FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

PhD Thesis

Jácome Armas

(Electro)Elasticity from Gravity

Academic advisors:
Niels A. Obers
Troels Harmark

Submitted: October 29, 2012

(Electro)Elasticity from Gravity

by

Jácome Armas

Submitted for the degree of Doctor of Philosophy in Physics
at the Niels Bohr Institute, Faculty of Science,
University of Copenhagen.

October 2012

Abstract

Gravitational physics in higher dimensions can be used as a laboratory for conducting experiments in material science. A laboratory can be set up whenever there exists a connection or duality between gravity and the physics of continuous media. Conducting experiments amounts to placing the experimenter on a surface in space-time whose task is to measure the fluxes of energy-momentum originating from a gravitational solution. Black branes - black holes infinitely extended along some directions - are the canonical example of gravity solutions that can be used as objects of study for material science. From this point of view, they belong to a rather unusual class of materials since, besides the fact that they cannot be experienced through the senses, they can be deformed in space as well as in time and exhibit simultaneously fluid and solid properties. In this thesis, we explore the *blackfold approach* to higher-dimensional black holes, an effective worldvolume theory that describes the dynamics of thin black branes. Within this framework, black holes made of wrapped black branes (blackfolds), can be viewed as materials, characterized by transport and response coefficients such as viscosities, Young modulus and piezoelectric moduli. The linkage between fluctuations of black branes and the physics of dissipative fluid flows has been established in the past few years, however, the connection between the bending of black branes and relativistic (electro)elasticity theory is genuinely new and has been developed here. In fact, what is realized is that depending on the type of deformation applied to the back brane, different behavior is exhibited. In this thesis, we explore this idea and show that the effective theory describing the dynamics of black branes can be reformulated in a way such that the (electro)elastic character of these materials becomes evident. Blackfolds can be seen as thin elastic branes and in the limit where they become infinitely thin we compute their modulus of hydrostatic compression and elasticity tensor. The elastic equilibrium condition for different configurations in non-trivial background space-times is obtained and known black hole solutions, such as Kerr-(Anti) deSitter black holes, are reproduced. These last are shown to suffer from a Gregory-Laflamme instability, which is verified by taking an ultra-spinning limit. The (electro)elastic character of neutral and charged dilatonic black strings is unravelled by measuring for the first time the elastic and piezoeletric moduli of materials that are deformed both in space as well as in time.

Resumé

I denne afhandling udforsker vi den såkaldte blackfold metode for sorte huller og braner i mere end fire dimensioner. Blackfold metoden går ud på at approksimere tynde sorte braner med en effektiv dynamisk world-volume teori. En blackfold er en sort bran der kan bøjes, vikles og bevæge sig dynamisk hvilket beskrives af blackfold metoden. I denne metode kan sorte braner beskrives som kontinuerte materialer og de har både hydrodynamiske og kontinuummekaniske karakteristika. Dette inkluderer både transport og responskoefficienter såsom viskositet, Young modulus og piezoelektriske moduli. Forbindelsen mellem fluktuationer af sorte braner og hydrodynamik er udviklet fornyligt over flere år mens forbindelsen til kontinuummekanik er ny og er blevet udviklet som en del af dette projekt. I denne afhandling udforsker vi denne ide og viser at den effektive teori der beskriver dynamikken af sorte braner kan formuleres på en måde sådan at de (elektro)elastiske karakteristika explicit kommer til syne. Blackfolds kan ses som værende tynde elastiske braner og i grænsen hvor de bliver arbitrært tynde udregner vi deres moduli for hydrodynamisk kompression og tensoren for elasticitet. Vi udregner betingelsen for elastisk ligevægt for forskellige konfigurationer i ikke-trivielle rum-tids baggrunde for kendte sorte huller såsom Kerr-(Anti)-de Sitter sorte hul. I dette specielle tilfælde er det sorte hul ustabilt med hensyn til Gregory-Laflamme instabiliteten, og dette kan ses ved at tage en ultraspin grænse. De (elektro)statiske karakteristika for neutrale og ladede sorte stringe bliver fundet ved at måle for første gang den elastiske og piezoelektriske moduli af materialer som deformeres både i rum såvel som i tid.

Acknowledgements

The work in this thesis was mostly carried out at the Niels Bohr Institute under the supervision of Niels Obers and Troels Harmark. I am extremely grateful for their guidance and support. I have also had the pleasure to work with a few amazing collaborators: Pawel Caputa, Joan Camps, Andreas Vigand Pedersen and Jakob Gath.

Part of this thesis was developed at the Tata Institute for Fundamental Research in India during the period of four months, where I was received by Shiraz Minwalla with great hospitality. I have enjoyed many discussions there with Jyotirmoy Bhattacharya.

During the course of this thesis I had the pleasure to meet many friends, and developed many other projects that contributed to this in an indirect way. I give my special thanks to these towards the end.

I am grateful to Fabrikken, Christiania, for providing me with a safe environment where I could live and work.

This thesis was financed by FCT Portugal.

List of Publications

1. J. Armas and T. Harmark, **'Uniqueness Theorem for Black Hole Space-Times with Multiple Disconnected Horizons,"** *JHEP* **05** (2010) 093, 0911.4654.
2. J. Armas, **"Maximal Analytic Extension and Hidden Symmetries of the Dipole Black Ring,"** *Class. Quantum Grav.* **28** (2011) 235014, 1011.5618.
3. J. Armas and N. A. Obers, **"Blackfolds in (Anti)-de Sitter Backgrounds,"** *Phys.Rev.* **D83** (2011) 084039, 1012.5081.
4. J. Armas, J. Camps, T. Harmark, and N. A. Obers, **"The Young Modulus of Black Strings and the Fine Structure of Blackfolds,"** *JHEP* **1202** (2012) 110, 1110.4835.
5. J. Armas, P. Caputa, and T. Harmark, **"Domain Structure of Black Hole Space-Times with a Cosmological Constant,"** *Phys.Rev.* **D85** (2012) 084019,
6. J. Armas, T. Harmark, N. A. Obers, M. Orselli, and A. V. Pedersen, **"Thermal Giant Gravitons,"** *Accepted for publication in JHEP*, 1207.2789.
7. J. Armas, J. Gath, and N. A. Obers, **"Black Branes as Piezoelectrics,"** *Accepted for publication in PRL*, 1209.2127.
8. J. Armas and N. A. Obers, **"Relativistic Elasticity of Stationary Fluid Branes,"** 1210.5197.
9. J. Armas, **"Conversations on Quantum Gravity,"** *In preparation*

Contents

Chapter 1

Introduction

Black holes continue to be extremely interesting and important objects of study of experimental and theoretical physics not only due to their complexity but also due to their weirdness. Experimentally (in the astrophysics sense) they pose highly difficult problems of observation and measurement, let alone the puzzling phenomena of quasars, supermassive black holes and black hole formation. Theoretically, besides the high level of fine mathematical detail necessary to explain experimental observations, they provide one of the most exciting puzzles that defy both gravitational physics and quantum mechanics.

According to general relativity, black holes hide a singularity behind the horizon where the strength of gravitational fields is beyond reasonable leading to a break down of known physics. In addition, a semi-classical analysis that attempts to incorporate a slight amount of quantum physics results in information loss, which has lead to the famous *information paradox* [1, 2, 3]. While there are many unfinished proposals to deal with these two problems, the most common belief is that only a consistent theory of quantum gravity will be able to resolve both the singularity and information loss puzzles [4]. Several attempts to unify gravity with quantum mechanics have been made and none so far could be experimentally verified. Among these, string theory [5, 6] has been the most exhaustively explored approach providing us with many insights into these puzzles and, in certain cases, able to touch "reality" in a qualitative way through one of its spinoffs - the AdS/CFT correspondence [7].

This thesis, as any other in the field of theoretical high energy physics, and allowing ourselves to portray it partly by a human body, has one foot in a mathematical development, another in a hunch (or in many cases belief) of what physical reality is supposed to be and a hand that barely touches reality through an intricate process of relations to other hands - parts of other bodies (theses). Let us be more explicit about this point. The AdS/CFT correspondence [7] is a holographic duality that relates, in its original form, type IIB string theory (our hunch/belief) on $AdS_5 \times S^5$ space-time - a product of five-dimensional Anti-de Sitter (AdS) and a five-dimensional sphere - and $\mathcal{N} = 4$ Super Yang-Mills (SYM) - a conformal quantum field theory that lives on the boundary of AdS. This field theory living on the boundary, in a certain regime, admits a hydrodynamic description which translates into fluctuations in the geometry of planar black holes in the bulk of AdS - this has been dubbed the *fluid/gravity correspondence* [8, 9, 10, 11, 12]. In this way, by studying higher-dimensional gravitational physics it is possible to obtain properties, such as the viscosity to entropy ratio, of quantum field theories in the regime where they are described by fluid configurations [8, 9]. While $\mathcal{N} = 4$ SYM is a theory that has a priori nothing to do with any particle theory observed in physical reality it seems to share similar properties with certain physical systems, in particular with the quark-gluon plasma extensively studied at RHIC [13]. In this context, the work presented in this thesis, which explores a method - *the blackfold approach* [14, 15, 16, 17, 18, 19] - that can be regarded as a generalization of the *fluid/gravity correspondence*, could possibly be used to predict by extrapolation (through an intricate process of relations) properties of real world physical models.

Despite the connection between the study of higher-dimensional gravity and real world physics not being straightforward, the linkage between gravitational physics and hydrodynamics and, more generally, material science deserves attention in its own right. Let us give an example. Within the framework of the *fluid/gravity correspondence* perturbations of black branes along boundary directions are mapped onto dissipative fluid configurations on the boundary. The physics of dissipative fluid flows can therefore be used to study the dynamics of black holes and, conversely, the analysis of perturbations of black holes can be used as a testing ground for developing the correct theory of hydrodynamics. As a matter of a fact, this has lead to the development of parity violating theories of hydrodynamics and to a rigorous development of superfluid dynamics [20, 21, 22, 20, 23, 24]. Another example, pertinent to this thesis, relates the bending of black brane geometries along directions transverse to the worldvolume with the physics of elastic solids and piezoelectrics [25, 18, 26, 19]. Connections as these can be used to conduct experiments, as if a laboratory was given to us, and can also lead to a formal development of relativistic (electro)elasticity theory - a subject which is also far from being established in full generality. The important lesson to take from these examples is that different long-wavelength perturbations of black branes are described by different long-wavelength physics.

The main subject of this thesis is to establish the connection between gravitational physics and the theory of (electro)elasticity. However, it can be considered, in a broader context, as an exploration of higher dimensional black hole physics despite its applications to the fundamental problems of string theory and, in holographic setups, to the construction of toy models that can mimic properties of quantum matter. From a purely gravitational perspective, the interest in gravitational physics in more than three spatial dimensions has increased considerably in the past few years due to the fact that as the space-time dimension D is increased the space of possible solutions becomes more intricate, rich and complex [27, 28, 29]. To what concerns black hole solutions of Einstein equations, many more horizon topologies (single or multiple) are allowed than those restricted by uniqueness theorems in $D = 4$ [30, 31, 32, 33, 34, 35, 36]. The non-linearity of Einstein equations also becomes aggravated in higher dimensions and solution generating techniques are scarcer and more evolved, hence so is the number of exact and analytic solutions. Effective descriptions of black holes in a corner of phase space and perturbative methods for solving Einstein equations are needed in order to study the vast landscape of black hole solutions and their dynamics. Within the realm of higher dimensional gravity, there are a few interesting avenues of research:

- **Analysis of known solutions:** In $D = 5$ dimensions new and exotic, exact and analytic solutions such as black rings, black saturns, di-rings, bi-rings exhibit complicated metric structures that are hard to analytically analyze due to their dependence on a large number of parameters. The study of their geodesic structure, possible extensions across Killing horizons, causal relations and properties such as regularity, stable causality or even global hyperbolicity is still in many cases left untouched while in others some considerable amount of work has been done [37, 38, 39, 40, 41].

- **Classification and uniqueness:** In $D = 4$ one can describe black holes in vacuum Einstein gravity just by their asymptotic charges and show uniqueness of the Kerr solution [42, 43, 44]. In $D \geq 5$ black hole uniqueness theorems must be at least supplemented by the rod/domain structure as the asymptotic charges are not sufficient [45, 46, 47, 48, 49]. In $D = 5$ and $D = 11$ uniqueness theorems have been written down for certain types of theories where an integrable subspace of Einstein equations exists, which normally involves a high degree of symmetry [47, 50, 51, 52, 53]. In non-asymptotically flat space-times uniqueness of black holes has not been accomplished even in $D = 4$[1] but a complete topological and geometric description has been provided and serves as a first step for a uniqueness theorem [49].

- **Construction of solutions:** Due to the non-linearity and non-integrability of Einstein equations in general, solution generating techniques in higher dimensions are few. Whenever Einstein equations are integrable in a certain sub-sector, one can generate new black hole solutions using, for example, the rod structure [31, 32, 33, 34, 35, 36]. In vacuum, the only non-trivial, exact and analytic black hole solutions so far known are the $5D$ black ring [30, 31] and the higher-dimensional Myers-Perry (MP) solution [55] or composites with multiple horizons made out of these two [32, 33, 34, 35]. In (Anti)-de Sitter space-time for example, the only non-trivial black hole solution is the higher-dimensional Kerr-(A)dS solution [56, 57]. It is clear that even just a method for solving perturbatively in a derivative expansion Einstein equations in higher-dimensions can be useful to probe and study the higher dimensional phase space and properties of black holes.

- **Effective descriptions:** Due to the complexity of solution space in higher dimensions and the difficulty in finding exact analytic solutions, effective descriptions in certain regions of parameter space can be used to study the dynamics of black holes. Examples of these effective theories are: the *membrane paradigm* [58, 59, 60, 61, 62], which describes the behavior of black hole horizons; the *fluid/gravity correspondence*

[1] Uniqueness in AdS of static black holes has been shown under certain assumptions [54].

[8, 9, 10, 11, 12], which provides an effective long-wavelength description for the behavior of perturbations of black branes along boundary directions; and the *blackfold approach* [14, 15, 16, 17, 18, 19], an effective theory for the dynamics of thin black branes, which characterizes the behavior of strained black brane geometries along worldvolume directions and orthogonal to these. All these approaches show that the physical properties of black holes in certain regimes can be captured by the physics of fluid flows that either live on the stretched horizon (*membrane paradigm*), on the boundary (*fluid/gravity correspondence*) or in an intermediate region (*blackfold approach*).

While all these possibilities must be investigated to correctly understand the consequences of higher dimensionality in gravitational physics, it is the last two avenues of these four that we take in this thesis by exploring and applying the *blackfold approach* to higher-dimensional black holes. Below, we provide the insight underlying this development, briefly describe the method, summarize some of its accomplishments and give an overview of the main contributions of the work presented here together with the structure of this thesis.

1.1 The blackfold approach

In higher-dimensional gravity black holes admit regimes where two widely separated horizon length scales exist simultaneously. Two such well known examples are 5-dimensional black rings [30] and MP black holes [55]. A 5-dimensional black ring has $S^1 \times S^2$ horizon topology with rotation along the S^1 and is characterized by a horizon size r_0 and a radius R. When set to rotate very fast its radius increases rapidly and one obtains a hierarchy of scales $r_0 \ll R$. Taking a near horizon limit at a fixed angle of the S^1 direction leads to the metric of a boosted black string [63]. If we now focus on the case of a singly-spinning MP black hole in dimensions higher than six, the horizon topology is S^{D-2} under mild rotation. Increasing its angular momentum results in an horizon topology for which a S^{D-4}-sphere of radius r_0 is fibered over a disk D parametrized by (ρ, θ) of radius R satisfying $r_0 \ll R$. Again approaching the disk at any fixed angle θ leads to the metric of a boosted black membrane [64, 25].

These two geometries are in fact the only known non-trivial, exact and analytic back hole solutions of vacuum Einstein equations in higher dimensions and, as explained above, have something in common: both exhibit regimes where their horizon is characterized by two distinct and widely separated length scales. Bearing this in mind, one is lead to postulate the existence of other black hole solutions that show this same common feature and to devise a method capable of searching for them. Such method is called the *blackfold approach* and consists in wrapping (or bending) the metric of a boosted black brane along an arbitrary horizon geometry and solving perturbatively Einstein equations order-by-order in the parameter $\varepsilon \equiv r_0/R$. We proceed by describing in detail how this method works.

Description of the method

The *blackfold approach* was first used in the context of black rings in a variety of different settings [14, 65, 66]. Recently, it has been generalized for a very large class of black holes which are locally described by boosted Schwarzschild branes [18]. Its application requires solving Einstein equations by means of a matched asymptotic expansion (MAE), in which there are two different coordinate regions: the region near the horizon for which $r \ll R$, and the far region for which $r \gg r_0$ where the weak field approximation is valid. The applicability of a MAE requires the existence of an overlap region $r_0 \ll r \ll R$ which is guaranteed whenever $r_0 \ll R$. Focusing on neutral (uncharged) black branes, in the near region the metric is a perturbation of the boosted black brane while in the far region the gravitational field is well described by the linear approximation sourced by the appropriate blackfold effective stress-energy tensor. Each region feeds the other with boundary conditions. We summarize the procedure to first order in ε:

- **0th order (near/far):** In the near region the geometry is locally that of a boosted Schwarzschild membrane

$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \left(1 - \frac{r_0^n}{r^n}\right)^{-1} dr^2 + r^2 d\Omega_{n+1}^2 \ , \tag{1.1.1}$$

  where the vectors u^a are related to the boost velocities and satisfy $u^a u_a = -1$. In the far region the metric $g^{\mu\nu}$ is that of the ambient spacetime we choose our black hole to tend asymptotically to.

- **1st order (far):** For an observer sitting in the far region the gravitational field is that of an object with the shape of the horizon of the black hole we want to construct and locally described by the stress-energy tensor that sources the metric (1.1.1). This stress tensor $\hat{T}^{\mu\nu}$ can be written in the perfect fluid form [16]. Requiring the solvability of Einstein equations in the linearized regime $\Box \bar{h}_{\mu\nu} = -16\pi G \hat{T}_{\mu\nu}$ implies that only sources satisfying

$$\nabla_\nu \hat{T}^{\mu\nu} = 0 \tag{1.1.2}$$

  can be consistently coupled to the gravitational field [14, 18]. Here we have assumed the absence of couplings to other background fields. The set of equations that result from (1.1.2) constitute the blackfold effective equations of motion (EOMs), which we will analyze in detail below, and have been derived directly from Einstein equations for objects with no boundaries and whose local geometry is given by (1.1.1)[2].

[2] Even though that Eq. (1.1.2) has only been proven to be the correct requirement for constructing black hole spacetimes in these cases, the correct matching with many different analytic solutions makes it a strong case to assume its validity in full generality.

- **1st order (near):** The geometry in the near region is the $1/R$ perturbation to the Schwarzschild black membrane (1.1.1) that is found using the **1st order (far)** metric as a boundary condition. As shown in [18], it has locally the general form[3]

$$ds^2 = \left(\eta_{ab} - 2K_{ab}{}^i y_i + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \left(1 - \frac{r_0^n}{r^n}\right)^{-1} dr^2 + r^2 d\Omega_{n+1}^2 + h_{\mu\nu}(y^i) dx^\mu dx^\nu \ , \tag{1.1.3}$$

  where $r = \sqrt{y^i y_i}$ and $K_{ab}{}^i$ is the extrinsic curvature tensor of a surface with the same topology as that of the horizon. The metric is regular in and outside the horizon as long as Eq. (1.1.2) is satisfied.

It is expected that this procedure can be continued order-by-order by measuring the corrected effective stress-energy tensor from (1.1.3) and proceeding as above. The reader might find certain similarities between the procedure implemented here and the procedure carried out in the context of the *fluid/gravity correspondence* [8, 9, 10, 11, 12]. There, the brane fluctuations are of the hydrodynamic type and introduced by perturbing the fields that characterize the fluid flows, in particular, they do not break the spherical symmetry of the brane (1.1.1).

On the other hand, for most practical purposes one does not necessarily need to produce a metric. It is possible to scan for a wide variety of horizon topologies, to study thermodynamic and dynamical stability of the solutions or to evaluate their conserved charges simply by studying the effective EOMs (1.1.2) which also suffer corrections order-by-order. It is the theory defined by these equations that is called the blackfold effective theory and, being a truncation of Einstein equations, describes the dynamics of thin black branes to leading order in ε. As mentioned above, because the effective stress-energy tensor is, to leading order, of the perfect fluid form, Eqs. (1.1.2) present a generalization of the usual relativistic fluid mechanics since the fluid is in these situations confined to a dynamical surface and include the *fluid/gravity correspondence* effective description as a particular case.

Applications and structure of the thesis

The method described above has been applied in different settings and the EOMs analyzed in different situations. We summarize the several directions directions that have arise from this exploration:

- **Analysis of the EOMs:** The effective blackfold EOMs (1.1.2) require a generalization of usual fluid mechanics and hence must be studied in its own right. For stationary fluid configurations it is shown that the system inherits both fluid (intrinsic) properties as well as elastic solid (extrinsic) properties. A rigorous formal development of this with applications beyond the scope of the *blackfold approach* is presented in Sec. 2. The procedure of Sec. 1.1 can also be applied to charged branes. In this context it has

[3] A generalization of this metric to Einstein-Maxwell-Dilaton theories will be presented in [67].

required the development of a theory that describes stationary charged perfect fluids living on a $(p+1)$-dimensional surface with $q=p$ or $q=0$ brane charge [76, 69] as well as the study of anisotropic fluids with $q=1$ brane charge, in the latter case, fluids with conserved string number [69]. If external background fields are present, the theory must be further developed. An example, pertinent to AdS/CFT applications was explored in [68].

- **Constructing metrics/solutions:** The construction of metrics to order ε by direct application of the method of Sec. 1.1 has been applied to black rings in flat, (A)dS and Taub-Nut asymptotics [14, 65, 66]. An application to charged strings in Einstein-Maxwell-Dilaton (EMD) theory is considered in Sec. 4. Construction of solutions by studying the EOMs has lead to a variety of new exotic horizons such as helical rings and strings, odd-spheres with or without different types of charges, black cylinders and the higher-dimensional Kerr-Newman solution [63, 69]. In Sec. 3 we present some examples of these in (A)dS backgrounds and show the existence of ultra-spinning regimes in the Kerr-(A)dS black hole, a study motivated by the recovery of known solutions from the *blackfold approach*.

- **Fine structure corrections and transport/response coefficients:** 'Fine structure' corrections is the terminology introduced in [25] to characterize the corrections that come from pushing the method of Sec. 1.1 to next order when backreaction can be neglected. In these situations the EOMs take the same form as in (1.1.2) but the effective stress tensor is modified. This includes the case of viscous corrections as well as of curvature corrections. Viscous corrections have been considered in the context of *fluid/gravity correspondence* [8, 9] and also in Minkowski space-time [17] and allow for the measurement of transport coefficients such as shear and bulk viscosities. In the case of charged branes, they allow for the measurement of conductivities [20, 21]. Curvature corrections, which require a multipole expansion of the stress-energy tensor, have been considered for black strings and branes in asymptotically flat space as well as for charged black strings [25, 18, 26]. A rigorous analysis of how the EOMs get modified to next to leading order is given in Sec. 4 together with the example of the measurement of two new response coefficients of black branes: the elastic and piezoelectric moduli.

- **Thermal brane probes:** The EOMs given in (1.1.2) can be used to probe space-times at finite temperature and in the context of AdS/CFT can be seen as a generalization of the DBI action at finite temperatures. The novelty of the method, as compared to previous attempts to thermalize certain configurations, is the requirement of the brane probe to be in thermodynamic equilibrium with the background. This method has been used to construct thermal versions of the BIon solution [70, 71, 72, 73] as well of thermal Wilson loops [74] and giant gravitons [68].

The following sections are adapted versions of the papers [19, 75, 26, 25].

Chapter 2

Stationary fluids on dynamical surfaces

In this section we review and reinterpret the effective worldvolume theory describing stationary fluid configurations confined to a $(p+1)$-dimensional dynamical surface, parametrized by the mapping functions X^{μ} embedded in an ambient D-dimensional space-time with metric $g_{\mu\nu}$ [14, 15]. When the surface and the space-time are of equal dimension we obtain the ordinary description of fluid mechanics. We work under the assumption that the fluid is in local thermodynamic equilibrium. This is achieved when the mean free path characterized by the inverse of the local temperature $\mathcal{T}(\sigma^a)$ is much smaller than the radius of curvature of the embedding geometry $R(\sigma^a)$, i.e.,

$$\frac{1}{\mathcal{T}(\sigma^a)} \ll R(\sigma^a) \ , \tag{2.0.1}$$

where $\sigma^a,\ a = 0, ..., p$ are the coordinates that parametrize the worldvolume $\mathcal{W}_{p+1}$ traced out by the surface in space-time and endowed with metric $\gamma_{ab} = g_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu},\ \mu, \nu = 0, ..., D-1$, with Lorentzian signature. We further assume that the surface is infinitely thin. Finite thickness effects will be considered in Sec. 4. Secs. 2.1 and 2.2 review the worldvolume effective theory and the intrinsic fluid dynamics following Refs. [15, 16, 69, 76], while the remaining reinterpret the extrinsic dynamics in terms of relativistic elasticity theory.

2.1 The worldvolume effective theory

Assuming the fluid not to backreact onto the background, the usual equations of fluid mechanics can be derived by imposing conservation of the stress-energy tensor $\hat{T}^{\mu\nu}$ [77]

$$\nabla_{\nu}\hat{T}^{\mu\nu} = \hat{\mathcal{F}}^{\mu} \ , \tag{2.1.1}$$

where we have included the possibility of an external force $\hat{\mathcal{F}}^{\mu}$ as opposed to Eq. (1.1.2). The dynamics of fluids living on dynamical surfaces also follows from (2.1.1) but the stress-energy tensor characterizing these configurations is confined to the surface in the following way:

$$\hat{T}^{\mu\nu}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}T^{\mu\nu}(\sigma^a)\frac{\delta^D(x^\mu - X^\mu(\sigma^a))}{\sqrt{-g(x^\alpha)}} \ . \tag{2.1.2}$$

Here one should interpret $\hat{T}^{\mu\nu}(x^\alpha)$ as the full spacetime stress-tensor while the components $T^{\mu\nu}(\sigma^a)$ with support on $\mathcal{W}_{p+1}$ should be seen as worldvolume densities of stress-energy. We assume the force term $\hat{\mathcal{F}}^\mu(x^\alpha)$ to be of the same form as (2.1.2) with appropriate worldvolume density $\mathcal{F}^\mu(\sigma^a)$. Introducing (2.1.2) into Eq. (2.1.1) results in a worldvolume theory where the densities $T^{\mu\nu}(\sigma^a)$ only have components tangential to the worldvolume

$$\perp^\rho{}_\nu T^{\mu\nu} = 0 \ , \tag{2.1.3}$$

where the orthogonal projector to the worldvolume is defined as $\perp^{\rho\nu} = g^{\rho\nu} - \gamma^{\rho\nu}$ with $\gamma^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu$ being the push forward of the induced metric. The equations of motion derivable from Eq. (2.1.1) can be written in the form

$$\gamma^\rho{}_\nu \nabla_\rho T^{\mu\nu} = \mathcal{F}^\mu \ , \tag{2.1.4}$$

together with the boundary condition

$$T^{\mu\nu}\hat{n}_\nu|_{\partial\mathcal{W}_{p+1}} = 0 \ , \tag{2.1.5}$$

where $\hat{n}_\nu$ is a unit normal vector orthogonal to the worldvolume boundary. Imposing the constraint (2.1.3) on $T^{\mu\nu}$ requires $T^{\mu\nu} = T^{ab}u^\mu_a u^\nu_b$ with $u^\mu_a = \partial_a X^\mu$ which can then be used to split Eq. (2.1.4) into two sets of equations by projecting along tangential and orthogonal directions to the worldvolume:

$$D_b T^{ab} = u^a_\mu \mathcal{F}^\mu \ , \tag{2.1.6}$$

$$T^{ab}K_{ab}{}^\rho = \perp^\rho_\mu \mathcal{F}^\mu \ . \tag{2.1.7}$$

Here D_a is the covariant derivate with respect to the induced metric γ_{ab} of the surface on which the fluid lives. Eqs. (2.1.6)-(2.1.7) were first derived by Carter [77]. The first equation expresses the conservation of worldvolume stress-energy while the second can be interpreted as the balance of forces acting on the fluid in orthogonal directions. We will in the remaining sections take T^{ab} to be of the perfect fluid form and $\mathcal{F}^\mu$ to be vanishing but we note that this worldvolume effective theory is valid for any type of material one would like to describe.

Fluid and elastic interpretation

Before continuing further, we would like to point out in which sense Eqs. (2.1.6)-(2.1.7) encode simultaneously fluid and elastic behavior. Assuming $\mathcal{F}^{\mu} = 0$, Eqs. (2.1.6)-(2.1.7) reduce to

$$D_a T^{ab} = 0 \quad , \quad T^{ab} K_{ab}{}^{\rho} = 0 \ . \tag{2.1.8}$$

When the stress-energy tensor density T^{ab} is assumed to be of the perfect fluid form, as in the case of planar AdS branes or Schwarzschild branes, the first set of equations give rise to the usual energy density continuity and Euler equations of a perfect fluid. When considering, for example, the type of (intrinsic) hydrodynamic perturbations of black hole horizons encountered in the context of the *fluid/gravity correspondence*, this set of equations can be derived as constraint equations directly from Einstein equations [8]. One can think of perturbations of this type as fluctuations in the fields that characterize the material - in this case the fluid - that lives on a space-time surface. Proceeding order-by-order in perturbation theory results in dissipative corrections to the stress-energy tensor. The second set of equations in (2.1.8) is of extrinsic nature and is associated with deformations of the geometry (surface) on which the fluid flows. As in the case of hydrodynamic fluctuations, this set is also directly derivable as constraint equations from Einstein equations [14, 18] when deformations, for example bending, of black brane geometries are considered, as we will show in Sec. (4). One of the main goals of this thesis is to show that this extrinsic set of equations can be viewed as a relativistic generalization of elasticity theory of thin branes.

The reader may be familiar with Eqs. (2.1.8) when dealing with Dirac branes where $T^{ab} = T_{D_p}\gamma^{ab}$ with T_{D_p} being the tension of the p-brane but we note that they have a direct non-relativistic analog when considering deformations of thin membranes. To be precise, suppose that we are given a thin elastic membrane of thickness r_0 and subject it to external forces applied at its circumference causing it to stretch in all directions. Assuming the material to behave elastically, generating internal stresses σ^{ab} that encode its Hookean response to the stretching, the equations of motion that govern its mechanical equilibrium can be obtained by varying the free energy [78]

$$F[X^{\mu}] = \frac{1}{2}\int_{Vol} dV \sigma^{ab} U_{ab} \ , \tag{2.1.9}$$

where X^{μ} is the set of mapping functions that parametrize the position of the membrane in the ambient space while U^{ab} is the strain tensor. The resulting set of equations is exactly (2.1.8) with T^{ab} replaced by σ^{ab} and with the indices a, b only running through the spatial directions. Within this perspective, a response of the Dirac brane type can be seen as the isotropic stretching of a p-brane.

2.2 Intrinsic fluid dynamics

Our case study is that of a perfect fluid with energy density $\epsilon(\sigma^a)$ and pressure $P(\sigma^a)$ with associated local entropy $s(\sigma^a)$ and corresponding local temperature $\mathcal{T}(\sigma^a)$. The stress-energy tensor is given by

$$T^{ab} = (\epsilon + P)u^a u^b + P\gamma^{ab} \ . \tag{2.2.1}$$

Here the fields u^a denote the fluid velocities normalized such that $u^a u_a = -1$. Assuming local thermodynamic equilibrium the first law of thermodynamics must be satisfied

$$d\epsilon = \mathcal{T} ds \ , \tag{2.2.2}$$

where the infinitesimal differentials are taken along worldvolume directions. In addition these quantities are supplemented by the Gibbs-Duhem relations

$$\epsilon + P = \mathcal{T}s \ , \ dP = sd\mathcal{T} \ . \tag{2.2.3}$$

The fluid dynamical equations follow directly from the intrinsic equation (2.1.6), which upon contraction along directions tangential and orthogonal to the fluid flows and using the above local thermodynamic relations can be written as the conservation of the entropy current J_s^a,

$$D_a J_s^a = 0 \ , \ J_s^a = su^a \ , \tag{2.2.4}$$

and the Euler force equations

$$\hat{\perp}^{ab}\mathcal{T}s(\dot{u}_b + \partial_b \text{ln}\mathcal{T}) = 0 \ , \tag{2.2.5}$$

where $\hat{\perp}^{ab} = \gamma^{ab} + u^a u^b$. Eqs. (2.2.4)-(2.2.5) are subject to the boundary condition (2.1.5) now in the form:

$$T^{ab}u_a^{\mu}\hat{n}_b|_{\partial\mathcal{W}_{p+1}} = 0 \ . \tag{2.2.6}$$

Focusing on stationary fluid configurations, these two equations are solved by requiring the fluid velocities u^a to be proportional to a worldvolume Killing vector field $\mathbf{k}$ of the form [79]

$$\mathbf{k} = \xi + \Omega_i \chi_i \ , \tag{2.2.7}$$

where ξ is a worldvolume timelike Killing vector, χ_i are spacelike ones and i labels spatial worldvolume directions. Without loss of generality we choose

$$u^a = \frac{\mathbf{k}^a}{|\mathbf{k}|} \ . \tag{2.2.8}$$

The global fluid temperature T appears as an integration constant from Eq. (2.2.5) and it is related to the local temperature $\mathcal{T}$ via a local redshift factor,

$$T = |\mathbf{k}|\mathcal{T} \ , \tag{2.2.9}$$

while the total entropy, assuming $\mathbf{k}_a$ to be hypersurface orthogonal with respect to the worldvolume metric, can be obtained by integrating the entropy current over the spatial part of the worldvolume $\mathcal{B}_p$,

$$S = -\int_{\mathcal{B}_p} dV_{(p)} s u^a n_a \ . \tag{2.2.10}$$

Here we introduced the spatial measure $dV_{(p)}$ on the worldvolume and defined an orthogonal vector n_a to a worldvolume spacelike hypersurface in the manner

$$n^a = \frac{\xi^a}{R_0} \ , \tag{2.2.11}$$

where R_0 is the norm of the timelike Killing vector field ξ on the worldvolume.

2.3 Extrinsic elastic dynamics

In this section we analyze the extrinsic dynamics of fluid configurations described by Eq. (2.1.7). We begin by defining the state of strain of the brane and the strain tensor and then proceed to describe the equations of mechanical equilibrium.

The state of strain and the strain tensor

The fluid configuration studied in the previous section lives on an infinitely thin surface described by the induced metric $\gamma_{\mu\nu}$. We are interested in examinaning the thermodynamic properties of such a fluid when the geometry is deformed along orthogonal directions. The metric $\gamma_{\mu\nu}$ measures distances between neighboring points on the embedding surface, therefore, working under the assumption that the surface is thin and hence that variations in distances measured with $\perp_{\mu\nu}$ can be ignored[1], the metric $\gamma_{\mu\nu}$ describes the local state of strain of the brane.

Let us define the state of strain of the brane prior to a deformation (unstrained state) by $\bar{\gamma}_{\mu\nu}$. The length of infinitesimal spacetime distances along the surface is thus of the form

$$\bar{ds}^2 = \bar{\gamma}_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.3.1}$$

[1]This is the usual assumption of classical elasticity theory when considering deformations of thin membranes and small strains and stresses [78].

After a deformation, the state of strain is no longer described by $\bar{\gamma}_{\mu\nu}$ but instead by the actual value of $\gamma_{\mu\nu}$, hence the length of the infinitesimal element is changed to

$$ds^2 = \gamma_{\mu\nu}dx^\mu dx^\nu \ . \tag{2.3.2}$$

Assuming the strains and stresses involved to be small, the difference between the length of the line elements of the strained and unstrained case along an arbitrary orthogonal vector N_ρ is given by:

$$\Delta s^2 = ds^2 - \bar{ds}^2 = \pounds_N \gamma_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.3.3}$$

Since we are only interested in changes along worldvolume directions we project this measure along those directions,

$$\Delta s^2|_{\mathcal{W}_{p+1}} = \gamma_\mu^\lambda \gamma_\nu^\rho \pounds_N \gamma_{\lambda\rho} dx^\mu dx^\nu = -2N_\rho K_{\mu\nu}{}^\rho dx^\mu dx^\nu \ , \tag{2.3.4}$$

where we have used a mathematical identity described in [16]. Therefore, along any orthogonal direction N_ρ, the strain varies proportionally to the extrinsic curvature tensor $K_{\mu\nu}{}^\rho$ [2]. Since $K_{\mu\nu}{}^\rho$ satisfies the property $K_{\mu\nu}{}^\rho = u^a_\mu u^b_\nu K_{ab}{}^\rho$, we define the Lagrangian strain tensor [80] for the brane as

$$U_{ab} = -\frac{1}{2}\left(\gamma_{ab} - \bar{\gamma}_{ab}\right) \ , \tag{2.3.5}$$

which for infinitesimal deformations reads

$$dU_{ab} = -\frac{1}{2} d\gamma_{ab} = N_\rho K_{ab}{}^\rho \ . \tag{2.3.6}$$

Equations of extrinsic dynamics

The extrinsic dynamics of the fluid living on the dynamical surface are described by the extrinsic equation (2.1.7) which can be written in the form [16]

$$\mathcal{T} s \perp^\rho{}_\mu \dot{u}^\mu + P K^\rho = 0 \ , \tag{2.3.7}$$

relating the acceleration of the fluid along orthogonal directions to the action of the mean extrinsic curvature $K^\rho \equiv \gamma^{ab} K_{ab}{}^\rho$. Here, $\dot{u}^\mu$ is the fluid acceleration defined as $\dot{u}^\mu \equiv u^\nu \nabla_\nu u^\mu$. We now make the assumption of the existence of a background Killing vector $\mathbf{k}^\mu$ whose pullback onto the worldvolume coincides with the worldvolume Killing vector field $\mathbf{k}^a$. Under this assumption we can write

$$\mathcal{T} s \dot{u}_\mu = -s \partial_\mu \mathcal{T} \ . \tag{2.3.8}$$

[2]Preliminary arguments in this direction were first given in [25].

Introducing this into Eq. (2.3.7) and using the definition of K^ρ yields,

$$\perp^\rho{}_\mu s\partial^\mu \mathcal{T} = P\gamma^{ab}K_{ab}{}^\rho \ . \tag{2.3.9}$$

Contracting the last equation with an arbitrary orthogonal vector N_μ leads to

$$sd\mathcal{T} = -\frac{1}{2}\sigma^{ab}d\gamma_{ab} \ , \tag{2.3.10}$$

where the infinitesimal differentials denote a variation along orthogonal directions and where we have defined the pressure tensor σ^{ab} as:

$$\sigma^{ab} = P\gamma^{ab} \ . \tag{2.3.11}$$

The r.h.s. of Eq. (2.3.10) can be written as $\sigma^{ab}dU_{ab}$ by recognizing the strain tensor defined in (2.3.5). We now define the elastic solid density ρ as

$$\rho = \epsilon + P = \mathcal{T}s \ , \tag{2.3.12}$$

and hence interpret s as an average particle density satisfying the conservation law (2.2.4) and $\mathcal{T}$ as a mass function. In terms of infinitesimal variations along orthogonal directions to the worlvolume we find

$$\begin{aligned} d\rho &= \mathcal{T}ds + sd\mathcal{T} \\ &= \mathcal{T}ds + \sigma^{ab}dU_{ab} = \mathcal{T}ds - Pd\mathcal{V} \ , \end{aligned} \tag{2.3.13}$$

where we have used the mathematical identity obtained in [16] for the relative change in the local volume element

$$d\mathcal{V} \equiv \frac{\delta_N\sqrt{-\gamma}}{\sqrt{-\gamma}} = \frac{1}{2}\gamma^{ab}d\gamma_{ab} \ . \tag{2.3.14}$$

Eq. (2.3.13) is exactly the relation that an elastic solid density ρ should respect under hydrostatic compression [78] and expresses the fact that the system described by this set of equations accounts for only changes in volume but not in shape. This is due to the assumption of an infinitely thin surface since effects due to bending or torsion would require a varying concentration of material across transverse directions. Along orthogonal directions we can thus write the equivalent relations

$$\left(\frac{\partial\rho}{\partial U_{ab}}\right)_s = \sigma^{ab}, \ \left(\frac{\partial\rho}{\partial\mathcal{V}}\right)_s = -P \ . \tag{2.3.15}$$

Here, as well as in (2.3.13), we have assumed that all strain components dU_{ab} are linearly independent. This is not necessarily the case and we will deal with linear dependence towards the end of this section. We note that the first equality in (2.3.15) only constrains the components of the pressure tensor which are contracted with non-vanishing components of the strain tensor in (2.3.10). In order to make further contact with relativistic elasticity theory [80, 81, 82] note that by means of the definition (2.3.12) the above thermodynamic quantities and stresses can be derived from the mass function $\mathcal{T}$, for example

$$s\left(\frac{\partial \mathcal{T}}{\partial \mathcal{V}}\right) = -P \ . \tag{2.3.16}$$

Furthermore, we rewrite the stress-energy tensor (2.2.1) as

$$T^{ab} = \rho u^a u^b + \sigma^{ab} \ , \tag{2.3.17}$$

motivating the interpretation of a solid at rest which has suffered hydrostatic compression along all worldvolume directions. The definition and interpretation of Eq. (2.3.12) together with the relations (2.3.13) and the decomposition (2.3.17) is one of the central results of this section as they express the elastic character of the relativistic material since the components of the pressure tensor σ^{ab} involved in the extrinsic dynamics (2.3.10) can be obtained from a single potential ρ - a required condition in relativistic elasticity theory [83, 80, 84, 82, 85]. It is possible to consider a more general elastic potential which is also better suited when we consider charged branes. This will be the aim of the next section. For now, we focus on the case for which the strain components in (2.3.13) can be linearly dependent. In such situations, given the independent components of the strain tensor $U_{\tilde{a}\tilde{b}}$ we write (2.3.13) as

$$d\rho = \mathcal{T} ds + \tilde{\sigma}^{\tilde{a}\tilde{b}} dU_{\tilde{a}\tilde{b}} \ , \tag{2.3.18}$$

where we have introduced the effective pressure tensor along the component $(\tilde{a}, \tilde{b})$,

$$\tilde{\sigma}^{\tilde{a}\tilde{b}} = \perp^{\tilde{a}\tilde{b}}{}_{ab}\sigma^{ab} \ , \qquad \perp^{\tilde{a}\tilde{b}}{}_{ab} = \frac{\partial \gamma_{ab}}{\partial \gamma_{\tilde{a}\tilde{b}}} \ . \tag{2.3.19}$$

Here the operator $\perp^{\tilde{a}\tilde{b}}{}_{ab}$ acts as a projector onto the linearly independent subspace of the components of strain. Using Eqs. (2.3.18)-(2.3.19) we can write

$$\left(\frac{\partial \rho}{\partial U_{\tilde{a}\tilde{b}}}\right)_s = \tilde{\sigma}^{\tilde{a}\tilde{b}} \ . \tag{2.3.20}$$

We wish to rewrite an effective form for the stress-energy tensor (2.3.17) along the directions $(\tilde{a}, \tilde{b})$. To this aim we note that from Eqs. (2.3.7)-(2.3.8) together with the identity $u^\mu u^\nu K_{\mu\nu}{}^\rho = \perp^\rho{}_\mu \dot{u}$ [16] and the Gibbs-Duhem relation (2.2.3) one finds the expression

$$\rho u^a u^b = 2\left(\frac{\partial P}{\partial \gamma_{ab}}\right) \ , \tag{2.3.21}$$

which is only valid along transverse directions. Acting with the projector $\perp^{\tilde{a}\tilde{b}}{}_{ab}$ on both sides of (2.3.21) and using (2.3.19) we can write the effective stress-energy tensor along transverse directions as

$$\tilde{T}^{\tilde{a}\tilde{b}} = 2\left(\frac{\partial P}{\partial \gamma_{\tilde{a}\tilde{b}}}\right) + \tilde{\sigma}^{\tilde{a}\tilde{b}} \ , \tag{2.3.22}$$

which must satisfy the constraint:

$$\tilde{T}^{\tilde{a}\tilde{b}} = 0 \ . \tag{2.3.23}$$

Imposing (2.3.23) leads directly to the extrinsic equations of motion (2.3.7).

2.4 Elastic free energy, elasticity tensor and charges

Using the identity (2.2.3) it is possible to rewrite Eq. (2.3.10) only in terms of the pressure P as

$$dP = -\frac{1}{2}\sigma^{ab}d\gamma_{ab} \ , \tag{2.4.1}$$

or alternatively, along orthogonal directions and for the independent components of $d\gamma_{\tilde{a}\tilde{b}}$,

$$-2\left(\frac{\partial P}{\partial \gamma_{\tilde{a}\tilde{b}}}\right) = \tilde{\sigma}^{\tilde{a}\tilde{b}} \ . \tag{2.4.2}$$

One can view this equation as a balance of forces between the pressure tensor and the internal stresses generated by a variation in volume[3]. Moreover, at the level of uncharged fluid branes, Eq. (2.4.2) is equivalent to Eq. (2.3.20) but this is not so in general. In fact, the pressure P, for reasons that will become apparent, provides a more general elastic potential which from now on we take to be the canonical one. From Eq. (2.4.1) it is possible to define the bulk modulus or modulus of hydrostatic compression K that measures the material response to variations in volume through the relation

$$\frac{1}{K} = \left(\frac{\partial \mathcal{V}}{\partial P}\right)_T = -\frac{1}{P} \ . \tag{2.4.3}$$

The definition (2.4.3) has a direct classical analog [78]. Eq. (2.4.1) can be integrated to an action [16]

$$I[X^\mu] = \int_{\mathcal{W}_{p+1}} \sqrt{-\gamma}\, P \ , \tag{2.4.4}$$

[3]For the fluid branes arising from a gravitational dual analyzed in [16] the pressure tensor takes the interpretation of gravitational tension acting as a compression force while the first term in Eq. (2.4.2) takes the interpretation of a centripetal force acting outwards the surface when the fluid is rotating. This is clear from the r.h.s. of (2.3.21) since it is proportional to a density ρ and two copies of the four-velocity u^a.

which resembles the usual action for fluids living on a fixed background, the difference being that γ is the volume measure on the worldvolume instead of being on the space-time [86]. On the other hand, the worldvolume being a surface of co-dimension higher than zero, allows for elastic behavior which can be seen from the variation of the integrand along orthogonal directions

$$\begin{aligned} -d(\sqrt{-\gamma}P) &= -\sqrt{-\gamma}\,dP - d\left(\sqrt{-\gamma}\right)P \\ &= \sqrt{-\gamma}\left(-sd\mathcal{T} - Pd\mathcal{V}\right) \ , \end{aligned} \tag{2.4.5}$$

where we have used the Gibbs-Duhem relation (2.2.3). Eq. (2.4.5) has direct analogy with the variational form of the Helmholtz free energy of an elastic solid, exhibiting the same local thermodynamic properties [78]. The action (2.4.4) demands to be interpreted as the solid free energy $F[X^\mu] \equiv -I[X^\mu]$ when the material has suffered hydrostatic compression. To motivate this interpretation even further we assume in the present moment the fluid to be barotropic, characterized by an equation of state $\epsilon = wP$. This together with (2.2.3) and (2.3.12) allows us to rewrite the solid free energy as

$$F[X^\mu] = -\frac{1}{w+1}\int_{\mathcal{W}_{p+1}} \sqrt{-\gamma}\,\rho \tag{2.4.6}$$

which is the usual action for relativistic elastic media [87, 81, 82]. The assumption of barotropy can be relaxed in order to obtain an expression of the form of (2.4.6) as we show in Sec. 2.6 when considering charged fluids. We note that the action (2.4.4) is very general extending also to charged fluid branes [70, 69, 76, 74, 68], while the action (2.4.6) is dependent on the equation of state. For this reason we have chosen the elastic potential P to be the canonical one instead of ρ.

The elasticity tensor

In elasticity theory, given the potential P from which the pressure tensor $\tilde{\sigma}^{\tilde{a}\tilde{b}}$ can be obtained, it is possible to define a deformation tensor $\tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}$ as the variation of the pressure tensor with respect to the state of strain [80, 81, 82]. For the fluid branes considered here this has the general form

$$\tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \equiv \left(\frac{\partial\tilde{\sigma}^{\tilde{a}\tilde{b}}}{\partial U_{\tilde{c}\tilde{d}}}\right) = \left(\frac{\partial^2 P}{\partial U_{\tilde{a}\tilde{b}}\partial U_{\tilde{c}\tilde{d}}}\right) \ , \tag{2.4.7}$$

satisfying the properties $\tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = \tilde{K}^{(\tilde{a}\tilde{b})(\tilde{c}\tilde{d})} = \tilde{K}^{\tilde{c}\tilde{d}\tilde{a}\tilde{b}}$. Note that in the second equality in (2.4.7) we have imposed the constraint (2.4.2). Given a certain stationary fluid brane in a certain prestrained state satisfying (2.4.2), one can apply a deformation taking the configuration to another strained state. The definition (2.4.7) implies that variations of the pressure tensor along orthogonal directions are related to the deformation tensor as

$$d\tilde{\sigma}^{\tilde{a}\tilde{b}} = \tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}dU_{\tilde{c}\tilde{d}} \ . \tag{2.4.8}$$

In order to give a concise explicit expression for (2.4.7) we introduce an effective pressure $\tilde{P}(\gamma^{\tilde{a}\tilde{b}})$ along a direction $(\tilde{a},\tilde{b})$ such that

$$\tilde{\sigma}^{\tilde{a}\tilde{b}} = \tilde{P}(\gamma^{\tilde{a}\tilde{b}})\gamma^{\tilde{a}\tilde{b}} \ . \tag{2.4.9}$$

Using then the definition (2.4.7) we obtain the general expression

$$\tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = -2\left(\left(\frac{\partial \tilde{P}}{\partial \gamma_{\tilde{a}\tilde{b}}}\right)\gamma^{\tilde{c}\tilde{d}} - \tilde{P}\gamma^{\tilde{a}(\tilde{c}}\gamma^{\tilde{d})\tilde{b}}\right) \ . \tag{2.4.10}$$

This has the expected form of the deformation tensor of a material that is responding to stretching or compression signaling the fact that since the surface is taken to be infinitely thin, only variations in volume can be accounted for. The second term in (2.4.10) is the usual term when the pressure $\tilde{P}$ is constant while the first term arises due to pressure variations. For Dirac branes the first term on the r.h.s. of (2.4.10) vanishes expressing isotropic compression at constant pressure.

The relativistic elasticity tensor $\tilde{E}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}$ can be expressed in an analogous way with respect to the stress-energy tensor (2.3.22) [80, 81, 82]. In orthogonal directions to the worldvolume, the elasticity tensor describes variations of $\tilde{T}^{\tilde{a}\tilde{b}}$ such that

$$d\tilde{T}^{\tilde{a}\tilde{b}} = \tilde{E}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} dU_{\tilde{c}\tilde{d}} \ . \tag{2.4.11}$$

According to (2.4.11) the elasticity tensor takes the following generic form

$$\tilde{E}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \equiv \left(\frac{\partial \tilde{T}^{\tilde{a}\tilde{b}}}{\partial U_{\tilde{c}\tilde{d}}}\right) = \left(\tilde{K}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} - \left(\frac{\partial^2 P}{\partial U_{\tilde{a}\tilde{b}} U_{\tilde{c}\tilde{d}}}\right)\right) \ , \tag{2.4.12}$$

where in the second equality we have used (2.4.10). The tensor $\tilde{E}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}}$ satisfies the usual properties of an elasticity tensor $\tilde{E}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = \tilde{E}^{(\tilde{a}\tilde{b})(\tilde{c}\tilde{d})} = \tilde{E}^{\tilde{c}\tilde{d}\tilde{a}\tilde{b}}$. We note that the Eqs. (2.4.8), (2.4.11) express linear Hokean deformations of the pressure and stress-energy tensors. We will evaluate (2.4.12) explicitly for neutral black branes in Sec. 3.

Conserved charges

The action (2.4.4) has a thermodynamic interpretation [16]. To make it precise, we note that from Eqs. (2.1.4) with vanishing external force one can construct a set of conserved surface currents $T^{\mu\nu}\mathbf{k}_\mu$, such that

$$\gamma^{\rho}{}_{\nu}\nabla_{\rho}(T^{\mu\nu}\mathbf{k}_\mu) = 0 \ , \tag{2.4.13}$$

where $\mathbf{k}_\mu$ should be interpreted here as a generic space-time Killing vector field. The total energy M and angular momentum J^i of the fluid brane can then be computed by integrating the surface currents over the spatial part of the worldvolume in the following way:

$$M = \int_{\mathcal{B}_p} dV_{(p)} T^{ab} \xi_a n_b \ , \ \ J^i = -\int_{\mathcal{B}_p} dV_{(p)} T^{ab} \chi^i_a n_b \ . \tag{2.4.14}$$

Here we have assumed $\mathbf{k}_\mu$ to be hypersurface orthogonal with respect to the space-time metric and also $\mathbf{k}_a$ to be hypersurface orthogonal with respect to the worldvolume metric[4]. To proceed further, we introduce the fluid Gibbs free energy density $\mathcal{G}$ as

$$\mathcal{G} = \epsilon - \mathcal{T} s = -P \ , \tag{2.4.15}$$

which has the following thermodynamic properties along orthogonal directions

$$d\mathcal{G} = -s d\mathcal{T} = P d\mathcal{V} \ . \tag{2.4.16}$$

From Eq. (2.4.16) we conclude that deformations of $\mathcal{G}$ along orthogonal directions cause the material to stretch or compress. After integrating the density (2.4.15) over the worldvolume one finds the relation

$$F[X^\mu] = -\int_{\mathcal{W}_{p+1}} \sqrt{-\gamma}\, \mathcal{G} = -\left(M - \sum_i \Omega_i J^i - TS \right) \ . \tag{2.4.17}$$

Extremizing (2.4.17) while keeping the set of potentials T, Ω_i fixed implies the first law of thermodynamics to be satisfied for the fluid branes:

$$dM = \sum_i \Omega_i dJ^i + T dS \ . \tag{2.4.18}$$

This formula can interpreted as a prediction [16], namely, that a stationary fluid configuration living on a particular dynamical surface must globally satisfy the first law of thermodynamics.

2.5 Deformations of fluid branes and elastic waves

The considerations of the previous sections are based on the action (2.4.4) being only a function of the state of strain given a fixed temperature T and angular velocities Ω_i. Therefore, providing the pressure P as a function of the independent components of the state of strain γ_{ab}, i.e., $P(\gamma_{ab})$, deformations of the embedding geometry by explicit variation of the embedding map $X^\mu(\sigma^a)$ can be analyzed through variations of the induced metric γ_{ab}. These take the form [70]

$$\delta\gamma_{ab} = g_{\mu\nu,\lambda} \partial_a X^\mu \partial_b X^\nu \delta X^\lambda + g_{\mu\lambda} (\partial_a X^\mu \partial_b \delta X^\lambda + \partial_b X^\mu \partial_a \delta X^\lambda) \ . \tag{2.5.1}$$

[4]Other cases where $\mathbf{k}_\mu$ or $\mathbf{k}_a$ are not hypersurface orthogonal have been considered in [66, 68].

Hence, searching for the extrema of the action (2.4.4) implies that

$$T^{ab}\delta\gamma_{ab} = 0 \ , \tag{2.5.2}$$

where the effective stress-energy tensor is obtained in the usual way

$$T^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta I}{\delta\gamma_{ab}} \ . \tag{2.5.3}$$

For the present case, using (2.5.3), the stress-energy tensor has the general form

$$T^{ab} = 2\frac{\partial P}{\partial\gamma_{ab}} + \sigma^{ab} \ , \tag{2.5.4}$$

and is generally different from σ^{ab}. Note that (2.5.4) is equal to (2.3.17) when the relation (2.3.21) is used. The stress-tensor (2.5.4) obtained from the action is only valid along orthogonal directions and can be regarded as an off-shell form of (2.3.17) since a priori one can compute (2.5.4) without the knowledge of the extrinsic curvature of the embedding. The equations of motion obtained from Eq. (2.5.2) when projected along tangential and orthogonal directions to the worldvolume give rise to the intrinsic and extrinsic equations (2.1.8) [70] together with the boundary term

$$\left[\sqrt{-\gamma}T^{ab}u^{\mu}_{a}\hat{n}_{b}\delta X_{\mu}\right]|_{\partial\mathcal{W}_{p+1}} \ , \tag{2.5.5}$$

yielding the boundary condition (2.1.5) and hence vanishing independently of the initial and final configurations by construction. It is also possible to define an off-shell form of the elasticity tensor (2.4.12) using (2.5.4), this is given by

$$E^{abcd} = 2\left(P\gamma^{a(c}\gamma^{d)b} - \left(\frac{\partial P}{\partial\gamma_{ab}}\right)\gamma^{cd} - 2\left(\frac{\partial^2 P}{\partial\gamma_{ab}\partial\gamma_{cd}}\right)\right) \ . \tag{2.5.6}$$

Projecting (2.5.6) using (2.3.19) for a particular embedding surface results in (2.4.12).

The speed of elastic waves

The dynamical properties of fluid branes as elastic materials can also be seen by applying a small perturbation to the brane geometry. To this aim we follow [16]. Assuming the material to be initially at rest $u^a = (1, 0, ...)$, we introduce a perturbation in the mapping functions δX^{μ} and initial pressure P such that

$$\delta P \ , \ \delta\rho = \frac{d\rho}{dP}\delta P, \ \delta u^a = (0, v^i), \ \delta X^{\mu} = \xi^{\mu} \ . \tag{2.5.7}$$

Using the form of the stress tensor (2.2.1) together with the equation of extrinsic dynamics (2.1.7) we find

$$\left((\rho - P)\partial_t^2 - K\partial_i^2\right)\xi^{\mu} = 0 \ , \tag{2.5.8}$$

and hence conclude that elastic waves propagate at the speed

$$c_\perp^2 = \frac{K}{\rho - P} \ , \tag{2.5.9}$$

where we used the definitions of the modulus of rigidity (2.4.3) and of the solid density (2.3.12). The above result is what is expected from a relativistic solid which has been subject to hydrostatic compression [88, 81].

2.6 Charged fluid branes

Fluid configurations carrying a q-charge are not only characterized by the stress tensor (2.1.2) but also by a totally anti-symmetric current tensor $\hat{J}^{\mu_1 \cdots \mu_{q+1}}$ which is confined to the worldvolume surface

$$\hat{J}^{\mu_1 \cdots \mu_{q+1}}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\, J^{\mu_1 \cdots \mu_{q+1}}(\sigma^a) \frac{\delta^D(x^\mu - X^\mu(\sigma^a))}{\sqrt{-g(x^\alpha)}} \tag{2.6.1}$$

and must satisfy current conservation

$$\nabla_{\mu_1} \hat{J}^{\mu_1 \cdots \mu_{q+1}} = 0 \ , \tag{2.6.2}$$

in the absence of external couplings. The set of effective worldvolume equations that result from (2.6.2) imply that the current is purely tangential

$$J^{\mu_1 \cdots \mu_{q+1}} = u^{\mu_1}_{a_1} \ldots u^{\mu_{q+1}}_{a_{q+1}} J^{a_1 \ldots a_{q+1}} \ , \tag{2.6.3}$$

and

$$D_{a_1} J^{a_1 \ldots a_{q+1}} = 0 \ , \ \ J^{a_1 \ldots a_{q+1}} n_{a_1}|_{\partial \mathcal{W}_{p+1}} = 0 \ . \tag{2.6.4}$$

We will now apply these equations to the fluid configurations with $q = p$ charge studied in [76] and with $q = 0$ charge studied in [69, 76].

$q = p$ worldvolume charge

The fluids carrying $q = p$ worldvolume charge studied in [76] are characterized by a worldvolume current density $J^{a_1 \ldots a_{p+1}}$ of the form

$$J = \mathcal{Q}_p \hat{V}_{p+1} \ , \tag{2.6.5}$$

where $\hat{V}_{p+1}$ is the $(p+1)$-volume form of the embedding surface and $\mathcal{Q}_p$ is the charge density. The worldvolume conservation equation (2.6.4) then implies

$$\partial_a \mathcal{Q}_{p+1} = 0 \ . \tag{2.6.6}$$

Therefore for $q = p$ charges the charge density $\mathcal{Q}_p$ is not allowed to vary along worldvolume directions, hence the total charge of the configuration Q_p equals the charge density $\mathcal{Q}_p$. Given this, the fluid does not carry any additional extra degrees of freedom associated with the charge and as such the Gibbs-Duhem relations presented in (2.2.3) still hold for the case at hand. In this way, as long as all extrinsic variations are performed while the charge Q_p is kept constant, the results of the previous sections hold. However, it is possible to introduce a chemical potential Φ_p conjugate to $\mathcal{Q}_p$ [76], giving rise to a well defined global quantity

$$\Phi_{\rm H}^{(p)} = -\int_{\mathcal{B}_p} dV_{(p)} \Phi_p u^a n_a \ . \tag{2.6.7}$$

Using Φ_p we can define the Gibbs free energy of the fluid as

$$\mathcal{G} = \epsilon - \mathcal{T}s - \Phi_p \mathcal{Q}_p = -P - \Phi_p \mathcal{Q}_p \ , \tag{2.6.8}$$

where in the second equality we have made use of the relations (2.2.3). Therefore, for hydrodynamical fluctuations where the charge Q_p is kept constant due to (2.6.6) we find

$$d\mathcal{G} = -s d\mathcal{T} - \mathcal{Q}_p d\Phi_p \ , \tag{2.6.9}$$

and hence for orthogonal variations that keep the potential (2.6.7) constant, ie.,

$$d\Phi_p = -\Phi_p d\mathcal{V} \ , \tag{2.6.10}$$

the action can be recast as (2.4.17) noting that along those directions $d\mathcal{G} = -\mathcal{G} d\mathcal{V}$. From here one can define an electroelastic modulus of rigidity describing the deformation of the charge potential as

$$\frac{1}{K_{\rm E}} = \left(\frac{\partial \mathcal{V}}{\partial \Phi_p} \right)_{\mathcal{T}} = -\frac{1}{\Phi_p} \ . \tag{2.6.11}$$

As far as the authors are aware, Eq. (2.6.11) does not have a classical analog. We suspect that this modulus of electroelastic rigidity is associated with fluctuations of the charge density in transverse directions in the same way as the isothermal permittivity is associated with worldvolume fluctuations [76]. We further note that it is possible to define an electroelasticity tensor associated with Φ_p as in (2.4.12) but its meaning is unclear. A better understanding of this is presently lacking.

Finally, extremizing (2.4.17) at fixed T, Ω_i, $\Phi_{\rm H}^{(p)}$ implies the first law of thermodynamics [76]

$$dM = \sum_i \Omega_i dJ^i + T dS + \Phi_{\rm H}^{(p)} dQ_p \ , \tag{2.6.12}$$

to be satisfied.

$q = 0$ worldvolume charge

Stationary fluids carrying a $q = 0$ brane charge were analyzed in [69, 76] and have quite different thermodynamic properties than the $q = p$ case. These are instead characterized by the worldvolume particle current[5]

$$J^a = \mathcal{Q}u^a \ , \tag{2.6.13}$$

and hence must satisfy (2.6.4):

$$D_a\left(\mathcal{Q}u^a\right) = 0 \ . \tag{2.6.14}$$

The crucial difference with the $q = p$ case is that now the charge density $\mathcal{Q}$ is allowed to vary along the worldvolume and hence adds extra degrees of freedom to the system. Local thermodynamic equilibrium implies

$$d\epsilon = \mathcal{T}ds + \Phi d\mathcal{Q} \tag{2.6.15}$$

while the thermodynamic Gibbs-Duhem relations (2.2.3) are now changed to

$$\epsilon + P = \mathcal{T}s + \Phi\mathcal{Q} \quad , \quad dP = sd\mathcal{T} + \Phi d\mathcal{Q} \ . \tag{2.6.16}$$

The intrinsic equations of motion (2.1.6) again lead to conservation of the entropy current as in (2.2.4) while the Euler equations (2.2.5) are modified to [69]

$$P^{ab}\mathcal{T}s(\dot{u}_b + \partial_b \ln\mathcal{T}) - \mathcal{Q}\Phi\left(\hat{K}^a - P^{ab}\partial_b\ln\Phi\right) = 0 \ , \tag{2.6.17}$$

where we have introduced the mean curvature of the worldlines embedded in $\mathcal{W}_{p+1}$,

$$\hat{K}^a = u^b u^c D_b\left(u_c u^a\right) \ . \tag{2.6.18}$$

Assuming the solution to be stationary, we take the fluid velocities to be aligned with a worldvolume Killing vector field as in (2.2.8). This ensures that the first term in Eq. (2.6.17) vanishes leading to the same relation between local and global fluid temperatures (2.2.9). The second term in Eq. (2.6.17) can be dealt with by imposing the mean curvature due to the dissolved 0-charge on the worldvolume to balance the gradient of the chemical potential [69], yielding

$$\hat{K}^a = P^{ab}\partial_b\ln\Phi \ . \tag{2.6.19}$$

On the other hand, the extrinsic equation (2.3.8) now becomes

$$\mathcal{T}s{\perp^\rho}_\nu\dot{u}^\mu + \mathcal{Q}{\perp^\rho}_\nu\partial^\nu\Phi + PK^\rho = 0 \ , \tag{2.6.20}$$

[5] From hereon we omit the index in $\mathcal{Q}_p$ and Φ_p.

which after using (2.2.8) and contracting with an arbitrary orthogonal vector N_ρ leads to

$$P d\mathcal{V} + s d\mathcal{T} + \mathcal{Q} d\Phi = 0 \ . \tag{2.6.21}$$

This can be integrated to the action (2.4.4). In order to make further contact with electroelasticity we define the solid density

$$\rho = \mathcal{T} s + \Phi \mathcal{Q} \ , \tag{2.6.22}$$

which under deformations along the extrinsic directions satisfies

$$d\rho = \mathcal{T} ds - P d\mathcal{V} + \Phi d\mathcal{Q} \ . \tag{2.6.23}$$

This is the thermodynamic relation that a solid charged under a particle current should respect. From here, as before, we can obtain useful relations along orthogonal directions:

$$\left(\frac{\partial \rho}{\partial s}\right)_{\mathcal{V},\mathcal{Q}} = \mathcal{T} \ , \ \left(\frac{\partial \rho}{\partial \mathcal{V}}\right)_{s,\mathcal{Q}} = -P \ , \ \left(\frac{\partial \rho}{\partial \mathcal{Q}}\right)_{s,\mathcal{V}} = \Phi \ . \tag{2.6.24}$$

Unlike the neutral case, not everything can be derived from a mass function. In fact, if we use the definition (2.6.22) into (2.6.24) we find the identities

$$\begin{aligned} \left(\frac{\partial \Phi}{\partial \mathcal{T}}\right)_{\mathcal{V},\mathcal{Q}} = \left(\frac{\partial \Phi}{\partial \mathcal{T}}\right)_{s,\mathcal{V}} &= -\frac{s}{\mathcal{Q}} \ , \\ s\left(\frac{\partial \mathcal{T}}{\partial \mathcal{V}}\right)_{s,\mathcal{Q}} + \mathcal{Q}\left(\frac{\partial \Phi}{\partial \mathcal{V}}\right)_{s,\mathcal{Q}} &= -P \ . \end{aligned} \tag{2.6.25}$$

The Gibbs free energy introduced in (2.6.8) for charged fluids is now, due to (2.6.16), equal to the pressure as in (2.4.15). Hence, if one wishes to specify an equation of state of the form

$$\epsilon = wP + \Phi \mathcal{Q} \ , \tag{2.6.26}$$

one finds the action (2.4.6):

$$F[X^\mu] = -\frac{1}{w+1}\int_{\mathcal{W}_{p+1}} \sqrt{-\gamma}\,(\rho - \Phi\mathcal{Q}) \ . \tag{2.6.27}$$

When written in terms of the solid density ρ, the extra term on the r.h.s. of the action (2.6.27) acts as an external force. Variation of (2.6.27) keeping $T, \Omega_i, \Phi_{\rm H}$ constant leads to the first law of thermodynamics (2.6.12), but now with global charge [69]:

$$Q = -\int_{\mathcal{B}_p} dV_{(p)} \mathcal{Q} u^a n_a \ . \tag{2.6.28}$$

Chapter 3

Constructing black hole solutions

In this section we apply the considerations of the previous sections, as an example, to the case of neutral black p-branes wrapped on a generic submanifold (blackfolds) studied in [14, 16, 63, 65, 75]. Such branes are characterized by a stress-energy tensor of the type presented in (2.1.2) and hence encompassed within the framework put forth above. We begin by describing the effective blackfold fluid characterizing these branes and then write down the elasticity tensor as well as an action of the thermodynamic type for blackfold objects We finish by obtaining the elastic equilibrium condition of black rings and black odd-spheres in asymptotically flat space by direct use of the extrinsic equation (2.4.2) and then proceed to construct and analyze in more detail based on the action principle several black holes in (A)dS and their ultra-spinning limits.

3.1 The effective blackfold fluid

The blackfold effective fluid characterizing neutral black branes is of the perfect fluid type (2.2.1) and characterized by the equation of state [16]

$$\epsilon = -(n+1)P \ . \tag{3.1.1}$$

The local thermodynamic quantities associated with the fluid living on the brane (ϵ, P, $\mathcal{T}$, s) can all be described in terms of the brane thickness r_0 in the following way:

$$\frac{\epsilon}{n+1} = -P = \frac{\Omega_{(n+1)} r_0^n}{16\pi G} \ \ , \ \ \mathcal{T} = \frac{n}{4\pi r_0} \ \ , \ \ s = \frac{\Omega_{(n+1)} r_0^{n+1}}{4G} \ . \tag{3.1.2}$$

Here n is the number of transverse directions given by $n = D - p - 3$. By inspection of the above expressions it is easily observed that the Gibbs-Duhem relations (2.2.3) are satisfied. The thickness r_0, as well as the fluid velocities u^a are allowed to be functions of σ^a and hence to vary along worldvolume directions. From (3.1.2) one concludes that the requirement of local thermodynamic equilibrium (2.0.1) implies the hierarchy of scales

$$r_0(\sigma^a) \ll R(\sigma^a) \ , \tag{3.1.3}$$

or in other words, that the black branes being wrapped on the submanifold $\mathcal{W}_{p+1}$ are thin compared to the curvature radius of the submanifold.

3.2 Elasticity tensor of blackfolds and thermodynamic action

Here we write down the elasticity tensor for neutral blackfolds using Eq. (2.5.6) and then an action of the thermodynamic type. We begin by noting that from Eq. (2.2.9) and using the quantities (3.1.2) we obtain a relation between $|\mathbf{k}|$ and r_0

$$r_0 = \left(\frac{n}{4\pi T}\right)|\mathbf{k}| = \lambda|\mathbf{k}| \ , \tag{3.2.1}$$

where we have defined $\lambda = n/4\pi T$. Using this, the pressure P can be expressed in terms of the induced metric γ_{ab} as

$$P(\gamma_{ab}) = -\frac{\Omega_{(n+1)}}{16\pi G}\lambda^n| - \gamma_{ab}\mathbf{k}^a\mathbf{k}^b|^{\frac{n}{2}} \ . \tag{3.2.2}$$

Hence, the derivative of the pressure with respect to the induced metric is simply given by

$$\left(\frac{\partial P}{\partial \gamma_{ab}}\right) = -\frac{n}{2}Pu^a u^b \ . \tag{3.2.3}$$

Using the expression for the effective stress-energy tensor derived in (2.5.4) we arrive at

$$T^{ab} = P\left(-nu^a u^b + \gamma^{ab}\right) \ , \tag{3.2.4}$$

agreeing with (2.2.1) when the equation of state (3.1.1) is introduced. In order to continue further, it is necessary to evaluate the second derivative of P with respect to the state of strain. This yields:

$$\left(\frac{\partial^2 P}{\partial \gamma_{ab}\partial \gamma_{cd}}\right) = \frac{n(n-2)}{4}Pu^a u^b u^c u^d \ . \tag{3.2.5}$$

Therefore, using Eq. (2.5.6) we obtain the off-shell elasticity tensor for neutral blackfolds:

$$E^{abcd} = P\left(2\gamma^{a(c}\gamma^{d)b} + nu^a u^b \gamma^{cd} - n(n-2)u^a u^b u^c u^d\right) \ . \tag{3.2.6}$$

This is not manifestly invariant under $(a, b) \to (c, d)$ as discussed around Eq. (2.4.12) but it becomes so after the equations of motion (2.4.2) are imposed. We note that (3.2.6) has the same structure as the Young modulus tensor measured for black branes in [25, 18, 26][1]. Moreover, it satisfies the properties

$$E^{abcd}\gamma_{cd} = P\left(2\gamma^{ab} + n(n+p-1)u^a u^b\right) \quad , \quad E^{abcd}\gamma_{ab}\gamma_{cd} = P\left(2(p+1) - n(n+p-1)\right) \ . \tag{3.2.7}$$

Thermodynamic action

Here we rewrite the action (2.4.4), adapted to blackfold objects, in terms of the total entropy (2.2.10). Using the Gibbs-Duhem relations (2.2.3) and also (3.1.1) we obtain the relation

$$P = -\frac{1}{n}\mathcal{T}s \ , \tag{3.2.8}$$

which when introduced into the action (2.4.4) leads to

$$I[X^\mu] = -\frac{T}{n}\int_{\mathcal{B}_p} dV_{(p)} s u^a n_a = \frac{TS}{n} \ . \tag{3.2.9}$$

Therefore we see that extremizing the blackfold action at constant temperature and angular velocities is to extremize the black hole entropy.

3.3 Physical properties and Smarr relations

Given the specific fluid properties of Sec. 3.1 it is convenient to rewrite the thermodynamic quantities presented in Sec. 2.4 in a way more adapted to the construction of black hole solutions. To the that aim we note that the Killing vector fields ξ and χ_i introduced in Sec. (2.2.7) and given the assumptions of Sec. (2.3) are now generators of asymptotic time translations and of asymptotic rotations respectively of the background space-time. We then define the redshift factor R_0 between infinity and the blackfold worldvolume and the proper radii R_i of the orbits generated by χ_i along the worldvolume as the norm of this set of commuting Killing vectors on the worldvolume:

$$R_0 = \sqrt{-\xi^2}|_{\mathcal{W}_{p+1}} \ , \ R_i = \sqrt{\chi_i^2}|_{\mathcal{W}_{p+1}} \ . \tag{3.3.1}$$

From here, it follows that $\mathbf{k}$ given in Eq. (2.2.7) can be expressed in a more convenient way as

$$\mathbf{k} = R_0\sqrt{1-V^2} \ , \tag{3.3.2}$$

[1]There is a difference between the structure of (3.2.6) and the Young modulus measured in [25, 18, 26], namely that here there is no ambiguity in the choice of worldvolume surface since we take the surface to be infinitely thin.

where the velocity field V is defined as

$$V^2 = \frac{1}{R_0^2}\sum_i \Omega_i^2 R_i^2 \ . \tag{3.3.3}$$

The horizon thickness r_0 (3.2.1) can be related to the velocity field in the following way:

$$r_0(\sigma^a) = \frac{nR_0(\sigma^a)}{2\kappa}\sqrt{1-V^2(\sigma^a)} \ , \tag{3.3.4}$$

where we have introduced the surface gravity κ of the black hole space-time we ought to construct. With this we can rewrite the action (3.2.9) as

$$I[X^\mu] = \lambda \int_{\mathcal{B}_p} dV_{(p)} R_0 |\mathbf{k}|^n \ . \tag{3.3.5}$$

Here λ is the constant introduced in Eq. (3.2.1) and since it does not play a role in the variation of (3.3.5) we henceforth omit it in the remaining parts of this section.

The physical properties of the resulting blackfold solutions can then be easily computed. The total mass M and angular momenta J_i read

$$M = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n-2}{2}}(n+1-V^2) \ , \tag{3.3.6}$$

$$J_i = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n n\Omega_i \int_{\mathcal{B}_p} dV_{(p)} R_0^{n-1}(1-V^2)^{\frac{n-2}{2}} R_i^2 \ , \tag{3.3.7}$$

while the entropy is given by

$$S = \frac{\Omega_{(n+1)}}{4G}\left(\frac{n}{2\kappa}\right)^{n+1} \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n}{2}} \ . \tag{3.3.8}$$

Furthermore, the total integrated tension [63] takes the form

$$\mathcal{T} = \frac{\Omega_{(n+1)}}{16\pi G}\left(\frac{n}{2\kappa}\right)^n \int_{\mathcal{B}_p} dV_{(p)} R_0^{n+1}(1-V^2)^{\frac{n-2}{2}}(p-(n+p)V^2) \ . \tag{3.3.9}$$

Using the explicit expressions (3.3.6)-(3.3.9) and also that $T = \frac{\kappa}{2\pi}$ one finds that these physical quantities satisfy the Smarr relation[2]

$$(D-3)M = (D-2)\left(\sum_i \Omega_i J_i + TS\right) + \mathcal{T} \ . \tag{3.3.10}$$

For asymptotically flat black hole solutions of the vacuum Einstein equations the tension $\mathcal{T}$ [89] must vanish (see [63]), but this is generally not true in an (A)dS$_D$ background. In fact, for the thin black rings constructed in [65] this is not the case, nor, as we will see, for any of the other A(dS) blackfold solutions found in this thesis.

[2]This relation was first derived in [89] for flat black branes of vacuum gravity in D dimensions.

3.4 Black holes in flat space

In this section we apply the results of the previous sections to re-derive the equilibrium condition of thin black rings and black odd 3-spheres [63] in asymptotically flat space parametrized by coordinates $(t, r, \theta, \phi, \psi, x^i)$. We write the induced metric on the worldvolume as

$$\gamma_{ab} d\sigma^a d\sigma^b = -d\tau^2 + R^2 \left(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\psi^2 \right) \ , \tag{3.4.1}$$

where R is the radius of the sphere (or the ring). The map onto the ambient space-time is simply

$$t = \tau, \ r = R, \ x^i = 0 \ , \tag{3.4.2}$$

while the remaining coordinates coincide. The non-vanishing components of the extrinsic curvature are

$$K_{\theta\theta}{}^r = -R \quad K_{\phi\phi}{}^r = -R\cos^2\theta, \quad K_{\psi\psi}{}^r = -R\sin^2\theta \ , \tag{3.4.3}$$

while the mean extrinsic curvature vector reads

$$K^r = -\frac{p}{R} \ , \ p = 1, 3 \ . \tag{3.4.4}$$

Black rings

Black rings are described by setting $p = 1$ and $\theta = 0$ in the formulae above. This leads to the induced metric and extrinsic curvature

$$\gamma_{ab} d\sigma^a d\sigma^b = -d\tau^2 + R^2 d\phi^2 \ , \ K_{\phi\phi}{}^r = -R \ , \ K^r = -\frac{1}{R} \ . \tag{3.4.5}$$

The worldvolume Killing vector field takes the form

$$\mathbf{k} = \partial_\tau + \Omega \partial_\phi \ , \tag{3.4.6}$$

which we use to write the pressure P as

$$P = -\frac{\Omega_{(n+1)}}{16\pi G} \lambda^n | - \gamma_{\tau\tau} - \gamma_{\phi\phi}\Omega^2|^{\frac{n}{2}} \ . \tag{3.4.7}$$

Since the only independent and non-zero component of the strain tensor is $d\gamma_{\phi\phi}$ we can use Eq.(2.4.2) to obtain the equilibrium

$$-2 \left(\frac{\partial P}{\partial \gamma_{\phi\phi}} \right) - \sigma^{\phi\phi} = 0, \tag{3.4.8}$$

which in turn implies, using the result (3.2.3),

$$\Omega^2 R^2 = \frac{1}{n+1} \ . \tag{3.4.9}$$

This equilibrium condition has been derived previously in [63] also using the action (3.2.9), here we have merely performed a different derivation. From Eq. (3.4.8) we see that there is only one non-vanishing component of strain, hence the only on-shell component of the elasticity tensor is given by (3.2.6):

$$E^{\phi\phi\phi\phi} = 2\left(\frac{n+1}{n}\right) P\gamma^{\phi\phi}\gamma^{\phi\phi}. \tag{3.4.10}$$

Black odd-spheres

Black odd 3-spheres are described by setting $p = 3$. We assume them to be rotating with equal angular velocity Ω along both directions (ϕ, ψ). This allows us to write the worldvolume Killing vector field in the form

$$\mathbf{k} = \partial_\tau + \Omega\left(\partial_\phi + \partial_\psi\right) \ , \tag{3.4.11}$$

and hence the pressure as

$$P = -\frac{\Omega_{(n+1)}}{16\pi G}\lambda^n| - \gamma_{\tau\tau} - \gamma_{\theta\theta}\Omega^2|^{\frac{n}{2}} \ . \tag{3.4.12}$$

There are now three non-vanishing components of the strain, $d\gamma_{\theta\theta}$, $d\gamma_{\phi\phi}$ and $d\gamma_{\psi\psi}$. The last two can be expressed in terms of $d\gamma_{\theta\theta}$ in the following way:

$$d\gamma_{\phi\phi} = \cos^2\theta d\gamma_{\theta\theta} \ , \ d\gamma_{\psi\psi} = \sin^2\theta d\gamma_{\theta\theta} \ . \tag{3.4.13}$$

Using now Eq. (2.4.2) for the only independent component $d\gamma_{\theta\theta}$ we obtain the equilibrium equation

$$-2\left(\frac{\partial P}{\partial \gamma_{\theta\theta}}\right) - \tilde{\sigma}^{\theta\theta} = 0 \ , \tag{3.4.14}$$

where the effective stress along $d\gamma_{\theta\theta}$ is given by

$$\tilde{\sigma}^{\theta\theta} = P\left(\gamma^{\theta\theta} + \cos^2\theta\gamma^{\phi\phi} + \sin^2\theta\gamma^{\psi\psi}\right) = p\, P\gamma^{\theta\theta} \ . \tag{3.4.15}$$

Solving Eq.(3.4.14) results in the equilibrium condition

$$\Omega^2 R^2 = \frac{p}{n+p} \ , \tag{3.4.16}$$

which has been derived previously in [63]. Even though we only considered the cases $p = 1, 3$, the result (3.4.16) is valid for all p [63]. Eq. (2.4.2) could have been solved using any of the other two non-vanishing components of strain $U_{\phi\phi}$,$U_{\psi\psi}$, in fact, it is easy to see that the effective pressure tensor has in this case the generic form $\tilde{\sigma}^{ab} = \tilde{P}\gamma^{ab}$ where $\tilde{P} = p\,P$. Using expression (2.4.12) for the on-shell value of the elasticity tensor we find the non-vanishing components:

$$\tilde{E}^{\theta\theta\theta\theta} = 2\left(\frac{n+p}{n}\right)\tilde{P}\gamma^{\theta\theta}\gamma^{\theta\theta} \ , \ \tilde{E}^{\phi\phi\phi\phi} = 2\left(\frac{n+p}{n}\right)\tilde{P}\gamma^{\phi\phi}\gamma^{\phi\phi} \ , \ \tilde{E}^{\psi\psi\psi\psi} = 2\left(\frac{n+p}{n}\right)\tilde{P}\gamma^{\psi\psi}\gamma^{\psi\psi},$$
$$\tilde{E}^{\theta\theta\phi\phi} = 2\frac{p}{n}\tilde{P}\gamma^{\theta\theta}\gamma^{\phi\phi} \ , \ \tilde{E}^{\theta\theta\psi\psi} = 2\frac{p}{n}\tilde{P}\gamma^{\theta\theta}\gamma^{\psi\psi} \ , \ \tilde{E}^{\phi\phi\psi\psi} = 2\frac{p}{n}\tilde{P}\gamma^{\phi\phi}\gamma^{\psi\psi} \ . \tag{3.4.17}$$

3.5 Black holes in (Anti)-de Sitter space

We now consider black holes in (A)dS space-time constructed from wrapping neutral black branes. Since (A)dS naturally introduces its own length scale set by the cosmological constant Λ, we consider blackfolds in the regime

$$r_0(\sigma^a) \ll \min(R, |\Lambda|^{-\frac{1}{2}}) \ , \tag{3.5.1}$$

so that neither the curvature of the worldvolume nor the curvature set by the (A)dS radius are felt locally at the blackfold. This implies, in particular, that locally the blackfold is still described by the asymptotically flat neutral black brane solution of (1.1.1).

It will be useful to make use of the AdS metric written in terms of two different coordinate systems. We first write the metric for global AdS$_D$ space-time in the form

$$ds^2 = -\mathcal{V}(r)dt^2 + \frac{dr^2}{\mathcal{V}(r)} + r^2 d\Omega_{D-2}^2 \ , \quad 0 \leq r \leq \infty \ , \quad \mathcal{V}(r) = 1 + \frac{r^2}{L^2} \ . \tag{3.5.2}$$

It will also be convenient to work with a metric that highlights the existent $U(1)$ symmetries of the background space-time. This new metric can be obtained by introducing a new radial coordinate ρ defined as

$$r = \frac{\rho}{1 - \frac{\rho^2}{4L^2}} \ , \tag{3.5.3}$$

thus bringing the AdS$_D$ metric (3.5.2) into homogenous (spatially conformally flat) coordinates

$$ds^2 = -F(\rho)dt^2 + H(\rho)^{-1}(d\rho^2 + \rho^2 d\Omega_{D-2}^2) \ , \quad 0 \leq \rho \leq 2L, \tag{3.5.4}$$

$$F(\rho) = \left(\frac{1 + \frac{\rho^2}{4L^2}}{1 - \frac{\rho^2}{4L^2}}\right)^2 \ , \quad H(\rho) = \left(1 - \frac{\rho^2}{4L^2}\right)^2 \ . \tag{3.5.5}$$

The AdS radius L is related to the cosmological constant Λ by

$$\Lambda = \frac{(D-2)(D-1)}{L^2} \ , \tag{3.5.6}$$

and thus the range of validity (3.5.1) of the results in this section can be recast as $r_0 \ll \min(R, L)$. The dS_D metric in both coordinate systems can be obtained by performing a Wick rotation such that $L \to iL$ in the metrics (3.5.2) and (3.5.4).

3.5.1 Blackfolds with odd-sphere horizon topology

In Ref. [63] the *blackfold approach* was used to construct a class of novel black holes in D-dimensional flat spacetime with horizon topology

$$(\Pi_{p_a=\mathrm{odd}} S^{p_a}) \times s^{n+1} \ , \quad \sum_{a=1}^{l} p_a = p \ . \tag{3.5.7}$$

This class contains not only the family of thin black rings with horizon topology $S^1 \times s^{n+1}$ but also single (and the product of) odd-spheres with S^{2k+1} horizon geometry. In this section we generalize these results to (A)dS_D spacetime, and furthermore study the thermodynamic stability of these new solutions.

Black S^{2k+1}-folds in AdS_D

The first step for constructing a stationary blackfold solution is to embed the spatial world-volume $\mathcal{B}_p$, $p = 2k+1$ into the background space. In this case we want to wrap the spatial world-volume on a S^{2k+1} sphere embedded into a $(2k+2)$-dimensional spatially conformally flat subspace of AdS_D spacetime (3.5.4). The appropriate part of the background metric can be conveniently expressed as

$$ds^2_{2k+2} = H(\rho)^{-1} \left(d\rho^2 + \rho^2 \sum_{i=1}^{k+1} (d\mu_i^2 + \mu_i^2 d\phi_i^2) \right) \ , \quad \sum_{\mu=i}^{k+1} \mu_i^2 = 1 \ , \tag{3.5.8}$$

so that the S^{2k+1} is parameterized by $k+1$ Cartan angles ϕ_i and k independent director cosines μ_i. It is then natural to choose a gauge in which the worldvolume $\mathcal{B}_{2k+1}$ is specified by the embedding scalar $\rho = \bar{R}(\{\mu_i\})$ and the spatial worldvolume coordinates

$$\{\mu_i, \ i = 1, ..., k\}, \ \{\phi_i, \ i = 1, ..., k+1\} \ . \tag{3.5.9}$$

In order to construct the action for these blackfolds one needs the induced metric on the worldvolume. In terms of the Cartan angles and director cosines this metric takes the form

$$
\begin{aligned}
ds^2_{2k+1} = H(\bar{R}(\mu_i))^{-1} \sum_{i,j=1}^{k} \left(\left(\delta_{ij} + \frac{\mu_i \mu_j}{\mu^2_{k+1}} \right) \bar{R}(\mu_i)^2 + \partial_i \bar{R}(\mu_i) \partial_j \bar{R}(\mu_j) \right) d\mu_i d\mu_j \\
+ H(\bar{R}(\mu_i))^{-1} \bar{R}(\mu_i)^2 \sum_{i=1}^{k+1} \mu_i^2 d\phi_i^2 \ .
\end{aligned}
\tag{3.5.10}
$$

Since, in order to have a stationary blackfold, the corresponding Killing vector must generate isometries of the worldvolume, the horizon Killing vector takes the form

$$
\mathbf{k} = \frac{\partial}{\partial t} + \sum_{i=1}^{k+1} \Omega_i \frac{\partial}{\partial \phi_i} \ . \tag{3.5.11}
$$

The redshift factor R_0 and the proper radii R_i of the orbits generated by $\frac{\partial}{\partial \phi_i}$ are given respectively by $R_0 = \sqrt{F(\bar{R}(\mu_i))}$ and $R_i = H(\bar{R}(\mu_i))^{-\frac{1}{2}} \bar{R}(\mu_i)$, while the velocity field (3.3.3) becomes

$$
V(\mu_i)^2 = \frac{\bar{R}(\mu_i)^2}{(1 + \frac{\bar{R}(\mu_i)^2}{4L^2})^2} \sum_{i=1}^{k+1} \mu_i^2 \Omega_i^2 \ . \tag{3.5.12}
$$

We recall that the functions F and H entering the background metric are defined in (3.5.5).

For simplicity we restrict to round odd-spheres, so that we take the scalar $\bar{R}$ to be constant. Furthermore, we are interested in the maximally symmetric case for which the S^{2k+1} sphere is rotating with equal angular velocity Ω in all $k+1$ directions ϕ_i It follows that the action (3.3.5) reduces to an $\bar{R}$-dependent potential of the form

$$
I[\bar{R}] = \Omega_{(p)} \sqrt{F(\bar{R})}\, H(\bar{R})^{-\frac{p}{2}} \bar{R}^p (F(\bar{R}) - \Omega^2 H(\bar{R})^{-1} \bar{R}^2)^{\frac{n}{2}} \ , \tag{3.5.13}
$$

where $p = 2k+1$ and $\Omega_{(p)}$ is the area of the S^{2k+1} sphere. A nicer form of the action can be obtained by performing the inverse transformation between the coordinate systems (3.5.2) and (3.5.4). Thus defining $R = \bar{R}/(1 - \frac{\bar{R}^2}{4L^2})$, the action (3.5.13) above becomes[3]

$$
I[R] = \Omega_{(p)} R_0 R^p (R_0^2 - \Omega^2 R^2)^{\frac{n}{2}} \ , \tag{3.5.14}
$$

where now $R_0 = \sqrt{\mathcal{V}(R)}$, with $\mathcal{V}$ defined in (3.5.2). Varying this action with respect to R we obtain the equilibrium condition for Ω

$$
\Omega^2 = \frac{1 + \mathbf{R}^2}{R^2} \frac{p + \mathbf{R}^2 (p + n + 1)}{(n+p) + \mathbf{R}^2 (n + p + 1)} \ , \tag{3.5.15}
$$

[3]In fact, in this highly symmetrical case we could have simply used the form of the metric (3.5.2) and obtained this action straight away.

where we have defined the dimensionless parameter $\mathbf{R} = \frac{R}{L}$. It is straightforward to check that the limit $L \to \infty$ gives the result obtained in [63] for S^{2k+1}-folds constructed in a Minkowski background and that the special case of a black ring in AdS$_D$ ($p = 1$) agrees with the one obtained in [65]. It should also be noted that the inverse of the relation (3.5.15) above in terms of R is single valued for a fixed value of Ω and is valid for all values of L.

Physical properties: The physical properties for the odd-sphere AdS blackfolds are straightforwardly obtained using equations (3.3.6)-(3.3.9). Setting $V_{(p)} = R^p \Omega_{(p)}$ for the volume of the S^{2k+1} sphere we find

$$M = \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} r_0^n (1 + \mathbf{R}^2)^{\frac{3}{2}} (1 + n + p) \ , \tag{3.5.16}$$

$$S = \frac{\Omega_{(n+1)} V_{(p)}}{4G} r_0^{n+1} \sqrt{\frac{\mathbf{R}^2 + (n+p)(1+\mathbf{R}^2)}{n}} \ , \ T = \frac{n}{4\pi r_0} \sqrt{\frac{n\left(1+\mathbf{R}^2\right)}{\left(1+\mathbf{R}^2\right)(n+p) + \mathbf{R}^2}} \ , \tag{3.5.17}$$

$$J_i = \frac{2}{p+1} \frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} r_0^n R \sqrt{\left(p + \mathbf{R}(n+p+1)\right)\left(\left(1+\mathbf{R}^2\right)(n+p) + \mathbf{R}^2\right)} \ , \tag{3.5.18}$$

$$\Omega_i = \Omega \ , \quad i = 1, ..., k+1 \ . \tag{3.5.19}$$

Moreover, the total tension $\mathcal{T}$ becomes

$$\mathcal{T} = -\frac{\Omega_{(n+1)} V_{(p)}}{16\pi G} r_0^n (1 + \mathbf{R}^2)^{\frac{1}{2}} \mathbf{R}^2 (n + p + 1) \ , \tag{3.5.20}$$

showing explicitly that in AdS spacetime this quantity is not necessarily zero. The physical quantities above can be shown to satisfy the Smarr relation (4.1.35).

The general product of odd-spheres in AdS$_D$

The class discussed above is part of a larger one in which in which the spatial worldvolume $\mathcal{B}_p$ is a product of l round odd-spheres embedded as in Sec. 3.5.1 above. We label the different spheres by an index $a = 1, ..., l$ and denote $\bar{R}_a$ as the corresponding (constant) radius of each S^{p_a}, where p_a is an odd integer. For the sake of simplicity we choose for each sphere the angular velocity associated with each Cartan angle direction to be equal

$$\Omega_i^{(a)} = \Omega^{(a)} \ , \quad \forall i = 1, ..., \frac{p_a + 1}{2} \ . \tag{3.5.21}$$

To embed $\mathcal{B}_p$ we consider a conformally flat $(p+l)$-dimensional subspace of AdS_D with the metric

$$H(\rho)^{-1}\sum_{a=1}^{l}\left(d\rho_a^2+\rho_a^2 d\Omega_{p_a}^2\right)\ ,\quad \rho^2=\sum_{a=1}^{l}\rho_a^2\ ,\quad \sum_{a=1}^{l}p_a=p\ . \tag{3.5.22}$$

Again we choose the Cartan angles and director cosines of each S^{p_a} sphere as the spatial worldvolume coordinates and take $\rho_a=\bar{R}_a$ as the embedding scalars. The transverse space is $(n+2-l)$-dimensional, hence we require that $l\leq n+2$.

Defining $\bar{R}^2=\sum_{a=1}^{l}\bar{R}_a^2$, the action (3.5.13) can be generalized to an $\bar{R}_a$-dependent potential

$$I[\{\bar{R}\}]=\Pi_{b=1}^{l}\Omega_{(p_b)}\sqrt{F(\bar{R})}\,\bar{R}_b^{p_b}H(\bar{R})^{-\frac{p_b}{2}}\left(F(\bar{R})-H(\bar{R})^{-1}\sum_{a=1}^{l}(\Omega^{(a)}\bar{R}_a)^2\right)^{\frac{n}{2}}\ . \tag{3.5.23}$$

Introducing new scalars R_a as $R_a=\bar{R}_a/(1-\frac{\bar{R}_a^2}{4L^2})$ the previous action can be put in a simpler form

$$I[\{R\}]=\Pi_{b=1}^{l}\Omega_{(p_b)}R_0R_b^{p_b}\left(R_0^2-\sum_{a=1}^{l}(\Omega^{(a)}R_a)^2\right)^{\frac{n}{2}}\ , \tag{3.5.24}$$

where $R_0=1-\frac{R^2}{L^2}$, with $R^2=\sum_{a=1}^{l}R_a^2$. Varying this action with respect to each of the scalars R_a gives rise to l coupled equations for each of the angular velocities $\Omega^{(a)}$. The equilibrium condition can then be found to be

$$\Omega^{(a)}=\frac{1+\mathbf{R}^2}{R_a}\sqrt{\frac{p_a+\mathbf{R}_a^2(1+n+p)}{(n+p)+\mathbf{R}^2(1+n+p)}}\ , \tag{3.5.25}$$

where we have defined $\mathbf{R}_a=R_a/L$. It easy to check that in the limit $L\to\infty$ the above condition agrees with that of [63] while the particular case $l=1$ agrees with (3.5.15).

The physical properties for these blackfolds are also easily computed. In fact, the expressions for M, S, T, $\mathcal{T}$ coincide with those in (3.5.16), (3.5.17), (3.5.20) for a single odd-sphere if we define the volume of $\mathcal{B}_p$ as $V_{(p)}=\Pi_a V_{(p_a)}$ while the angular momenta and angular velocities read

$$J_i^{(a)}=\frac{2}{p_a+1}\frac{\Omega_{(n+1)}V_{(p)}}{16\pi G}r_0^n n\Omega_i^{(a)}\frac{R_a^2}{(1+\mathbf{R}^2)^{\frac{1}{2}}(1-V^2)}\ ,\quad \Omega_i^{(a)}=\Omega^{(a)}\ , \tag{3.5.26}$$

where the velocity field is given by $V=\sum_{a=1}^{l}R_a\Omega^{(a)}/(1+\mathbf{R}^2)^{\frac{1}{2}}$. The Smarr relation (4.1.35) can also be verified for this case.

Black S^{2k+1}-folds in dS_D

The equilibrium condition for odd-sphere blackfolds in a de Sitter background dS_D can be easily obtained from those in (3.5.15) by performing the Wick rotation $L \to iL$, leading to

$$\Omega^2 = \frac{1-\mathbf{R}^2}{R^2}\frac{\mathbf{R}^2(n+p+1)-p}{\mathbf{R}^2(n+p+1)-(n+p)} \ . \tag{3.5.27}$$

Since Ω^2 might become negative for certain values of the parameters we must impose the condition $\Omega^2 \geq 0$, which implies that the ratio $\mathbf{R}$ should be constrained to the region

$$\sqrt{\frac{n+p}{n+p+1}} \leq \mathbf{R} \leq 1 \ \vee \ \sqrt{\frac{p}{n+p+1}} < \mathbf{R} \ . \tag{3.5.28}$$

Hence black S^{2k+1}-folds in dS_D do not exist for all values of $\mathbf{R}$. Moreover, a static solution always exists if[4]

$$\mathbf{R}^2 = \frac{p}{n+p+1} \ . \tag{3.5.29}$$

The physical properties of these solutions can be obtained from those given in Sec. 3.5.1 by taking into account the same Wick rotation.

3.5.2 Ultra-spinning and "ultra-spinning" Kerr-(A)dS$_D$ black holes as blackfolds

Blackfold solutions in flat space with $\mathcal{B}_p$ an even-dimensional ellipsoidal ball have been shown to exist in Refs. [15, 63]. These have event horizon with S^{D-2} topology due to the fact that the transverse S^{n+1} is non-trivially fibered over the ellipsoid, becoming zero size at the boundary. In fact, the physical properties of these even-ball blackfolds have been shown to exactly reproduce those of ultra-spinning MP black holes [64].

Despite the fact that ultra-spinning regimes have not been found for spinning black holes in AdS$_D$, it has been pointed out in [65] that the Kerr-AdS$_D$ black hole with an appropriate choice of mass and rotation parameters m, a has an "ultra-spinning" regime which shares many of the same properties with the ultra-spinning regime of the MP black hole. As an example, the transverse and parallel size of the horizon of the single spinning Kerr-AdS$_D$ black hole in $D \geq 6$ behave in the limit $a \to L$ as

$$l_\perp \sim r_+ \ , \quad l_\parallel \sim \left(\frac{r_+^2+a^2}{\Xi}\right)^{\frac{1}{2}} \ , \quad \Xi \equiv 1-\frac{a^2}{L^2} \ , \tag{3.5.30}$$

where r_+ is the event horizon radius. For fixed mass the ratio $\frac{l_\parallel}{l_\perp}$ diverges like $\sim \Xi^{-\frac{D-1}{2(D-5)}}$, meaning that the horizon pancakes out along the plane of rotation. Thus this limit could in

[4]The case for which $\mathbf{R}=1$ leads to a blackfold with vanishing horizon and vanishing physical properties.

principle be captured by an even-dimensional ellipsoidal ball $\mathcal{B}_p$, in particular a disk in the case of one plane of rotation. Moreover, the Kerr-AdS$_D$ solution has a BPS bound $J \leq LM$ [90] restricting the rotation parameter in such a way that $a \leq L$. Thus it is clear that at fixed mass M and fixed L one cannot simply take $a \to \infty$ and obtain an ultra-spinning limit as in the asymptotically flat case since the bound would be violated. However, as we will show below, it is possible to take a limit in which $a \to \infty$ and simultaneously taking $L \to \infty$ while keeping the ratio $\frac{a}{L}$ constant. This limit amounts to considering a very thin black hole compared to the scale L, set by the cosmological constant, while keeping a of the same order of magnitude as L, *i.e.*, making the black hole simultaneously thin compared to the parallel section of the horizon. Thus the resulting limit is not asymptotically flat. Furthermore, we will show that this limit can also be captured by the same blackfold $\mathcal{B}_p$.

In this section we will start by solving the action for a worldvolume with even-dimensional ball geometry in an AdS$_D$ background and compute the physical properties of these solutions. Subsequently we will identify the properties of this solution with both the ultra-spinning and "ultra-spinning" regimes of the Kerr-(A)dS$_D$ black hole. At the end of this section we will generalize these results to a dS$_D$ background.

Even-ball blackfolds in AdS$_D$

The starting point for constructing these blackfolds is to consider a planar $2k$-fold embedded into a $(2k+1)$-dimensional spatially conformally flat subspace of AdS$_D$, which can be equipped with the metric

$$ds^2_{2k+1} = H(\rho)^{-1}\left(dz^2 + \sum_{i=1}^{k}(d\rho_i^2 + \rho_i^2 d\phi_i^2) + \sum_{j=1}^{D-2(k+1)} dx_j^2\right) \ , \quad \rho^2 = \sum_{i=1}^{k}\rho_i^2 + \sum_{j=1}^{D-2(k+1)} x_j^2 \ . \tag{3.5.31}$$

It is then natural to choose the embedding of $\mathcal{B}_{2k}$ as

$$z = Z(\rho_i),\ x_j = 0,\ j = 1, ..., D-2(k+1),\ \{\rho_i = \sigma_i,\ \phi_i = \sigma_{i+1},\ i = 1, ..., k\} \ , \tag{3.5.32}$$

with the function $Z(\rho_i)$ to be determined. The Killing vector field that generates the isometries of the worldvolume is of the form

$$\mathbf{k} = \frac{\partial}{\partial t} + \sum_{i}^{k}\Omega_i\frac{\partial}{\partial\phi_i} \ . \tag{3.5.33}$$

Thus the action (3.3.5) takes the simple form

$$I[Z(\rho_i)] = \Omega_{(p)}\int\prod_{i=1}^{k} d\rho_i R_0\prod_{j=1}^{k} R_i\sqrt{1+\partial_{\rho_i}Z(\rho_i)}\left(R_0^2 - \sum_{i=1}^{k}R_i^2\Omega_i^2\right)^{\frac{n}{2}} \ , \tag{3.5.34}$$

where $R_0 = \sqrt{F(\rho)}$ and $R_i = \sqrt{H(\rho)^{-1}}\rho_i$. Varying this action with respect to $Z(\rho_i)$ and analyzing the resulting equation leads to the conclusion that $Z = 0$ is a blackfold solution. In the asymptotically flat case [63], any plane $Z =$ const. is a valid solution, while in the present case due to the AdS$_D$ potential only $Z = 0$ is a solution.

In what follows we will focus on the case of singly-spinning blackfolds ($k = 1$) with angular momentum along the ϕ_1 direction and deal with the general case in App. B. In this case the worldvolume velocity field is given by

$$V(\rho) = \frac{\rho}{1 + \frac{\rho^2}{4L^2}}\Omega \ , \tag{3.5.35}$$

where $R = R_1$, $\Omega = \Omega_1$ and $\rho = \rho_1$. Since V cannot exceed the speed of light $V = 1$, we find that ρ is bounded by the maximum value

$$\rho_{\rm max} = 2L(L\Omega - \sqrt{L^2\Omega^2 - 1}) \ . \tag{3.5.36}$$

The other value for ρ at which $V = 1$, which has a plus sign in front of the square root, can be discarded since in this coordinate system (see (3.5.4)) spatial infinity is reached at $\rho = 2L$.

Since the argument of the square root in (3.5.36) must be positive definite we obtain a constraint

$$0 \leq \alpha \leq 1 \ , \quad \alpha \equiv (L^2\Omega^2)^{-1} \ . \tag{3.5.37}$$

In terms of the parameter α defined above we can now distinguish three different situations:
(i) $\alpha = 0$. In this case $\rho_{\rm max} \to \frac{1}{\Omega}$. This is the asymptotically flat space case and so we correctly recover the ultra-spinning MP black hole where $\rho_{\rm max} \sim a$, so that the blackfold has the shape of a disc with radius a (see [15, 63]).
(ii) $\alpha = 1$. In this case $\rho_{max} \to 2L$, so the disc extends all the way to spatial infinity. As we show below, this corresponds to the "ultra-spinning" limit taken in [65] where $a \to L$.
(iii) $0 < \alpha < 1$. In this case $\rho_{\rm max} < 2L$, so that the disc is cut at some value of ρ and does not reach spatial infinity. As we will see below, this corresponds to a new ultra-spinning limit of the Kerr-AdS$_D$ black hole.

Before taking these limits it is useful to compute the physical properties of this blackfold. These are easily obtained using the equations (3.3.6)-(3.3.9), yielding

$$M = \frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{r}_+^n}{(1-\alpha)^2}\left(1 + \frac{(n+1)(1-\alpha)}{2}\right)\frac{1}{\Omega^2} \ , \tag{3.5.38}$$

$$J = \frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{r}_+^n}{(1-\alpha)^2}\frac{1}{\Omega^3} \ , \quad S = \frac{\Omega_{(D-2)}}{4G}\frac{\hat{r}_+^{n+1}}{(1-\alpha)}\frac{1}{\Omega^2} \ , \tag{3.5.39}$$

$$\mathcal{T} = -\alpha\frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{r}_+^n}{(1-\alpha)^2}\frac{1}{\Omega^2} \ , \tag{3.5.40}$$

where we have defined $\hat{r}_+ = \frac{n}{2\kappa}$. It is worthwhile to notice that the tension $\mathcal{T}$ vanishes only if $\alpha = 0$, in agreement with the flat space result. Also, when α lies within the region $0 < \alpha \leq 1$ the tension is non-zero and hence the blackfold does not describe an asymptotically flat solution. Moreover, it is straightforward to check that the quantities above satisfy the Smarr relation (4.1.35).

We also note that by defining $r = \rho/(1 - \frac{\rho^2}{4L^2})$ the thickness r_0 becomes

$$r_0 = \frac{n}{2\kappa}\sqrt{1 - r^2\Omega^2(1-\alpha)} \ , \tag{3.5.41}$$

so that in terms of this coordinate we now have $r_{\rm max} = \rho_{\rm max}/(1 - \frac{\rho^2_{\rm max}}{4L^2})$. The thickness remains finite for all values of α since when $\alpha = 1$ and $r_{max} \to \infty$, $R_0 \to \infty$ but $r_0 \to 0$. Thus the blackfold is always in the regime $r_0 \ll L$.

We will now proceed to identify the physical properties of the disc blackfold given above with those corresponding to the two different limits of the Kerr-AdS$_D$ black hole.

$\alpha = 1$: the "ultra-spinning" limit: This limit was found in Ref. [65] and amounts to taking $a \to L$, and hence $\Omega \to \frac{1}{L}, \alpha \to 1$ while keeping $\hat{\mu} = \frac{2m}{L^2(1-\alpha)^2}$ finite, i.e., sending $m \to 0$. The resulting metric near the rotation axis can be expressed in appropriately rescaled coordinates as

$$ds^2 = \Xi^{\frac{4}{D-5}}\left(-\left(1 - \frac{\hat{\mu}}{\hat{r}^{D-5}}\right)d\hat{t}^2 + \left(1 - \frac{\hat{\mu}}{\hat{r}^{D-5}}\right)^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega^2_{D-4} + d\sigma^2 + \sigma^2 d\phi^2\right) \ , \tag{3.5.42}$$

where Ξ is given in (3.5.30). This metric describes the geometry of a flat black membrane with an overall conformal factor. Its physical properties can be summarized as follows

$$M = \frac{\Omega_{(D-2)}}{8\pi G}\hat{\mu}L^2 \ , \tag{3.5.43}$$

$$S = \frac{\Omega_{(D-2)}}{4G}r_+^{D-4}\frac{r_+^2 + L^2}{(1-\alpha)} \ , \quad T = \frac{D-5}{4\pi r_+} \ , \tag{3.5.44}$$

$$J = \frac{\Omega_{(D-2)}}{8\pi G}\hat{\mu}L^3 \ , \quad \Omega = \frac{1}{L} \ , \tag{3.5.45}$$

$$r_+ = \left(\frac{2m}{L^2}\right)^{\frac{1}{(D-5)}} \ . \tag{3.5.46}$$

It is easy to check that with the identification $\hat{r}_+ = r_+$ and using $\Omega = L^{-1}$ the blackfold physical properties (3.5.38)-(3.5.40) found above exactly reproduce the properties (3.5.43)-(3.5.45) of this "ultra-spinning" limit (note that $n = D - 5$). To see this one also needs to use the fact that our blackfold is a valid solution only in the regime $L \gg r_+$, *i.e.*, the entropy becomes[5]

$$S = \frac{\Omega_{(D-2)}}{4G}\frac{r_+^{D-4}}{(1-\alpha)}L^2 \ . \tag{3.5.47}$$

Moreover a straightforward computation shows that the thickness for this solution behaves like $r_0(\theta) = r_+ \cos\theta$. However, since in this limit $m \to 0$, it follows from (3.5.46) that this implies $r_+ \to 0$ and thus $r_0 \to 0$. This is actually a prediction from the blackfold side since by taking Eq. (3.5.41) we see that in the case $\alpha = 1$, $r_0 = r_+$ except when $r \to r_{\text{max}}$ in which case $r_0 = 0$, therefore such solution would only be regular if $r_+ = 0, \forall r$.

$0 \leq \alpha < 1$: the ultra-spinning limit: This limit resembles very closely that of the ultra-spinning MP black hole. To see this, start with the metric of the singly-spinning Kerr-AdS$_D$ black hole [56] in spheroidal coordinates $(t, r, \theta, \phi, \Omega_{D-4})$ (see App. C for the multi-spin case). The ultra-spinning limit of the Kerr-AdS$_D$ black holes is defined as

$$a \to \infty \ , \quad L \to \infty \ , \quad m \to \infty \ , \tag{3.5.48}$$

keeping $\alpha = a^2/L^2$ and $\hat{\mu} = 2m/a^2$ fixed. Consider in this limit the metric near the axis of rotation by defining a new coordinate $\sigma = a\sin\theta$ which remains finite as the axis is approached, i.e. as $\theta \to 0$. Then the metric takes the form of that of a flat black membrane

$$ds^2 = -\left(1 - \frac{\hat{\mu}}{r^{D-5}}\right)dt^2 + \left(1 - \frac{\hat{\mu}}{r^{D-5}}\right)^{-1}dr^2 + r^2 d\Omega_{D-4}^2 + \frac{1}{1-\alpha}\left(d\sigma^2 + \sigma^2 d\phi^2\right) \ . \tag{3.5.49}$$

The difference between this metric and the flat space case ($\alpha = 0$) resides in the last term, which is multiplied by the factor $(1-\alpha)^{-1}$. In fact, since we are free to rescale the coordinate σ by a factor of $\sqrt{(1-\alpha)^{-1}}$, we can eliminate the factor in front of the line-element of the two-plane (σ, ϕ). However we are only allowed to do this if $0 \leq \alpha < 1$ since if $\alpha = 1$ the metric diverges and if $\alpha > 1$ the metric changes signature. In summary, the limit above is only valid if α lies within the range $0 \leq \alpha < 1$ as claimed in the discussion below (3.5.37).

The physical properties of the Kerr-AdS$_D$ solution in the limit (3.5.48) can be easily obtained from [91]

$$M = \frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{\mu}}{(1-\alpha)^2}\left(1 + \frac{(D-4)(1-\alpha)}{2}\right)a^2 \ , \tag{3.5.50}$$

$$S = \frac{\Omega_{(D-2)}}{4G}r_+^{D-4}\frac{a^2}{(1-\alpha)} \ , \quad T = \frac{D-5}{4\pi r_+} \ , \tag{3.5.51}$$

[5] In fact, as $a \to L$, the horizon size r_+ approaches zero and hence $S \to 0$. The tension $\mathcal{T}$ remains finite in this limit.

$$J = \frac{\Omega_{(D-2)}}{8\pi G}\frac{\hat{\mu}}{(1-\alpha)^2}a^3 \ , \quad \Omega = \frac{1}{a} \ , \tag{3.5.52}$$

$$r_+ = \left(\frac{2m}{a^2}\right)^{\frac{1}{(D-5)}} \ . \tag{3.5.53}$$

It is then seen that with the identification $\hat{r}_+ = r_+$ and using $\Omega = a^{-1}$, we can reproduce from the *blackfold approach* (Eqs. (3.5.38)-(3.5.40)) the thermodynamic quantities given in (3.5.50)-(3.5.52). Furthermore, the thickness of this black membrane is given by $r_0(\theta) = r_+ \cos\theta$. By looking at Eq. (3.5.41) and defining a new coordinate $\theta = \arcsin\left(r\Omega\sqrt{1-\alpha}\right)$ the two expressions for the thickness exactly match.

We note that the resulting metric does not represent an asymptotically flat solution. This is clear from the fact that there is a non-vanishing tension, as seen in (3.5.40). Another way is by looking at the Quantum Statistical Relation that these black holes must satisfy, this relation reads [91]

$$M - TS - \sum_i \Omega_i J_i = T I_D \ , \tag{3.5.54}$$

where the Euclidean action I_D in this limit reduces to

$$I_D = \frac{1}{4T}\frac{\Omega_{(D-2)}}{(1-\alpha)}m \ . \tag{3.5.55}$$

We can see that there is a factor of $(1-\alpha)^{-1}$ in the expression above, and one may check that only for $\alpha = 0$ does one recover the Euclidean action for the asymptotically flat case.[6] This limit thus represents an asymptotically AdS_D solution.

The existence of the ultra-spinning limit of Kerr-AdS_D black holes described above provides non-trivial information on the stability properties of these black holes. In the asymptotically flat case Ref. [64] showed that ultra-spinning MP black holes become membrane-like suggesting that these should exhibit a GL-type instability [92][7], as confirmed in [94]. Similarly, our analysis thus predicts that Kerr-(A)dS_D black holes for $D \geq 6$ suffer from an ultra-spinning GL-type instability when $\Omega L > 1$. This is in agreement with the recent numerical analysis of Ref. [95] for the singly-spinning case in AdS. More generally, it follows From App. C that in the multi-spin case there is an ultra-spinning GL instability when $\Omega_i L > 1$.

Even-ball blackfolds in dS_D

In this section we want to generalize the results of Sec. 3.5.2 to a dS_D background. By performing a Wick rotation $L \to iL$ the action (3.5.34) takes the same form but now with

[6] To compare this result with the one obtained in [15] note that the parameter μ in [15] is related to m by $\mu = 2m$.

[7] See [93] for a review on the GL instability.

different functions $F(\rho), H(\rho)$ which transform accordingly. $Z = 0$ is still a valid blackfold solution and the velocity field attains the velocity of light at a maximum value of

$$\rho_{\max} = 2L(-L\Omega + \sqrt{L^2\Omega^2 + 1}) \ . \tag{3.5.56}$$

There is thus no upper bound on the parameter $\alpha = (L^2\Omega^2)^{-1}$ and hence α is free to take any value in the interval

$$\alpha \geq 0 \ . \tag{3.5.57}$$

In terms of α we can now distinguish two different regimes:
(i) $\alpha = 0$. This is the flat space case as noted previously in Sec. 3.5.2.
(ii) $\alpha > 0$. In this case $\rho_{max} \leq 2L$, and so the disc is cut at some value of ρ in general but reaching the cosmological horizon when $\alpha \to \infty$ and hence $\Omega = 0$ for which case the solution is static[8]. As we will see below, this case ($\alpha > 0$) corresponds to the ultra-spinning limit of the Kerr-dS$_D$ black hole.

We have not mentioned here any "ultra-spinning" regime. This is because the Kerr-dS$_D$ does not show such special behavior when $a \to L$. To see this it suffices to look at Eq. (3.5.30) and keep in mind that the ratio $\frac{l_\parallel}{l_\perp} \sim \Xi^{-\frac{D-1}{2(D-5)}}$ remains finite since now $\Xi = 1 + \alpha$, so that the event horizon does not pancake out along the plane of rotation.

The ultra-spinning limit of the Kerr-dS$_D$ black hole can be obtained by performing the same Wick rotation on the metric (3.5.49)

$$ds^2 = -\left(1 - \frac{\hat{\mu}}{r^{D-5}}\right)dt^2 + \left(1 - \frac{\hat{\mu}}{r^{D-5}}\right)^{-1}dr^2 + r^2 d\Omega^2_{D-4} + \frac{1}{1+\alpha}\left(d\sigma^2 + \sigma^2 d\phi^2\right) \ . \tag{3.5.58}$$

It is then obvious that this metric is valid for all values of $\alpha \geq 0$. Moreover, it is also a straightforward exercise to show that the physical properties of this solution matches those of the even-ball blackfold. Finally, as in the AdS case, it follows that Kerr-dS$_D$ black holes for $D \geq 6$ have an ultra-spinning GL instability.

3.5.3 Rings and helices

In [63] blackfold solutions were found in $D \geq 5$ with exotic horizons and a single axial $U(1)$ isometry. These helical black rings and helical black strings constitute the first examples of asymptotically flat black holes that saturate[9] the rigidity theorem [97]. In this section we address the question whether helical rings can be attained as well in (A)dS$_D$ spacetime. We will show that solutions describing helical rings with these symmetries, which are valid in the regime $r_0 \ll L$, can also be constructed in these backgrounds. On the other hand the question wether or not helical strings can be constructed in these backgrounds remains open as it would require a different starting point from that of an asymptotically flat black brane (see (1.1.1)).

[8]We are grateful to Roberto Emparan for pointing this out.
[9]Ref. [96] found evidence for another example, in the context of time-independent perturbations at the onset of instabilities of higher-dimensional black holes.

Helical black rings in (A)dS$_D$

In order to construct the action for these blackfolds it is convenient to write the metric of a $2N$-dimensional spatially conformally flat subspace of (A)dS$_D$ spacetime in such a way that all its $U(1)^N$ symmetries are explicit

$$ds^2_{D-1} = H(\rho)^{-1}\left(\sum_{i=1}^{N}(\rho_i^2 + \rho_i^2 d\phi_i^2) + \sum_{j=1}^{D-(2N+1)} dx_j^2\right) \ , \quad \rho^2 = \sum_{i=1}^{N}\rho_i^2 + \sum_{j=1}^{D-(2N+1)} x_j^2 \ , \quad (3.5.59)$$

where we have the constraint $n \geq (2N-1)$. To embed the black 1-fold worldvolume $\mathcal{B}_1$ we set $x_j = 0, \forall j$ and choose the set of scalars $\rho_i = \bar{R}_i$ and the spatial worldvolume coordinate σ such that

$$\{\phi_i = n_i\sigma \ , \quad 0 \leq \sigma \leq 2\pi, \ i = 1, ..., N\} \ , \quad (3.5.60)$$

where we assume without loss of generality that $n_i \geq 0$. The numbers n_i must be integers in order for the ring to close up on itself and the smallest of them (n_{min}) must be coprime with all the remaining ones to avoid multiple covering of the ring. The Killing vector field must be such that all the $U(1)^N$ symmetries generate the isometry of the worldvolume, giving

$$\mathbf{k} = \frac{\partial}{\partial t} + \sum_{i=1}^{N}\Omega_i\frac{\partial}{\partial\phi_i} \ . \quad (3.5.61)$$

The ratios between the angular velocities must be rational such that

$$\left|\frac{\Omega_i}{\Omega_j}\right| = \frac{n_i}{n_j}, \ \forall i, j \ , \quad (3.5.62)$$

and hence we can simply set $|\Omega_i| = \Omega n_i$. The action then takes the simple form

$$I[\{R\}] = 2\pi R_0 R\left(R_0^2 - R^2\Omega^2\right)^{\frac{n}{2}} \ , \quad (3.5.63)$$

with $R_0^2 = F(\rho)$, $R^2 = H(\rho)^{-1}\sum_{i=1}^{N} n_i^2\bar{R}_i^2$ and $\rho^2 = \sum_{i=1}^{N}\bar{R}_i^2$. The general form of the action (3.5.63) was given in Ref. [15]. The action depends on the single scalar R with R_0 a function of R, which should be taken into account when varying as well as the fact that the variation should be orthogonal to the helix. As a result one single equation is found.

A more convenient action to work with can be obtained by making the following redefinition

$$R_i = \frac{\bar{R}_i}{1 \pm \frac{\sum_{i=1}^{N}\bar{R}_i^2}{4L^2}} \ . \quad (3.5.64)$$

The action (3.5.63) then takes the same form but now with $R_0^2 = \mathcal{V}(r)$, $R^2 = \sum_{i=1}^{N} n_i^2 R_i^2$ and $r^2 = \sum_{i=1}^{N} R_i^2$. Varying this with respect to R leads to [63]

$$\Omega^2 = \frac{R_0^2}{R^2}\frac{1 + (n+1)\frac{d\ln R_0}{d\ln R}}{n + 1 + \frac{d\ln R_0}{d\ln R}} \ . \quad (3.5.65)$$

In the case at hand for AdS$_D$ the solution becomes

$$\Omega^2 = \frac{1+\mathbf{R}^2}{R^2}\frac{(1+\mathbf{R}^2)(n+2)-(n+1)}{(1+\mathbf{R}^2)(n+2)-1} \ , \tag{3.5.66}$$

where we have defined $\mathbf{R}^2 = L^{-2}\sum_{i=1}^{N} R_i^2$. This agrees with the result for planar black rings $n_i = 1,\ \forall i$ of (3.5.15) with $p = 1$. The equilibrium condition for helical rings and planar rings is exactly the same but with a more complicated expression for R in the former case. The only difference resides in the fact that in the planar case specifying R immediately specifies R_0 for these backgrounds while for the helical case one needs to specify R and R_0 independently since the latter is a function of $\sum_{i=1}^{N} R_i^2$. Note that it follows from (3.5.66) that static helical black rings can exist in dS$_D$ provided

$$\mathbf{R}^2 = \frac{1}{n+2} \ , \tag{3.5.67}$$

which is the same condition as for static planar rings in dS$_D$ and hence independent of the integers n_i. Accordingly, (3.5.67) can also be obtained from (3.5.29) in the special case of $p = 1$.

We now proceed by describing the physical quantities of the helical AdS black rings

$$M = \frac{\Omega_{(n+1)}}{8G}(n+2)r_0^n(1+\mathbf{R}^2)^{\frac{3}{2}}\sqrt{\sum_{i=1}^{N} n_i^2 R_i^2} \ , \tag{3.5.68}$$

$$J_i = \pm\frac{\Omega_{(n+1)}}{8G}r_0^n((1+\mathbf{R}^2)(n+2)-1)\sqrt{1-\frac{n}{(1+\mathbf{R}^2)(n+2)-1}}\,n_i R_i^2 \ , \tag{3.5.69}$$

$$S = \frac{\pi\Omega_{(n+1)}}{2G}r_0^{n+1}\sqrt{\frac{(1+\mathbf{R}^2)(n+2)-1}{n}}\sqrt{\sum_{i=1}^{N} n_i^2 R_i^2} \ . \tag{3.5.70}$$

These quantities agree with the ones computed in [63] for helical rings in flat space (when taking $L \to \infty$) and with the ones computed in [65] for planar rings in (A)dS (when taking $n_i = 1,\ \forall i$). For completeness we also give the tension for these helical rings

$$\mathcal{T} = -\frac{\Omega_{(n+1)}}{8G}(n+2)r_0^n(1+\mathbf{R}^2)^{\frac{1}{2}}\mathbf{R}^2\sqrt{\sum_{i=1}^{N} n_i^2 R_i^2} \ . \tag{3.5.71}$$

As expected, the tension vanishes only in the asymptotically flat case when $\mathbf{R} \to 0$. It can be shown that these physical properties satisfy the Smarr relation (4.1.35).

Helical rings in different backgrounds

We would now like to give a few comments on helical rings in different background geometries. In fact it seems likely that helical rings can exist in any spherically symmetric background of the form (3.5.2) since these can always be put into coordinates for which the potential $\mathcal{V}(r)$ is constant along the ring. As a matter of a fact, Eq. (3.5.65) first derived in [63] holds for any 1-fold assuming only the existence of a background timelike and spacelike Killing vector, hence a valid solution should exist for such backgrounds. This leads us to the following conjecture

Conjecture 3.5.1 *Neutral helical black ring solutions exist in any background with spherical symmetry in the regime* $r_0 \ll |\Lambda|^{-\frac{1}{2}}$.

As an example of a different spherically symmetric background we take the Schwarzschild-Tangherlini solution in D dimensions (Sch_D) as the background and try to construct a helical black Saturn.[10] The Sch_D metric can be written as in (3.5.2) but with

$$\mathcal{V}(r) = 1 - \left(\frac{\mu}{r}\right)^{D-3} , \quad \mu \leq \ r \leq \infty \ . \tag{3.5.72}$$

By performing the transformation $r = (1 + \frac{\mu^{D-3}}{4\rho^{D-3}})^{\frac{2}{D-3}}\rho$ one can bring the Schwarzschild metric to the form (3.5.4) with

$$F(\rho) = \frac{(1 - \frac{\mu^{D-3}}{4\rho^{D-3}})^2}{(1 + \frac{\mu^{D-3}}{4\rho^{D-3}})^2} , \quad H(\rho) = (1 + \frac{\mu^{D-3}}{4\rho^{D-3}})^2 , \quad \left(\frac{5}{4}\mu^{\frac{D-3}{2}}\right)^{\frac{2}{D-3}} \leq \rho \leq \infty \ . \tag{3.5.73}$$

Using the embedding (3.5.60) the action reduces to (3.5.63) but with $R_0^2 = \mathcal{V}(r)$ given by (3.5.72). The solution can be obtained from (3.5.65) and reads

$$\Omega^2 = \frac{(1 - \mathbf{m}^{n+1})}{R^2} \frac{(1 - \mathbf{m}^{n+1})(2 - (n+1)^2) + (n+1)^2}{(n+1)\mathbf{m}^{n+1}} , \tag{3.5.74}$$

where we have defined the parameter $\mathbf{m} = \frac{\mu}{\sum_i^N R_i^2}$ and used the fact that in this case $D = n+4$. One can go even further and perform the same calculation for a general potential of the form $\mathcal{V}(r)$ with $r^2 = \sum_{i=1}^N R_i^2$, the equilibrium condition for Ω is given by the relation

$$\Omega^2 = \frac{R_0^2}{R^2} \frac{2R_0^2 + (n+1)R_0^{2'}\sqrt{\sum_{i=1}^N R_i^2}}{2R_0^2(n+1) + R_0^{2'}\sqrt{\sum_{i=1}^N R_i^2}} , \tag{3.5.75}$$

where $R_0^{2'} \equiv \partial_r R_0^2$. This generalizes the equilibrium condition obtained in [65] for planar rings in backgrounds of this form.

[10] The black Saturn solution in five dimensions was constructed in Ref. [32]. See also Refs. [14, 65, 63, 29] for results on black Saturns in higher dimensions.

Chapter 4

Linear (electro)elastic deformations and spin

In this section we push the method described in Sec. 1.1 one order further by deriving the modified EOMs (1.1.2) and measuring the corrected stress-tensor from the metric (1.1.3) in the case of a bent string $p = 1$. This is done by doing a multipole expansion of the stress tensor $\hat{T}^{\mu\nu}$ (finite thickness corrections) as opposed to viscous corrections when hydrodynamic perturbations are introduced. It is convenient to introduce an analogy to develop intuition about the physics of this type of corrections. Consider a dielectric object with electric charge q under the influence of an electric field $\vec{E}$. In the point-like approximation the charge density of the object is $\rho(x) = q\,\delta^{D-1}(x)$ and its equation of motion is

$$m\vec{a} = q\vec{E}\,. \tag{4.0.1}$$

For a real material the electric field $\vec{E}$ causes a charge redistribution and induces an electric dipole, which to lowest order in $\vec{E}$ is given by linear response

$$\vec{d} = \kappa\vec{E}\,, \tag{4.0.2}$$

where κ depends on the material (and could be a matrix). The object is no longer an electric monopole,

$$\rho(x) = q\,\delta^{D-1}(x) - \vec{d}\cdot\vec{\partial}\left(\delta^{D-1}(x)\right)\,, \tag{4.0.3}$$

and the equations of motion for this pole-dipole object now read

$$m\vec{a} = q\vec{E} + \vec{d}\cdot\vec{\partial}\,\vec{E}\,. \tag{4.0.4}$$

If the induced dipole is small with respect to the scale in which the electric field varies, the second term in the r.h.s of this equation is a small perturbation. Here we present the gravitational analog of this equation that applies to black branes under bending by the action of the extrinsic curvature and use these results to exhibit the elastic character of black strings by measuring their elastic moduli - a relativistic generalization of the Young modulus. Finite thickness corrections also encode intrinsic spin degrees of freedom and we use this to correctly describe doubly-spinning MP black holes. The procedure can also be applied to charged branes in which case they will develop piezoelectric properties. In line with this we measure the piezolectric moduli of charged dilatonic black strings in Einstein-Maxwell-Dilaton (EMD) theory.

4.1 Dynamics and conserved charges of pole-dipole branes

Here we briefly review the equations of motion for p-dimensional objects in the pole-dipole approximation, i.e., when the stress-energy tensor is expanded to first order in a Dirac-delta series. Following closely the work done in Ref. [98], it is shown how to iteratively account for higher-pole deformations to the stress-energy tensor $\hat{T}^{\mu\nu}$ while the extra symmetries that this object exhibits are commented upon. The equations of motion are then presented in their original form, as derived in [98], which, when applied to black p-branes, are collectively called blackfold pole-dipole equations. In search of a clearer physical interpretation, we introduce a new set of quantities that make apparent the physics involved. Towards the end of this section, we provide a characterization of these p-branes in terms of well defined physical quantities and, in the particular case of blackfold constructions, of well defined thermodynamic properties. In Sec. 4.1.4 we derive the equations of motion for branes carrying a 0-brane charge.

4.1.1 Stress-energy tensor and extra symmetries

The stress-energy tensor is a well-localized object on the brane and can be consistently expanded into a Dirac delta function series around the embedding surface $x^\mu = X^\mu(\sigma^a)$. Schematically, the expansion has the following form[1]:

$$\hat{T}^{\mu\nu}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\left[B^{\mu\nu}(\sigma^a)\frac{\delta^{(D)}(x^\alpha - X^\alpha)}{\sqrt{-g}} - \nabla_\rho\left(B^{\mu\nu\rho}(\sigma^a)\frac{\delta^{(D)}(x^\alpha - X^\alpha)}{\sqrt{-g}}\right) + ...\right]. \tag{4.1.1}$$

In the context of electrodynamics, (4.1.1) corresponds to the usual multipole expansion of a charge distribution. For the series (4.1.1) to be well defined we must require $\hat{T}^{\mu\nu}$ to fall off exponentially to zero as we move away from the surface $x^\mu = X^\mu(\sigma^a)$, which implies that each of the coefficients $B^{\mu\nu\alpha_1...\alpha_k}$ must become smaller and smaller at each order k of the expansion. At order $k = 0$ the only non-vanishing coefficient is $B^{\mu\nu}$, resulting in $\hat{T}^{\mu\nu}$ acquiring the form of (2.1.2) and, by means of Eq. (1.1.2), leading to the equations

[1] Note that we slightly changed notation with respect to (2.1.2) to match the original work of [98]. Here the monopole stress-energy tensor $B^{\mu\nu}$ is the same as $T^{\mu\nu}$ in (2.1.2).

of motion for single-pole branes as presented in (2.3.7). In this section we are concerned with truncating the expansion (4.1.1) to order $k = 1$ and obtaining, in the same way, the equations of motion for pole-dipole branes moving in curved backgrounds. Truncation of the series is a covariant operation and can be done at any arbitrary order. As it stands, (4.1.1) is written in a manifestly invariant way both under spacetime diffeomorphisms and worldvolume reparametrizations.

These are not the only gauge redundancies that $\hat{T}^{\mu\nu}$ possesses since it is also invariant under two other gauge transformations, which were coined by the authors of [98] as 'extra symmetry 1' and 'extra symmetry 2'. As these symmetries play an important role in understanding the physics of pole-dipole branes, we proceed by describing their action on the B-tensors.

Extra symmetry 1

This additional gauge freedom arises naturally in the expansion (4.1.1) due to the $p+1$ δ-functions and $p+1$ integrations that were introduced solely with the purpose of making the full expression covariant. Specifically, derivatives along the worldvolume directions are integrated out, implying that there are redundant components of $B^{\mu\nu\rho}$. Physically, this is a consequence of the fact that the multipole expansion is an expansion in derivatives transverse to the brane, rather than longitudinal. The invariance of the stress-energy tensor under this symmetry is defined by its action on the $B^{\mu\nu}$ and $B^{\mu\nu\rho}$ tensors as

$$\delta_1 B^{\mu\nu} = -\nabla_a \epsilon^{\mu\nu a}, \qquad \delta_1 B^{\mu\nu\rho} = \epsilon^{\mu\nu a} u_a^{\rho} \ , \tag{4.1.2}$$

with $\epsilon^{\mu\nu a} = \epsilon^{\nu\mu a}$ being free parameters except at the boundary of the worldvolume where they are required to obey,

$$\hat{n}_a \epsilon^{\mu\nu a}|_{\partial \mathcal{W}_{p+1}} = 0 \ , \tag{4.1.3}$$

where $\hat{n}^a$ is the unit normal vector to the brane boundary (see App. A for details). Using the transformation laws (4.1.2) one can easily check that the purely tangential components to the worldvolume of $B^{\mu\nu\rho}$ are in fact a gauge artifact,

$$\delta_1 (B^{\mu\nu\rho} u_\rho^a) = \epsilon^{\mu\nu a} \ . \tag{4.1.4}$$

Hence, the components $B^{\mu\nu a}$ can be gauged away everywhere except at the boundary where the parameters $\epsilon^{\mu\nu a}$ cannot be freely chosen. This implies that there are degrees of freedom that live exclusively on the boundary of the worldvolume, for which a physical interpretation will be given in the next section.

Extra symmetry 2

The stress-energy tensor $T^{\mu\nu}$ has been expanded around the surface $x^\mu = X^\mu(\sigma^a)$ as in (4.1.1) but since we are dealing with objects of finite thickness there is freedom in choosing a different worldvolume. In physical terms, the finite thickness of the brane allows for different choices of worldvolume surfaces[2]. This redundancy is an exact symmetry of the full series expansion (4.1.1) but only an approximate one of the truncated series to order $k = 1$. This is because neglecting higher order terms in the expansion (4.1.1) already constrains the allowed choices of worldvolumes. In particular, choosing the surface $X^\alpha(\sigma^a)$ to lie outside the localized matter would require a non-zero contribution from the higher order B-tensors. Therefore, we choose the surface to lie within the localized matter and assume the following hierarchy of scales:

$$B^{\mu\nu} = \mathcal{O}_0, \qquad B^{\mu\nu\rho} = \mathcal{O}_1, \qquad B^{\mu\nu\rho\lambda} = \mathcal{O}_2, \quad \dots \,. \tag{4.1.5}$$

In this way, we can define the action of 'extra symmetry 2' as

$$X'^\alpha(\sigma^a) = X^\alpha(\sigma^a) + \epsilon^\alpha(\sigma^a) \;, \tag{4.1.6}$$

where ϵ^α is constrained by the requirement that the transformed B-tensors obey $B_{k+1} = \mathcal{O}_{k+1}$. In both the single-pole and pole-dipole cases this implies $\epsilon^\alpha = \mathcal{O}_1$. The action of (4.1.6) to order $k = 1$ demands the following transformation rule for the B-tensors:

$$\delta_2 B^{\mu\nu} = -B^{\mu\nu} u^a_\rho \nabla_a \epsilon^\rho - 2B^{\lambda(\mu}\Gamma^{\nu)}_{\lambda\rho}\epsilon^\rho, \qquad \delta_2 B^{\mu\nu\rho} = -B^{\mu\nu}\epsilon^\rho \;, \tag{4.1.7}$$

where we have ignored contributions of $\mathcal{O}_2$ and higher. In the single-pole approximation we find $\delta_2 X^\alpha = 0$ and $\delta_2 B^{\mu\nu} = 0$, emphasizing the fact that there is no freedom in choosing the worldvolume surface for the object as they are infinitely thin.

4.1.2 Equations of motion and physical interpretation

The equations of motion (EOMs) for probe pole-dipole branes moving in a curved background spacetime can be obtained by solving Eq. (1.1.2) using the stress-energy tensor given in (4.1.1) truncated to order $k = 1$. The derivation of these equations is somewhat involved and we refer to [98] for an extensive detailed analysis.

It is convenient to decompose the objects $B^{\mu\nu}$ and $B^{\mu\nu\rho}$ into tangential and orthogonal components to the worldvolume, the latter being subjected to the constraint equation [98]

$$\perp^\nu{}_\lambda \perp^\sigma{}_\rho B^{\mu(\lambda\rho)} = 0 \;, \tag{4.1.8}$$

while the former is not altogether independent and bares a relation with $B^{\mu\nu\rho}$ which will be described below. This suggests the following decomposition[3]:

$$B^{\mu\nu} = B^{\mu\nu}_\perp + 2u^{(\mu}_b B^{\nu)b}_\perp + u^\mu_a u^\nu_b B^{ab}, \qquad B^{\mu\nu\rho} = 2u^{(\mu}_b B^{\nu)\rho b}_\perp + u^\mu_a u^\nu_b B^{\rho ab}_\perp + u^\rho_a B^{\mu\nu a} \;, \tag{4.1.9}$$

[2] In the particle case there is a natural choice of reference frame which is the centre of mass.

[3] A subindex $_\perp$ on a tensor indicates that all μ,ν type of indices are orthogonal, e.g., $B^{a\mu}_\perp = \perp^\mu{}_\nu B^{a\nu}_\perp$. Details can be found in App. A.

that can be shown to obey the properties $B_{\perp}^{(\mu\nu)a} = B_{\perp}^{\mu[ab]} = B^{[\mu\nu]a} = 0$. Due to 'extra symmetry 1', described in the previous section, the last components in the decomposition of $B^{\mu\nu\rho}$ are left neither parallel nor perpendicular to the worldvolume, as $B^{\mu\nu a}$ can be gauged away in the bulk of the brane. A convenient form of the EOMs can be obtained by defining a new set of tensors

$$S^{\mu\nu a} = B_{\perp}^{\mu\nu a} + u_b^{[\mu} B_{\perp}^{\nu]ba}, \qquad N^{\mu\nu a} = B^{\mu\nu a} + u_b^{(\mu} B_{\perp}^{\nu)ba} \ , \tag{4.1.10}$$

which are, respectively, anti-symmetric and symmetric in the first two indices μ, ν. In terms of these it is straightforward to check that $B^{\mu\nu\rho}$ can be recast as

$$B^{\mu\nu\rho} = 2u_a^{(u} S^{\nu)\rho a} + N^{\mu\nu a} u_a^{\rho} \ . \tag{4.1.11}$$

Furthermore, the interdependence between the orthogonal components of $B^{\mu\nu}$ and the quantities $S^{\mu\nu a}$ and $N^{\mu\nu a}$ is expressed through the relations

$$B_{\perp}^{\mu\nu} = \perp^{\mu}{}_{\lambda} \perp^{\nu}{}_{\rho} \nabla_a N^{\lambda\rho a}, \qquad B_{\perp}^{\mu a} = u_{\lambda}^{a} \perp^{\mu}{}_{\rho} \nabla_b \left(S^{\lambda\rho b} + N^{\lambda\rho b} \right) \ , \tag{4.1.12}$$

while the tangential components B^{ab} describe the monopole contribution to the intrinsic stress-energy tensor of the brane.

Parametrizing the EOMs using (4.1.11)-(4.1.12) yields two sets of bulk equations: a partial conservation equation of the brane worldvolume currents $S^{\mu\nu a}$,

$$\perp^{\mu}{}_{\lambda} \perp^{\nu}{}_{\rho} \nabla_a S^{\lambda\rho a} = 0 \ , \tag{4.1.13}$$

and the equation that describes the motion of the pole-dipole brane

$$\nabla_b \left(m^{ab} u_a^{\mu} - 2u_{\lambda}^{b} \nabla_a S^{\mu\lambda a} + u_c^{\mu} u_{\rho}^{c} u_{\lambda}^{b} \nabla_a S^{\rho\lambda a} \right) - u_a^{\nu} S^{\lambda\rho a} R^{\mu}{}_{\nu\lambda\rho} \ = 0 \ , \tag{4.1.14}$$

where we have defined, for later convenience, the worldvolume tensor m^{ab} through the formula

$$m^{ab} = B^{ab} - u_{\rho}^{a} u_{\lambda}^{b} \nabla_c N^{\rho\lambda c} \ . \tag{4.1.15}$$

Eqs. (4.1.13)-(4.1.14) reduce to those of a spinning point particle as obtained by Papapetrou in [99] when $p = 0$. In order to highlight the physical meaning of Eq. (4.1.14), we project it along the tangential and orthogonal directions to the worldvolume. This operation leads to the intrinsic and extrinsic worldvolume equations:

$$\nabla_b m^{ab} = 2\nabla_b \left(u_{\rho}^{[b} K^{a]}{}_{c\lambda} S^{\rho\lambda c} \right) - 2u_{\lambda}^{b} K^{a}{}_{b\rho} \nabla_c S^{\rho\lambda c} - u_{\mu}^{a} u_c^{\nu} S^{\rho\lambda c} R^{\mu}{}_{\nu\lambda\rho} \ , \tag{4.1.16}$$

$$m^{ab} K_{ab}{}^{\rho} = \left(2\perp^{\rho}{}_{\lambda} K^{b}{}_{b\nu} + u_{\nu}^{c} u_{\lambda}^{b} K_{bc}{}^{\rho} \right) \nabla_a S^{\nu\lambda a} + 2u_{\nu}^{b} \perp^{\rho}{}_{\lambda} \nabla_b \nabla_a S^{\nu\lambda a} - u_a^{\nu} \perp^{\rho}{}_{\mu} S^{\sigma\lambda a} R^{\mu}{}_{\nu\lambda\sigma} \ . \tag{4.1.17}$$

In this way, it is clear that Eq. (4.1.16) can be interpreted as an equation for the conservation of the intrinsic monopole stress-energy tensor B^{ab}, which can be violated due to the higher order dipole contributions[4], while Eq. (4.1.17) is the generalized geodesic equation for a pole-dipole p-dimensional object, in contrast with the single-pole case (2.3.7).

In turn, the EOMs that govern the brane dynamics (4.1.13)-(4.1.14) everywhere inside $\mathcal{W}_{p+1}$ are supplemented by well defined boundary conditions derived from solving Eq. (1.1.2),

$$
\begin{aligned}
S^{\mu\nu a}\hat{n}_a\hat{n}_\nu|_{\partial\mathcal{W}_{p+1}} &= 0 \\
\perp^{\mu}{}_{\lambda}\perp^{\nu}{}_{\rho}S^{\lambda\rho a}\hat{n}_a|_{\partial\mathcal{W}_{p+1}} &= 0 \\
\left[\nabla_{\hat{i}}\left(N^{\hat{i}\hat{j}}v^{\mu}_{\hat{j}} + 2S^{\mu\nu a}\hat{n}_a v^{\hat{i}}_{\nu}\right) - \hat{n}_b\left(m^{ab}u^{\mu}_a - 2u^b_\lambda\nabla_a S^{\mu\lambda a} + u^{\mu}_c u^c_\rho u^b_\lambda \nabla_a S^{\rho\lambda a}\right)\right]|_{\partial\mathcal{W}_{p+1}} &= 0 \ ,
\end{aligned}
\tag{4.1.18}
$$

where $v^{\hat{j}}_{\nu}$ are the boundary coordinate vectors (see App. A for details) and we have defined $N^{\hat{i}\hat{j}} = N^{\mu\nu a}\hat{n}_a v^{\hat{i}}_{\mu} v^{\hat{j}}_{\nu}$. These quantities appear only in the boundary conditions and nowhere else and, as mentioned in the previous section while discussing 'extra symmetry 1', contain the degrees of freedom that live exclusively on the boundary. In full generality, the tensors m^{ab}, $S^{\mu\nu a}$ and $N^{\hat{i}\hat{j}}$ characterize the internal structure of the brane and play a crucial role in describing its dynamics. We will proceed by analyzing their physical meaning and of the resulting EOMs.

Physical interpretation

Ref. [98] introduced the $S^{\mu\nu a}$ and m^{ab} quantities, in terms of which the equations of motion of pole-dipole branes simplify considerably. In the following we give an interpretation of $S^{\mu\nu a}$, and see that it contains two types of contributions: the genuine intrinsic transverse angular momenta, and the dipole moment of the distribution of worldvolume stress-energy[5]. Thus, besides generalising Papapetrou's equations, Eqs. (4.1.16), (4.1.25) include dipole interactions analogous to those in (4.0.4).

We begin by analyzing which components of $S^{\mu\nu a}$ are involved in the description of the intrinsic angular momenta. To this end, we assume to be working in flat spacetime written in Cartesian coordinates and focus on uniform p-branes extended along the $x^0, ..., x^p$ directions. Evaluating the total angular momentum on the transverse plane labeled by the indices μ, ν leads to

$$
J^{\mu\nu}_{\perp} = \int_{\Sigma} d^{D-1}x\left(\hat{T}^{0\mu}x^{\nu} - \hat{T}^{0\nu}x^{\mu}\right) = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}\left(2B^{0\mu\nu}_{\perp}\right) + \text{boundary terms} \ , \tag{4.1.19}
$$

[4] Even though the conservation of the monopole stress-energy tensor is not necessarily guaranteed, in the cases studied here we always find that B^{ab} is conserved.

[5] In the case of the 0−brane the dipole can be gauged away and, as demonstrated in the original work of Corinaldesi and Papapetrou [99, 100], $S^{\mu\nu a}$ describes only spin degrees of freedom.

where Σ is a constant time slice in the bulk spacetime. At this point, we ignore the boundary terms, which only depend on the components $B^{\mu\nu a}$, but we will consider them towards the end of this section. From (4.1.19), we can see that the monopole contribution to the intrinsic stress-energy tensor B^{ab} does not play a role in (4.1.19) and hence a p-brane when treated in the single-pole approximation can never carry intrinsic angular momenta. Furthermore, only the components $B_{\perp}^{a\mu\nu}$ contain information about the spin of the object. This suggests the introduction of a current density of transverse angular momenta as

$$j^{a\mu\nu} = 2u^a_\rho {\perp^\mu}_\sigma {\perp^\nu}_\lambda B^{\rho[\sigma\lambda]} = 2B_{\perp}^{a\mu\nu} \ , \tag{4.1.20}$$

where both indices μ, ν are orthogonal to the worldvolume.

On the other hand, there is another source of $B^{\mu\nu\rho}$ which is of a different nature than transverse angular momenta. It arises from the fact that, since we are probing the finite thickness of the brane, we need to take into account corrections to the intrinsic stress-energy tensor T^{ab} due to the dipole-type effects. This is characterized by the integral[6],

$$D^{ab\rho} = \int_\Sigma d^{D-1}x\hat{T}^{ab}x^\rho = \int_\Sigma d^{D-1}x\hat{T}^{\mu\nu}u^a_\mu u^b_\nu x^\rho = \int_{B_p} d^p\sigma\sqrt{-\gamma}B_{\perp}^{\rho ab} + \text{boundary terms} \ , \tag{4.1.21}$$

where x^ρ is an orthogonal coordinate to the worldvolume. $D^{ab\rho}$ captures the dipole moment of the distribution of worldvolume stress-energy. As in the case of intrinsic angular momenta, we introduce a current density that describes such deformations to the intrinsic stress-energy tensor by

$$d^{ab\rho} = u^a_\mu u^b_\nu {\perp^\rho}_\lambda B^{\mu\nu\lambda} = B_{\perp}^{\rho ab} \ , \tag{4.1.22}$$

where the index ρ is orthogonal to the worldvolume $\mathcal{W}_{p+1}$.

Our aim now is to recast the EOMs, including the boundary conditions, in terms of these newly defined quantities. Using the definitions of the current densities (4.1.20) and (4.1.22), we can rewrite the tensors introduced in (4.1.10) as

$$S^{\mu\nu a} = \frac{1}{2}j^{a\mu\nu} - d^{ab[\mu}u_b^{\nu]}, \qquad N^{\mu\nu a} = B^{\mu\nu a} + d^{ab(\mu}u_b^{\nu)} \ . \tag{4.1.23}$$

We note that we have not been concerned so far with giving a physical interpretation to the components $B^{\mu\nu a}$. This is because, due to 'extra symmetry 1', we can gauge them away

[6] As a matter of a fact, this is the usual notion of an electric induced dipole. In electrostatics, given a density of charge $\rho(x)$, the dipole can computed as,

$$\vec{D} = \int_\Sigma d^{D-1}x\vec{x}\rho(x) \ .$$

everywhere in the bulk while on the boundary we will have to deal with $N^{\hat{i}\hat{j}}$ as we will see below. In turn, the current conservation equation (4.1.13) becomes:

$$\frac{1}{2}\perp^{\mu}{}_{\lambda}\perp^{\nu}{}_{\rho}\nabla_a j^{a\rho\lambda} + d^{ab[\nu}K_{ab}{}^{\mu]} = 0 \ . \tag{4.1.24}$$

This equation can be interpreted as the balance between orbital angular momentum and intrinsic angular momentum. Nevertheless, as it will be argued in Sec. 4.2, for blackfold-type objects, the dipole current $d^{ab\rho}$ is induced by the extrinsic curvature. In all such situations, the second term in Eq. (4.1.24) vanishes, leading to a conserved spin current which can be naturally interpreted as a 0-brane particle current on the worldvolume.

We now turn our attention to the intrinsic and extrinsic Eqs. (4.1.16)-(4.1.17), which can be rewritten using (4.1.20) and (4.1.22) as

$$D_a m^{ab} = K_c{}^b{}_\mu \left(K_a{}^c{}_\lambda j^{a\mu\lambda} + \nabla_a d^{ac\mu}\right) + \nabla_c K_a{}^{[b}{}_\mu d^{c]a\mu} - u^b_\mu u^\nu_a \left(\frac{1}{2}j^{a\lambda\rho} - d^{ab[\lambda}u^{\rho]}_b\right) R^\mu{}_{\nu\lambda\rho} \ , \tag{4.1.25}$$

$$\begin{aligned} m^{ab}K_{ab}{}^\rho \;=\; & \nabla_b\left(j^{a\lambda\rho}K^b{}_{a\lambda}\right) + u^\rho_c K^a{}_{b\lambda}K^c{}_{a\sigma}j^{b\lambda\sigma} - K_{ac}{}^\rho K^{(c}{}_{b\lambda}d^{a)b\lambda} - \perp^\rho{}_\sigma \nabla_b\nabla_a d^{ab\sigma} \\ & - u^\nu_a \perp^\rho{}_\mu \left(\frac{1}{2}j^{a\lambda\sigma} - d^{ab[\lambda}u^{\sigma]}_b\right) R^\mu{}_{\nu\lambda\sigma} \ . \end{aligned} \tag{4.1.26}$$

These equations provide the gravitational analog of Eq. (4.0.4) for p-branes. Written in this way it is apparent that, besides spin interactions, we also have couplings to the dipole current $d^{ab\rho}$. For blackfold objects these interactions can be interpreted as elastic forces, for which a derivation in terms of high-pressure elasticity theory [80] can in principle be accomplished and will be presented elsewhere. This point will be further motivated in Sec. 4.2. Finally, the boundary conditions (4.1.18) take the form:

$$\begin{aligned} d^{ab\mu}\hat{n}_a\hat{n}_b|_{\partial\mathcal{W}_{p+1}} &= 0 \\ j^{a\mu\nu}\hat{n}_a|_{\partial\mathcal{W}_{p+1}} &= 0 \\ \left[\nabla_{\hat{i}}\left(N^{\hat{i}\hat{j}}v^\mu_{\hat{j}} - d^{ab\mu}\hat{n}_a v^{\hat{i}}_b\right) - \hat{n}_b\left(m^{ab}u^\mu_a - j^{a\mu\lambda}K_a{}^b{}_\lambda - K_a{}^{(c}{}_\lambda d^{b)a\lambda}u^\mu_c - \nabla_a d^{ab\mu}\right)\right]|_{\partial\mathcal{W}_{p+1}} &= 0 \ , \end{aligned} \tag{4.1.27}$$

where $N^{\hat{i}\hat{j}}v^\mu_{\hat{j}}$ is in fact

$$N^{\hat{i}\hat{j}}v^\mu_{\hat{j}} = B^{\lambda\nu a}\hat{n}_a v^{\hat{i}}_\lambda v^{\hat{j}}_\nu v^\mu_{\hat{j}} \ . \tag{4.1.28}$$

Eqs. (4.1.24)-(4.1.27) when applied to black p-branes constitute the blackfold equations in the pole-dipole approximation and will be analyzed in detail in particular cases throughout the course of this work.

We end this section by briefly commenting on the physical interpretation of the coefficients $N^{\hat{i}\hat{j}}$. As remarked in [98], $N^{\hat{i}\hat{j}}$ characterizes the tangential components to the worldvolume of the brane thickness. The reason why these components drop out of the bulk equations is because thickening the brane along tangential directions does not affect the brane interior but if the brane has a boundary then it will be affected by such process. Essentially, $N^{\hat{i}\hat{j}}$ is a correction to the intrinsic stress-energy tensor of the brane boundary $T^{\hat{i}\hat{j}}$ and can be read off from an analytic solution by evaluating the stress-energy tensor on the boundary of the brane surface. Since for all the cases analyzed here $B^{\mu\nu a}$ vanishes everywhere, including at the boundary, we assume $B^{\mu\nu a} = 0$ from hereon.

4.1.3 Physical properties

Branes in the pole-dipole approximation can carry different conserved charges and this section is devoted to describing them. In the case of blackfolds, where these charges acquire a thermodynamic interpretation, we present a method for obtaining the remaining thermodynamic quantities involved.

As mentioned in Sec. 1.1, fine structure corrections do not violate stress-energy conservation (1.1.2). Hence, since we are working in the probe approximation, associated with any background Killing vector field $\mathbf{k}^\mu$, there exists a conserved current given by $T^{\mu\nu}\mathbf{k}_\nu$ and thus the conserved charges can be obtained in the usual way. In general, we have a charge $Q_{\mathbf{k}}$ given by

$$|Q_{\mathbf{k}}| = \int_{\mathcal{B}_p} dV_{(p)} B^{\mu\nu} n_\mu \mathbf{k}_\nu + \int_{\mathcal{B}_p} dV_{(p)} B^{\mu\nu\rho} \nabla_\rho \left(n_\mu \mathbf{k}_\nu \right) \ , \tag{4.1.29}$$

where n^μ is the normal vector to a constant time slice of the background space-time and is defined as

$$n^\mu = \frac{\xi^\nu}{R_0} \ . \tag{4.1.30}$$

In writing Eq. (4.1.29) we have assumed that ξ^ν is hypersurface orthogonal with respect to the background spacetime and that ξ^ν is parallel to the worldvolume timelike Killing vector field, which is also hypersurface orthogonal with respect to the worldvolume metric, i.e., $\xi^\mu = u^\mu_a \xi^a$.

Using the decompositions (4.1.9) and Eqs. (4.1.20),(4.1.22) we can write down general expressions for the physical quantities in terms of the spin and dipole currents. The total mass, associated with ξ^ν, reads[7]

$$M = \int_{\mathcal{B}_p} dV_{(p)} B^{ab} n_a \xi_b + \int_{\mathcal{B}_p} dV_{(p)} d^{ab\rho} \xi_a u^\mu_b \partial_\rho n_\mu \ . \tag{4.1.31}$$

[7] Here we have used the Killing equation $\nabla_{(\mu}\mathbf{k}_{\mu)} = 0$ to exchange covariant derivatives for partial derivatives.

The first term above describes the contribution to the total mass arising from a monopole source of stress-energy, while the second is the extra contribution coming from a dipole distribution. Similarly, the total angular momentum along a worldvolume spatial direction i associated with the rotational Killing vector field χ^i takes the form

$$J^i = -\int_{\mathcal{B}_p} dV_{(p)} B^{ab} n_a \chi^i_b - \frac{1}{2}\int_{\mathcal{B}_p} dV_{(p)} d^{ab\rho} u^\mu_b \left(\xi_a \partial_\rho \frac{\chi^i_\mu}{R_0} + \chi^i_a \partial_\rho n_\mu \right) . \quad (4.1.32)$$

Moreover, the transverse angular momentum associated with the rotational Killing vector field $\chi^\alpha_\perp$ is given by

$$J^\alpha_\perp = -\int_{\mathcal{B}_p} dV_{(p)} \left(\frac{1}{2}\nabla_a j^{a\lambda\rho} + u^\lambda_b \nabla_a d^{ab\rho} \right) n_\lambda \chi_{\perp\rho} - \frac{1}{4}\int_{\mathcal{B}_p} dV_{(p)} j^{a\mu\rho} u^\nu_a \left(\xi_\nu \partial_\rho \frac{\chi^\alpha_{\perp\mu}}{R_0} + \chi^\alpha_{\perp\mu} \partial_\rho n_\nu \right) . \quad (4.1.33)$$

We see that the spin current $j^{a\mu\rho}$ only plays a role in the transverse angular momenta. On the other hand, the dipole current $d^{ab\rho}$ does influence all charges, including $J^\alpha_\perp$ (which corresponds to orbital angular momentum). This is because, in general, $j^{a\mu\rho}$ is not a conserved current due to Eq. (4.1.24). In the cases considered here, the first term in Eq. (4.1.33) always vanishes, as the spin current is always conserved.

Thermodynamic quantities

When considering the dynamics of black p-branes, the conserved charges have a thermodynamic interpretation. Furthermore, a local temperature and entropy density can be assigned to the p-dimensional object from which one can then compute a global temperature and total area. As we will be mainly concerned with blackfolds constructed from wrapped Schwarzschild and Myers-Perry p-branes on curved submanifolds, the knowledge of these quantities amounts to perform a near-horizon computation which is beyond the scope of this thesis. Instead, we present here an alternative method to obtain the on-shell temperature and entropy.

Focusing for the moment on asymptotically flat spacetimes, the method consists of using the first law of black hole thermodynamics

$$dM = \sum_i \Omega_i dJ^i + \sum_\alpha \Omega_\alpha dJ^\alpha_\perp + TdS , \quad (4.1.34)$$

together with the Smarr relation

$$(n+p)M = (n+p+1)\left(\sum_i \Omega_i J^i + \sum_\alpha \Omega_\alpha J^\alpha_\perp + TS \right) , \quad (4.1.35)$$

in order to determine the two unknown quantities T and S. Since, in general, all physical quantities depend on a set of intrinsic parameters, such as r_0 and R, which we collectively call by Φ^w, then, by means of Eq. (4.1.35) the product TS can be determined, as all other quantities involved can be evaluated using the expressions (4.1.31)-(4.1.33). Inserting this result into Eq. (4.1.34) leads to a set of equations, one for each Φ^w, which take the following form:

$$\frac{1}{T}\frac{\partial T}{\partial \Phi^w} = \frac{1}{TS}\left(\frac{\partial TS}{\partial \Phi^w} + \sum_i \Omega_i \frac{\partial J^i}{\partial \Phi^w} + \sum_\alpha \Omega_\alpha \frac{\partial J_\perp^\alpha}{\partial \Phi^w} - \frac{\partial M}{\partial \Phi^w}\right) . \tag{4.1.36}$$

Solving this set of equations yields the temperature of the black object and hence also the entropy by Eq. (4.1.35). The method just described provides the knowledge of T and S up to a constant, which can later be fixed by demanding the correct behavior in the infinitely thin limit, or in the single-pole approximation, where all quantities can be unambiguously determined using the results of Ref. [16].

We conclude this section by briefly considering blackfolds in (A)dS backgrounds in situations where the dipole current $d^{ab\rho}$ vanishes. In such cases the total integrated tension $\mathcal{T}$, given by

$$\mathcal{T} = -\int_{\mathcal{B}_p} dV_{(p)} R_0 B^{\mu\nu}\left(h_{\mu\nu} - n_\mu n_\nu\right) - \int_{\mathcal{B}_p} dV_{(p)} R_0 B^{\mu\nu\rho}\nabla_\rho\left(h_{\mu\nu} - n_\mu n_\nu\right) , \tag{4.1.37}$$

plays a role in the thermodynamics and must be added to the rhs of Eq. (4.1.35). These considerations will be used later in Sec. (4.4) and App. (E) in order to make contact with the thermodynamic properties of doubly-spinning Myers-Perry and higher-dimensional Kerr-(A)dS black holes.

4.1.4 Pole-dipole branes carrying a 0-brane charge

Here we consider finite thickness corrections to the worldvolume theory of charged branes described in the end of Sec. 2.6. Besides being described by a stress-energy tensor (4.1.1), these are characterized by an additional monopole source of electric current that must suffer multipole corrections. We will be interested in applying this theory to black branes that are electrically charged under a 2-form field strength $F^{\mu\nu}(x^\alpha)$ which develop worldvolume electric dipoles due to the action of bending. In direct analogy with classical electrodynamics we expand the electric current in a Dirac-delta function series[8]:

$$\hat{J}^\mu(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\left[J^\mu_{(0)}(\sigma^a)\frac{\delta^{(D)}(x^\alpha - X^\alpha)}{\sqrt{-g}} - \nabla_\rho\left(J^{\mu\rho}_{(1)}(\sigma^a)\frac{\delta^{(D)}(x^\alpha - X^\alpha)}{\sqrt{-g}}\right) + ...\right] . \tag{4.1.38}$$

[8]Here we also changed notation with respect to (2.6.1). Here the components $J^\mu_{(0)}$ are the same as the components J^μ in (2.6.1).

In parallel with (4.1.1), this displays general covariance via 'extra symmetry 1', $\delta_1 J^{\mu\rho} = -\epsilon^{\mu a} u_a^\rho$ and $\delta_1 J^\mu = -\nabla_a \epsilon^{\mu a}$, and has the same ambiguity as the stress-energy tensor under small displacements of the embedding surface expressed by 'extra symmetry 2':

$$\begin{aligned}\delta_2 J^\mu_{(0)} &= -J^\mu_{(0)} u_\rho^a \nabla_a \epsilon^\rho - J^\lambda_{(0)} \Gamma^\mu_{\rho\lambda} \epsilon^\rho \, , \\ \delta_2 J^{\mu\rho}_{(1)} &= -J^\mu_{(0)} \epsilon^\rho \, .\end{aligned} \tag{4.1.39}$$

In the probe approximation, Eq. (1.1.2) is now supplemented with current conservation (2.6.2). Following the same method as in [98], we can solve Eq. (2.6.2) decomposing the monopole part of the current as

$$J^\mu_{(0)} = u_a^\mu J^a_{(0)} + J^\mu_\perp, \quad J^\mu_\perp = \perp^\mu{}_\lambda J^\lambda_{(0)} \, , \tag{4.1.40}$$

and the 2-index structure $J^{\mu\rho}_{(1)}$ as

$$J^{\mu\rho}_{(1)} = m^{[\mu\rho]} + u_a^\mu p^{a\rho} + J^{\mu a}_{(1)} u_a^\rho \, , \tag{4.1.41}$$

where we have used 'extra symmetry 1' to gauge away some of the components. Here $m^{[\mu\rho]} = \perp^{[\mu}{}_\nu \perp^{\rho]}{}_\lambda J^{\nu\lambda}_{(1)}$ is an additional contribution to the electric current due to the motion in transverse directions, $p^{a\rho} = u^a_\mu \perp^\rho{}_\nu J^{\mu\nu}_{(1)}$ is the electric dipole moment while the components $J^{\mu a}_{(1)}$ are pure gauge and can be set to zero. The extra components of the monopole part of the current are not independent and can be related to the dipolar part by the relation $J^\mu_\perp = \perp^\mu{}_\lambda \nabla_a \left(2 J^{(\lambda a)}_{(1)} - J^{(ab)}_{(1)} u_b^\lambda \right)$. Consequently, current conservation (2.6.2) results in the worldvolume conservation equation

$$\nabla_a \left(\mathcal{J}^a + p^{b\rho} \nabla_b u^a_\rho \right) = 0 \, , \tag{4.1.42}$$

where we have defined the worldvolume tensor $\mathcal{J}^a = J^a_{(0)} - \nabla_b J^{(ab)}_{(1)}$. In the case of worldvolumes with boundaries Eq. (4.1.42) must be supplement by additional boundary conditions [67]. When applied to the special case $p = 0$, these equations reduce to those derived for the charged spinning point particle [101, 102], though the complete decomposition (4.1.41), crucial for its physical interpretation, has not been considered in the literature.

4.2 Black string elasticity and thin black rings

The purpose of this section is to study how a Schwarzschild black string reacts to its bending. This is done using an explicit solution describing a black string of thickness r_0 bent on a circle of radius R, to first order in r_0/R. Very similarly to what happens to elastic rods, the effective blackfold stress-energy tensor distribution acquires a dipole contribution, exhibiting an effective elastic behavior characterized by a response coefficient: the Young modulus.

4.2.1 Measuring the dipole from the approximate analytic solution

This section uses the solutions constructed in [14] to compute the dipole contribution to the effective stress-energy distribution induced by the bending of strings. This reference computes, to first order in r_0/R, the spacetime of a bent Schwarzschild string, where r_0 is the thickness of the string and R is the radius of curvature of the circle on which the string is bent by means of the method described in Sec. 1.1. The resulting bent metric takes the local form of (1.1.3). Here we are interested in pushing the method one step further, i.e., **2nd far**. Let us summarize the order-by-order logic, now adapted to the specific case of a black string:

- **0th (near/far):** The geometry is that of the straight boosted Schwarzschild black string (1.1.1):

$$\begin{aligned} ds^2 =& \frac{dr^2}{1-\frac{r_0^n}{r^n}} + r^2\left(d\theta^2 + \sin^2\theta d\Omega_{(n)}^2\right) \\ &- \left(1-\frac{r_0^n}{r^n}\right)(\cosh\alpha\, dt + \sinh\alpha\, dz)^2 + (\cosh\alpha\, dz + \sinh\alpha\, dt)^2 \ , \end{aligned} \tag{4.2.1}$$

  where we have parametrized $u^a = [\cosh\alpha, -\sinh\alpha]$.

- **1st (far):** The far field is described by the ADM stress-energy tensor of the string (4.2.1), which takes the perfect fluid form and is localized on the string worldvolume, $T_{\mu\nu} = B_{\mu\nu}\delta^{n+2}(r)$:

$$B_{tt} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}\left(n\cosh^2\alpha + 1\right) \ , \tag{4.2.2a}$$

$$B_{tz} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G} n\cosh\alpha\sinh\alpha \, , \tag{4.2.2b}$$

$$B_{zz} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}\left(n\sinh^2\alpha - 1\right) \ . \tag{4.2.2c}$$

  now along a bent manifold with curvature $1/R$. Only sources satisfying Eq. (1.1.2) can be coupled to the gravitational field. For the case of the ring they reduce to constancy of r_0 and $B_{zz} = 0$, which sets the rapidity to

$$\sinh^2\alpha = \frac{1}{n} \, . \tag{4.2.3}$$

  This is interpreted as the value at which the centrifugal repulsion compensates the tension.

- **1st (near):** The geometry in the near zone is a $1/R$ perturbation to the Schwarzschild black string, that is found using **1st (far)** as a boundary condition and has the local form (1.1.3).
- **2nd (far):** From the metric (1.1.3) obtained in **1st (near)** one can compute the corrected effective stress-energy tensor. The corrected stress-energy tensor would need to satisfy the corrected effective equation of motion given in (4.1.24)-(4.1.27), which would modify (4.2.3)...

and so on. [14] went to the step **1st (near)**. In this section we push that analysis to **2nd (far)** by reading the correction to the stress-energy source, which is of dipole type, and which will have the equations derived in Sec. 4.1 as effective equation of motion.

The solution of a bent black string in flat space at **1st (near)** was written down in [14] as

$$g_{tt} = -1 + \frac{n+1}{n}\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}a(r)\,, \tag{4.2.4a}$$

$$g_{tz} = \frac{\sqrt{n+1}}{n}\left[\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}b(r)\right]\,, \tag{4.2.4b}$$

$$g_{zz} = 1 + \frac{1}{n}\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}c(r)\,, \tag{4.2.4c}$$

$$g_{rr} = \left(1 - \frac{r_0^n}{r^n}\right)^{-1}\left[1 + \frac{\cos\theta}{R}f(r)\right]\,, \tag{4.2.4d}$$

$$g_{ij} = \hat{g}_{ij}\left[1 + \frac{\cos\theta}{R}g(r)\right]\,, \tag{4.2.4e}$$

where $\hat{g}_{ij}$ is the metric of a round $n+1$ sphere of radius r in polar coordinates:

$$\hat{g}_{ij}dx^i dx^j = r^2 d\Omega_{(n+1)}^2 = r^2\left(d\theta^2 + \sin^2\theta\, d\Omega_{(n)}^2\right)\,. \tag{4.2.5}$$

As expected, the limit $R \to \infty$ corresponds to a black string boosted in the z direction with rapidity $\sinh\alpha = 1/\sqrt{n}$, according to (4.2.3).

The regular solution to the Einstein equations found in [14] has the following large r asymptotics

$$a(r) = \left[k_1(1+n) - n(4+3n)\xi(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}\left(r^{-(n+2)}\right)\,, \tag{4.2.6a}$$

$$b(r) = \frac{r_0^n}{r^{n-1}} + \left[k_1 n - 2n^2\xi(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}\left(r^{-(n+2)}\right)\,, \tag{4.2.6b}$$

$$c(r) = 2r + \frac{1}{n}\frac{r_0^n}{r^{n-1}} + k_1\frac{r_0^{n+2}}{r^{n+1}} - \frac{n^2+2n-1}{2n^2(n-2)}\frac{r_0^{2n}}{r^{2n-1}} + k_2\frac{r_0^{2n+2}}{r^{2n+1}} + \mathcal{O}\left(r^{-(2n+2)}\right) , \quad (4.2.6c)$$

$$f(r) = -\frac{2}{n}r + \frac{1}{n^2}\frac{r_0^n}{r^{n-1}} + \left[(k_1+2k_2)n - (4n^2+7n+4)\xi(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}\left(r^{-(n+2)}\right) , \quad (4.2.6d)$$

$$g(r) = -\frac{2}{n}r + \frac{1}{n^2}\frac{r_0^n}{r^{n-1}} + \left[\frac{2(k_1-k_2n)}{1+n} + (n-4)\xi(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}\left(r^{-(n+2)}\right) , \quad (4.2.6e)$$

where we defined

$$\xi(n) = -\frac{2^{-\frac{4+n}{n}}}{n+1}\frac{\Gamma\left(\frac{2n+1}{n}\right)\Gamma\left(-\frac{n+2}{2n}\right)}{\Gamma\left(-\frac{n+1}{n}\right)\Gamma\left(\frac{n+2}{2n}\right)} , \quad (4.2.7)$$

which is zero for $n=1$ and divergent for $n=2$. In the latter case this is because the corresponding terms in the expansion should have logarithm contributions on top the polynomial ones. The interest of this thesis is in the $n>2$ cases, which are those in which the step **2nd (far)** can be interpreted in the probe approximation, as discussed in the introduction. In what follows we assume $n>2$, i.e., black rings in more than six dimensions.

It is worth mentioning various things about expressions (4.2.6). First, note that all functions are expanded to order $r^{-(n+2)}$ except for $c(r)$, which is expanded to order $r^{-(2n+2)}$.

Second, there is residual gauge freedom in the coordinate system in which the metric reads (4.2.4) and, because of that, $c(r)$ is essentially unfixed by the Einstein equations. This shows up in the unfixed k_1 and k_2 parameters, which parametrize gauge ambiguities. $c(r)$ is not completely free because we demand that at large r the metric goes to a known metric in a familiar coordinate system. That known metric is the one that one recovers if one keeps only terms in this expansions up to order r^{-n} in (4.2.4). The geometry to order r^{-n} is just the linearized gravitational field of the blackfold distributional stress-energy tensor sitting on a circle of radius R, to first order in $1/R$.

To be more specific with this interpretation, let us decompose the metric (4.2.4) in the following way:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(\mathrm{M})}_{\mu\nu} + h^{(\mathrm{D})}_{\mu\nu} + \mathcal{O}\left(r^{-(n+2)}\right) . \quad (4.2.8)$$

Here $\eta_{\mu\nu}$ is flat space in cylindrical coordinates

$$\eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\Omega^2_{(n)}) , \quad (4.2.9)$$

and $h^{(\mathrm{M})}_{\mu\nu}$, which is the correction to the flat metric in the **(1st far)** step, includes two types of contributions; There is a $1/R$ change of coordinates of flat space such that $r=0$ corresponds to a circle, which are the terms not multiplying any r_0 in (4.2.6). The other type of terms, which are the ones multiplied by r_0^n, correspond to the linearized gravitational field sourced by the monopole blackfold stress-energy tensor.

$h_{\mu\nu}^{(D)}$ are the order $r^{-(n+1)}$ terms in the expansion. All these terms have

$$h_{\mu\nu}^{(D)} \propto \frac{r_0^{n+2}}{R}\frac{\cos\theta}{r^{n+1}}\,, \tag{4.2.10}$$

which is the field of a dipole source located at $r = 0$ in the spacetime (4.2.9)[9], namely

$$\Delta h_{\mu\nu}^{(D)} \propto \frac{r_0^{n+2}}{R}\partial_{(r\cos\theta)}\delta^{n+2}(r)\,, \tag{4.2.11}$$

where $\partial_{(r\cos\theta)}$ points outwards from the worldvolume, such that $\partial_{(r\cos\theta)}\, r = \cos\theta$. The metric (4.2.4) at order $r^{-(n+1)}$ is the linearized solution of

$$G_{\mu\nu}[g] = 8\pi G\left(B_{\mu\nu}\delta^{n+2}(r) - d_{\mu\nu}{}^{(r\cos\theta)}\partial_{(r\cos\theta)}\delta^{n+2}(r)\right)\,. \tag{4.2.12}$$

The goal of this section is to determine $d_{\mu\nu}{}^{(r\cos\theta)}$, which is the dipole source to the far field, and because it is proportional to $1/R$, it is induced by the bending of the monopole source. To find it, we note that in TT gauge, the Einstein equations for $h_{\mu\nu}^{(D)}$ linearize to

$$\Delta\bar{h}_{\mu\nu}^{(D)} = 16\pi G\, d_{\mu\nu}{}^{(r\cos\theta)}\partial_{(r\cos\theta)}\delta^{n+2}(r)\,, \tag{4.2.13}$$

for

$$\bar{h}_{\mu\nu}^{(D)} = h_{\mu\nu}^{(D)} - \frac{h^{(D)}}{2}\eta_{\mu\nu}\,, \qquad h^{(D)} = h_{\mu\nu}^{(D)}\eta^{\mu\nu}\,, \tag{4.2.14}$$

such that[10]

$$\bar{h}_{\mu\nu}^{(D)} = \frac{16\pi G}{\Omega_{n+1}}\frac{\cos\theta}{r^{n+1}}d_{\mu\nu}{}^{(r\cos\theta)}\,. \tag{4.2.15}$$

It has been argued in Sec. 4.1 that $d_{\mu\nu}{}^{(r\cos\theta)}$ has $\mu\nu$ indices parallel to the worldvolume, which is located at $r = 0$. μ and ν can thus only be t or z. This implies that in TT gauge

$$h_{rr}^{(D)} = \frac{h_{\Omega\Omega}^{(D)}}{r^2}\,, \qquad h_{tt}^{(D)} - h_{zz}^{(D)} = n\, h_{rr}^{(D)}\,, \tag{4.2.16}$$

which is satisfied for

$$k_1 = \frac{2n}{1-n}\left(k_2 - 2(n+1)\xi(n)\right)\,, \tag{4.2.17}$$

[9] Note that the field of the dipole is insensitive to the fact that $r = 0$ is now a circle of curvature $1/R$. This is because the metric (4.2.4) is valid only up to $1/R^2$ corrections. Being the dipole source already a $1/R$ effect, accounting for the fact that the dipole source sits on a circle of curvature $1/R$ is a $1/R^2$ effect.

[10] We used that in D spacetime dimensions $\Delta_{(D-1)}r^{-(D-3)} = -(D-3)\Omega_{(D-2)}\delta^{D-1}(r)\,.$

which, of course, is a gauge choice. After this gauge fixing, it is straightforward to read ${d_{\mu\nu}}^{(r\cos\theta)}$ from $\bar{h}^{(\mathrm{D})}_{\mu\nu}$. For the sake of interpretation it is convenient to first redefine[11],

$$k_2 = (n+3)\,\xi(n) + \frac{(n-1)}{2n}\,\tilde{k}_2\,. \tag{4.2.18}$$

The dipole contribution is found to be

$${d_{tt}}^{(r\cos\theta)} = -\frac{\Omega_{(n+1)} r_0^n}{16\pi G}\frac{r_0^2}{R}(n^2+3n+4)\xi(n) - \tilde{k}_2\frac{r_0^2}{R}\,B_{tt}\,, \tag{4.2.19a}$$

$${d_{tz}}^{(r\cos\theta)} = -\tilde{k}_2\frac{r_0^2}{R}\,B_{tz}\,, \tag{4.2.19b}$$

$${d_{zz}}^{(r\cos\theta)} = \frac{\Omega_{(n+1)} r_0^n}{16\pi G}\frac{r_0^2}{R}(3n+4)\xi(n)\,, \tag{4.2.19c}$$

where B_{ab} in these expressions are (4.2.2) and are evaluated at equilibrium (4.2.3). We remind the reader that these expressions are valid for $n > 2$.

Some of these terms are not gauge invariant, as they depend on $\tilde{k}_2$. This is the expected 'extra symmetry 2' ambiguity in the dipole under changes of the representative worldvolume surface, see Sec. 4.1. Indeed, $\tilde{k}_2 \to \tilde{k}_2 + \delta$ picks a worldvolume outwards by $\delta\, r_0^2/R$. ${d_{zz}}^{(r\cos\theta)}$ being unambiguously defined fits nicely with the fact that the equilibrium condition for the ring is precisely $B_{zz} = 0$, which renders this symmetry (4.1.7) trivial for this component of the dipole.

4.2.2 More general backgrounds

The calculation of Sec. 4.2.1 can be generalized to black strings lying on flat submanifolds with a more general extrinsic curvature than just non-vanishing ${K_{zz}}^{(r\cos\theta)}$. Ref. [65] studied the gravitational field of a black string lying on the $r = 0$ submanifold of

$$\begin{aligned} ds^2 = &-\left(1 + C_t\frac{2r\cos\theta}{R}\right)dt^2 + \left(1 + C_z\frac{2r\cos\theta}{R}\right)dz^2 \\ &+ \left(1 - \frac{C_t + C_z}{n}\frac{2r\cos\theta}{R}\right)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\Omega^2_{(n)})\right] + \mathcal{O}\left((r\cos\theta)^2\right)\,. \end{aligned} \tag{4.2.20}$$

Note that the metric induced on the worldvolume is flat, and the non-vanishing components of its extrinsic curvature read

$${K_{tt}}^{(r\cos\theta)} = \frac{C_t}{R}\,, \qquad {K_{zz}}^{(r\cos\theta)} = -\frac{C_z}{R}\,. \tag{4.2.21}$$

[11] See Eq. (4.2.23) for a motivation for this.

This family of worldvolumes can be used to perturbatively study black rings in (A)dS space or in Schwarzschild-Tangherlini black hole backgrounds (in which case one is studying Black Saturns in a perturbative regime).

The equilibrium condition for the boosted string lying on $r = 0$ solving (2.1.7) reads

$$\sinh^2\alpha = \frac{C_z + (n+1)C_t}{n(C_z - C_t)}\,, \tag{4.2.22}$$

and the linear blackfold stress-energy tensor takes the form (4.2.2) with the boost (4.2.22). A calculation very similar to the one explained in Sec. 4.2.1 on the approximate solution found in [65] reveals the induced dipole induced by such extrinsic curvature. To write it down in a compact way it is useful to first consider

$$K_{ab}{}^{\rho}d^{ab}{}_{\rho} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}\frac{r_0^2}{R}\left[-(3n+4)(C_t^2 + C_z^2) - 2(n^2+3n+4)C_tC_z\right]\xi(n)\,, \tag{4.2.23}$$

which is $\tilde{k}_2-$invariant thanks to the leading extrinsic equation of motion, $K_{ab}{}^{\rho}B^{ab} = 0$. The symmetry of this expression under $C_t - C_z$ exchange motivates a parametrization of the gauge ambiguity[12] such that this symmetry is apparent in the $d^{ab\mu}$ object,

$$k_2 = \frac{\left[(n+1)C_t + (n+3)C_z\right]\left(2C_z + (n+2)C_t\right)}{2(C_z - C_t)}\,\xi(n) + \frac{(n-1)\left[C_z + (n+1)C_t\right]}{2n(C_z - C_t)}\,\tilde{k}_2\,. \tag{4.2.24}$$

The induced dipole of a black string on this submanifold reads

$$d_{tt}{}^{(r\cos\theta)} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}\frac{r_0^2}{R}\left[-C_t(3n+4) - C_z(n^2+3n+4)\right]\xi(n) - \tilde{k}_2\frac{r_0^2}{R}B_{tt}\,, \tag{4.2.25a}$$

$$d_{tz}{}^{(r\cos\theta)} = -\tilde{k}_2\frac{r_0^2}{R}B_{tz}\,, \tag{4.2.25b}$$

$$d_{zz}{}^{(r\cos\theta)} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}\frac{r_0^2}{R}\left[C_z(3n+4) + C_t(n^2+3n+4)\right]\xi(n) - \tilde{k}_2\frac{r_0^2}{R}B_{zz}\,. \tag{4.2.25c}$$

4.2.3 The Young Modulus of Black Strings

The study of how strains induce stresses is the subject of elasticity theory. It is well known from elementary elasticity theory [78] that the bending of an elastic rod induces a stress on

[12] Here, as in Eq. (4.2.6c), k_2 is defined as the coefficient of the r_0^{2n+2}/r^{2n+1} term in the large r expansion of the $c(r)$ function, appearing in Eq. (B.15) in [65].

it that has opposite signs on the inner and outer side. This is so because under bending, the inner side is compressed and the outer side is stretched. Thus, in classical elasticity theory bent rods develop dipoles of stress. The geometric object capturing how strain varies on directions transverse to a bent rod is the extrinsic curvature as explained in Sec. 2.3.

Black strings also develop dipoles when bent, exhibiting elastic behavior. In the linear (Hookean) regime, the response coefficient relating stress and strain in non-relativistic elasticity theory is the Young modulus. The black strings we have studied are in such linear regime, as their dipole is a small deformation, of order $1/R$.

To characterize the elastic behavior of black strings we need to introduce a relativistic generalization of the Young modulus. Let us start by going back to classical elasticity theory in flat space, in which a bent rod develops, in the case $C_t = 0$ and $C_z = 1$, the stress

$$T_{zz} = Y \frac{r\cos\theta}{R}\,, \tag{4.2.26}$$

where Y is the Young modulus. If the rod has, to first approximation, a circular cross-section of radius r_0, the dipole of stress reads

$$d_{zz}{}^{(r\cos\theta)} = \int T_{zz}\, r\cos\theta \left(r^{n+1}\sin^n\theta\, dr\, d\theta\, d\Omega_{(n)}\right) = \frac{r_0^{n+4}}{(n+2)(n+4)} \frac{\Omega_{(n+1)}}{\Omega_{(n)}} \left(\frac{Y}{R}\right). \tag{4.2.27}$$

Motivated by this we now define the Young modulus of black strings through the formula

$$d_{ab}{}^{\rho} = \left[-\frac{r_0^{n+4}}{(n+2)(n+4)} \frac{\Omega_{(n+1)}}{\Omega_{(n)}}\right] Y_{ab}{}^{cd} K_{cd}{}^{\rho}\,. \tag{4.2.28}$$

For relativistic matter we need to use a tensor, accounting for necessary anisotropy of the worldvolume directions: one of the a, b directions is timelike and the rest are spacelike.

It is not the purpose of this work to carry out a deep study of the properties of the relativistic Young modulus that we just defined, $Y_{ab}{}^{cd}$. We note, however, that it should display the 'extra symmetry 2' ambiguity that $d_{ab}{}^{\rho}$ enjoys. By construction, for the result (4.2.25) $\tilde{k}_2 = 0$ yields

$$Y^{ttzz} = Y^{zztt}\big|_{\tilde{k}_2=0}\,, \tag{4.2.29}$$

which is a desirable property of such tensors in non-relativistic anisotropic media.

We close this section by collecting the measured components of Y^{abcd} from (4.2.25) at $\tilde{k}_2 = 0$:

$$\begin{aligned} Y^{tttt} = Y^{zzzz} &= \frac{\Omega_{(n)}(n+2)(n+4)}{16\pi G\, r_0^2}(3n+4)\xi(n)\,, \qquad Y^{tztt} = Y^{tzzz} = 0\,, \\ Y^{ttzz} = Y^{zztt} &= -\frac{\Omega_{(n)}(n+2)(n+4)}{16\pi G\, r_0^2}(n^2+3n+4)\xi(n)\,. \end{aligned} \tag{4.2.30}$$

The addition of flat directions to (4.2.4) or the corresponding case of (4.2.20) does not threaten the solutions to Einstein equations we have been considering in this section, and does not change the final result (4.2.30). Thus, much like the equation of state $\varepsilon = -(n+1)P$ is a property only of the codimension of black $p-$branes, so is the Young modulus, and expressions (4.2.30) are also valid for $p > 1$. Also, because calculations in this section are linear in $1/R$, one can include the effect of a more general extrinsic curvature by just adding the effects of different components up; Considerations in this section are also valid, for example, for black tori [63], in which the ρ index in $K_{ab}{}^{\rho}$ may not be aligned in all ab components of the extrinsic curvature.

4.2.4 Dipole corrections to the equilibrium of thin black rings

One of the main applications of the *blackfold approach* has been the construction of approximate black hole solutions. The dipole corrections to the stress-energy tensor of black branes that we have found modify the effective blackfold equations of motion, and can be used to compute next to leading order contributions to these approximate solutions. In this subsection we consider corrections to the approximate construction of thin black rings in flat space.

The relevant blackfold equation for stationary black rings is the extrinsic equation (2.1.7). It reads

$$B^{ab}K_{ab}{}^{\rho} = 0\,. \tag{4.2.31}$$

In flat space, where in the thin limit black rings live on flat submanifolds with $K_{zz}{}^{\rho}$, these equations become, as discussed around Eq. (4.2.3),

$$B_{zz} = 0 \tag{4.2.32}$$

We shall consider corrections to homogeneous, stationary configurations in backgrounds of the type (4.2.20). In these backgrounds, the only non-vanishing Christoffel symbols at $r = 0$ (apart from the usual coordinate pathology on the transverse sphere) have two parallel indices to the worldvolume:

$$u_a^{\nu}\, u_b^{\rho}\, \Gamma^{\mu}_{\nu\rho} = K_{ab}{}^{\mu}\,, \qquad \text{and} \qquad u_{\mu}^{a}\, u_b^{\nu}\, \Gamma^{\mu}_{\nu\rho} = -K^{a}{}_{b\rho}\,. \tag{4.2.33}$$

These, together with homogeneity, imply

$$\nabla_c d^{ab\mu} = -u_d^{\mu} K_c{}^{d}{}_{\rho} d^{ab\rho}\,, \qquad \nabla_c j^{a\mu\nu} = -u_d^{\mu} K_c{}^{d}{}_{\rho} j^{a\rho\nu} - u_d^{\nu} K_c{}^{d}{}_{\rho} j^{a\mu\rho}\,. \tag{4.2.34}$$

As has been stressed, this includes black rings in (A)dS and Black Saturns [32] with a static central black hole, but excludes other cases, as that of Black Rings in Taub-NUT [66].

Under these assumptions, the only non-trivial equation of motion is the extrinsic one (4.1.17), which reduces to

$$m^{ab}K_{ab}{}^{\mu} - \perp^{\mu}{}_{\sigma} u_a^{\nu} S^{\lambda\rho a} R^{\sigma}{}_{\nu\lambda\rho} = 0\,. \tag{4.2.35}$$

In terms of B^{ab}, $j^{a\mu\nu}$ and $d^{ab\rho}$ these become

$$\left(B^{ab} + d^{ca}{}_{\rho} K_c{}^{b\rho}\right) K_{ab}{}^{\mu} - \perp^{\mu}{}_{\sigma} u_a^{\nu} \left(\frac{1}{2} j^{a\lambda\rho} - d^{ab\lambda} u_b^{\rho}\right) R^{\sigma}{}_{\nu\lambda\rho} = 0 \,. \tag{4.2.36}$$

We will now use this formula to draw some predictions, but it is worth noticing that we will not have complete predictability. The reason is that, because the dipole is an induced effect, the $d^{ab\rho}$ corrections that are derived from it are two orders away from the leading order in r_0/R, instead of just one, and a complete accounting of effects at this order would require extending the calculations of [14, 65] to one order beyond.

For rings in flat space or (A)dS there are no intrinsic angular momentum corrections to (4.2.31). This is expected, as these should be insensitive to the orientation of the intrinsic angular momentum (which is in a plane transverse to that of the ring), and thus should be a j^2 contribution. When $j^{a\mu\nu} = 0$, the leading correction for rings in flat space to (4.2.31) is the dipole correction

$$B_{zz} = -d^{zz(r\cos\theta)} K^{z}{}_{z(r\cos\theta)} \,, \tag{4.2.37}$$

which implies that the critical boost is

$$\sinh^2\alpha = \frac{1}{n} + \frac{r_0^2}{R^2}\frac{3n+4}{n}\xi(n) \,. \tag{4.2.38}$$

In the large n limit this becomes

$$\sinh^2\alpha = \frac{1}{n} + \frac{r_0^2}{R^2}\left(\frac{3}{n^2} + \dots\right) + \dots \,, \tag{4.2.39}$$

highlighting the fact that, at large n, the corrections to the single pole account of this type of black holes are further suppressed in $1/n$. Arguments along these lines have been given in the past [79] stating that this supression is due to the shorter range of the gravitational interaction at large n.

The correction term in (4.2.38) is two powers in r_0/R away from the leading contribution. This prevents us from computing the conserved charges and thermodynamic properties of these black rings, as we do not have enough data to carry this out; a naive calculation of these properties to the order at which the result (4.2.38) is relevant, that is to $(r_0/R)^2$, gives $\tilde{k}_2-$dependent charges, which is unphysical. Only the next order computation in the MAE will cancel the gauge dependence by introducing the unknown ambiguous part in B_{ab}, which by Eq. (4.1.7) will be

$$B_{ab} \to B_{ab} + \left(B_{ab} K^{c}{}_{c\rho} + 2B^{c}{}_{(a} K_{b)c\rho}\right) \epsilon^{\rho} \,, \tag{4.2.40}$$

and is of order $(r_0/R)^2$.

Note, however, that the leading equilibrium condition of black rings in flat space, $B_{zz} = 0$, implies $\delta_2 B_{zz} = 0$. One then expects Eq. (4.2.38) to hold to order $(r_0/R)^2$, and we trust this equation to the order we have written it. This is not the case for rings in the more general backgrounds (4.2.20), and this is why we do not write down a corrected equilibrium condition for (4.2.22). In conclusion, we cannot at this point predict corrections to the conserved charges of the black rings at order $(r_0/R)^2$. However, we would like to stress that at order r_0/R we do have a prediction, namely that the black rings do not receive corrections for $n > 2$.

4.3 Black string electroelasticity and charged black rings

In this section we generalize the prescription of Sec. 4.2 to charged dilatonic black strings. In addition to the elastic moduli, we find a new response coefficient of black branes making a solid connection between the physics of piezoelectrics and gravity. This new effect can be intuitively understood if one imagines slightly curving a charged black string of finite thickness inducing a higher concentration of charged black material in the inner surface and a depletion in the outer surface. A varying concentration of matter due to the compression and stretching of the material on opposite sides induces a bending moment of dipolar character as in classical Hookean elasticity theory and, since the matter is charged, it also induces an electric dipole moment that describes the response of the charged string to the mechanical stress. Electric fields induced by mechanical stresses are the basic feature of piezoelectrics and their behavior is governed by the physics of electroelastic materials.

4.3.1 Electroelasticity of black branes

The dipole moment of worldvolume stress-energy $d^{ab\rho}$ introduced in (4.1.22) is not a priori constrained. Under the expectation that bent black branes will behave like elastic solids, as it was observed in (4.2.28), we assume the following relation between the dipole moment and the strain in transverse directions ${K_{cd}}^{\rho}$:

$$d^{ab\rho} = \tilde{Y}^{abcd} {K_{cd}}^{\rho} \ , \tag{4.3.1}$$

where $\tilde{Y}^{abcd}$ are the elastic moduli that characterize the brane response to the bending[13]. The assumption (4.3.1) has a direct analogy with Hookean classical elasticity theory. In the following we make a further assumption, namely that a bent charged brane will behave like a piezoelectric material in the manner dictated by linear electroelasticity theory:

$$p^{a\rho} = \tilde{\kappa}^{abc} {K_{bc}}^{\rho} \ , \tag{4.3.2}$$

where $\tilde{\kappa}^{abc}$ are the piezoelectric moduli that capture the response of the bent charged material due to electroelastic deformations. In the case of a point particle Eq. (4.3.2) should

[13] Here we use $\tilde{Y}^{abcd}$ instead of Y^{abcd} as given in (4.2.28) to make the notation less cumbersome.

be interpreted as the effect of polarization due to acceleration. To further motivate this interpretation we consider the equations of motion for a spinless charged string bent over a circle of radius R in Minkowski space-time:

$$D_a m^{ab} = 0,\ m^{ab} K_{ab}{}^{\rho} = 0,\ D_a J^a = 0 \ , \tag{4.3.3}$$

where $m^{ab} = B^{ab} + \tau^{ab}$ and $J^a = J^a_{(0)} + \Upsilon^a_{(1)}$ are the effective worldvolume stress-energy tensor and current respectively. The linear electroelastic corrections are

$$\tau^{ab} = \tilde{Y}^{abcd} K_{cd}{}^{\rho} K_{\rho} \ , \quad \Upsilon^a_{(1)} = \tilde{\kappa}^{abc} K_{bc}{}^{\rho} K_{\rho} \ , \tag{4.3.4}$$

which arise along the direction set by the mean extrinsic curvature K^{ρ}. Eqs. (4.3.3) are the analog of Eq. (4.2.35) for pole-dipole branes with 0-brane charge in flat space. Below, we provide explicit examples of charged dilatonic branes that satisfy these requirements.

4.3.2 Measurement of Piezoelectric Moduli

In order to measure the piezoelectric moduli from a gravitational solution we consider asymptotically flat charged dilatonic black branes in EMD theory with action

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - 2(\partial\phi)^2 - \frac{e^{-2a\phi}}{4} F^2 \right] \ . \tag{4.3.5}$$

In Sec. 4.2 we measured the elastic moduli of uncharged black branes by extracting the coefficients $d^{ab\rho}$ from the stress-energy tensor measured far away from the brane horizon where the weak field approximation is valid and using Eq. (4.3.1). Similarly, we extract the coefficients $p^{a\rho}$ by determining how the gauge field A_{μ} falls off at infinity and obtain $\tilde{\kappa}^{abc}$ using Eq. (4.3.2). In general, the current $\hat{J}^{\mu}$ and the gauge field are related through the equation of motion

$$\nabla_{\nu} \left(e^{-2a\phi} F^{\mu\nu} \right) = 16\pi G \hat{J}^{\mu} \, , \tag{4.3.6}$$

which in Lorenz gauge, $\nabla^{\mu} A_{\mu} = 0$, and for asymptotically flat space-times gives rise to the linearized equation

$$e^{-2a\phi_0} \partial^{\nu} \partial_{\nu} A^{\mu} = -16\pi G \hat{J}^{\mu} \, , \tag{4.3.7}$$

where ϕ_0 is the value of the dilaton far away from the brane horizon. The dipole term $p^{a\rho}$ can then be obtained by introducing the expansion (4.1.38) into Eq. (4.3.7).

Charged dilatonic black strings

We now focus on a large class of asymptotically flat charged dilatonic black strings carrying 0-brane charge which can be obtained by performing an uplift of the neutral black string solution in $D = n + 4$ dimensions, followed by a boost with rapidity α and a Kaluza-Klein (KK) reduction along a Killing direction. The resulting metric is given by [69]

$$ds^2 = -\frac{f}{h^A}dt^2 + h^B\left(\frac{dr^2}{f} + r^2 d\Omega^2_{(n+1)} + dz^2\right) , \tag{4.3.8}$$

with $f(r) = 1 - \frac{r_0^n}{r^n}$ and $h(r) = 1 + \frac{r_0^n}{r^n}\sinh^2\alpha$, and the gauge and dilaton fields read

$$A_t(r) = -\sqrt{N}\frac{r_0^n}{r^n h(r)}\sinh\alpha\cosh\alpha , \tag{4.3.9}$$

$$\phi(r) = -\frac{1}{4}Na\log h(r) . \tag{4.3.10}$$

This solution generating technique leaves the horizon regular and yields a specific value for the dilaton coupling $a^2 = \frac{2(n+3)}{n+2}$ as well as parameters $A = \frac{n+1}{n+2}$, $B = \frac{1}{n+2}$ and $N = A+B = 1$. After KK reduction, the rapidity α gains the interpretation of a charge parameter. The bent version of (4.3.8)-(4.3.10) can be obtained in a similar fashion using its neutral counterpart [14] as a seed solution, i.e., the solution presented in the beginning of Sec. 4.2. Since we only need to know how the dipole corrections to the fields $g_{\mu\nu}$, A_μ, ϕ behave at infinity we focus on the large r-asymptotics. For use below we recall the decomposition of the metric components of the neutral bent black string given in (4.2.8). We now parametrize the dipole contribution as $h^{(D)}_{\mu\nu} = f^{(D)}_{\mu\nu}\varepsilon\, r_0^{n+1}\left(\frac{\cos\theta}{r^{n+1}}\right)$, with coefficients $f^{(D)}_{\mu\nu}$ obtained from Sec. 4.2.1,

$$f^{(D)}_{tt} = (n+1)\tilde{k}_2 - n(n+2)\xi(n) , \tag{4.3.11}$$

$$f^{(D)}_{tz} = -\sqrt{n+1}\,\tilde{k}_2 , \quad f^{(D)}_{zz} = \tilde{k}_2 + 2n\xi(n) , \tag{4.3.12}$$

$$f^{(D)}_{rr} = f^{(D)}_{\Omega\Omega} = \tilde{k}_2 - (n+4)\xi(n) . \tag{4.3.13}$$

where $\tilde{k}_2$ is the residual gauge freedom associated with the 'extra symmetry 2' and $\xi(n)$ as given in (4.2.7). Turning to the charged case, we adopt a similar decomposition for the gauge field

$$A_\mu = A^{(M)}_\mu + A^{(D)}_\mu + \mathcal{O}\left(r^{-n-2}\right) . \tag{4.3.14}$$

Defining the asymptotic coefficients $a^{(D)}_\mu$ of the gauge field by $A^{(D)}_\mu = a^{(D)}_\mu \varepsilon\, r_0^{n+1}\left(\frac{\cos\theta}{r^{n+1}}\right)$, one then finds after KK reduction,

$$\begin{aligned} a^{(D)}_t &= -\sinh\alpha_k\cosh\alpha_k\, f^{(D)}_{tt} \\ a^{(D)}_z &= -\sinh\alpha_k\, f^{(D)}_{tz} , \end{aligned} \tag{4.3.15}$$

where $n\sinh^2\alpha = (n+1)\sinh^2\alpha_k$. Using this on the left hand side of Eq. (4.3.7) together with the expansion (4.1.38) and the fact that $\phi_0 = 0$ yields

$$\nabla^2_\perp A^{(D)}_\nu = 16\pi G p_\nu{}^{r_\perp}\partial_{r_\perp}\delta^{(n+2)}(r) , \tag{4.3.16}$$

where the Laplacian operator is taken along transverse directions to the worldvolume and $r_\perp = r\cos\theta$. For the given configuration at hand one finds the electric dipole moment

$$p_a{}^{r_\perp} = -\frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G R} a_a^{(D)} \quad , \quad a = t, z\,. \tag{4.3.17}$$

The dipole moment of stress-energy tensor $d^{ab\rho}$ can also be obtained using the expansion (4.1.1) and the linearized equation

$$\nabla_\perp^2 \bar{h}_{\mu\nu}^{(D)} = 16\pi G d_{\mu\nu}{}^{r_\perp} \partial_{r_\perp} \delta^{(n+2)}(r)\,, \tag{4.3.18}$$

where the dipole perturbation $\bar{h}_{\mu\nu}^{(D)}$ of the bent charged black string is defined in analogy with (4.2.8). This leads to the components

$$d_{tt}{}^{r_\perp} = -\frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G R}\left(\cosh^2\alpha_k f_{tt}^{(D)} + f_{\Omega\Omega}^{(D)}\right)\,, \tag{4.3.19}$$

$$d_{tz}{}^{r_\perp} = -\frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G R}\left(\cosh\alpha_k f_{tz}^{(D)}\right)\,, \tag{4.3.20}$$

$$d_{zz}{}^{r_\perp} = -\frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G R}\left(f_{zz}^{(D)} - f_{\Omega\Omega}^{(D)}\right)\,, \tag{4.3.21}$$

expressed in terms of the asymptotic coefficients (4.3.11) of the neutral bent black string solution.

Response coefficients and corrections to black rings

The leading order effective worldvolume stress-energy tensor is of the perfect fluid form [76, 69],

$$T^{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n(1 + N\sinh^2\alpha) u^a u^b - \eta^{ab}\right)\,, \tag{4.3.22}$$

where u^a is the local boost on the string. Using this in the second equation of (4.3.3) one finds the leading order critical boost $u^a = [\cosh\beta_{(0)}, -\sinh\beta_{(0)}]$ with $\sinh^2\beta_{(0)} = (n\cosh^2\alpha)^{-1}$. To lowest order in ε the electric current takes the form $J^a_{(0)} = \mathcal{Q} u^a$ with

$$\mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N}\, r_0^n \sinh\alpha\cosh\alpha \tag{4.3.23}$$

being the charge density to the same order. The piezoelectric moduli are then obtained from Eqs. (4.3.2) and (4.3.17) as

$$\tilde{\kappa}^{tzz} = \left(\tilde{k}_2 - \frac{2(n+2)}{n+1}\xi(n)\right) r_0^2 J^t_{(0)}\,, \tag{4.3.24}$$

$$\tilde{\kappa}^{zzz} = -\tilde{k}_2 r_0^2 J^z_{(0)}\,, \tag{4.3.25}$$

expressed in terms of the critical current. Similarly, the elastic moduli can be obtained using Eqs. (4.3.1) and (4.3.19)

$$\tilde{Y}^{ttzz} = \tilde{k}_2 r_0^2 T^{tt}_{(0)} - \frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G}\left(\frac{n^2(n+2)}{n+1}\sinh^2\alpha + n^2 + 3n + 4\right)\xi(n)\,, \quad (4.3.26)$$

$$\tilde{Y}^{tzzz} = -\tilde{k}_2 r_0^2 T^{tz}_{(0)}\,, \quad (4.3.27)$$

$$\tilde{Y}^{zzzz} = \frac{\Omega_{(n+1)} r_0^{n+2}}{16\pi G}(3n+4)\,\xi(n)\,, \quad (4.3.28)$$

expressed in terms of the critical stress-energy tensor. The neutral case obtained in Sec. 4.2 is reproduced when α is taken to zero. Using Eqs. (4.3.3) we can compute the correction to the critical boost of a thin charged dilatonic black ring by bending the charged dilatonic black string described by the response coefficients (4.3.24)- (4.3.25) and (4.3.26)-(4.3.28). This yields:

$$\sinh^2\beta = (n\cosh^2\alpha)^{-1}\left(1 + \varepsilon^2(3n+4)\xi(n)\right)\ , \quad (4.3.29)$$

and constitute a prediction for $n > 2$ where the probe approximation is valid. The result (4.3.29) reduces to (4.2.38) when the charge parameter α is taken to zero.

4.4 Spin corrections for Myers-Perry black holes

Myers-Perry (MP) black holes exhibit ultra-spinning regimes where the horizon pancakes along one of the planes of rotation [64], a behavior which was realized also to be shown by higher-dimensional Kerr-(A)dS black holes, as derived in Sec. (3.5.2). Both of these cases can be captured within the blackfold framework (see Sec. (3.5.2)). The usual method, as explained in Sec. (3.5.2), consists in focusing near the axis of rotation and taking a limit such that the horizon looks like a boosted Schwarzschild membrane. On the other hand, having been able to describe this particular limit using the *blackfold approach* implies that the entire horizon must locally have the geometry of a boosted Schwarzschild membrane. This has not been considered in the literature so far and in App. D we show how to fill this gap by precisely taking a regular ultra-spinning limit everywhere over the horizon.

This section begins with a demonstration of the existence of a similar regime for MP black holes with one non-zero transverse angular momentum. By taking the limit in which the horizon flattens out along one of the planes of rotation and approaching any point on the horizon it is shown that the horizon geometry is locally that of a boosted MP membrane. The extended blackfold formalism presented in Sec. 4.1 should be able to capture this behavior and, in fact, by reducing Eqs. (4.1.24)-(4.1.27) to the case under consideration together with a detailed analysis of the thermodynamic properties of such blackfold geometry, this is shown to be the case. The same type of analysis can be carried out for higher dimensional Kerr-(A)dS black holes and it is presented in App. E.

4.4.1 Refined ultra-spinning limit

Consider the metric of a MP black hole with two angular momenta in $n+5$ dimensions [55]

$$\begin{aligned} ds^2 &= -dt^2 + \sum_{i=1}^{2}\left[a_i^2 d\mu_i^2 + (r^2+a_i^2)\mu_i^2 d\phi_i^2\right] + \frac{\mu}{r^{n+2}\Pi F}(dt - \sum_{i=1}^{2} a_i \mu_i^2 d\phi_i^2)^2 \\ &\quad + \frac{\Pi F}{\Pi - \frac{\mu}{r^{n+2}}} dr^2 + r^2\left[d\theta^2 + \cos^2\theta(d\psi^2 + \cos^2\psi d\Omega^2_{(n-1)})\right] , \end{aligned} \tag{4.4.1}$$

where,

$$\mu_1 = \sin\theta, \qquad \mu_2 = \cos\theta\sin\psi \ , \tag{4.4.2}$$

$$\Pi = \prod_{i=1}^{2}(1+\frac{a_i^2}{r^2}), \qquad F = 1 - \sum_{i=1}^{2}\frac{a_i^2\mu_i^2}{r^2+a_i^2} \ . \tag{4.4.3}$$

The event horizon is located at $r = r_+$ with r_+ being the largest positive real root of $r^{n+2}\Pi - \mu = 0$. For clarity of notation, in what follows we label a_1 and a_2 as $a_1 \equiv a$ and $a_2 \equiv b$. The aim of this section is to show that there exists an allowed region of parameters where, near the horizon, the metric (4.4.1) looks locally like a boosted MP membrane. To this end, we take the ultra-spinning limit in the first angular momentum parametrized by a, that is, $a \gg r_+$. Since we want to capture the dynamics on the transverse plane b is kept finite such that $a \gg b$. Within such restricted phase space the blackfold looks much like the one encountered in the singly-spinning case[14]: a disc with center at $\theta = 0$, radius a and boundary at $\theta = \pi/2$.

The assumption of this hierarchy of scales is sufficient to recover the metric of a black membrane near the center of the disc but not everywhere close to the horizon. Instead, the requirement $r \ll a\cos\theta$ is needed. Under this assumption we find

$$\Pi \simeq \frac{a^2}{r^4}(r^2+b^2), \qquad F \simeq 1 - \mu_1^2 - \frac{b^2\mu_2^2}{r^2+b^2}, \qquad g_{rr} \simeq \frac{(1-\mu_1^2)(r^2+b^2) - b^2\mu_2^2}{r^2+b^2-\frac{\tilde{\mu}}{a^2 r^{n-2}}} \ ,$$

$$\frac{\mu}{r^{n+2}\Pi F} \simeq \frac{\mu}{a^2 r^{n-2}[(1-\mu_1^2)(r^2+b^2) - b^2\mu_2^2]} \ .$$

Now we introduce the coordinate ρ_1 as

$$\rho_1 = a\sin\theta \ , \tag{4.4.4}$$

which can be seen as the radius on the disc. Furthermore, assuming also that $b \ll a\cos\theta$ we obtain

$$\sum_{i=1}^{2} a_i^2 d\mu_i^2 + r^2 d\theta^2 + r^2\cos^2\theta d\psi^2 \simeq d\rho_1^2 + \cos^2\theta(r^2 + b^2\cos^2\psi)d\psi^2 \ . \tag{4.4.5}$$

[14] See App. D for details.

As we are after a description of the local geometry of the horizon we need to approach it at any point, thus we consider the metric near a fixed angle θ_* (i.e., at a given radius of the disc $\rho_1 = \rho_{1*}$). The necessary requirements become $r \ll a\cos\theta_*$ and $b \ll a\cos\theta_*$. To make contact with the metric of a MP membrane written in its usual form, it is convenient to define

$$\tilde{r} = r\cos\theta_*, \qquad \tilde{b} = b\cos\theta_*, \qquad \tilde{r}_0^n = \frac{\mu(\cos\theta_*)^n}{a^2} \ ,$$

$$\Sigma = \tilde{r}^2 + \tilde{b}^2\cos^2\psi, \qquad \Delta = \tilde{r}^2 + \tilde{b}^2 - \frac{\tilde{r}_0^n}{\tilde{r}^{n-2}} \ . \tag{4.4.6}$$

Using these definitions we obtain

$$g_{rr}dr^2 \simeq \frac{\Sigma}{\Delta}d\tilde{r}^2, \qquad \frac{\mu}{r^{n+2}\Pi F} \simeq \frac{\tilde{r}_0^n}{\cos^2\theta_*\tilde{r}^{n-2}\Sigma} \ . \tag{4.4.7}$$

Next, we introduce the coordinate z by

$$z = \rho_{1*}\phi_1 = a\sin\theta_*\phi_1 \ , \tag{4.4.8}$$

which parametrizes the angular direction on the disc. The metric (4.4.1) near the horizon is now seen to become:

$$\begin{aligned} ds^2 &= -dt^2 + d\rho_1^2 + dz^2 + \frac{\tilde{r}_0^n}{\Sigma\tilde{r}^{n-2}}\left(\frac{dt}{\cos\theta_*} - \tan\theta_* dz - \tilde{b}\sin^2\psi d\phi_2\right)^2 + \frac{\Sigma}{\Delta}d\tilde{r}^2 + \Sigma d\psi^2 \\ &\quad +(\tilde{r}^2 + \tilde{b}^2)\sin^2\psi d\phi_2^2 + \tilde{r}^2\cos^2\psi d\Omega_{n-1}^2 \ . \end{aligned} \tag{4.4.9}$$

This geometry corresponds to the Myers-Perry membrane,

$$\begin{aligned} ds^2 &= -dt^2 + d\rho_1^2 + dz^2 + \frac{\tilde{r}_0^n}{\Sigma\tilde{r}^{n-2}}(dt - \tilde{b}\sin^2\psi d\phi_2)^2 + \frac{\Sigma}{\Delta}d\tilde{r}^2 + \Sigma d\psi^2 \\ &\quad +(\tilde{r}^2 + \tilde{b}^2)\sin^2\psi d\phi_2^2 + \tilde{r}^2\cos^2\psi d\Omega_{n-1}^2 \ , \end{aligned} \tag{4.4.10}$$

boosted along the z direction with the boost

$$\begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\cos\theta_*} & -\tan\theta_* \\ -\tan\theta_* & \frac{1}{\cos\theta_*} \end{pmatrix} \begin{pmatrix} \tilde{t} \\ \tilde{z} \end{pmatrix} \ . \tag{4.4.11}$$

Applying the boost (4.4.11) to (4.4.10) and removing the tildes from $\tilde{t}$ and $\tilde{z}$ leads to the metric (4.4.9). In turn, (4.4.11) corresponds to the Lorentz boost[15]

$$V = \sin\theta_* = \frac{\rho_1}{a}, \qquad \tilde{\gamma} = \frac{1}{\sqrt{1-V^2}} = \frac{1}{\cos\theta_*}, \qquad \tilde{\gamma}V = \tan\theta_* \ . \tag{4.4.12}$$

[15]The relation between θ_* and the rapidity η is $\tan\theta_* = \sinh\eta$.

In the regime under consideration the MP black hole given in (4.4.1) has angular velocity along the ϕ_1 direction,

$$\Omega_1 \simeq \frac{1}{a} \ . \tag{4.4.13}$$

Thus, in the blackfold description, the ultra-spinning MP black hole with transverse angular momentum is a MP membrane which is rigidly rotating with constant angular velocity,

$$V = \rho_1 \Omega_1 \ , \tag{4.4.14}$$

as in the singly-spinning case.

4.4.2 Blackfold pole-dipole equations with non-zero transverse angular momentum

As seen above, there exists a limit in which the MP black hole can be locally seen as a black membrane, hence, it should be possible to describe it using the formalism of Sec. 4.1. Here, we construct such blackfold geometry by solving Eqs.(4.1.24)-(4.1.27) for a disc-like topology with internal spin current.

Consider $n+5$-dimensional flat spacetime with metric

$$ds^2 = -dt^2 + d\rho_1^2 + \rho_1^2 d\phi_1^2 + ds_\perp^2 + \sum_{i=1}^{n} dx_i^2 \ , \tag{4.4.15}$$

where $ds_\perp^2$ is the metric on the transverse 2-plane written in the form

$$ds_\perp^2 = d\rho_2^2 + \rho_2^2 d\phi_2^2 \ . \tag{4.4.16}$$

In this we embed a membrane as

$$t = \sigma^0, \qquad \rho_1 = \sigma^1, \qquad \phi_1 = \sigma^2, \qquad \rho_2 = 0, \qquad x_i = 0 \ , \tag{4.4.17}$$

which gives rise to the induced metric

$$\gamma_{ab} d\sigma^a d\sigma^b = -dt^2 + d\rho_1^2 + \rho_1^2 d\phi_1^2 \ . \tag{4.4.18}$$

The stress-energy tensor of the boosted MP membrane can be read off from (4.4.10) and takes the perfect fluid form (2.2.1), with energy density and pressure given by

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G}(n+1)\tilde{r}_0^n, \qquad P = -\frac{\Omega_{(n+1)}}{16\pi G}\tilde{r}_0^n \ . \tag{4.4.19}$$

Since we are trying to construct a black hole solution with a horizon Killing vector field of the form

$$\mathbf{k} = \frac{\partial}{\partial t} + \Omega_1 \frac{\partial}{\partial \phi_1} + \Omega_2 \frac{\partial}{\partial \phi_2} \ , \tag{4.4.20}$$

with constant Ω_1 and Ω_2[16], then, the fluid velocity must be of the form $u^a = k^a/|k|$, with non-vanishing components

$$u^t = \tilde{\gamma}, \qquad u^{\rho_1} = 0, \qquad u^{\phi_1} = \tilde{\gamma}\Omega_1, \qquad \tilde{\gamma} = \frac{1}{\sqrt{1-\rho_1^2\Omega_1^2}} \ . \tag{4.4.21}$$

Moreover, the boosted MP membrane spin current can also be obtained from (4.4.10) and in this coordinate system simply reads

$$j^{a\nu\rho} = \frac{\Omega_{(n+1)}}{8\pi G}\frac{\tilde{b}\tilde{r}_0^n u^a}{\rho_2}\delta^{\nu}_{\rho_2}\delta^{\rho}_{\phi_2} \ . \tag{4.4.22}$$

Since the embedding (4.4.17) is completely flat we must have ${K_{ab}}^{\rho} = 0$. According to the considerations of Sec. 4.2 this immediately implies $d^{ab\rho} = 0$. Physically, this is so because there is no bending taking place and hence no elastic forces playing a role. Also, there is no extra contribution to the boundary stress-energy tensor from (4.4.10) leading to a vanishing $B^{\mu\nu a}$ everywhere. This is most likely due to the fact that the blackfold boundary is regular, containing no extra stress-energy sources. Finally, we need to know the transverse angular velocity Ω_2 as a function of $\tilde{r}_0^n$ and $\tilde{b}$. This can be seen as an additional equation of state and is given by

$$\Omega_2(\tilde{r}_0^n, \tilde{b}) = \frac{1}{\tilde{\gamma}}\frac{\tilde{b}}{\tilde{r}_+(\tilde{r}_0^n, \tilde{b})^2 + \tilde{b}^2} \ , \tag{4.4.23}$$

where $\tilde{r}_+(\tilde{\mu}, b)$ is found as the highest real root of $\Delta(\tilde{r})$, see (4.4.6), thus,

$$\tilde{r}_+^2 + \tilde{b}^2 = \frac{\tilde{r}_0^n}{\tilde{r}_+^{n-2}} \ . \tag{4.4.24}$$

With this in hand, the pole-dipole blackfold equations (4.1.24)-(4.1.27), reduce in this case to:

$$B^{ab}{K_{ab}}^{\rho} = 0 \ , \quad D_a B^{ab} = 0 \ , \quad \perp^{\mu}_{\lambda}\perp^{\mu}_{\rho}\nabla_a j^{a\nu\rho} = 0 \ , \quad B^{ab}u^{\mu}_a n_b|_{\partial\mathcal{W}_3} = 0 \ , \quad j^{a\mu\nu}n_a|_{\partial\mathcal{W}_3} = 0 \ . \tag{4.4.25}$$

The first and last equations in (4.4.25) are trivially satisfied due to the vanishing of the extrinsic curvature tensor and to the vanishing of the ρ_1 component of the fluid velocity u^a respectively. The boundary condition $B^{ab}u^{\mu}_a n_b|_{\partial\mathcal{W}_3} = 0$ requires that, at the boundary, the fluid must be moving with the speed of light, i.e., $\rho_1|_{\partial\mathcal{W}_3} = \Omega_1^{-1}$.

Furthermore, assuming that ϵ and P only depend on ρ_1, we find from the conservation of the stress-energy tensor

$$D_a B^{ab} = (\epsilon + P)\dot{u}^b + \partial^b P \ , \tag{4.4.26}$$

[16] The constancy of Ω_1 and Ω_2 need not be imposed. In fact, it is a consequence of the requirement of stationarity. A derivation of these conditions can be accomplished and it will be presented elsewhere.

which is solved by,

$$\tilde{r}_0(\rho_1) = \tilde{r}_0(0)\sqrt{1-\rho_1^2\Omega_1^2}\ . \tag{4.4.27}$$

Since $j^{a\nu\rho}$ is independent of t and ϕ_1, the conservation equation $\perp^{\mu}_{\lambda}\perp^{\mu}_{\rho}\nabla_a j^{a\nu\rho}=0$ is trivially obeyed, *i.e.*, it does not constrain $\tilde{b}$ as a function of ρ_1. According to (4.4.20) we now use the constancy of Ω_2 over the blackfold. As a function of ρ_1 we see from Eq. (4.4.27) that $\tilde{r}_0^n$ is proportional to $\gamma(\rho_1)^{-n}$. Using Eq. (4.4.23) and Eq. (4.4.24) we see that both $\tilde{b}$ and $\tilde{r}_0^n$ should be proportional to $\gamma(\rho_1)^{-1}$. Thus, we find

$$\tilde{b}(\rho_1) = \tilde{b}(0)\sqrt{1-\rho_1^2\Omega_1^2}\ . \tag{4.4.28}$$

In order to show that this blackfold solution corresponds indeed to the limit taken above in the doubly-spinning MP metric (4.4.1) we proceed by computing its thermodynamic properties.

4.4.3 Thermodynamic quantities

The thermodynamic properties for the analytic solution given in (4.4.1) in the ultra-spinning regime can be obtained from reference [91] and read,

$$M = \frac{\Omega_{(n+3)}}{16\pi G}\mu(n+3), \qquad J_1 = \frac{\Omega_{(n+3)}}{8\pi G}\mu a, \qquad J_2 = \frac{\Omega_{(n+3)}}{8\pi G}\mu b\ , \tag{4.4.29}$$

with angular velocities

$$\Omega_1 = \frac{1}{a_1}, \qquad \Omega_2 = \frac{b}{r_+^2+b^2}\ , \tag{4.4.30}$$

while the temperature and entropy are given by,

$$S = \frac{\Omega_{(n+3)}}{4G}\frac{a^2\mu}{r_+}, \qquad T = \frac{1}{4\pi r_+}\left(n - \frac{2b^2}{r_+^2+b^2}\right)\ . \tag{4.4.31}$$

We want to show that this is correctly reproduced from the blackfold description. To this aim, we use the expressions (4.1.31)-(4.1.33) together with (4.4.22) and find,

$$M = \frac{\Omega_{(n+3)}}{16\pi G}\frac{\tilde{r}_0^n(0)}{\Omega_1^2}(n+3), \qquad J^1 = \frac{\Omega_{(n+3)}}{8\pi G}\frac{\tilde{r}_0^n(0)}{\Omega_1^2}\frac{1}{\Omega_1}, \qquad J_\perp^2 = \frac{\Omega_{(n+3)}}{8\pi G}\frac{\tilde{r}_0^n(0)}{\Omega_1^2}\tilde{b}(0)\ . \tag{4.4.32}$$

We see that we find perfect agreement between the above quantities and those presented in (4.4.29)-(4.4.30) if we identify $\tilde{r}_0^n(0)/\Omega_1^2=\mu$, $\Omega_1=a^{-1}$ and $\tilde{b}(0)=b$. To compute the entropy and temperature we use the method described in Sec. 4.1.3. We first start by computing the product TS using the Smarr relation (4.1.35), yielding,

$$TS = \frac{\Omega_{(n+3)}}{16\pi G}\frac{\tilde{r}_0^n(0)}{\Omega_1^2}\left(n - \frac{2\tilde{b}(0)^2}{r_+^2+\tilde{b}(0)^2}\right)\ . \tag{4.4.33}$$

All the quantities given in (4.4.32) and the product above can be parametrized in terms of r_+ and $b(0)$ using Eq. (4.4.24). According to Eq. (4.1.36) we obtain a set of two equations, for which the solution is,

$$T = \frac{\tilde{C}}{r_+}\left(n - \frac{2\tilde{b}(0)^2}{r_+^2 + \tilde{b}(0)^2}\right) \ . \tag{4.4.34}$$

The constant $\tilde{C}$ can be fixed by requiring the right result in the infinitely thin limit, i.e., when $\tilde{b}(0) = 0$. This implies $\tilde{C} = 1/4\pi$. It is easy to check that this matches with the temperature given in (4.4.31). The entropy then follows by using Eq. (4.4.33). In this way, we have indeed shown that the ultra-spinning MP black hole with one transverse angular momentum is accurately described within the blackfold framework in the pole-dipole approximation.

Chapter 5

Overview & open problems

This thesis has developed and explored a connection between gravitational physics and relativistic (electro)elasticity theory using the *blackfold approach* to higher-dimensional black holes. Within this framework, the dynamics of thin black branes are described by an effective theory of fluids living on dynamical surfaces whose thickness can be accounted for in a long wavelength derivative expansion. Fluids living on thin surfaces of co-dimension higher than zero were shown to behave like fluids along worldvolume directions and like elastic solids along transverse directions to the worldvolume. The fact that fluid behavior takes place along worldvolume directions was already known [8, 9, 17] but solid behavior along transverse directions, even to leading order in the derivative expansion, is genuinely new. Such behavior, as surprising as it might sound, can be intuitively understood. Given a perfect fluid with density ϵ and pressure P living on a two-dimensional sphere embedded in four-dimensional space-time and applying a deformation to the surface such that the radius of the sphere increases by a small amount, then the density and pressure will drop (or increase) due to the increase in volume. This is what we observe here. To leading order the surface on which the fluid lives is infinitely thin and hence only variations in volume (stretching and compressing) can be accounted for. The important quantities that quantify this are the modulus of compression K (2.4.3) and the elasticity tensor E^{abcd} (3.2.6), which are properties of the material composed of wrapped black branes. It is interesting to note that the structure of the elasticity tensor (3.2.6) is similar to the structure of Young modulus obtained from (1.1.3) in [18]

$$\tilde{Y}^{abcd} = -P\, f(n) \left(\frac{1}{n+2} \gamma^{a(c}\gamma^{d)b} + 2u^{(a}\gamma^{c)(d}u^{d)} + \frac{3n+4}{n+2} u^a u^b u^c u^d \right) , \tag{5.0.1}$$

where $f(n)$ is a function that depends on the geometric properties of the brane. This means that, to what concerns the type of elastic responses that an object can exhibit, deforming a neutral blackfold to leading order or bending a black brane to order ε is equivalent: in both cases, the reaction is that of a material characterized by a varying modulus of compression.

Relativistic elasticity theory has been considered in the literature by a number of authors [83, 87, 103, 84, 104, 81, 82], one of the most influential works being that conducted by Carter and Quintana [80]. The main focus of these works has been on describing elastic solids which are space-filling, contrary to being confined to a dynamical surface. However, the main difference compared to the work presented here is not only related to the existence of an embedding surface but also to the usual assumption regarding the form of the pressure (2.3.11) and elasticity (2.4.12) tensors:

$$\sigma^{ab}u_b = 0 \quad , \quad E^{abcd}u_d = 0 \; . \tag{5.0.2}$$

This orthogonality condition with respect to the fluid flows seems to have classical roots, namely, that an elastic material can be deformed in space but not in time. Not imposing (5.0.2) has been the main point of departure in this work. From the point of view of material science, materials that do not satisfy (5.0.2) might belong to a rather unusual class, but from the point of view of gravitational physics, in which black holes in certain regimes may take the effective description as a fluid brane, they are relevant for the analysis of their dynamics. Their study can lead to the formulation of a corner of relativistic elasticity theory that has not been previously explored (e.g. by applying extrinsic perturbations to black brane geometries) in the same way that the *fluid/gravity correspondence* led to a more general formulation of dissipative corrections to fluid and superfluid hydrodynamics [20, 21, 22, 20, 23, 24]. In fact, here we have already taken a few steps in this direction, not only by measuring the elasticity tensor (3.2.6), but also by measuring the Young modulus and piezoelectric moduli for neutral and charged dilatonic black strings (4.2.30), (4.3.26), (4.3.24). These new effects can also be easily understood if one imagines slightly curving a charged black string of finite thickness inducing a higher concentration of charged black material in the inner surface and a depletion in the outer surface. A varying concentration of matter due to the compression and stretching of the material on opposite sides induces a bending moment of dipolar character as in classical Hookean elasticity theory and, since the matter is charged, it also induces an electric dipole moment that describes the response of the charged string to the mechanical stress. Electric fields induced by mechanical stresses are the basic feature of piezoelectrics and their behavior is governed by the physics of electroelastic materials.

The material scientist must already be dazzled by the usage of gravitational physics as a laboratory where experiments with strange materials can take place, but even more would he become if he knew that we are still far from exhibiting all the possible connections between gravity and material science. There exists a vast landscape of other transport and response coefficients that can be obtained from perturbations of black branes besides those associated with viscous flows and stationary (electro)elastic deformations. Along worldvolume or boundary directions we expect that perturbations of black branes will reveal the viscoelastic character of the fluid flows by looking at shorter timescales [105] while along transverse directions time-dependent perturbations will lead us to the physics of nemetics (or liquid crystals). In general, fluid and solid behavior will appear along both directions. Moreover, when placing black branes on ambients with non-vanishing background fields one expects the material to behave like a dieletric (dimagnetic) allowing for the measurement of electric (magnetic) susceptibilities.

We finalize this thesis by discussing two possible research directions that can be considered the *holly grails* of the *blackfold approach*:

- **The metric to order ε^2:** The metric to order ε^2 would provide us with great insight on the structure of the derivative expansion when applying the MAE procedure. It could in principle aid us in showing that the expansion is well defined to all orders in ε and hence formally establish a connection between fluid mechanics on dynamical surfaces and black hole solutions. This is of course expected to be the case, since we were able to reproduce the thermodynamics of singly and doubly-spinning Kerr-(A)dS black holes in Sec. 3 and App. E. Moreover, the metric to order ε^2 would also allow us to further develop the worldvolume effective theory of Sec. 4 in the same spirit of Sec. 2 when finite size corrections are taken into account. This could lead to a further development of relativistic elasticity theory for materials that can be deformed in time as well as in space.

- **The bending of the $D3$-brane:** The bending of the $D3$-brane would show what the interpretation of the Young modulus and piezoelectric moduli is in the context of the AdS/CFT correspondence. We expect these effects to appear as a dipole deformation of the transverse sphere, which would correspond to deforming the S^5 of $AdS_5 \times S^5$. The dual interpretation of this would be that of sourcing the scalars of $\mathcal{N} = 4$ SYM which transform in the fundamental representation of the $SO(6)$ R-symmetry group.

These problems we expect to tackle head-on in the near future.

Special thanks

This thesis is greatly indebted to a few people and experiences.

A few months after starting my PhD I soon realized that to deal with frustration, difficulties and hard work I had to spend at least half of the time exploring other parts of life. During the course of this thesis I co-created and managed a cultural and scientific newspaper as well as a cultural association in the Azores. I co-funded and co-managed a cocktail bar and art gallery for one year in Copenhagen. I wrote a poetry book that won a prize. I co-managed a cinema/theater in Christiania for about two years. I created the most successful scientific lecture program in Denmark and probably beyond which has been running for three years. I organized more than 100 events. I joined the art group Live Art Installations and developed underwater lamps and lighting systems for floating platforms. I edited two music albums and gave many concerts. I sailed across the Atlantic in a 10 meter, 70 years old boat. I almost finished a collection of 30 interviews to theoretical physicists across the world (only 5 left!). I became scientific and technologic advisor for the Azorean government. Besides many other things. Without these activities and the people involved in them I couldn't have found the inspiration, creativity and energy to pursue my work in academia.

I have benefited greatly by having Niels as supervisor. Besides all the great things that supervisors don't usually do and he did I am extremely grateful for his understanding and support of my ideas and decisions. The best way I have to describe his character is by recalling the moment when I told him that I was going to open a bar. He said: great. One year later I told him that I was going to close the bar. He said: I think that's a good idea. And he was always right.

I have also benefited a great deal by having Troels as co-supervisor. Having many things in common, such as the sense of humor, made it extremely fun every time I went into his office. I am thankful for his patience with my stupidity and, more honestly, for not being mad at me even when I screwed up very badly.

My life at the institute could have been much worse if Costas wasn't there. He is definitely the coolest dude around who can make his tongue speak at least 5 languages and his fingers bake 10 pieces of bread at the same time.

I also thank Kim for all the great discussions and his willing to help me with all sorts of different matters. I thank Anna-Maria and Tarja for making my life easier and for being responsible for turning the working environment into something fun. I thank Pawel for being a great friend as well as collaborator and not giving up on me.

I thank my friend Jesper for being supportive and to show me that one should stand up for one's own ideas. I thank Rasmus for being just great in every possible way. I thank Lucas, My, Carmen, Lars, Christina and Christo for having established something worth millions. I thank Barbara, Olga, Di, Lene and Indigo for their teachings. I thank Rafat for teaching me the way of the Sitar and to Gunjan for showing me India. I thank Sara for many things but in particular for introducing me to Auroville, as well as to Rolf and Brigitte. I thank Pedro Solà and Henrik Schutze for their spirit. I thank Johan for being there. I thank Sara, Peter, Sofie, Hjalte, Ulla, Ella, Cale, Hadrian, Karla, Peter, Momas, Frederike and a lot to my brother Thomas for creating a safe environment for me to live in. I thank Pipaluk too, Fabrikken and Christiania. I thank Maria for her patience and support.

I specially thank my family for pretty much everything.

Appendix A

Notation

In this appendix we collect some conventions about notation. This thesis deals with submanifolds $\mathcal{W}_{p+1}$ of a $(D = p + n + 3)-$dimensional spacetime, described by the mapping functions $X^\mu(\sigma^a)$ from the worldvolume, parametrized by the coordinates σ^a, to the ambient spacetime, with coordinates x^μ.

$g_{\mu\nu}$ is the background metric while $\mathcal{W}_{p+1}$ inherits the metric

$$\gamma_{ab} = u_a^\mu \, g_{\mu\nu} \, u_b^\nu \,, \qquad u_a^\mu = \partial_a X^\mu \,. \tag{A.0.1}$$

Here, μ, ν indices are raised and lowered with $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, and a, b indices with γ_{ab} and its inverse γ^{ab}.

To project any spacetime tensor along tangential directions to the worldvolume we can use u_μ^a, while for directions orthogonal to $\mathcal{W}_{p+1}$ we define the projector,

$$\perp_{\mu\nu} = g_{\mu\nu} - u_\mu^a \, \gamma_{ab} \, u_\nu^b \,. \tag{A.0.2}$$

A subindex $_\perp$ on a tensor indicates that all μ, ν type of indices are orthogonal, e.g.,

$$B_\perp^{a\mu} = \perp^\mu{}_\nu B_\perp^{a\nu} \,. \tag{A.0.3}$$

$j^{a\mu\nu}$ and $d^{ab\mu}$ defined in (4.1.20) and (4.1.22) are $_\perp$-objects, but we do not write the $_\perp$ subindex on them in order to avoid cluttering.

∇_μ is the covariant derivative on the ambient space compatible with $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are its Christoffel symbols. D_a is the intrinsic covariant derivative on $\mathcal{W}_{p+1}$ compatible with γ_{ab} and $\{ {}^{\,a}_{b\,c} \}$ are its Christoffel symbols. Our convention for the Riemann curvature tensor is

$$R^\mu{}_{\lambda\nu\rho} = \Gamma^\mu_{\lambda\rho,\nu} - \Gamma^\mu_{\lambda\nu,\rho} + \Gamma^\mu_{\sigma\nu}\Gamma^\sigma_{\lambda\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma{}_{\lambda\nu} \,. \tag{A.0.4}$$

The operator ∇_a is defined to be compatible both with γ_{ab} and $g_{\mu\nu}$ such that, for instance,

$$\nabla_a v_\mu{}^\nu{}_b{}^c = u_a^\rho \, \partial_\rho \, v_\mu{}^\nu{}_b{}^c - u_a^\rho \, \Gamma^\sigma_{\rho\mu} \, v_\sigma{}^\nu{}_b{}^c + u_a^\rho \, \Gamma^\nu_{\rho\sigma} \, v_\mu{}^\sigma{}_b{}^c - \Gamma^d_{ab}, v_\mu{}^\nu{}_d{}^c + \Gamma^c_{ad} v_\mu{}^\nu{}_b{}^d \,. \tag{A.0.5}$$

For a submanifold tensor, $\nabla_c t^{a\cdots}{}_{b\ldots} = D_c t^{a\cdots}{}_{b\ldots}$. Moreover, the extrinsic curvature of $\mathcal{W}_{p+1}$ can be written as

$$\nabla_a u_b^\rho = K_{ab}{}^\rho \,, \tag{A.0.6}$$

with $K_{ab}{}^\rho$ being also a $\perp$−object.

The boundary of the submanifold is described by $\sigma^a = \zeta^a(\lambda)$ and normal vector $\hat{n}^\mu = u_a^\mu \hat{n}^a$ with unit norm. We introduce the coordinate vectors $v_{\hat{i}}^a$ as

$$v_{\hat{i}}^a = \frac{\partial \zeta^a}{\partial \lambda^{\hat{i}}} \,, \tag{A.0.7}$$

satisfying the properties $v_{\hat{i}}^\mu = u_a^\mu v_{\hat{i}}^a$ and $\hat{n}_a v_{\hat{i}}^a = 0$ such that the induced metric on the boundary takes the form $\hat{h}_{\hat{i}\hat{j}} = \gamma_{ab}(\zeta) v_{\hat{i}}^a v_{\hat{j}}^b$. $\nabla_{\hat{i}}$ is the boundary covariant derivative compatible with the metric $\hat{h}_{\hat{i}\hat{j}}$.

Appendix B

Multi-spin Kerr-(A)dS$_D$ black holes as blackfolds

In this appendix we generalize the even-ball (A)dS blackfold construction of Sec. 3.5.2, which focused on the case of a spinning disc, to include the multi-spinning case.

In this case the $(2k)$-ball rotates rigidly in k independent two-planes and the total velocity is given by

$$V = \left(1 + \frac{\sum_{i=1}^{N} \rho_i^2}{4L^2}\right)^{-1} \sqrt{\sum_{i=1}^{N} \rho_i^2 \Omega_i^2} \ . \tag{B.0.1}$$

The boundary of the ball is given by the locus where $V = 1$, which is solved by

$$\sum_{i=1}^{k} \Omega_i^2 r_i^2 (1 - \alpha_i) = 1 \ , \tag{B.0.2}$$

where we have defined $r_i = \rho_i/(1 - \frac{\sum_i^N \rho_i^2}{4L^2})$ and $\alpha_i = (\Omega_i^2 L^2)^{-1}$. According to the parameters α_i we can distinguish three different cases:

(i) $\alpha_i = 0, \forall i$. This corresponds to the even-ball blackfold construction of the ultra-spinning MP black hole constructed and discussed in Refs. [15, 63].

(ii) $\alpha_i = 1, \forall i$. This corresponds to the "ultra-spinning" limit $\Omega_i \to L^{-1}$ given in [65].

(iii) $0 < \alpha_i < 1, \forall i$. This corresponds to a new ultra-spinning limit of the Kerr-AdS$_D$ black hole presented in App. C.

The physical properties of the even-ball blackfolds can be computed from (3.3.6)-(3.3.9) and read

$$M = \frac{\Omega_{(D-2)} \hat{r}_+^n}{8\pi G \prod_j (1 - \alpha_j)} \left(\sum_{j=1}^{k} \frac{1}{1 - \alpha_j} + \frac{n+1}{2} \right) \prod_j \frac{1}{\Omega_j^2} \ , \tag{B.0.3}$$

$$J_i = \frac{\Omega_{(D-2)} \hat{r}_+^n \prod_j \frac{1}{\Omega_j^2}}{8\pi G \prod_j (1 - \alpha_j)} \frac{1}{1 - \alpha_i} \frac{1}{\Omega_i} \ , \ , \quad S = \frac{\Omega_{(D-2)} \hat{r}_+^{n+1}}{4G} \prod_j \left(\frac{1}{1 - \alpha_j} \frac{1}{\Omega_j^2} \right) \tag{B.0.4}$$

$$\mathcal{T} = -\frac{\Omega_{(D-2)}\hat{r}_+^n}{8\pi G \prod_j (1-\alpha_j)} \left(\sum_{j=1}^{k} \frac{1}{1-\alpha_j} - k \right) \prod_j \frac{1}{\Omega_j^2} \ , \tag{B.0.5}$$

where $\hat{r}_+ = n/2\kappa$. These expressions reduce to the results (3.5.38)-(3.5.40) for the for the singly-spinning case. The quantities above can be shown to satisfy the Smarr relation (4.1.35). The horizon thickness r_0 is given by

$$r_0 = \frac{n}{2\kappa} \sqrt{1 - \sum_{i=1}^{k} \Omega_i^2 r_i^2 (1-\alpha_i)} \ . \tag{B.0.6}$$

We can identify these even-ball blackfolds with two different limits of the Kerr-AdS$_D$ as follows:

$\alpha_i = 1, \ \forall i$: the "ultra-spinning" limit

A detailed study of this limit has been presented in Ref. [65]. The limit amounts to taking $a_i \to L, \ \forall i$ while keeping $\hat{\mu} \equiv \frac{2m}{L^{2k}\prod_{i=1}^{k}\Xi_i}$ finite. The resulting metric gives a flat membrane with metric as in (3.5.42) where the horizon size r_+ shrinks to zero. Its physical properties are the same as for the ultra-spinning case (C.0.8)-(C.0.10) in the limit $r_+ \ll L$ but now with $a_i = L, \ \forall i$. With the identifications

$$\Omega_i = \frac{1}{L}, \ \forall i \ , \quad \hat{r}_+ = r_+ = \left(\frac{2m}{L^{2k}} \right)^{\frac{1}{D-2k-3}} \ , \tag{B.0.7}$$

the physical properties of this solution exactly match those of the blackfolds (B.0.3), (B.0.4).

$0 \leq \alpha_i < 1, \ \forall i$: the ultra-spinning limit

A careful calculation of this limit is given in App. C. It amounts to taking k number of spins to infinity while keeping the ratios $\alpha_i = \frac{a_i^2}{L^2}$ and $\hat{\mu} = \frac{2m}{\prod_{i=1}^{k} a_i^2}$ finite. The resulting metric is presented in (C.0.7) and has the geometry of a flat black membrane near the axes of rotation. With the identifications

$$\Omega_i = \frac{1}{a_i}, \ \forall i \ , \quad \hat{r}_+ = r_+ = \left(\frac{2m}{\prod_{i=1}^{k} a_i^2} \right)^{\frac{1}{D-2k-3}} \ , \tag{B.0.8}$$

the physical properties (C.0.8)-(C.0.10) match precisely those of the blackfolds (B.0.3), (B.0.4).

Appendix C

The ultra-spinning Kerr-(A)dS$_D$ black hole

In this appendix we take the ultra-spinning limit of the Kerr-(A)dS$_D$ black hole with an arbitrary number of spins in $D \geq 6$. We focus here on the AdS case since we can obtain its dS counterpart by performing a Wick rotation.

Defining $N = \frac{D-1}{2} \bmod 2$ as the number of two-planes, ϵ by the relation $D = 2N + 1 + \epsilon$, with ϵ being either 1 for even dimensions or 0 for odd dimensions, and introducing $N + \epsilon$ direction cosines μ_i obeying the constraint $\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1$, the metric of this spacetime can be conveniently written in Boyer-Lindquist coordinates as [57]

$$
\begin{aligned}
ds^2 = &- W(1+\frac{r^2}{L^2})dt^2 + \frac{2m}{U}\left(W dt - \sum_{i=1}^{N}\frac{a_i\mu_i^2 d\varphi_i}{\Xi_i}\right)^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2 \\
&+ \frac{U dr^2}{V-2m} + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 - \frac{1}{L^2 W(1+\frac{r^2}{L^2})}\left(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2 ,
\end{aligned}
\tag{C.0.1}
$$

where

$$
W = \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{\Xi_i} , \quad U = r^{\epsilon}\sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{w=1}^{N}(r^2+a_w^2) , \tag{C.0.2}
$$

$$
V = r^{\epsilon-2}(1+\frac{r^2}{L^2})\prod_{i=1}^{N}(r^2+a_i^2) , \quad \Xi_i = 1 - \frac{a_i^2}{L^2} . \tag{C.0.3}
$$

The horizon r_+ sits at the largest positive real root of $V - 2m = 0$. We assume that the parameters are chosen in a such a way that a horizon exists. In fact a horizon always exists for odd D if any two of the spin parameters a_i vanish, while for even D its existence is guaranteed if only one vanishes.

We now take an k number of spin parameters a_i to be very large as compared to the remaining $N-k$ ones, i.e.,

$$\begin{aligned} &a_j \to \infty,\ \ j=1,...,k\ , \\ &a_l \text{ finite},\ \ l=k+1,...,N\ . \end{aligned} \tag{C.0.4}$$

Furthermore we take L to be of an equal magnitude as compared to the a_j parameters such that the ratios $\alpha_j = a_j^2/L^2$ remain constant, i.e.

$$\begin{aligned} L &\to \infty\ , \\ \Xi_j &\to 1-\alpha_j\ ,\ \ \Xi_l \to 0\ . \end{aligned} \tag{C.0.5}$$

Moreover we take $m \to \infty$ such that the ratio $\hat{\mu} = \frac{2m}{\prod_j a_j^2}$ remains finite and define new coordinates $\sigma_j = a_j \mu_j$ that remain finite as we approach $\mu_j \to 0$. The remaining μ_l stay finite and satisfy $\sum_l \mu_l^2 + \mu_{N+1}^2 = 1$. In this limit we find that the metric function W behaves like $W \to 1$ while the remaining metric functions become

$$\begin{aligned} U &\to r^{\epsilon}\frac{\mu_l^2}{r^2+a_l^2}\prod_l (r^2+a_l^2)^2 \prod_j a_j^2 \equiv r^{\epsilon}\hat{F}\hat{\Pi}\prod_j a_j^2\ , \\ V &\to r^{\epsilon-2}\prod_l (r^2+a_l^2)^2\prod_j a_j^2 \equiv r^{\epsilon-2}\hat{\Pi}\prod_j a_j^2\ , \end{aligned} \tag{C.0.6}$$

where we have assumed the summation convention over j and l. The limiting metric then reads

$$\begin{aligned} ds^2 = &-dt^2 + r^2 d^2\mu_{N+1} + (r^2+a_l^2)(d\mu_l^2 + \mu_l^2 d\varphi_l^2) + \frac{\hat{\mu} r^{-\epsilon}}{\hat{F}\hat{\Pi}}(dt + a_l\mu_l^2\varphi_l^2)^2 + \frac{r^2\hat{F}\hat{\Pi}}{\hat{\Pi}-\hat{\mu}r^{2-\epsilon}}dr^2 \\ &+ \frac{1}{1-\alpha_j}(d\sigma_j^2 + \sigma_j^2\varphi_j^2)\ . \end{aligned} \tag{C.0.7}$$

This limit looks just like the ultra-spinning limit of the MP black hole [64] except for the extra factor $(1-\alpha_j)^{-1}$ in the last term. Indeed since the parameters α_j must lie within the range $0 \le \alpha_j < 1$, since otherwise the metric either changes signature or diverges, we can rescale σ_j such that $\sigma_j \to \hat{\sigma}_j = \sqrt{(1-\alpha_j)^{-1}}\sigma_j$, and hence the metric above will be that of a rotating black $2k$-brane with rotation along the spherical $S^{D-(2k+1)}$ sections of the horizon.

The physical properties of this solution can be easily computed from [91] with the result

$$M = \frac{\Omega_{(D-2)}\hat{\mu}}{8\pi G\prod_j(1-\alpha_j)}\left(\sum_{j=1}^{k}\frac{1}{1-\alpha_j} - \frac{D-2k-1}{2}\right)\prod_j a_j^2\ , \tag{C.0.8}$$

$$S = \frac{\Omega_{(D-2)} r_+^{2(N-k)-1+\epsilon}}{4G} \prod_j \frac{a_j^2}{1-\alpha_j^2} \ , \quad T = \frac{2(N-k-1)+\epsilon}{4\pi r_+} \ , \tag{C.0.9}$$

$$J_i = \frac{\Omega_{(D-2)} \hat{\mu} \prod_j a_j^2}{8\pi G \prod_j (1-\alpha_j)} \frac{a_i}{1-\alpha_i} \ , \quad \Omega_i = \frac{1}{a_i} \ , \tag{C.0.10}$$

$$r_+ = \left(\frac{2m}{\prod_j a_j^2} \right)^{\frac{1}{D-2k-3}} \ , \quad i = 1, ..., k \ . \tag{C.0.11}$$

These correctly reduce to the results (3.5.50)-(3.5.53) for the singly-spinning case. The quantities in (C.0.8)-(C.0.11) obey the Quantum Statistical Relation

$$M - \sum_j \Omega_j J_j - TS = \frac{\Omega_{(D-2)}}{8 \prod_j (1-\alpha_j)} \hat{\mu} \prod_j a_j^2 \ , \tag{C.0.12}$$

and thus only when $\alpha_j = 0, \forall j$ does one recover the flat space result.

We conclude this appendix by noting that there are only three ways of making the Kerr-AdS$_D$ black hole ultra-spin in the sense that $J \to \infty$. This is obvious by looking at the general expression for J_i

$$J_i = \frac{\Omega_{(D-2)} m}{4\pi \prod_w^N \Xi_w} \frac{a_i}{\Xi_i} \ . \tag{C.0.13}$$

One way is to send $a_i \to \infty$ but keeping Ξ_i finite since otherwise $J_i \to 0$. This can only be done by simultaneously sending $L \to \infty$. Another way is to send $a_i \to L$ leading to $\Xi_i \to 0$, if then one sends $m \to 0$, J_i remains finite and we obtain the "ultra-spinning" limit of (3.5.42). If otherwise we keep m finite we obtain the rotating hyperboloid membrane of [65]. This limit has the consequence that the horizon size is kept finite but the mass diverges, i.e., $M \to \infty$ and hence it is not ultra-spinning in the sense that $J \gg M$. A third possibility would be to naively send $m \to \infty$ but this results in a black hole with infinite horizon radius, which is senseless unless we simultaneously send $a_i \to \infty$ and $L \to \infty$ and thus recover the limit taken here. Moreover, since we have taken $L \to \infty$ while keeping r_+ finite, the ultra-spinning limit of the Kerr-(A)dS black hole exists only in the regime $r_+ \ll L$ and hence can be fully captured by the *blackfold approach*.

Appendix D

Ultra-spinning Myers-Perry black holes revisited

The purpose of this section is to instructively show how to take the blackfold limit for singly-spinning MP black holes as it was done in Sec. 4.4.1 for its doubly-spinning counterpart. Bearing this in mind, we consider the MP metric with a single angular momentum in $n+5$ dimensions

$$ds^2 = -dt^2 + \frac{\mu}{r^n \Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{(n+1)} \ , \tag{D.0.1}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - \frac{\mu}{r^n} \ . \tag{D.0.2}$$

The horizon radius r_+ is given by the largest positive real root of $\Delta(r) = 0$. The ultra-spinning limit is attained when $r_+ \ll a$ [64]. From the definition of Δ above we see that in this limit $r_+^n \simeq \mu/a^2$. From the $n+1$-sphere metric in (D.0.1) we see that the radius of the $(n+1)$-sphere is

$$r_0 = r_+\cos\theta = \left(\frac{\mu}{a^2}\right)^{\frac{1}{n}}\cos\theta \ . \tag{D.0.3}$$

Hence, the blackfold is a rotating disc of radius a with its center and boundary located at $\theta = 0$ and $\theta = \pi/2$ respectively. We can see this directly from the metric (D.0.1) above. Sufficiently close to the horizon, the metric (D.0.1) should locally be that of a boosted black membrane. Close to the center of the disc we need $r \ll a$. However, this is not sufficiently close everywhere on the blackfold. We thus introduce the coordinate

$$\rho_1 = a\sin\theta \ . \tag{D.0.4}$$

This can be seen as the radius on the disc. In terms of this we can write the thickness as

$$r_0(\rho_1) = r_+\sqrt{1 - \frac{\rho_1^2}{a^2}} \ . \tag{D.0.5}$$

For a given point on the disc we should require that the distance scale over which the thickness change is much larger than the thickness of the disc. Thus, we should require

$$r_0 \ll \frac{1}{|r_0''|} \ . \tag{D.0.6}$$

In terms of the horizon radius and rotation parameter, this requirement becomes:

$$r_+ \ll a\sqrt{1-\frac{\rho_1^2}{a^2}} \ . \tag{D.0.7}$$

This tells us, for each point on the disc, how widely separated the scales must be for the blackfold approximation to be valid. To see the boosted black membrane from the metric (D.0.1), given a radius ρ_1, we need

$$r \ll a\sqrt{1-\frac{\rho_1^2}{a^2}} \ . \tag{D.0.8}$$

This implies $r \ll a\cos\theta$. Now, consider a given point with radius $\rho_1 = \rho_{1*}$ at the disc, corresponding to the angle θ_* with $\rho_{1*} = a\sin\theta_*$. Hence, we require $r \ll a\cos\theta_*$. In order to make a more clear contact with the metric of a Schwarzschild membrane we define

$$r_{0*} = r_+\cos\theta_*, \qquad \tilde{r} = r\cos\theta_*, \qquad z = \rho_{1*}\phi \ . \tag{D.0.9}$$

In the approximation $r \ll a\cos\theta_*$ the metric near $\rho = \rho_{1*}$ becomes:

$$ds^2 = -dt^2 + d\rho_1^2 + dz^2 + (1-f)\left(\frac{dt}{\cos\theta^*} - \tan\theta_* dz\right)^2 + \frac{d\tilde{r}^2}{f} + \tilde{r}^2 d\Omega_{(n+1)}^2 \ , \tag{D.0.10}$$

with

$$f \equiv 1 - \frac{r_{0*}^n}{\tilde{r}^n} \ . \tag{D.0.11}$$

This corresponds to boosting the static black Schwarzschild membrane

$$ds^2 = -dt^2 + dz^2 + (1-f)dt^2 + d\rho_1^2 + \frac{d\tilde{r}^2}{f} + \tilde{r}^2 d\Omega_{(n+1)}^2 \ , \tag{D.0.12}$$

along z with the boost (4.4.11) with corresponding Lorentz boost parameter (4.4.12). The angular velocity in this ultra-spinning regime is also given by (4.4.13) and hence we see that in the blackfold description the ultra-spinning MP black hole is a black membrane which is rigidly rotating with constant angular velocity given by $V = \rho_1\Omega_1$.

Blackfold equations with zero transverse angular momentum

To facilitate comparison with the doubly-spinning case of Sec. (4.4.2) we present here a brief analysis of the blackfold equations in the single-pole approximation.

We begin by embedding the disc in the background (4.4.15) using the mapping functions (4.4.17), leading to the induced metric (4.4.18). Next, we read off the static black membrane stress-energy tensor from the metric (D.0.10), which has the form of (2.2.1). The fluid velocity is given by the pullback of the background Killing vector field

$$\mathbf{k} = \frac{\partial}{\partial t} + \Omega_1 \frac{\partial}{\partial \phi_1} \ , \tag{D.0.13}$$

which gives rise to the same non-vanishing components as those given in (4.4.21).

Now, we analyze the blackfold equations (4.1.24)-(4.1.27) adapted to the current situation. As the embedding is flat, the extrinsic curvature ${K_{ab}}^{\mu}$ vanishes and thus the extrinsic equation $T^{ab}{K_{ab}}^{\mu} = 0$ is trivially satisfied. The remaining non-trivial equations read

$$D_a B^{ab} = 0, \qquad B^{ab} u^{\mu}_a n_b|_{\partial \mathcal{W}_3} = 0 \ . \tag{D.0.14}$$

The conservation of the intrinsic stress-energy tensor, assuming that ϵ and P only depend on ρ_1, implies

$$r_0(\rho_1) = r_0(0)\sqrt{1 - \rho_1^2 \Omega_1^2} \ . \tag{D.0.15}$$

Furthermore the boundary condition $B^{ab} u^{\mu}_a n_b|_{\partial \mathcal{W}_3} = 0$ is again satisfied provided $\rho_1|_{\partial \mathcal{W}_3} = \Omega_1^{-1}$, hence the disc has a radius of $\rho_1 = \Omega_1^{-1}$ and is moving at the speed of light there. This is the result obtained in [15, 63] and matches the thickness (D.0.5) upon the identification $r_0(0) = (\mu/a_1^2)^{\frac{1}{n}}$ and $\Omega_1 = a_1^{-1}$.

Appendix E

Spin corrections for Kerr-(A)dS black holes

Here we study the same limiting behavior of higher-dimensional Kerr-(A)dS black holes as in Sec. 4.4 for MP black holes and show that it can be described using the blackfold formalism. Due to the similarity between both cases we refer to Sec. (4.4) for a more extensive analysis.

In spheroidal coordinates, the metric of the Kerr-AdS black hole with two angular momenta in even dimensions is given by[1] [56, 57]

$$
\begin{aligned}
ds^2 = & -W\left(1+\frac{r^2}{L^2}\right)dt^2 + \frac{\mu}{U}\left(dt - \sum_{i=1}^{2}\frac{a_i\mu_i^2}{\Xi_i}d\phi_i\right)^2 + \frac{U}{V-\mu}dr^2 \\
& + \sum_{i=1}^{2}\frac{r^2+a_i^2}{\Xi_i}\left(d\mu_i^2 + \mu_i^2\left(d\phi_i + \frac{\sqrt{\alpha_i}}{L}dt\right)^2\right) + r^2\left(\sum_{i=3}^{(n+5)/2}d\mu_i^2 + \sum_{i=3}^{N}\mu_i^2 d\phi_i^2\right) \\
& + \frac{1}{L^2W(1+\frac{r^2}{L^2})}\left(\sum_{i=1}^{2}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i + r^2\sum_{i=3}^{(n+5)/2}\mu_i d\mu_i\right)\ ,
\end{aligned}
\tag{E.0.1}
$$

where

$$
\begin{aligned}
W = 1 + \sum_i \alpha_i\frac{\mu_i^2}{\Xi_i}, \qquad U = r^{n-2}\left(1 - \sum_{i=1}^{2}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\right)\prod_{j=1}^{2}(r^2+a_j^2), \\
V = r^{n-2}(1+\frac{r^2}{L^2})\prod_{i=1}^{2}(r^2+a_i^2)\ ,
\end{aligned}
\tag{E.0.2}
$$

[1] The same analysis also holds in the case of odd dimensions.

with μ_1 and μ_2 as given in Eq. (4.4.2) and

$$\frac{U}{V-\mu} = \frac{\left(1-\sum_{i=1}^{2}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\right)}{(1+\frac{r^2}{L^2})-\frac{\mu}{r^{n-2}\prod_{j=1}^{2}(r^2+a_j^2)}}, \qquad \alpha_i = \frac{a_i^2}{L^2}, \qquad \Xi_i = 1-\alpha_i \ . \tag{E.0.3}$$

The horizon r_+ is given by the largest positive real root of $V(r)-\mu=0$. For clarity of notation we set $a_1 \equiv a$ and $a_2 \equiv b$. The ultra-spinning limit is attained when $r_+, b \ll a, L$ with $0 \leq \alpha_1 < 1$ and $r \ll a, L$ with r finite[2], in a similar fashion as in the singly-spinning case, see Sec. 3. However, for the metric to look locally like a boosted MP membrane, we furthermore need $r \ll \frac{a}{\sqrt{\Xi_1}}\cos\theta$ and $b \ll \frac{a}{\sqrt{\Xi_1}}\cos\theta$.

Under these assumptions, it is easy to show that the last term in (E.0.1) is subleading while the functions V, U, W reduce to

$$W \to 1, \qquad U \to r^{n-2}a^2\left(1-\mu_1^2-\frac{a_i^2\mu_i^2}{r^2+a_i^2}\right)(r^2+b^2), \qquad V \to r^{n-2}a^2(r^2+b^2) \ . \tag{E.0.4}$$

In order to parametrize the membrane in a convenient way, we introduce the coordinates

$$\rho_1 = \frac{a}{\sqrt{\Xi_1}}\sin\theta, \qquad z = \rho_*\phi_1 \ . \tag{E.0.5}$$

Then, near a fixed angle θ_*, the metric (E.0.1) is seen to reduce again to that of a MP membrane (4.4.10) but with Lorentz boost,

$$V = \sin\theta_* = \sqrt{\Xi_1}\,\Omega_1\rho_1, \qquad \tilde{\gamma} = \frac{1}{\sqrt{1-\Xi_1\rho_1^2\Omega_1^2}}, \qquad \Omega_1 = \frac{1}{a} \ . \tag{E.0.6}$$

Blackfold pole-dipole equations with non-zero transverse angular momentum

We now want to describe the above limiting behavior of the higher-dimensional Kerr-(A)dS black holes using the *blackfold approach*. It is convenient, in order to highlight the existent 2-planes of the background spacetime, to write the metric of AdS in conformally flat coordinates

$$ds^2 = -F(\rho)dt^2 + H(\rho)^{-1}\left(d\rho_1^2+\rho_1^2d\phi_1^2+ds_\perp^2+\sum_{i=1}^{n}dx_i^2\right), \quad \rho^2 = \rho_1^2+\rho_2^2+\sum_{i=1}^{n}x_i^2 \ , \tag{E.0.7}$$

where $ds_\perp^2$ is given by (4.4.16) and

$$F(\rho) = \left(\frac{1+\frac{\rho^2}{4L^2}}{1-\frac{\rho^2+}{4L^2}}\right)^2, \qquad H(\rho) = 1-\frac{\rho^2}{4L^2} \ . \tag{E.0.8}$$

[2] In the case of higher-dimensional Kerr-dS black holes, the parameter α_1 is instead constrained by $\alpha_i \geq 0$.

Note that this coordinate system differs from the one used in (E.0.1). To see how to translate from one coordinate system to the other see Sec. 3. To embed the membrane in this background we chose the embedding coordinates (4.4.17), which give rise to the induced metric

$$\gamma_{ab}d\sigma^a d\sigma^b = -F(\rho_1)dt^2 + H(\rho_1)^{-1}\left(d\rho_1^2 + \rho_1^2 d\phi_1^2\right) \ . \tag{E.0.9}$$

Again, all extrinsic curvature components vanish since the embedding is flat. The stress-energy tensor is still given by (2.2.1) but now with boost velocities,

$$u^t = \tilde{\gamma}, \qquad u^{\rho_1} = 0, \qquad u^{\phi_1} = \tilde{\gamma}\Omega_1, \qquad \tilde{\gamma} = \frac{1}{\sqrt{1 - \Xi_1 \frac{\rho_1^2}{1-\frac{\rho_1^2}{4L^2}}\Omega_1^2}} \ . \tag{E.0.10}$$

Furthermore the spin current in this coordinate system reads

$$j^{a\nu\rho} = \frac{\Omega_{(n+1)}}{8\pi G}\frac{\tilde{b}\tilde{r}_0^n H(\rho)}{\rho_2}u^a \delta^\nu_{\rho_2}\delta^\rho_{\phi_2} \ , \tag{E.0.11}$$

while the dipole current and $B^{\mu\nu a}$ components vanish. Due to the vanishing of the extrinsic curvature and of all the contractions involving the Riemann tensor in Eqs. (4.1.24)-(4.1.27), the blackfold equations reduce to Eqs. (4.4.25) as in the flat space case. Solving the bulk equations requires:

$$\tilde{r}_0(\rho_1) = \tilde{r}_0(0)\sqrt{1 - \Xi_1\frac{\rho_1^2}{1-\frac{\rho_1^2}{4L^2}}\Omega_1^2}, \qquad \tilde{b}(\rho_1) = \tilde{b}(0)\sqrt{1 - \Xi_1\frac{\rho_1^2}{1-\frac{\rho_1^2}{4L^2}}\Omega_1} \ . \tag{E.0.12}$$

Moreover the boundary condition $B^{ab}u_a^\mu n_b|_{\partial\mathcal{W}_3} = 0$ implies that the disc has a maximum radius given by $\rho_1|_{\partial\mathcal{W}_3} = 2L(L\Omega_1 - \sqrt{L^2\Omega_1^2 - 1})$. We note that in the singly-spinning case where $\tilde{b}(0) = 0$ the thickness $\tilde{r}_0$ obtained in Sec. 3 coincides with the one given in (E.0.12) and agrees with the analytic solution (E.0.1) upon the identification $\tilde{r}_0^n(0)/\Omega_1^2 = \mu$ and $\Omega_1^{-1} = a_1$.

Thermodynamic quantities

The thermodynamical quantities of the analytic solution (E.0.1) in the ultra-spinning regime can be obtained from [91] and read

$$M = \frac{\Omega_{(n+3)}}{16\pi G}\frac{\mu}{\Xi_1^2}\left(2 + \Xi_1(n+1)\right), \qquad J^1 = \frac{\Omega_{(n+3)}}{8\pi G}\frac{\mu}{\Xi_1^2}a, \qquad J_\perp^2 = \frac{\Omega_{(n+3)}}{8\pi G}\frac{\mu}{\Xi_1}b \ , \tag{E.0.13}$$

with

$$\Omega_1 = \frac{1}{a}, \qquad \Omega_2 = \frac{b}{r_+^2 + b^2} \ , \tag{E.0.14}$$

while the entropy and temperature are given by

$$S = \frac{\Omega_{(n+3)}}{4G}\frac{a^2\mu}{\Xi_1 r_+}, \qquad T = \frac{1}{4\pi r_+}\left(n - \frac{2b^2}{r_+^2 + b^2}\right) . \tag{E.0.15}$$

Evaluating the conserved charges using Eqs. (4.1.31)-(4.1.33) results in

$$M = \frac{\Omega_{(n+3)}}{16\pi G}\frac{\tilde{r}_0^n(0)}{\Xi_1^2\Omega_1^2}\left(2 + \Xi_1(n+3)\right), \qquad J^1 = \frac{\Omega_{(n+3)}}{8\pi G}\frac{\tilde{r}_0^n(0)}{\Xi_1\Omega_1^2}\frac{1}{\Omega_1}, \qquad J_\perp^2 = \frac{\Omega_{n+3)}}{8\pi G}\frac{\tilde{r}_0^n(0)}{\Xi_1\Omega_1^2}\tilde{b}(0) . \tag{E.0.16}$$

We can straightforwardly check that these results agree with the ones presented in (E.0.13)-(E.0.14) upon the identification $\tilde{r}_0^n(0)/\Omega_1^2 = \mu$, $\Omega_1^{-1} = a_1$, and $\tilde{b}(0) = b$. Moreover, in the AdS case the tension (4.1.37) is non-zero and reads

$$\mathcal{T} = -\alpha\frac{\Omega_{(n+3)}}{8\pi G}\frac{\tilde{r}_0^n(0)}{\Xi_1^2\Omega_1^2} . \tag{E.0.17}$$

Using the Smarr relation (4.1.35) to compute the product TS and then the first law of black hole thermodynamics we can exactly reproduce the temperature and entropy as given in (E.0.15) in the same way we did for MP black holes. This leads one to conclude that higher-dimensional Kerr-(A)dS black holes in the ultra-spinning regime are correctly captured by the blackfold pole-dipole equations.

Bibliography

[1] S. Hawking, "Particle Creation by Black Holes," *Commun.Math.Phys.* **43** (1975) 199–220.

[2] S. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys.Rev.* **D14** (1976) 2460–2473.

[3] S. D. Mathur, "What Exactly is the Information Paradox?," *Lect.Notes Phys.* **769** (2009) 3–48, `arXiv:0803.2030 [hep-th]`.

[4] J. Armas, "Conversations on Quantum Gravity," *In preparation* .

[5] J. Polchinski, "What is string theory?," `arXiv:hep-th/9411028 [hep-th]`.

[6] D. Gross, "Where do we stand in fundamental string theory," *Phys.Scripta* **T117** (2005) 102–105.

[7] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200 [hep-th]`.

[8] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," *JHEP* **0802** (2008) 045, `arXiv:0712.2456 [hep-th]`.

[9] P. Kovtun, D. Son, and A. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," *Phys.Rev.Lett.* **94** (2005) 111601, `arXiv:hep-th/0405231 [hep-th]`. An Essay submitted to 2004 Gravity Research Foundation competition.

[10] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, *et al.*, "Local Fluid Dynamical Entropy from Gravity," *JHEP* **0806** (2008) 055, `arXiv:0803.2526 [hep-th]`.

[11] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," *JHEP* **0812** (2008) 116, `arXiv:0809.4272 [hep-th]`.

[12] V. E. Hubeny, S. Minwalla, and M. Rangamani, "The fluid/gravity correspondence," arXiv:1107.5780 [hep-th].

[13] G. Policastro, D. Son, and A. Starinets, "The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma," *Phys.Rev.Lett.* **87** (2001) 081601, arXiv:hep-th/0104066 [hep-th].

[14] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers, and M. J. Rodriguez, "The Phase Structure of Higher-Dimensional Black Rings and Black Holes," *JHEP* **10** (2007) 110, arXiv:0708.2181 [hep-th].

[15] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "World-Volume Effective Theory for Higher-Dimensional Black Holes," *Phys. Rev. Lett.* **102** (2009) 191301, arXiv:0902.0427 [hep-th].

[16] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Essentials of Blackfold Dynamics," *JHEP* **03** (2010) 063, arXiv:0910.1601 [hep-th].

[17] J. Camps, R. Emparan, and N. Haddad, "Black Brane Viscosity and the Gregory-Laflamme Instability," *JHEP* **05** (2010) 042, arXiv:1003.3636 [hep-th].

[18] J. Camps and R. Emparan, "Derivation of the blackfold effective theory," *JHEP* **1203** (2012) 038, arXiv:1201.3506 [hep-th].

[19] J. Armas and N. A. Obers, "Relativistic Elasticity of Stationary Fluid Branes," arXiv:1210.5197 [hep-th].

[20] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, *et al.*, "Hydrodynamics from charged black branes," *JHEP* **1101** (2011) 094, arXiv:0809.2596 [hep-th].

[21] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, "Fluid dynamics of R-charged black holes," *JHEP* **0901** (2009) 055, arXiv:0809.2488 [hep-th].

[22] D. T. Son and P. Surowka, "Hydrodynamics with Triangle Anomalies," *Phys.Rev.Lett.* **103** (2009) 191601, arXiv:0906.5044 [hep-th].

[23] C. P. Herzog, N. Lisker, P. Surowka, and A. Yarom, "Transport in holographic superfluids," *JHEP* **1108** (2011) 052, arXiv:1101.3330 [hep-th].

[24] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, "A Theory of first order dissipative superfluid dynamics," arXiv:1105.3733 [hep-th].

[25] J. Armas, J. Camps, T. Harmark, and N. A. Obers, "The Young Modulus of Black Strings and the Fine Structure of Blackfolds," *JHEP* **1202** (2012) 110, arXiv:1110.4835 [hep-th].

[26] J. Armas, J. Gath, and N. A. Obers, "Black Branes as Piezoelectrics," arXiv:1209.2127 [hep-th].

[27] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," *Living Rev.Rel.* **11** (2008) 6, arXiv:0801.3471 [hep-th].

[28] N. A. Obers, "Black Holes in Higher-Dimensional Gravity," *Lect.Notes Phys.* **769** (2009) 211–258, arXiv:0802.0519 [hep-th].

[29] R. Emparan and P. Figueras, "Multi-black rings and the phase diagram of higher-dimensional black holes," *JHEP* **1011** (2010) 022, arXiv:1008.3243 [hep-th].

[30] R. Emparan and H. S. Reall, "A rotating black ring in five dimensions," *Phys. Rev. Lett.* **88** (2002) 101101, arXiv:hep-th/0110260.

[31] A. A. Pomeransky and R. A. Sen'kov, "Black ring with two angular momenta," arXiv:hep-th/0612005.

[32] H. Elvang and P. Figueras, "Black Saturn," *JHEP* **05** (2007) 050, arXiv:hep-th/0701035.

[33] J. Evslin and C. Krishnan, "The Black Di-Ring: An Inverse Scattering Construction," arXiv:0706.1231 [hep-th].

[34] H. Iguchi and T. Mishima, "Black di-ring and infinite nonuniqueness," *Phys. Rev.* **D75** (2007) 064018, arXiv:hep-th/0701043.

[35] H. Elvang and M. J. Rodriguez, "Bicycling Black Rings," *JHEP* **04** (2008) 045, arXiv:0712.2425 [hep-th].

[36] J. Evslin, "Geometric Engineering 5d Black Holes with Rod Diagrams," *JHEP* **09** (2008) 004, arXiv:0806.3389 [hep-th].

[37] P. T. Chrusciel and J. Cortier, "Maximal analytic extensions of the Emparan-Reall black ring," arXiv:0807.2309 [gr-qc].

[38] P. T. Chrusciel, J. Cortier, and A. G.-P. Gomez-Lobo, "On the global structure of the Pomeransky-Senkov black holes," *Adv.Theor.Math.Phys.* **14** (2010) , arXiv:0911.0802 [gr-qc].

[39] P. T. Chrusciel and S. J. Szybka, "Stable causality of the Pomeransky-Senkov black holes," *Adv.Theor.Math.Phys.* **15** (2011) 175–178, arXiv:1010.0213 [hep-th].

[40] P. T. Chrusciel, M. Eckstein, and S. J. Szybka, "On smoothness of Black Saturns," *JHEP* **1011** (2010) 048, arXiv:1007.3668 [hep-th].

[41] J. Armas, "Maximal Analytic Extension and Hidden Symmetries of the Dipole Black Ring," *Class.Quant.Grav.* **28** (2011) 235014, arXiv:1011.5618 [hep-th].

[42] W. Israel, "Event horizons in static vacuum space-times," *Phys.Rev.* **164** (1967) 1776–1779.

[43] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom," *Phys.Rev.Lett.* **26** (1971) 331–333.

[44] D. Robinson, "Uniqueness of the Kerr black hole," *Phys.Rev.Lett.* **34** (1975) 905–906.

[45] T. Harmark, "Stationary and axisymmetric solutions of higher-dimensional General Relativity," *Phys. Rev.* **D70** (2004) 124002, arXiv:hep-th/0408141.

[46] T. Harmark and P. Olesen, "On the structure of stationary and axisymmetric metrics," *Phys.Rev.* **D72** (2005) 124017, arXiv:hep-th/0508208 [hep-th].

[47] S. Hollands and S. Yazadjiev, "Uniqueness theorem for 5-dimensional black holes with two axial Killing fields," *Commun. Math. Phys.* **283** (2008) 749–768, arXiv:0707.2775 [gr-qc].

[48] T. Harmark, "Domain Structure of Black Hole Space-Times," *Phys. Rev.* **D80** (2009) 024019, arXiv:0904.4246 [hep-th].

[49] J. Armas, P. Caputa, and T. Harmark, "Domain Structure of Black Hole Space-Times with a Cosmological Constant," *Phys.Rev.* **D85** (2012) 084019, arXiv:1111.1163 [hep-th].

[50] S. Hollands and S. Yazadjiev, "A Uniqueness theorem for 5-dimensional Einstein-Maxwell black holes," *Class. Quant. Grav.* **25** (2008) 095010, arXiv:0711.1722 [gr-qc].

[51] S. Hollands and S. Yazadjiev, "A uniqueness theorem for stationary Kaluza-Klein black holes," arXiv:0812.3036 [gr-qc].

[52] J. Armas and T. Harmark, "Uniqueness Theorem for Black Hole Space-Times with Multiple Disconnected Horizons," *JHEP* **05** (2010) 093, arXiv:0911.4654 [hep-th].

[53] S. Hollands, "Black hole uniqueness theorems and new thermodynamic identities in eleven dimensional supergravity," *Class.Quant.Grav.* **29** (2012) 205009, arXiv:1204.3421 [gr-qc].

[54] M. T. Anderson, P. T. Chrusciel, and E. Delay, "Nontrivial, static, geodesically complete, vacuum space-times with a negative cosmological constant," *JHEP* **0210** (2002) 063, arXiv:gr-qc/0211006 [gr-qc].

[55] R. C. Myers and M. J. Perry, "Black Holes in Higher Dimensional Space-Times," *Ann. Phys.* **172** (1986) 304.

[56] S. W. Hawking, C. J. Hunter, and M. Taylor, "Rotation and the AdS/CFT correspondence," *Phys. Rev.* **D59** (1999) 064005, `arXiv:hep-th/9811056`.

[57] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope, "The general Kerr-de Sitter metrics in all dimensions," *J. Geom. Phys.* **53** (2005) 49–73, `arXiv:hep-th/0404008`.

[58] T. Damour, "Black Hole Eddy Currents," *Phys.Rev.* **D18** (1978) 3598–3604.

[59] M. Parikh and F. Wilczek, "An Action for black hole membranes," *Phys.Rev.* **D58** (1998) 064011, `arXiv:gr-qc/9712077 [gr-qc]`.

[60] K. Thorne, D. Macdonald, and R. Price, "Black holes: The Membrane paradigm," *Yale University Press* (1986) .

[61] C. Eling and Y. Oz, "Relativistic CFT Hydrodynamics from the Membrane Paradigm," *JHEP* **1002** (2010) 069, `arXiv:0906.4999 [hep-th]`.

[62] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, "Wilsonian Approach to Fluid/Gravity Duality," *JHEP* **1103** (2011) 141, `arXiv:1006.1902 [hep-th]`.

[63] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "New Horizons for Black Holes and Branes," *JHEP* **04** (2010) 046, `arXiv:0912.2352 [hep-th]`.

[64] R. Emparan and R. C. Myers, "Instability of ultra-spinning black holes," *JHEP* **0309** (2003) 025, `arXiv:hep-th/0308056 [hep-th]`.

[65] M. M. Caldarelli, R. Emparan, and M. J. Rodriguez, "Black Rings in (Anti)-deSitter space," *JHEP* **11** (2008) 011, `arXiv:0806.1954 [hep-th]`.

[66] J. Camps, R. Emparan, P. Figueras, S. Giusto, and A. Saxena, "Black Rings in Taub-NUT and D0-D6 interactions," *JHEP* **0902** (2009) 021, `arXiv:0811.2088 [hep-th]`.

[67] J. Armas, J. Gath, and N. A. Obers, "Forthcoming," *Forthcoming* (2012) .

[68] J. Armas, T. Harmark, N. A. Obers, M. Orselli, and A. V. Pedersen, "Thermal Giant Gravitons," `arXiv:1207.2789 [hep-th]`.

[69] M. M. Caldarelli, R. Emparan, and B. Van Pol, "Higher-dimensional Rotating Charged Black Holes," *JHEP* **1104** (2011) 013, `arXiv:1012.4517 [hep-th]`.

[70] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Heating up the BIon," *JHEP* **1106** (2011) 058, `arXiv:1012.1494 [hep-th]`.

[71] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Thermodynamics of the hot BIon," *Nucl.Phys.* **B851** (2011) 462–480, arXiv:1101.1297 [hep-th].

[72] V. Niarchos and K. Siampos, "M2-M5 blackfold funnels," *JHEP* **1206** (2012) 175, arXiv:1205.1535 [hep-th].

[73] V. Niarchos and K. Siampos, "Entropy of the self-dual string soliton," arXiv:1206.2935 [hep-th].

[74] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Thermal string probes in AdS and finite temperature Wilson loops," *JHEP* **1206** (2012) 144, arXiv:1201.4862 [hep-th].

[75] J. Armas and N. A. Obers, "Blackfolds in (Anti)-de Sitter Backgrounds," *Phys.Rev.* **D83** (2011) 084039, arXiv:1012.5081 [hep-th].

[76] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Blackfolds in Supergravity and String Theory," *JHEP* **1108** (2011) 154, arXiv:1106.4428 [hep-th].

[77] B. Carter, "Essentials of classical brane dynamics," *Int. J. Theor. Phys.* **40** (2001) 2099–2130, gr-qc/0012036.

[78] L. Landau and E. M. Lifshitz, "Theory of elasticity," *Course of Theoretical Physics* **Vol. 7** (1959) 134.

[79] M. M. Caldarelli, O. J. C. Dias, R. Emparan, and D. Klemm, "Black Holes as Lumps of Fluid," *JHEP* **04** (2009) 024, arXiv:0811.2381 [hep-th].

[80] B. Carter and H. Quintana, "Foundations of general relativistic high-pressure elasticity theory," *Proc.Roy.Soc.Lond.* **A331** (1972) 57–83.

[81] M. Karlovini and L. Samuelsson, "Elastic stars in general relativity. 1. Foundations and equilibrium models," *Class.Quant.Grav.* **20** (2003) 3613–3648, arXiv:gr-qc/0211026 [gr-qc].

[82] R. Beig and B. G. Schmidt, "Relativistic elasticity," *Class.Quant.Grav.* **20** (2003) 889–904, arXiv:gr-qc/0211054 [gr-qc].

[83] W. C. Hernandez, "Elasticity Theory in General Relativity,"*Phys. Rev. D* **1** (Feb, 1970) 1013–1018. http://link.aps.org/doi/10.1103/PhysRevD.1.1013.

[84] S. Sokolov, "Equations of motion of elastic continuum in gravitational field," *IFVE-94-9* (1994) .

[85] S. Sokolov, "Elasticity Theory in GR," http://web.ihep.su/library/pubs/tconf03/c2-4.htm.

[86] J. D. Brown, "Action functionals for relativistic perfect fluids," *Class.Quant.Grav.* **10** (1993) 1579–1606, `arXiv:gr-qc/9304026 [gr-qc]`.

[87] D. Salt, "Action principles for the elastic solid and the perfect fluid in general relativity," *J. Phys. A.* **4** (1971) 501.

[88] B. Carter, "Speed of sound in a high-pressure general-relativistic solid," *Phys. Rev. D* **7** (Mar, 1973) 1590–1593. `http://link.aps.org/doi/10.1103/PhysRevD.7.1590`.

[89] T. Harmark and N. A. Obers, "General definition of gravitational tension," *JHEP* **05** (2004) 043, `arXiv:hep-th/0403103`.

[90] P. T. Chrusciel, D. Maerten, and P. Tod, "Rigid upper bounds for the angular momentum and centre of mass of non-singular asymptotically anti-de Sitter space-times," *JHEP* **0611** (2006) 084, `arXiv:gr-qc/0606064 [gr-qc]`.

[91] G. W. Gibbons, M. J. Perry, and C. N. Pope, "The First Law of Thermodynamics for Kerr-Anti-de Sitter Black Holes," *Class. Quant. Grav.* **22** (2005) 1503–1526, `arXiv:hep-th/0408217`.

[92] R. Gregory and R. Laflamme, "Black strings and p-branes are unstable," *Phys.Rev.Lett.* **70** (1993) 2837–2840, `arXiv:hep-th/9301052 [hep-th]`.

[93] T. Harmark, V. Niarchos, and N. A. Obers, "Instabilities of black strings and branes," *Class.Quant.Grav.* **24** (2007) R1–R90, `arXiv:hep-th/0701022 [hep-th]`.

[94] O. J. Dias, P. Figueras, R. Monteiro, J. E. Santos, and R. Emparan, "Instability and new phases of higher-dimensional rotating black holes," *Phys.Rev.* **D80** (2009) 111701, `arXiv:0907.2248 [hep-th]`.

[95] O. J. Dias, P. Figueras, R. Monteiro, and J. E. Santos, "Ultraspinning instability of anti-de Sitter black holes," *JHEP* **1012** (2010) 067, `arXiv:1011.0996 [hep-th]`.

[96] O. J. Dias, P. Figueras, R. Monteiro, H. S. Reall, and J. E. Santos, "An instability of higher-dimensional rotating black holes," *JHEP* **1005** (2010) 076, `arXiv:1001.4527 [hep-th]`.

[97] S. Hollands, A. Ishibashi, and R. M. Wald, "A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric," *Commun. Math. Phys.* **271** (2007) 699–722, `arXiv:gr-qc/0605106`.

[98] M. Vasilic and M. Vojinovic, "Classical spinning branes in curved backgrounds," *JHEP* **0707** (2007) 028, `arXiv:0707.3395 [gr-qc]`.

[99] A. Papapetrou, "Spinning test particles in general relativity. 1.," *Proc.Roy.Soc.Lond.* **A209** (1951) 248–258.

[100] E. Corinaldesi and A. Papapetrou, "Spinning test particles in general relativity. 2.," *Proc.Roy.Soc.Lond.* **A209** (1951) 259–268.

[101] W. Dixon, "Dynamics of extended bodies in general relativity: Equations of Motion," *Trans. R. Soc.* **A277** (1974) 59.

[102] J. Souriou, "Modele de particule spin dans le champ electromagnetique et gravitationnel," *Ann. Inst. H. Poincare* **20** (1974) 315.

[103] G. A. Maugin, "Exact relativistic theory of wave propagation in prestressed nonlinear elastic solids," *Annales de l'institut Henri Poincaré (A) Physique théorique* **28** (1978) no. 2, 155–185. `http://eudml.org/doc/75976`.

[104] J. Kijowski and G. Magli, "Relativistic elastomechanics is a gauge type theory," `arXiv:hep-th/9411212 [hep-th]`.

[105] M. Fukuma and Y. Sakatani, "Conformal higher-order viscoelastic fluid mechanics," *JHEP* **1206** (2012) 102, `arXiv:1204.6288 [hep-th]`.