tilde{Z} = |\tilde{Z}|$ without loss of generality.

One can show that Q_1^I and $\bar{Q}_{\dot{2}}^I$ lower the spin of a massive state by a half, while Q_2^I and $\bar{Q}_{\dot{1}}^I$ raise the spin by a half. It follows that $\alpha_2, \alpha^\dagger_{\dot{1}}, \beta_2$ and $\beta^\dagger_{\dot{1}}$ raise the spin by a half, while the others lower it by a half.

When $M > |\tilde{Z}|$ we then have four creation operators, two that raise the spin and two that lower the spin by a half, respectively $\alpha^\dagger_{\dot{1}}, \beta^\dagger_{\dot{2}}$ and $\alpha^\dagger_{\dot{2}}, \beta^\dagger_{\dot{1}}$. Acting on a state with spin 0, one can then construct a massive vector multiplet with 8 fermionic and 8 bosonic degrees of freedom.

On the other hand, massive states with mass

$$M = |\tilde{Z}|, \tag{2.20}$$

will only have two creation operators, as the $b^\dagger$'s create zero-norm states. In total there will only be 2 fermionic and 2 bosonic degrees of freedom. In this case we say that the states saturate the Bogomolnyi bound, or that they are BPS states[3]. In general BPS multiplets have 2^N states, while a general massive multiplet has 2^{2N} states.

The above reasoning can be generalized to higher dimensions, in particular to the cases that we are interested in, namely the settings of type II string theories, the a-chiral IIA

[2] We are free to change the basis in this way, simply because these are central charges and commute with the rest of the generators.

[3] This term stems from a similar bound, the Bogomolnyi-Prasad-Sommerfeld bound encountered in the study of monopoles in supersymmetric gauge theories.

and the chiral IIB. These posses $N = 2$ supersymmetry, which is the maximum allowed supersymmetry in 10 dimensions, due to the constraint limit of a maximum number of 32 real supercharges[4]. From now on when we say supercharges we mean strictly *real* supercharges.

It is not at all obvious that $N = 2$ SUSY in 10 dimensions has a minimum of 32 supercharges. Naively one might think it has 2×64 supercharges, since a Dirac spinor in D dimensions will have $2^{D/2}$ complex components[5], i.e 64 real components. This number can however be lowered. It turns out that for even D we can have Majorana spinors, that is, Dirac spinors satisfying the Majorana condition $\Psi = \Psi^C = \mathcal{C}\bar{\Psi}^T$, thus halving the number of components. This brings the number of supercharges down to 2×32 which is still greater than 32. However, for $D = 10$ it turns out that in addition to the Majorana condition, one can simultaneously project out half of the charges using Γ_{11}, the product of all the 10-dimensional Dirac gamma matrices, that satisfy $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \cdot \mathbb{1}$. The spinors transforming in this chiral representation and simultaneously satisfying the Majorana condition are referred to as chiral-Majorana spinors. We see now how two chiral-Majorana spinors contain 2×8 complex components in total, giving rise to $2 \times 2 \times 8 = 32$ supercharges. Indeed type IIA and IIB SUSY have two such chiral-Majorana spinors (also called Majorana-Weyl spinors) each with 8 complex components [47]. Similarly, quite remarkably, beyond 11 dimensions one cannot have supersymmetry with 32 supercharges or less. Indeed $N = 1$ SUSY in 11 dimensions is the maximum number of dimensions with a minimum of 32 supercharges.

The reason for the upper limit of 32 supercharges goes hand in hand with the upper limit on the helicity of the allowed field content of the theory. Roughly speaking, we do not know of any interacting field theories with a finite number of fields with helicity greater than 2. It is easy to see why we cannot have more than 32 supercharges in the case of four dimensions. In this case the $N = 8$ SUSY generators raise or lower the helicity (spin in the massive case) by a 1/2, thus we have four raising operators and four lowering operators. Acting with a raising operator four times on a state of helicity zero gives a state with helicity 2, and likewise, lowering four times gives a state of helicity -2. Clearly we can not have more than a total of 8 SUSY generators in four dimensions, i.e 32 supercharges as this would necessarily imply states of helicity greater than 2. I turns out to be the general case for higher dimensional SUSY as well, that 32 supercharges is the upper limit if we wish to restrict our field content to states with at most helicity 2 [47].

We will go ahead and simply write down the important results, which can be found in [51]. The IIA supersymmetry algebra has a single non-chiral Majorana spinor worth of supersymmetry charges. That is, it is a combination of a left-handed and a right-handed chiral-Majorana spinor, denoted $(1, 1)$ SUSY. The anti-commutators read

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = (\mathcal{C}\Gamma^M)_{\alpha\beta}P_M + (\mathcal{C}\Gamma_{11})_{\alpha\beta}Z + (\Gamma^M\Gamma_{11}\mathcal{C})_{\alpha\beta}Z_M + (\Gamma^{MN}\mathcal{C})_{\alpha\beta}Z_{MN} \\ + (\Gamma^{MNPQ}\Gamma_{11}\mathcal{C})_{\alpha\beta}Z_{MNPQ} + (\Gamma^{MNPQR}\mathcal{C})_{\alpha\beta}Z_{MNPQR},\end{aligned} \tag{2.21}$$

where Γ are 10-dimensional gamma matrices, and ones with several indices are antisymmetrized products thereof. Furthermore $\mathcal{C}$ is the charge conjugation matrix for 10-dimensional Majorana spinors, and the Z's are the different p-form central charges which are carried by BPS states[6]. For type IIB, the supercharges, are packaged into two chiral-Majorana spinors Q^I_α of the same chirality. We may take these to be left-handed and denote this $(2, 0)$ SUSY.

[4] We do not know of any consistent interacting field theories with a finite number of fields with helicity > 2.

[5] When D is odd it will be $2^{(D-1)/2}$ complex components.

[6] As elaborated on in detail in [35], the different p-form charges Z, Z^+ are interpreted as charges carried by p-branes. In particular

$$\frac{1}{p!}Z_{i_1\cdots i_p}v^{i_1\cdots i_p} = \frac{1}{\Omega_{\tilde{n}+1}}\int_{S^{\tilde{n}+1}}\tilde{F},$$

where the p-brane is aligned in the directions given by the spatial p-form v.

The anti-commutators read

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} = \delta^{ij}(\mathcal{P}\Gamma^M\mathcal{C})_{\alpha\beta}P_M + (\mathcal{P}\Gamma^M\mathcal{C})_{\alpha\beta}Z^{ij}_M + \epsilon^{ij}(\mathcal{P}\Gamma^{MNP}\mathcal{C})_{\alpha\beta}Z_{MNP} \\ + \delta^{ij}(\mathcal{P}\Gamma^{MNPQR}\mathcal{C})_{\alpha\beta}(Z^+)_{MNPQR} + (\mathcal{P}\Gamma^{MNPQR}\mathcal{C})_{\alpha\beta}(Z^+)^{ij}_{MNPQR},\end{aligned} \tag{2.22}$$

where $\mathcal{P} = \frac{1}{2}(1+\Gamma_{11})$ is the chiral projector. In both (2.21) and (2.22) the maximum number of supercharges on the LHS is 528 which is broken down into the direct sums of charges on the right hand side, respectively as follows $528 = 10 \oplus 1 \oplus 10 \oplus 45 \oplus 210 \oplus 252$, and $528 = 10 \oplus 2 \times 10 \oplus 120 \oplus 126 \oplus 2 \times 126$.

One can again boost to a frame $P^N = M\delta^N_0$ as we did to get (2.16). When the dust settles one again finds that it is necessary to demand

$$M \geq |Z| \tag{2.23}$$

in order to avoid negative-norm states. As we mentioned in the example of $N = 2$ SUSY in four dimensions $M = |Z|$ gives us short BPS multiplets. Since the relation $M = |Z|$ is at the level of the supersymmetry algebra it is believed to be protected from renormalization (assuming unbroken supersymmetry) despite the fact that both $|Z|$ and M are in general renormalized. This is reasonable as one does not expect the degrees of freedom to somehow jump from e.g 4 to 16 at a certain value of g_s [6]. This allows for the extrapolation between a microscopic description in the low string coupling $g_s \ll 1$ regime to a macroscopic description of black holes in a regime where $g_s \gg 1$. As we shall see in chapter 4 this remains one of the most successful arguments for an entropy matching for certain black holes in the context of string theory.

In the next section we will introduce p-brane solitons as solutions of SUGRA, these serve as useful building blocks from which one may construct various black holes in lower dimensions by wrapping the branes around compact dimensions [43]. Thanks to the BPS condition (saturation) and the fact that it is protected from renormalization, we can then extrapolate between the strongly coupled SUGRA regime to the weakly coupled string perturbation theory regime. As we will motivate briefly in section 2.4, the SUGRA p-branes may be identified with a collection of Dp-branes at weak coupling, suggesting the underlying microscopic degrees of freedom. What is going to be really astonishing is that this $g_s \ll 1$ microscopic description in terms of Dp-branes produces an entropy matching the one prescribed by Bekenstein and Hawking.

2.3 Solitons in Supergravity

In the literature and as reviewed in [6, 58], solutions to the equations of motion that are regular, stable and have finite energy are referred to as solitons or being solitonic[7]. Of particular interest are the supersymmetric solitons which as implied by the invariance under part of the supersymmetry are BPS. Certain extremal black holes fall into this category, i.e they are supersymmetric solitons of the SUGRA action, and the central charges in the BPS bound are identified with the charges of the black hole.

2.3.1 Reissner-Nordström as a SUGRA Soliton

It is instructive to consider the embedding of a familiar black hole solution into a supersymmetric background. In particular, consider the unique solution to $D = 4$ Einstein-Maxwell theory, the Reissner-Nordström solution. We embed this solution into $N = 2$ supergravity, to this end it is necessary to add two gravitini to fill the supergravity multiplet. Indeed the

[7]Strictly speaking, by soliton, one is referring to the subset of non-singular solutions, and one calls the singular ones elementary. In this thesis we will use soliton to refer to both regular and singular solutions.

SUGRA multiplet contains degrees of freedom corresponding to a graviton (helicity ± 2), a gauge boson (helicity ± 1) and two gravitini (helicity $\pm 3/2$), thus the Reissner-Nordström solution is only missing out on the gravitini. What we would like to show is that in order for the Reissner-Nordström solution to be a supersymmetric soliton of the SUGRA action, it has to be extremal [41].

The well known Reissner-Nordström metric with mass parameter M and charge parameter Q reads

$$ds^2_{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.24}$$

The outer and inner horizons are respectively

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.25}$$

We require $M \geq |Q|$ to avoid a naked singularity, and notice that for $M = |Q|$, $r_+ = r_-$, i.e the horizon is degenerate, and hence the surface gravity $\kappa = 0$. In other words one gets the extremal Reissner-Nordström solution for $M = |Q|$. In isotropic coordinates ($r \to \rho = r - M$) the metric reads

$$ds^2_{\mathrm{ERN}} = -\left(1 + \frac{M}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{M}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2). \tag{2.26}$$

We will see that (2.26) is the only metric in the ($D = 4$) Reissner-Nordström family of solutions that preserves supersymmetry. We do this in a roundabout way, starting with a general symmetric ansatz for the metric

$$ds^2 = e^{2A(r)} dt^2 + e^{2B(r)} d\vec{x}^2, \qquad r = \sqrt{\delta_{ij} x^i x^j}, \tag{2.27}$$

and an electric type ansatz for the gauge field

$$A_0 = e^{C(r)} - 1 \implies F_{i0} = \partial_i C(r) e^{C(r)}. \tag{2.28}$$

One could now calculate the curvature scalars, and solve Einstein-Maxwell equations of motion, and arrive at the Reissner-Nordström solution. However, we would like to impose supersymmetry invariance prior to solving the equations of motion, which simplifies the calculation greatly, and goes to show how supersymmetry invariance, i.e BPS saturation, is related to extremal Reissner-Nordström.

In order to preserve half of the supersymmetry, half of the super-transformations must act trivially on the field content of the solution. In our case, and as is usual, the background is free of fermions, and therefore all super-variations of the bosons are trivially zero for all super-transformations. The only variations we have to check explicitly are the variations of the gravitini [41]

$$\delta_\epsilon \psi_{\mu A} = \nabla_\mu \epsilon_A - \tfrac{1}{4} F^-_{ab} \gamma^a \gamma^b \gamma_\mu \varepsilon_{AB} \epsilon^B = 0, \tag{2.29}$$

where

$$F^\pm_{\mu\nu} = \tfrac{1}{2}(F_{\mu\nu} \pm i \star F_{\mu\nu}), \qquad F^\pm_{ab} = F^\pm_{\mu\nu}\, e^\mu_a\, e^\nu_b, \tag{2.30}$$

and $A, B = 1, 2$ are spinor indices, where the position of the index denotes the chirality $\gamma_5 \epsilon^A = \epsilon^A, \gamma_5 \epsilon_A = -\epsilon_A$. In order to satisfy this equation, i.e preserve half the SUSY, it turns out that the three functions A, B and C are all determined by a single harmonic function H. This is precisely, and only then realized by the single centered, that is, the extremal Reissner-Nordström solution, i.e

$$e^{A(r)} = 1 + M/r, \tag{2.31}$$

or the multi-centered solution, which is really just several copies of the extremal Reissner-Nordström solution positioned arbitrarily in spacetime

$$e^{A(|\vec{x}|)} = \sum_i \left(1 + \frac{M}{|\vec{x} - \vec{x}_i|}\right). \tag{2.32}$$

The multi-centered solution is only possible due to the force balance; gravitational attraction balances electro-magnetic repulsion, due to the BPS saturation. These black holes will therefore not interact with each other, and hence the multi-centered solution is static and stable!

2.3.2 More General p-brane Solitons

We are now warmed up, and ready to tackle more general p-brane solitons of $N = 2$ supergravity, i.e SUGRA. The steps involved are more or less proceeding in the same spirit as the example of the Reissner-Nordström black hole. The major difference is that calculations get more tedious as the field content of the action is more involved. Just like we looked upon the Reissner-Nordström black hole as a point particle or 0-brane, coupling electrically to the gauge field via $F = \mathrm{d}A$, we now consider generalizations thereof, namely higher dimensional extended fundamental objects, branes, and more generally p-branes that couple to $(p+1)$-forms.

In general p-branes are charged by coupling to $(p+1)$-form gauge fields $C_{(p+1)}$. It is really simply a matter of extending the familiar notion of a charged point particle (0-brane), which we know couples to a one-form gauge field $A_{(1)}$. The equations of motion for such a charge point particle are derived from the action

$$\begin{aligned} S_{\text{pp}} &= -m \int_\gamma \sqrt{\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} d\tau + q \int_\gamma d\tau A_\mu \dot{X}^\mu \\ &= -m \int_\gamma ds + q \int_\gamma A, \end{aligned} \tag{2.33}$$

where

$$\dot{X} \equiv \frac{dX}{d\tau} \tag{2.34}$$

and, m and q are respectively the mass and charge of the point particle, and in this last line it is understood that the integrands such as A are pulled back onto the world-line γ. This is the probe action for the 0-brane.

Conceptually, the generalization to a p-branes is rather straight forward, however whereas the action of a point particle is simply the proper length, for a higher dimensional object it is a little less straight forward, indeed for a one-brane (string) it is the proper area (Nambu-Goto action). For Dp-branes we not only have world volume bosons (embedding coordinates), but also a $U(1)$ gauge field. As explained in [50], studying the beta function and using the same kind of argument as was used for the low-energy effective SUGRA action, one arrives at the action for a p-brane probing a closed string background $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$, which reads

$$S_{\text{D}p\text{-probe}} = S_{\text{DBI}} + Q_p \int_{V_{p+1}} C_{(p+1)}, \tag{2.35}$$

where

$$S_{\text{DBI}} = -T_p \int d\xi^{p+1} e^{-\tilde{\Phi}} \sqrt{-\det\left(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab}\right)}. \tag{2.36}$$

Extra terms can be added to describe interactions between several p-branes. The terms in (2.36) respectively correspond to fluctuations of the brane, the $U(1)$ gauge field, and coupling with open strings that are charged under the Kalb-Ramond field.

It is $S_{p\text{-probe}}$ that acts as a source for the R-R gauge field $C_{(p+1)}$ in the SUGRA action

$$S_{\mathrm{II}} = \frac{1}{2\kappa_0^2}\int d^{10}X\sqrt{-G}e^{-2\Phi}\left(R + 4(\mathrm{d}\Phi)^2 - \tfrac{1}{12}H^2 - \tfrac{1}{2(p+2)!}F^2_{(p+2)}\right). \tag{2.37}$$

We consider general p-branes as they naturally couple to the field strengths $F_{(p+2)} = dC_{(p+1)}$ already present in the massless spectrum of superstring theory. Of course, even more apparent, we have the fundamental string itself, that couples electrically to the Kalb-Ramond two form gauge field B, which follows from the term $H = \mathrm{d}B$ present in the SUGRA action. We call it fundamental to distinguish it from the R-R string that is electrically charged under $C_{(2)}$. We should point out that when in IIB the self dual five-form R-R field strength is present, one has to supplement the action (2.37) (at the level of the equations of motion) with a self duality constraint.

What about magnetic charges, indeed there are magnetic p-branes that are magnetically charged with respect to dual field strengths $\star F = \mathrm{d}\tilde{C}$. However, we will not dwell on the details thereof, we will only mention that as an important example, the dual object to the fundamental string is the magnetic NS-NS 5-brane to distinguish it from the R-R 5-brane [6].

Let us construct some p-brane solitons, i.e generalize the technique used to find the solution for the 0-brane, the Reissner-Nordström black hole. In general one starts with a p-brane ansatz that reads

$$ds^2 = e^{2A(r)}d\vec{x}^2 + e^{2B(r)}d\vec{y}^2, \tag{2.38}$$

where $\vec{x}$ denote the longitudinal coordinates, i.e the coordinates along the brane: $\vec{x} = x^\mu\partial_\mu, \mu = 0,\ldots,p$, and the remaining $D-p-1$ coordinates, the transverse coordinates are denoted by $\vec{y}$, and $r = \sqrt{\vec{y}^2}$. We imply furthermore that the world volume metric $d\vec{x}^2$ be Poincaré invariant and the transverse metric be $SO(D-p-1)$ invariant, i.e

$$d\vec{y}^2 = dr^2 + d\Omega^2_{D-p-2}. \tag{2.39}$$

This finalizes the ansatz for the metric, furthermore we have the ansatz for the dilaton $\phi = \phi(r)$ and the R-R gauge field (electric) ansatz with the only non-zero component $C_{01\cdots p} = e^{C(r)}$. For a magnetic ansatz it is more convenient to deal with the field strength $F_{(p+2)}$ which should be proportional to the volume form of S^{p+2}, i.e the magnetic ansatz involves no unspecified function. We proceed with the electric ansatz following reference [6].

As we pointed out for the 0-brane, the Reissner-Nordström black hole, we could proceed to compute the Christoffel symbols and solve the equations of motion directly, however, we consider solutions that are invariant under a fraction of the supersymmetry transformations, and it turns out to be much more economical to satisfy this condition first. In short, one finds that the four arbitrary functions A, B, C and ϕ all reduce to one function via the requirement of supersymmetry. We will not dive into the details, but merely note that the procedure is similar albeit more complex than our previous discussion concerning the Reissner-Nordström black hole. At the end of the day, the p-brane solutions are specified by the single function $H_p(r)$, which needs to be harmonic

$$H_p(r) = e^{-C(r)} = e^{-C_0} + \frac{Q_p}{r^{D-p-3}}, \tag{2.40}$$

where $C_0 \equiv \lim_{r\to\infty} C(r)$ and the possibility of $C_0 \neq 0$ allows for non-trivial vev ϕ_0 of the dilaton. In general for the solution to be asymptotically Minkowskian, one has for $D = 10$ in the Einstein frame

$$A = \frac{7-p}{16}(C-C_0), \qquad B = -\frac{p+1}{16}(C-C_0), \qquad \phi = \phi_0 + \frac{p-3}{4}(C-C_0). \tag{2.41}$$

An important example is the R-R three-brane, which has constant dilaton. It serves a central role in a canonical example of the AdS/CFT correspondence, namely the type IIB on $\mathrm{AdS}_5 \times S^5/(\mathcal{N}=4)$ SYM_4. Specifically as evident from the harmonic function

$$H_3 = 1 + \frac{Q_3}{r^4}, \tag{2.42}$$

that appears in the spacetime metric, it is evident that the limit $Q_3 \to \infty$ would result in a non-asymptotically flat spacetime, an alternative interpretation of this limit is taking $r \to 0$. Indeed in this limit one gets precisely $\mathrm{AdS}_5 \times S^5$ [6]. As we will discuss in limited detail in the next section, R-R charged p-branes have a microscopic description in terms of Dp-branes arising in the perturbative sector of string theory. In this case the R-R three-brane with charge Q_3 is identified, in the above limit $Q_3 \to \infty$ as Q_3 coincident D3-branes. The world-volume theory of such a stack of D3-branes is found to coincide with the large N limit of $U(N)$ $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions. That is a four-dimensional CFT with $\mathcal{N}=4$ supersymmetry [6]. The relation between black holes, p-branes and the conjectured gauge/gravity duality is nicely reviewed in [1].

2.4 Black Holes as Strings and Dp-branes

We have seen that p-branes are the generalized black object possible in higher dimensions, we can think of them as extended black holes. It turns out that one can construct lower dimensional black holes from p-branes in higher dimensions by wrapping the branes around compact dimensions. Even more interesting black holes can be constructed by putting different branes together as described in [43]. We will not go into the details, but we will eventually give a brief example in chapter 3. What we would like to address in this section is the extent to which p-branes and indeed the low-energy-effective actions (SUGRA) are a valid descriptions of black holes, and what might happen when we go away from the regime of validity. We shall see that there is a non-trivial correspondence point.

Clearly the SUGRA action is only valid when the sigma-model loop-counting parameter $\alpha' = \ell_s^2$ is small compared to a suitable measure of the spacetime curvature. For the d-dimensional Schwarzschild black hole, an invariant measure of the curvature is (and evaluates at the horizon to)

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big|_{r=r_H} = \frac{12}{r_H^4}, \qquad r_H^{d-3} = \frac{16\pi G_d M}{(d-2)\Omega_{d-2}} \sim g_s^2 \ell_s^{d-2} M. \tag{2.43}$$

This suggests that as longs as $\ell_s \ll r_H$ we can trust the supergravity description. However, when $r_H \sim \ell_s$ it gets murkier, one goes over to a regime where stringy effects matter [43]. The question is what kind of stringy degrees of freedom take over.

For the neutral black holes, the simplest neutral string theory object which only carries mass is the fundamental string. Following [43], it is instructive to consider how the entropy of a single fundamental string behaves near a supposed correspondence point where $r_H \sim \ell_s$. From string perturbation theory it is well know that the degeneracy of states scales as e^{m/m_0}, where $m_0 \sim 1/\ell_s$ is the Hagedorn temperature. One then finds (using that $\ell_s^2 m^2 = N$, where N is the oscillator number) that the entropy associated with the fundamental string goes as

$$S_{\text{string}} \sim \log(e^{m/m_0}) \sim m\ell_s. \tag{2.44}$$

This is a result from perturbation theory, hence only valid for $g_s \sim 0$. We will currently work under the assumption $g_s \sim 0$, the validity of which we will address in a moment.

The mass of the fundamental string should scales as the mass of the black hole at the correspondence point, thus

$$M \sim \frac{r_H^{d-3}}{g_s^2 \ell_s^{d-2}} \sim m. \tag{2.45}$$

Now using this we find that the ratio of the entropies scales as

$$\frac{S_{\mathrm{BH}}}{S_{\mathrm{string}}} \sim \frac{r_H}{\ell_s}. \tag{2.46}$$

This again suggests that when $r_H \sim \ell_s$ there exists a correspondence between the fundamental string and the neutral black hole. Furthermore, noting that $S \sim \sqrt{N}$ and $S \sim 1/g_s^2$, we see that for large N, $g_s \sim 0$, implying that our assumption of $g_s \sim 0$ is valid for large N. Thus the correspondence principle is applicable for large N.

When it comes to black holes carrying R-R charges, i.e constructed out of R-R branes, a similar correspondence is evident [34], this time involving also open strings ending on so called Dp-branes.

Dp-branes were realized as p-dimensional hypersurfaces on which open strings with $(D-p)$ Dirichlet boundary conditions could end. The remaining p coordinates have Neumann boundary conditions, i.e string endpoints are constrained to the p-dimensional world-volume of the brane. In this sense, open strings with such boundary conditions allow for the inclusion of so called Dp-branes in string perturbation theory.

We may ask whether Dp-branes are related to the classical R-R p-brane solutions of the low-energy effective SUGRA actions. There is good reason to believe that Dp-branes in perturbation theory describe the general brane object at low string coupling $g_s \sim 0$, while R-R p-branes describe a collection of coincident Dp-branes at large coupling $g_s \gg 1$ [34]. The idea then is that R-R p-branes have charge Q_p proportional to the number of fundamental Dp-branes that constitute them.

In general, the correspondence principle as elaborated in [34] suggests that we can identify strings and Dp-branes as the microscopic constituents of black holes, and general black p-branes. This principle along side with the important non-renormalization theorem for BPS states allows for the exact extrapolation of results calculated in the low string coupling, i.e Dp-brane picture, to the regime of high string coupling where the classical black p-branes are the valid description. As we shall see in the following sections the counting of the degeneracy of states for certain extremal BPS black holes agrees exactly with the Bekenstein-Hawking entropy.

3

— Entropy Through the Looking Glass —

In the previous chapter we introduced supergravity as a low-energy effective string theory. We also introduced several results such as the BPS condition which is protected by a non-renormalization theorem. In this chapter we will finally discuss an example in which these results can be applied to get an entropy from the perturbative string theory regime, that matches with the classical Bekenstein-Hawking area law for the entropy (1.15). However, of equal importance, and of central importance to this thesis, is the idea of attributing the entropy to microscopic degrees of freedom associated with a dual CFT.

3.1 Entropy Prelude

The concept of entropy has been refined classically and quantum mechanically in statistical physics, where it has been identified in the microcanonical ensemble to be the logarithm of the number of accessible microstates Ω

$$S = k_{\mathrm{B}} \ln \Omega, \tag{3.1}$$

and in general, given a density matrix, a corresponding density operator $\hat{\rho}$ the above is generalized to the Von Neumann entropy

$$S = -\operatorname{Tr}(\hat{\rho} \ln \hat{\rho}). \tag{3.2}$$

If S_{BH} really is the entropy we should associate with black holes, then for a complete understanding of the entropy, an understanding of the nature of the microstates comprising it is needed.

We mainly focus our attention on string theory and its applications to black hole physics. However, having mentioned the Von Neumann entropy formula it is appropriate to mention ”entanglement-entropy”, entropy attributed to correlations between states separated by a domain wall, in the case of black holes the horizon. This entropy is very different from the kind of entropy that one usually considers, i.e one related to the random fluctuations within the bulk of a system due to a finite temperature, instead entanglement-entropy resides in the correlations across the boundary (the domain wall) [23]. When introducing the Planck length as a cut-off, the Von Neumann entropy associated with the correlations across the horizon scales as the area of the horizon, however, crucially it does not seem to be able to fix the constant of proportionality [56].

Although various approaches to microscopic derivations have managed to show that the entropy scales as the area of the horizon like the above mentioned entanglement-entropy, but also other approaches such as attributing the entropy to the thermal atmosphere just outside the horizon, it seems that at present they are unable to fix the constant of proportionality [56].

Without further delay, let us move on to the successful microscopic entropy-counting facilitated by string theory for certain extremal (BPS) black holes. To understand the derivation of entropy it is necessary to be aware of the preliminary result known as the Cardy formula that gives the entropy at high-temperature for any unitary 2D conformal field theory. Aspects of conformal field theories and specifically a derivation of the Cardy formula is reserved for chapter 4.

3.2 Microscopic Counting for BPS Black Holes

We will now take a look at an example where the counting of Dp-brane states derives the correct entropy (coinciding with the Bekenstein-Hawking area law). The general procedure relies on the identification of Dp-brane constituents as the appropriate objects for describing the black hole (wrapped p-branes) at strong coupling. Then by the non-renormalization theorem that applies for BPS states, the degeneracy of the states in the D-brane worldvolume theory should agree with the degeneracy at high string coupling.

The simplest, and first studied case, seems to be the five-dimensional three-charged black hole constructed from D1, D3 and D5-branes. Such black holes were considered by Strominger and Vafa in [49]. The charges are carried by p-branes ($p = 1, 3, 5$), which are described by Dp-branes at low string coupling.

In [49] Strominger and Vafa consider 10-dimensional type IIB string theory with 5 compact dimensions over $K3 \times S^1$. In this theory one can construct black holes with both axionic and electric charge respectively (Q_F and Q_H) from Dp-branes $p = 1, 3, 5$ wrapping the $K3 \times S^1$.

They restrict to extremal black holes whose microstates are BPS specifically preserving a quarter of the $N = 4$ D-brane worldvolume supersymmetry. In this case one can make use of the non-renormalization theorem (that BPS remains BPS) when extrapolating between a regime of strong and weak string coupling. At low string coupling it is the D-brane description that is valid, and it is in this regime that the counting of states is performed.

Then proceeding by the simplification where $K3$ is taken to be very small, the theory is effectively described by the conformal field theory on the cylinder $S^1 \times \mathbb{R}$ whose target-space manifold is given by the symmetric product of $K3$ surfaces [49]. The number of real dimensions of this manifold turn out to be $4(\frac{1}{2}Q_F^2 + 1)$ thus dictating a central charge $c = 6(\frac{1}{2}Q_F^2 + 1)$ for the effective conformal field theory [58].

To get the entropy it is then simply a matter of applying the Cardy formula

$$S = 2\pi\sqrt{\frac{nc}{6}}, \tag{3.3}$$

where n is the lowest level L_0 (the left-movers do not contribute). In the case under consideration $n = Q_H$. For the Cardy formula to be applicable we need to have Q_H much larger than Q_F. Inserting the expressions for n and c one finds that the statistical entropy reads

$$S = 2\pi\sqrt{Q_H(\tfrac{1}{2}Q_F^2 + 1)}. \tag{3.4}$$

For large Q_F this matches with the entropy prescribed by the Bekenstein-Hawking formula (1.15) which for the black hole under consideration equates to

$$S_{\mathrm{BH}} = 2\pi\sqrt{\tfrac{1}{2}Q_H Q_F^2}. \tag{3.5}$$

Although there is not precise agreement, the results are valid for large Q_F and that is when the Bekenstein-Hawking result is reliable [49]. In the literature this is considered the first real example of a microscopic derivation of entropy [38].

Obviously we have left out most of the details which go beyond the scope of the thesis. We mention it as an important example, illustrating the level at which string theory actually succeeds at deriving the correct entropy. We should also note the more intuitive approaches taken in a similar case described in detail in [9, 39, 55]. It is overall a slightly different perspective, where the effective conformal field theory of the D1-D5 system (with Q_5 D5-branes wrapped around T^5 and Q_1 D-strings wrapped around one of the compact dimensions belonging to T^5) comprises a total of $4Q_1Q_5$ superconformal fields with only the massless modes excited and carrying left moving momentum. This corresponds to $4Q_1Q_5$ bosonic and

$4Q_1Q_5$ fermionic species with energy $E = N/R_9$ (ie. at level N). This dictates a central charge $c_L = \frac{3}{2} \times 4Q_1Q_5$ and that we in the Cardy formula should substitute $h_{\text{eff}} = N$, giving us the entropy

$$S = 2\pi\sqrt{Q_1Q_5N} \tag{3.6}$$

which coincides nicely with the Bekenstein-Hawking result [39]. In [40] they also sketch a quick derivation of this five-dimensional example, and extend it to a four-dimensional black hole.

Without going into the details, we merely note that for certain near-extremal black holes, a similar counting of microscopic states gives the correct entropy. In particular employing the dilute gas regime for the near-extremal black holes, effectively a decoupling limit, one finds agreement with the area law for the entropy [43].

3.3 Entropy Without Counting

We shall now see that a derivation of microscopic entropy does not require knowledge of the precise nature of the underlying microscopic degrees of freedom. This is perhaps not that surprising: At the end of the day, the much utilized Cardy formula only requires that the underlying microscopic degrees of freedom comprise a two-dimensional unitary conformal field theory, and utilizes merely the effective central charges and conformal weights to determine the asymptotic density of states. What is more surprising is the means by which one can extract the effective central charge without knowledge of the UV completion, and furthermore that the Cardy entropy coincides with (1.15).

In the following we briefly discuss the work of Brown and Henneaux [8] and a more recent, yet similar approach [27], that identifies the relevant central charges, without referring to a specific UV completion such as string theory, but simply by analyzing the asymptotic symmetry group of respectively AdS_3 and the near-horizon extremal Kerr (NHEK) geometry.

Brown and Henneaux have demonstrated that *any* consistent theory of quantum gravity on AdS_3 is holographically dual to a two-dimensional conformal field theory [8]. Specifically, they found that the algebra satisfied by the generators of the asymptotic symmetry group could be identified with the Virasoro algebra with central charges $c_L = c_R = \frac{3\ell}{2G}$. More recently this has been found to apply to any black hole whose near-horizon geometry is locally AdS_3 up to global identifications [48]. As an instructive example, the BTZ black hole which is locally AdS_3, is dual to a two-dimensional conformal field theory with central charges $c_L = c_R = \frac{3\ell}{2G}$, where ℓ is the AdS_3 curvature radius and G is the three-dimensional Newton constant [8]. The conformal weights of the dual field theory are respectively n_R, n_L and relate to the mass M and angular momentum J as described in [48] yielding the entropy via Cardy's formula

$$S = \pi\sqrt{\frac{\ell(\ell M + J)}{2G}} + \pi\sqrt{\frac{\ell(\ell M - J)}{2G}}, \tag{3.7}$$

which is found to be in complete agreement with the Bekenstein-Hawking area law.

As another interesting example, we can derive the statistical entropy for the five-dimensional black hole considered in [49] (which we discussed previously had a microscopic interpretation in terms of wound Dp-branes) by identifying the effective three-dimensional BTZ black hole [48]. Again using that the central charge is the one given by Brown and Henneaux in [8], we can apply Cardy's formula and as noted in [48] one gets precise agreement with the area law.

Lastly we point out that in the recent paper [27], the Kerr/CFT correspondence, the authors more or less repeat the analysis done by Brown and Henneaux in [8], but now for the near-horizon extremal Kerr (NHEK) geometry. They find that extremal Kerr is dual to a chiral two-dimensional CFT with left central charge $c_L = 12J$, and after identifying the correct CFT temperature, they find precise agreement between the entropy prescribed by Cardy's formula and the Bekenstein-Hawking area law.

Crucially all of these examples as noted in [27] require no knowledge of the ultraviolet degrees of freedom that dictate the UV completion of the theory, in other words we do not need to know about the underlying theory in detail. It is enough to assert the existence of a conformal field theory, which we may then conjectured to be dual to the quantum gravity theory in question. This is corroborated by the matching of the CFT entropy and the semi-classical Bekenstein-Hawking area law.

Indeed as reviewed in [38] the entropy matching for the BTZ black hole, or black holes with an effective BTZ description, is not surprising as the effective partition function for the BTZ black holes matches with the partition function of a CFT, so called $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

We see that even though we can only utilize the D-brane picture for extremal and sometimes near-extremal black holes, we can, as exemplified with the BTZ black hole, identify a dual CFT whose central charge derives the correct entropy via Cardy's formula. In this sense, as far as this thesis is concerned, string theory is currently the most promising UV completion of quantum gravity, however we do not need to worry about the UV completion in order to derive the entropy for black holes, we simply need to identify an underlying conformal symmetry, that would hint at a dual CFT description.

At this point in the thesis, we thus part with our idealistic extremal black holes, and proceed to discuss more recent development in the realm of non-extremal black holes (i.e more realistic black holes). It has become increasingly apparent that an entropy as prescribed by the Cardy formula matches with Bekenstein Hawking entropy, given an appropriate dual CFT description of the degrees of freedom. Before going into the details of explicit examples and the general outlines of this procedure, we would like to review CFTs, just like we reviewed SUSY prior to the example displayed in the previous section. Unlike our discussion of SUSY/BPS-bound we will devote an entire chapter to CFTs, simply because it is more central to the focus of the thesis.

4

Conformal Symmetry and CFTs

In many cases we are able to identify the black hole entropy with the entropy of a supposed dual conformal field theory (CFT), this is to a large extent in the spirit of AdS/CFT. In this chapter we discuss conformal symmetry and some of its implications for CFTs, in particular for two-dimensional CFTs. It is important to note that to extract the entropy of a two-dimensional CFT, we do not need to know the precise details of the CFT: In the appropriate limit, the Cardy formula gives us the entropy as a function of the central charges and the lowest weight. Indeed we shall take a close look at the derivation of the Cardy formula in the last section, but we start off with general considerations of CFTs, for the most part drawing from [7, 26, 50].

4.1 Symmetry Prelude

It seems only fair to take a moment to briefly address the notion of a symmetry in general. Symmetries, as always, are enjoyed by systems that are invariant under the associated transformation. For example, Minkowski space-time enjoys Poincaré symmetry, that is the metric is invariant under Lorentz transformations and general space-time translations. A theory that enjoys conformal symmetry is characterized by being invariant under coordinate transformations that effectively result in a local rescaling of the metric. For instance the world-sheet field theory living on the freely propagating relativistic string, is as seen by the Polyakov action (2.1), invariant under a local Weyl rescaling of the world-sheet metric

$$g_{\mu\nu}(\sigma) \to \Omega^2(\sigma) g_{\mu\nu}(\sigma). \tag{4.1}$$

It is easy to see that the factors cancel

$$\sqrt{\Omega^4(\sigma)} \times \Omega^{-2}(\sigma) = 1, \tag{4.2}$$

leaving the action invariant. Clearly the world-sheet field theory is a two-dimensional CFT, *classically*. A theory that enjoys a symmetry classically, may not successfully be quantized in a manner that retains that symmetry at the quantum level. Simply put, conformal symmetry, while plain and simple classically, may not even be supported in the quantized theory. In fact, as an example the bosonic string only retains its classical symmetries at the quantum level if the dimensionality of the target space (the background) is precisely 26. Similarly the superstring requires $D = 10$.

4.2 Conformal Transformations

A uniquely specifying feature of general conformal transformations, are that they leave all local angles between tangent vectors of intersecting curves, unaltered, i.e conformal transformations preserve angles locally. Which is quite transparent noting that the angle θ between two vectors u and v is given by

$$\cos\theta = \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{g_{\rho\sigma} u^\rho u^\sigma \, g_{\kappa\gamma} v^\kappa v^\gamma}}, \tag{4.3}$$

and after a rescaling $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ the angle θ' is given by

$$\cos\theta' = \frac{\Omega^2(x) g_{\mu\nu} u^\mu v^\nu}{\sqrt{\Omega^4(x)\, g_{\rho\sigma} u^\rho u^\sigma\, g_{\kappa\gamma} v^\kappa v^\gamma}} = \cos\theta, \tag{4.4}$$

that is the two angles are the same, regardless of the rescaling function $\Omega^2(x)$

$$\theta = \theta'. \tag{4.5}$$

To efficiently and abstractly deal with the physical implications of symmetry invariance we find it tremendously useful to work with the concept of a symmetry group. The symmetry group is in general a group in the strict mathematical sense, i.e closure, associativity, etc., and contains all the transformations that characterize the symmetry (i.e leave the system invariant). When the symmetries are continuous, the generators form a Lie algebra. The key to this abstraction, is that any transformation in the group can be constructed by compounding the generators. It is crucial to keep in mind that the Lie algebra is only a local statement. Indeed, only a finite-dimensional subset of the infinite-dimensional space of local conformal transformations in two-dimensions are globally well defined.

It is instructive to derive the generators of conformal transformations and their Lie algebra. All groups will have an identity, that acts trivially. We can derive the properties of the generators in general, by considering deviations from the identity element to lowest oder. To this end we begin by inspecting how the infinitesimal transformations act on a vector x^μ specifying a point in the coordinate system. The vector is mapped to itself by the identity, and gets a correction proportional to the infinitesimal parameter when acting on it with a transformation deviating infinitesimally from the identity

$$x^\mu \to \bar{x}^\mu = x^\mu + \epsilon^\mu(x). \tag{4.6}$$

The above resulting infinitesimal change in x^μ is completely general. We have not specified the nature of the transformations yet. To find out the nature of the vectors ϵ^μ under conformal transformations, we need to consider how the metric transforms under the same general transformation induced by the infinitesimal ϵ^μ. To this end we proceed, noting that in general the metric transforms as a (0,2) tensor, thus

$$g_{\mu\nu}(x) dx^\mu dx^\nu \to g_{\bar{\mu}\bar{\nu}}(\bar{x}) d\bar{x}^{\bar{\mu}} d\bar{x}^{\bar{\nu}} = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial \bar{x}^{\bar{\mu}}} \frac{\partial x^\nu}{\partial \bar{x}^{\bar{\nu}}} d\bar{x}^{\bar{\mu}} d\bar{x}^{\bar{\nu}}. \tag{4.7}$$

We have

$$x^\mu = \bar{x}^\mu - \epsilon^\mu(\bar{x}) \implies \frac{\partial x^\mu}{\partial \bar{x}^{\bar{\mu}}} = \delta^\mu_{\bar{\mu}} - \partial_{\bar{\mu}} \epsilon^\mu(\bar{x}), \tag{4.8}$$

Note that to lowest order in ϵ we have

$$\epsilon(x) = \epsilon(\bar{x}). \tag{4.9}$$

We thus find that

$$g_{\bar{\mu}\bar{\nu}}(\bar{x}) = g_{\mu\nu}(x) \left(\delta^\mu_{\bar{\mu}} - \partial_{\bar{\mu}} \epsilon^\mu(x)\right) \left(\delta^\nu_{\bar{\nu}} - \partial_{\bar{\nu}} \epsilon^\nu(x)\right) \tag{4.10}$$

and keeping only the terms to leading order in ϵ yields

$$g_{\mu\nu}(\bar{x}) = g_{\mu\nu}(x) - \partial_\mu \epsilon_\nu(x) - \partial_\nu \epsilon_\mu(x). \tag{4.11}$$

Clearly, for the transformation to be a conformal transformation, the change in the metric needs to be cast in the form of a local rescaling

$$g_{\mu\nu}(\bar{x}) = \Omega^2(x) g_{\mu\nu}(x). \tag{4.12}$$

This requirement imposes the condition $\partial_{(\mu}\epsilon_{\nu)} \propto g_{\mu\nu}$, and the constant of proportionality is fixed by simply requiring equality of the traces on both sides of the equation.

$$2\partial_\mu \epsilon^\mu = 2\partial \cdot \epsilon = \text{const} \times D \tag{4.13}$$

where D is the dimensionality of the metric. Thus

$$\partial_{(\mu}\epsilon_{\nu)} = \tfrac{2}{D}(\partial \cdot \epsilon) g_{\mu\nu} \implies \Omega^2 = 1 - \tfrac{2}{D}\partial \cdot \epsilon. \tag{4.14}$$

When the metric is simply Minkowskian, this restricts the $\epsilon(x)$ to be at most quadratic in x as it can be shown to follow from the above relation for $\epsilon(x)$ that for $D > 2$ the third derivatives must vanish. One can then split up the different forms of $\epsilon(x)$ into the independent generators of

$$\epsilon^\mu(x) = a^\mu \qquad \text{(translations)} \tag{4.15}$$
$$\epsilon^\mu(x) = \omega^\mu{}_\nu x^\nu \qquad \text{(rotations)} \tag{4.16}$$
$$\epsilon^\mu(x) = \lambda x^\mu \qquad \text{(scaling)} \tag{4.17}$$
$$\epsilon^\mu(x) = b^\mu x^2 - 2x^\mu b \cdot x \qquad \text{(special conf.)} \tag{4.18}$$

We just mention them here for completeness, while our main focus will be on the case $D = 2$, which we address in the next section.

4.3 Conformal Algebra in Two Dimensions

We will from now on restrict ourselves to two dimensions, and flat space with Euclidean signature, that is, we consider the case where

$$g_{\mu\nu} = \delta_{\mu\nu}, \qquad \mu, \nu = 1, 2. \tag{4.19}$$

We are free to go to Euclidean space, by a Wick rotation, at the end of the day we can undo the Wick rotation and recover the physical setting i.e Minkowski space

$$x^0 \to -ix^2, \quad x^1 \to x^1, \tag{4.20}$$

where x^0 is taken to be the timelike coordinate, and x^2 is the "Euclidean-time". Euclidean space is chosen so that the formalism becomes more elegant. It is easy to see that the condition set on the $\epsilon^\mu(x)$, then become

$$\partial_1\epsilon_2 + \partial_2\epsilon_1 = 0, \qquad (\delta_{12} = \delta_{21} = 0), \tag{4.21}$$
$$\partial_1\epsilon_1 - \partial_2\epsilon_2 = 0, \qquad (\delta_{11} = \delta_{22} = 1). \tag{4.22}$$

These equations are none other than the Cauchy-Riemann equations for the complex function

$$\epsilon(z) = \epsilon_1(x^1, x^2) + i\epsilon_2(x^1, x^2), \quad z = x^1 + ix^2. \tag{4.23}$$

Any complex function satisfying these conditions is holomorphic. Furthermore we observe that $f(z) = z + \epsilon(z)$, is also holomorphic. Therefore we identify the set of infinitesimal conformal transformations in two-dimensions as the set of transformations, one for every holomorphic function $f(z)$

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}). \tag{4.24}$$

It is convenient to at the same time work with $\bar{z}$ the complex conjugate[1], since

$$x^1 = \tfrac{1}{2}(z + \bar{z}), \quad x^0 = -ix^2 = -\tfrac{1}{2}(z - \bar{z}). \tag{4.25}$$

[1] Clearly it should be $\bar{z} \to f(\bar{z})$ and not $\bar{f}(\bar{z})$, this is however standard notation found in the references, an abuse of notation, that nevertheless makes book keeping easier, we should just think of it as a single bar i.e $\overline{f(z)} = \bar{f}(\bar{z}) = f(\bar{z})$.

When treating z and $\bar{z}$ as the independent variables[2] we have that

$$\epsilon(z) = \epsilon_1(x^1, x^2) + i\epsilon_2(x^1, x^2), \tag{4.26}$$
$$\bar{\epsilon}(\bar{z}) = \epsilon_1(x^1, x^2) - i\epsilon_2(x^1, x^2) \tag{4.27}$$

are the independent local infinitesimals. Here $\bar{\epsilon}(\bar{z})$ is necessarily anti-holomorphic, satisfying a slightly different version of the Cauchy-Riemann equations where the signs have been switched (it is equivalent to being holomorphic with respect to $\bar{z}$). As a side note, one often uses the terms, left and right-movers for respectively holomorphic and anti-holomorphic in analogy with the terminology used for excitations of the string.

A function that is holomorphic is also analytic (in the relevant domain), these two properties are interchangeable, and therefore we are free to power series expand around any point in its domain, i.e

$$\epsilon(z) = \sum_{-\infty}^{+\infty} \epsilon_n f_n(z), \qquad f_n(z) \equiv -z^{n+1}, \tag{4.28}$$
$$\bar{\epsilon}(\bar{z}) = \sum_{-\infty}^{+\infty} \bar{\epsilon}_n \bar{f}_n(\bar{z}), \qquad \bar{f}_n(\bar{z}) \equiv -\bar{z}^{n+1}, \tag{4.29}$$

where ϵ_n and $\bar{\epsilon}_n$ are infinitesimal constant parameters. In this way we find a natural basis in which we can express any infinitesimal conformal coordinate transformation $\{\epsilon(z), \bar{\epsilon}(\bar{z})\}$. Furthermore we readily identify the corresponding generators that should exponentiate and act on $z, \bar{z}$ as follows

$$z \to e^{\ell_n} z = z + f_n(z) + \cdots \implies \ell_n = -z^{n+1}\partial_z, \tag{4.30}$$
$$\bar{z} \to e^{\bar{\ell}_n} \bar{z} = \bar{z} + \bar{f}_n(\bar{z}) + \cdots \implies \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{4.31}$$

The commutators are readily computed, one finds

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \qquad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \qquad [\ell_m, \bar{\ell}_n] = 0, \tag{4.32}$$

that is, we have two copies of the Witt algebra.

The infinite set of generators, is what is remarkable about infinitesimal local conformal transformations in two dimensions. In two dimensions, imposing conformal symmetry therefore turns out to be rather restrictive, and a great deal can be deduced about conformal quantum field theories in two dimensions, simply from the conformal symmetry. Clearly for $D > 2$ we have a finite set of generators as indicated by the list (4.15 - 4.18).

We should stress that the generators are not *all* well defined globally. To address the issue one needs to consider not $\mathbb{C}$ but the Riemann sphere $S^2 \sim \mathbb{C} \cup \infty$. Specifically generators ℓ_n become singular at $z = 0$ for $n < -1$ and at $z = \infty$ for $n > 1$, and the same applies for the anti-holomorphic (barred) generators, in terms of $\bar{z}$. The only globally well defined conformal generators are then $\ell_0, \ell_{-1}, \ell_{+1}$ and $\bar{\ell}_0, \bar{\ell}_{-1}, \bar{\ell}_{+1}$.

It is instructive to get a geometric intuition of the transformations generated by this subset of the globally defined ℓ's and $\bar{\ell}$'s. It is straight forward to see how ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations, while for ℓ_0 and $\bar{\ell}_0$ it is useful to introduce r and ϕ by

$$z = re^{i\phi}. \tag{4.33}$$

Then it becomes clear that $\ell_0 + \bar{\ell}_0$ generates dilations, while $\ell_0 - \bar{\ell}_0$ generates rotations. We are then left with ℓ_{+1} and $\bar{\ell}_{+1}$ which generate the special conformal transformations. All together these generate transformations of the form

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{C}. \tag{4.34}$$

[2] Thus treating $\mathbb{C}^2$ instead of $\mathbb{R}^2$, and one needs to keep in mind that we are really only concerned with $z^* = \bar{z}$, i.e the z under the bar is the same z.

These transformations comprise the so called Möbius group, $PSL(2,\mathbb{C})$ which is isomorphic to

$$SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{4.35}$$

The modular invariance ($PSL(2,\mathbb{Z})$) of a two-dimensional CFT on a torus will facilitate a crucial step in the derivation of the Cardy formula which is at the end of this chapter.

4.4 The Energy-Momentum Tensor

For any conformal field theory the trace of the energy-momentum tensor has to vanish. This follows from scale invariance alone. It is instructive to see how this comes about.

Specifically a scaling transformation

$$x^\mu \to x^\mu + (\lambda - 1)x^\mu = \lambda x^\mu \tag{4.36}$$

results in a change of the metric

$$\delta g_{\mu\nu} = 2(\lambda - 1)g_{\mu\nu}. \tag{4.37}$$

To see how this relates to the energy-momentum tensor, we note that for a diffeomorphism invariant theory it is readily identified with the functional derivative of the action with respect to the metric

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\mu\nu}}. \tag{4.38}$$

Now varying the action with respect to the metric and plugging in the above metric variation, as well as using the explicit form of the energy-momentum tensor, we find

$$\delta S = \int d^2x \frac{\partial S}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = -\frac{1}{4\pi}\int d^2x\sqrt{g}\, T^{\mu\nu}2(\lambda-1)g_{\mu\nu} = -\frac{\lambda-1}{2\pi}\int d^2x\sqrt{g}\, T^\mu{}_\mu, \tag{4.39}$$

thus

$$T^\mu{}_\mu = 0 \tag{4.40}$$

if scaling is a symmetry of the action. Therefore in any conformal field theory the trace of the energy-momentum tensor has to vanish. It is curious how restrictive this condition turns out to be upon quantization.

Restricting back to the case we investigated previously, i.e conformal symmetry in two-dimensional Euclidean space, in terms of complex coordinates z and $\bar{z}$, we note that the metric reads

$$ds^2 = \tfrac{1}{2}dz \otimes d\bar{z} + \tfrac{1}{2}d\bar{z} \otimes dz, \quad \delta_{z\bar{z}} = \delta_{\bar{z}z} = \tfrac{1}{2}, \quad (\delta^{z\bar{z}} = \delta^{\bar{z}z} = 2). \tag{4.41}$$

Thus for the energy-momentum tensor to be trace less, we have to require

$$T_{z\bar{z}} + T_{\bar{z}z} = 0, \tag{4.42}$$

and since it is symmetric this implies

$$T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{4.43}$$

Conservation of the energy-momentum tensor furthermore restricts the tensor to obey

$$\partial_\mu T^{\mu\nu} = 0 \implies \partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0, \tag{4.44}$$

from which it follows that

$$T_{zz} \equiv T(z), \quad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}), \tag{4.45}$$

i.e, they are respectively holomorphic and anti-holomorphic functions. $T(z)$ and $\bar{T}(\bar{z})$ specify the energy-momentum tensor completely, as the other components are trivial.

4.5 Approaching Operators: The OPE

In conformal field theories, it is common to refer to pretty much any local expression (operator) $f(z,\bar{z})$ as a field. We will of course stick to this nomenclature throughout our discussion of CFT. The crucial nature of conformal invariance, is very much due to scale invariance. As a result CFTs can not have massive excitations, there is simply no length scale, i.e inverse mass scale to be defined. As such, the degrees of freedom are all massless. Furthermore there is no such thing as a scattering matrix, since the s-matrix is only definable when the concept of separation between states is meaningful, clearly this is not the case for scale invariant theories. Therefore, instead of scattering amplitudes, we focus on the local relations between fields, i.e correlation functions between fields (whereby fields we are really restricting to *local* operators in a CFT).

Precisely because we are interested in correlation functions between local operators, we turn to the Operator Product Expansion (OPE) of such time-ordered products of fields. The OPE gives us the precise behavior of such expressions. We are taking the limit $|z-w| \to 0$ of local operator insertions at points z and w. The OPE asserts that in this limit the product can be expressed as a linear combination of operators inserted at one of the points. Suppressing the $\langle\ \rangle$ around products of operators, and implying time-ordering, we write the OPE between two operators O_i and O_j as

$$O_i(z,\bar{z})O_j(w,\bar{w}) = \sum_k C_{ijk}(z-w,\bar{z}-\bar{w})O_k(w,\bar{w}). \tag{4.46}$$

The weights C_{ijk} take their functional form to ensure translational invariance.

Using Ward identities for the conformal currents, and invoking other consistency arguments, such as equality of the dimensionality of fields, one can deduce the general forms of OPEs. Of particular importance are the ones involving the energy-momentum tensors $T(z), \bar{T}(\bar{z})$. In particular the OPEs give us a way of labeling operators with conformal weights h and $\tilde{h}$. We say that an operators $O(z,\bar{z})$ has weight $(h,\tilde{h})$ if

$$T(z)O(w,\bar{w}) = \cdots + h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w} + \cdots \tag{4.47}$$

$$T(\bar{z})O(w,\bar{w}) = \cdots + \tilde{h}\frac{O(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} O(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots \tag{4.48}$$

where '$\cdots$' indicate possible other singular as well as non-singular terms. The conformal weights h and $\tilde{h}$ are related to the more intuitive spin s (eigenvalue under rotation), and scaling dimension Δ (eigenvalue under scaling)

$$s = h - \tilde{h}, \qquad \Delta = h + \tilde{h}. \tag{4.49}$$

One also refers to local operators O whose OPE with the energy-momentum tensor takes the form

$$T(z)O(w,\bar{w}) = h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w} + \text{non-singular} \tag{4.50}$$

$$T(\bar{z})O(w,\bar{w}) = \tilde{h}\frac{O(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} O(w,\bar{w})}{\bar{z}-\bar{w}} + \text{non-singular} \tag{4.51}$$

as *primary* fields. These are fields from which all other's can be generated by acting with raising operators in the conformal algebra. They are so called highest weight states, annihilated by lowering operators in the appropriate representation. Finally we give the OPE of the energy-momentum tensor with itself, the TT-OPE reads

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{non-singular}, \tag{4.52}$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \text{non-singular.} \tag{4.53}$$

Of particular importance, and of special interest to us, are the central charges c and $\tilde{c}$, that when turned on, ensure that the energy-momentum tensor is not a primary field. An often used example, and indeed a very useful study, is that of the conformal mapping

$$w \to z = e^{-iw}, \quad w = \sigma + i\tau, \quad \sigma \in [0, 2\pi) \tag{4.54}$$

that maps points on the cylinder w to the plane z. Indeed, working out in detail how the energy-momentum tensor transforms under such a mapping (which of course lives in the space of conformal transformations), gives the result

$$T_{\text{cyl}}(w) = -z^2 T_{\text{plane}}(z) + \tfrac{c}{24}, \tag{4.55}$$

and equivalent for $\bar{T}(\bar{z})$. From this one identifies the negative Casimir energy associated with the cylinder to

$$-2\pi(c+\tilde{c})/24. \tag{4.56}$$

Another important appearance that c and $\tilde{c}$ make is in the expectation value of the trace of the energy-momentum tensor

$$\langle T^{\mu}{}_{\mu}\rangle = -\frac{c}{12}R \tag{4.57}$$

where R is the Ricci scalar[3]. In general this equality is referred to as the Weyl anomaly, and for good reason, we recall that the trace must vanish in any conformal, i.e 'Weyl' invariant theory, and here we see that the charges c and $\tilde{c}$ as well as R have appeared on the RHS instead of *zero*, hence making conformal invariance at the quantum level, rather restrictive! Moreover, it is this Weyl anomaly that breaks the symmetry of the classical theory upon quantization.

As suggested by these remarks, the central charges are very 'central'. They encode key features of the conformal theory at hand. As a major goal of this chapter, we wish to heuristically derive, and argue for its intimate relation to entropy. However, for both fluidity and to some extent completeness, we wish to discuss a particularly nice quantization procedure, well suited for 2D CFTs.

4.6 Quantizing a 2D CFT

Indeed it is upon quantizing a theory with conformal invariance that the fun and games begin. The tricky part of any classical theory, is whether or not its symmetries survive quantization. Indeed there turn out to be very few unitary quantum field theories that are conformal at the quantum level in dimensions greater than two, a unique and important example is $\mathcal{N}=4$ super-Yang-Mills which is a superconformal field theory in four-dimensional spacetime. There are however many two-dimensional unitary CFTs.

What we would like to in this section, is to shed light on the Virasoro algebra, that appears as a centrally extended Witt algebra, with central charges. The central charges are very important for our purposes, and will appear as key ingredients in the CFT tools that we will employ throughout the remainder of this thesis.

Radial quantization, as it's name suggest involves a mapping of the space under consideration such that time ordering translates to radial ordering. Using the same mapping as in the previous section,

$$w \to z = e^{-iw}, \quad w = \sigma + i\tau, \quad \sigma \in [0, 2\pi), \tag{4.58}$$

[3] Why c and not $\tilde{c}$? Actually, we could just as well have written $\tilde{c}$. In fact it only makes sense to have $c = \tilde{c}$ when $R \neq 0$, else one has a so called gravitational anomaly.

we relate the theory on the cylinder naturally mapped by $w = \tau + i\sigma$ to the theory on the plane z and vice versa. On the cylinder, time τ is open and extends from $-\infty$ to ∞ while space σ is compact. Note that due to the factor i in front of the argument in the exponential, constant τ maps to circles with radius e^{τ}, while σ parameterizes these circles. In this way constant time, i.e space-like slices of the cylinder, are mapped to circles of a given radius in the plane as illustrated in figure 4.1.

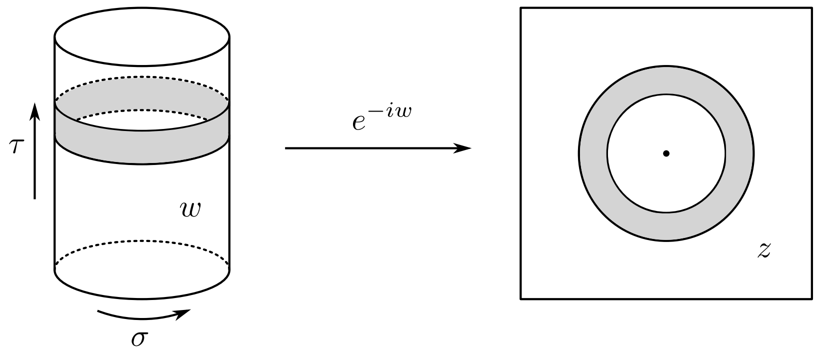

Figure 4.1: Illustrating the mapping of the cylinder to the plane. The shaded region on the cylinder gets mapped onto the shaded region in the plane.

Due to the periodic nature of the compact direction σ it is natural to write the Fourier decomposition

$$T_{\text{cyl}}(w) = -\sum_{n=-\infty}^{+\infty} L_n\, e^{-iwn} + \frac{c}{24} \tag{4.59}$$

where the signs and the constant central charge term, are added such that we from (4.55) get as a definition for the operators L_n:

$$T_{\text{plane}}(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}. \tag{4.60}$$

Of course we implicitly have identical expressions for the anti-holomorphic (right-moving) functions $\bar{T}(\bar{z})$. It follows from the Cauchy integral formula, that upon integrating both sides around a suitable contour enclosing the singularity, we effectively invert the equation, and arrive at an expression for L_n in terms of $T(z)$, where we now drop the subscript *plane*.

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z), \qquad \tilde{L}_m = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}) \tag{4.61}$$

Now we can in principle dive into evaluating the commutator, between modes, L_m and L_n. Let us do just that! In order to do so leniently, let us play with the commutator of contour integrals.

$$[\oint dz, \oint dw] \tag{4.62}$$

Without further ado, we choose to perform the z contour while keeping w fixed. Due to the radial ordering, translating into, operators to the left are at greater radii, and operators to the right are at smaller radii, we get a contour containing w subtracted by a contour not containing w. Deforming the contours, we identify performing the z integration while keeping w fixed, as the single contour tightly wound around w.

As shown in figure 4.2, the commutator amounts to taking the z contour tightly around the point w. Clearly, acting on some expression $\star$ with the commutator, one has

$$[\oint dz, \oint dw][\star] = 2\pi i \oint dw\, \text{Res}[\star]\Big|_{z=w}. \tag{4.63}$$

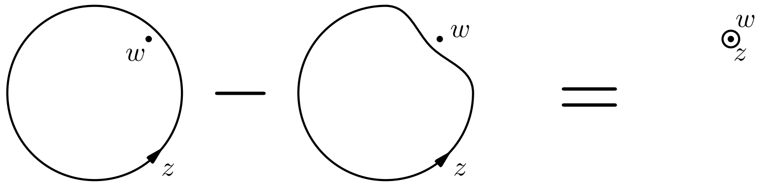

Figure 4.2: The difference of the contours gives a contour winding anticlockwise tightly around w.

Thus we find from the expressions for the modes L_n in terms of the energy-momentum tensor, that

$$[L_n, L_m] = \frac{1}{2\pi i}\oint dw \,\mathrm{Res}[z^{n+1}w^{m+1}T(z)T(w)]\Big|_{z=w}. \tag{4.64}$$

Given the TT-OPE, and residue calculus, we get contributions only from the singular terms in the OPE. To ease reading of this section we display the TT-OPE once again

$$T(z)T(w) = \frac{c/2}{(z-w)^3} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots. \tag{4.65}$$

Evaluating the residues associated with each of the poles (of different order) at w then yields

$$\begin{aligned}[L_n, L_m] = \frac{1}{2\pi i}\oint dw \,\Big(&\frac{c}{12}(n+1)n(n-1)w^{n+m-1} \\ &+ 2(n+1)w^{n+m+1}T(w) + w^{n+m+2}\partial T(w)\Big).\end{aligned} \tag{4.66}$$

Integrating by parts to move the derivative in front of $T(w)$ to act on the factor w^{n+m+2}. The term involving the central charge only contributes when $n+m=0$, as that is when it has a simple pole at the origin. The other terms are readily identified with a multiple of one of the modes, and we end up with having found that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{(n-m),0} \tag{4.67}$$

and it then follows in exactly the same way that

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{\tilde{c}}{12}n(n^2-1)\delta_{(n-m),0}, \tag{4.68}$$

and since z and $\bar{z}$ can be treated as independent

$$[L_n, \tilde{L}_m] = 0. \tag{4.69}$$

These commutators specify the algebra, namely the Virasoro algebra, the centrally extended Witt algebra.

4.7 The Cardy Formula

All this business with CFT has up to this point, not really presented us with any unique tools specific to black hole physics. In this section we shall remedy this, by introducing the Cardy formula. We will discover that the central charges encode information about the CFTs asymptotic density of states (density of states for high energy), which in turn gives us the entropy.

One can readily compute the central charges of a theory with N non-interacting free bosonic degrees of freedom to be $c = \tilde{c} = N$. Clearly this indicates that c in some sense encodes the number of degrees of freedom in the CFT. Of course this is simply a vague

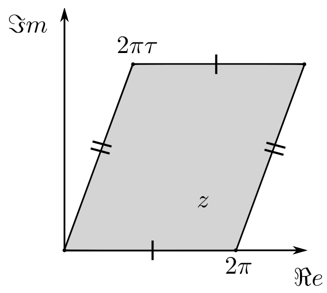

Figure 4.3: This parallelogram illustrates the fundamental domain of the torus, corresponding to a given modular parameter τ.

suggestive argument, which gets firmer ground in the derivation of the Cardy formula that is to follow.

We will not go into extreme detail, and this is merely an outline of the derivation. In short it comes about from a saddle-point approximation of the CFTs partition function on a torus, which is only valid in the regime $c_{\text{eff}}h_{\text{eff}} \gg 1$ and $24h_{\text{eff}}/c_{\text{eff}} \gg 1$. These are the effective charges and conformal weights which will be defined in a moment. In sketching the derivation we found appendix B of [42] very useful.

One considers a CFT on a torus with modular parameter τ, (not to be confused with the timelike coordinate on the cylinder), that is, we compactify the complex z plane by the identifications

$$z \sim z + 2\pi, \qquad z \sim z + 2\pi\tau, \qquad \tau \equiv \tau_1 + i\tau_2 \tag{4.70}$$

where both τ_1 and τ_2 are taken to be positive reals. It is instructive to visualize these identifications as a parallelogram in the complex plane, by simply shifting the real interval $[0, 2\pi)$ by $2\pi\tau$ as illustrated in figure 4.3.

Working in Euclidean signature, the periodicity allows us to identify the correlation function with the partition function at inverse temperature given by the periodicity in the time like coordinate. In terms of the independent moduli τ and $\bar{\tau}$, the partition function reads [7]

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \mathrm{Tr}\left(e^{2\pi i\tau(L_0 - c/24)}e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \tilde{c}/24)}\right) \\ &= \sum_{\phi}\langle\phi|\left(e^{2\pi i\tau(L_0 - c/24)}e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \tilde{c}/24)}\right)|\phi\rangle \\ &= \sum_{h,\tilde{h}}\rho(h,\tilde{h})e^{2\pi i\tau(h - c/24)}e^{-2\pi i\bar{\tau}(\tilde{h} - \tilde{c}/24)}, \end{aligned} \tag{4.71}$$

where $\rho(h, \tilde{h})$ is the density of states, i.e the number of states with given conformal weights $h, \tilde{h}$. The partition function is identical to a discrete Laplace transformation of the density of states. Introducing q and $\bar{q}$

$$q = e^{2\pi i\tau}, \quad \bar{q} = e^{-2\pi i\bar{\tau}}, \tag{4.72}$$

one has

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}}\rho(h,\tilde{h})q^{(h-c/24)}\bar{q}^{(\tilde{h}-\tilde{c}/24)}. \tag{4.73}$$

By the inverse transformation we find

$$\rho(h,\tilde{h}) = \frac{1}{(2\pi i)^2}\oint dq \oint d\bar{q}\, q^{-(h-c/24+1)}\bar{q}^{-(\tilde{h}-\tilde{c}/24+1)}Z(\tau,\bar{\tau}) \tag{4.74}$$

$$= \oint_C d\tau \oint_{\tilde{C}} d\bar{\tau}\, e^{-2\pi i\tau(h-c/24)}e^{2\pi i\bar{\tau}(\tilde{h}-\tilde{c}/24+1)}Z(\tau,\bar{\tau}), \tag{4.75}$$

where C and $\tilde{C}$ are respectively the contours around the origins of the complex planes mapped by q and $\bar{q}$.

In order to use the method of steepest descent, i.e approximating the integral by the saddle-point, we need the integrand to factor into a slowly varying prefactor and a rapidly varying phase. This can be arranged!

For simplicity we continue the calculation, only for the τ dependence, suppressing $\bar{\tau}$. The calculation for the $\bar{\tau}$ dependence follows identically. We would then like to assert that $Z(\tau)$ is a slowly varying factor, as we approach high temperature, i.e when $1/\tau_2 \to \infty$ in other words, when $\tau \to i0$. Now is the time to invoke a trick, namely the modular invariance of the CFT (4.34), specifically the invariance under $\tau \to -1/\tau$, $\bar{\tau} \to -1/\bar{\tau}$ which implies that the partition function better be invariant as well. Suppressing the $\bar{\tau}$ dependence, we then write, by modular invariance, that

$$Z(\tau) = Z(-1/\tau). \tag{4.76}$$

We have then, quite remarkably, that the high temperature limit is identified with the low temperature limit. In other words $\tau \to i0$ has translated into the limit $-1/\tau \to -i\infty$. Looking back at the τ dependence of $Z(\tau)$, we see that indeed, in this limit $Z(-1/\tau)$ exponentially suppresses all states with weight different from a minimum value h_{min}. We therefore write

$$Z(-1/\tau) = e^{-2\pi i(h_{\text{min}}-c/24)/\tau}\tilde{Z}(-1/\tau), \qquad \tilde{Z}(-1/\tau) = \text{Tr}\left[e^{-2\pi i(L_0-h_{\text{min}})/\tau}\right]. \tag{4.77}$$

In this way we have managed to split the partition function into a slowly varying factor $\tilde{Z}(-1/\tau)$ and a rapidly oscillating phase. Let us write down the integral once more, now explicitly in terms of the slowly varying and rapidly oscillating factors, and suppressing $\bar{\tau}$ dependence,

$$\rho(h) = \oint_C d\tau e^{-2\pi i f(\tau)}\tilde{Z}(-1/\tau), \qquad f(\tau) \equiv (h - c/24)\tau + (h_{\text{min}} - c/24)/\tau. \tag{4.78}$$

So far no approximations have been made, it is an exact statement. However, we would like to find a closed form for the asymptotic density of states, and we do so, by a saddle-point approximation. The function $\tilde{Z}(-1/\tau)$ is by construction slowly varying and in fact equal to unity in the limit $\tau \to i0$. We therefore pull this factor of 1 outside the integral, effectively replacing $\tilde{Z}(-1/\tau)$ by 1. The integral that remains is the one of the oscillating phase, which in accordance with the saddle-point approximation is approximated by only taking into account the contributions coming from the minimum of $f(\tau)$. The minimum of $f(\tau)$ is clearly at

$$\tau \equiv \tau_0 = i\sqrt{\frac{c/24 - h_{\text{min}}}{h - c/24}} \equiv i\sqrt{\frac{c_{\text{eff}}}{24h_{\text{eff}}}}, \tag{4.79}$$

where in order to reduce the number of variables we have introduced $c_{\text{eff}} = c - 24h_{\text{min}}$ and $h_{\text{eff}} = h - c/24$. We notice that in order to justify the limit $\tau \to i0$ we must have

$$24h_{\text{eff}}/c_{\text{eff}} \gg 1. \tag{4.80}$$

We must also have

$$h_{\text{eff}}c_{\text{eff}} \gg 1, \tag{4.81}$$

in order for the rapidly oscillating phase to be rapidly oscillating and to be dominating in the integral.

Expanding $f(\tau)$ around its minimum

$$f(\tau) = f(\tau_0) + \tfrac{1}{2}f''(\tau_0)(\tau - \tau_0)^2 + \cdots, \tag{4.82}$$

we get to leading order in $(\tau - \tau_0)$, by evaluating the Gaussian integral

$$\rho(h) \approx \exp\left(2\pi\sqrt{\frac{c_{\text{eff}}h_{\text{eff}}}{6}}\right)\left(\frac{c_{\text{eff}}}{96h_{\text{eff}}^3}\right)^{1/4}. \tag{4.83}$$

It is possible to take higher order terms into account, but it is usually sufficient to stop at this order. In fact, one is usually satisfied with identifying the entropy with

$$S = \log\rho(h,\tilde{h}) = 2\pi\sqrt{\frac{c_{\text{eff}}h_{\text{eff}}}{6}} + 2\pi\sqrt{\frac{\tilde{c}_{\text{eff}}\tilde{h}_{\text{eff}}}{6}}, \tag{4.84}$$

i.e ignoring the logarithmic correction(s). Here we have included the full expression in terms of the right-moving degrees of freedom as well. This is referred to as the Cardy formula.

Transforming to the canonical ensemble [15], we can express the entropy (the Cardy formula) in terms of left and right temperatures defined by

$$\left(\frac{\partial S}{\partial h_{\text{eff}}}\right)_{\tilde{h}_{\text{eff}}} = \frac{1}{T_L}, \qquad \left(\frac{\partial S}{\partial \tilde{h}_{\text{eff}}}\right)_{h_{\text{eff}}} = \frac{1}{T_R}. \tag{4.85}$$

To save writing subscript 'eff', let us dub

$$c_L \equiv c_{\text{eff}}, \qquad c_R \equiv \tilde{c}_{\text{eff}}. \tag{4.86}$$

Then, in terms of the above defined left and right effective temperatures, we have the Cardy formula expressed as

$$S = \tfrac{1}{3}\pi^2(c_L T_L + c_R T_R). \tag{4.87}$$

The region of validity in terms of these temperatures, as dictated by $h_{\text{eff}}c_{\text{eff}} \gg 1$ and $24h_{\text{eff}}/c_{\text{eff}} \gg 1$ reads

$$T_L \gg 1/2\pi, \qquad c_L T_L \gg \sqrt{6}/\pi, \tag{4.88}$$

and of course the same applies for the right movers

$$T_R \gg 1/2\pi, \qquad c_R T_R \gg \sqrt{6}/\pi. \tag{4.89}$$

5

Subtracted Geometry - Hidden Symmetry

In this chapter we briefly review hidden conformal symmetry for the Kerr black hole and focus on the subtracted geometry approach to revealing hidden conformal symmetries for asymptotically flat black holes in the STU model. We will see that the black hole thermodynamics are independent of a certain warp factor Δ_0. Essentially it is the insensitivity of black hole thermodynamics to changes in Δ_0, that allows for subtracted geometry. We will derive a separability condition for general warp factors Δ that gives the radial equation a hypergeometric form. We find a warp factor of particular interest, namely Δ_{NHEK}. Furthermore we study the large r behavior of subtracted geometries and address the extent to which we may interpret subtracted geometries as placing the black hole in a confining box. Finally we present the generators of $SL(2,\mathbb{R})^2$ relevant in the minimally subtracted case.

5.1 Introducing Subtracted Geometry

The entropy matching is striking for *extremal* black holes, and is suggestive of a dual CFT description. The Kerr/CFT correspondence conjectured for extremal Kerr black holes is a prime example. In [27] the near-horizon extremal Kerr (NHEK) geometry with enhanced isometry group $SL(2,\mathbb{R}) \times U(1)$ is found to possess a set of BCs for metric fluctuations, whose asymptotic symmetry group (ASG) enhances the $U(1)$ to a single copy of the Virasoro algebra with central charge $c_L = 12J$. It would be a triumph to achieve similar results for black holes away from extremality. However, this has shown to be problematic; the symmetries inherent in the near-horizon region for extremal geometries are in general not present away from extremality. As an example, the NHEK region which facilitates the Kerr/CFT correspondence in [27] is no longer present. Instead, the near-horizon geometry is simply Rindler space, for which a dual CFT is currently out of reach [11]. It is also clear that non-extremal Kerr black holes have both the left and right moving sector turned on, while extremal Kerr is chiral. The fact that there are no consistent boundary conditions that allow for both left and right movers [25] goes to show how the methods of [27] cannot be applied for non-extremal black holes.

The trouble with a CFT description away from extremality is the lack of symmetry in the geometry. Conformal symmetry of the geometry is sufficient for conformal symmetry of the inhabiting fields, however it is not necessary. When field equations have the symmetry, while the geometry does not, we call it hidden. Indeed there is a hidden conformal symmetry for Kerr black holes as noted in [11]. The symmetry is apparent for massless scalar fields[1] whose wave equation (in a certain regime[2]) is solved by hypergeometric functions. Hypergeometric functions transform under $SL(2,\mathbb{R})$ indicating conformal symmetry.

In the Kerr case [11] the hidden symmetry is only apparent in a certain limit when the offending term proportional to ω^2 can be discarded. It would be great if such a simplification could be circumvented. This is what is achieved in the subtracted geometry approach to hidden conformal symmetry.

[1]This also appears for higher spin and fermionic cousins.

[2]The offending term is proportional to ω^2, the authors consider a regime in which this term is safely ignored

The so called *subtracted* geometries were introduced by Mirjam Cvetiĉ and Finn Larsen in [19, 20]. They find that the thermodynamic potentials of the black hole and the causal structure are independent of a certain warp factor Δ_0. They interpret this independence to mean that the warp factor encodes information about the spacetime surrounding the black hole, suggesting that one may consider alternate warp factors to study the same black hole.

They also argue more generally that it is the negative specific heat characteristic of generic black holes that obstructs a CFT interpretation, since unitary CFTs always have positive specific heat [20]. Furthermore the fact that this is related to the physical coupling between the internal structure of the black hole and modes that escape to infinity, suggests that one needs to put the black hole in a reflecting cavity (a system in equilibrium) in order to study the black hole by itself. Indeed the subtraction procedure effectively facilitates this, by altering the asymptotic form of the metric, in particular one finds that the subtracted geometries are asymptotically conical [16]. Furthermore one finds that it is necessary to alter the matter as the subtracted geometry by itself does not satisfy the equations of motion. One may then loosely interpret this matter as supporting the confining box.

A guiding principle in the subtraction procedure is the requirement of separability motivated by the separability of the massless wave equation for the original black hole. Generically one therefore looks for warp factors that render the wave equation separable. However, more importantly the goal is to identify warp factors that in addition give a wave equation that can be mapped to the hypergeometric equation, thus revealing hidden conformal symmetry.

Before diving into the details of [19] we give a list of the requirements imposed on the possible generic warp factors Δ that define a subtracted geometry a priori:

1. Δ must render the massless wave equation separable.

2. Δ needs to give the massless wave equation a hypergeometric form.

3. Δ cannot contain singularities distinct from the event horizons and $r = \infty$.

It is important to stress that whatever warp factor we choose, the thermodynamics are implicitly unchanged as shown in [19, 20], and that the above restrictions are imposed to ensure separability and hypergeometric structure in the wave equation. Actually the last point regarding the singularities is not only motivated physically, but also goes hand in hand with a hypergeometric radial equation, we will get to these salient points eventually.

We also stress that it is the $SL(2,\mathbb{R})$ invariance of the hypergeometric radial equation that indicates conformal symmetry. Indeed this invariance is the precursor for identifying a Virasoro algebra in the general subtracted geometry [20].

In what follows we will be considering a particular family of four-dimensional black holes in the $\mathcal{N} = 2$ minimal supergravity coupled to three vector multiplets, the STU model. We note that Cvetič and Larsen originally considered five-dimensional black holes in [20]. A lot of the work in [20] parallels that in [19] where they consider four-dimensional black holes, and we will stick to the latter in this thesis. The four-dimensional black holes are of course of special interest due to them being four-dimensional.

5.2 The Four-Dimensional Black Holes

In [19] the stage is set by the general rotating four-dimensional black holes from string theory with four independent $U(1)$ charges Q_I, $I = 0, 1, 2, 3$, mass M and angular momentum J

parameterized by m, a and δ_I via

$$G_4 M = \tfrac{1}{4} m \sum_{I=0}^{3} \cosh 2\delta_I, \tag{5.1}$$

$$G_4 Q_I = \tfrac{1}{4} m \sinh 2\delta_I, \tag{5.2}$$

$$G_4 J = ma(\Pi_c - \Pi_s), \tag{5.3}$$

where G_4 is the four-dimensional Newton constant, and

$$\Pi_c \equiv \prod_{I=0}^{3} \cosh \delta_I, \qquad \Pi_s \equiv \prod_{I=0}^{3} \sinh \delta_I. \tag{5.4}$$

We note that the δ_I are dimensionless, while both m and a have dimensions of length.

The metric reads

$$ds_4^2 = -\Delta_0^{-1/2} G(dt + \mathcal{A})^2 + \Delta_0^{1/2} \left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2\theta \, d\phi^2 \right), \tag{5.5}$$

where

$$X = r^2 - 2mr + a^2, \tag{5.6}$$

$$G = r^2 - 2mr + a^2 \cos^2\theta, \tag{5.7}$$

$$\mathcal{A} = \frac{2ma \sin^2\theta}{G} [(\Pi_c - \Pi_s) r + 2m\Pi_s] d\phi, \tag{5.8}$$

$$\begin{aligned} \Delta_0 = {} & \prod_{I=0}^{3} (r + 2m \sinh^2\delta_I) + 2a^2 \cos^2\theta \Bigg[r^2 + mr \sum_{I=0}^{3} \sinh^2\delta_I + 4m^2 (\Pi_c - \Pi_s)\Pi_s \\ & - 2m^2 \sum_{I<J<K} \sinh^2\delta_I \sinh^2\delta_J \sinh^2\delta_K \Bigg] + a^4 \cos^4\theta. \end{aligned} \tag{5.9}$$

The original warp factor Δ_0 simplifies greatly when all charges vanish, then we have the purely rotating geometry with

$$\Delta_0 = (r^2 + a^2 \cos^2\theta)^2. \tag{5.10}$$

A derivation of this family of black holes and the accompanying matter fields can be found in [13]. Details regarding the STU model can be found in appendix C.

Having set the stage, we investigate the physical and thermodynamic quantities of these black holes. What we need to show is that the physical and thermodynamical properties ascribed to the black hole, are independent of Δ_0.

We start by identifying the ergosphere, this is the region outside the event horizon, and enclosed by the static-limit surface. The static limit surface is the locus of points in the spacetime for which trajectories along the Killing direction parameterized by t cease to be time-like, i.e where $g_{tt} = 0$. It is clear from (5.5) that this surface is where $G = 0$. The event horizon, in the clever coordinates chosen, can simply be identified as the constant r hypersurface that is everywhere null. This is where $g^{rr} = 0$, and hence where $X = 0$. As a check we see that the points for which $X = 0$ are inside the static limit surface $G = 0$. Specifically the outer and inner horizons are two-spheres of radius

$$r_\pm = m \pm \sqrt{m^2 - a^2}. \tag{5.11}$$

When moving on to study the thermodynamic potentials, we will keep it slightly more general, considering arbitrary[3] $X, \Delta_0, \mathcal{A}_\phi$ and G specified via (5.7). Following [19], we write the metric in the form

$$ds_4^2 = \Delta_0^{1/2}\left(\frac{dr^2}{X} + X\frac{\sin^2\theta}{G}d\phi^2\right) + \left[\Delta_0^{1/2}d\theta^2 - \Delta_0^{-1/2}G(dt + \mathcal{A}_\phi d\phi)^2\right], \tag{5.12}$$

where it is now apparent that when near the horizon $X \sim 0$ and $G \sim -a^2\sin^2\theta$, the part enclosed in round brackets has Lorentzian signature, and indeed can be interpreted as Rindler space:

$$\frac{dr^2}{X} - \frac{X}{a^2}d\phi^2. \tag{5.13}$$

Wick rotating to Euclidean signature $\tilde{\phi} = i\phi$ and employing a new radial coordinate $\rho = 2\sqrt{X}$, we find

$$\left.\frac{d\rho^2}{(\partial_r X)^2}\right|_{r=r_+} + \rho^2\frac{d\tilde{\phi}^2}{4a^2}. \tag{5.14}$$

Now $\tilde{\phi}$ must have period

$$\beta_\phi = \left.\frac{4\pi a}{\partial_r X}\right|_{r=r_+}, \tag{5.15}$$

to avoid a conical singularity. To determine the Euclidean periodicity of the asymptotic time t, we use the fact that β_ϕ should take on the same value everywhere on the event horizon. Furthermore, seeing that the geometry of the event horizon is encoded in the square brackets of equation (5.12), we see that

$$dt + \mathcal{A}_\phi\, d\phi \tag{5.16}$$

needs to be unchanged as ϕ is periodically identified. This determines the Euclidean periodicity of t (and thus the Hawking temperature) to

$$\beta_H = -\mathcal{A}_\phi\Big|_{r=r_+}\beta_\phi. \tag{5.17}$$

Since $\mathcal{A}_\phi$ may in general depend non-trivially on θ, it is not immediately clear that the inverse temperature is constant over the horizon, as it should be. To remedy this it is useful to introduce the reduced potential

$$\mathcal{A}_{\text{red}} \equiv \frac{G}{a\sin^2\theta}\mathcal{A}_\phi, \tag{5.18}$$

which depends on r only, and at the horizon coincides with $-a\mathcal{A}_\phi$, whose θ independence at $r = r_+$ is unclear in general, but certainly holds when $\mathcal{A}_\phi$ is given by (5.8). In general we should have that $\mathcal{A}_\phi$ reduces to an expression independent of θ at the horizon $r = r^+$. Thus we express the thermodynamic potentials in terms of a, in general unspecified reduced potential $\mathcal{A}_{\text{red}}$ which in our case is defined by (5.18).

We also have that these periodicities are related to the rotational velocity Ω_H, that is

$$\Omega_H = \frac{\beta_\phi}{\beta_t} = -\left.\frac{1}{\mathcal{A}_\phi}\right|_{r=r_+}. \tag{5.19}$$

Finally the area of the outer horizon r_+ is found by the integral

$$A_+ = \int_{r=r_+}\sqrt{-G\mathcal{A}_\phi^2}\,d\phi\,d\theta = \int |\mathcal{A}_{\text{red}}|\Big|_{r=r_+}\sin\theta\,d\theta\,d\phi = 4\pi\,|\mathcal{A}_{\text{red}}|\Big|_{r=r_+}, \tag{5.20}$$

[3]We will however restrict to non-extremal black holes, for which $X = 0$ is a simple pole in $X(r)$.

where the measure is read off from (5.12). This concludes our short inspection of the thermodynamics. The main lesson is that the locations of the static-limit surface as well as the event horizon are independent of Δ_0. Moreover the thermodynamic quantities; Hawking temperature, horizon area (entropy) and the rotational velocity of the horizon are all independent of Δ_0. This is what justifies the interpretation of Δ_0 as encoding information about the geometry in the vicinity of the black hole, i.e its asymptotics, rather than information about the black hole itself.

5.3 Separability of the Massless Wave Equation

We want to find out the restrictions set on possible warp factors Δ, when demanding a separable wave equation. We begin by showing that the wave equation is separable for the original warp factor Δ_0, and afterwards find a condition for separability for general Δ. We will also write down the radial and angular equations that result from separation of variables.

The wave equation, specifically the Klein-Gordon equation for a scalar field of mass $\tilde{M}$ reads

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Phi - \tilde{M}^2\Phi = 0. \tag{5.21}$$

To start off with, we again only employ the generic dependence on the various functions, i.e in terms of the functions X, Δ and $\mathcal{A}_\phi$ and using the relation

$$G = X - a^2\sin^2\theta. \tag{5.22}$$

We employ Δ instead of Δ_0, leaving open the possibility of $\Delta \neq \Delta_0$. In terms of these functions the metric components read

$$g_{\mu\nu} = \Delta^{-1/2}\begin{bmatrix} -G & & & -G\mathcal{A}_\phi \\ & \frac{\Delta}{X} & & \\ & & \Delta & \\ -G\mathcal{A}_\phi & & & \frac{\Delta X}{G}\sin^2\theta - G\mathcal{A}_\phi^2 \end{bmatrix}. \tag{5.23}$$

Due to the block diagonal like form of the corresponding matrix, we find the determinant to be simply the product

$$g = g_{rr}g_{\theta\theta}(g_{tt}g_{\phi\phi} - g_{t\phi}g_{\phi t}) = -\Delta\sin^2\theta. \tag{5.24}$$

Similarly, finding the inverse amounts to inverting g_{rr} and $g_{\theta\theta}$ while the $(t-\phi)$ block, is also easily inverted

$$g^{\mu\nu} = \Delta^{1/2}\begin{bmatrix} \frac{G}{\Delta}\frac{\mathcal{A}_\phi^2}{X\sin^2\theta} - \frac{1}{G} & & & -\frac{G\mathcal{A}_\phi}{\Delta X\sin^2\theta} \\ & \frac{X}{\Delta} & & \\ & & \frac{1}{\Delta} & \\ -\frac{G\mathcal{A}_\phi}{\Delta X\sin^2\theta} & & & \frac{G}{\Delta X\sin^2\theta} \end{bmatrix}. \tag{5.25}$$

Since the components of the metric only depend on r and θ, we get the Laplacian

$$g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu})\partial_\nu, \tag{5.26}$$

with the last term only getting contributions from terms with $\mu = r, \theta$. Computing gives

$$g^{\mu\nu}\partial_\mu\partial_\nu = \Delta^{-1/2}\left[-\frac{\Delta}{G}\partial_t^2 + X\partial_r^2 + \partial_\theta^2 + \frac{G}{X\sin^2\theta}\left(\mathcal{A}_\phi\partial_t - \partial_\phi\right)^2\right], \tag{5.27}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu})\partial_\nu = \Delta^{-1/2}\left[(\partial_r X)\partial_r + \frac{1}{\sin\theta}(\partial_\theta\sin\theta)\partial_\theta\right]. \tag{5.28}$$

Thus we find that the wave equation reads

$$\Delta^{-1/2}\left[-\frac{\Delta}{G}\partial_t^2+\partial_r X\partial_r+\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta+\frac{G}{X\sin^2\theta}\big(\mathcal{A}_\phi\partial_t-\partial_\phi\big)^2\right]\Phi-\tilde{M}^2\Phi=0. \quad (5.29)$$

In general it is the first and last term inside the square brackets of (5.29) that prevent separability, i.e these terms generally prevent scalar probes of the form

$$\Phi(t,r,\theta,\phi)=R(r)Y(\theta,\phi,t), \quad (5.30)$$

since the effective potential may couple the different coordinates "$V(x,y)\neq V_x(x)+V_y(y)$".

However, using $G=X-a^2\sin^2\theta$ and employing the reduced potential (5.18), the Laplacian simplifies to

$$\Delta^{-1/2}\left[-\frac{1}{X}(\mathcal{A}_{\mathrm{red}}\partial_t+a\partial_\phi)^2+\partial_r X\partial_r+\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta+\frac{1}{\sin^2\theta}\partial_\phi^2+\frac{\mathcal{A}_{\mathrm{red}}^2-\Delta}{G}\partial_t^2\right]. \quad (5.31)$$

Furthermore, plugging in the explicit expressions for G and $\mathcal{A}_\phi$, and setting $\Delta=\Delta_0$, we see that the massless wave equation becomes separable, since

$$\mathcal{A}_{\mathrm{red}}=2m[(\Pi_c-\Pi_s)r+2m\Pi_s] \quad (5.32)$$

and

$$\Delta_0-\mathcal{A}_{\mathrm{red}}^2=G\Bigg[r^2+2mr\left(1+\sum_{I=0}^{3}s_I^2\right)+8m^2(\Pi_c-\Pi_s)\Pi_s$$
$$-4m^2\sum_{I<J<K}s_I^2s_J^2s_K^2+a^2\cos^2\theta\Bigg], \quad (5.33)$$

where $s_I^2\equiv\sinh^2\delta_I$. Clearly the terms potentially spoiling separability are now seen not to do so. Equation (5.33) is simply a polynomial in r with "constant" term depending on θ (the factor of G in front is divided out in the Laplacian (5.31)). Thus the original warp factor Δ_0 indeed realizes a separable massless wave equation.

From what we have found so far we can conclude that for an arbitrary warp factor Δ, the massless wave equation is separable provided that

$$\frac{\Delta-\mathcal{A}_{\mathrm{red}}^2}{G}=f_r(r)+f_\theta(\theta), \quad (5.34)$$

where f_r and f_θ are arbitrary up to some constraints which we will elaborate on in a moment.

Indeed, to investigate the situation further, we should separate the differential equation via the method of separation of variables. The general warp factor Δ is taken to be independent of t and ϕ, in general we therefore have that ∂_t and ∂_ϕ are Killing vectors of the spacetime (5.5), and expanding in eigenmodes

$$\Phi(t,r,\theta,\phi)=\Phi(r,\theta)e^{-i\omega t+im_\phi\phi}, \quad (5.35)$$

the Laplacian (5.31) is found to read

$$\Delta^{-1/2}\left[\frac{1}{X}(\mathcal{A}_{\mathrm{red}}\omega-am_\phi)^2+\partial_r X\partial_r+\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta-\frac{m_\phi^2}{\sin^2\theta}+\frac{\Delta-\mathcal{A}_{\mathrm{red}}^2}{G}\omega^2\right]. \quad (5.36)$$

If this is separable, we should be able to extract two independent differential equations, one for the r dependence and one for θ dependence. Writing

$$\Phi(r,\theta)=R(r)S(\theta), \quad (5.37)$$

noting that the overall factor of $\Delta^{-1/2}$ plays no role for massless scalars, and dividing by $R(r)$ and $S(\theta)$, one finds that the massless wave equation, (5.29) with $\tilde{M} = 0$, implies

$$X\frac{R''}{R} + X'\frac{R'}{R} + \frac{S''}{S} + \frac{\cos\theta}{\sin\theta}\frac{S'}{S} + V(r,\theta) = 0, \tag{5.38}$$

where

$$V(r,\theta) = \frac{1}{X}(\mathcal{A}_{\text{red}}\omega - am_\phi)^2 - \frac{m_\phi^2}{\sin^2\theta} + \frac{\Delta - \mathcal{A}_{\text{red}}^2}{G}\omega^2 \equiv V_r(r) + V_\theta(\theta), \tag{5.39}$$

and primes on R and S denote differentiation with respect to r and θ respectively. Clearly all terms are either constants or depend solely on either the radial or angular coordinate given that Δ satisfies (5.34). This allows us to separate r and θ dependence such that equation (5.38) takes the form

$$X\frac{R''}{R} + X'\frac{R'}{R} + V_r(r) = -\frac{S''}{S} - \frac{\cos\theta}{\sin\theta}\frac{S'}{S} - V_\theta(\theta). \tag{5.40}$$

As usual the fact that this equation needs to hold for all r while θ is kept fixed and vice versa, implies that each side equates to the same separation constant, which we denote K. The angular and radial equation then read

$$S'' + \frac{\cos\theta}{\sin\theta}S' + (V_\theta + K)S = 0, \tag{5.41}$$

and

$$XR'' + X'R' + (V_r - K)R = 0, \tag{5.42}$$

respectively.

From this analysis, it is clear that the massless Klein-Gordon equation is separable, however the massive Klein-Gordon equation is in general not separable. This is related to the fact that there is no Exact Killing-Stackel Tensor (EKST) for the general four-charged black holes that we are considering, instead one has a Conformal Killing-Stackel Tensor (CKST) [37], hence a weaker form of separability. Notably however, setting all charges equal one gets the familiar Kerr-Newman black holes. Both the Kerr-Newman black holes and the two-charge black holes (setting the charges equal in pairs) exhibit EKSTs; both the massless and massive Klein-Gordon equations are separable [37]. In this thesis we will focus on the massless Klein-Gordon equation, for the four-charged rotating black holes (5.5).

5.4 Singular Points of the Radial Equation

We have already seen that requiring separability of the massless wave equation gives rise to the relation (5.34). Now we would like to identify the subset of warp factors that in addition give us a hypergeometric radial equation. It is well know that any linear ordinary differential equation with three regular singular points can be mapped to the hypergeometric equation, thus we proceed to analyze under what circumstances our radial equation has three regular singular points.

We recall that a differential equation of the form

$$f''(x) + p_0(x)f'(x) + p_1(x)f(x) = 0 \tag{5.43}$$

has in general regular points, but also possibly singular points, which are identified as being either regular or irregular [28]. Consider a point $x = p$, then (5.43) is regular at p if $p_0(x)$

and $p_1(x)$ do not diverge at $x = p$, while it is singular otherwise. Indeed if either $p_0(x)$ or $p_1(x)$ diverge, that is

$$p_0(x) \sim \frac{1}{(x-p)^\alpha}, \qquad p_1(x) \sim \frac{1}{(x-p)^\beta} \qquad \text{for } |x-p| \to 0 \tag{5.44}$$

then p is *regular* singular when $\alpha \leq 1$ and $\beta \leq 2$, otherwise when $\alpha > 1$ and/or $\beta > 2$, p is an *irregular* singular point. For completeness, if the singular behavior is not of the form (5.44), then it is an essential singularity.

To analyze the possible singular points of the radial equation we consider the normalized radial equation (divide (5.42) by X):

$$R'' + \frac{X'}{X}R' + \frac{V_r - K}{X}R = 0. \tag{5.45}$$

We immediately see two singular points, the values of r for which $X = 0$, namely

$$r_\pm = m \pm \sqrt{m^2 - a^2}. \tag{5.46}$$

The factor multiplying R' has a pole of order one. Furthermore, noting that

$$V_r = \frac{(\mathcal{A}_{\text{red}}\omega - am_\phi)^2}{X} + (\text{regular at } r_+ \text{ and } r_-) \tag{5.47}$$

and that the numerator of the first term is simply a polynomial in r with roots distinct from $r_\pm$, we see that the factor multiplying R has a second order pole at r_+ and r_-. Thus r_+ and r_- are *regular* singular points.

To investigate the singular nature of $r = \infty$ we employ the coordinate transformation

$$\frac{1}{w} = r \tag{5.48}$$

such that $w = 0$ coincides with $r = \infty$. Derivatives with respect to r then transform into derivatives with respect to w via the chain rule

$$\frac{d}{dr}f(r) = \frac{d}{dw}f(r)\left(\frac{dr}{dw}\right)^{-1} \implies f' = -w^2\dot{f} \tag{5.49}$$

where primes denote differentiation with respect to r and dots with respect to w. Repeated use of the chain rule gives

$$f'' = w^4\ddot{f} + 2w^3\dot{f}. \tag{5.50}$$

After applying this transformation to the radial equation (5.45) we get

$$\ddot{R} + 2\left(\frac{a^2w - m}{a^2w^2 - 2mw + 1}\right)\dot{R} + \frac{1}{w^2}\left(\frac{V_r - K}{a^2w^2 - 2mw + 1}\right)R = 0. \tag{5.51}$$

The factor multiplying $\dot{R}$ is seen to be regular at $w = 0$, while the factor in front of R is looking singular. Indeed, since (for Δ_0) $V_r \sim r^2$ for large r (look at (5.39)), we can pull out an extra factor of $1/w^2$ revealing a fourth order pole. From now on when we write $\sim r^\alpha$ we mean the power-law behavior for *large* r.

Clearly $r = \infty$ is an irregular singular point, thus for the original warp factor Δ_0, we get a radial equation with two *regular* singular points r_+ and r_-, and an *irregular* singular point $r = \infty$. We would have been happy if $r = \infty$ was a regular singular point, however, since it is irregular we cannot map the radial equation to the hypergeometric equation.

We note that the singular point at infinity is only irregular as a consequence of $\Delta_0 \sim r^4$ for large r, which in turn results in potential $V_r \sim r^2$. Therefore we should be able to reduce the order of this pole by replacing the original warp factor by a "subtracted" warp factor Δ. If we choose a $\Delta \sim r^3$, infinity remains irregular, while for $\Delta \sim r^2$, and indeed for $\Delta \sim r^\gamma$ with $\gamma \leq 2$ the singularity at $r = \infty$ becomes regular. As we shall see we can only have $\gamma = 1$ or $\gamma = 2$ as we are not at liberty to introduce new singularities in the metric. Furthermore, upon closer inspection these are the only cases for which we can satisfy (5.34).

Let $f_r(r) = \text{const.} \times r^\alpha$, then the condition of separability (5.34) implies that

$$\Delta = \cdots + \text{const.} \times r^\alpha + \text{const.} \times r^{\alpha+1} + \text{const.} \times r^{\alpha+2}. \tag{5.52}$$

To ensure at most $\Delta \sim r^2$ we see that we need $\alpha \leq 0$. For $\alpha < 0$, Δ would be singular at $r = 0$ thanks to the term $\sim r^\alpha$. The only possible choice is therefore $\alpha = 0$, thus proving that f_r must be a constant which we are free to absorb with $f_\theta(\theta)$. We may now write the "hybrid" separability condition as

$$\frac{\Delta - \mathcal{A}_{\text{red}}^2}{G} = f_\theta(\theta). \tag{5.53}$$

As we claimed in the previous paragraph, it is now clear that we can only have $\Delta \sim r^\gamma$, $\gamma = 1, 2$, as both G and $\mathcal{A}_{\text{red}}^2$ are second order polynomials in r (with the exception of a "constant" θ dependent term in G).

We move on to verify the result of [19]; that there is a unique $\Delta \sim r$ satisfying (5.53). We will also see that for this particular warp factor the angular equation simplifies to spherically symmetric form. After this we will identify the broader class of warp factors when allowing for $\Delta \sim r^2$.

5.5 The Minimal Warp Factor $\bar{\Delta} \sim r$

Explicitly the condition for separability (5.53) reads

$$\frac{\Delta - 4m^2\big[(\Pi_c - \Pi_s)^2 r^2 + 4m\Pi_s(\Pi_c - \Pi_s)r + 4m^2\Pi_s^2\big]}{r^2 - 2mr + a^2\cos^2\theta} = f_\theta(\theta). \tag{5.54}$$

For $\bar{\Delta} \sim r$, the condition for separability is rather restrictive. Note that we are unable to effect the overall coefficient multiplying r^2 in the numerator, i.e it is fixed to $-4m^2(\Pi_c - \Pi_s)^2$, thus the condition implies

$$f_\theta(\theta) = -4m^2(\Pi_c - \Pi_s)^2. \tag{5.55}$$

Solving for $\bar{\Delta}$ gives

$$\bar{\Delta} = (2m)^3(\Pi_c^2 - \Pi_s^2)r + (2m)^4\Pi_s^2 - (2m)^2(\Pi_c - \Pi_s)^2 a^2\cos^2\theta. \tag{5.56}$$

For this particular warp factor

$$V_r(r) = \frac{\big(2m\big[(\Pi_c - \Pi_s)r + 2m\Pi_s\big]\omega - am_\phi\big)^2}{r^2 - 2mr + a^2} - 4m^2(\Pi_c - \Pi_s)^2\omega^2, \tag{5.57}$$

$$V_\theta(\theta) = -\frac{m_\phi^2}{\sin^2\theta}. \tag{5.58}$$

We see that for such an angular potential V_θ, the angular equation (5.41) reads

$$S'' + \frac{\cos\theta}{\sin\theta}S' - \frac{m_\phi^2}{\sin^2\theta}S = -KS. \tag{5.59}$$

Writing $\chi(\theta, \phi) = S(\theta)e^{im_\phi\phi}$, and combining the θ differentials this angular equation takes on the familiar spherically symmetric form

$$\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\chi(\theta, \phi) = -K\,\chi(\theta, \phi). \tag{5.60}$$

Where $K = \ell(\ell+1)$ for integer ℓ, as usual. Furthermore, for the radial equation the now regular singular point $r = \infty$ is found to have indices $(\ell, -\ell-1)$. We briefly elaborate on how.

When using Frobenius's method to solve the radial equation around the regular singular point $r = \infty$, a series expansion multiplied by $w^{-\gamma}$ is used (where $w = 1/r$). To zeroth order, this exponent γ has to satisfy the so called indicial equation. For the radial equation (5.51), performing such a series expansion around $r = \infty$, i.e $w = 0$, gives the indicial equation

$$\gamma(\gamma+1) - K = 0. \tag{5.61}$$

This has roots

$$\gamma_{\pm} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + K} = -\tfrac{1}{2} \pm \left(\ell + \tfrac{1}{2}\right), \tag{5.62}$$

and hence

$$\gamma_+ = \ell, \qquad \gamma_- = -\ell - 1. \tag{5.63}$$

5.6 General Warp Factors

We just saw that there is a unique $\bar{\Delta} \sim r$ that gives a separable massless wave equation. Now, we investigate the possible $\Delta \sim r^2$, and see what restrictions (5.53) imposes in this case. A priori, the constraint (5.53) should be less restrictive in this case, due to the extra degree of freedom associated with the ability to tune coefficients multiplying terms $\sim r^2$ in Δ.

We start by inspecting the explicit version of (5.53)

$$\frac{\Delta - 4m^2\left[(\Pi_c - \Pi_s)^2 r^2 + 4m\Pi_s(\Pi_c - \Pi_s)r + 4m^2\Pi_s^2\right]}{r^2 - 2mr + a^2\cos^2\theta} = f_\theta(\theta). \tag{5.64}$$

In general, this implies that separability amounts to tuning coefficients in Δ such that the numerators factorizes as $G \times$ (separable). It is easy to see that the warp factor has to depend on θ, indeed we need a term proportional to $\cos^2\theta$. We restrict the r dependence to polynomials that go like r^2 for large r. Indeed as we showed in section 5.4 Δ must be a polynomial in r (with the exception of θ dependent "constant") of at most quadratic degree.

We are therefore restricted to warp factors that take the form

$$\Delta = Ar^2 + Br + C + D\cos^2\theta, \tag{5.65}$$

where A, B, C and D are real constants. For the numerator to factorize as $G \times f_\theta$ one must have

$$A - 4m^2(\Pi_c - \Pi_s)^2 = f_\theta, \tag{5.66}$$

$$B - 16m^3\Pi_s(\Pi_c - \Pi_s) = -2m\, f_\theta, \tag{5.67}$$

$$C - 16m^4\Pi_s^2 = 0, \tag{5.68}$$

$$D = a^2\, f_\theta. \tag{5.69}$$

We identify the minimally subtracted warp factor $\bar{\Delta}$ when $f_\theta = -4m^2(\Pi_c - \Pi_s)^2$, and furthermore, see that this allows us to express the general $\Delta \sim r^2$ as

$$\Delta = \bar{\Delta} + [f_\theta + 4m^2(\Pi_c - \Pi_s)^2]G. \tag{5.70}$$

We introduce $\Theta \equiv f_\theta + 4m^2(\Pi_c - \Pi_s)^2$ in terms of which (5.70) reads

$$\Delta = \bar{\Delta} + \Theta G, \tag{5.71}$$

where now $\Theta = 0$ corresponds to the minimally subtracted warp factor.

We have now established the general class of warp factors that give us a separable massless wave equation whose radial equation takes on hypergeometric form (5.71). We note that in [19] they say that Θ is a constant, clearly our analysis shows that it may depend non-trivially on θ. Whether this went unnoticed or there is an underlying reason to discard θ dependence is not clear from [19]. We continue under the assumption that the result (5.71) is correct.

We are mostly interested in the cases $\Delta \sim r^2$ for large r since the minimally subtracted case with $\Delta = \bar{\Delta}$ has been studied extensively in the literature [2, 16, 19, 20, 54]. Furthermore as we will show (and as was pointed out in [19]), considering the case $\delta_I = 0$, the NHEK limit on $\bar{\Delta}$ does not coincide with the NHEK limit on Δ_0, thus suggesting perhaps, that $\bar{\Delta}$ is not an ideal choice. It would indeed be interesting to try and find warp factors (5.71) that give the same NHEK limit as Δ_0.

5.7 NHEK Limit on Warp Factor

Restricting to the case $\delta_I = 0$, we find that the original warp factor simplifies to (5.10). This is the purely rotating four-dimensional Kerr black hole. In [27] they study the near horizon region of such an extremal Kerr black hole, and furthermore identify boundary conditions for metric fluctuations whose ASG suggest a dual chiral CFT description. The NHEK limit originally taken in [3] plays a central role in the analysis of [27]. When applied to the Kerr geometry, the NHEK geometry is seen as a warped AdS_3 geometry with radius $\ell^2 = 2m^2$ and warp factor

$$\Omega = \tfrac{1}{2}(1 + \cos^2\theta). \tag{5.72}$$

As far as the warp factors are concerned, the NHEK limit is realized via the scaling limit

$$\sqrt{m^2 - a^2} = \epsilon\lambda, \qquad r - m = U\lambda, \qquad \lambda \to 0, \tag{5.73}$$

where ϵ and U are dimensionfull constants and λ is the dimensionless scaling parameter. In this limit, we find that the warp factor

$$\sqrt{\Delta_0} \to \frac{\ell^2}{2}(1 + \cos^2\theta). \tag{5.74}$$

On the other hand, for $\delta_I = 0$, the minimally subtracted warp factor (5.56) simplifies to

$$\bar{\Delta} = 4m^2(2mr - a^2 \cos^2\theta). \tag{5.75}$$

This time, the NHEK limit (5.73) gives

$$\sqrt{\bar{\Delta}} \to \ell^2\sqrt{1 + \sin^2\theta}, \tag{5.76}$$

which is clearly different from the NHEK limit of Δ_0. In [19] they suggest that this may not be contradictory, but may simply suggest that there are two valid CFT descriptions for rapidly spinning (i.e extremal/near-extremal Kerr) black holes. Nevertheless, it may be worth while to search for other warp factors for which the NHEK limit coincides with (5.74).

Sticking to the case of vanishing charges, we have that the general $\Delta \sim r^2$ warp factor reads

$$\Delta = 4m^2(2mr - a^2\cos^2\theta) + \Theta(\theta)(r^2 - 2mr + a^2\cos^2\theta). \tag{5.77}$$

In the NHEK limit, ignoring for now what happens to Θ we get

$$\Delta \to 4m^4(2 - \cos^2\theta) - m^2\sin^2\theta\,\Theta(\theta). \tag{5.78}$$

Interestingly we see that there is a $\Theta(\theta)$ such that $\Delta \to 0$, namely

$$\Theta(\theta) = 4m^2 \left(1 + \frac{1}{\sin^2\theta}\right). \tag{5.79}$$

Furthermore we readily see that the correct NHEK limit is achieved with the choice

$$\begin{aligned}\Theta(\theta) &= 4m^2 \left[1 + \frac{1}{\sin^2\theta} - \frac{(1+\cos^2\theta)^2}{4\sin^2\theta}\right] \\ &= 7m^2 + m^2\cos^2\theta,\end{aligned} \tag{5.80}$$

In this expression all the terms are proportional to m^2, however it is clear that we could also have chosen to make them proportional to a^2 since in the NHEK limit $\sqrt{m^2 - a^2} \to 0$. However, as we will elaborate on in limited detail in section 6.1, the warp factor cannot depend on θ when $a = 0$, therefore the angular term in (5.80) should be proportional to a^2 not m^2, while the constant term is still ambiguous, therefore we restrict to Θ of the form

$$\Theta(\theta) = Am^2 + (7 - A)a^2 + a^2\cos^2\theta, \qquad A \in \mathbb{R}. \tag{5.81}$$

For a given A such a Θ corresponds to a warp factor

$$\begin{aligned}\Delta_{\text{NHEK};A} = (Am^2 &+ (7 - A)a^2 + a^2\cos^2\theta)r^2 \\ &+ [(4 - A)m^2 + (A - 7)a^2 - a^2\cos^2\theta](2mr - a^2\cos^2\theta),\end{aligned} \tag{5.82}$$

whose NHEK limit coincides with the NHEK limit on Δ_0. To be clear (5.81) is the most general choice given that the angular term should vanish when a vanishes, clearly for all choices of A the NHEK limit on such a Θ gives $7m^2 + m^2\cos^2\theta$, where $m = a$ as a result of the limit.

To the best of the authors knowledge such $\Delta_{\text{NHEK};A}$ have not been observed in the literature. In order to take such warp factors seriously, one would have to show that there exists matter consistent with the STU model that support the corresponding four-dimensional black hole geometries. In [19] they found matter consistent with the STU model that supports the geometry with warp factor $\bar{\Delta}$ in the static case, and in [16] they extended this to the general rotating case by employing a suitable scaling limit. However, $\bar{\Delta}$ is a warp factor that goes like r for large r, while our proposed $\Delta_{\text{NHEK};A}$ go like r^2 for large r as long as a and m are non-vanishing. Although we will not manage to establish matter for this warp factor in this thesis, we will make progress in that direction by finding matter both in the static and rotating case, that supports a warp factor that goes like r^2 for large r, namely the warp factor $\Delta_- = \mathcal{A}^2_{\text{red}}$. We will get to that shortly, first we address the asymptotic behavior for subtracted geometries.

5.8 Asymptotic Behavior

The subtraction process alters the asymptotic behavior. This, as we have seen, is a consequence of demanding that the radial equation has a regular singular point at $r = \infty$. However it is also desirable from a purely physical point of view. As a great example, the Hawking-Page phase transition between an asymptotically AdS-Schwarzschild black hole and thermal AdS, was discovered by studying the Schwarzschild black hole in confinement, in particular by placing it in an asymptotically AdS spacetime [31]. The reason behind studying such a confined black hole is to avoid the problem of negative specific heat; the black holes Hawking temperature increases with a decrease in mass. We therefore take a closer look at the asymptotic behavior for the subtracted geometries, and furthermore briefly address the related confinement properties.

For a general warp factor the asymptotic behavior of (5.5) reads

$$ds^2 \sim -\Delta^{-1/2} r^2 dt^2 + \Delta^{1/2} \frac{1}{r^2}(dr^2 + r^2 d\Omega_2^2), \tag{5.83}$$

which readily follows from the definitions of the various functions involved in the line element for our family of black holes.

The minimally subtracted warp factor $\bar{\Delta}$, goes like $\bar{\Delta} \sim (2m)^3(\Pi_c - \Pi_s)^2 r$. Introducing $R = 4\ell^{3/4} r^{1/4}$ where $\ell^3 \equiv (2m)^3(\Pi_c - \Pi_s)^2$, the asymptotic reads

$$ds^2 \sim -\left(\frac{R}{4\ell}\right)^6 dt^2 + dR^2 + \left(\frac{R}{4}\right)^2 d\Omega_2^2, \tag{5.84}$$

which has the apparent scaling symmetry $ds^2 \to \lambda^2 ds^2$ implemented by $R \to \lambda R$ and $t \to \lambda^{-2} t$. Here the non-standard scaling of time is pointed out to be reminiscent of the Lifshitz symmetry found in recent developments of holography in condensed matter systems [19].

For the more general warp factors $\Delta \sim r^2$ one has the behavior $\Delta \sim \Theta(\theta) r^2$ for large r. With this behavior it seems appropriate to introduce $R = 2\Theta^{1/4} r^{1/2}$, for which the behavior reads

$$ds^2 \sim -\frac{R^2}{4\Theta} dt^2 + \left(dR - \tfrac{1}{4} R\, \partial_\theta(\log\Theta)\, d\theta\right)^2 + \tfrac{1}{4} R^2\, d\Omega_2^2. \tag{5.85}$$

For constant Θ we have

$$ds^2 \sim -\left(\frac{R}{2\ell}\right)^2 dt^2 + dR^2 + \left(\frac{R}{2}\right)^2 d\Omega_2^2. \tag{5.86}$$

with $\ell \equiv \Theta^{1/2}$. This time the scaling is implemented by $R \to \lambda R$ and $t \to t$.

It is interesting to note that for Θ's of the form (5.81), $\partial_\theta \Theta = -2a^2 \cos\theta \sin\theta$. This expression vanishes at $\theta = 0$ and $\theta = \frac{\pi}{2}$, implying that $\partial_\theta(\log\Theta)$ in (5.85) vanishes. Furthermore since $a \leq m$ to avoid a naked singularity, we have that $\Theta > 0$ for all θ provided that A in (5.81) is chosen such that the constant term in (5.81) is positive for $m \geq a > 0$.

In [16] they note that these metrics asymptote to a form where the spatial part for fixed θ is a flat two-dimensional cone. This they refer to as the metric being asymptotically conical (AC). Indeed, the relevant two-dimensional cone for the spatial part of (5.86) restricted to the equatorial plane ($\theta = \frac{\pi}{2}$) reads

$$ds^2_{\text{equ}} = dR^2 + \tfrac{1}{4} R^2\, d\phi^2. \tag{5.87}$$

This is the metric of a two-dimensional cone with deficit angle

$$2\pi(1 - \tfrac{1}{2}). \tag{5.88}$$

While the cone in the case $\Delta \sim r$ has deficit angle

$$2\pi(1 - \tfrac{1}{4}), \tag{5.89}$$

which is a greater angle of deficit implying a "steeper" cone. Indeed for $\Delta \sim r^4$ the metric is no longer asymptotically conical, but as expected, asymptotically flat. We see that $\Delta \sim r^2$ is closer to being asymptotically flat, than $\bar{\Delta} \sim r$.

The fact that $\Delta \sim r^2$ subtracted geometries are AC with angular deficit less than for the case $\bar{\Delta} \sim r$ suggests that those geometries are less confining as illustrated in figure 5.1. An idea behind the subtracted geometry approach is that one necessarily needs to study the black hole in a "confining box", a system in equilibrium for which the specific heat is no loner negative, making it possible for a CFT description. As elaborated on in [16], the confining property is determined by the exponent p when writing the general AC metric of the form

$$ds^2 = -\left(\frac{R}{R_0}\right)^{2p} dt^2 + B^2\, dR^2 + R^2\, d\Omega_2^2. \tag{5.90}$$

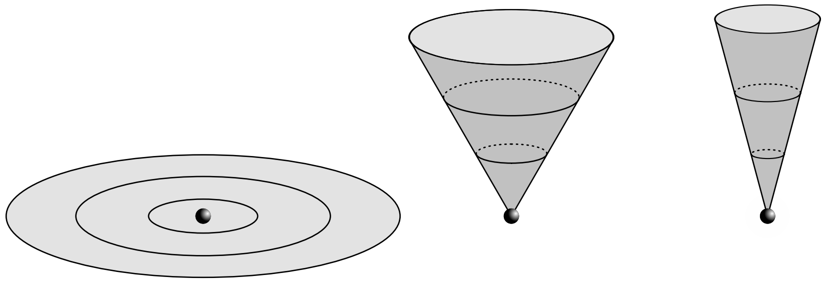

Figure 5.1: Illustrating the spatial part of the asymptotic spacetime (restricted to $\theta = \frac{\pi}{2}$) for different power-law fall off of the warp factor. The black dot represents the throat region connection the asymptotic region with the horizon.

Clearly $B = 2, R_0 = \ell$ and $p = 1$ gives (5.86) up to an overall constant factor. In [16] they argue that for $p > 1$ these geometries have confining properties, and indeed since $p = 3$ for $\bar{\Delta} \sim r$ the minimally subtracted geometry displays the desired confinement properties, akin to the properties of asymptotically AdS spacetimes. It seems however that for the general warp factors $\Delta \sim r^2$ we may not have such confining properties since $p = 1$. This challenges the usefulness of subtracted geometries with $\Delta \sim r^2$.

To make this discussion quantitative, they [16] consider null geodesics whose spatial projections are geodesics of the optical metric

$$ds_o^2 = B^2 \left(\frac{R_0}{R}\right)^{2p} dR^2 + R^2 \left(\frac{R_0}{R}\right)^{2p} \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{5.91}$$

which is directly read off from (5.90) setting $ds^2 = 0$, and identifying $dt^2 = ds_o^2$. To analyze $R = \infty$ we employ $\tilde{R} = R^{1-p} R_0^p$ in terms of which the optical metric reads

$$ds_o^2 = \tilde{B}^2\, d\tilde{R}^2 + \tilde{R}^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \qquad \tilde{B} \equiv \frac{B}{|1-p|}. \tag{5.92}$$

This is clearly the metric of a cone over a two-sphere. We note that $R = \infty$ coincides with $\tilde{R} = 0$ for $p > 1$. Furthermore outwardly directed null-geodesics that are not strictly directed radially (in the R direction) will necessarily not reach $\tilde{R} = 0$ ($R = \infty$), but instead wind around $\tilde{R} = 0$ (the tip of the cone) for a finite time and head back. When such geodesics are radially directed they will however hit $\tilde{R} = 0$ in finite time.

We see that the cone in question has angular deficit

$$2\pi \left(1 - \frac{|1-p|}{B}\right). \tag{5.93}$$

The special case $p = 1$, which we recall is the case for warp factors $\Delta \sim r^2$, is seen to give a deficit angle of 2π, however the radial coordinate $\tilde{R}$ is ill-defined. Going back to (5.91) and employing the radial coordinate $\check{R} = R_0 \log(R/R_0)$, we find that the optical metric reads

$$ds_o^2\big|_{p=1} = B^2\, d\check{R}^2 + R_0^2\, (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{5.94}$$

It is clear that we do not have a cone, instead as suggested by the fact that the angular deficit limits to 2π and now clarified by the above form of the optical metric, we have $R \times S^2$, and it seems that null-geodesics are free to escape toward infinity regardless of whether they are strictly radially outward directed or not.

Furthermore, according to the Tolman red shifting law the temperature of thermal radiation for metrics of the form (5.90) falls off as $-1/\sqrt{g_{00}} = R^{-p}$, implying that for $p > 1$ the entropy outside a shell of radius R_s is finite while for $p = 1$ for instance, which is the case for $\Delta \sim r^2$, we have that the entropy outside a shell of radius R_s is infinite. This is apparent from the behavior of $\int_{R_s}^{\infty} R^{-p}\, dR$.

Before ending this section it is worthwhile to mention the alternative viewpoint of the "confinement" properties. Indeed surprisingly little is required to show that the metrics (5.90) for $p > 1$ are conformal to $\mathrm{AdS}_2 \times S^2$. We rescale the time coordinate and employ a new radial coordinate

$$\tau = \left(\tfrac{\ell}{R_0}\right)^p \frac{t}{\tilde{B}}, \qquad \rho = \frac{R^{p-1}}{\ell^{p-2}}, \tag{5.95}$$

where $\tilde{B}$ was defined in (5.92) and ℓ will be identified with the radius of AdS_2. In terms of the new coordinates (5.90) reads

$$ds^2 = \tilde{B}^2 \left(\frac{\rho}{\ell}\right)^{\frac{2}{p-1}} \left(ds^2_{\mathrm{AdS}} + \frac{\ell^2}{\tilde{B}^2}\, d\Omega_2^2\right), \tag{5.96}$$

where

$$ds^2_{\mathrm{AdS}} = -\frac{\rho^2}{\ell^2}\, d\tau^2 + \frac{\ell^2}{\rho^2}\, d\rho^2. \tag{5.97}$$

Thus it is evident that the asymptotically conical metrics (5.90) are conformal to $\mathrm{AdS}_2 \times S^2$ where the sphere has radius $\ell/\tilde{B}$. This nicely shows that null-geodesics are confined just like in AdS spacetimes. However this form of the metric is ill-defined for $p = 1$, thus again leaving the discussion open for warp factors $\Delta \sim r^2$.

We conclude this section on the note that confinement properties for warp factors $\Delta \sim r^2$ seem to be significantly weaker than for $\bar{\Delta} \sim r$. Indeed it seems unclear whether or not we can interpret subtracted geometries with $\Delta \sim r^2$ as putting the original black hole in a confining box. A thorough investigation of the confinement properties for $\Delta \sim r^2$ is left for future study. In particular it would be illuminating to study the radial coordinate of geodesics, and the corresponding effective potential.

5.9 Hidden Conformal Symmetry

There are several means by which to reveal explicitly the hidden conformal symmetry of a geometry. As we have pointed out, a precursor is for the scalar wave equation to be hypergeometric, however, it is instructive as well as clarifying to see explicit evidence. In [19] they show how the subtracted geometry uplifts to a five-dimensional geometry which is locally $\mathrm{AdS}_3 \times S^2$, and as we mentioned in section 3.3, all locally AdS_3 spacetimes have a dual CFT description. Another way to go about it, is to construct the generators of the rigid conformal transformations $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ as in [11]. In [19] they accomplish this by comparing the Laplacian on global AdS_3

$$\ell^2 \nabla^2 = \frac{1}{\sinh 2\rho} \partial_\rho \sinh 2\rho\, \partial_\rho - \frac{\partial_\tau^2}{\cosh^2\!\rho} + \frac{\partial_\sigma^2}{\sinh^2\!\rho} \tag{5.98}$$

with the Laplacian of the wave equation in the minimally subtracted geometry (5.31) with $\Delta = \bar{\Delta}$. It is straight forward to check that these Laplacians are identified if we identify the coordinates as follows

$$\sinh^2\!\rho = \frac{r - r_+}{r_+ - r_-}, \tag{5.99}$$

$$\sigma - \tau = -\frac{2\pi i}{\beta_L}\left(t - \frac{\beta_R}{\beta_H \Omega_H}\phi\right), \tag{5.100}$$

$$\sigma + \tau = -\frac{2\pi i}{\beta_H \Omega_H}\phi, \tag{5.101}$$

where

$$\frac{\beta_L}{2\pi} = m(\Pi_c - \Pi_s), \qquad \frac{\beta_R}{2\pi} = \frac{m^2}{\sqrt{m^2-a^2}}(\Pi_c + \Pi_s). \tag{5.102}$$

AdS_3 is a maximally symmetric spacetime with isometry group $SL(2,\mathbb{C}) \approx SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$, the generators of these rigid conformal transformations can be represented using the global coordinates as

$$\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2 = e^{\pm(\tau+\sigma)}(\mp i\partial_\rho + \tanh\rho\,\partial_\tau + \coth\rho\,\partial_\sigma), \tag{5.103}$$

$$\mathcal{R}_3 = \partial_\tau + \partial_\sigma, \tag{5.104}$$

$$\mathcal{L}_\pm, \mathcal{L}_3 \text{ are given by } \mathcal{R}_\pm, \mathcal{R}_3 \text{ with } \tau \to -\tau. \tag{5.105}$$

The generators $\mathcal{R}_i$ and $\mathcal{L}_i$ satisfy the $SL(2,\mathbb{R})$ algebra

$$[\mathcal{G}_i, \mathcal{G}_j] = 2i\epsilon_{ijk}(-)^{\delta_{k3}}\mathcal{G}_k, \tag{5.106}$$

and the Casimir $\mathcal{H}^2 = \mathcal{G}_1^2 + \mathcal{G}_2^2 - \mathcal{G}_3^2$ is precisely the Laplacian (5.98). Noting that

$$\partial_\sigma = \tfrac{i}{2\pi}\beta_H\Omega_H\partial_\phi + \tfrac{i}{2\pi}(\beta_R + \beta_L)\partial_t, \tag{5.107}$$

$$\partial_\tau = \tfrac{i}{2\pi}\beta_H\Omega_H\partial_\phi + \tfrac{i}{2\pi}(\beta_R - \beta_L)\partial_t, \tag{5.108}$$

we see how in the coordinates t, r, ϕ we have

$$\mathcal{H}^2 = \ell^2\nabla^2 = \partial_r X \partial_r - \frac{1}{X}(\mathcal{A}_{\text{red}}\partial_t + a\partial_\phi)^2 + 4m^2(\Pi_c - \Pi_s)^2\partial_t^2, \tag{5.109}$$

which is the Laplacian (5.31) with warp factor $\Delta = \bar{\Delta}$. However, due to the periodicity in ϕ, the generators $\mathcal{R}_\pm$ and $\mathcal{L}_\pm$ are globally ill-defined. We may however interpret this in terms of a thermal CFT on a torus with temperatures T_R and T_L for which the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ is spontaneously broken down to the $U(1)_L \times U(1)_R$ subgroup generated by $(\mathcal{L}_3, \mathcal{R}_3)$ [11]. In terms of the useful coordinates

$$t^\pm = \tfrac{i}{2}(\sigma \pm \tau) \tag{5.110}$$

we have $\mathcal{R}_3 = i\partial_t^+, \mathcal{L}_3 = i\partial_t^-$. Furthermore, we see that the identification $\phi \equiv \phi + 2\pi$ forces

$$t^+ \equiv t^+ + \frac{4\pi^2}{\beta_H\Omega_H}, \tag{5.111}$$

$$t^- \equiv t^- - \frac{4\pi^2}{\beta_H\Omega_H}\frac{\beta_R}{\beta_L}. \tag{5.112}$$

For a CFT with left and right temperatures T_R, T_L the fundamental domain reads

$$t^+ \equiv t^+ + 4\pi^2 T_R, \tag{5.113}$$

$$t^- \equiv t^- - 4\pi^2 T_L, \tag{5.114}$$

from which we read off

$$T_R = \frac{1}{\beta_H\Omega_H}, \qquad T_L = \frac{1}{\beta_H\Omega_H}\frac{\beta_R}{\beta_L}. \tag{5.115}$$

Recalling chapter 4, specifically the Cardy formula that gave the entropy for a unitary 2D CFT on a torus

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R), \tag{5.116}$$

we may now wonder whether we could reproduce the entropy (5.20)

$$\left.\frac{\pi \mathcal{A}_{\text{red}}}{G_4}\right|_{r=r_+} = \frac{2\pi m}{G_4}\left((\Pi_c + \Pi_s)m + (\Pi_c - \Pi_s)\sqrt{m^2 - a^2}\right) \tag{5.117}$$

via (5.116) for some central charges c_L, c_R. Indeed it turns out that for $c_L = c_R = 12J$, where (5.3)

$$G_4 J = ma(\Pi_c - \Pi_s) \tag{5.118}$$

gives the correct entropy [19]. This is striking, as it coincides precisely with the central charge $c_L = 12J$ established in [27]. It would be interesting to see if a similar analysis is possible for subtracted warp factors $\Delta \sim r^2$.

6

Supporting Matter - Static Case

In general, the subtracted geometry where Δ_0 has been replaced by Δ, does not satisfy the equations of motion. As an example, the Kerr geometry $\delta_I = 0$ is originally a vacuum solution. However, when Δ_0 is replaced by $\bar{\Delta}$, this is no longer the case: The Einstein tensor no longer vanishes. In general any matter whose energy momentum tensor equates to the Einstein tensor will support the geometry, however, one also needs to ensure that the matter solves its own equations of motion. It can be challenging to identify matter that both supports the geometry and solves the matter equations of motion.

To address the situation we begin with a thorough analysis of Einstein's field equations in the static case for generic warp factors that only dependent on r. We verify our computations by checking that the original matter solves its equation of motion and supports the original geometry. We then move on to establish matter for the minimally subtracted case in section 5.9.3 following [19]. Finally in section 5.9.4 we address the situation for the general class of warp factors (5.71). In particular we find the warp factor $\Delta = \mathcal{A}_{\text{red}}^2$ as the simplest member of (5.71) that goes like r^2 for large r, in hindsight this is rather obvious (look at equation (5.53)), finding matter that supports this subtracted geometry is nonetheless instructive and relevant toward identifying matter for generic warp factors.

6.1 Einstein's Field Equations

In the absence of rotation, i.e $a = 0$, the line element (5.5) with functions (5.6 - 5.9) is diagonal and reads

$$ds^2 = -\frac{r(r-2m)}{\Delta^{1/2}}\,dt^2 + \Delta^{1/2}\left(\frac{dr^2}{r(r-2m)} + d\theta^2 + \sin^2\theta\,d\phi^2\right). \tag{6.1}$$

In [19] the warp factor Δ depends on r only, but is otherwise unspecified. For the minimally subtracted warp factor $\bar{\Delta}$, this is clearly the case, since then $\Theta(\theta)$ is a constant and the angular term being proportional to a vanishes in the static case. However, the condition of separability (5.53) in the static case reads

$$\frac{\Delta - 4m^2\left[(\Pi_c - \Pi_s)^2 r^2 + 4m\Pi_s(\Pi_c - \Pi_s)r + 4m^2\Pi_s^2\right]}{r^2 - 2mr} = \Theta(\theta), \tag{6.2}$$

which still allows for $\Delta = \Delta(r,\theta)$ when $\Delta \sim r^2$ for large r. Even though there exists static charged black holes that are not necessarily spherically symmetric [45], in our case it is safe to assume that the matter content of the action discards this possibility, and we consider spherically symmetric geometries where necessarily $\Delta = \Delta(r)$. Thus even though the separability condition allows for $\Delta(r,\theta)$ generically, we argue that this is unphysical in the static case, and therefore restrict our discussion to $\Delta(r)$ when $a = 0$.

Computing the Einstein tensor, with $\Delta \equiv e^{-4U}$, then gives

$$-r(r-2m)G_{rr} = G_{\theta\theta} = \frac{G_{\phi\phi}}{\sin^2\theta} = (r\partial_r U + 1)\big((r-2m)\partial_r U + 1\big) \tag{6.3}$$

$$G_{tt} = r(r-2m)e^{4U}\left[G_{\theta\theta} + 2r(r-2m)\Big(\partial_r^2 U - (\partial_r U)^2\Big)\right] \tag{6.4}$$

In [19] they use the standard orthonormal frame, i.e they make use of the vielbeins

$$e^a{}_\mu = |g_{\mu\mu}|^{1/2}\delta^a{}_\mu, \qquad g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu e^b{}_\nu \tag{6.5}$$

and one finds that the non-zero components of $G_{ab} = G_{\mu\nu}\, e_a{}^\mu\, e_b{}^\nu$ read

$$\begin{aligned} G_{\hat\theta\hat\theta} &= e^{2U} G_{\theta\theta}, \qquad & G_{\hat r\hat r} &= r(r-2m)e^{2U} G_{rr}, \\ G_{\hat\phi\hat\phi} &= \frac{e^{2U}}{\sin^2\theta} G_{\phi\phi}, \qquad & G_{\hat t\hat t} &= \frac{e^{-2U}}{r(r-2m)} G_{tt}, \end{aligned} \tag{6.6}$$

where the hats indicate that these are components in the vielbein basis. In orthonormal frame we thus have

$$-G_{\hat r\hat r} = G_{\hat\theta\hat\theta} = G_{\hat\phi\hat\phi} = e^{2U}(r\partial_r U + 1)\big((r-2m)\partial_r U + 1\big), \tag{6.7}$$

$$G_{\hat t\hat t} = G_{\hat\theta\hat\theta} + 2e^{2U} r(r-2m)\big(\partial_r^2 U - (\partial_r U)^2\big). \tag{6.8}$$

Now that we have the LHS of Einstein's field equations (the Einstein tensor), it remains to calculate the RHS, i.e the energy-momentum tensor. The energy-momentum tensor is given by varying the action with respect to the metric. In the static case it turns out to be sufficient to deal with the truncated (pseudoscalar free) STU Lagrangian, which reads

$$\mathcal{L} = \frac{1}{16\pi G_4}\left(R - \tfrac{1}{2}\sum_{i=1}^{3}\nabla_\mu\eta_i\nabla^\mu\eta_i - \tfrac{1}{4}e^{-\eta_0}F^0_{\mu\nu}F^{0\mu\nu} - \tfrac{1}{4}e^{-\eta_0}\sum_{i=1}^{3} e^{2\eta_i}F^i_{\mu\nu}F^{i\mu\nu}\right), \tag{6.9}$$

where $\eta_0 \equiv \eta_1 + \eta_2 + \eta_3$, and the action is given by

$$S = \int d^4x\sqrt{|g|}\mathcal{L}. \tag{6.10}$$

As noted in [2], solutions to the truncated action still have to satisfy the pseudoscalar equations of motion to be consistent, a requirement that amounts to

$$-fH_{ij} \star F^0 \wedge F^j + \tfrac{1}{2}C_{ijk}F^j \wedge F^k = 0, \tag{6.11}$$

where H_{ij} is the metric on the scalar moduli space, $H_{ii} = (h^i)^{-2}$, $h^i = e^{\frac{1}{3}\eta_0 - \eta_i}$ and $f = e^{-\frac{1}{3}\eta_0}$, see appendix C.3. Here we are restricting lowercase Latin indices to $i, j, k, \ldots = 1, 2, 3$ while uppercase cover the full range $I, J, K, \ldots = 0, 1, 2, 3$, we will stick to this index convention throughout.

In [2] they solve the constraint (6.11) by taking F^i purely magnetic and F^0 purely electric. In order to avoid confusion we will put tildes over the dual field strengths. In particular reference [19] employs purely electric fields, in which case the Lagrangian density reads

$$\mathcal{L} = \frac{1}{16\pi G_4}\left(R - \tfrac{1}{2}\sum_{i=1}^{3}\nabla_\mu\eta_i\nabla^\mu\eta_i - \tfrac{1}{4}e^{-\eta_0}F^0_{\mu\nu}F^{0\mu\nu} - \tfrac{1}{4}e^{\eta_0}\sum_{i=1}^{3} e^{-2\eta_i}\tilde F^i_{\mu\nu}\tilde F^{i\mu\nu}\right), \tag{6.12}$$

where

$$\tilde F^i = -e^{\eta_0 - 2\eta_i} \star F^i. \tag{6.13}$$

The dualization prescription is the one that interchanges the Bianchi identity with the equations of motion for the given field strength. For the moment we stick to F^i so that we can compare with [2].

Having introduced the truncated action, we proceed to compute the energy-momentum tensor:

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{6.14}$$

where S_M is the matter part of the action, given by the terms in the Lagrangian that depend on the matter, in this case all terms other than R. Keep in mind the relation $\delta S = \int d^4x \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu}$. The variation reads

$$\delta S_M = \int d^4x \left(\delta(\sqrt{|g|}) \mathcal{L}_M + \sqrt{|g|} \delta \mathcal{L}_M \right), \tag{6.15}$$

where $\mathcal{L}_M$ is the matter part of $\mathcal{L}$, and

$$\delta \sqrt{|g|} = -\tfrac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu}, \tag{6.16}$$

$$\delta \mathcal{L}_M = -\frac{1}{32\pi G_4} \left(\sum_{i=1}^{3} \partial_\mu \eta_i \partial_\nu \eta_i + e^{-\eta_0} F^{0\lambda}_{\mu} F^{0}_{\lambda\nu} + e^{-\eta_0} \sum_{i=1}^{3} e^{2\eta_i} F^{i\lambda}_{\mu} F^{i}_{\lambda\nu} \right) \delta g^{\mu\nu}. \tag{6.17}$$

This amounts to

$$\begin{aligned} \delta S_M = -\frac{1}{32\pi G_4} \int d^4x \sqrt{|g|} \Bigg[& \sum_{i=1}^{3} \left(\partial_\mu \eta_i \partial_\nu \eta_i - \frac{g_{\mu\nu}}{2} (\partial \eta_i)^2 \right) + e^{-\eta_0} \left(F^{0\lambda}_{\mu} F^{0}_{\nu\lambda} - \frac{g_{\mu\nu}}{4} (F^0)^2 \right) \\ & + e^{-\eta_0} \sum_{i=1}^{3} e^{2\eta_i} \left(F^{i\lambda}_{\mu} F^{i}_{\nu\lambda} - \frac{g_{\mu\nu}}{4} (F^i)^2 \right) \Bigg] \delta g^{\mu\nu}, \end{aligned} \tag{6.18}$$

where

$$(\partial \eta_i)^2 \equiv \partial_\mu \eta_i \partial^\mu \eta_i, \qquad (F^I)^2 \equiv F^I_{\mu\nu} F^{I\mu\nu}. \tag{6.19}$$

Thus the energy momentum tensor reads

$$\begin{aligned} 8\pi G_4 T_{\mu\nu} = \frac{1}{2} \Bigg[& \sum_{i=1}^{3} \left(\partial_\mu \eta_i \partial_\nu \eta_i - \frac{g_{\mu\nu}}{2} (\partial \eta_i)^2 \right) + e^{-\eta_0} \left(F^{0\lambda}_{\mu} F^{0}_{\nu\lambda} - \frac{g_{\mu\nu}}{4} (F^0)^2 \right) \\ & + e^{-\eta_0} \sum_{i=1}^{3} e^{2\eta_i} \left(F^{i\lambda}_{\mu} F^{i}_{\nu\lambda} - \frac{g_{\mu\nu}}{4} (F^i)^2 \right) \Bigg], \end{aligned} \tag{6.20}$$

in agreement with [2].

In addition to Einsteins equations we have the equations of motion for the scalar and vector matter. From the action it follows that they read

$$0 = \nabla_\mu \nabla^\mu \eta_i + \tfrac{1}{4} e^{-\eta_0} F^0_{\mu\nu} F^{0\mu\nu} + \tfrac{1}{4} e^{-\eta_0} \sum_{j=1}^{3} e^{2\eta_j} (1 - 2\delta_{ij}) F^j_{\mu\nu} F^{j\mu\nu}, \tag{6.21}$$

$$0 = \nabla_\mu \left(e^{-\eta_0} F^{0\mu\nu} \right), \tag{6.22}$$

$$0 = \nabla_\mu \left(e^{-\eta_0 + 2\eta_i} F^{i\mu\nu} \right). \tag{6.23}$$

In [19] they consider spherically symmetric matter, i.e $\eta_i = \eta_i(r)$, and only electric fields, as they are implicitly employing $\tilde{F}^i$ in place of F^i, however they suppress the tildes. The spherical ansatz for the electric gauge fields thus reads

$$A^0 = A^0(r)\, dt, \qquad \tilde{A}^i = A^i_t(r)\, dt. \tag{6.24}$$

For this kind of matter, the non-zero components of the energy-momentum tensor read

$$8\pi G_4 T_{tt} = \tfrac{1}{4}X^2 e^{4U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}Xe^{2U}\left(e^{-\eta_0}(F^0_{tr})^2 + \sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2\right), \tag{6.25}$$

$$8\pi G_4 T_{rr} = \tfrac{1}{4}\sum_{i=1}^{3}(\partial_r\eta_i)^2 - \frac{\tfrac{1}{4}e^{-2U}}{X}\left(e^{-\eta_0}(F^0_{tr})^2 + \sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2\right), \tag{6.26}$$

$$8\pi G_4 T_{\theta\theta} = -\tfrac{1}{4}X\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-2U}\left(e^{-\eta_0}(F^0_{tr})^2 + \sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2\right), \tag{6.27}$$

$$8\pi G_4 T_{\phi\phi} = -\tfrac{1}{4}X\sin^2\theta\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-2U}\sin^2\theta\left(e^{-\eta_0}(F^0_{tr})^2 + \sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2\right). \tag{6.28}$$

In orthonormal frame (indicated by hatted indices), we find

$$8\pi G_4 T_{\hat{t}\hat{t}} = \tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 + \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2, \tag{6.29}$$

$$8\pi G_4 T_{\hat{r}\hat{r}} = \tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 - \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 - \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2, \tag{6.30}$$

$$8\pi G_4 T_{\hat{\theta}\hat{\theta}} = -\tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 + \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2, \tag{6.31}$$

$$8\pi G_4 T_{\hat{\phi}\hat{\phi}} = -\tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 + \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2. \tag{6.32}$$

It is rather apparent that three of the four non-trivial Einstein equations are linearly dependent: $-G_{\hat{r}\hat{r}} = G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}}$. Solving Einstein's equations for the geometry then boils down to solving

$$G_{\hat{\theta}\hat{\theta}} = -\tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 + \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2, \tag{6.33}$$

$$G_{\hat{t}\hat{t}} = \tfrac{1}{4}Xe^{2U}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2 + \tfrac{1}{4}\sum_{i=1}^{3} e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2. \tag{6.34}$$

We can isolate an equation for the scalar matter, and one for the vector matter

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} + G_{\hat{\theta}\hat{\theta}}) = 8\pi G_4 T^{\text{vector}}_{\hat{t}\hat{t}}, \tag{6.35}$$

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} - G_{\hat{\theta}\hat{\theta}}) = 8\pi G_4 T^{\text{scalar}}_{\hat{t}\hat{t}}. \tag{6.36}$$

6.2 Matter Supporting the Original Geometry

Before we go on to find the matter that supports the minimally subtracted geometry, let us verify that

$$e^{-\eta_i} = h_i\sqrt{\frac{h_0}{h_1h_2h_3}}, \qquad A^I_t = \frac{2m\sinh\delta_I\cosh\delta_I}{h_I}, \tag{6.37}$$

where

$$h_I = r + 2m\sinh^2\delta_I, \qquad \text{and} \qquad i = 1,2,3, \quad I = 0, i, \tag{6.38}$$

is the matter that supports the original geometry with standard warp factor (in the static case)

$$\Delta_0 = \prod_{I=0}^{3} h_I, \qquad U_0 = -\tfrac{1}{4}\log(h_0h_1h_2h_3). \tag{6.39}$$

For this warp factor we find

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} + G_{\hat{\theta}\hat{\theta}}) = \tfrac{1}{4}e^{2U_0}\left[r(r-2m)\sum_{I=0}^{3}\frac{1}{h_I^2} - 2(r-m)\sum_{I=0}^{3}\frac{1}{h_I} + 4\right], \tag{6.40}$$

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} - G_{\hat{\theta}\hat{\theta}}) = \tfrac{1}{4}r(r-2m)e^{2U_0}\left[\sum_{I=0}^{3}\frac{1}{h_I^2} - \tfrac{1}{4}\left(\sum_{I=0}^{3}\frac{1}{h_I}\right)^2\right]. \tag{6.41}$$

The vector matter gives

$$8\pi G_4 T^{\text{vector}}_{\hat{t}\hat{t}} = \tfrac{1}{4}e^{2U_0}\left[\frac{(2m)^2\sinh^2\delta_0\cosh^2\delta_0}{h_0^2} + \sum_{i=1}^{3}\frac{(2m)^2\sinh^2\delta_i\cosh^2\delta_i}{h_i^2}\right]. \tag{6.42}$$

Clearly we have

$$\frac{(2m)^2\sinh^2\delta_I\cosh^2\delta_I}{h_I^2} = \frac{r(r-2m)}{h_I^2} - \frac{2(r-m)}{h_I} + 1. \tag{6.43}$$

We thus see that the vector matter supports its "half" of the Einstein equations. It is similarly straight forward to verify that the scalar matter supports the other "half".

We note that the equations of motion (6.21 - 6.23) change slightly in terms of $\tilde{F}^i$. Equation (6.22) is unaltered while the other two read

$$0 = \nabla_\mu\nabla^\mu\eta_i + \tfrac{1}{4}e^{-\eta_0}F^0_{\mu\nu}F^{0\mu\nu} - \tfrac{1}{4}\sum_{j=1}^{3}e^{\eta_0-2\eta_j}(1-2\delta_{ij})\tilde{F}^j_{\mu\nu}\tilde{F}^{j\mu\nu}, \tag{6.44}$$

$$0 = \nabla_\mu\left(e^{\eta_0-2\eta_i}\tilde{F}^{i\mu\nu}\right). \tag{6.45}$$

6.3 Matter Supporting the Minimally Subtracted Geometry

We proceed to find matter that support the subtracted geometry with the minimally subtracted warp factor $\bar{\Delta} = e^{-4\bar{U}}$, keep in mind $a = 0$ for the moment. For

$$\bar{U} = -\tfrac{1}{4}\log\left((2m)^3[(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2]\right), \tag{6.46}$$

the equations (6.35 - 6.36) read

$$\tfrac{1}{4}e^{2\bar{U}}\left[3 + \left(\frac{2m\Pi_c\Pi_s}{(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2}\right)^2\right] = \tfrac{1}{4}e^{-\eta_0}(F^0_{rt})^2 + \tfrac{1}{4}\sum_{i=1}^{3}e^{\eta_0-2\eta_i}(\tilde{F}^i_{rt})^2, \tag{6.47}$$

$$\frac{3}{4}\left(\frac{(\Pi_c^2-\Pi_s^2)}{(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2}\right)^2 = \sum_{i=1}^{3}(\partial_r\eta_i)^2. \tag{6.48}$$

It is easy to see that the differential equation for the scalars η_i under the ansatz $\eta_i \equiv \eta$, implies

$$\eta_i = C \pm \tfrac{1}{2}\log\left[(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2\right], \tag{6.49}$$

where C is an integration constant. For the vector equation, we see that we need to be able to pull out a factor of $e^{2\bar{U}}$ on the RHS. Recalling that $\eta_0 = \eta_1+\eta_2+\eta_3$, we see that a solution is[1]

$$\eta_i = -\tfrac{1}{2}\log\left((2m)^3[(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2]\right), \tag{6.50}$$

$$F^i_{tr} = e^{-\frac{1}{2}\eta_i+\bar{U}} = 1, \tag{6.51}$$

$$F^0_{tr} = \frac{\Pi_c\Pi_s}{(2m)^2\left[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2\right]^2}, \tag{6.52}$$

when one chooses $C = -\frac{1}{2}\log\left[(2m)^3\right]$.

This matter is not unique, more general matter is given in [2], in terms of magnetic charges B_i, $i = (1,2,3)$. Dualizing the magnetic field strengths of [2] gives us the more general matter from the electric perspective chosen in [19]:

$$\eta_i = -\tfrac{1}{2}\log\left((2m)^3[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2]\right) + \sum_{j\neq i}\log B_j, \tag{6.53}$$

$$F^i_{tr} = \frac{1}{B_i}, \tag{6.54}$$

$$F^0_{tr} = \frac{\Pi_c\Pi_s B_1B_2B_3}{(2m)^2[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2]^2}. \tag{6.55}$$

6.4 Toward Matter for General Warp Factors

We would like to investigate whether or not there can exist matter supporting geometries with more general warp factors $\Delta \sim r^2$ for large r. Eventually we would like to be even more specific, and see whether or not we can find matter supporting a $\Delta \sim r^2$ whose NHEK limit coincides with the NHEK limit on Δ_0 (recall our discussion from section 5.7). In order to investigate the latter, we inevitably need to consider the case with rotation, which is more complex. To start off with, we therefore stick to the simpler static case and try our luck with the general warp factors (5.71) when $a = 0$:

$$\Delta = (2m)^3[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2] + r(r-2m)\Theta(\theta). \tag{6.56}$$

Indeed, when $\Theta(\theta)$ is angularly dependent, the warp factor necessarily depends on θ as well. This is incompatible with a static and stationary black hole, which (as we pointed out at the beginning of section 6.1) is necessarily spherically symmetric. Thus in the static case we are naturally restricted to $\Theta = \Theta_0$ a constant.

It is not difficult to see that for the general constant $\Theta = \Theta_0$, the angular equation (5.41) simplifies to the familiar spherically symmetry form. Furthermore $\Delta = \Delta(r)$, i.e only depends on the radial coordinate, implying that the non-vanishing components of the Einstein tensor are again (6.7 - 6.8) now with

$$U = -\tfrac{1}{4}\log\left((2m)^3[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2] + \Theta_0\, r(r-2m)\right). \tag{6.57}$$

Using a spherical ansatz for the matter, finding matter again boils down to solving (6.33 - 6.34) of which the linear combinations

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} + G_{\hat{\theta}\hat{\theta}}) = 8\pi G_4 T^{\text{vector}}_{\hat{t}\hat{t}}, \tag{6.58}$$

$$\tfrac{1}{2}(G_{\hat{t}\hat{t}} - G_{\hat{\theta}\hat{\theta}}) = 8\pi G_4 T^{\text{scalar}}_{\hat{t}\hat{t}}, \tag{6.59}$$

[1] There is a slight typo in the expression for the scalar fields η_i in [19].

are nicer to consider.

Employing (6.57) the LHSs evaluate to

$$\left(\frac{\frac{1}{2}e^{U}}{(2m)^3[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2]+\Theta_0\, r(r-2m)}\right)^2\Big[3-\Theta_0^2\, r^2(r-2m)^2 \\ +2(2m)^3\Theta_0(r-m)[(\Pi_c^2-\Pi_s^2)r^2+4m\Pi_s^2(r-m)]-(2m)^8\Pi_c^2\Pi_s^2\Big], \tag{6.60}$$

and after dividing the scalar equation by $\frac{1}{4}Xe^{2U}$ its LHS reads

$$\left(\frac{1}{(2m)^3[(\Pi_c^2-\Pi_s^2)r+2m\Pi_s^2]+\Theta_0\, r(r-2m)}\right)^2\Big[\Theta_0^2\big(r(r-2m)+3m^2\big) \\ +(2m)^3\,\Theta_0\left[(\Pi_c^2-\Pi_s^2)r-(3\Pi_c^2+\Pi_s^2)m\right]+\tfrac{3}{4}(2m)^6(\Pi_c^2-\Pi_s^2)^2\Big]. \tag{6.61}$$

We now have numerators on the LHS that are no longer constants depending on the black hole parameters, as they were for $\Delta=\bar{\Delta}$, instead we have quadratic polynomials in r. When setting $\Theta_0=0$ the above match with the LHSs of (6.47 - 6.48) as they should. After dividing the scalar equation by e^{2U}, the RHSs of both vector and scalar equations are independent of the warp factor, and are thus identical to the RHSs of (6.47 - 6.48).

6.5 The Warp Factor $\Delta_-=\mathcal{A}^2_{\mathbf{red}}$ and its Matter

As corroborated by the ansatz $\eta_i\equiv\eta$, for which the RHS of the scalar equation reads $3(\partial_r\eta)^2$, it would be advantageous to have the LHS of the scalar equation be the square of some function, say $F^2(r)$. This would give a particularly clean linear ODE after taking the root:

$$F(r)=3\partial_r\eta_i. \tag{6.62}$$

Collecting powers of r in the numerator of (6.61)

$$\Theta_0^2\, r^2+2m\,\Theta_0\big((2m)^2(\Pi_c^2-\Pi_s^2)-\Theta_0\big)r+3m^2\Theta_0^2 \\ -8m^4\Theta_0(3\Pi_c^2+\Pi_s^2)+\tfrac{3}{4}(2m)^6(\Pi_c^2-\Pi_s^2)^2, \tag{6.63}$$

we find that for special Θ:

$$\Theta_{0(\pm)}\equiv 4m^2(\Pi_c\pm\Pi_s)^2 \tag{6.64}$$

the polynomial is a complete square, thus providing our sought after simplification for the scalar equation. For $\Theta_{0(\pm)}$ the warp factor reads

$$\Delta_\pm\equiv 4m^2\big((\Pi_c\pm\Pi_s)r\mp 2m\Pi_s\big)^2\equiv e^{-4U_\pm}, \tag{6.65}$$

and the Einstein equations for the matter simplify greatly

$$\tfrac{1}{4}e^{2U_\pm}\left(2\mp\frac{8m^2\Pi_c\Pi_s}{\big[(\Pi_c\pm\Pi_s)r\mp 2m\Pi_s\big]^2}\right)=\tfrac{1}{4}e^{-\eta_0}(F^0_{tr})^2+\tfrac{1}{4}\sum_{i=1}^{3}e^{\eta_0-2\eta_i}(\tilde{F}^i_{tr})^2, \tag{6.66}$$

$$\left(\frac{\Pi_c\pm\Pi_s}{(\Pi_c\pm\Pi_s)r\mp 2m\Pi_s}\right)^2=\sum_{i=1}^{3}(\partial_r\eta_i)^2. \tag{6.67}$$

Inspecting the vector equation, wee see that the possible minus sign in front of the second term on the LHS could give rise to an imaginary field strength depending on the sign of Π_s. Clearly if $\Pi_s>0$ we should choose Δ_-, while if $\Pi_s<0$ we should choose Δ_+ to avoid imaginary field strengths. We also note that $\Delta_-=\mathcal{A}^2_{\rm red}$ and thus trivially ensures separability of the wave equation. Note also that (6.65) are expressions valid in the static case only, when $a \neq 0$ the warp factor Δ_- still coincides with $\mathcal{A}^2_{\mathrm{red}}$, however, Δ_+ acquires angular dependence

$$\Delta_+ = 4m^2\big((\Pi_c + \Pi_s)r - 2m\Pi_s\big)^2 + 16m^2\Pi_c\Pi_s a^2\cos^2\theta. \tag{6.68}$$

We shall finish off this section by finding matter that supports the subtracted geometry with warp factor Δ_- in the static case. To start off our search for matter, we employ the same simplifying ansatz as was done in [19], namely

$$\eta_1 = \eta_2 = \eta_3 \equiv \eta(r). \tag{6.69}$$

Under this ansatz, the scalar equation simplifies to a linear ODE, which is easily integrated. Choosing integration constant $-\frac{1}{3}\sqrt{3}\log(2m)$, the solution reads (up to a sign)

$$\eta = -\tfrac{1}{6}\sqrt{3}\log\left((2m)^2\big[(\Pi_c - \Pi_s)r + 2m\Pi_s\big]^2\right) = \tfrac{2}{3}\sqrt{3}U_-. \tag{6.70}$$

When we now attempt to identify possible field strengths for the vector matter, we run into problems that seem to be related to the irrational factor $\sqrt{3}$. It seems that the easies way to avoid an irrational factor is to employ the alternative ansatz $\eta_1 \equiv \eta(r)$ and η_2, η_3 constants. This works out, but we will not carry it out in detail, instead we draw from the work done in [2].

In [2] a four-parameter family is constructed that is a solution to the STU model with truncated Lagrangian (6.9), that is, it is a family of static black holes. This family contains both the original and the minimally subtracted geometries of [19], but also warp factors that go like r^2 for large r. Maybe some of these warp factors are separable, and perhaps one coincides with Δ_-.

For clarity we display the parameterization of the warp factor in the family studied in [2]

$$\Delta = \frac{\sqrt{q_0^2B_1^2B_2^2B_3^2}}{(2m)^4}\prod_{I=0}^{3}\frac{a_I^2 r + 2m}{\sqrt{1+a_I^2}}, \tag{6.71}$$

here q_0 and B_1, B_2, B_3 are respectively an electric and three magnetic charges, and $a_I, I = 0,1,2,3,4$ are the four parameters. We provide a derivation of this family in chapter 7.

It is now rather straight forward to see that for

$$a_0 = a_1 = \sqrt{\frac{\Pi_c - \Pi_s}{\Pi_s}}, \qquad a_2 = a_3 = 0 \tag{6.72}$$

and

$$q_0 = -16m^4\frac{\Pi_c\Pi_s}{B_1B_2B_3}, \tag{6.73}$$

one has

$$\Delta = \frac{\sqrt{q_0^2B_1^2B_2^2B_3^2}}{(2m)^4}\prod_{I=0}^{3}\frac{a_I^2 r + 2m}{\sqrt{1+a_I^2}} = 4m^2\big((\Pi_c - \Pi_s)r + 2m\Pi_s\big)^2, \tag{6.74}$$

which is precisely the subtracted warp factor Δ_- given in (6.65). This is strikingly similar to the a's that give the minimally subtracted warp factor in [2], and we note that the charge q_0 matches precisely, take a look at (7.56).

With a's defined in (6.72), the scalar matter is readily transcribed from [2], specifically the equations (7.51 - 7.54), and reads

$$\eta_1 = -\tfrac{1}{2}\log\left(4m^2\big((\Pi_c - \Pi_s)r + 2m\Pi_s\big)^2\right) + \log(B_2B_3), \tag{6.75}$$

$$\eta_2 = -\tfrac{1}{2}\log(16m^4\Pi_c\Pi_s) + \log(B_1B_3), \tag{6.76}$$

$$\eta_3 = -\tfrac{1}{2}\log(16m^4\Pi_c\Pi_s) + \log(B_1B_2). \tag{6.77}$$

Similarly the vector matter reads[2]

$$F^0_{tr} = \frac{16m^4\Pi_c\Pi_s}{q_0}\, e^{4U_-}, \tag{6.78}$$

$$F^1_{tr} = \frac{16m^4\Pi_c\Pi_s}{B_1}\, e^{4U_-}, \tag{6.79}$$

$$F^2_{tr} = \frac{1}{B_2}, \qquad F^3_{tr} = \frac{1}{B_3}. \tag{6.80}$$

We got the electric F^i's by dualizing the magnetic fields of [2]. Equation (6.66) was explicitly checked and found to hold for these field strengths, clearly the scalars solve (6.67). Furthermore, all the equations of motion were found to hold. This is not surprising as this particular matter and geometry is a member of the four-parameter family studied in [2].

Here we took a close look at $\Delta_- = \mathcal{A}^2_{\text{red}}$ in particular, however we could also have considered Δ_+ for which one equally well can find matter using the interpolating solution of [2]. Indeed in this case one gets Δ_+ for a's

$$a_0 = a_1 = \sqrt{-\frac{\Pi_c + \Pi_s}{\Pi_s}}, \qquad a_2 = a_3 = 0. \tag{6.81}$$

6.6 Matter for General Warp Factors

It is instructive to work the other way around, i.e start with the family of warp factors (6.71), and see what constraints (5.53) imposes on the a's. This will turn out to give us a larger class of warp factors for which one has supporting matter given by the interpolating solution of [2].

We first note that imposing $\Delta \sim r^2$ for large r amounts to forcing two of the parameters in (6.71) to be zero. Without loss of generality, let us consider the case

$$a_0 \neq 0, \qquad a_1 \neq 0, \qquad a_2 = a_3 = 0, \tag{6.82}$$

then (6.71) reads

$$\Delta = \frac{\sqrt{q_0^2 B_1^2 B_2^2 B_3^2}}{4m^2}\,\frac{(a_0^2 r + 2m)(a_1^2 r + 2m)}{\sqrt{(1+a_0^2)(1+a_1^2)}}. \tag{6.83}$$

Comparing this with (6.57) and recall $\Delta = e^{-4U}$, we find that imposing separability and a hypergeometric radial equation restricts the four-parameters a_0 and a_1 via three equations

$$4m^2\Theta_0 = a_0^2 a_1^2 \sqrt{\frac{Q^2}{(1+a_0^2)(1+a_1^2)}}, \tag{6.84}$$

$$16m^4(\Pi_c^2 - \Pi_s^2) - 4m^2\Theta_0 = (a_0^2 + a_1^2)\sqrt{\frac{Q^2}{(1+a_0^2)(1+a_1^2)}}, \tag{6.85}$$

$$16m^4\Pi_s^2 = \sqrt{\frac{Q^2}{(1+a_0^2)(1+a_1^2)}}, \tag{6.86}$$

where

$$Q \equiv q_0 B_1 B_2 B_3. \tag{6.87}$$

Treating a_0, a_1 and Q as unknowns, we proceed to solve in terms of Θ_0, Π_c, Π_s and m.

[2] We note that there is also the more compact way to write the field strengths F^i: $F^i_{tr} = B_i\, e^{-\eta_0 + 2\eta_i + 2U_-}$.

Adding the three equations (6.84 - 6.86) we find

$$16m^4\Pi_c^2 = \sqrt{Q^2(1+a_0^2)(1+a_1^2)}. \tag{6.88}$$

Utilizing this relation we eliminate Q from the first two equations. Suitable linear combinations of these and a bit of manipulation gives

$$\frac{\Theta_0}{4m^2\Pi_s^2} = a_0^2 a_1^2, \tag{6.89}$$

$$\frac{\Pi_c^2}{\Pi_s^2} = (1+a_0^2)(1+a_1^2). \tag{6.90}$$

Using the first of these to eliminate either a_0 or a_1 from the last equation, we get a quadratic equation for respectively a_1^2 and a_0^2. Since a_0 and a_1 appear on equal footing, we get the same quadratic equation

$$a_i^4 - \left(\frac{\Pi_c^2 - \Pi_s^2}{\Pi_s^2} - \frac{\Theta_0}{4m^2\Pi_s^2}\right) a_i^2 + \frac{\Theta_0}{4m^2\Pi_s^2} = 0, \qquad i = 0, 1. \tag{6.91}$$

The solutions read

$$(a_i^2)_\pm = \frac{1}{2}\left(\frac{\Pi_c^2 - \Pi_s^2}{\Pi_s^2} - \frac{\Theta_0}{4m^2\Pi_s^2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\Pi_c^2 + \Pi_s^2}{\Pi_s^2} - \frac{\Theta_0}{4m^2\Pi_s^2}\right)^2 - \frac{\Pi_c^2}{\Pi_s^2}}. \tag{6.92}$$

Thus given a Θ_0 we can find a's and a Q via (6.88) that produce the subtracted warp factor (6.57) via (6.71). Most significantly, for any warp factors (6.71) there exists matter that supports it [2], given explicitly by the parameters a_i and the charges baked into Q.

As is easily checked, when plugging $\Theta_{0(\pm)}$ into (6.92) the square root term vanishes, thus we drop the $\pm$ label on the a_i^2 in this particular case and find

$$a_i^2\Big|_{\Theta_{0(+)}} = -\frac{\Pi_c + \Pi_s}{\Pi_s}, \qquad a_i^2\Big|_{\Theta_{0(-1)}} = \frac{\Pi_c - \Pi_s}{\Pi_s}. \tag{6.93}$$

Clearly a_i for $\Theta_{0(-)}$ and $\Theta_{0(+)}$ coincide with (6.81) and (6.72) respectively up to an overall sign. We furthermore observe that when $\Pi_s > 0$, the a_i corresponding to $\Theta_{0(+)}$ are strictly imaginary, while the a_i for $\Theta_{0(-)}$ are strictly real, and vice versa when $\Pi_s < 0$. Furthermore we note the general observation that $(a_i^2)_\pm$ are only strictly real provided that

$$\Theta_0 \leq 4m^2(\Pi_c \pm \Pi_s)^2 = \Theta_{0(\pm)}. \tag{6.94}$$

The main result here, is that we have found that a significant sub-class of the warp factors (6.57) are supported by the matter presented in the family of solutions in [2]. Furthermore, the bound (6.94) suggests that it is only when Θ_0 satisfies (6.94) that we have sensible matter supporting the geometry. It would be interesting to investigate the properties of the matter when Θ_0 does not satisfy (6.94), we defer this to future work.

Recall section 5.7 where we found a class of warp factors $\Delta_{\mathrm{NHEK};A}$ for which the NHEK limit coincided with the NHEK limit on Δ_0. In particular the corresponding Θ's are given by (5.81) which reads $\Theta = Am^2$ when $a = 0$. Now the bound (6.94) suggests that we should restrict to $A \leq 4$. Furthermore the case $A = 4$ might be of particular interest since such a Θ coincides with Δ_- for non-charged and static geometries; (5.5) with $\delta_I = 0$ and $a = 0$.

7

Irrelevant Deformations

In sections 6.5 and 6.6 we made use of the four-parameter family of solutions whose warp factors read (6.71). This family is derived in [2], where the Einstein equations and matter equations of motion are diagonalized and solved explicitly. In this chapter we go through the steps of [2] that lead to the four-parameter family. We will also focus on the core analysis of that paper, namely the identification of irrelevant operators in a 2D CFT that are dual to the sources that start the flow from the minimally subtracted static geometry to the original static geometry. Furthermore we will extend parts of this analysis to the case $\Delta \sim r^2$ by repeating a similar analysis for $\Delta_- = \mathcal{A}^2_{\text{red}}$.

7.1 The Static Four-Parameter Family

Since we are restricting to the static case, we again deal with the truncated Lagrangian (6.9), and the relevant action reads

$$S = -\frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}\left[R - \tfrac{1}{4}e^{-\eta_0}F^0_{\mu\nu}F^{0\mu\nu} - \tfrac{1}{2}\sum_{i=1}^{3}\left(\nabla_\mu\eta_i\nabla^\mu\eta_i + \tfrac{1}{2}e^{2\eta_i-\eta_0}F^i_{\mu\nu}F^{i\mu\nu}\right)\right]. \quad (7.1)$$

Note that in order to satisfy the original constraint $h^1h^2h^3 = 1$, the new scalars are constrained by

$$\eta_0 = \eta_1 + \eta_2 + \eta_3, \quad (7.2)$$

see appendix C.3.

The equations of motion read

$$0 = \nabla_\mu\nabla^\mu\eta_i + \frac{1}{4}\left[e^{-\eta_0}F^0_{\mu\nu}F^{0\mu\nu} + e^{-\eta_0}\sum_{j=1}^{3}(1-2\delta_{ij})e^{2\eta_j}F^j_{\mu\nu}F^{j\mu\nu}\right], \quad (7.3)$$

$$0 = \nabla_\mu\left(e^{-\eta_0}F^{0\mu\nu}\right), \quad (7.4)$$

$$0 = \nabla_\mu\left(e^{-\eta_0+2\eta_i}F^{i\mu\nu}\right), \quad (7.5)$$

$$\begin{aligned}G_{\mu\nu} = {} & \tfrac{1}{2}\sum_{i=1}^{3}\left(\nabla_\mu\eta_i\nabla^\mu\eta_i - \tfrac{1}{2}g_{\mu\nu}\nabla_\lambda\eta_i\nabla^\lambda\eta_i\right) + \tfrac{1}{2}e^{-\eta_0}\left(F^{0\rho}_{\mu}F^0_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F^0_{\lambda\rho}F^{0\lambda\rho}\right) \\ & + \tfrac{1}{2}e^{-\eta_0}\sum_{i=1}^{3}e^{2\eta_i}\left(F^{i\rho}_{\mu}F^i_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F^i_{\lambda\rho}F^{i\lambda\rho}\right). \end{aligned} \quad (7.6)$$

Since we are dealing with the static case, we employ a spherically symmetric ansatz for the metric. In addition, to fulfill the constraint equation (6.11), we adopt an electric ansatz for F^0 and a magnetic ansatz for F^i [2]. The ansatz reads

$$ds^2 = -\frac{G(r)}{\sqrt{\Delta(r)}}\,dt^2 + \sqrt{\Delta(r)}\left(\frac{dr^2}{X(r)} + d\theta^2 + \frac{X(r)}{G(r)}\sin^2\theta\,d\phi^2\right), \quad (7.7)$$

$$A^0 = A^0_t(r)\,dt, \quad (7.8)$$

$$A^i = B_i\cos\theta\,d\phi, \quad (7.9)$$

$$\eta_i = \eta_i(r), \quad (7.10)$$

where the constants $B^i, i = 1, 2, 3$ are the magnetic charges. We note that the ansatz for the metric is identical to (5.5) with vanishing angular momentum ($a = 0$).

The functions $G(r)$, $X(r)$ and $A_t^0(r)$ are for the moment arbitrary, and will be fixed as we go along. A constraint between G and X can already be inferred from the (r, θ) component of (7.6), which is seen to imply

$$-\frac{\cos\theta}{2\sin\theta}\left(\frac{X'G - XG'}{XG}\right) = 0, \tag{7.11}$$

where primes denote differentiation with respect to r, from which it follows that

$$G = \gamma X, \tag{7.12}$$

for some constant γ. Plugging this back into the ansatz one finds a further constraint on X''. This can be seen after writing out (7.6) for G_{rr} and $G_{\theta\theta}$. Indeed (7.6) only comprises 4 non-trivial equations:

$$G_{tt} = -\tfrac{1}{4}g_{tt}g^{rr}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}g^{rr}(F_{rt}^0)^2 - \tfrac{1}{4}e^{-\eta_0}g_{tt}g^{\theta\theta}g^{\phi\phi}\sum_{i=1}^{3}e^{2\eta_i}(F_{\theta\phi}^i)^2, \tag{7.13}$$

$$G_{rr} = \tfrac{1}{4}\sum_{i=1}^{3}(\partial_r\eta_i)^2 + \tfrac{1}{4}e^{-\eta_0}g^{tt}(F_{rt}^0)^2 - \tfrac{1}{4}e^{-\eta_0}g_{rr}g^{\theta\theta}g^{\phi\phi}\sum_{i=1}^{3}e^{2\eta_i}(F_{\theta\phi}^i)^2, \tag{7.14}$$

$$G_{\theta\theta} = -\tfrac{1}{4}g_{\theta\theta}g^{rr}\sum_{i=1}^{3}(\partial_r\eta_i)^2 - \tfrac{1}{4}e^{-\eta_0}g_{\theta\theta}g^{rr}g^{tt}(F_{rt}^0)^2 + \tfrac{1}{4}e^{-\eta_0}g^{\phi\phi}\sum_{i=1}^{3}e^{2\eta_i}(F_{\theta\phi}^i)^2, \tag{7.15}$$

$$G_{\phi\phi} = -\tfrac{1}{4}g_{\phi\phi}g^{rr}\sum_{i=1}^{3}(\partial_r\eta_i)^2 - \tfrac{1}{4}e^{-\eta_0}g_{\phi\phi}g^{rr}g^{tt}(F_{rt}^0)^2 + \tfrac{1}{4}e^{-\eta_0}g^{\theta\theta}\sum_{i=1}^{3}e^{2\eta_i}(F_{\theta\phi}^i)^2. \tag{7.16}$$

The RHSs were evaluated using the fact that the metric is diagonal, that $F_{rt}^0 = -F_{tr}^0$ and $F_{\theta\phi}^i = -F_{\phi\theta}^i$ are the only non-vanishing components of the field strengths, and that the scalars $\eta_i(r)$ only depend on r. The Einstein tensor was computed and the non-zero components read

$$G_{tt} = -\tfrac{1}{16}\frac{\gamma X}{\Delta^3}\left[-16\Delta^2 - 7X(\Delta')^2 + 8X\Delta''\Delta + 4X'\Delta'\Delta\right], \tag{7.17}$$

$$G_{rr} = \tfrac{1}{16}\frac{1}{\Delta^2 X}\left[-16\Delta^2 - X(\Delta')^2 + 4X'\Delta'\Delta\right], \tag{7.18}$$

$$G_{\theta\theta} = -\tfrac{1}{16}\frac{1}{\Delta^2}\left[-X(\Delta')^2 - 8\Delta^2 X'' + 4X'\Delta'\Delta\right], \tag{7.19}$$

$$G_{\phi\phi} = -\tfrac{1}{16}\frac{\sin^2\theta}{\Delta^2\gamma}\left[-X(\Delta')^2 - 8\Delta^2 X'' + 4X'\Delta'\Delta\right]. \tag{7.20}$$

Notably X'' only appears in (7.19 - 7.20). Using that the metric is diagonal we see from (7.14) and (7.15) that

$$G_{rr} + g^{\theta\theta}g_{rr}G_{\theta\theta} = 0. \tag{7.21}$$

Using (7.18) and (7.19), this constraint reads

$$8\Delta^2 X'' - 16\Delta^2 = 0, \tag{7.22}$$

and hence

$$X'' = 2. \tag{7.23}$$

Therefore without loss of generality G and X are fixed to

$$G = X = r^2 - 2mr. \tag{7.24}$$

The function $A_t^0(r)$, or rather F_{rt}^0, is fixed by (7.4). Using $\eta_0 = \eta_0(r)$, and employing the ansatz for F_{rt}^0 we find

$$\nabla_\mu(e^{-\eta_0}F^{0\mu\nu}) = -\partial_r\eta_0 e^{-\eta_0}F^{0rt} + e^{-\eta_0}\left(\partial_r F^{0rt} + \Gamma^\mu_{\mu\lambda}F^{0\lambda\nu} + \Gamma^\nu_{\mu\lambda}F^{0\mu\lambda}\right). \tag{7.25}$$

The last term inside the parenthesis vanishes since the connection is symmetric while F^0 is antisymmetric. Furthermore, computing the middle term yields

$$\Gamma^\mu_{\mu\lambda}F^{0\lambda\nu} = \tfrac{1}{2}g^{\mu\nu}\partial_r g_{\mu\nu}F^{0rt} = \tfrac{1}{2}\frac{\Delta'}{\Delta}F^{0rt}. \tag{7.26}$$

Upon lowering of the indices on F^0, which only changes the equation by an overall sign, the equation amounts to

$$\partial_r F_{rt}^0 - \partial_r\left(\eta_0 - \log(\sqrt{\Delta})\right)F_{rt}^0 = 0, \tag{7.27}$$

which gives

$$F_{rt}^0 = q_0\frac{e^{\eta_0}}{\sqrt{\Delta}}, \tag{7.28}$$

where q_0 is the integration constant. Note that the equation for F^i is solved by the ansatz (7.9), and does not require any adjustment.

With the arbitrary functions of the ansatz now fixed, we recast the non-trivial equations of motion (7.3) and (7.6). We start with the scalar equation (7.3). The first term is easily computed using that $g^{rr} = X/\sqrt{\Delta}$

$$\nabla_\mu\nabla^\mu\eta_i = \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r)\eta_i = \frac{(r(r-2m)\eta_i')'}{\sqrt{\Delta}}. \tag{7.29}$$

The contractions of the $F's$ read

$$F_{\mu\nu}^0F^{0\mu\nu} = 2g^{rr}g^{tt}(F_{rt}^0)^2 = -2q_0^2\frac{e^{2\eta_0}}{\Delta}, \tag{7.30}$$

and noting that $F^i_{\theta\phi} = -B_i\sin\theta$

$$F_{\mu\nu}^iF^{i\mu\nu} = 2g^{\theta\theta}g^{\phi\phi}(F^i_{\theta\phi})^2 = 2\frac{B_i^2}{\Delta}. \tag{7.31}$$

Substituting these into (7.3) yields

$$0 = \big(r(r-2m)\eta_i'\big)' - \frac{e^{\eta_0}}{2\sqrt{\Delta}}\left[q_0^2 + \sum_{j=1}^3(2\delta_{ij}-1)e^{-2(\eta_0-\eta_j)}B_j^2\right]. \tag{7.32}$$

Finally we take a closer look at (7.6), which written out reads on the RHS (7.13 - 7.16), and on the LHS (7.17 - 7.20). Given $\gamma = 1$ and $X'' = 2$, it is straight forward to see that (7.13 - 7.16) are degenerate. We can therefore reduce from 4 to 2 equations, and following the paper, we choose the (r,r) component and a linear combination of (t,t) and (ϕ,ϕ), namely

$$g^{tt}g_{\phi\phi}G_{tt} + G_{\phi\phi} = -\tfrac{1}{2}g_{\phi\phi}g^{rr}\sum_{i=1}^3(\partial_r\eta_i)^2. \tag{7.33}$$

Thus the four equations (7.6) reduce to the following two equations

$$\begin{aligned} 0 = &\left(\frac{\Delta'}{2\Delta}\right)^2 - \frac{2(r-m)}{r(r-2m)}\frac{\Delta'}{\Delta} + \frac{4}{r(r-2m)} + (\eta_1')^2 + (\eta_2')^2 + (\eta_3')^2 \\ &- \frac{e^{\eta_0}}{r(r-2m)\sqrt{\Delta}}\left[q_0^2 + \sum_{i=1}^3 e^{-2(\eta_0-\eta_i)}B_i^2\right], \end{aligned} \tag{7.34}$$

and

$$0 = \frac{\Delta''}{\Delta} - \frac{3}{4}\left(\frac{\Delta'}{\Delta}\right)^2 + \left(\eta_1'\right)^2 + \left(\eta_2'\right)^2 + \left(\eta_3'\right)^2. \tag{7.35}$$

In total we now have five coupled non-linear differential equations; three in (7.32) and the above two. Quite remarkably, it turns out that we can diagonalize this set of equations. In [2] they simply introduce the fields ϕ_0, ϕ_i

$$\phi_0 = \tfrac{1}{2}\log\left(\frac{\Delta}{m^4}\right) - \eta_1 - \eta_2 - \eta_3, \tag{7.36}$$

$$\phi_1 = \tfrac{1}{2}\log\left(\frac{\Delta}{m^4}\right) - \eta_1 + \eta_2 + \eta_3, \tag{7.37}$$

$$\phi_2 = \tfrac{1}{2}\log\left(\frac{\Delta}{m^4}\right) + \eta_1 - \eta_2 + \eta_3, \tag{7.38}$$

$$\phi_3 = \tfrac{1}{2}\log\left(\frac{\Delta}{m^4}\right) + \eta_1 + \eta_2 - \eta_3, \tag{7.39}$$

in terms of which the equations are diagonalized.

One way realize this is by subtracting (7.35) from (7.34), which gets rid of the $(\eta_i')^2$ terms. After multiplying by an overall factor of $r(r-2m)$ the resulting equation reads

$$0 = r(r-2m)\left[\left(\frac{\Delta'}{\Delta}\right)^2 - \frac{\Delta''}{\Delta}\right] - 2(r-m)\frac{\Delta'}{\Delta} + 4 - \frac{e^{\eta_0}}{\sqrt{\Delta}}\left[q_0^2 + \sum_{i=1}^{3} e^{-2(\eta_0-\eta_i)}B_i^2\right]. \tag{7.40}$$

Furthermore, the three first terms in (7.40) can be compactly written in a form similar to the first term in (7.32). Using this to rewrite the equation, and multiplying by an overall factor of $-\frac{1}{2}$, we arrive at

$$0 = \left(r(r-2m)\left(\tfrac{1}{2}\log(\alpha^2\Delta)\right)'\right)' - 2 + \frac{e^{\eta_0}}{2\sqrt{\Delta}}\left[q_0^2 + \sum_{i=1}^{3} e^{-2(\eta_0-\eta_i)}B_i^2\right], \tag{7.41}$$

where α is some unspecified constant. To ease the reading, we reproduce (7.32)

$$0 = \left(r(r-2m)\eta_i'\right)' - \frac{e^{\eta_0}}{2\sqrt{\Delta}}\left[q_0^2 + \sum_{j=1}^{3}(2\delta_{ij}-1)e^{-2(\eta_0-\eta_j)}B_j^2\right]. \tag{7.42}$$

We now see that it is tempting to try with fields of the form

$$\phi = C_0 \cdot \tfrac{1}{2}\log(\alpha^2\Delta) + C_1\cdot\eta_1 + C_2\cdot\eta_2 + C_3\cdot\eta_3 \tag{7.43}$$

where C_0, C_1, C_2, C_3 are constants to be specified. The differential equations, in terms of these new fields, are then linear combinations (specified by the C's) of (7.41) and the three equations that comprise (7.32). This gives equations for the ϕ's of the form

$$0 = \left(r(r-2m)\phi'\right)' + \cdots. \tag{7.44}$$

We need to tune the constants (C's) in such a way that we can express $\cdots$ completely in terms of the ϕ's and the charges q_0, B_i. Writing it out, we find

$$\begin{aligned} \cdots = {} & \tfrac{1}{2}\alpha q_0^2(C_0 - C_1 - C_2 - C_3)\exp\{-\tfrac{1}{2}\log(\alpha^2\Delta) + \eta_1 + \eta_2 + \eta_3\} \\ + {} & \tfrac{1}{2}\alpha B_1^2(C_0 - C_1 + C_2 + C_3)\exp\{-\tfrac{1}{2}\log(\alpha^2\Delta) + \eta_1 - \eta_2 - \eta_3\} \\ + {} & \tfrac{1}{2}\alpha B_2^2(C_0 + C_1 - C_2 + C_3)\exp\{-\tfrac{1}{2}\log(\alpha^2\Delta) - \eta_1 + \eta_2 - \eta_3\} \\ + {} & \tfrac{1}{2}\alpha B_3^2(C_0 + C_1 + C_2 - C_3)\exp\{-\tfrac{1}{2}\log(\alpha^2\Delta) - \eta_1 - \eta_2 + \eta_3\} \\ - {} & 2C_0. \end{aligned} \tag{7.45}$$

Clearly, if we wish to write the above in terms of ϕ, ϕ must satisfy

$$\phi \sim \tfrac{1}{2}\log(\alpha^2\Delta). \tag{7.46}$$

In general the linear combination of terms appearing in the exponentials must be proportional to the ϕ's or else diagonalization is not possible. We see to our relief that the fields in the exponents are all linearly independent, and furthermore, that the $C's$ corresponding to a given exponent, ensure that the prefactors in front of the others vanish.

To clarify, focusing our attention on the first term in (7.45), where all η's come with the same positive weight, the exponent is identical to the ϕ with $C_1 = C_2 = C_3 = -C_0$, which is a constraint which simultaneously ensures that the other three terms vanish. Thus, setting $\alpha = 1/m^2$ for dimensional reasons, we have found the ϕ's of (7.36 - 7.39) that diagonalize the set of coupled non-linear differential equations.

From all this work we can easily read off the new decoupled equations

$$0 = \big(r(r-2m)\phi_0'(r)\big)' + 2\left(\frac{q_0^2}{m^2}e^{-\phi_0(r)} - 1\right), \tag{7.47}$$

$$0 = \big(r(r-2m)\phi_i'(r)\big)' + 2\left(\frac{B_i^2}{m^2}e^{-\phi_i(r)} - 1\right). \tag{7.48}$$

Solutions to these equations give us solutions $\eta_i(r)$, $\Delta(r)$, and $F^0(r)$ given by (7.28) that solve the non-linear equations that we started with.

In [2] they have obtained general solutions to these decoupled equations involving two integration constants. Of those solutions only a specific subset are found to be regular at the horizon, effectively fixing one of the integration constants. The regular solutions read

$$\phi_0^{\text{reg}}(r) = \log\left[\frac{q_0^2}{4m^4}\frac{(a_0^2 r + 2m)^2}{1+a_0^2}\right], \tag{7.49}$$

$$\phi_i^{\text{reg}}(r) = \log\left[\frac{B_i^2}{4m^4}\frac{(a_i^2 r + 2m)^2}{1+a_i^2}\right]. \tag{7.50}$$

Here a_0 and a_i are the four integration constants that parametrize the family of static black hole solutions. The solutions in terms of η_i and Δ read

$$\Delta(r) = \frac{\sqrt{q_0^2 B_1^2 B_2^2 B_3^2}}{16m^4}\prod_{I=0}^{3}\frac{a_I^2 r + 2m}{\sqrt{1+a_I^2}}, \tag{7.51}$$

$$e^{2\eta_1(r)} = \left|\frac{B_2 B_3}{q_0 B_1}\right|\sqrt{\frac{(1+a_0^2)(1+a_1^2)}{(1+a_3^2)(1+a_2^2)}\frac{(a_2^2 r+2m)(a_3^2 r+2m)}{(a_0^2 r+2m)(a_1^2 r+2m)}}, \tag{7.52}$$

$$e^{2\eta_2(r)} = \left|\frac{B_3 B_1}{q_0 B_2}\right|\sqrt{\frac{(1+a_0^2)(1+a_2^2)}{(1+a_1^2)(1+a_3^2)}\frac{(a_3^2 r+2m)(a_1^2 r+2m)}{(a_0^2 r+2m)(a_2^2 r+2m)}}, \tag{7.53}$$

$$e^{2\eta_3(r)} = \left|\frac{B_1 B_2}{q_0 B_3}\right|\sqrt{\frac{(1+a_0^2)(1+a_3^2)}{(1+a_2^2)(1+a_1^2)}\frac{(a_1^2 r+2m)(a_2^2 r+2m)}{(a_0^2 r+2m)(a_3^2 r+2m)}}. \tag{7.54}$$

7.2 Interpolating: Subtracted and Original Geometry

The main feature of the four-parameter family of solutions (7.51 - 7.54), is that it contains both the original and the subtracted geometries of [19]. Comparing (5.9) with (7.51), it is easy to see that in the static case the warp factor (7.51) coincides with the original warp factor (5.9) when

$$a_I^{\text{orig}} = \frac{1}{\sinh\delta_I}, \qquad q_0^{\text{orig}} = m\sinh(2\delta_0), \qquad B_i^{\text{orig}} = m\sinh(2\delta_i). \tag{7.55}$$

Similarly one readily obtains the minimal warp factor in the static case (5.56) when

$$a_0^{\text{subt}} = \sqrt{\frac{\Pi_c^2 - \Pi_s^2}{\Pi_s^2}}, \qquad a_i^{\text{subt}} = 0, \qquad q_0^{\text{subt}} = -\frac{16m^4\Pi_c\Pi_s}{B_1B_2B_3}. \tag{7.56}$$

Furthermore, as indicated in [2] figure 1 and our own version figure 7.1, these geometries are smoothly connected in parameter space: By dialing the a_I's, this family of solutions interpolates between the original and the subtracted geometries of [19]. It is also apparent from (7.51) that there exist choices of a_I that give warp factors $\Delta \sim r^2$. We made us of this in sections 6.5 - 6.6 to identify matter supporting the subtracted geometry with warp factor $\Delta_- = \mathcal{A}_{\text{red}}^2$, recall the choices (6.72) and (6.81) respectively coinciding with Δ_- and Δ_+.

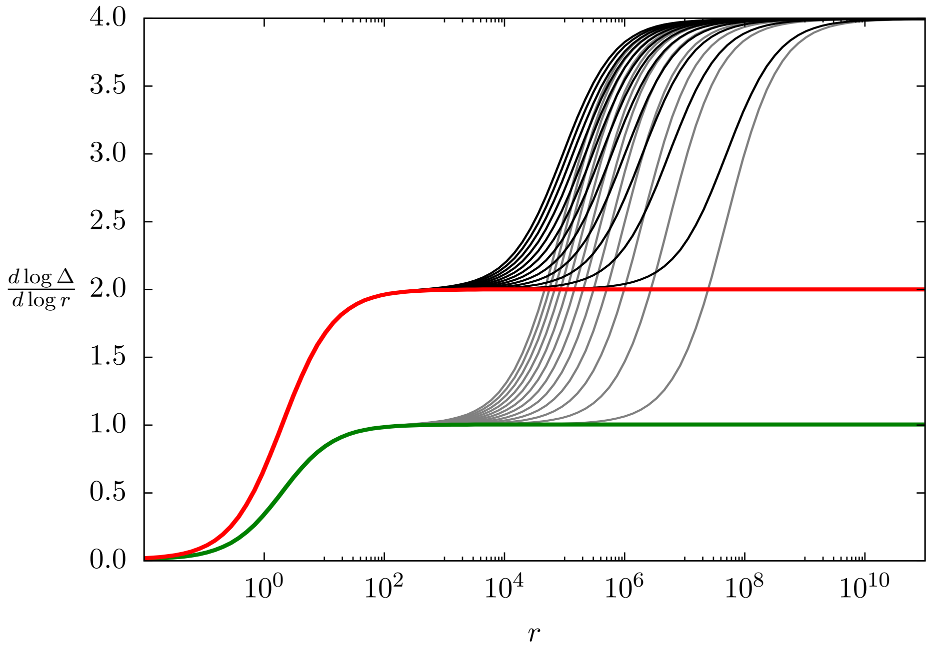

Figure 7.1: Log plot of $\frac{d\log\Delta}{d\log r}$ for the warp factors (7.51). The thick lines leveling out to 2.0 and 1.0 for large r show respectively the behavior of general $\Delta \sim r^2$ and the minimal warp factor $\bar{\Delta}$. The curves departing from the thicker ones show the behavior of the original warp factor Δ_0 where successive curves to the right correspond to smaller values of a_i. Specifically we have for the curves departing the upper solid line $i = 2, 3$ and for the lower $i = 1, 2, 3$. From (7.55) it is apparent that the original warp factor and subtracted warp factors agree over a broader range as the magnetic charges are increased (B_2, B_3 for $\Delta \sim r^2$ and B_1, B_2, B_3 for $\Delta \sim r$). This figure is an adaptation of figure 1. in [2]

7.3 Irrelevant Deformations: Summary of the Procedure

In [2] the sources that turn on the flow to the original geometry are found to correspond to irrelevant operators in a CFT dual to the theory. To arrive at this conclusion, the 4D theory is uplifted to 5D using that the STU model is obtainable by dimensional reduction of a Lagrangian that we denote $\mathcal{L}_5$, see appendix C. The 4D subtracted solution is found to uplift to a geometry that asymptotes to $\text{AdS}_3\times\text{S}^2$ allowing for the standard application of the AdS/CFT correspondence. Firstly though the AdS_3 part is isolated by dimensionally reducing over S^2.

A consistent Kaluza-Klein reduction is performed giving rise to a 3D theory with scalar content $\Psi(x), \Phi(x), U(x)$ where $x = (r, t, z)$. The 3D equations of motion simplify when considering constant scalars. In particular Einstein's field equations simplify allowing for the identification of an effective cosmological constant, thus the space-time is found to be locally AdS_3. Furthermore, crucially, there is a unique solution for the values of the fields Ψ, Φ and U that coincide with the subtracted solution. Finally linearizing the bulk fields around this subtracted solution one finds that the perturbations δF are all solutions to the same equation, and using the dictionary of AdS/CFT, these perturbations are found to be irrelevant operators of the dual CFT with conformal dimension $\Delta = 4$.

In the sections that follow we reproduce some of the main results of [2] going into a little more detail in the analysis. We start with the linearized analysis, that is a study of the equations (7.47 - 7.48) for perturbations around the subtracted fields corresponding to the subtracted geometry and matter. The resulting differential equations for the linearized perturbations are solvable analytically. Thereafter the sources that turn on these linear perturbations are found by expanding the fields around the subtracted solution in terms of the parameters a_I. From comparing the results of the linearized analysis and the perturbation in a_I one identifies the sources. The remaining sections then cover the uplift and subsequent reduction that give the sources a 3D interpretation, and finally dual CFT operators, that are found to be irrelevant deformations. At the very end we repeat the analysis of sections 7.4 and 7.5 for the subtracted geometry with Δ_-.

7.4 Linearized Analysis for $\bar{\Delta}$

In terms of the fields ϕ_I, $I = 0, 1, 2, 3$ that diagonalize the non-linear equations of motion (7.47 - 7.48), linearization is simple. We would like to linearize around the minimally subtracted solution following [2], i.e consider

$$\phi_I = \phi_I^{\text{subt}} + \delta\phi_I, \tag{7.57}$$

where $\delta\phi_I$ are the linearized perturbations. Plugging such a ϕ_I into the decoupled equations of motion (7.47 - 7.48) gives

$$r(r-2m)\delta\phi_0'' + 2(r-m)\delta\phi_0' - 2\frac{q_0^2 B_1^2 B_2^2 B_3^2}{\bar{\Delta}^2}\delta\phi_0 = 0, \tag{7.58}$$

$$r(r-2m)\delta\phi_i'' + 2(r-m)\delta\phi_i' - 2\delta\phi_i = 0. \tag{7.59}$$

We start deriving the first of these. Plugging $\phi_0 = \phi_0^{\text{subt}} + \delta\phi_0$ into (7.47), and using that ϕ_0^{subt} is a solution gives

$$r(r-2m)\delta\phi_0'' + 2(r-m)\delta\phi_0' - \frac{2q_0^2}{m^2}e^{-\phi_0^{\text{subt}}}\delta\phi_0 = 0. \tag{7.60}$$

Plugging in the explicit expression for ϕ_0^{subt} gives

$$\frac{2q_0^2}{m^2}e^{-\phi_0^{\text{subt}}} = 4m^2\frac{1 + (a_0^{\text{subt}})^2}{\left((a_0^{\text{subt}})^2 r + 2\right)^2}. \tag{7.61}$$

Comparing with (7.51) and noting that for the minimally subtracted geometry only $a_0^{\text{subt}} \neq 0$, we can write the above in terms of the subtracted warp factor and the charges of the subtracted geometry. We find

$$r(r-2m)\delta\phi_0'' + 2(r-m)\delta\phi_0' - 2\frac{q_0^2 B_1^2 B_2^2 B_3^2}{\bar{\Delta}^2}\delta\phi_0 = 0, \tag{7.62}$$

where it is understood that the charges and the warp factor are those corresponding to the minimally subtracted geometry.

Similarly, plugging $\phi_i = \phi_i^{\text{subt}} + \delta\phi_i$ into (7.48) gives

$$r(r-2m)\delta\phi_i'' + 2(r-m)\delta\phi_i' - \frac{2B_i^2}{m^2}e^{-\phi_i^{\text{subt}}}\delta\phi_i = 0. \tag{7.63}$$

The prefactor in the last term is simpler this time, since $a_i^{\text{subt}} = 0$. It reduces to a factor of 1, and we have

$$r(r-2m)\delta\phi_i'' + 2(r-m)\delta\phi_i' - 2\delta\phi_i = 0. \tag{7.64}$$

The linearized equations (7.58 - 7.59) simplify in terms of a new radial coordinate

$$x = r/m - 1, \tag{7.65}$$

for which (letting primes denote differentiation with respect to the argument)

$$\delta\phi'(r) = \frac{1}{m}\delta\phi'(x), \qquad \delta\phi''(r) = \frac{1}{m^2}\delta\phi''(x), \tag{7.66}$$

and we find that (7.59) is a Legendre equation

$$(1-x^2)\delta\phi_i''(x) - 2x\delta\phi_i'(x) + 2\delta\phi_i(x) = 0. \tag{7.67}$$

The general solution is well known and reads

$$\delta\phi_i(x) = \alpha_i x + \beta_i\left(\frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1\right), \tag{7.68}$$

where α_i and β_i are real constants. The attentive reader may note however that the sign of the argument of the logarithm is opposite of that given in [2] (typo). This has no effect however, as they set $\beta_i = 0$ to impose regularity of the solutions at $x = 1$ corresponding to the horizon $r = 2m$. Going back to the original radial coordinate r we have thus

$$\delta\phi_i(r) = \alpha_i\left(\frac{r}{m} - 1\right). \tag{7.69}$$

After the same change of variable (7.58) reads

$$(1-x^2)\delta\phi_0''(x) - 2x\delta\phi_0'(x) + 8\left(\frac{\Pi_c\Pi_s}{(\Pi_c^2 - \Pi_s^2)(1+x) + 2\Pi_s^2}\right)^2\delta\phi_0(x). \tag{7.70}$$

The last term can be simplified by introducing

$$b \equiv \frac{\Pi_c^2 - \Pi_s^2}{2\sqrt{2}\Pi_c\Pi_s}, \qquad c \equiv \frac{\Pi_c^2 + \Pi_s^2}{2\sqrt{2}\Pi_c\Pi_s}, \tag{7.71}$$

and we end up with

$$(1-x^2)\delta\phi_0''(x) - 2x\delta\phi_0'(x) + \frac{1}{(bx+c)^2}\delta\phi_0(x) = 0. \tag{7.72}$$

The solution that is regular at the horizon reads

$$\delta\phi_0(x) = \alpha_0\frac{cx+b}{bx+c} = \alpha_0\frac{(\Pi_c^2+\Pi_s^2)r - 2m\Pi_s^2}{(\Pi_c^2-\Pi_s^2)r + 2m\Pi_s^2}. \tag{7.73}$$

7.5 Perturbation Theory around $\bar{\Delta}$ - Identifying Sources

In [2] they linearize the solution ϕ_I around the values a_I^{subt}, in order to identify the sources that start the flow. Starting with ϕ_i and linearizing around $a_i^2 = 0$ they find

$$\phi_i = \phi_i^{\text{subt}} + a_i^2 \left(\frac{r}{m} - 1 \right) + \cdots , \tag{7.74}$$

and recognize the second term on the right-hand side as corresponding to the linearized perturbation that we found in the previous section. This is found by a Taylor expansion of

$$\phi_i = \log \left[\frac{B_i^2}{4m^4} \frac{(a_i^2 r + 2m)^2}{1 + a_i^2} \right] , \tag{7.75}$$

around $a_i^2 = 0$. Continuing to higher orders, one readily notices that all higher order terms will not be linear in r, therefore by comparison with the previous section (7.69) we establish

$$\alpha_i = (a_i^{\text{orig}})^2 = \frac{1}{\sinh^2 \delta_i} . \tag{7.76}$$

For ϕ_0, the Taylor expansion of

$$\phi_0 = \log \left[\frac{q_0^2}{4m^4} \frac{(a_0^2 r + 2m)^2}{1 + a_0^2} \right] \tag{7.77}$$

around $a_0^2 = (a_0^{\text{subt}})^2$ gives

$$\phi_0 = \phi_0^{\text{subt}} + (a_0^2 - (a_0^{\text{subt}})^2) \frac{\Pi_s^2}{\Pi_c^2} \left[\frac{(\Pi_c^2 + \Pi_s^2) r - 2m\Pi_s^2}{(\Pi_c^2 - \Pi_s^2) r + 2m\Pi_s^2} \right] + \cdots . \tag{7.78}$$

Noting again that none of the higher order terms are proportional to r, leads us via comparison with (7.73) to conclude that the source reads

$$\alpha_0 = \left((a_0^{\text{orig}})^2 - (a_0^{\text{subt}})^2 \right) \frac{\Pi_s^2}{\Pi_c^2} . \tag{7.79}$$

These sources are not in general infinitesimal, however they become infinitesimal in the limit of large charges δ_i. In this limit the sources take on simpler forms. It is easy to see from

$$\alpha_i = \frac{1}{\sinh^2 \delta_i} \tag{7.80}$$

that in the limit where the three charges are all large

$$\sinh^2 \delta_i = \tfrac{1}{4}(e^{2\delta_i} + e^{-2\delta_i} + 2) \approx \tfrac{1}{4} e^{2\delta_i} \quad \Longrightarrow \quad \alpha_i \approx 4e^{-2\delta_i} . \tag{7.81}$$

For the sources α_0 we have

$$\begin{aligned} \alpha_0 &= \left(\frac{1}{\sinh^2 \delta_0} - \frac{\Pi_c^2 - \Pi_s^2}{\Pi_s^2} \right) \frac{\Pi_s^2}{\Pi_c^2} && (7.82) \\ &= \frac{\sinh^2 \delta_1 \sinh^2 \delta_2 \sinh^2 \delta_3}{\cosh^2 \delta_1 \cosh^2 \delta_2 \cosh^2 \delta_3} - 1 && (7.83) \\ &\approx 4 \sum_{i=1}^{3} e^{-2\delta_i} , && (7.84) \end{aligned}$$

here δ_0 is kept fixed while the other three charges $\delta_i \gg \delta_0$, where $i = 1, 2, 3$.

7.6 Kaluza-Klein Reduction

The realization is essentially the one that the STU model being considered is obtainable by dimensionally reducing the 5D theory (see appendix C) with Lagrangian

$$\mathcal{L}_5 = R_5 \star_5 \mathbb{1} - \tfrac{1}{2}H_{ij} \star_5 \mathrm{d}h^i \wedge \mathrm{d}h^j - \tfrac{1}{2}H_{ij} \star_5 \tilde{F}^i \wedge \tilde{F}^j + \tfrac{1}{6}C_{ijk}\tilde{F}^i \wedge \tilde{F}^j \wedge \tilde{A}^k, \tag{7.85}$$

where H_{ij} and C_{ijk} are defined as previously. The 5D line element is related to the 4D line element by the Kaluza-Klein (KK) ansatz

$$ds_5^2 = f^2(dz + A^0)^2 + f^{-1}ds_4^2, \tag{7.86}$$

and the 5D vector fields related to the 4D pseudo scalars and vector fields via

$$\tilde{A}^i = \chi^i(dz + A^0) + A^i. \tag{7.87}$$

This allows for the uplifting of the 4D geometry and matter fields to 5D, explicitly we have the 5D line element reading

$$ds_5^2 = e^{\frac{\eta_0}{3}}\sqrt{\Delta}\left(\frac{dr^2}{X} - \frac{G}{\Delta}dt^2 + \frac{e^{-\eta_0}}{\sqrt{\Delta}}(dz + A^0)^2\right) + e^{\frac{\eta_0}{3}}\sqrt{\Delta}\, ds^2(S^2). \tag{7.88}$$

For the minimally subtracted geometry on has $e^{\frac{\eta_0}{3}} = \bar{\Delta}^{-1/2}$ and as shown in [19] the 5D line element is locally of the form $\mathrm{AdS}_3 \times S^2$.

Following [2] we proceed to reduce from the 5D theory to a 3D theory choosing a KK ansatz reading

$$ds_5^2 = ds^2_{\text{string}}(M) + e^{2U(x)}ds^2(Y), \tag{7.89}$$

$$\tilde{F}^i = -B_i \sin\theta \, d\theta \wedge d\phi, \tag{7.90}$$

$$\Psi = \Psi(x), \tag{7.91}$$

$$\Phi = \Phi(x), \tag{7.92}$$

where M denotes the $(2+1)$-dimensional external manifold with coordinates $x = \{r, t, z\}$ and Y represents the two-sphere of radius ℓ_S with coordinates $y = \{\theta, \phi\}$. It is furthermore assumed that A^0 has no legs on the two-sphere (i.e it is purely electric). The orientation is chosen such that the volume form of the internal two-sphere reads

$$\mathrm{vol}_2 = \ell_S^2 \sin\theta \, d\theta \wedge d\phi. \tag{7.93}$$

The explicit calculations can be found in appendix E, we find that the five-dimensional scalar equations of motion reduce to the three-dimensional equations

$$0 = \mathrm{d}(\star_3 \mathrm{d}\Psi) + 2\mathrm{d}U \wedge \star_3 \mathrm{d}\Psi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Psi}\mathrm{vol}_3, \tag{7.94}$$

$$0 = \mathrm{d}(\star_3 \mathrm{d}\Phi) + 2\mathrm{d}U \wedge \star_3 \mathrm{d}\Phi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta_\Phi}\mathrm{vol}_3. \tag{7.95}$$

Similarly, the Einstein equations reduce to

$$\begin{aligned} R_{ab} =& 2(\nabla_b\nabla_a U + \nabla_a U \nabla_b U) \\ &+ \frac{1}{2}\left[\nabla_a\Psi\nabla^a\Psi + \nabla_a\Phi\nabla^a\Phi - \tfrac{1}{3}\eta_{ab}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi)\right], \end{aligned} \tag{7.96}$$

and

$$\nabla_a \nabla^a U + 2\nabla_a U \nabla^a U - \frac{e^{-2U}}{\ell_S^2} + \tfrac{1}{3} e^{-4U} \sum_{i=1}^{3} \frac{B_i^2}{\ell_S^4} H_{ii} = 0, \tag{7.97}$$

where (7.96) and (7.97) stem from the components of the Einstein equations in M and Y respectively.

Finally from the fact that all reference to the internal two-sphere drops out, we see that the reduction is indeed consistent. Furthermore as noted in [2], the equations of motion (7.94 - 7.97) can be derived from the string frame action

$$S_{\text{string}} = -\frac{1}{16\pi G_3} \int d^3x \sqrt{|g|} e^{2U} \Bigg[R + \frac{2}{\ell_S^2} e^{-2U} - \tfrac{1}{2} e^{-4U} \sum_{i=1}^{3} \frac{B_i^2}{\ell_S^4} H_{ii}(\Psi, \Phi) \\ + 2(\nabla U)^2 - \tfrac{1}{2}(\nabla \Psi)^2 - \tfrac{1}{2}(\nabla \Phi)^2 \Bigg]. \tag{7.98}$$

In order to pass to Einstein frame, the metric is simply rescaled by the transformation

$$ds^2_{\text{string}}(M) = e^{-4U} ds^2_{(E)}(M), \tag{7.99}$$

which implies that

$$\sqrt{|g|} = e^{-6U} \sqrt{|g_{(E)}|}, \tag{7.100}$$

and that

$$R = e^{4U} \left[R_{(E)} + 8\Box_{(E)} U - 8(\nabla_{(E)} U)^2 \right]. \tag{7.101}$$

From this it follows that the effective action in Einstein frame reads

$$S_{(E)} = -\frac{1}{16\pi G_3} \int d^3x \sqrt{|g_{(E)}|} \Bigg[R_{(E)} + \frac{2}{\ell_S^2} e^{-6U} - \tfrac{1}{2} e^{-8U} \sum_{i=1}^{3} \frac{B_i^2}{\ell_S^4} H_{ii}(\Psi, \Phi) \\ - 6(\nabla_{(E)} U)^2 - \tfrac{1}{2}(\nabla_{(E)} \Psi)^2 - \tfrac{1}{2}(\nabla_{(E)} \Phi)^2 \Bigg]. \tag{7.102}$$

7.7 Explicit Expressions for the 3D Fields

We now derive the results found in appendix **A.3** of [2]; the explicit expressions for the scalars h^i apparent in both the 5D and the 3D equations of motion, as well as the expressions for the 3D fields U, Ψ and Φ. We begin by casting the fields h^i in terms of the fields ϕ_I that diagonalize the coupled equations.

From

$$h^i = e^{\frac{1}{3}\eta_0 - \eta_i} \tag{7.103}$$

and from (7.36 - 7.39) one readily finds

$$\begin{aligned} h^1 &= \exp\left[\tfrac{1}{6}(2\phi_1 - \phi_2 - \phi_3)\right], \\ h^2 &= \exp\left[\tfrac{1}{6}(2\phi_2 - \phi_1 - \phi_3)\right], \\ h^3 &= \exp\left[\tfrac{1}{6}(2\phi_3 - \phi_1 - \phi_2)\right]. \end{aligned} \tag{7.104}$$

Plugging in the regular solutions (7.50) one then finds

$$\begin{aligned} h^1(r) &= \left(\frac{B_1^2}{|B_2 B_3|} \frac{\sqrt{(1+a_2^2)(1+a_3^2)}}{1+a_1^2} \frac{(a_1^2 r + 2m)^2}{(a_2^2 r + 2m)(a_3^2 r + 2m)} \right)^{1/3}, \\ h^2(r) &= \left(\frac{B_2^2}{|B_1 B_3|} \frac{\sqrt{(1+a_1^2)(1+a_3^2)}}{1+a_2^2} \frac{(a_2^2 r + 2m)^2}{(a_1^2 r + 2m)(a_3^2 r + 2m)} \right)^{1/3}, \\ h^3(r) &= \left(\frac{B_3^2}{|B_1 B_2|} \frac{\sqrt{(1+a_1^2)(1+a_2^2)}}{1+a_3^2} \frac{(a_3^2 r + 2m)^2}{(a_1^2 r + 2m)(a_2^2 r + 2m)} \right)^{1/3}. \end{aligned} \tag{7.105}$$

Next we consider the 3D fields. Staring with U, we deduce from the KK ansatz (7.89) and the general form of the 5D line element (7.86) that

$$e^{2U} = e^{\eta_0/3}\frac{\sqrt{\Delta}}{\ell_S^2}. \tag{7.106}$$

Now from (7.36 - 7.39) we find

$$U = \tfrac{1}{6}(\phi_1 + \phi_2 + \phi_3) + \log(m/\ell_S) \tag{7.107}$$

and plugging in the regular solutions (7.50) we find

$$U(r) = \tfrac{1}{3}\log\left[\frac{1}{\sqrt{(1+a_1^2)(1+a_2^2)(1+a_3^2)}}\left(\frac{a_1^2}{2m}r+1\right)\left(\frac{a_2^2}{2m}r+1\right)\left(\frac{a_3^2}{2m}r+1\right)\right]. \tag{7.108}$$

We will not bother going through the other fields, the procedure is very similar, and the results can be found in appendix **A.3** of [2].

7.8 Subtracted Solution for 3D Fields

The subtracted solution is the member of the general family of solutions with $a_i = 0$. We will simply note down the form that the fields h^i and U take on in this case. We find plugging $a_i = 0$ into the findings of the previous section that

$$h^1 = \left(\frac{B_1^2}{|B_2 B_3|}\right)^{1/3}, \qquad h^2 = \left(\frac{B_2^2}{|B_1 B_3|}\right)^{1/3}, \qquad h^3 = \left(\frac{B_3^2}{|B_1 B_2|}\right)^{1/3}, \tag{7.109}$$

and

$$\bar{U} = 0. \tag{7.110}$$

Plugging this into the 3D equation of motion for U gives the relation

$$\ell_S^2 = \tfrac{1}{3}\sum_{i=1}^{3} B_i^2 H_{ii}, \tag{7.111}$$

which given $H_{ii} = (h^i)^{-2}$ gives the result

$$\ell_S = (B_1 B_2 B_3)^{1/3}. \tag{7.112}$$

We see now that the term

$$\frac{2}{\ell_s^2}e^{-6U} - \tfrac{1}{2}e^{-8U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_s^4}H_{ii}(\Psi,\Phi) \tag{7.113}$$

in equation (7.102) is constant for the subtracted solution, and as in [2], dubbing this constant $-2\Lambda_{\text{eff}}$, we see that the subtracted geometry is a solution to Einstein's equations with an effective cosmological constant Λ_{eff}. All solutions of Einstein's equations with negative cosmological constant in three-dimensions are locally AdS_3, thus paving the road for a two-dimensional CFT interpretation of the subtracted geometry.

7.9 Linearizing 3D Fields

Linearizing the 3D fields Ψ, Φ, U around the subtracted solution, one finds that their fluctuations all satisfy the same equation

$$\nabla_\mu \nabla^\mu \delta F - \frac{8}{L^2}\delta F = 0, \tag{7.114}$$

where L is the effective AdS$_3$ length and F denotes any of the fields Ψ, Φ, U. We will later need that

$$L = 2e^{3\bar{U}}\ell_S, \qquad \ell_S = (B_1 B_2 B_3)^{1/3}. \tag{7.115}$$

We will derive this result for Ψ, the derivation follows similarly for Φ and U. It comes about by linearizing their equations of motion around the subtracted solution. The equations of motion read

$$0 = \nabla_\mu \nabla^\mu \Psi - \tfrac{1}{2}e^{-8U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}(\Psi,\Phi)}{\delta \Psi}, \tag{7.116}$$

$$0 = \nabla_\mu \nabla^\mu \Phi - \tfrac{1}{2}e^{-8U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}(\Psi,\Phi)}{\delta \Phi}, \tag{7.117}$$

$$0 = \nabla_\mu \nabla^\mu U - \frac{e^{-6U}}{\ell_S^2} + \tfrac{1}{3}e^{-8U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi). \tag{7.118}$$

The subtracted solution $\{\bar{\Psi}, \bar{\Phi}, \bar{U}\}$ reads

$$e^{\bar{U}} = \left(\frac{B_1 B_2 B_3}{\ell_S^2}\right)^{1/3} = 1, \qquad e^{\bar{\Psi}} = \left(\frac{B_1^2}{B_2 B_3}\right)^{1/\sqrt{6}}, \qquad e^{\bar{\Phi}} = \left(\frac{B_3}{B_2}\right)^{1/\sqrt{2}}. \tag{7.119}$$

Furthermore we need

$$h^1 = e^{\sqrt{2/3}\Psi}, \qquad h^2 = e^{-\sqrt{1/6}\Psi - \sqrt{1/2}\Phi}, \qquad h^3 = e^{-\sqrt{1/6}\Psi + \sqrt{1/2}\Phi}, \tag{7.120}$$

since $H_{ii} = (h^i)^{-2}$

$$H_{11} = e^{-\sqrt{8/3}\Psi}, \qquad H_{22} = e^{\sqrt{2/3}\Psi + \sqrt{2}\Phi}, \qquad H_{33} = e^{\sqrt{2/3}\Psi - \sqrt{2}\Phi}. \tag{7.121}$$

We start by inserting $\Psi = \bar{\Psi} + \delta\Psi$ into its equation of motion (7.116)

$$0 = \nabla_\mu \nabla^\mu \bar{\Psi} + \nabla_\mu \nabla^\mu \delta\Psi - \tfrac{1}{2}e^{-8\bar{U}}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Psi}\bigg|_{\Psi=\bar{\Psi}+\delta\Psi}. \tag{7.122}$$

Note that we are evaluating on the subtracted background, i.e we have $\Phi = \bar{\Phi}$ and $U = \bar{U}$. We need to evaluate the functional derivative of H_{ii} with respect to Ψ. Using (7.121) we see that these functional derivatives will all be of the form $\exp(F)$, a linearization of which reads

$$e^{\bar{F}+\delta F} = e^{\bar{F}}e^{\delta F} \approx e^{\bar{F}} + e^{\bar{F}}\delta F. \tag{7.123}$$

Clearly the $e^{\bar{F}}$ together with $\nabla_\mu \nabla^\mu \bar{F}$ will vanish, since $\bar{F}$ is a solution to the equation of motion. We use F here since this argument is general and applies for all the three fields. For Ψ we get explicitly

$$0 = \nabla_\mu \nabla^\mu \delta\Psi - \frac{e^{-8\bar{U}}}{2\ell_S^4}\left(\tfrac{8}{3}B_1^2 e^{-\sqrt{8/3}\bar{\Psi}} + \tfrac{2}{3}B_2^2 e^{\sqrt{2/3}\bar{\Psi}+\sqrt{2}\bar{\Phi}} + \tfrac{2}{3}B_3^2 e^{\sqrt{2/3}\bar{\Psi}-\sqrt{2}\bar{\Phi}}\right)\delta\Psi. \tag{7.124}$$

To evaluate the parenthesis, we use (7.119), and find

$$0 = \nabla_\mu \nabla^\mu \delta\Psi - \frac{8}{L^2}\delta\Psi. \tag{7.125}$$

From this one can read off the conformal weights of the dual operators, which are found using the standard dictionary to be "$\Delta = 4$" for each of the three fields [2], from which one draws the conclusion that they are *irrelevant*.

The standard dictionary (operator-state map) states that a scalar field in the bulk of AdS_{d+1} is dual to an operator at the boundary with scaling dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2}, \tag{7.126}$$

where d is the dimensionality of the dual CFT, m is the mass of the scalar field and ℓ is the curvature radius of AdS [1]. In our case it is straight forward to see how this equates to $\Delta = 4$, for the CFT operators dual to the 3D scalar perturbations $\delta\Phi, \delta\Psi$ and δU, in the AdS_3 bulk theory.

7.10 Linearizing around Δ_-

We now repeat section 7.4 for the subtracted geometry with warp factor Δ_-, i.e

$$\phi_I = \phi_I^{\mathrm{subt}} + \delta\phi_I, \tag{7.127}$$

where ϕ_I are given as

$$\phi_I(r) = \log\left[\frac{Q_I^2}{4m^4}\frac{(a_I^2 r + 2m)^2}{1+a_I^2}\right], \qquad Q_0 \equiv q_0, \qquad Q_i \equiv B_i. \tag{7.128}$$

The decoupled equations of motion for these fields reads

$$(r(r-2m)\phi_I'(r))' + 2\left(\frac{Q_I^2}{m^2}e^{-\phi_I(r)} - 1\right) = 0. \tag{7.129}$$

Plugging in $\phi_I = \phi_I^{\mathrm{subt}} + \delta\phi_I$ gives

$$(r(r-2m)\delta\phi_I')' - 2\frac{Q_I^2}{m^2}e^{-\phi_I^{\mathrm{subt}}}\delta\phi_I = 0. \tag{7.130}$$

Explicitly in terms of the a's this reads

$$(r(r-2m)\delta\phi_I')' - 8m^2\frac{(1+a_I^2)}{(a_I^2 r + 2m)^2}\delta\phi_I = 0. \tag{7.131}$$

For $I = 2, 3$ we then get that the solutions for $\delta\phi_I$ coincide with the ones for the minimally subtracted case studied in [2], i.e, after changing variable $x = r/m - 1$, the equation for $\delta\phi_i$ for $i = 2, 3$ is seen to be the Legendre equation. The solutions that are regular at the horizon read

$$\delta\phi_i = \alpha_i\left(\frac{r}{m} - 1\right), \qquad i = 2, 3. \tag{7.132}$$

For $I = 0, 1$ and inserting

$$a_0 = a_1 = \sqrt{\frac{\Pi_c - \Pi_s}{\Pi_s}}, \tag{7.133}$$

we get

$$(r(r-2m)\delta\phi_I')' - 2\frac{|q_0 B_1 B_2 B_3|}{\Delta_-}\delta\phi_I = 0, \tag{7.134}$$

where we expressed the coefficient in front of $\delta\phi_I$ in terms of the warp factor and the charges, to aid comparison with (7.58) for the minimally subtracted geometry where the coefficient reads

$$2\frac{q_0^2 B_1^2 B_2^2 B_3^2}{\bar{\Delta}^2}. \tag{7.135}$$

To solve (7.134) we again perform the coordinate change $x = r/m - 1$, and thereafter employ

$$b = \frac{\Pi_c - \Pi_s}{2\sqrt{2\Pi_c\Pi_s}}, \qquad c = \frac{\Pi_c + \Pi_s}{2\sqrt{2\Pi_c\Pi_s}}, \tag{7.136}$$

in terms of which (7.134) reads

$$(1 - x^2)\delta\phi_I''(x) - 2x\delta\phi_I'(x) + \frac{1}{(bx + c)^2}\delta\phi_I(x) = 0. \tag{7.137}$$

The solutions that are regular at the horizon read

$$\delta\phi_I(x) = \alpha_I \frac{cx + b}{bx + c} = \alpha_I \frac{(\Pi_c + \Pi_s)r - 2m\Pi_s}{(\Pi_c - \Pi_s)r + 2m\Pi_s}, \qquad I = 0, 1. \tag{7.138}$$

7.11 Perturbation Theory around Δ_-

We now linearize around the values a_I^{subt} that give us Δ_-. We start with ϕ_i for $i = 2, 3$

$$\phi_i^{\text{subt}} = \log\left[\frac{B_i^2}{m^2}\right], \qquad i = 2, 3. \tag{7.139}$$

Making a Taylor expansion around $a_i^2 = (a_i^{\text{subt}})^2 = 0$ we find

$$\phi_i = \phi_i^{\text{subt}} + a_i^2\left(\frac{r}{m} - 1\right) + \cdots \tag{7.140}$$

which agrees with the minimally subtracted case. We proceed on to $I = 0, 1$

$$\phi_I^{\text{subt}} = \log\left[\frac{Q_I^2}{m^2}\frac{\Delta_-}{|q_0 B_1 B_2 B_3|}\right], \qquad Q_I = q_0, B_1, \qquad I = 0, 1. \tag{7.141}$$

Now the Taylor expansion around $a_I^2 = (a_I^{\text{subt}})^2 = (\Pi_c - \Pi_s)/\Pi_s, I = 0, 1$ gives

$$\phi_I = \phi_I^{\text{subt}} + (a_I^2 - (a_I^{\text{subt}})^2)\frac{\Pi_s}{\Pi_c}\frac{(\Pi_c + \Pi_s)r - 2m\Pi_s}{(\Pi_c - \Pi_s)r + 2m\Pi_s} + \cdots. \tag{7.142}$$

Essentially, we are lead to the suspicion that employing Δ_- in place of $\bar{\Delta}$ does not change things a whole lot. As noted in [2], one is already lead to the suspicion that in the minimally subtracted case, the three $\delta\phi_i, i = 1, 2, 3$ correspond to irrelevant perturbations while $\delta\phi_0$ seems to be associated with a marginal perturbation. This suspicion is supported by the large r behavior of the perturbations $\delta\phi_I$. Similarly, in the case of Δ_- our linear analysis leads us to suspect that now two of the modes $\delta\phi_i, i = 2, 3$ correspond to irrelevant perturbations while the modes $\delta\phi_I, I = 0, 1$ seem marginal. The validity of the dual conformal field theory, if any, will then still be limited to the same extent as in the case of the minimally subtracted geometry. We still require $\alpha \ll 1$ (now α being the smallest of α_2, α_3). However, for the Δ_- case we do then have the freedom to consider two finite charges q_0 and B_1, while we need to have B_2 and B_3 large. So in that sense it is an improvement since in the minimally subtracted case, all three magnetic charges have to be large. As we noted earlier, we necessarily have two a's turned off, in order to at most have $\Delta \sim r^2$ for large r. Thus it seems that allowing for two finite charges is the best one can do in the subtracted geometry approach, for the family of static black holes discussed. Extending the analysis in this chapter to the rotating case is of particular interest, this has been achieved by the use of Harrison transformations in [17].

8

Supporting Matter - Rotating Case

The interpolating black hole family (7.51 - 7.54) established in [2] for the four charge static black holes in the STU model, is useful in that it readily gives the linear perturbations needed to start the interpolating flow between the subtracted and original geometry. This allows for a detailed comparison of the subtracted and original geometry, which is highly valuable in an attempt to understand the extent to which a dual CFT description is viable.

In the rotating case, we do not seem to have the luxury of a simple interpolating solution, however there is still a powerful approach at our disposal [1]. Indeed the scaling limits discussed in [16] that extract the minimally subtracted geometry from the original geometry, is not limited to the static case. We will review the scaling limit on the minimally subtracted geometry, and implement a similar scaling limit that extracts our warp factor Δ_- from the original: We successfully employ the scaling limit in the rotating case, and verify that the matter supports the geometry with warp factor Δ_-.

8.1 Scaling Limits in the Static Case

In [16] they obtain the minimally subtracted geometry of [19] in a scaling limit of the original geometry. Since their notation is slightly different, we will firstly point out how the parameters relate to the notation used in [2], which is the notation we stick to when discussing the static case.

The charge parameters δ_i coincide with the ones employed herein, while the δ_4 of [16] is instead referred to as δ_0. Furthermore, crucially, the scalar fields are related non trivially

$$\varphi_1 \leftrightarrow \eta_2, \qquad \varphi_2 \leftrightarrow \eta_1, \qquad \varphi_3 \leftrightarrow \eta_3. \tag{8.1}$$

For the vector matter we have in the static case

$$F_1 \leftrightarrow F^1, \qquad F_2 \leftrightarrow -e^{-\eta_0+2\eta_2} \star F^2, \qquad \mathcal{F}_1 \leftrightarrow F^3, \qquad \mathcal{F}_2 \leftrightarrow F^0, \tag{8.2}$$

where we are relating field strengths as defined in [16] on the LHSs with those in [2] on the RHSs. In the static case $F^i, i = 1, 2, 3$ are purely magnetic, while F^0 is purely electric.

We follow the first example of [16]. Considering the static case $a = 0$, and without loss of generality taking three of the charges equal $\tilde{\delta}_i \equiv \tilde{\delta}, i = 1, 2, 3$ we take the scaling limit with $\epsilon \to 0$

$$\begin{gathered}\tilde{r} = r\epsilon, \qquad \tilde{t} = t\epsilon^{-1}, \qquad \tilde{m} = m\epsilon, \\ 2\tilde{m}\sinh^2\tilde{\delta} \equiv Q = 2m\epsilon^{-1/3}(\Pi_c^2 - \Pi_s^2)^{1/3}, \qquad \sinh^2\tilde{\delta}_0 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2},\end{gathered} \tag{8.3}$$

where the "tilde" coordinates and parameters of the *scaled* solution are related to those of the (in this case) minimally subtracted geometry in the static case. Applying the scaling limit (8.3) to the original warp factor (6.39), one readily finds

$$\Delta_0 \to \bar{\Delta} = (2m)^3[(\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2]. \tag{8.4}$$

[1] We note that in [17] Harrison transformations provide a four-parameter family that interpolates between the original and minimally subtracted geometry in the rotating case.

Furthermore, employing this scaling limit on the equations for the matter supporting the original geometry, one finds the matter that supports the subtracted geometry.

We wish to find a similar scaling limit that gives us

$$\Delta_0 \to \Delta_-. \tag{8.5}$$

Inspired by the example of [16], we set the charges equal in pairs, i.e

$$\tilde{\delta}_2 = \tilde{\delta}_3 = \tilde{\delta}, \qquad \tilde{\delta}_0 = \tilde{\delta}_1 = \tilde{\delta}_f. \tag{8.6}$$

The subscript f indicates the finiteness of those parameters. It seems one has to impose that the two finite charges coincide, while the two large charges are set identical without loss of generality. We find that employing the scaling limit $\epsilon \to 0$

$$\begin{gathered}\tilde{r} = r\epsilon, \qquad \tilde{t} = t\epsilon^{-1}, \qquad \tilde{m} = m\epsilon, \\ 2\tilde{m}\sinh^2\tilde{\delta} \equiv Q = 2m\epsilon^{-1}(\Pi_c - \Pi_s), \qquad \sinh^2\tilde{\delta}_f = \frac{\Pi_s}{\Pi_c - \Pi_s},\end{gathered} \tag{8.7}$$

on the original warp factor gives

$$\Delta_0 = \epsilon^2(r + 2m\sinh^2\tilde{\delta}_f)^2(r\epsilon + Q)^2 \;\to\; 4m^2\big[(\Pi_c - \Pi_s)r + 2m\Pi_s\big]^2. \tag{8.8}$$

For the scalar matter

$$e^{-\eta_i} = h_i\sqrt{\frac{h_0}{h_1h_2h_3}}, \qquad h_I = r + 2m\sinh^2\delta_I, \qquad I = 0,1,2,3, \tag{8.9}$$

one finds that the scaling limit gives

$$\eta_1 = -\tfrac{1}{2}\log\left(4m^2[(\Pi_c - \Pi_s)r + 2m\Pi_s]^2\right) + \log Q^2, \tag{8.10}$$

$$\eta_2 = \eta_3 = 0, \tag{8.11}$$

and for the vector matter

$$A_t^I = \frac{2m\sinh\delta_I\cosh\delta_I}{h_I}, \tag{8.12}$$

one finds

$$A^0 = A^1 = \frac{2mQ\sqrt{\Pi_c\Pi_s}}{(\Pi_c - \Pi_s)\Delta_-^{1/2}}\,dt, \tag{8.13}$$

$$A^2 = A^3 = -\frac{r}{Q}\,dt. \tag{8.14}$$

The resulting field strengths read

$$F^0_{tr} = F^1_{tr} = \frac{4m^2Q\sqrt{\Pi_c\Pi_s}}{\Delta_-}, \tag{8.15}$$

$$F^2_{tr} = F^3_{tr} = \frac{1}{Q}. \tag{8.16}$$

From the field strengths we can read off the charges

$$-q_0 = B_1 = \frac{4m^2\sqrt{\Pi_c\Pi_s}}{Q}, \qquad B_2 = B_3 = Q. \tag{8.17}$$

This agrees with (6.73 - 6.80), and it follows that the scaling limit has given us matter and geometry that solves the Einstein equations and the matter equations of motion, as it is a member of the family studied in [2].

Note that taking $\Pi_s \to -\Pi_s$ and repeating the above, the limit gives the subtracted geometry with warp factor Δ_+, and the matter is identical up to a change in the sign of Π_s. Notably the gauge fields A^0 and A^1 become proportional to $\sqrt{-\Pi_c\Pi_s}$.

8.2 Scaling Limit for $\bar{\Delta}$ in the Rotating Case

We found matter supporting the subtracted geometries in the statics case, both via directly solving the equations of motion, and also now in the last section by applying a scaling limit on the original geometry and matter. To find matter that supports the subtracted geometry in the rotating case is rather difficult. Indeed it is not straight forward to simply find a solution to the equations of motion. However, employing a scaling limit on the original matter that supports the original geometry with rotation, is significantly easier. In reference [16] they employed such a limit and found matter supporting the minimally subtracted geometry in the rotating case. We will attempt the same for the rotating subtracted geometry with warp factor Δ_-.

Now that rotation is turned on it is no longer sufficient to deal with the truncated pseudoscalar-free STU model, indeed we require in general non-vanishing pseudoscalars in the rotating case. To warm up we quickly go through the steps taken in [16], where the relevant Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_4 = R \star \mathbb{1} - \tfrac{1}{2} \star d\varphi_i \wedge d\varphi_i - \tfrac{1}{2} e^{2\varphi_i} \star d\chi_i \wedge d\chi_i - \tfrac{1}{2} e^{-\varphi_1} \Big(e^{\varphi_2 - \varphi_3} \star F_1 \wedge F_1 \\ + e^{\varphi_2 + \varphi_3} \star F_2 \wedge F_2 + e^{-\varphi_2 + \varphi_3} \star \mathcal{F}_1 \wedge \mathcal{F}_1 + e^{-\varphi_2 - \varphi_3} \star \mathcal{F}_2 \wedge \mathcal{F}_2 \Big) \\ - \chi_1 (F_1 \wedge \mathcal{F}_1 + F_2 \wedge \mathcal{F}_2),\end{aligned} \tag{8.18}$$

in terms of field strengths

$$F_1 = dA_1 - \chi_2 d\mathcal{A}_2, \tag{8.19}$$
$$F_2 = dA_2 + \chi_2 d\mathcal{A}_1 - \chi_3 dA_1 + \chi_2 \chi_3 d\mathcal{A}_2, \tag{8.20}$$
$$\mathcal{F}_1 = d\mathcal{A}_1 + \chi_3 d\mathcal{A}_2, \tag{8.21}$$
$$\mathcal{F}_2 = d\mathcal{A}_2. \tag{8.22}$$

This Lagrangian is considered as opposed to the one in [2] since the matter solutions supporting the original geometry derived in [13] are solutions with this parameterization of the action. We show how this Lagrangian relates to the triality invariant form used in [2] in appendix C.2.

From now on we stick to the notation used in [16]. Again, without loss of generality they consider the case where three of the charges are equal, thus

$$\star F_1 = F_2 = \star \mathcal{F}_1 \equiv F = dA, \qquad \mathcal{F}_2 \equiv \mathcal{F} = d\mathcal{A}. \tag{8.23}$$

The original matter supporting that geometry is rather non-trivial, now that rotation is turned on. That matter is however, readily transcribed from [13], and with

$$\tilde{\delta}_1 = \tilde{\delta}_2 = \tilde{\delta}_3 = \tilde{\delta}, \qquad \tilde{\delta}_4 = \tilde{\delta}_0, \tag{8.24}$$

and employing the abbreviations

$$\tilde{s} = \sinh \tilde{\delta}, \quad \tilde{c} = \cosh \tilde{\delta}, \qquad \tilde{s}_0 = \sinh \tilde{\delta}_0, \quad \tilde{c}_0 = \cosh \tilde{\delta}_0, \tag{8.25}$$

the original warp factor reads

$$\begin{aligned}\Delta_0 = (\tilde{r} + 2\tilde{m}\tilde{s}^2)^3 (\tilde{r} + 2\tilde{m}\tilde{s}_0^2) + \tilde{a}^4 \cos^4\theta \\ + 2\tilde{a}^2 \cos^2\theta \left[\tilde{r}^2 + \tilde{m}\tilde{r}(3\tilde{s}^2 + \tilde{s}_0^2) + 4\tilde{m}^2 \tilde{c}_0 \tilde{s}_0 \tilde{c}^3 \tilde{s}^3 - 2\tilde{m}^2 (\tilde{s}^6 + 3\tilde{s}_0^2 \tilde{s}^4 + 2\tilde{s}_0^2 \tilde{s}^6) \right],\end{aligned} \tag{8.26}$$

the scalar matter reads[2]

$$\chi_1 = -\chi_2 = \chi_3 = \frac{2\tilde{m}\tilde{a}\cos\theta}{(\tilde{r}+2\tilde{m}\tilde{s}^2)^2+\tilde{a}^2\cos^2\theta}\,\tilde{c}\tilde{s}(\tilde{c}\tilde{s}_0-\tilde{s}\tilde{c}_0), \tag{8.27}$$

$$e^{\varphi_i} = \frac{(\tilde{r}+2\tilde{m}\tilde{s}^2)^2+\tilde{a}^2\cos^2\theta}{\tilde{\Delta}_0^{1/2}}, \quad i=1,2,3, \tag{8.28}$$

and the gauge potentials read

$$\begin{aligned} A = \frac{2\tilde{m}}{\tilde{\Delta}_0}\Big\{ & \big[(\tilde{r}+2\tilde{m}\tilde{s}^2)^2(\tilde{r}+2\tilde{m}\tilde{s}_0^2)+\tilde{r}\tilde{a}^2\cos^2\theta\big] \\ & \times\big[\tilde{c}\tilde{s}\,d\tilde{t}-\tilde{a}\sin^2\theta\;\tilde{c}\tilde{s}(\tilde{c}\tilde{c}_0-\tilde{s}\tilde{s}_0)d\phi\big] \\ & +2\tilde{m}\tilde{a}^2\cos^2\theta(e\,d\tilde{t}-\tilde{a}\sin^2\theta\;\tilde{s}_0\tilde{c}\tilde{s}^2\,d\phi)\Big\}, \end{aligned} \tag{8.29}$$

$$\begin{aligned} \mathcal{A} = \frac{2\tilde{m}}{\tilde{\Delta}_0}\Big\{ & \big[(\tilde{r}+2\tilde{m}\tilde{s}^2)^3+\tilde{r}\tilde{a}^2\cos^2\theta\big] \\ & \times\big[\tilde{c}_0\tilde{s}_0\,d\tilde{t}-\tilde{a}\sin^2\theta(\tilde{c}^3\tilde{s}_0-\tilde{s}^3\tilde{c}_0)d\phi\big] \\ & +2\tilde{m}\tilde{a}^2\cos^2\theta\big(e_0\,d\tilde{t}-\tilde{a}\sin^2\theta\;\tilde{s}^3\tilde{c}_0\,d\phi)\Big\}, \end{aligned} \tag{8.30}$$

where

$$e = \tilde{s}^2\tilde{c}^2\tilde{c}_0\tilde{s}_0(\tilde{c}^2+\tilde{s}^2)-\tilde{s}^3\tilde{c}(\tilde{s}^2+2\tilde{s}_0^2+2\tilde{s}^2\tilde{s}_0^2), \tag{8.31}$$

$$e_0 = \tilde{s}^3\tilde{c}^3(\tilde{c}_0^2+\tilde{s}_0^2)-\tilde{c}_0\tilde{s}_0(3\tilde{s}^4+2\tilde{s}^6). \tag{8.32}$$

Applying the scaling limit (8.3) along with

$$\tilde{a} = a\epsilon, \tag{8.33}$$

to the geometry and the matter, one gets the minimally subtracted geometry (with rotation) and matter.

In the static case, as pointed out in [16], as far as the geometry is concerned, scaling only effects the warp factor. This is rather straight forward to see, however, when rotation is turned on we need to explicitly check that the reduced potential $\mathcal{A}_{\rm red}$ is unaffected by the scaling limit. Performing the limit we find that

$$\mathcal{A}_{\rm red} = 2\tilde{m}\big[(\tilde{c}^3\tilde{c}_0-\tilde{s}^3\tilde{s}_0)\tilde{r}+2\tilde{m}\tilde{s}^3\tilde{s}_0\big] \to 2m\left[(\Pi_c-\Pi_s)r+2m\Pi_s\right], \tag{8.34}$$

thus showing that the scaling limit only effects the matter and the warp factor, leaving the metric otherwise unaltered.

We were able to verify most of the matter as displayed in reference [16], however, our result for the time component of the gauge field A differs. Indeed the time component requires special care, as in the limit one has to go to second order since to leading order one literally has $A_t \to 1/\epsilon$ (a constant). One needs to be very precise in order to capture all the terms that contribute. For instance, it becomes necessary to expand the original warp factor in powers of ϵ, i.e we cannot simply plug in the subtracted warp factor before taking the limit. We find that we can reproduce the result in [16] if we in the limit let

$$\cosh\tilde{\delta} = \sinh\tilde{\delta}, \tag{8.35}$$

[2]There seems to have been a slight sign mistake in [16], according to [13] we should have $-\chi_2$ in (8.27).

but of course to higher order in epsilon, this is incorrect, instead one has

$$\cosh\tilde{\delta} = \sinh\tilde{\delta} + \frac{\frac{1}{2}}{\sinh\tilde{\delta}} + \cdots . \tag{8.36}$$

According to our calculations the t component should read

$$A_t = -\frac{r}{Q} - \frac{(2m)^2(\Pi_c - \Pi_s)[(\Pi_c - \Pi_s)r + 2m\Pi_s]a^2\cos^2\theta}{Q\Delta}. \tag{8.37}$$

Our calculations agreed with the matter in [16] otherwise. We leave explicit checking of the equations of motion for this matter for future work.

8.3 Scaling Limit for Δ_- in the Rotating Case

We now repeat the steps taken in the previous section, but for our warp factor Δ_-. We will be taking the scaling limit (8.7) supplemented with $\tilde{a} = a\epsilon$ [3]. That scaling limit requires that we set the two finite charges equal, while the other two are set equal without loss of generality

$$F_2 = \star\mathcal{F}_1 \equiv F = dA, \qquad \star F_1 = \mathcal{F}_2 \equiv \mathcal{F} = d\mathcal{A}, \tag{8.38}$$

$$\tilde{\delta}_2 = \tilde{\delta}_3 = \tilde{\delta}, \qquad \tilde{\delta}_1 = \tilde{\delta}_4 = \tilde{\delta}_0. \tag{8.39}$$

We then transcribe from [13] that the original warp factor reads

$$\Delta_0 = \left((\tilde{r} + 2\tilde{m}\tilde{s}^2)(\tilde{r} + 2\tilde{m}\tilde{s}_0^2) + \tilde{a}^2\cos^2\theta\right)^2 \tag{8.40}$$

and the matter reads

$$\chi_2 = \frac{2\tilde{m}(\tilde{s}^2 - \tilde{s}_0^2)\tilde{a}\cos\theta}{(\tilde{r} + 2\tilde{m}\tilde{s}^2)^2 + \tilde{a}^2\cos^2\theta}, \qquad \chi_1 = \chi_3 = 0, \tag{8.41}$$

$$e^{\varphi_2} = \frac{(\tilde{r} + 2\tilde{m}\tilde{s}^2)^2 + \tilde{a}^2\cos^2\theta}{\tilde{\Delta}_0^{1/2}}, \qquad \varphi_1 = \varphi_3 = 0, \tag{8.42}$$

$$A = \frac{2\tilde{m}}{\tilde{\Delta}_0^{1/2}}\,\tilde{c}\tilde{s}(\tilde{r} + 2\tilde{m}\tilde{s}_0^2)\left[d\tilde{t} - \tilde{a}\sin^2\theta\, d\phi\right], \tag{8.43}$$

$$\mathcal{A} = \frac{2\tilde{m}}{\tilde{\Delta}_0^{1/2}}\,\tilde{c}_0\tilde{s}_0(\tilde{r} + 2\tilde{m}\tilde{s}^2)\left[d\tilde{t} - \tilde{a}\sin^2\theta\, d\phi\right]. \tag{8.44}$$

Taking the scaling limit (8.7) supplemented with $\tilde{a} = a\epsilon$, one finds that the geometry is unaltered aside from the intended change in the warp factor

$$\Delta_0 \to \Delta_- = (2m)^2\left[(\Pi_c - \Pi_s)r + 2m\Pi_s\right]^2, \tag{8.45}$$

and matter

$$\chi_2 = \frac{2m(\Pi_c - \Pi_s)a\cos\theta}{Q^2}, \qquad \chi_1 = \chi_3 = 0, \tag{8.46}$$

$$e^{\varphi_2} = \frac{Q^2}{\Delta_-^{1/2}}, \qquad \varphi_1 = \varphi_3 = 0, \tag{8.47}$$

[3]Note that in our current notation we will be using $\tilde{\delta}_0$ instead of $\tilde{\delta}_f$.

$$A = -\frac{r}{Q}\,dt - \frac{2m(\Pi_c - \Pi_s)a^2\cos^2\theta}{Q\Delta_-^{1/2}}\,dt + \frac{2m(\Pi_c - \Pi_s)a\sin^2\theta}{Q}\,d\phi, \tag{8.48}$$

$$\mathcal{A} = \frac{2mQ\sqrt{\Pi_c\Pi_s}}{(\Pi_c - \Pi_s)\Delta_-^{1/2}}\,dt - \frac{(2m)^3\sqrt{\Pi_c\Pi_s}(\Pi_c - \Pi_s)a\sin^2\theta}{Q\Delta_-^{1/2}}\,d\phi. \tag{8.49}$$

From these potentials, when setting $a = 0$ we read off the non-zero field strength components

$$F_{tr} = \frac{1}{Q}, \qquad \mathcal{F}_{tr} = \frac{4m^2Q\sqrt{\Pi_c\Pi_s}}{\Delta_-}.$$

This agrees on the nose with (8.15 - 8.16). Similarly the scalar matter for $a = 0$ coincides with the scalar matter (8.10 - 8.11).

Instead of checking the equations of motion for this matter and geometry, we note that there is a more natural scaling limit from the point of view of the two-magnetic two-electric formulation in [13]. Indeed in [13] they remark on the simplified solution obtained when setting the charges equal in pairs, but not like we did before, instead the more natural choice

$$A_1 = \mathcal{A}_1, \qquad A_2 = \mathcal{A}_2, \tag{8.50}$$

$$\tilde\delta_1 = \tilde\delta_3 = \tilde\delta, \qquad \tilde\delta_2 = \tilde\delta_4 = \tilde\delta_0, \tag{8.51}$$

where A_1 and $\mathcal{A}_1$ carry magnetic charge, while A_2 and $\mathcal{A}_2$ carry electric charge. The original warp factor for this pairing of charges coincides with the previous one, while the matter now reads

$$\chi_1 = \frac{2\tilde m\,(\tilde s_0^2 - \tilde s^2)\tilde a\cos\theta}{(\tilde r + 2\tilde m\tilde s^2)^2 + \tilde a^2\cos^2\theta}, \qquad \chi_2 = \chi_3 = 0, \tag{8.52}$$

$$e^{\varphi_1} = \frac{(\tilde r + 2\tilde m\tilde s^2)^2 + \tilde a^2\cos^2\theta}{\tilde\Delta_0^{1/2}}, \qquad \varphi_2 = \varphi_3 = 0, \tag{8.53}$$

$$A_2 = \mathcal{A}_2 = \frac{2\tilde m}{\tilde\Delta_0^{1/2}}\,\tilde c_0\tilde s_0(\tilde r + 2\tilde m\tilde s^2)\left[d\tilde t - \tilde a\sin^2\theta\,d\phi\right], \tag{8.54}$$

$$A_1 = \mathcal{A}_1 = \frac{2\tilde m}{\tilde\Delta_0^{1/2}}\,\tilde c\tilde s\left[\tilde a\,d\tilde t - ((\tilde r + 2\tilde m\tilde s^2)(\tilde r + 2\tilde m\tilde s_0^2) + \tilde a^2)\,d\phi\right]\cos\theta. \tag{8.55}$$

Applying the scaling limit, one finds

$$\chi_1 = -\frac{2m(\Pi_c - \Pi_s)a\cos\theta}{Q^2}, \qquad \chi_2 = \chi_3 = 0, \tag{8.56}$$

$$e^{\varphi_1} = \frac{Q^2}{\Delta_-^{1/2}}, \qquad \varphi_2 = \varphi_3 = 0, \tag{8.57}$$

$$A_2 = \mathcal{A}_2 = \frac{2mQ\sqrt{\Pi_c\Pi_s}}{(\Pi_c - \Pi_s)\Delta_-^{1/2}}\,dt - \frac{(2m)^3\sqrt{\Pi_c\Pi_s}\,(\Pi_c - \Pi_s)a\sin^2\theta}{Q\Delta_-^{1/2}}\,d\phi, \tag{8.58}$$

$$A_1 = \mathcal{A}_1 = \frac{Q}{\Delta_-^{1/2}}\,a\cos\theta\,dt - Q\cos\theta\,d\phi. \tag{8.59}$$

This matter solves all the equations of motion as shown in appendix D.

Unfortunately it seems impossible to find a scaling limit for Δ_+ when rotation is turned on, simply because the expression (8.40) mixes the angular term with terms proportional to r, r^2. It could however be that we can find supporting matter using so called Harrison transformations. We will discuss this further in relation to $\Delta_{\mathrm{NHEK};A}$ at the end of this chapter.

8.4 Geometrical Interpretation for Graviphoton

The original metric (5.5) can be rewritten in a form where the naive $G = 0$ poles are explicitly canceled [19]

$$\begin{aligned}\Delta^{-1/2} ds_4^2 &= -\frac{G}{\Delta}\left(dt + \frac{a\sin^2\theta}{G}\mathcal{A}_{\mathrm{red}} d\phi\right)^2 + \frac{X\sin^2\theta}{G}d\phi^2 + \frac{dr^2}{X} + d\theta^2 \\ &= \frac{dt^2}{\mathcal{R}} - \frac{1}{\mathcal{R}\Delta}\left(\mathcal{A}_{\mathrm{red}}dt + \mathcal{R}\, a\sin^2\theta\, d\phi\right)^2 + \frac{dr^2}{X} + d\theta^2 + \sin^2\theta\, d\phi^2,\end{aligned} \tag{8.60}$$

where

$$\mathcal{R} \equiv \frac{\mathcal{A}_{\mathrm{red}}^2 - \Delta}{G}. \tag{8.61}$$

For the warp factors (5.71), we thus have $\mathcal{R} = 4m^2(\Pi_c - \Pi_s)^2 - \Theta$. With $\Delta = \bar{\Delta}$ one has $\mathcal{R} = 4m^2(\Pi_c - \Pi_s)^2$, which is precisely what they consider in [19]. Furthermore they cancel the term

$$-\bar{\Delta}\mathcal{B}^2 \equiv -\frac{1}{\mathcal{R}\bar{\Delta}}\left(\mathcal{A}_{\mathrm{red}}dt + \mathcal{R}\, a\sin^2\theta\, d\phi\right)^2 \tag{8.62}$$

by introducing an auxiliary coordinate α taking part in the five-dimensional line element

$$ds_5^2 = \bar{\Delta}(d\alpha + \mathcal{B})^2 + \bar{\Delta}^{-1/2}ds_4^2. \tag{8.63}$$

This should be compared with the five-dimensional line element (C.14).

Of special interest is the benefited geometrical interpretation of the gauge field F^0 which is now seen to coincide with $d\mathcal{B}$, i.e the gauge field derived from the potential

$$\mathcal{B} = \frac{[(\Pi_c - \Pi_s)r + 2m\Pi_s]\, dt + 2m(\Pi_c - \Pi_s)^2 a\sin^2\theta\, d\phi}{(\Pi_c - \Pi_s)\bar{\Delta}}. \tag{8.64}$$

This coincides with the gravi-photon field strength in the static case as remarked in [19]. Inspecting the gravi-photon gauge field from [16]

$$\mathcal{A} = \frac{Q^3[(2m)^2\Pi_c\Pi_s + (\Pi_c - \Pi_s)^2 a^2\cos^2\theta]}{2m(\Pi_c^2 - \Pi_s^2)\bar{\Delta}}\, dt + \frac{Q^3 2m(\Pi_c - \Pi_s)a\sin^2\theta}{\bar{\Delta}}\, d\phi \tag{8.65}$$

we see indeed that when $a = 0$, $\partial_r\mathcal{A}_t$ matches with $\partial_r\mathcal{B}_t$ (up to an overall factor Q^3 which is set to 1 in [19]). Furthermore, shifting $\mathcal{A}_t$ by a constant we find

$$\mathcal{A}_t + \frac{Q^3}{2m(\Pi_c^2 - \Pi_s^2)} = \mathcal{B}_t, \tag{8.66}$$

and the ϕ components match without shifting, thus validating the geometrical interpretation in the minimally subtracted case.

We wish to achieve a similar geometrical interpretation when we consider the geometry with warp factor $\Delta = \Delta_-$. In this case, plugging Δ_- into (8.61) gives $\mathcal{R} = 0$, and thus it seems perhaps problematic. However, comparing

$$\mathcal{B} = \frac{\mathcal{A}_{\mathrm{red}}dt + \mathcal{R}a\sin^2\theta d\phi}{\Delta\sqrt{\mathcal{R}}} \tag{8.67}$$

with (8.49) and recalling that $Q = 2m\epsilon^{-1}(\Pi_c - \Pi_s) \gg 2m(\Pi_c - \Pi_s)$ we may then hope to capture the divergence of $1/\sqrt{\mathcal{R}}$ as $\mathcal{R} \to 0$ in factors of Q. We take the limit $\lambda \to 0$

$$\Delta = \mathcal{A}_{\rm red}^2 - G\lambda^2, \qquad \mathcal{R} = \lambda^2. \tag{8.68}$$

Taking this limit on (8.60) gives the same metric as when applying the scaling limit with $\epsilon \to 0$. Applying this limit to $\mathcal{B}$ we find [4]

$$\mathcal{B} = \frac{dt}{\Delta_-^{1/2}\lambda}. \tag{8.69}$$

If we knew the precise relation between λ and the scaling parameter ϵ, it should be possible to write $\mathcal{B}$ in terms of Q, m, Π_c and Π_s, and hopefully one would find agreement with (8.49). However, it seems that it is not possible to relate the scaling parameters λ and ϵ. Indeed taking the ϵ limit of $\mathcal{R}$ one finds the finite r dependent result

$$\lim_{\epsilon\to 0}\mathcal{R} = -4m\big((\Pi_c - \Pi_s)r + 2m\Pi_s\big) = -2\mathcal{A}_{\rm red}. \tag{8.70}$$

In retrospect it is clear that (C.14) for the subtracted geometry with $\Delta = \Delta_-$ does not coincide with (8.63) with Δ_- in place of $\bar{\Delta}$. This is easy to see noting that for the case Δ_-, the only non-zero φ_i is φ_1 see (8.57) and we have

$$ds_5^2 = \Delta_-^{1/3}(d\alpha + \mathcal{B})^2 + \Delta_-^{-1/6}ds_4^2, \tag{8.71}$$

where we have for simplicity set all factors of Q to one. It is clear that it is this line element that is relevant for the uplift of the subtracted geometry with $\Delta = \Delta_-$. It therefore seems like quite a coincidence that in the limit $\mathcal{R} = \lambda^2 \to 0$ we got an expression for $\mathcal{B}_t$, that aside from an overall unspecified constant, coincides nicely with the t-component of the gravi-photon field (8.49). A detailed investigation of a geometrical interpretation of the gravi-photon in the case $\Delta = \Delta_-$ is left for future work.

We note that since (when discarding the term $\sim 1/Q$) $\hat{\mathcal{A}}^2$ in (8.58) is purely electric, i.e has no legs on the sphere directions parameterized by θ, ϕ, we could hope that a similar analysis to that carried out for $\bar{\Delta}$ in section 7.6 could be carried out. We did not have time to pursue this, however it could give valuable clues regarding the extension of the work [2] to the generally rotating case.

8.5 No Scaling Limit for NHEK-Like Warp Factor

Seeing how successfully we got matter for the subtracted geometry with warp factor Δ_- in the general rotating case by employing the scaling limit (8.7), we would now like to identify a similar scaling limit for the general warp factors (5.71) where Θ may now depend on θ. We pointed out that we could only have constant $\Theta = \Theta_0$ in the static case, but in the general rotating case we may have an angularly dependent Θ, indeed both the original and the minimal warp factor depend on θ when $a \neq 0$.

We recall our proposed warp factors $\Delta_{\rm NHEK;A}$ for the purely rotating case (5.82), that unlike the corresponding minimally subtracted warp factor $\bar{\Delta}$, give us the correct NHEK limit. It is clear that a scaling limit $\tilde{r} = r\epsilon, \tilde{a} = a\epsilon, \epsilon \to 0$ as used in section 8.2, will not give us $\Delta_{\rm NHEK;A}$ when applied to

$$\Delta_0 = (r^2 + a^2\cos^2\theta)^2. \tag{8.72}$$

However including the charges might improve the situation, since the original warp factor (charges equal in pairs) reads

$$\Delta_0 = \left((\tilde{r} + 2\tilde{m}\tilde{s}^2)(\tilde{r} + 2\tilde{m}\tilde{s}_0^2) + \tilde{a}^2\cos^2\theta\right)^2. \tag{8.73}$$

[4] We are dropping the term proportional to λ.

On the face of it, this seems rather promising, however, in the scaling limit we employed in section 8.2, we had in particular $\tilde{a} = a\epsilon$, and in the limit we took $\epsilon \to 0$. This is bad, as we then loose the terms involving $\cos^2\theta$ and $\cos^4\theta$. We could try with a scaling limit that does not effect a, i.e $\tilde{a} = a$. Then according to a scaling limit akin to (8.7), where we generalize in terms of arbitrary constants σ and σ_0

$$\tilde{m}\tilde{s}^2 = \sigma^2\epsilon^{-1}, \qquad \tilde{s}_0^2 = \sigma_0^2, \tag{8.74}$$

we get

$$\Delta_0 \to \left(2\sigma^2 mr + 4\sigma_0^2\sigma^2 m^2 + a^2\cos^2\theta\right)^2. \tag{8.75}$$

This is close, however, there are no clever choices for the constants σ and σ_0 that make $\Delta_0 \to \Delta_{\text{NHEK};A}$. In particular, we see that there is no way that (8.75) can give a term $\sim r^2\cos^2\theta$. Thus it seems that we do not have a scaling limit at our disposal. In [46, 54], the road from original geometry to subtracted geometry has been facilitated by so called Harrison transformations. It may be that despite the lack of a scaling limit, one may find a suitable Harrison transformation that when applied to the original solution would give a geometry with warp factor $\Delta_{\text{NHEK};A}$. We have not had the time to study Harrison transformations in detail, however they are clearly relevant, and applicable to the study of subtracted geometry, and should be incorporated in future work.

9

Conclusion and Outlook

In this thesis we have provided a brief historical account of black hole physics, along with a short review of modern black hole physics, with special focus on understanding aspects of the entropy matching in the context of string theory. It is apparent that during the last few decades we have unraveled a stringy description that works wonders for certain extremal BPS black holes as we pointed out in chapter 3. Unfortunately the arguments that allow for the extrapolation between low coupling $g_s \ll 1$ perturbative string theory description in terms of Dp-branes breaks down for non-extremal black holes. However, we have been making progress during the last couple of years.

Indeed as we have reviewed in this thesis, chapter 5, the appearance of hidden conformal symmetry, which is made manifest in the subtracted geometry, suggests that a CFT plays a central role even for non-extremal black holes. The extent to which the CFT description is accurate, remains an open problem. This has however been illuminated recently by the identification of irrelevant operators in the CFT dual that start the flow from subtracted to original geometry, thus suggesting that the dual CFT description is only valid in the limit of large magnetic charges [2], which as shown in figure 7.1 corresponds to an increased region of overlap of the original and subtracted warp factors. The work of [2] has recently been extended to the rotating case in [17] using Harrison transformations. The CFT deformations remain irrelevant however, and as a whole [2, 17] suggest that in general for non-extremal black holes we need to supplement the CFT description with deformations: $\mathrm{CFT} + \mathcal{O}$.

In some sense it is to be expected that the CFT dual to the minimally subtracted geometry is only an approximation, just like the subtracted geometry itself differs from the original geometry. The discrepancy between subtracted and original geometry has become even more apparent in the recent study of entanglement entropy. Specifically, in [21] the entanglement entropy for the original and the minimally subtracted geometry is computed. The entropies agree at tree-level, however the logarithmic corrections in both cases differ. As they point out, the logarithmic correction for the minimally subtracted geometry has the opposite sign of the correction for the original geometry, and there may be a Harrison transformation that gives a geometry where the logarithmic correction vanishes. Whether such a Harrison transformation would be related to a subtracted geometry or not remains to be studied.

Although the CFT dual to the minimally subtracted geometry does not give a complete description of the original geometry, there may still be a lot to learn from this approach. In this thesis we have investigating warp factors $\Delta \sim r^2$ corresponding to subtracted geometries. Unlike the case $\Delta \sim r$ for which there is only a single unique warp factor $\bar{\Delta}$ that maintains a separable wave equation and gives a hypergeometric radial equation, we have several possible warp factors $\Delta \sim r^2$: $\Delta = \bar{\Delta} + \Theta\, G$. Notably we find a set of candidate warp factors $\Delta_{\mathrm{NHEK};A}$, where the NHEK limit on each $\Delta_{\mathrm{NHEK};A}$ for a given $A \in \mathbb{R}$ coincides with the NHEK limit on Δ_0, section 5.7. Furthermore we manage to find matter for a large class of warp factors $\Delta \sim r^2$ in the static case 6.6, and make progress toward finding matter that supports a $\Delta_{\mathrm{NHEK};A}$ by finding matter supporting $\Delta_- = \mathcal{A}_{\mathrm{red}}^2 \sim r^2$ in the rotating case. Our suggested $\Delta_{\mathrm{NHEK};4}$ coincides with Δ_- for the uncharged, static case, thus if there exists matter supporting $\Delta_{\mathrm{NHEK};A}$ it should show likeness to the matter supporting Δ_- in the limit of vanishing charges and vanishing angular momentum.

Initially we hoped that perhaps $\Delta \sim r^2$ being more closely related with $\Delta_0 \sim r^4$, than $\bar{\Delta} \sim r$, could give rise to a CFT dual that would better describe the states of the original black hole. However, as suggested by sections 7.4 - 7.5, even though two of the fields now seem to be irrelevant deformations, as opposed to the three irrelevant deformations in the minimally subtracted case, the CFT dual if any, is still only a reasonable description in a regime where $\alpha \ll 1$ section 7.5. On the positive side, the case $\Delta \sim r^2$ does provide a valid description for two finite charges, as opposed to only one finite electric charge which is the case for the minimally subtracted case.

For the minimally subtracted geometry, [19] nicely shows how the geometry uplifts to a geometry that is locally $\mathrm{AdS}_3 \times S^2$, thus making a dual CFT description apparent. Although we have not investigated the extent to which the uplifted geometry for warp factors $\Delta \sim r^2$ similarly factors into an $\mathrm{AdS}_3 \times S^2$ locally, we point out the possible geometrical interpretation of the gravi-photon field for the matter supporting $\Delta_- = \mathcal{A}^2_{\mathrm{red}}$ in section 8.4.

We have also looked at the asymptotic behavior of subtracted geometries. In general it may be that the usefulness of subtracted geometries is related to the fact that they alter the asymptotics, circumventing the problem of negative specific heat. Furthermore as we elaborate on in section 5.8, this is related to confinement of matter fields, which are found to be confined for the minimally subtracted case akin to how matter is confined in AdS space. However, we find that subtracted geometries with $\Delta \sim r^2$ may not be as confining, indeed as suggested by our analysis in section 5.8 it seems that such subtracted geometries may not be confining.

As we mentioned in the last paragraph above, subtracted geometries with $\Delta \sim r^2$ may not have the desired confining properties. Whether this is correct, and more specifically, whether or not this renders such subtracted geometries useless definitely needs to be answered. To analyze this, it could be fruitful to study geodesics in the asymptotic regions described by the asymptotically conical metrics with $p = 1$.

Another important work that we have deferred to future investigation is the extent to which the subtracted geometries $\Delta \sim r^2$ such as Δ_- uplift to a 5D geometry that is locally $\mathrm{AdS}_3 \times S^2$ like the minimally subtracted geometry. In general we did not establish direct evidence for a CFT dual for any of the subtracted geometries with $\Delta \sim r^2$, which although suggested by a hypergeometric radial equation, nevertheless deserves a detailed quantitative analysis.

As an important result in this thesis, we propose a candidate warp factor Δ_{NHEK} for future study. This warp factor needs to have supporting matter in order to be of physical relevance. We found matter for Δ_-, suggesting that we may find similar matter for Δ_{NHEK}, this should be a primary objective in the extension of this work. There are several fruitful directions that could aid in identifying matter. First and foremost, even though we ruled out the usefulness of a scaling limit, it may be that the more general Harrison transformations could relate Δ_0 to Δ_{NHEK} and give supporting matter. From references that employ such Harrison transformations [17, 46, 54] it is clear that it is possible to generate $\Delta \sim r^2$ from Harrison transformations in general, and unlike the scaling limits, the Harrison transformations may be able to generate the desired angular terms that we failed to generate in a scaling limit in section 8.5.

Aside from the proposed warp NHEK-like warp factors $\Delta_{\mathrm{NHEK};A}$, the general $\Delta \sim r^2$, if CFT duals are identified, could be interesting cases of subtracted geometry. Perhaps in the future one could extend some of the more recent work centered around $\bar{\Delta}$, to subtracted geometries with $\Delta = \bar{\Delta} + \Theta\, G, \Theta \neq 0$. In particular the recent work [21], in which the entropy of the CFT dual was found to deviate from the original black hole geometry beyond first order when computing the entanglement entropy. Perhaps these deviations would be suppressed for subtracted geometries with $\Delta \sim r^2$ warp factors.

It would also be interesting to investigate in detail, why the scaling limits as employed in [16] work. It is tempting to propose that they may be related to Harrison transformations, which would answer this question, as Harrison transformations are particular solution generating techniques employed in the three-dimensional coset model that results when reducing the STU model along time, as in [13].

Lastly, we point out the work done in [12] in which the general subtracted geometries with warp factors $\Delta \sim r^2$ as we have studied, may be of interest. The idea is that extremal subtracted geometries, so called subttractors, are boundary cases for the attractor basin studied in the attractor mechanism. This mechanism is characterized by the moduli scalars taking on a specific value at the double degenerate horizon, while they may upon perturbations flow to very different asymptotic values. This mechanism is interesting for various reasons, notably that it hints at the horizon region being responsible for the characteristics of black holes.

A

Form Notation

Let ω be a differential form of degree p, i.e a p-form. If we wished to explicitly specify the degree of the form we would in this case write $\omega_{(p)}$.

In general we express ω in terms of its components $\omega_{\mu_1\cdots\mu_p}$:

$$\omega = \frac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{A.1}$$

where the wedges are defined to be antisymmetric tensor products. Let A be a p-form and B be a q-form, then the wedge product is defined such that

$$(A\wedge B)_{\mu_1\cdots\mu_p\,\nu_1\cdots\nu_q} = \frac{(p+q)!}{p!\,q!}\,A_{[\mu_1\cdots\mu_p}B_{\nu_1\cdots\nu_q]}, \tag{A.2}$$

where the antisymmetrization is done with weight one, i.e

$$A_{[\mu_1\cdots\mu_p}B_{\nu_1\cdots\nu_q]} \equiv \frac{1}{(p+q)!}\sum_{\text{perm.}} \pm A_{\mu_1\cdots\mu_p}B_{\mu_{(p+1)}\cdots\mu_{(p+q)}}, \tag{A.3}$$

with the sum going over all permutations with positive or negative weight for respectively even and odd permutations, e.g

$$\sum_{\text{perm.}} \pm A_{\mu\nu} = A_{\mu\nu} - A_{\nu\mu}. \tag{A.4}$$

The Hodge dual is represented in D dimensions by $\star_D$ and often we will suppress the subscript D when it is evident what the dimension should be. We define the D-dimensional hodge dual (we suppress the subscript D) to work on ω such that the components of $\star\,\omega$ read

$$(\star\,\omega)_{\mu_1\cdots\mu_{(D-p)}} = \frac{1}{p!}\,\epsilon_{\mu_1\cdots\mu_{(D-p)}\,\nu_1\cdots\nu_p}\,\omega^{\nu_1\cdots\nu_p}, \tag{A.5}$$

where $\epsilon_{\mu_1\cdots\mu_D}$ are the components of the volume form ϵ defined by the completely antisymmetric symbol[1] $\varepsilon_{\mu_1\cdots\mu_D}$ and the metric determinant g:

$$\epsilon_{\mu_1\cdots\mu_D} = \sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_D}. \tag{A.6}$$

Raising the indices with the inverse metric one finds

$$\epsilon^{\mu_1\cdots\mu_D} = \frac{(-1)^t}{\sqrt{|g|}}\varepsilon^{\mu_1\cdots\mu_D}, \tag{A.7}$$

where the components $\varepsilon^{\mu_1\cdots\mu_D}$ are identical to $\varepsilon_{\mu_1\cdots\mu_D}$, and t is the number of timelike directions, i.e the number of negative diagonal entries in the metric.

It is standard to use this compact notation when writing out Lagrangian densities of actions which would otherwise have a lot of indices. Noting that

$$\star\,1 = \frac{1}{D!}\epsilon_{\mu_1\cdots\mu_D}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D} = \sqrt{|g|}\,dx^0\wedge dx^1\wedge\cdots\wedge dx^D \tag{A.8}$$

[1] In e.g four dimensions it is defined by requiring that $\varepsilon_{0123} = +1$.

we see that the Einstein-Hilbert action $S = \int \mathcal{L}$ can be written compactly with

$$\mathcal{L} = R \star 1. \tag{A.9}$$

Furthermore, which is not too difficult to show, the relation

$$\star A \wedge B = \star B \wedge A = \frac{1}{p!} A^{\mu_1 \cdots \mu_p} B_{\mu_1 \cdots \mu_p} \star 1 \tag{A.10}$$

is valid for any p-forms A, B. This allows one to compactly write the kinetic terms for gauge fields, e.g for a one form gauge field $A_{(1)}$ the kinetic term usually reads $\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where $F_{(2)} = dA_{(1)}$. In form notation (noting the above relation for p-forms A, B) we can write such a kinetic term simply as

$$\tfrac{1}{2} \star F_{(2)} \wedge F_{(2)}, \tag{A.11}$$

which at first seems like equally much work, however it pays off tremendously for higher degree field strengths, and indeed makes certain structures in the Lagrangian much more transparent.

Without having introduced the exterior derivative d we just used it to construct the field strength $F_{(2)}$ from $A_{(1)}$. The exterior derivative is simply defined to act on our p-form as follows

$$(d\omega)_{\mu_1 \cdots \mu_{(p+1)}} = (\partial \wedge \omega)_{\mu_1 \cdots \mu_{(p+1)}}, \tag{A.12}$$

thus

$$(dA_{(1)})_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{A.13}$$

as one is used to.

While we are at it, there is a neat way to write the covariant divergence of ω, namely

$$\star\, d \star \omega = (-1)^{t+p(D-p+1)-1}\, \nabla^\nu \omega_{\nu\, \mu_2 \cdots \mu_p}\, dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \tag{A.14}$$

the sign out front is rather involved, but usually not terribly important. We see that the special cases $p = 1, 2$ give signs that coincide nicely with $(A.7)$ and $(A.8)$ in [2].

We give a quick derivation of (A.14). Clearly

$$\begin{aligned}(d \star \omega)_{\nu\mu_1 \cdots \mu_{D-p}} &= \tfrac{D-p+1}{p!}\, \partial_{[\nu}\, \epsilon_{\mu_1 \cdots \mu_{D-p}]\, \nu_1 \cdots \nu_p}\, \omega^{\nu_1 \cdots \nu_p} \\ &= \tfrac{D-p+1}{p!}\, (-1)^{D-p}\, \epsilon_{[\mu_1 \cdots \mu_{D-p}\, |\nu_1 \cdots \nu_p|}\, \left(\tfrac{1}{2} g^{\rho\sigma}\, \partial_{\nu]}\, g_{\rho\sigma} + \partial_{\nu]}\right) \omega^{\nu_1 \cdots \nu_p},\end{aligned} \tag{A.15}$$

where the indices isolated inside $|\cdots|$ are not part of the antisymmetrization. In the last line we used that

$$\partial_\nu |g| = |g| g^{\rho\sigma}\, \partial_\nu\, g_{\rho\sigma}, \tag{A.16}$$

and that when pulling the ν index all the way to the right we get the appropriate sign in front. Now taking the Hodge dual of this, we get

$$(\star\, d \star \omega)_{\rho_1 \cdots \rho_{p-1}} = \tfrac{1}{p!(D-p)!}\, \epsilon_{\rho_1 \cdots \rho_{p-1}}{}^{\nu\, \mu_1 \cdots \mu_{D-p}}\, \epsilon_{\mu_1 \cdots \mu_{D-p}\, \nu_1 \cdots \nu_p}\, \left(\tfrac{1}{2} g^{\rho\sigma}\, \partial_\nu\, g_{\rho\sigma} + \partial_\nu\right) \omega^{\nu_1 \cdots \nu_p}, \tag{A.17}$$

where we used the antisymmetry of the left most epsilon tensor to remove the antisymmeterization, and furthermore got rid of the sign factor by moving ν to the left. To perform the contraction over the indices on the two epsilon tensors, we want to use that

$$\epsilon^{\mu_1 \cdots \mu_p\, \alpha_1 \cdots \alpha_{D-p}}\, \epsilon_{\mu_1 \cdots \mu_p\, \beta_1 \cdots \beta_{D-p}} = (-1)^t p!(D-p)!\, \delta^{[\alpha_1}_{\beta_1} \cdots \delta^{\alpha_{D-p}]}_{\beta_{D-p}}. \tag{A.18}$$

To this end one has to move the indices $\mu_1 \cdots \mu_{D-p}$ past the p indices on the left-most epsilon tensor inducing a sign $(-1)^{p(D-p)}$. One also has to raise the lower indices on this epsilon tensor hence

$$\begin{aligned}
(\star\, d \star \omega)_{\rho_1\cdots\rho_{p-1}} &= (-1)^{t+p(D-p)} g_{\mu_1\rho_1}\cdots g_{\mu_{p-1}\rho_{p-1}}\, \delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_{p-1}}_{\nu_{p-1}}\,\delta^{\nu]}_{\nu_p}\left(\tfrac{1}{2}g^{\rho\sigma}\,\partial_\nu\, g_{\rho\sigma}+\partial_\nu\right)\omega^{\nu_1\cdots\nu_p} \\
&= (-1)^{t+p(D-p)} g_{\mu_1\rho_1}\cdots g_{\mu_{p-1}\rho_{p-1}}\left(\tfrac{1}{2}g^{\rho\sigma}\,\partial_\nu\, g_{\rho\sigma}+\partial_\nu\right)\omega^{\mu_1\cdots\mu_{p-1}\,\nu} \\
&= (-1)^{t+p(D-p+1)-1}\left(\tfrac{1}{2}g^{\rho\sigma}\,\partial_\nu\, g_{\rho\sigma}+\partial_\nu\right)\omega^{\nu}{}_{\rho_1\cdots\rho_{p-1}} \\
&= (-1)^{t+p(D-p+1)-1}\nabla^\nu\,\omega_{\nu\rho_1\cdots\rho_{p-1}}. \qquad \text{(A.19)}
\end{aligned}$$

To get the second line we contracted the δ's with the ω, and used the antisymmetry of ω. The third line induces a sign by interchanging indices on ω. Lastly we use an identity holding for any covariant divergence of a p-form defined via the affine connection Γ.

Another useful identity is

$$\star\star\,\omega = (-1)^{t+p(D-p)}\omega, \qquad \text{(A.20)}$$

which is relatively simple to work out in comparison to the derivation we just showed above.

Let us now consider Maxwell's equations, now in the light of form notation. Firstly, let us write down Stoke's Theorem for a $(D-1)$-form A which reads

$$\int_M dA = \int_{\partial M} A, \qquad \text{(A.21)}$$

where M is a D-dimensional manifold with boundary ∂M. We will use this in a moment. The Maxwell equations in four dimensions in index notation read

$$\partial_{[\mu}F_{\nu\rho]} = 0, \qquad \text{(A.22)}$$

$$\nabla^\mu F_{\mu\nu} = -J_\nu. \qquad \text{(A.23)}$$

In form notation we may write more compactly

$$dF = 0, \qquad \text{(Bianchi identity)} \qquad \text{(A.24)}$$

$$d\star F = -\star J, \qquad \text{(Equation of motion)} \qquad \text{(A.25)}$$

where the last line implies that in four dimensions

$$\star\, d\star F = -J, \qquad \text{(A.26)}$$

since $\star\star J = J$ in four dimensions. It is now easy to construct a conserved charge, namely

$$Q = -\int_\Sigma \star J = \int_\Sigma d\star F = \int_{\partial\Sigma}\star F, \qquad \text{(A.27)}$$

where Σ is a spacial slice at constant time.

We close this appendix by considering the electric-magnetic duality, which in effect replaces F by its dual $\tilde{F}$ such that now the Bianchi identity for F implies the equation of motion for $\tilde{F}$ and vice versa. For the moment setting $J = 0$ (vacuum) we see that indeed

$$\tilde{F} = \star F \qquad \Leftrightarrow \qquad \star\tilde{F} = -F, \qquad \text{(A.28)}$$

leaves the vacuum Maxwell equations unaltered, as the Bianchi identity takes over the role of the Equations of motion and vice versa.

When $J \neq 0$ we have to include a magnetic current $\tilde{J}$ to make the Maxwell equations invariant under this electro-magnetic duality. We may ascribe a magnetic charge to F, defined by

$$\tilde{Q} = \int_{\partial\Sigma} F. \qquad \text{(A.29)}$$

This charge is conserved thanks to the Bianchi identity, i.e it is a conserved topological quantum number. On the other hand the conventional electric charge is conserved thanks to the equation of motion, and is a so called Noether charge [6].

B

Dimensional Reduction

B.1 Motivating Dimensional Reduction

Kaluza-Klein reduction, or more generally dimensional reduction, is a procedure that allows you to go from a theory in a D-dimensional spacetime $\mathsf{M}\times\mathsf{K}$, to a theory in the d-dimensional spacetime M. The degrees of freedom associated with K are frozen out by compactifying the space to a scale so small that upon applying suitable boundary conditions the massive modes diverge and the only physically realizable modes are the massless ones.

On our first encounter with this idea it is pedagogical to consider a massless scalar $\hat{\phi}(x,z)$ abiding the appropriate Klein-Gordon equation

$$\hat{\Box}^2\hat{\phi}=0. \tag{B.1}$$

In what follows we will investigate how the field and its equation of motion are expressed in one dimension less. We shall reduce from $D+1$ down to D dimensions by compactifying one of the flat space-like directions with topology R onto a circle S^1 of radius R. The compactification is realized as the identification $z\sim z+2\pi R$, furthermore we choose periodic boundary conditions for $\hat{\phi}(x,z)$

$$\hat{\phi}(x,z)=\hat{\phi}(x,z+2n\pi R),\quad n\in\mathbb{N}. \tag{B.2}$$

This allows for the Fourier decomposition of ϕ into its Fourier modes

$$\hat{\phi}(x,z)=\sum_n\phi_n(x)e^{inz/R}, \tag{B.3}$$

effectively giving

$$\left(\Box^2-\frac{n^2}{R^2}\right)\phi_n(x)=0, \tag{B.4}$$

which is the Klein-Gordon equation for the modes $\phi_n(x)$ with mass term $|n|/R$. Evidently the procedure has led to a mass which should be thought of as energy arising from the quantized momentum along z (acting on $\hat{\phi}(x,z)$ with the momentum operator $-i\hbar\partial_z$ clearly yields the eigenvalue $p_z=\hbar n/R$).

Although the equation of motion we started with and the one we ended up with are not very different, the first one is cleaner as it does not contain a mass term. This is a general result of dimensional reduction; the original (higher dimensional) equations of motion will dimensionally reduce to more complex equations. This is to be expected as translational degrees of freedom associated with the extra dimensionality need to be captured by additional fields in the lower dimensional theory. This will become very evident in the next section where we perform a standard Kaluza-Klein type reduction of pure Einstein gravity, which even though we will truncate away all massive modes, gives a much richer theory.

It is common practice to also truncate away the massive modes. In the scalar example we just considered, dimensional reduction gave us an infinite collection of massive fields in addition to a massless field. It is now reasonable to consider the truncated lower dimensional theory, i.e only keeping the massless mode, as the mass $|n|/R$ for the massive modes diverges as $R\to 0$. It is easy to argue taking $R\to 0$ when $D+1$ is greater than the apparent number of dimensions in our universe, say 4 for instance. The philosophy is simply that the radius of compactification is so small that for all intents and purposes it plays no role as a dimension. We are in a quite literal sense, extracting the effect of extra invisible dimensions.

B.2 Dimensional Reduction of Einstein Gravity

We will now perform dimensional reduction of Einstein gravity in $D+1$ dimensions to D dimensions, by circle reduction. We shall let the compactification be of the direction parameterized by the z coordinate of the $(D+1)$-dimensional theory onto a circle of radius R via the identification $z \sim z + 2\pi R$. We will be following [44], a reference which we found particularly useful.

Before we get started, we need a way to distinguish between objects belonging to the $(D+1)$-dimensional theory and the theory in D dimensions. Let us place hats on objects that are part of the $(D+1)$-dimensional theory while objects without hats will be objects belonging to the D-dimensional perspective. We will also need to distinguish between two types of index labels, and we shall employ $M, N, \ldots$ and $\mu, \nu, \ldots$ as follows by the example

$$M = 0, 1, 2, \ldots, D, \qquad \mu = 0, 1, 2, \ldots, D-1. \tag{B.5}$$

We denote the coordinates that span the $(D+1)$-dimensional space by

$$\hat{x}^M = x^\mu \, \delta_\mu^M + z \, \delta_z^M. \tag{B.6}$$

where x^μ are the coordinates that "survive" the dimensional reduction.

Now, just like we did for the scalar field considered in the previous section, we impose periodic boundary conditions on the metric components and expand in Fourier modes

$$\hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x}) = \sum_n g^{(n)}_{\hat{\mu}\hat{\nu}}(x) e^{inz/R}, \tag{B.7}$$

where x and $\hat{x}$ denote respectively the set of coordinates in D and $D+1$ dimensions. We then proceed by making the reduction ansatz $\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}}(x)$, that is, we take it to be independent of z. This is really something that goes hand in hand with truncating away the massive modes. In general such a truncation may not be consistent, in the sense that the lower dimensional theory, may not have equations of motion that are consistent with the higher dimensional theory. In the case of circle reduction, the group theory argument mentioned in [44] is enough to convince us of the consistency of this truncation.

Now, the realization to be made is that from the D-dimensional perspective, the $(D+1)$-dimensional metric components $\hat{g}_{MN}(x)$ break up into

$$\hat{g}_{\mu\nu}(x), \quad \hat{g}_{\mu z}(x), \quad \hat{g}_{zz}(x). \tag{B.8}$$

Note here that z is not an index label but a value, it is a fixation of the index $N = z$. Also note that due to the symmetry of the metric we only need to specify the one, $\hat{g}_{\mu z} = \hat{g}_{z\mu}$. This separation of the metric in $D+1$ dimensions into these three objects in the D-dimensional perspective results as a consequence of our Greek index labels being unable to take on the value z.

In principle we could simply use $\hat{g}_{\mu\nu}, \hat{g}_{\mu z}, \hat{g}_{zz}$ to parameterize our D-dimensional theory, however they are not a very nice choice as they do not recognize the underlying symmetry of the problem [44]. A more natural route is to parameterize the line element as

$$d\hat{s}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi}(dz + \mathcal{A})^2, \tag{B.9}$$

where α and β are constants that will be fixed later on, and $\mathcal{A} = \mathcal{A}_\mu dx^\mu$.

All we need to do is express the $(D+1)$-dimensional Einstein-Hilbert action in terms of the metric $g_{\mu\nu}$, dilaton ϕ and gauge field $\mathcal{A}_\mu$ (respectively the Kaluza-Klein scalar and potential). To this end one has to compute the Riemann tensor and from it the Ricci scalar. This is quite a lengthy and tedious calculation, however we will find that it is simplified greatly in terms of an orthonormal non-coordinate basis, in terms of which the metric takes the canonical form, i.e flat.

We follow the usual convention and write the basis one-forms as

$$\hat{e}^A = \hat{e}_M{}^A\, d\hat{x}^M. \tag{B.10}$$

We stop for a moment to address the appearance of the label on the vielbein, which we shall treat as an index, and to distinguish it from $M, N, \dots$ we call the latter *curved* while the labels on the vielbeins are referred to as *flat*. To be precise, we shall let $A, B, \dots, L$ be the flat indices of $D+1$ dimensions, while the lower-case letters $a, b, \dots$ will be the flat indices of the D-dimensional theory.

These vielbeins will provide us with an orthonormal basis provided that

$$\hat{g}^{MN}\, \hat{e}_M{}^A \hat{e}_N{}^B = \hat{\eta}^{AB}. \tag{B.11}$$

In matrix notation

$$\hat{g}_{MN} = e^{2\beta\phi} \begin{pmatrix} e^{2(\alpha-\beta)\phi} g_{\mu\nu} + \mathcal{A}_\mu \mathcal{A}_\nu & \mathcal{A}_\mu \\ \mathcal{A}_\nu & 1 \end{pmatrix}, \tag{B.12}$$

and we readily identify the following suitable vielbeins

$$\hat{e}^a = e^{\alpha\phi} e^a, \quad \hat{e}^z = e^{\beta\phi}(dz + \mathcal{A}), \tag{B.13}$$

where we introduced the D-dimensional ones $e^a = e_\mu{}^a dx^\mu$, which provide an orthonormal basis for the metric $g_{\mu\nu}$.

Instead of the standard covariant derivative involving the affine connection Γ, we must instead adopt the spin-connection when dealing with spinors on spacetime, which is just a way of formulating the covariant derivative in terms of our vielbein basis. The spin-connection $\omega_\mu{}^a{}_b$ is defined as follows

$$\nabla_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b, \tag{B.14}$$

$$\nabla_\mu X_a = \partial_\mu X_a - \omega_\mu{}^b{}_a X_b. \tag{B.15}$$

This is really similar to how our usual affine connection $\Gamma^\rho_{\mu\nu}$ is defined, the difference is that we now have two different bases; flat and curved. Using the above definition it is not that difficult to find the expression for the spin-connection in terms of Γ and the vielbeins. Comparing the explicit expressions for $(\nabla_\mu X^a) dx^\mu \otimes \hat{e}_{(a)}$ and $(\nabla_\mu X^\nu) dx^\mu \otimes \partial_\nu$ one finds

$$\omega_\mu{}^a{}_b = e_\nu{}^a\, e^\lambda{}_b\, \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b\, \partial_\mu e_\lambda{}^a. \tag{B.16}$$

However, for our purposes it is the defining relations for the torsion T and the curvature R in terms of the spin-connection that will be particularly useful

$$\hat{T}^A = d\hat{e}^A + \hat{\omega}^A{}_B \wedge \hat{e}^B, \tag{B.17}$$

$$\hat{\Theta}^A{}_B = d\hat{\omega}^A{}_B + \hat{\omega}^A{}_C \wedge \hat{\omega}^C{}_B, \tag{B.18}$$

where the curved indices on $\hat{T}$ and $\hat{\Theta}$ have been suppressed, it is understood that they are both two-forms.

We could now directly compute the components of $\hat{\omega}$ and from the Cartan structure equations (B.17 - B.18) read off the components of $\hat{\Theta}$, from there on it is furthermore straight forward to read off the components of the Riemann tensor noting that

$$\hat{\Theta}^A{}_B = \tfrac{1}{2} \hat{R}^A{}_{BCD}\, \hat{e}^C \wedge \hat{e}^D. \tag{B.19}$$

Computing the spin-connection directly from the components of the Christoffel connection gives

$$\hat{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha\phi}\big(\partial^b\phi\,\hat{e}^a - \partial^a\phi\,\hat{e}^b\big) - \tfrac{1}{2}e^{(\beta-2\alpha)\phi}\mathcal{F}^{ab}\hat{e}^z, \tag{B.20}$$

$$\hat{\omega}^{az} = -\beta e^{-\alpha\phi}\partial^a\phi\,\hat{e}^z - \tfrac{1}{2}e^{(\beta-2\alpha)\phi}\mathcal{F}^a{}_b\hat{e}^b, \tag{B.21}$$

where $\mathcal{F} = d\mathcal{A}$ and $\omega^{zz} = 0$ due to antisymmetry. This is a rather tedious exercise, a generally easier and more satisfying approach to finding the spin-connection, is to try to solve the torsion-less condition (B.17) with $\hat{T}^A = 0$, as we are in general considering spacetimes without torsion. From the first of the Cartan structure equations we see that $\hat{T}^A = 0$ implies

$$\hat{\omega}^A{}_B \wedge \hat{e}^B = -d\hat{e}^A. \tag{B.22}$$

From the relations

$$\hat{e}^a = e^{\alpha\phi}e^a, \qquad \hat{e}^z_M\, d\hat{x}^M = e^{\beta\phi}(\mathcal{A}_\mu\, dx^\mu + dz), \tag{B.23}$$

and the antisymmetry of the spin-connection we establish the relation

$$\hat{\omega}^z{}_A \wedge \hat{e}^A = e^{\alpha\phi}\,\hat{\omega}^z{}_a \wedge e^a. \tag{B.24}$$

Now the torsion-less condition (B.22) implies

$$\hat{\omega}^z{}_a \wedge e^a = -e^{-\alpha\phi}\, d(e^{\beta\phi}(\mathcal{A} + dz)). \tag{B.25}$$

Letting the exterior derivative work on the parenthesis, and recalling that the Leibniz rule implies in general

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p\omega \wedge \mathrm{d}\eta \tag{B.26}$$

for a p-form ω and a q-form η, we find explicitly

$$\hat{\omega}^z{}_a \wedge e^a = -e^{(\beta-\alpha)\phi}(\beta\, d\phi \wedge (\mathcal{A} + dz) + \mathcal{F}), \tag{B.27}$$

Rewriting the RHS to

$$e^{(\beta-\alpha)\phi}(\beta\,\partial_a\phi\,(\mathcal{A} + dz) \wedge e^a + \tfrac{1}{2}\mathcal{F}_{ab}\, e^b \wedge e^a), \tag{B.28}$$

we find that

$$\hat{\omega}^z{}_a = e^{(\beta-\alpha)\phi}(\beta\,\partial_a\phi\,(\mathcal{A} + dz) + \tfrac{1}{2}\mathcal{F}_{ab}\, e^b). \tag{B.29}$$

We proceed to calculate $\hat{\omega}^a{}_b$. Again we start by

$$\hat{\omega}^a{}_B \wedge \hat{e}^B = -d\hat{e}^a, \tag{B.30}$$

from which we get

$$e^{\alpha\phi}\hat{\omega}^a{}_b \wedge e^b = -e^{\beta\phi}\hat{\omega}^a{}_z \wedge (\mathcal{A} + dz) - e^{\alpha\phi}(\alpha\, d\phi \wedge e^a + de^a). \tag{B.31}$$

Now we can use what we already found, namely the components $\hat{\omega}^z{}_a$. Using the antisymmetry, and appropriate raising and lowering of indices we find

$$\hat{\omega}^a{}_z = -e^{(\beta-\alpha)\phi}(\beta\,\partial^a\phi\,(\mathcal{A} + dz) + \tfrac{1}{2}\mathcal{F}^a{}_b\, e^b) \tag{B.32}$$

Wedging that with $\mathcal{A} + dz$ we find that (B.31) gives

$$\hat{\omega}^a{}_b \wedge e^b = \tfrac{1}{2}e^{2(\beta-\alpha)\phi}\mathcal{F}^a{}_b\, e^b \wedge (\mathcal{A} + dz) - \alpha\, d\phi \wedge e^a - de^a. \tag{B.33}$$

Using the torsion-free condition on the un-hatted spin-connection, i.e $\omega^a{}_b \wedge e^b = -de^a$, and rewriting the other terms, we find

$$\hat{\omega}^a{}_b \wedge e^b = -\tfrac{1}{2} e^{2(\beta-\alpha)\phi}\, \mathcal{F}^a{}_b\, (\mathcal{A} + dz) \wedge e^b + \alpha\, \partial_b \phi\, e^a \wedge e^b + \omega^a{}_b \wedge e^b. \tag{B.34}$$

This suggest that

$$\hat{\omega}^a{}_b = \omega^a{}_b - \tfrac{1}{2} e^{2(\beta-\alpha)\phi}\, \mathcal{F}^a{}_b\, (\mathcal{A} + dz) + \alpha\, \partial_b \phi\, e^a - \alpha\, \partial^a \phi\, e_b, \tag{B.35}$$

where the extra term proportional to ϕ was added to ensure antisymmetry of the connection. The extra term that we added vanishes upon wedging the connection with e^b, we thus see that the above is also compatible with the torsion less condition as it should. We happily note that the expressions for the spin-connection agree with those computed earlier.

Now we proceed to calculate the components of the Curvature two-form and then we will make use of (B.19) to read off the components of the Riemann tensor. We see that we want in the end to write the curvature two form as an expression multiplying a wedge product between two hatted vielbeins. To this end we note that the components of the spin-connection in terms of hatted vielbeins read

$$\hat{\omega}^a{}_b = \omega^a{}_b + \alpha e^{-\alpha\phi} (\partial_b \phi \hat{e}^a - \partial^a \phi \hat{e}_b) - \tfrac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_b \hat{e}^z, \tag{B.36}$$
$$\hat{\omega}^z{}_a = \beta e^{-\alpha\phi} \partial_a \phi\, \hat{e}^z + \tfrac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}_{ab}\, \hat{e}^b. \tag{B.37}$$

Calculating the curvature $\hat{\Theta}^M{}_N$ is in principle straight forward

$$\hat{\Theta}^a{}_b = d\hat{\omega}^a{}_b + \hat{\omega}^a{}_s \wedge \hat{\omega}^s{}_b + \hat{\omega}^a{}_z \wedge \hat{\omega}^z{}_b, \tag{B.38}$$
$$\hat{\Theta}^a{}_z = d\hat{\omega}^a{}_z + \hat{\omega}^a{}_s \wedge \hat{\omega}^s{}_z. \tag{B.39}$$

We present the different terms:

$$\begin{aligned} d\hat{\omega}^a{}_b = \Big\{ & e^{-2\alpha\phi} \partial_c\, \omega_d{}^a{}_b + \alpha e^{-2\alpha\phi} [\partial_c \partial_b \phi\, \delta^a_d - \partial_c \partial^a \phi\, \eta_{bd} - \omega_c{}^a{}_d\, \partial_b \phi + \omega_{cbd}\, \partial^a \phi] \\ & - \tfrac{1}{4} e^{2(\beta-2\alpha)\phi} \mathcal{F}^a{}_b\, \mathcal{F}_{cd} \Big\}\, \hat{e}^c \wedge \hat{e}^d + \tfrac{1}{2} e^{(\beta-3\alpha)\phi} [2(\beta-\alpha) \partial_d \phi\, \mathcal{F}^a{}_b + \partial_d \mathcal{F}^a{}_b]\, \hat{e}^z \wedge \hat{e}^d, \end{aligned} \tag{B.40}$$

$$\begin{aligned} d\hat{\omega}^a{}_z = & -\tfrac{1}{2} e^{(\beta-3\alpha)\phi} \big\{ \beta \partial^a \phi\, \mathcal{F}_{cd} + (\beta-\alpha) \partial_c \phi\, \mathcal{F}^a{}_d + \partial_c\, \mathcal{F}^a{}_d - \omega_c{}^s{}_d\, \mathcal{F}^a{}_s \big\}\, \hat{e}^c \wedge \hat{e}^d \\ & + \beta e^{-2\alpha\phi} [(\beta-\alpha) \partial_d \phi\, \partial^a \phi + \partial_d \partial^a \phi]\, \hat{e}^z \wedge \hat{e}^d, \end{aligned} \tag{B.41}$$

$$\begin{aligned} \hat{\omega}^a{}_s \wedge \hat{\omega}^s{}_b = e^{-2\alpha\phi} \Big\{ & \omega_c{}^a{}_s\, \omega_d{}^s{}_b + \alpha [\omega_d{}^s{}_b (\partial_s \phi\, \delta^a_c - \partial^a \phi\, \eta_{cs}) + \omega_c{}^a{}_s (\partial_b \phi\, \delta^s_d - \partial^s \phi\, \eta_{bd})] \\ & + \alpha^2 (\partial_s \phi\, \delta^a_c - \partial^a \phi\, \eta_{sc})(\partial_b \phi\, \delta^s_d - \partial^s \phi\, \eta_{bd}) \Big\}\, \hat{e}^c \wedge \hat{e}^d \\ - \tfrac{1}{2} e^{(\beta-3\alpha)\phi} \Big\{ & \alpha (\partial_b \phi\, \delta^s_d - \partial^s \phi\, \eta_{bd}) \mathcal{F}^a{}_s - \alpha (\partial_s \phi\, \delta^a_d - \partial^a \phi\, \eta_{ds}) \mathcal{F}^s{}_b \\ & + \omega_d{}^s{}_b\, \mathcal{F}^a{}_s - \omega_d{}^a{}_s\, \mathcal{F}^s{}_b \Big\}\, \hat{e}^z \wedge \hat{e}^d, \end{aligned} \tag{B.42}$$

$$\begin{aligned} \hat{\omega}^a{}_s \wedge \hat{\omega}^s{}_z = & -\tfrac{1}{2} e^{(\beta-3\alpha)\phi} \Big\{ \alpha (\partial_s \phi\, \delta^a_c - \partial^a \phi\, \eta_{sc}) \mathcal{F}^s{}_d + \omega_c{}^a{}_s\, F^s{}_d \Big\}\, \hat{e}^c \wedge \hat{e}^d \\ & + \Big\{ \beta e^{-2\alpha\phi} [\alpha (\partial_s \phi\, \partial^s \phi\, \delta^a_d - \partial^a \phi\, \partial_d \phi) + \omega_d{}^a{}_s\, \partial^s \phi] \\ & + \tfrac{1}{4} e^{2(\beta-2\alpha)\phi} \mathcal{F}^a{}_s\, \mathcal{F}^s{}_d \Big\}\, \hat{e}^z \wedge \hat{e}^d, \end{aligned} \tag{B.43}$$

$$\hat{\omega}^a{}_z \wedge \hat{\omega}^z{}_b = -\tfrac{1}{4} e^{2(\beta-2\alpha)\phi} \mathcal{F}^a{}_c F_{bd}\, \hat{e}^c \wedge \hat{e}^d - \tfrac{1}{2} \beta e^{(\beta-3\alpha)\phi} (\partial^a \phi\, \mathcal{F}_{bd} - \partial_b \phi\, \mathcal{F}^a{}_d)\, \hat{e}^z \wedge \hat{e}^d. \tag{B.44}$$

Now that we have obtained $\hat{\Theta}^M{}_N$ we can start to read off the components of the $(D+1)$-dimensional Riemann tensor $\hat{R}^M{}_{NPQ}$ from (B.19). Reading off the components is not as simple as identifying whatever is contracting with the wedged vielbeins on the LHS with that contracting the vielbeins on the RHS. We can only make the identification between the antisymmetrized expressions, where the antisymmetrization is in the Latin indices that take part in the contraction with the vielbeins. To clarify, consider a two form $T = \frac{1}{2} T_{ab}\, \hat{e}^a \wedge \hat{e}^b$, furthermore let A bad B be one forms such that

$$\tfrac{1}{2} T_{ab}\, \hat{e}^a \wedge \hat{e}^b = A_a B_b\, \hat{e}^a \wedge \hat{e}^b \tag{B.45}$$

Then in general $T_{ab} \neq 2A_a B_b$, however since the wedge product is an antisymmetrized product, we can identify after antisymmetrizing in a and b, i.e

$$T_{ab} = A_a B_b - A_b B_a. \tag{B.46}$$

Apart from this subtlety, the calculations is otherwise rather mundane. Instead of presenting the rather large expressions for the components of the Riemann tensor we give the components of the Ricci tensor

$$\begin{aligned} \hat{R}_{ab} &= e^{-2\alpha\phi}(R_{ab} - \alpha^2(D-1)(D-2)\partial_a\phi\,\partial_b\phi - \alpha\Box\phi\,\eta_{ab}) - \tfrac{1}{2} e^{2(\beta-2\alpha)\phi} \mathcal{F}^c{}_a \mathcal{F}_{cb} \\ &\quad - e^{-2\alpha\phi}[(D-2)\alpha + \beta](\nabla_d\,\partial_b\phi + \alpha(\partial\phi)^2\,\eta_{bd}), \end{aligned} \tag{B.47}$$

$$\hat{R}_{az} = \hat{R}_{za} = -\tfrac{1}{2} e^{(\beta-3\alpha)\phi}\big([3(\beta-\alpha) + (D-1)\alpha]\partial_c\phi\,\mathcal{F}^c{}_a + \nabla_c\,\mathcal{F}^c{}_a\big), \tag{B.48}$$

$$\hat{R}_{zz} = -\beta e^{-2\alpha\phi}\big((\beta + (D-2)\alpha)(\partial\phi)^2 + \Box\phi\big) + \tfrac{1}{4} e^{2(\beta-2\alpha)\phi}\mathcal{F}^2. \tag{B.49}$$

Before we go on to contract once more to get the Ricci scalar $\hat{R}$ we would like to make a general observation. In the $(D+1)$-dimensional Einstein-Hilbert action we have

$$\sqrt{-\hat{g}}\hat{R}. \tag{B.50}$$

We can easily compute $\hat{g}$ in terms of the determinant of the D-dimensional metric by the observation

$$\hat{g} = \det(\hat{e}\hat{\eta}\hat{e}) = -\det(\hat{e})^2 = -e^{2(D\alpha+\beta)\phi}\det(e)^2 = e^{2(D\alpha+\beta)\phi} g, \tag{B.51}$$

we find

$$\sqrt{-\hat{g}} = e^{(D\alpha+\beta)\phi}\sqrt{-g}. \tag{B.52}$$

Without doing the full computation we see quite immediately that the $(D+1)$-dimensional Ricci scalar $\hat{R}$ will look like

$$\hat{R} = e^{-2\alpha\phi}(R + \cdots) \tag{B.53}$$

implying that the Lagrangian density in the D-dimensional perspective takes the form

$$\mathcal{L} = e^{((D-2)\alpha+\beta)\phi}\sqrt{-g}(R + \cdots). \tag{B.54}$$

To get the standard normalization of this D-dimensional Einstein-Hilbert term we restrict ourselves to $\beta = -(D-2)\alpha$, which is ok, at least for $D > 2$ [44]. This is in order to arrange for the Einstein-frame form of the gravitational action in $D+1$ dimensions to go over to the Einstein-frame form of the action in D dimensions. With β satisfying this relation, we find that the components of $\hat{R}_{MN}$ now read

$$\hat{R}_{ab} = e^{-2\alpha\phi}(R_{ab} - \alpha^2(D-1)(D-2)\partial_a\phi\,\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}) - \tfrac{1}{2} e^{-2D\alpha\phi}\mathcal{F}^c{}_a F_{cb}, \tag{B.55}$$

$$\hat{R}_{az} = \hat{R}_{za} = \tfrac{1}{2} e^{(D-3)\alpha\phi}\,\nabla^c\big(e^{-2(D-1)\alpha\phi}\mathcal{F}_{ac}\big), \tag{B.56}$$

$$\hat{R}_{zz} = (D-2)\alpha\, e^{-2\alpha\phi}\Box\phi + \tfrac{1}{4} e^{-2D\alpha\phi}\mathcal{F}^2. \tag{B.57}$$

If we now calculate the Ricci scalar $\hat{R} = \hat{R}_{zz} + \eta^{ab}\hat{R}_{ab}$ we find

$$\hat{R} = e^{-2\alpha\phi}\big(R - \alpha^2(D-1)(D-2)(\partial\phi)^2 - 2\alpha\,\Box\phi\big) - \tfrac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2. \tag{B.58}$$

In addition to a nicely normalized Hilbert term, one likes to get a kinetic term with canonical normalization, that is $-\frac{1}{2}\sqrt{-g}(\partial\phi)^2$ in the action. We see that we can achieve this by fixing α to satisfy

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}. \tag{B.59}$$

With this choice of α, and discarding the term $\sim \Box\phi$ as it is a total derivative of ϕ and thus gives no contribution to the equations of motion, we find that the $(D{+}1)$-dimensional Einstein gravity when dimensionally reduced along a circle of space-like extension, is reincarnated as the following D-dimensional Einstein-Maxwell-dilaton theory, whose action reads

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}\big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2\big). \tag{B.60}$$

The story is similar, but with subtle differences, when considering the same procedure but in the string frame, [43]. Denoting the $(D+1)$-dimensional metric in string frame $\hat{G}_{MN}$, we would introduce the Kaluza-Klein scalar σ and potential A by

$$d\hat{s}_s^2 = ds_s^2 + e^{2\sigma}(A+dz)^2, \qquad \hat{G}_{MN} = \begin{pmatrix} G_{\mu\nu} + e^{2\sigma}A_\mu A_\nu & e^{2\sigma}A_\mu \\ e^{2\sigma}A_\nu & e^{2\sigma} \end{pmatrix}. \tag{B.61}$$

We would again make use of the vielbeins, defined such that

$$\hat{G}_{MN} = \hat{E}^A_M \hat{E}^B_N \hat{\eta}_{AB}, \qquad G_{\mu\nu} = E^a_\mu E^b_\nu \eta_{ab} \tag{B.62}$$

and this time find

$$\hat{E}^A_M = \begin{pmatrix} E^a_\mu & e^\sigma A_\mu \\ 0 & e^\sigma \end{pmatrix}, \qquad \hat{E}^M_A = \begin{pmatrix} E^\mu_a & 0 \\ -A_a & e^{-\sigma} \end{pmatrix}, \tag{B.63}$$

such that

$$\hat{G}^{MN} = \hat{E}^M_A \hat{E}^N_B \hat{\eta}^{AB} = \begin{pmatrix} G^{\mu\nu} & -A^\mu \\ -A^\nu & e^{-2\sigma} + A^2 \end{pmatrix}. \tag{B.64}$$

The resulting action will have a Lagrangian of the form

$$\mathcal{L} = \sqrt{-\hat{G}}e^{-2\hat{\Phi}}(\hat{R} + \cdots) = \sqrt{-G}e^{\sigma - 2\hat{\Phi}}(R + \cdots), \tag{B.65}$$

since $\det\hat{G} = -(\det\hat{E})^2 = -(\det E)^2 e^{2\sigma} = Ge^{2\sigma}$. Therefore to get the string frame action in D dimensions, we need to define the D-dimensional dilaton by

$$\Phi = \hat{\Phi} - \tfrac{1}{2}\sigma. \tag{B.66}$$

When the dust settles, one finds

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-G}e^{-2\Phi}(R + 4\partial_\mu\Phi\partial^\mu\Phi - \partial_\mu\sigma\partial^\mu\sigma - \tfrac{1}{4}e^{2\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}), \tag{B.67}$$

where[1]

$$\frac{1}{\kappa_D^2} = \frac{2\pi R}{\kappa_{D+1}^2} \tag{B.68}$$

We can be sure that this general circle reduction along with the implied truncation of massive modes (going hand in hand with the z independence of the Kaluza-Klein ansatz) is consistent. It nicely rests on the observation that the massive Fourier modes are all doublets, i.e modes of the form $\sim e^{\pm imz}$, while the massless mode is a singlet: There is no means by which a combination of doublets will result in a singlet or vice versa, and it is in this sense that this group theory inspired argument assures us that this truncation is consistent [44].

[1]Note that in this expression R is the radius of the circle of compactification, and not the Ricci scalar.

B.3 Reduction of n-form Field Strength

As we saw in the first section of this chapter, a scalar reduces rather trivially, at least under circle reduction. However an n-form field strength, $\hat{F}_{(n)}$ is a bit more involved, however, it is straight forward in comparison with the reduction of gravity.

Let $\hat{A}_{(n-1)}$ be a gauge potential in $(D+1)$-dimensions with field strength $\hat{F}_{(n)} = d\hat{A}_{(n-1)}$. Such a gauge potential decomposes as

$$\hat{A}_{(n-1)} = A_{(n-1)} + A_{(n-2)} \wedge dz. \tag{B.69}$$

in the D-dimensional perspective. Again this is a result of the inability of the D-dimensional index labels $\mu, \nu, \ldots$ or $a, b, \ldots$ to take on the value z. Now, when it comes to parameterizing the D-dimensional field strengths we could make the naive choice

$$\hat{F}_{(n)} = dA_{(n-1)} + dA_{(n-2)} \wedge dz, \tag{B.70}$$

however, like for the metric, the straight forward choice is not as good as it gets. We are much better off adding (and subtracting) the term $dA_{(n-2)} \wedge \mathcal{A}_{(1)}$:

$$\hat{F}_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (\mathcal{A}_{(1)} + dz) \tag{B.71}$$

and then define the D-dimensional field strengths

$$F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)}, \tag{B.72}$$

$$F_{(n-1)} = dA_{(n-2)}, \tag{B.73}$$

where $\mathcal{A}_{(1)}$ is the Kaluza-Klein potential that we obtained from circle reduction of $(D+1)$-dimensional Einstein gravity.

Finally the relation between the $D+1$ and D-dimensional field strengths is found by

$$\begin{aligned}
\hat{F}_{(n)} &= \frac{1}{n!}\hat{F}_{A_1\cdots A_n}\,\hat{e}^{A_1} \wedge \cdots \wedge \hat{e}^{A_n} \\
&= \frac{1}{n!}e^{n\alpha\phi}\,\hat{F}_{a_1\cdots a_n}\,\hat{e}^{a_1} \wedge \cdots \wedge \hat{e}^{a_n} + \frac{e^{(\beta+(n-1)\alpha)\phi}}{(n-1)!}\hat{F}_{a_1\cdots a_{n-1}\,z}\,\hat{e}^{a_1} \wedge \cdots \wedge \hat{e}^{a_{n-1}} \wedge (\mathcal{A}_{(1)} + dz) \\
&\equiv \frac{1}{n!}F_{a_1\cdots a_n}\,\hat{e}^{a_1} \wedge \cdots \wedge \hat{e}^{a_n} + \frac{1}{(n-1)!}F_{a_1\cdots a_{n-1}}\,\hat{e}^{a_1} \wedge \cdots \wedge \hat{e}^{a_{n-1}} \wedge (\mathcal{A}_{(1)} + dz),
\end{aligned} \tag{B.74}$$

where in the second line we used (B.13), and in the last line we used the decomposition of $\hat{F}_{(n)}$ into the field strengths $F_{(n)}$ and $F_{(n-1)}$. Thus

$$\hat{F}_{a_1\cdots a_n} = e^{-n\alpha\phi}F_{a_1\cdots a_n}, \qquad F_{a_1\cdots a_{n-1}\,z} = e^{(D-n-1)\alpha\phi}F_{a_1\cdots a_{n-1}} \tag{B.75}$$

where we used $\beta = -(D-2)\alpha$.

Using form notation, we see from how the $(D+1)$-dimensional field strength reduces, that if we have

$$-\tfrac{1}{2} \star \hat{F}_{(n)} \wedge \hat{F}_{(n)} \tag{B.76}$$

as the kinetic term for the gauge field in the $(D+1)$-dimensional Lagrangian, then upon circle reduction we obtain in the D-dimensional Lagrangian the kinetic terms

$$-\tfrac{1}{2}e^{-2(n-1)\alpha\phi} \star F_{(n)} \wedge F_{(n)} - \tfrac{1}{2}e^{2(D-n)\alpha\phi} \star F_{(n-1)} \wedge F_{(n-1)}, \tag{B.77}$$

where we stress that the hodge duals in this last expression differ from the $(D+1)$-dimensional one in the obvious way.

All this business with dimensional reduction will pay off later when we can use this to make contact between a four-dimensional super gravity theory, namely the STU model, and type IIB string theory in 10 dimensions. As an added benefit and indeed a motivation for performing reductions in its own right, is that we from a small set of higher dimensional theories can generate all kinds of interesting lower dimensional ones. We just mentioned the STU model, which we will see in the next section will arise upon several subsequent reduction steps of truncated type IIB SUGRA theory. Also, knowing that a theory say in D dimensions derives from one in $D + d$ dimensions via some dimensional reduction, say successively on circles or T^d if you like (for toroidal compactification), we know how the lower dimensional fields relate to the higher dimensional ones, and thus go the other way. This is known as uplifting, which is notably utilized in [2, 19, 20], where in particular the uplift of the subtracted geometry from 5 to 6 dimensions in [20] reveals the spacetime as being locally $\mathrm{AdS}_3 \times S^3$.

Another reference which uses several reduction steps, all the way down to 3 dimensions is [13]. The three-dimensional perspective reveals what are referred to as hidden symmetries in [54] that comes along with powerful machinery to generate solutions to the theory. In fact these solutions, in particular the gauge fields of [13] will be very useful to us when we extend the work of [16]. Indeed we should comment on the general property that upon multiple circle reductions or equivalently toroidal compactification the external symmetries of the original higher dimensional theory are to some extent reincarnated as internal symmetries, such as the $U(1)$ gauge symmetry associated with the Kaluza-Klein potential $\mathcal{A}_{(1)}$ when reducing on S^1. However it gets much richer for say n successive circle reductions [44], which not only gives the $U(1)^n$ internal symmetry but also other global symmetries, typically $GL(n, \mathbb{R})$. These symmetries are especially apparent among the scalars; both the pseudoscalars and the dilatons. We could go on into much more detail, which is very fruitful, but we leave the rest to be covered in the reference [44].

C

The STU Model

C.1 A Chain of Dimensional Reduction

In this section we follow the conventions of [54]. We will sketch the route from IIB string theory to the STU Model via a chain of dimensional reduction. Along the road we will present the details involved in the dualization of one of the field strengths.

A consistent truncation of type IIB SUGRA compactified over T^4, involves only the 10-dimensional graviton and RR two-form $C^{RR}_{[2]}$ [1]. The reduction ansatz reads

$$ds^2_{10,\text{string}} = ds^5_6 + e^{\frac{\Phi}{\sqrt{2}}} ds^2_4, \qquad \Phi_{10} = \frac{\Phi}{\sqrt{2}}, \qquad C^{RR}_{[2]} = C_{[2]}, \tag{C.1}$$

where ds^4 denotes the line element of the four-torus, and $C_{[2]}$ denotes the six-dimensional descendant of the ten-dimensional RR two-form. The reduction yields the six-dimensional theory, whose bosonic Lagrangian reads

$$\mathcal{L}_{6B} = R_6 \star_6 \mathbb{1} - \tfrac{1}{2} \star_6 d\Phi \wedge d\Phi - \tfrac{1}{2} e^{\sqrt{2}\Phi} \star_6 F_{[3]} \wedge F_{[3]}, \tag{C.2}$$

where $F_{[3]} = dC_{[2]}$. We see the field content is that of a graviton, a two-form and a dilaton. We proceed to five dimensions by circle reduction, employing the standard Kaluza-Klein ansatz.

$$ds^2_6 = e^{-\sqrt{\frac{3}{2}}\Psi}(dz_6 + A^1_{[1]})^2 + e^{\frac{1}{\sqrt{6}}\Psi} ds^2_5, \tag{C.3}$$

$$F_{[3]} = F^{(5\mathrm{d})}_{[3]} + dA^2_{[1]} \wedge (dz + A^1_{[1]}), \tag{C.4}$$

where

$$F^{(5\mathrm{d})}_{[3]} = dC^{(5\mathrm{d})}_{[2]} - dA^2_{[1]} \wedge dA^1_{[1]}. \tag{C.5}$$

We see that this Kaluza-Klein ansatz is a special case of the general circle reduction discussed in appendix B.2, and one finds that the five-dimensional Lagrangian reads

$$\begin{aligned}\mathcal{L}_5 = R_5 \star_5 \mathbb{1} - \tfrac{1}{2} \star_5 d\Phi \wedge d\Phi - \tfrac{1}{2} \star_5 d\Psi \wedge d\Psi - \tfrac{1}{2} e^{-2\sqrt{\frac{2}{3}}\Psi} \star_5 F^1_{[2]} \wedge F^1_{[2]} \\ - \tfrac{1}{2} e^{-\sqrt{\frac{2}{3}}\Psi + \sqrt{2}\Phi} \star_5 F^{(5\mathrm{d})}_{[3]} \wedge F^{(5\mathrm{d})}_{[3]} - \tfrac{1}{2} e^{\sqrt{\frac{2}{3}}\Psi + \sqrt{2}\Phi} \star_5 F^2_{[2]} \wedge F^2_{[2]},\end{aligned} \tag{C.6}$$

where $F^I_{[2]} \equiv dA^I_{[1]}$.

Next we dualize $F^{(5\mathrm{d})}_{[3]}$ in favor of a two-form $F^3_{[2]}$. Recall that in five dimensions the two-form $C^{(5\mathrm{d})}_{[2]}$ is dual to a one-form $A^3_{[1]}$. A dualization needs to exchange the roles of the Bianchi identity and the equations of motion for the gauge fields. A simple way to proceed is to add the Lagrange multiplier

$$\mathcal{L}_{LM} = A^3_{[1]} \wedge (dF^{(5\mathrm{d})}_{[3]} + F^2_{[2]} \wedge F^1_{[2]}), \tag{C.7}$$

[1] In this chapter we will use square brackets to indicate the degree of the differential form.

to the Lagrangian $\mathcal{L}_5$, since then the equations of motion for $A^3_{[1]}$ coincide precisely with the Bianchi identity for $F^{(5d)}_{[3]}$. The Bianchi identity is easily transcribed from the above Kaluza-Klein ansatz. One then proceeds by eliminating $F^{(5d)}_{[3]}$ from the Lagrangian $\mathcal{L}_5 + \mathcal{L}_{LM}$ via its algebraic equation of motion. Treating $F^{(5d)}_{[3]}$ as a fundamental field we find its algebraic equation of motion

$$F^3_{[2]} - e^{-\sqrt{\frac{3}{2}}\Psi+\sqrt{2}\Phi} \star_5 F^{(5d)}_{[3]} = 0. \tag{C.8}$$

The relevant part of $\mathcal{L}_5 + \mathcal{L}_{LM}$ reads

$$-\tfrac{1}{2}e^{-\sqrt{\frac{2}{3}}\Psi+\sqrt{2}\Phi} \star_5 F^{(5d)}_{[3]} \wedge F^{(5d)}_{[3]} + A^3_{[1]} \wedge (dF^{(5d)}_{[3]} + F^2_{[2]} \wedge F^1_{[2]}). \tag{C.9}$$

Taking special care with the signs and noting that

$$A^3_{[1]} \wedge dF^{(5d)}_{[3]} = -d(A^3_{[1]} \wedge F^{(5d)}_{[3]}) + dA^3_{[1]} \wedge F^{(5d)}_{[3]} \tag{C.10}$$

and that the total derivative can be discarded, we find that in terms of $F^3_{[2]}$ the new Lagrangian reads

$$\begin{aligned}\mathcal{L}'_5 = R_5 \star_5 \mathbb{1} - \tfrac{1}{2} \star_5 d\Phi \wedge d\Phi - \tfrac{1}{2} \star_5 d\Psi \wedge d\Psi - \tfrac{1}{2}e^{-2\sqrt{\frac{2}{3}}\Psi} \star_5 F^1_{[2]} \wedge F^1_{[2]} \\ - \tfrac{1}{2}e^{\sqrt{\frac{2}{3}}\Psi+\sqrt{2}\Phi} \star_5 F^2_{[2]} \wedge F^2_{[2]} - \tfrac{1}{2}e^{\sqrt{\frac{2}{3}}\Psi-\sqrt{2}\Phi} \star_5 F^3_{[2]} \wedge F^3_{[2]} + A^3_{[1]} \wedge F^2_{[2]} \wedge F^1_{[2]}.\end{aligned} \tag{C.11}$$

This is equivalent to five-dimensional $\mathrm{U}(1)^3$ supergravity [54] where the real special manifold is parameterized by

$$h^1 = e^{\sqrt{\frac{2}{3}}\Psi}, \quad h^2 = e^{-\sqrt{\frac{1}{6}}\Psi-\sqrt{\frac{1}{2}}\Phi}, \quad h^3 = e^{-\sqrt{\frac{1}{6}}\Psi+\sqrt{\frac{1}{2}}\Phi}. \tag{C.12}$$

A manifestly triality-invariant form may now be written as

$$\mathcal{L}''_5 = R_5 \star_5 \mathbb{1} - \tfrac{1}{2}H_{ij} \star_5 dh^i \wedge dh^j - \tfrac{1}{2}H_{ij} \star_5 F^i_{[2]} \wedge F^j_{[2]} + \tfrac{1}{6}C_{ijk}F^i_{[2]} \wedge F^j_{[2]} \wedge A^k_{[1]}, \tag{C.13}$$

where $C_{ijk} = |\varepsilon_{ijk}|$ and the only non-vanishing components of H are $H_{ii} = (h^i)^{-2}$.

Further reduction down to four-dimensions via the Kaluza-Klein ansatz

$$ds^2_5 = f^2(dz + \hat{A}^0_{[1]})^2 + f^{-1}ds^2_4 \tag{C.14}$$

$$A^i_{[1]} = \chi^i(dz + \hat{A}^0_{[1]}) + \hat{A}^i_{[1]}, \qquad i = 1,2,3, \tag{C.15}$$

one arrives at the STU model. We suppress the degree of the forms and all Hodge duals are implicitly in four dimensions, furthermore we remove the hats on the field strengths

$$\begin{aligned}\mathcal{L}_4 = R \star \mathbb{1} - \tfrac{1}{2}H_{ij} \star dh^i \wedge dh^j - \tfrac{3}{2}f^{-2} \star df \wedge df - \tfrac{1}{2}f^3 \star F^0 \wedge F^0 \\ - \tfrac{1}{2}f^{-2}H_{ij} \star d\chi^i \wedge \chi^j - \tfrac{1}{2}fH_{ij} \star (F^i + \chi^i F^0) \wedge (F^j + \chi^j F^0) \\ + \tfrac{1}{2}C_{ijk}\left(\chi^i F^j \wedge F^k + \chi^i\chi^j F^0 \wedge F^k + \tfrac{1}{3}\chi^i\chi^j\chi^k F^0 \wedge F^0\right).\end{aligned} \tag{C.16}$$

C.2 Two Magnetic two Electric

In [13, 16] they consider the STU Lagrangian of the form

$$\begin{aligned}\mathcal{L}_4 = R \star \mathbb{1} - \tfrac{1}{2} \star d\varphi_i \wedge d\varphi_i - \tfrac{1}{2}e^{2\varphi_i} \star d\chi_i \wedge d\chi_i - \tfrac{1}{2}e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3} \star \hat{F}_1 \wedge \hat{F}_1 \\ + e^{\varphi_2+\varphi_3} \star \hat{F}_2 \wedge \hat{F}_2 + e^{-\varphi_2+\varphi_3} \star \hat{\mathcal{F}}^1 \wedge \hat{\mathcal{F}}^1 + e^{-\varphi_2-\varphi_3} \star \hat{\mathcal{F}}^2 \wedge \hat{\mathcal{F}}^2\Big) \\ - \chi_1(\hat{F}_1 \wedge \hat{\mathcal{F}}^1 + \hat{F}_2 \wedge \hat{\mathcal{F}}^2),\end{aligned} \tag{C.17}$$

where

$$\hat{F}_1 = d\hat{A}_1 - \chi_2\, d\hat{\mathcal{A}}^2, \tag{C.18}$$
$$\hat{F}_2 = d\hat{A}_2 + \chi_2\, d\hat{\mathcal{A}}^1 - \chi_3\, d\hat{A}_1 + \chi_2\chi_3\, d\hat{\mathcal{A}}^2, \tag{C.19}$$
$$\hat{\mathcal{F}}^1 = d\hat{\mathcal{A}}^1 + \chi_3\, d\hat{\mathcal{A}}^2, \tag{C.20}$$
$$\hat{\mathcal{F}}^2 = d\hat{\mathcal{A}}^2. \tag{C.21}$$

We would like to see the connection between this form of the STU Lagrangian and the one used in [2]. To this end we dualize $\hat{F}_2$ by adopting the duel of $\hat{A}_2$ which we shall call B. The Bianchi identity for $\hat{F}_2$ reads

$$d\hat{F}_2 - d\chi_2 \wedge d\hat{\mathcal{A}}^1 + d\chi_3 \wedge \hat{A}_1 - d(\chi_2\chi_3) \wedge d\hat{\mathcal{A}}^2 = 0. \tag{C.22}$$

Thus we proceed to dualize by adding the Lagrange multiplier

$$B \wedge (d\hat{F}_2 - d\chi_2 \wedge d\hat{\mathcal{A}}^1 + d\chi_3 \wedge \hat{A}_1 - d(\chi_2\chi_3) \wedge d\hat{\mathcal{A}}^2), \tag{C.23}$$

to the above Lagrangian $\mathcal{L}_4$. Treating $\hat{F}_2$ as a fundamental field, we find its equation of motion reads

$$dB - e^{-\varphi_1+\varphi_2+\varphi_3} \star \hat{F}_2 - \chi_1 \hat{\mathcal{F}}^2 = 0. \tag{C.24}$$

Using this to eliminate $\hat{F}_2$ from the Lagrangian with the multiplier, we find, after some manipulation, that the collection of terms involving $\hat{F}_2$ can be rewritten to

$$-\tfrac{1}{2} e^{-\varphi_1+\varphi_2+\varphi_3} \star G \wedge G + dB \wedge (\chi_3 d\hat{A}_1 - \chi_2 d\hat{\mathcal{A}}^1 - \chi_2\chi_3 d\hat{\mathcal{A}}^2), \tag{C.25}$$

with

$$G = dB - \chi_1 \hat{\mathcal{F}}^2. \tag{C.26}$$

Furthermore we see that we can get the Lagrangian to coincide with the manifestly triality invariant form as in [54]. To do so we need to employ

$$\tilde{\chi}_1 = -\chi_1, \qquad \tilde{\chi}_2 = -\chi_2, \qquad \tilde{\chi}_3 = \chi_3, \tag{C.27}$$

$$A^0 = \hat{\mathcal{A}}^2, \qquad A^1 = B, \qquad A^2 = \hat{A}_1, \qquad A^3 = \hat{\mathcal{A}}^1, \tag{C.28}$$

and

$$h^i = e^{\varphi_0 - 2\varphi_i}, \qquad f = e^{-\frac{1}{3}\varphi_0}, \qquad \varphi_0 \equiv \varphi_1 + \varphi_2 + \varphi_3. \tag{C.29}$$

This is more or less remarked on in [18]. They consider a slightly different Lagrange multiplier, but it basically amounts to the same thing.

C.3 Truncated STU Model

In [2] they write the STU Lagrangian density as

$$\begin{aligned}\mathcal{L}_4 = {} & R \star \mathbb{1} - \tfrac{1}{2} H_{ij} \star dh^i \wedge dh^j - \frac{3}{2f^2} \star df \wedge df - \tfrac{1}{2} f^3 \star F^0 \wedge F^0 \\ & - \frac{1}{2f^2} H_{ij} \star d\chi^i \wedge d\chi^j - \tfrac{1}{2} f H_{ij} \star \left(F^i + \chi^i F^0\right) \wedge \left(F^j + \chi^j F^0\right) \\ & + \tfrac{1}{2} C_{ijk} \chi^i F^j \wedge F^k + \tfrac{1}{2} C_{ijk} \chi^i \chi^j F^0 \wedge F^k + \tfrac{1}{6} C_{ijk} \chi^i \chi^j \chi^k F^0 \wedge F^0,\end{aligned} \tag{C.30}$$

where f and h^i $(i = 1, 2, 3)$ are scalar fields, χ^i are pseudoscalars and F^0 and F^i are $U(1)$ gauge field strengths. H_{ij}, the metric on the scalar moduli space, is diagonal $H_{ii} = (h^i)^{-2}$, and C_{ijk} is a symmetric symbol whose only non-vanishing components are permutations of $C_{123} = 1$. The scalars h^i are furthermore constrained by the relation $h^1h^2h^3 = 1$ which needs to be fulfilled before varying the action.

In the static case, it is sufficient to consider the truncation $\chi^i = 0$. For this to be a consistent truncation, a constraint must be imposed among the gauge field strengths. Varying $\mathcal{L}_4$ with respect to χ^i, as in

$$\chi^i \to \chi^i + \delta\chi^i, \tag{C.31}$$

one finds

$$\mathcal{L}_4 \to \mathcal{L}_4 + \delta\mathcal{L}_4 \tag{C.32}$$

where

$$\begin{aligned}\delta\mathcal{L}_4 = &-\frac{1}{f^2}H_{ij}\nabla^\mu\chi^j\nabla_\mu\delta\chi^i - \tfrac{1}{2}fH_{ij}(\star F^0 \wedge F^j + 2\chi^j \star F^0 \wedge F^0 + \star F^j \wedge F^0)\delta\chi^i \\ &+ \tfrac{1}{2}C_{ijk}(F^j \wedge F^k + \chi^j F^0 \wedge F^k + \tfrac{1}{3}\chi^j\chi^k F^0 \wedge F^0)\delta\chi^i.\end{aligned} \tag{C.33}$$

This implies that the equations of motion for the pseudoscalars with all $\chi^i = 0$ reads

$$-\tfrac{1}{2}fH_{ij} \star F^0 \wedge F^j - \tfrac{1}{2}fH_{ij} \star F^j \wedge F^0 + \tfrac{1}{2}C_{ijk}F^j \wedge F^k = 0 \tag{C.34}$$

which after using

$$\star A \wedge B = \star B \wedge A, \tag{C.35}$$

one finds the constraint among the field strengths

$$-fH_{ij} \star F^0 \wedge F^j + C_{ijk}F^j \wedge F^k = 0. \tag{C.36}$$

Setting all $\chi^i = 0$

$$\mathcal{L}_4 = R \star \mathbb{1} - \tfrac{1}{2}H_{ij} \star dh^i \wedge dh^j - \frac{3}{2f^2} \star df \wedge df - \tfrac{1}{2}f^3 \star F^0 \wedge F^0 - \tfrac{1}{2}fH_{ij} \star F^i \wedge F^j. \tag{C.37}$$

And it is tedious although straight forward to verify that with

$$h^i = e^{\frac{1}{3}\eta_0 - \eta_i}, \qquad f = e^{-\frac{1}{3}\eta_0}, \qquad \eta_0 \equiv \eta_1 + \eta_2 + \eta_3, \tag{C.38}$$

one reproduces (6.9).

D

Checking EOMs for Δ_- Matter

Here we verify that the matter (8.56 - 8.59) satisfies the relevant equations of motion.

In [13] they give us the Lagrangian for the truncation $\chi_2 = \chi_3 = \varphi_2 = \varphi_3 = 0$

$$\begin{aligned}\mathcal{L}_4 = R \star \mathbb{1} - \tfrac{1}{2} \star d\varphi_1 \wedge d\varphi_1 - \tfrac{1}{2} e^{2\varphi_1} \star d\chi_1 \wedge d\chi_1 \\ - \tfrac{1}{2} e^{-\varphi_1} (\star F_1 \wedge F_1 + \star F_2 \wedge F_2) - \tfrac{1}{2} \chi_1 (F_1 \wedge F_1 + F_2 \wedge F_2)\end{aligned} \tag{D.1}$$

where $F_1 = \sqrt{2}\, dA_1$ and $F_2 = \sqrt{2}\, dA_2$. It is straight forward to verify that the equations of motion for the full bosonic STU Lagrangian are solved by solutions to this truncated Lagrangian, in other words, that it is a consistent truncation. The field strengths F_1 and F_2 enter on equal footing, and their equations of motion are identical. Letting $F = dA$ denote either of F_1 and F_2, the relevant part of the truncated Lagrangian written out in index notation reads

$$-\tfrac{1}{4} e^{-\varphi_1} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{8} \chi_1 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{D.2}$$

where we have dropped the overall volume form. Varying with respect to A_μ one finds the equations of motion

$$\nabla_\mu (e^{-\varphi_1} F^{\mu\nu}) - \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\, \partial_\mu \chi_1 = 0, \tag{D.3}$$

where the second term simplifies to a derivative only acting on χ_1 since F satisfying the Bianchi identity $dF = 0$.

Starting with $F = F_1$, we compute and find

$$\nabla_\mu (e^{-\varphi_1} F_1^{\mu\phi}) = -\frac{(2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos\theta}{Q \Delta_-^{3/2}} \sqrt{2}, \tag{D.4}$$

while the other three components vanish. The second term also vanishes for all but the ϕ-component which reads

$$-\epsilon^{\theta\phi t r} F_{1tr}\, \partial_\theta \chi_1 = \frac{(2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos\theta}{Q \Delta_-^{3/2}} \sqrt{2}, \tag{D.5}$$

thus F_1 solves its equation of motion.

For the other gauge field, it seems we really need to drop the term $\sim 1/Q$. We can argue that this term should not contribute, after all we take $Q \to \infty$ in the scaling limit. Then we find again that it is only the ϕ component of the equations of motion, that is non-trivial. One finds

$$\nabla_\mu (e^{-\varphi_1} F_2^{\mu\phi}) = -\frac{(2m)^3 a \sqrt{\Pi_c \Pi_s} (\Pi_c - \Pi_s)}{Q \Delta_-^{3/2}} \sqrt{2}, \tag{D.6}$$

and

$$-\epsilon^{\theta\phi t r} F_{2tr}\, \partial_\theta \chi_1 = \frac{(2m)^3 a \sqrt{\Pi_c \Pi_s} (\Pi_c - \Pi_s)}{Q \Delta_-^{3/2}} \sqrt{2}, \tag{D.7}$$

and thus F_2 solves its equation of motion as well.

Varying with respect to χ_1 we find that its equation of motion reads

$$\nabla_\mu (e^{2\varphi_1} \nabla^\mu \chi_1) + \tfrac{1}{8} \epsilon^{\mu\nu\rho\sigma} (F_{1\mu\nu} F_{1\rho\sigma} + F_{2\mu\nu} F_{2\rho\sigma}) = 0 \tag{D.8}$$

Since we are discarding the term $\sim 1/Q$ in $\hat{A}_2$ we only have F_{2tr} non-zero, thus the second term equates to

$$\epsilon^{tr\theta\phi} F_{1tr} F_{1\theta\phi} = -\frac{4mQ^2(\Pi_c - \Pi_s)a\cos\theta}{\Delta_-^{3/2}}. \tag{D.9}$$

Since φ_1 is a function of r only we find that the first term simplifies to

$$e^{2\varphi_1}\nabla_\mu\nabla^\mu\chi_1 = \frac{4mQ^2(\Pi_c - \Pi_s)a\cos\theta}{\Delta_-^{3/2}}, \tag{D.10}$$

and thus the equations of motion for χ_1 hold.

Varying with respect to φ_1 we find.

$$\nabla_\mu\nabla^\mu\varphi_1 - e^{2\varphi_1}\nabla_\mu\chi_1\nabla^\mu\chi_1 + \tfrac{1}{4}e^{-\varphi_1}(F_{1\mu\nu}F^{1\mu\nu} + F_{2\mu\nu}F^{2\mu\nu}) = 0 \tag{D.11}$$

The different terms evaluate to

$$\nabla_\mu\nabla^\mu\varphi_1 = -\frac{(2m)^2(\Pi_c - \Pi_s)}{\Delta_-^{3/2}}\left\{(\Pi_c - \Pi_s)(r^2 - a^2) + 4m\Pi_s(r - m)\right\}, \tag{D.12}$$

$$-e^{2\varphi_1}\nabla_\mu\chi_1\nabla^\mu\chi_1 = -\frac{(2m)^2a^2(\Pi_c - \Pi_s)^2\sin^2\theta}{\Delta_-^{3/2}}, \tag{D.13}$$

$$\tfrac{1}{4}e^{-\varphi_1}(F_{1\mu\nu}F^{1\mu\nu} + F_{2\mu\nu}F^{2\mu\nu}) = \frac{1}{\Delta_-^{1/2}} + \frac{(2m)^4\Pi_c\Pi_s - (2m)^2(\Pi_{\cdot} - \Pi_s)^2a^2\cos^2\theta}{\Delta_-^{3/2}}. \tag{D.14}$$

We find that the equation of motion for φ_1 holds.

Next up is checking the Einstein equations. To do that we need to equate the energy-momentum tensor, by varying the Lagrangian with respect to $g^{\mu\nu}$. We find

$$\begin{aligned} 8\pi G_4 T_{\mu\nu} = \frac{1}{2}\Big[&\partial_\mu\varphi_1\partial_\nu\varphi_1 - \frac{g_{\mu\nu}}{2}(\partial\varphi_1)^2 + e^{2\varphi_1}\left(\partial_\mu\chi_1\partial_\nu\chi_1 - \frac{g_{\mu\nu}}{2}(\partial\chi_1)^2\right) \\ &+ e^{-\varphi_1}\sum_{i=1}^{2}\left(F_{i\mu}{}^{\rho}F_{i\nu\rho} - \frac{g_{\mu\nu}}{4}(F_i)^2\right)\Big]. \end{aligned} \tag{D.15}$$

All components on the diagonal are non-zero, and the only off diagonal component that is non-zero is $T_{t\phi} = T_{\phi t}$. This also applies to the Einstein tensor for the metric with $\Delta = \Delta_-$. The five non-trivial Einstein equations were found to be satisfied by the matter (8.56 - 8.59). We thus concluding that the scaling limit has given matter supporting the subtracted geometry, and furthermore that this matter is consistent with the STU model. This is not the most general matter, and we leave it for future work to try and generalize to arbitrary charges.

E

Spherical Reduction of 5D EOMs

In section 7.6 we present the uplift from 4D to 5D and subsequent reduction to 3D. The uplifted line element reading (7.88) and the KK-ansatz (7.89 - 7.92). In the following we shall go through the technical steps of this spherical reduction (5D $\rightarrow$ 3D) at the level of the equations of motion.

We see that the magnetic field strengths $\tilde{F}^i$ of the ansatz are proportional to vol_2, from which it follows that its equation of motion is trivially satisfied. This implies that there will be no equation of motion for these fields in the lower-dimensional theory. For convenience we reproduce the equations of motion for the matter in the 5D theory [2]:

$$0 = \mathrm{d}\Big(H_{ij} \star_5 \tilde{F}^j\Big) - \tfrac{1}{2} C_{ijk} \tilde{F}^j \wedge \tilde{F}^k, \tag{E.1}$$

$$0 = \mathrm{d}(\star_5 \mathrm{d}\Psi) - \tfrac{1}{2} \frac{\delta H_{ij}}{\delta \Psi} \star_5 \tilde{F}^i \wedge \tilde{F}^j, \tag{E.2}$$

$$0 = \mathrm{d}(\star_5 \mathrm{d}\Phi) - \tfrac{1}{2} \frac{\delta H_{ij}}{\delta \Phi} \star_5 \tilde{F}^i \wedge \tilde{F}^j. \tag{E.3}$$

Here $\star_5$ denotes the five-dimensional hodge dual, and d the exterior derivative [1]. To get the effective 3D equations of motion for the scalar fields from (E.2 - E.3), they firstly make use of the identity

$$\star A \wedge B = \star B \wedge A = \frac{1}{p!} A^{\mu_1 \cdots \mu_p} B_{\mu_1 \cdots \mu_p} \mathrm{vol}_D, \tag{E.4}$$

valid for p forms A and B, to find

$$\star_5 \tilde{F}^i \wedge \tilde{F}^i = \frac{B_i^2}{\ell_S^4} e^{-2U(x)} \mathrm{vol}_3 \wedge \mathrm{vol}_2. \tag{E.5}$$

Let us derive the above result in some more detail. From the general result valid for any two p-forms, we get for the forms in our ansatz (7.90)

$$\star_5 \tilde{F}^i \wedge \tilde{F}^i = \frac{1}{2!} \tilde{F}^{i\mu\nu} F^i_{\mu\nu} \mathrm{vol}_5 = g_5^{\theta\theta} g_5^{\phi\phi} \big(\tilde{F}^i_{\theta\phi}\big)^2 \mathrm{vol}_5. \tag{E.6}$$

From the KK ansatz metric (7.89), we read off

$$g_5^{\theta\theta} = \frac{e^{-2U(x)}}{\ell_S^2}, \qquad g_5^{\phi\phi} = \frac{e^{-2U(x)}}{\ell_S^2 \sin^2\theta}. \tag{E.7}$$

Finally we also realize from the metric ansatz that we can express the 5D volume form as

$$\mathrm{vol}_5 = e^{2U(x)} \mathrm{vol}_3 \wedge \mathrm{vol}_2, \tag{E.8}$$

simply because $\sqrt{|g_5|} = \sqrt{|g_3|}\sqrt{|e^{4U(x)} g_2|}$. Piecing it all together we see how one arrives at (E.5).

Also present in the scalar equations of motion are the Laplacians. We take a closer look at how they evaluate given our KK ansatz. We notice that for scalar fields $F(x)$ that only

[1]Conventions for form notation can be found in appendix A.

depend on x i.e (r, t, z), the gradient $\mathrm{d}F$ only has support on M. For any one-form A with support on M only we find

$$\begin{aligned}\star_5 A &= \frac{1}{2!}\epsilon_{\nu_1\nu_2\,\mu\,\theta\,\phi}\, A^\mu\, dx^{\nu_1}\wedge dx^{\nu_2}\wedge d\theta\wedge d\phi \\ &= \frac{1}{2!}e^{2U(x)}\epsilon_{\nu_1\nu_2\,\mu}\, A^\mu\, dx^{\nu_1}\wedge dx^{\nu_2}\wedge \mathrm{vol}_2 \\ &= e^{2U(x)}\star_3 A\wedge \mathrm{vol}_2.\end{aligned} \tag{E.9}$$

Thus

$$\star_5\mathrm{d}\Psi = e^{2U(x)}\star_3 \mathrm{d}\Psi\wedge\mathrm{vol}_2, \tag{E.10}$$

and we find that the scalar Laplacian reads

$$\mathrm{d}\star_5\mathrm{d}\Psi = e^{2U(x)}\left[\mathrm{d}\star_3\mathrm{d}\Psi + 2\mathrm{d}U(x)\wedge\star_3\mathrm{d}\Psi\right]\wedge\mathrm{vol}_2, \tag{E.11}$$

where it is used that $\mathrm{d}(\mathrm{vol}_2)=0$, and using the product rule $\mathrm{d}(A\wedge B)=\mathrm{d}A\wedge B+(-1)^p A\wedge\mathrm{d}B$, where p is the degree of A.

Putting all of this together, one finds that the scalar equations subject to our KK ansatz give the following effective 3D equations of motion for Ψ and Φ

$$0 = \mathrm{d}(\star_3\mathrm{d}\Psi) + 2\mathrm{d}U\wedge\star_3\mathrm{d}\Psi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Psi}\mathrm{vol}_3, \tag{E.12}$$

$$0 = \mathrm{d}(\star_3\mathrm{d}\Phi) + 2\mathrm{d}U\wedge\star_3\mathrm{d}\Phi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Phi}\mathrm{vol}_3. \tag{E.13}$$

Using the general results

$$\star\,\mathrm{d}\star\phi = (-1)^{D-t}\nabla_\mu\nabla^\mu\phi\,\mathrm{vol}_D, \tag{E.14}$$

and again

$$\star A\wedge B = \star B\wedge A = \frac{1}{p!}A^{\mu_1\cdots\mu_p}B_{\mu_1\cdots\mu_p}\mathrm{vol}_D, \tag{E.15}$$

we see that the above equations of motion in component form read.

$$0 = \nabla_\mu\nabla^\mu\Psi + 2\nabla_\mu U\nabla^\mu\Psi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Psi}, \tag{E.16}$$

$$0 = \nabla_\mu\nabla^\mu\Phi + 2\nabla_\mu U\nabla^\mu\Phi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}\frac{\delta H_{ii}}{\delta\Phi}. \tag{E.17}$$

To reduce the 5D Einstein equations we study the spin-connection and the curvature two-form. We let $\hat{e}^M$ denote the 5D local Lorentz frame and $M, N, \ldots$ flat indices on the 5D manifold. Furthermore e^a and e^α are respectively the orthonormal frames for the manifolds M and Y, where by $a, b, \ldots$ we denote flat indices on M, and by $\alpha, \beta, \ldots$ flat indices on Y. The KK ansatz then gives by our choice of vielbeins

$$\hat{e}^a = e^a, \tag{E.18}$$

$$\hat{e}^\alpha = e^U e^\alpha. \tag{E.19}$$

We start by tackling the spin-connection for the 5D manifold, which we denote $\hat{\omega}^M{}_N$. We further denote the spin-connections for respectively M and Y by $\omega^a{}_b$ and $\omega^\alpha{}_\beta$. These are respectively understood to be one-forms, and we are for the moment using index free notation, to clarify

$$\hat{\omega}^M{}_N = \hat{\omega}_\mu{}^M{}_N dx^\mu. \tag{E.20}$$

In general the spin-connection can be derived from the connection, in this case the Christoffel (torsion-free) connection

$$\hat{\omega}_\mu{}^M{}_N = \hat{e}_\lambda{}^M \hat{e}^\nu{}_N \Gamma^\lambda_{\mu\nu} - \hat{e}^\sigma{}_N \partial_\mu \hat{e}_\sigma{}^M. \tag{E.21}$$

In the case when M and N are supported on the 3D manifold M we get

$$\hat{\omega}_\mu{}^a{}_b = e_\lambda{}^a e^\nu{}_b \Gamma^\lambda_{\mu\nu} - e^\sigma{}_b \partial_\mu e_\sigma{}^a = \omega_\mu{}^a{}_b, \tag{E.22}$$

where the last equality follows from the fact that e^a only have support on M, and the Christoffel symbols

$$\Gamma^\lambda_{\theta\nu} = \Gamma^\lambda_{\phi\nu} = 0, \qquad \nu, \lambda \in \{r, t, z\}. \tag{E.23}$$

Similarly, when M and N are supported on Y the 5D connection reads

$$\hat{\omega}_\mu{}^\alpha{}_\beta = e_\lambda{}^\alpha e^\nu{}_\beta \Gamma^\lambda_{\mu\nu} - e^\sigma{}_\beta \partial_\mu e_\sigma{}^\alpha = \omega_\mu{}^\alpha{}_\beta, \tag{E.24}$$

where the last equality follows from the fact that, again e^α are only supported on Y and furthermore

$$\Gamma^\lambda_{\mu\nu} = 0, \qquad \nu, \lambda \in \{\theta, \phi\}, \qquad \mu \in \{r, t, z\}. \tag{E.25}$$

Finally when there is mixed support, i.e M on Y and N on M, we get

$$\hat{\omega}_\mu{}^\alpha{}_a = e^U e_\lambda{}^\alpha e^\nu{}_a \Gamma^\lambda_{\mu\nu} - e^U e^\sigma{}_a \partial_\mu e_\sigma{}^\alpha = e_\lambda{}^\alpha e^\nu{}_a \Gamma^\lambda_{\mu\nu}. \tag{E.26}$$

We need to evaluate the contraction "$ee\Gamma$". The non-vanishing Christoffel symbols that contribute to this contraction are found to be

$$\Gamma^\theta_{\theta\nu} = \Gamma^\phi_{\phi\nu} = \partial_\nu U. \tag{E.27}$$

It follows that

$$\hat{\omega}^\alpha{}_a = P_a e^\alpha, \qquad P_a \equiv e^U \partial_a U. \tag{E.28}$$

There is a sleeker way of calculating the curvature two-form by simply solving the torsion-free condition

$$\hat{\omega}^M{}_N \wedge \hat{e}^N = -\mathrm{d}\hat{e}^M. \tag{E.29}$$

We go through such a calculation in appendix B.

Next we move on to calculate the curvature two form, which we denote Θ. The 5D curvature two form is then given by

$$\hat{\Theta}^M{}_N = \mathrm{d}\hat{\omega}^M{}_N + \hat{\omega}^M{}_P \wedge \hat{\omega}^P{}_N. \tag{E.30}$$

Again we go through the different component arrangements, in the same manner as when calculating the spin-connection.

$$\begin{aligned}\hat{\Theta}^a{}_b &= d\hat{\omega}^a{}_b + \hat{\omega}^a{}_M \wedge \omega_b{}^M \\ &= d\omega^a{}_b + \omega^a{}_c \wedge \omega_b{}^c + \hat{\omega}^a{}_\alpha \wedge \hat{\omega}_b{}^\alpha \\ &= \Theta^a{}_b + \hat{\omega}^a{}_\alpha \wedge \hat{\omega}_b{}^\alpha.\end{aligned} \tag{E.31}$$

We will now show that the last term on the last line vanishes

$$\hat{\omega}^a{}_\alpha \wedge \hat{\omega}_b{}^\alpha = (P^a e_\alpha) \wedge (P_b e^\alpha) = P^a P_b \eta_{\alpha\beta}(e^\beta \wedge e^\alpha) = 0, \tag{E.32}$$

which vanishes since $e^\beta \wedge e^\alpha$ is anti-symmetric in (α, β), and it is contracted with $\eta_{\alpha\beta}$ which is symmetric. Thus

$$\hat{\Theta}^a{}_b = \Theta^a{}_b. \tag{E.33}$$

Similarly one gets

$$\hat{\Theta}^{\alpha}{}_{\beta} = \Theta^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{a} \wedge \hat{\omega}_{\beta}{}^{a}. \tag{E.34}$$

The last term evaluates to

$$\hat{\omega}^{\alpha}{}_{a} \wedge \hat{\omega}_{\beta}{}^{a} = P_a P^a \eta_{\beta\gamma}(e^{\alpha} \wedge e^{\gamma}) = -P_a P^a \eta_{\beta[\gamma}\delta^{\alpha}{}_{\sigma]} e^{\sigma} \wedge e^{\gamma}. \tag{E.35}$$

Thus

$$\hat{\Theta}^{\alpha}{}_{\beta} = \Theta^{\alpha}{}_{\beta} - P_a P^a \eta_{\beta[\gamma}\delta^{\alpha}{}_{\sigma]} e^{\sigma} \wedge e^{\gamma}. \tag{E.36}$$

Finally the mixed term reads

$$\begin{aligned}
\hat{\Theta}^{\alpha}{}_{a} &= \mathrm{d}\hat{\omega}^{\alpha}{}_{a} + \hat{\omega}^{\alpha}{}_{M} \wedge \hat{\omega}^{M}{}_{a} \\
&= \mathrm{d}(P_a e^{\alpha}) + \omega^{\alpha}{}_{\beta} \wedge \hat{\omega}^{\beta}{}_{a} + \hat{\omega}^{\alpha}{}_{c} \wedge \omega^{c}{}_{a} \\
&= \mathrm{d}P_a \wedge e^{\alpha} + P_a \mathrm{d}e^{\alpha} + \omega^{\alpha}{}_{\beta} \wedge P_a e^{\beta} + P_c e^{\alpha} \wedge \omega^{c}{}_{a} \\
&= \mathrm{d}P_a \wedge e^{\alpha} + P_c e^{\alpha} \wedge \omega^{c}{}_{a} \\
&= (\mathrm{d}P_a - \omega^{c}{}_{a} P_c) \wedge e^{\alpha} \\
&= (\nabla_c P_a) e^{c} \wedge e^{\alpha} \\
&= \delta^{\alpha}{}_{\gamma} (\nabla_c P_a) e^{c} \wedge e^{\gamma}.
\end{aligned} \tag{E.37}$$

To get from the third to the fourth line in this calculation we used the torsion-free condition to eliminate the two terms in the middle. The expression can in the end be written in terms of the covariant derivative on the manifold M noting that

$$\nabla_{\mu} P_a = dP_a - \omega^{c}{}_{a} P_c, \qquad \nabla_c\, e^{c}{}_{\mu} = \nabla_{\mu}. \tag{E.38}$$

We can now proceed to use this to find the non-vanishing components of the Riemann tensor, via

$$\hat{\Theta}^{M}{}_{N} = \frac{1}{2!} \hat{R}^{M}{}_{NPQ}\, \hat{e}^{P} \wedge \hat{e}^{Q}. \tag{E.39}$$

Denoting the Riemann tensor of M by R and that of the compact manifold Y by $\mathcal{R}$ we also have

$$\Theta^{a}{}_{b} = \frac{1}{2!} R^{a}{}_{bcd}\, e^{c} \wedge e^{d}, \qquad \Theta^{\alpha}{}_{\beta} = \frac{1}{2!} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta}\, e^{\gamma} \wedge e^{\delta}. \tag{E.40}$$

Now we can readily read off the components of the 5D Riemann tensor by inspection. For instance, starting with

$$\hat{\Theta}^{a}{}_{b} = \frac{1}{2!} \hat{R}^{a}{}_{bPQ}\, \hat{e}^{P} \wedge \hat{e}^{Q} \tag{E.41}$$

we make use of the relation

$$\hat{\Theta}^{a}{}_{b} = \Theta^{a}{}_{b} = \frac{1}{2!} R^{a}{}_{bcd}\, e^{c} \wedge e^{d} = \frac{1}{2!} R^{a}{}_{bcd}\, \hat{e}^{c} \wedge \hat{e}^{d}, \tag{E.42}$$

and readily find that

$$\hat{R}^{a}{}_{bcd} = R^{a}{}_{bcd}. \tag{E.43}$$

Similarly, noting that in terms of the hatted vielbeins we have

$$\Theta^{\alpha}{}_{\beta} = \frac{1}{2!} e^{-2U} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta}\, \hat{e}^{\gamma} \wedge \hat{e}^{\delta}, \tag{E.44}$$

we make use of

$$\hat{\Theta}^{\alpha}{}_{\beta} = \Theta^{\alpha}{}_{\beta} - e^{-2U} P_a P^a \eta_{\beta[\gamma}\delta^{\alpha}{}_{\sigma]} \hat{e}^{\sigma} \wedge \hat{e}^{\gamma}, \tag{E.45}$$

to read off

$$\hat{R}^{\alpha}{}_{\beta\gamma\delta} = e^{-2U} \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} - 2e^{-2U} P_a P^a \delta^{\alpha}{}_{[\gamma}\eta_{\delta]\beta}. \tag{E.46}$$

Notice here that it is the relation $e^\alpha = e^{-U}\hat{e}^\alpha$ that gives us the extra factors of e^{-2U}.

There are two more non-vanishing components, both arising from

$$\begin{aligned}\hat{\Theta}^\alpha{}_a &= \delta^\alpha{}_\gamma(\nabla_c P_a)e^c \wedge e^\gamma \\ &= e^{-U}\delta^\alpha{}_\beta(\nabla_b P_a)\hat{e}^b \wedge \hat{e}^\beta \\ &= -e^{-U}\delta^\alpha{}_\beta(\nabla_b P_a)\hat{e}^\beta \wedge \hat{e}^b.\end{aligned} \tag{E.47}$$

We note that

$$\frac{1}{2!}R^\alpha{}_{a\beta b}\,\hat{e}^\beta \wedge \hat{e}^b + \frac{1}{2!}R^\alpha{}_{ab\beta}\,\hat{e}^b \wedge \hat{e}^\beta = \hat{\Theta}^\alpha{}_a, \tag{E.48}$$

where, because $\hat{\Theta}^\alpha{}_a$ involves a complete contraction of the hatted vielbeins we have to include two terms on the LHS. Using that the Riemann tensor is antisymmetric in its last two indices, this cancels the sign arising from swapping the order of the vielbeins in the second term on the LHS and we find

$$R^\alpha{}_{a\beta b}\,\hat{e}^\beta \wedge \hat{e}^b = \hat{\Theta}^\alpha{}_a, \tag{E.49}$$

from which we deduce

$$\hat{R}^\alpha{}_{a\beta b} = -\delta^\alpha{}_\beta e^{-U}\nabla_b P_a. \tag{E.50}$$

Lastly, using that the Riemann tensor with all indices lowered is also antisymmetric in its first two indices, we find

$$\hat{R}^a{}_{\alpha b\beta} = -\eta_{\alpha\beta}e^{-U}\nabla_b P^a. \tag{E.51}$$

These are the only non-vanishing components of the 5D Riemann tensor.

We now move on to calculate the Ricci tensor $\hat{R}_{MN} = \hat{R}^P{}_{MPN}$. Again going through the different types of component combinations:

$$\begin{aligned}\hat{R}_{ab} &= \hat{R}^M{}_{aMb} \\ &= \hat{R}^c{}_{acb} + \hat{R}^\alpha{}_{a\alpha b} \\ &= R^c{}_{acb} - \delta^\alpha{}_\alpha e^{-U}\nabla_b P_a \\ &= R_{ab} - 2e^{-U}\nabla_b P_a \\ &= R_{ab} - 2(\nabla_b\nabla_a U + \nabla_a U\nabla_b U).\end{aligned} \tag{E.52}$$

$$\begin{aligned}\hat{R}_{\alpha\beta} &= \hat{R}^M{}_{\alpha M\beta} \\ &= \hat{R}^a{}_{\alpha a\beta} + \hat{R}^\gamma{}_{\alpha\gamma\beta} \\ &= -\eta_{\alpha\beta}e^{-U}\nabla_a P^a + e^{-2U}\mathcal{R}^\gamma{}_{\alpha\gamma\beta} - 2e^{-2U}P_aP^a\delta^\gamma{}_{[\gamma}\eta_{\beta]\alpha} \\ &= -\eta_{\alpha\beta}e^{-U}\nabla_a P^a + e^{-2U}\mathcal{R}_{\alpha\beta} - \eta_{\alpha\beta}e^{-2U}P_aP^a \\ &= e^{-2U}\mathcal{R}_{\alpha\beta} - 2(\nabla_a U\nabla^a U)\eta_{\alpha\beta} - (\nabla_a\nabla^a U)\eta_{\alpha\beta}.\end{aligned} \tag{E.53}$$

$$\begin{aligned}\hat{R}_{a\alpha} &= \hat{R}^M{}_{aM\alpha} \\ &= \hat{R}^b{}_{ab\alpha} + \hat{R}^\beta{}_{a\beta\alpha} \\ &= 0.\end{aligned} \tag{E.54}$$

Now we are prepared to find the reduced Einstein equations. Recalling the 5D Einstein equation (using flat indices)

$$\begin{aligned}G_{MN} = \frac{1}{2}\Big[&\nabla_M\Psi\nabla_N\Psi - \tfrac{1}{2}\eta_{MN}\nabla_P\Psi\nabla^P\Psi + \nabla_M\Psi\nabla_N\Psi - \tfrac{1}{2}\eta_{MN}\nabla_P\Psi\nabla^P\Psi \\ &+ H_{ij}\left(\tilde{F}^i_M{}^P\tilde{F}^j_{NP} - \tfrac{1}{4}\eta_{MN}\tilde{F}^i_{PQ}\tilde{F}^{iPQ}\right)\Big],\end{aligned} \tag{E.55}$$

we proceed to find R_{ab} in terms of the 5D objects. From

$$\hat{R}_{ab} = R_{ab} - 2(\nabla_b \nabla_a U + \nabla_a U \nabla_b U), \tag{E.56}$$

we find that

$$R_{ab} = \hat{R}_{ab} + 2(\nabla_b \nabla_a U + \nabla_a U \nabla_b U). \tag{E.57}$$

Now we would like to establish the form of R_{ab} from the 5D Einstein equation. To this end we note that contracting both sides of

$$\hat{R}_{MN} - \tfrac{1}{2}\eta_{MN}\hat{R} = G_{MN} \implies \hat{R} - \tfrac{5}{2}\hat{R} = G, \tag{E.58}$$

from which we read off $\hat{R} = -\frac{2}{3}G$, thus

$$\hat{R}_{MN} = G_{MN} - \tfrac{1}{3}\eta_{MN}G, \tag{E.59}$$

and finally

$$\hat{R}_{ab} = G_{ab} - \tfrac{1}{3}\eta_{ab}G. \tag{E.60}$$

Taking the trace of the 5D Einstein tensor we find

$$\begin{aligned} G &= \tfrac{1}{2}\left[-\tfrac{3}{2}\nabla_P\Psi\nabla^P\Psi - \tfrac{3}{2}\nabla_P\Phi\nabla^P\Phi - \tfrac{1}{4}H_{ij}\tilde{F}^i_{PQ}\tilde{F}^{jPQ}\right] \\ &= \tfrac{1}{2}\left[-\tfrac{3}{2}\nabla_c\Psi\nabla^c\Psi - \tfrac{3}{2}\nabla_c\Phi\nabla^c\Phi - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}\right], \end{aligned} \tag{E.61}$$

where in the last line we have used that $\Psi = \Psi(x)$, $\Phi = \Phi(x)$ where $x = (r, t, z)$ and finally that H_{ii} are the only non-vanishing components, together with the explicit ansatz for $\tilde{F}^i$. Equating

$$R_{ab} = G_{ab} - \tfrac{1}{3}\eta_{ab}G \tag{E.62}$$

we finally get

$$\begin{aligned} R_{ab} =& 2(\nabla_b\nabla_a U + \nabla_a U\nabla_b U) \\ &+ \frac{1}{2}\left[\nabla_a\Psi\nabla^a\Psi + \nabla_a\Phi\nabla^a\Phi - \tfrac{1}{3}\eta_{ab}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi)\right]. \end{aligned} \tag{E.63}$$

The above is the reduced Einstein equations, where one can clearly see that all reference to the "internal" two-sphere has dropped out.

We finalize this lengthy reduction by looking at what we get from the components of the 5D Einstein equations that are in the directions of Y. Again we start off with the 5D Ricci tensor

$$\hat{R}_{\alpha\beta} = e^{-2U}\mathcal{R}_{\alpha\beta} - (\nabla_a\nabla^a U + 2\nabla_a U\nabla^a U)\eta_{\alpha\beta}, \tag{E.64}$$

together with the relation

$$\hat{R}_{\alpha\beta} = G_{\alpha\beta} - \tfrac{1}{3}\eta_{\alpha\beta}G. \tag{E.65}$$

Noting that the only non zero components of $H_{ij}\tilde{F}^{iP}_{\alpha}\tilde{F}^{i}_{\beta P}$ are (θ,θ) and (ϕ,ϕ) and read respectively

$$H_{ii}g^{\phi\phi}\big(\tilde{F}^i_{\theta\phi}\big)^2 = g^{\phi\phi}B_i^2\sin^2\theta, \qquad H_{ii}g^{\theta\theta}\big(\tilde{F}^i_{\phi\theta}\big)^2 = g^{\theta\theta}B_i^2\sin^2\theta, \tag{E.66}$$

and that we in turn can express these respectively as

$$\frac{e^{-4U}}{\ell_S^4}B_i^2 g_{\theta\theta}, \qquad \frac{e^{-4U}}{\ell_S^4}B_i^2 g_{\phi\phi}, \tag{E.67}$$

we find that we can rewrite it with flat indices as

$$H_{ij}\tilde{F}^{iP}_{\alpha}\tilde{F}^{i}_{\beta P} = e^{-4U}\eta_{\alpha\beta}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}. \tag{E.68}$$

We then find

$$G_{\alpha\beta} - \tfrac{1}{3}G = \tfrac{1}{3}e^{-4U}\eta_{\alpha\beta}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}, \tag{E.69}$$

and we get that the components of Einstein's equations read

$$e^{-2U}\mathcal{R}_{\alpha\beta} = \eta_{\alpha\beta}\left(\nabla_a\nabla^a U + 2\nabla_a U\nabla^a U + \tfrac{1}{3}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}\right). \tag{E.70}$$

Given $\mathcal{R}_{\alpha\beta} = \eta_{\alpha\beta}/\ell_S^2$ we clearly see that these equations reduce to

$$\nabla_a\nabla^a U + 2\nabla_a U\nabla^a U - \frac{e^{-2U}}{\ell_S^2} + \tfrac{1}{3}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii} = 0. \tag{E.71}$$

Finally from the fact that all reference to the internal two-sphere drops out, we see that the reduction is indeed consistent. To close off this lengthy procedure, we note as the authors have [2], that all four equations of motion can be derived from the string frame action

$$\begin{aligned} S_{\text{string}} = -\frac{1}{16\pi G_3}\int d^3x\sqrt{|g|}e^{2U}\Bigg[R + \frac{2}{\ell_S^2}e^{-2U} - \tfrac{1}{2}e^{-4U}\sum_{i=1}^{3}\frac{B_i^2}{\ell_S^4}H_{ii}(\Psi,\Phi) \\ + 2(\nabla U)^2 - \tfrac{1}{2}(\nabla\Psi)^2 - \tfrac{1}{2}(\nabla\Phi)^2\Bigg]. \end{aligned} \tag{E.72}$$

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