FACULTY OF SCIENCE
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Ph.D. Thesis

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Hydrodynamics and Elasticity of Charged Black Branes

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Submitted: September 1, 2013

Abstract

We consider long-wavelength perturbations of charged black branes to first order in a fluid-elastic derivative expansion. At first order the perturbations decouple and we treat the hydrodynamic and elastic perturbations separately. To put the results in a broader perspective, we present the first-order corrected dynamics of fluid branes carrying higher-form charge by obtaining the general form of their equations of motion to pole-dipole order in the absence of external forces. To monopole order, we characterize the corresponding effective theory of viscous fluid branes by writing down the general form of the first-order dissipative corrections in terms of the shear and bulk viscosities as well as the transport coefficient associated with charge diffusion. To dipole order, we furthermore, applying linear response theory, characterize the corresponding effective theory of stationary bent charged (an)isotropic fluid branes in terms of two sets of response coefficients, the Young modulus and the piezoelectric moduli. We subsequently consider a large class of examples in gravity of this effective theory. In particular, we consider dilatonic black p-branes in two different settings: charged under a Maxwell gauge field and charged under a $(p+1)$-form gauge field, including the D-branes and M-branes of type II string theory and M theory, respectively. Using familiar techniques, we compute the associated transport coefficients and uncover how the shear and bulk viscosities are modified in the presence of electric charge and a dilaton coupling. For the case of Maxwell black branes we furthermore compute the charge diffusion constant. We find that the shear viscosity to entropy bound is saturated and comment on proposed bounds for the bulk viscosity to entropy ratio. With the transport coefficients we compute the first-order dispersion relations of the effective fluid and analyze the dynamical stability of the black branes. We then focus on constructing stationary strained charged black brane solutions to first order in a derivative expansion. Using solution generating techniques and the bent neutral black brane as a seed solution, we obtain a class of charged black brane geometries carrying smeared Maxwell charge in Einstein-Maxwell-dilaton theory. In the specific case of ten-dimensional space-time we furthermore use T-duality to generate bent black branes with higher-form charge, including smeared D-branes of type II string theory. We compute the bending moment and the electric dipole moment which these solutions acquire due to the strain and uncover that their form is captured by classical electroelasticity theory. In particular, we find that the Young modulus and the piezoelectric moduli of the strained charged black brane solutions are parameterized by a total of four response coefficients, both for the isotropic as well as for the anisotropic cases.

Contents

Acknowledgements

I would like to express my thanks to my closest friends, colleagues, and family for all your help and support doing my time as a Ph.D. student, especially doing the period of writing this thesis. In particular, I would like to give my thanks to my advisor Niels A. Obers for your helpful guidance. I also want to give my thanks and respect to the two amazing persons: Jácome Armas and Andreas Vigand Pedersen.

List of Publications

[1] J. Armas, J. Gath, and N. A. Obers, "Black Branes as Piezoelectrics," *Phys.Rev.Lett.* **109** (2012) 241101, arXiv:1209.2127 [hep-th].

[2] J. Gath, J. Hartong, R. Monteiro, and N. A. Obers, "Holographic Models for Theories with Hyperscaling Violation," *JHEP* **1304** (2013) 159, arXiv:1212.3263 [hep-th].

[3] J. Gath and A. V. Pedersen, "Viscous Asymptotically Flat Reissner-Nordström Black Branes," arXiv:1302.5480 [hep-th].

[4] J. Armas, J. Gath, and N. A. Obers, "Electroelasticity of Charged Black Branes," arXiv:1307.0504 [hep-th].

Introduction

Black hole space-times in higher-dimensional Einstein gravity have been the subject of intense study. Indeed, the phase structure of these space-times has been found to be far more complex and intricate than their four dimensional counterparts [1–6]. Clearly, this originates from the considerable increase in degrees of freedom of the higher-dimensional theory, but the two pioneering findings that explicitly showed that the physics of higher-dimensional black holes was much richer came with the discovery of dynamical instabilities of extended black hole horizons [7] and the possibility of non-spherical horizon topology [8] as well as the fact that higher-dimensional black holes can not be fully characterized by their conserved charges [9]. This has laid the foundation for many advances in the classification and study of the properties of higher-dimensional black hole space-times.

As is well-known, general relativity is incorporated in the framework of string theory. The fact that it arises within the framework of a consistent quantum theory is perhaps the most salient feature, since it makes it a promising candidate for describing quantum gravity. At low energies string theory is described by higher-dimensional theories of gravity, specifically in terms of the various supergravities. Black hole space-times of Einstein gravity can therefore be regarded as a powerful tool to gain insights into the more fundamental theory as they arise in the consistent truncations of supergravity. Furthermore, since the space-time dimension is a dynamical concept in string theory much of the motivation for understanding extra space dimensions actually originates from string theory. Even though string theory may not be the true theory of nature, a noteworthy accomplishment is the first successful microscopic statistical counting of black hole entropy [10].

Another instance where black hole space-times play a significant role is in the context of gauge/gravity dualities. These dualities, that originated within the framework of string theory, are concrete realizations of the holographic principle in the sense that they relate two quantum theories of different dimensions i.e. higher-dimensional gravitational theories to (non-gravitational) gauge theories in space-times of one dimension less. In this context black holes are the necessary tool for studying finite temperature phases of the quantum field theory (QFT). The best established example of holography is the much acclaimed AdS/CFT correspondence [11–14], where conformal field theories (CFT) are dual to string theory on backgrounds that contain anti-de Sitter (AdS) space. In this setting black holes have been the key ingredient in making qualitative predictions of the very high temperature deconfinement phase of quantum chromodynamics (QCD) which to much success has been observed experimentally at high-energy colliders. However, there is also a wide-ranging effort to use holography in a more generic way. In particular, one of the most active directions is the application to the study of condensed matter systems [15–18], where gravitational models for hyperscaling-violating Lifshitz space-times have been proposed as gravity duals [19–22]. Finding gravitational models that possess specific properties of interest in this context has been studied in e.g. [23].

Even if string theory is replaced by an alternative framework, black hole space-times are still solutions of general relativity, albeit in higher dimensions, and as such they are among the most important Lorentzian manifolds. The study of higher-dimensional gravity is therefore also of genuine interest in the sense that solutions of different space-time dimensions can give valuable insights into the nature of gravity. For instance, understanding which properties of black holes are universal and which show dependence on the space-time dimension. By now black hole mechanics is established as a universal feature while the possible horizon topologies, maybe not surprisingly, evidently depend on the dimension. Indeed, as mentioned above, by changing the space-time dimension one will discover a rich structure of interconnecting phases with different horizon topologies and various stability issues.

An important feature that appears in higher dimensional gravity is the existence of spatial extended black objects i.e. black branes. Even though these have compact flat directions at infinity they are related to asymptotically flat black holes. This can be understood by using the basic intuition of curving a black string into the form of a black ring. A construction like this would make the black ring contract due to the tension of the black string, but with sufficient rotation this may be counterbalanced simply due to centrifugal repulsion. Although, quite basically put, this example provides us with two essential ingredients accounting for many of the novel features of higher dimensional black holes: extended horizons and rotation. In particular, higher dimensional black holes are known to exist in ultra-spinning regimes (i.e. no Kerr bound) [1, 8, 24, 25]. A regime which opens up for the possibility for having a horizon that has two widely separated length-scales associated with it. An observation that lead to the foundation and development of the blackfold approach [26, 27], where a black hole is regarded as a black brane whose world-volume spans a curved submanifold in a background space-time, i.e. a blackfold.

In fact, some black holes look locally like flat black branes in various regimes e.g. in their ultra-spinning or near-extremal limits. It is in these regimes that the long-wavelength physics of the black brane can be described by the blackfold approach by replacing the small-wavelength physics with an effective source. The effective source is thus characterized by a set of collective variables consisting partly of intrinsic parameters of the black brane and partly by the scalars that describe the position in the background space-time. To leading order, the effective source of the black brane takes the form of a fluid living on a flat surface, but one can continue to refine the description of the source by letting the collective variables fluctuate slowly over the world-volume of the black brane in a controlled manner. This will modify the effective source and therefore influence the long-wavelength physics.

Indeed, through the study of long-wavelength perturbations of black branes it has been shown that they behave much like any other type of continuous media whose dynamics is governed by specific effective theories. These properties arise from two types of deformations: time-(in)dependent fluctuations along the world-volume directions [28–33] and stationary perturbations along directions transverse to the world-volume [34–40]. The former is characterized by an effective theory of viscous fluid flows as is well known from the hydrodynamic limit of the AdS/CFT correspondence [28], while the latter is characterized by an effective theory of thin elastic branes [26, 34, 41, 42]. Both of these descriptions are unified in a general framework of fluids living on dynamical surfaces (fluid branes), which, when applied to black branes, is known collectively as the blackfold approach. The blackfold approach can therefore be understood as the effective theory that describe the long-wavelength perturbations of black branes. It is a general framework that accommodates both the fluid/gravity correspondence and membrane paradigm [43, 44]. In this sense, the fluid system does not need to live on the boundary of some space-time (fluid/gravity) nor on the black hole horizon (membrane paradigm), but can also live in an intermediate region.

The focus of this thesis is to obtain the transport and response coefficients that characterize the effective theory of charged black branes. As a starting point, we will in section 1 first review the first-order corrected dynamics of charged fluid branes by deriving the general form of their equations of motion in the absence of external forces. The derivation is specifically based on a multipole expansion of the stress-energy tensor developed by [45, 46] and contains the analysis of neutral branes [37] and branes with higher-form charges [39, 40]. The effective dynamics of the fluid brane is captured by the effective stress-energy tensor and current that, in general, are given by a fluid-elastic derivative expansion. Fluctuations along the world-volume directions of the brane give rise to fluid dynamic dissipative corrections while fluctuations in the directions transverse to the brane give rise to elastic contributions.

We will subsequently see how the effective theory of charged fluid branes can encompass the gravitational physics of the hydrodynamic and elastic sector of the long-wavelength perturbations of black branes under appropriate assumptions. In particular, we will consider a family of black p-branes of Einstein gravity coupled to a dilaton and a single $(q+1)$-form gauge field [47]. This family of solutions exist in any dimension and has both the dilaton coupling and the (integer) dimension q as free parameters. We will restrict the analysis to first-order corrections for which the two perturbative sectors decouple. This work is therefore separated into two main sections, summarized below: one part that considers hydrodynamic perturbations and one part that considers elastic perturbations. However, we mention that the interesting question about how and when higher-order perturbations couple has been partially answered for stationary black brane configurations in [42].

Hydrodynamic perturbations

In section 2, we consider dissipative correction to the leading order fluid characterizing the effective description of charged black branes. In particular, we compute the first-order hydrodynamic transport coefficients of dilatonic black branes in two different settings: first charged under a Maxwell gauge field ($q = 0$) and second charged under a $(p+1)$-form gauge field ($q = p$). This extends and incorporates the cases studied in [33, 43, 48] and will be presented as a part of a future publication [49]. In particular, we note that the former case contains the Reissner-Nordström black brane, while the latter case contains the supergravity descriptions of the D-branes and M-branes. For both cases the leading order effective stress-energy tensor takes the form of a boosted perfect fluid that accounts for the thermodynamics of the brane. We put emphasis on how the presence of charge modifies the fluid transport coefficients of the neutral black brane originally considered in [32] and in particular for the system with Maxwell charge we obtain the transport coefficient associated with charge diffusion. In detail, we consider long-wavelength fluctuations around the black brane solution in Einstein-Maxwell-dilaton theory and higher-form gauge field generalization thereof. We solve the full set of coupled equations to first order for both cases in the derivative expansion and compute the first-order corrected effective stress-energy tensor and current. This provides us with the charged generalizations of respectively the shear and bulk viscosities along with the charge diffusion constant.

Having computed the shear and bulk viscosities we find that the bound $\eta/s \geq 1/4\pi$ is saturated for both systems. This agrees with the expectation that this should hold for any two-derivative gravity theory [50, 51]. In addition, we consider the different proposals for a bulk viscosity to entropy bound [52–54]. Discrepancies are found for the Maxwell black brane which may not be surprising, since the bounds relies heavily on holographic considerations, but it is clear that in their current forms these bounds are not universal.

The transport coefficients also allows us to study the dynamical stability of the black branes. Indeed, by computing the speed of sound in the effective fluid of the neutral black brane ref. [32] was able to identify the unstable sound mode of the effective fluid with the Gregory-Laflamme (GL) instability [7, 55]. This simple computation can in fact already be carried out at the perfect fluid level [27]. The results of [32] allowed further refinement of this result and showed remarkable agreement with numerical data. We perform a similar computation for the $q = 0$ and $q = p$ systems, respectively, and obtain the speed of sound and next-to-leading order dispersion relations. We find that both systems suffers from a GL instability for large values of the dilaton coupling for all charge densities. For sufficiently small values of the dilaton coupling the instability continues to occur, but as the charge density is increased above their individual thresholds both systems appears to be stable, at least to next-to-leading order. The general feature is therefore that a sufficiently large charge density can stabilize the black branes. Although, both systems show the same overall behavior, the details of their individual behavior differ quite significantly as we shall see.

In many ways, studying intrinsic fluctuations of branes in the blackfold approach is similar in spirit to the well-known fluid/gravity correspondence of AdS/CFT [28]. We mention that the computation in fluid/gravity analogous to the Maxwell system with zero dilaton coupling (fluctuations of the AdS Maxwell black branes of co-dimension 1 with a Chern-Simons term) was carried out in the papers [30, 31, 56], and furthermore for Kaluza-Klein dilaton coupling in [53, 54]. However, we emphasize that our computation deals with asymptotically flat branes of general co-dimension and that the effective fluid stress-energy tensor has no direct interpretation as a dual fluid of a QFT. Also note that the fluid stress-energy tensor is not that of a conformal fluid.

Recently, a relation between black brane solutions in asymptotically AdS and Ricci-flat black brane solutions was established. This was done by constructing a map from asymptotically AdS solutions compactified on a torus to a corresponding Ricci-flat solution obtained by replacing the torus by a sphere [57]. This was used to take the general second order results of fluid/gravity [29] and map them to the second order blackfold stress-energy tensor. This gave even further improvement of the dispersion relation of the GL instability of the neutral black brane. In this work we will provide evidence that the map can be applied in more general cases between asymptotically AdS solutions and solutions that are asymptotically flat i.e in this case to theories that includes a dilaton and a gauge field (thus to theories that are not Ricci-flat). In particular, we show that the computed transport coefficients map exactly to results for the corresponding systems on the AdS side. The extension of the map will be made precise in [49].

Elastic perturbations

In section 3, we will consider stationary elastic perturbations along transverse directions to the world-volume of charged black branes. This type of deformation is achieved by breaking the symmetries of the transverse space to the brane world-volume directions in the same way that the circular cross-section of a rod is deformed when it is bent. Such perturbations have been studied in [34–38] for neutral black branes and in [39, 40] for charged asymptotically flat dilatonic black branes (on which this work is based). In these cases, to first order in the derivative expansion, the metric acquires a bending moment while, in the case of charged branes, the gauge field acquires an electric dipole moment which encode the brane response to applied strains. Recall, that as a consequence of placing a fluid on a dynamical surface embedded in a background space-time the induced metric changes when deformed along transverse directions and that change is the measure of the strain [37, 41].

According to the classical theory of elasticity, the bending moment encodes the response coefficients of the material to applied strains [58]. For a generic material these coefficients are a set of elastic moduli that are described by a tensor structure with the name of Young modulus. For the case of neutral black branes these have been measured in [37, 38] and have been recently classified using the general framework of [42]. If the material is electrically charged, according to the theory of electroelasticity, the gauge field will develop an electric dipole moment whose strength is proportional to a set of piezoelectric moduli [59]. This effect was first measured in [39] for asymptotically flat charged dilatonic black strings in Einstein-Maxwell-dilaton theory and later in [40] for the larger class of dilatonic black branes charged under higher-form gauge fields that we will consider here.

We will show that a large class of examples in gravity exhibit the electroelastic phenomena suggested by applying linear response theory to the dipole contributions we encounter in the general analysis of the equations of motion at pole-dipole order of charged fluid branes. In particular, we use standard solution generating techniques and the bent neutral black brane of [38] as a seed solution, to construct stationary strained charged black brane solutions to first order in a derivative expansion. In this way we first obtain a set of bent charged black brane solutions carrying smeared Maxwell ($q = 0$) charge in Einstein-Maxwell-dilaton theory with Kaluza-Klein coupling constant. The corresponding effective theory describing the perturbed solution is that of an isotropic fluid brane which has been subject to pure bending. In the specific case of ten-dimensional space-time we furthermore use T-duality to generate bent black branes charged under higher-form fields. This includes supergravity descriptions for type II Dq-branes smeared in $(p-q)$-directions, which in general are described by the effective theory of anisotropic p-branes carrying q-brane charge.

By measuring the bending moment and the electric dipole moment which these solutions acquire due to the strain, we uncover that their form is captured by classical electroelasticity theory. In particular, we obtain the Young modulus and piezoelectric moduli from the bent charged black brane solutions. We find that these are parameterized by a total of four response coefficients, both for the isotropic as well as anisotropic cases. These measurements constitute the first step in obtaining higher order corrections to the charged stationary black holes found in [47,60]. We emphasize that the resulting bending moment and electric dipole moment that we obtain are specific cases of dipole contributions that fit within the general framework of dipole-corrected equations of motion of fluid branes. Explicitly verifying that these first-order corrected equations of motion are satisfied for the cases we have considered is beyond the scope of the thesis.[1]

One motivation for bending charged black branes stems from the origin of the blackfold approach, namely, that it is an analytic method that can be used to construct approximate analytic black hole solutions by wrapping black branes along a submanifold with the desired topology. In this setting, the approach has provided a way to probe the vast space of possible higher-dimensional solutions. For example neutral black holes were found in Minkowski background in [34,36] and in (A)dS background in [35,61]. Charged black holes were considered in [47] and further in a supergravity setting in [60]. Earlier work where the blackfold methodology has been applied include [34,35,62–64]. Furthermore, the purpose of constructing explicit approximate curved black brane solutions in the context of the blackfold approach has been to show that perturbed solutions that satisfy the leading order blackfold equations given by Carter's equations [65] also have regular horizons. It was proven for neutral black branes in [38], but the analysis does not a priori apply to charged black branes, since they are solutions to a different set of equations of motion. We will find that all the charged solutions that are constructed in this work satisfy horizon regularity. However trivial, this provides evidence that it might also hold for more general charged black brane solutions. Finally, a valuable outcome of the blackfold approach is the connection between its effective theory and that of improved effective actions for QCD [66,67]. Uncovering the structure of the response coefficients for blackfold solutions therefore provide novel insights into the general structure of the these effective theories.

[1] See [37,42] for work in this direction for the case of bent neutral black strings.

The structure of this thesis is the following

Outline:

- In chapter 1, we present the first-order corrected dynamics of fluid branes carrying higher-form charge by considering the general form of their equations of motion to pole-dipole order. For infinity thin branes, we review the equations of motions and solve them under the assumption of stationarity. We furthermore introduce the transport coefficients and response coefficients that characterize the effective theory. In the last part, we introduce the family of Gibbons-Maeda black brane solutions of Einstein-Maxwell-dilaton theory and higher-dimensional generalizations thereof and show how to extract the effective stress-energy tensor and current.

  Significant parts of this chapter are based on [40].

- In chapter 2, we consider long-wavelength perturbations of charged dilatonic black branes along the directions of the world-volume. In particular, we consider two different settings, one in which the black brane is charged under a Maxwell gauge field and one where it is charged under a $(p+1)$-form gauge field. We compute the transport coefficients of the corresponding effective fluid and present the shear and bulk viscosities as well as the charge diffusion constant. Furthermore, we compute the next-to-leading order dispersion relations and consider the dynamical stability of the black branes.

  These results are not published yet, but will be included in a future publication [49]. They extend the analysis published in [33].

- In chapter 3, we review the perturbative framework for constructing bent black brane solutions in order to gain understanding of the following construction. We then construct strain black brane solutions by using a solution generating technique and the neutral black brane as seed. Subsequently, we extract the bending moment and electric dipole moment and compute the Young modulus and piezoelectric moduli.

  These results are based on [39, 40].

We conclude with a discussion of the results, possible generalizations and open problems.

- Appendix A contains the detailed computation of chapter 2 in the case of a dilatonic black p-brane charged under a $(p+1)$-form gauge field.

- Appendix B gives a detailed derivation of the equation of motion for fluid branes carrying Maxwell charge and string charge, respectively, as well as the conjectured form of the equation of motion for fluid branes charged under higher-form fields.

Chapter 1

Blackfolds

In this chapter, we will setup the effective theory for a charged fluid configuration living on a dynamical surface of small but finite thickness. In general, we will refer to fluids living on dynamical surfaces of arbitrary co-dimension embedded in a background space-time as fluid branes. We begin in section 1.1 by considering the pole-dipole expansion of the stress-energy tensor and current that characterize the fluid as well as the equations of motion governing these charged fluid branes. In section 1.2, we truncate the expansion at zeroth order and consider the case of fluid configurations living on a dynamical surface of vanishing thickness. At this order we consider a perfect fluid satisfying locally the first law of thermodynamics and study in detail their equations of motion for different instances of charge dissolved in their world-volume. We then briefly discuss the general form of first-order dissipative hydrodynamic corrections to these fluid configurations. In section 1.3 we focus on stationary fluid configurations and review the general solution of the hydrodynamic equations when stationarity is imposed as well as how conserved surface quantities can be constructed. In section 1.4, we look at the structures appearing in the decomposition of the effective stress-energy tensor and current at pole-dipole order and provide a physical interpretation of the bending moment, electrical dipole moment, and spin current. Lastly, in section 1.5, we introduce the black brane solutions in (super)gravity that we will consider in this work and describe how the effective sources that provide the input for the effective description in terms of fluid branes are obtained from these geometries.

1.1 Dynamics of charged pole-dipole branes

In this section, we obtain the equations of motion for charged pole-dipole branes by solving conservation equations for the effective stress-energy tensor and effective current that characterize the branes. For this purpose we consider a $(p+1)$-dimensional submanifold whose world-volume $\mathcal{W}_{p+1}$ is embedded in a D-dimensional background manifold with space-time metric $g_{\mu\nu}(x^{\alpha})$. The space-time coordinates are x^{α}, $\alpha = 0, ..., D-1$ and the dimension of the background manifold is parametrized by $D = p + n + 3$. The submanifold is therefore of co-dimension $n+2$. We will sometimes refer to the submanifold as a surface even though it is $(p+1)$-dimensional. The position of the surface is given by the embedding functions

$X^\mu(\sigma^a)$ with the world-volume coordinates σ^a, $a = 0, ..., p$. The indices μ, ν label space-time indices and a, b label directions along the world-volume. Associated to the geometry of the submanifold is the induced metric $\gamma_{ab} = g_{\mu\nu} u_a^\mu u_b^\nu$ with $u_a^\mu \equiv \partial_a X^\mu$ and the extrinsic curvature ${K_{ab}}^\rho \equiv \nabla_a u_b^\rho$ that encodes the profile of the embedding. Here we have defined the world-volume covariant derivative $\nabla_a \equiv u_a^\rho \nabla_\rho$ which acts on a generic space-time tensor $V^{c\mu}$ as

$$\nabla_a V^{c\mu} = \partial_a V^{c\mu} + \gamma_{ab}^c V^{b\mu} + \Gamma^\mu_{\lambda\rho} u_a^\lambda V^{c\rho} \quad , \tag{1.1}$$

where γ_{ab}^c are the Christoffel symbols associated with γ_{ab} and $\Gamma^\mu_{\lambda\rho}$ are the Christoffel symbols associated with $g_{\mu\nu}$. The extrinsic curvature is symmetric in its lower indices. Given the induced metric on $\mathcal{W}_{p+1}$ the first fundamental form of the submanifold is given by $\gamma^{\mu\nu} = \gamma^{ab} u_a^\mu u_b^\nu$. It has the action of projecting tensors onto the directions tangent to the submanifold. In addition, one can also form the orthogonal projection tensor $\perp_{\mu\nu} = g_{\mu\nu} - \gamma_{\mu\nu}$. We begin by considering the multipole expansion of the stress-energy tensor for curved branes.

1.1.1 Effective stress-energy tensor and current

Finite thickness effects of an object can be probed by bending it. Physically, this is because bending induces a varying concentration of matter along transverse directions to the brane world-volume resulting in a non-trivial bending moment [37]. If the brane was infinitely thin this effect would not be present. In order to include finite thickness effects in the brane dynamics one performs a multipole expansion of the stress-energy tensor in the manner [46]

$$T^{\mu\nu}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma} \left[T^{\mu\nu}_{(0)} \frac{\delta^D(x^\alpha - X^\alpha)}{\sqrt{-g}} - \nabla_\rho \left(T^{\mu\nu\rho}_{(1)} \frac{\delta^D(x^\alpha - X^\alpha)}{\sqrt{-g}} \right) + ... \right] \quad , \tag{1.2}$$

where we have omitted the explicit dependence of $T^{\mu\nu}_{(0)}, T^{\mu\nu\rho}_{(1)}$ and X^α on the world-volume coordinates σ^a. The stress-energy tensor (1.2) is characterized by two structures: $T^{\mu\nu}_{(0)}$ is a monopole source of stress-energy while $T^{\mu\nu\rho}_{(1)}$ encodes the dipole (finite thickness) effects. To each of these structures one associates an order parameter $\tilde{\varepsilon}$ such that $T^{\mu\nu}_{(0)} = \mathcal{O}(1)$ and $T^{\mu\nu\rho}_{(1)} = \mathcal{O}(\tilde{\varepsilon})$. Typically, for branes of thickness r_0 bent over a submanifold of characteristic curvature radius R, the parameter $\tilde{\varepsilon}$ has the form $\tilde{\varepsilon} = r_0/R$. If the expansion (1.2) is truncated to $\mathcal{O}(\tilde{\varepsilon})$, the stress-energy tensor is said to be expanded to pole-dipole order. When only hydrodynamic corrections are considered, the stress-energy tensor is localized on the surface described by $X^\mu(\sigma^a)$ due to the delta-function in (1.2), since in this case $T^{\mu\nu\rho}_{(1)} = 0$ while $T^{\mu\nu}_{(0)}$ receives viscous corrections order-by-order in a derivative expansion. The general form of the first-order hydrodynamic corrections are discussed in section 1.2.2. When elastic perturbations are considered the brane acquires a bending moment which is encoded in $T^{\mu\nu\rho}_{(1)}$. In this case, there exists an ambiguity in the position of the world-volume surface within a finite region of thickness r_0, which is parametrized by the 'extra symmetry 2' acting on $T^{\mu\nu}_{(0)}$ and $T^{\mu\nu\rho}_{(1)}$ under a $\mathcal{O}(\tilde{\varepsilon})$ displacement of the world-volume location $X^\alpha(\sigma^a) \to X^\alpha(\sigma^a) + \tilde{\varepsilon}^\alpha(\sigma^a)$ [46].

The equations of motion for an object with a stress-energy tensor of the type (1.2), assuming the absence of external forces and ignoring backreaction, follow from the conservation equation

$$\nabla_\nu T^{\nu\mu} = 0 \quad . \tag{1.3}$$

In section 1.1.2, we decompose $T_{(1)}^{\mu\nu\rho}$ and write down the equation of motion in a way adapted for the cases considered in this work.

If the brane is charged under a $(q+1)$ gauge field, it is also characterized by a total anti-symmetric current tensor $J^{\mu_1\cdots\mu_{q+1}}$ which can also be expanded in a Dirac-delta series as [39]

$$J^{\mu_1\cdots\mu_{q+1}}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\left[J_{(0)}^{\mu_1\cdots\mu_{q+1}}\frac{\delta^D}{\sqrt{-g}} - \nabla_\rho\left(J_{(1)}^{\mu_1\cdots\mu_{q+1}\rho}\frac{\delta^D}{\sqrt{-g}}\right) + ...\right] \quad , \tag{1.4}$$

where we have omitted the explicit dependence of $J_{(0)}^{\mu_1\cdots\mu_{q+1}}$ and $J_{(1)}^{\mu_1\cdots\mu_{q+1}\rho}$ on the world-volume coordinates σ^a and used the abbreviated form $\delta^D \equiv \delta^D(x^\alpha - X^\alpha(\sigma))$. As in the case of the stress-energy tensor (1.2) the structure $J_{(0)}^{\mu_1\cdots\mu_{q+1}}$ is a monopole source of a charged q-brane current while the structure $J_{(1)}^{\mu_1\cdots\mu_{q+1}\rho}$ encodes the finite thickness effects, including the electric dipole moment of the brane. Moreover, the structures involved in (1.4) follow the same hierarchy as in the case of the (1.2), i.e., $J_{(0)}^{\mu_1\cdots\mu_{q+1}} = \mathcal{O}(1)$ and $J_{(1)}^{\mu_1\cdots\mu_{q+1}\rho} = \mathcal{O}(\tilde{\varepsilon})$. The equations of motion for the current (1.4) follow from the conservation equation

$$\nabla_{\mu_1} J^{\mu_1\cdots\mu_{q+1}} = 0 \quad . \tag{1.5}$$

The equations of motion that follow from here have been previous derived by Dixon and Souriou in [68, 69] for charged point particles $(q = 0)$[1] in a different way than the one presented in [39]. In section 1.1.2 we will review these equations for p-branes carrying Maxwell charge $(q = 0)$. For details about the derivation we refer to appendix B where also the equations of motion for p-branes carrying q-brane charge with $q > 1$ are treated.

1.1.2 Decomposition and equations of motion

In order to write the equations of motion in a useful form we decompose $T_{(1)}^{\mu\nu\rho}$ into tangential and orthogonal parts with the help of the orthogonal projector $\perp^\mu{}_\nu$ such that

$$T_{(1)}^{\mu\nu\rho} = u_b{}^{(\mu} j^{|b|\nu)\rho} + u_a^\mu u_b^\nu d^{ab\rho} + u_a^\rho T_{(1)}^{\mu\nu a} \quad . \tag{1.6}$$

Here the vertical bars indicate that the index b is insensitive to the symmetrization which is done only over the space-time indices μ, ν. Moreover, $j^{b\nu\rho}$ are the components responsible for giving transverse motion (spin) to the brane and have been considered by Papapetrou when deriving the equations of motion for spinning point particles [71]. These have the properties $j^{b\nu\rho} = j^{b[\nu\rho]}$ and $u_\nu^a j^{b\nu\rho} = 0$. The components $d^{ab\rho}$ have the properties $d^{ab\rho} = d^{(ab)\rho}$ and $u_\rho^c d^{ab\rho} = 0$ and encode the bending moment of the brane. In the point particle case these components can be gauged away using the 'extra symmetry 2' [37], but not for the cases $p > 0$. The components $T_{(1)}^{\mu\nu a}$ can be gauged away everywhere on the world-volume using the 'extra symmetry 1' and can be set to zero at the boundary in the absence of additional boundary sources [46]. As we are only interested in bending corrections, i.e. $j^{b\nu\rho} = 0$, we can write the equations of motion as [37, 42]

$$\nabla_a \hat{T}^{ab} + u_\mu^b \nabla_a \nabla_c d^{ac\mu} = d^{ac\mu} R^b{}_{ac\mu} \quad , \tag{1.7}$$

[1] See ref. [70] for a recent review, including a treatment of the case when external forces are present.

$$\hat{T}^{ab}K_{ab}{}^{\rho} + \perp^{\rho}{}_{\mu}\nabla_a\nabla_b d^{ab\mu} = d^{ab\mu}R^{\rho}{}_{ab\mu} \quad , \tag{1.8}$$

where $\hat{T}^{ab} = T^{ab}_{(0)} + 2d^{(ac\mu}K^{b)}{}_{c\mu}$. Here $R^{\rho}{}_{\nu\lambda\mu}$ is the Riemann curvature tensor of the background space-time. If finite thickness effects are absent, $d^{ab\mu} = 0$, one recovers the equations of motion derived by Carter [65] and if one further takes $T^{ab}_{(0)}$ to be of the perfect fluid form with energy density and pressure of a black brane, these equations are the leading order blackfold equations [26, 27] which we will consider in section 1.2 in the case of perfect fluids with conserved q-brane charges on their world-volumes.

The equations (1.7)-(1.8) are relativistic generalizations of the equations of motion of thin elastic branes [42] and must be supplemented with the integrability condition $d^{ab[\mu}K_{ab}{}^{\rho]} = 0$ and boundary conditions

$$d^{ab\rho}\eta_a\eta_b|_{\mathcal{W}_{p+1}} = 0 \quad , \quad \left(\hat{T}^{ab}u^{\mu}_b - d^{ac\rho}K^b{}_{c\rho}u_b{}^{\mu} + \perp^{\mu}{}_{\rho}\nabla_b d^{ab\rho}\right)\eta_a|_{\mathcal{W}_{p+1}} = 0 \quad , \tag{1.9}$$

where η_a is a unit normal vector to the brane boundary. The equations (1.7)-(1.9) are also valid for charged (dilatonic) branes as long as couplings to external background fields are absent [47, 60, 72].

Pole-dipole p-branes carrying Maxwell charge ($q = 0$) are characterized by a current J^{μ} of the form (1.4). To write down the associated equations of motion for which the details are given in appendix B, we decompose $J^{\mu}_{(0)}$ and $J^{\mu\nu}_{(1)}$ in terms of tangential and orthogonal components such that

$$J^{\mu}_{(0)} = J^{a}_{(0)}u^{\mu}_a + J^{\mu}_{\perp(1)} \quad , \quad J^{\mu\nu}_{(1)} = m^{\mu\nu} + u^{\mu}_a p^{a\nu} + J^{\mu a}_{(1)}u^{\nu}_a \quad , \tag{1.10}$$

where $m^{\mu\nu}$ is transverse in both indices and satisfies $m^{\mu\nu} = m^{[\mu\nu]}$, while $p^{a\rho}$ is transverse in its space-time index. The structure $J^{\mu a}_{(1)}$ is neither parallel nor orthogonal to the world-volume and satisfies $J^{[ab]}_{(1)} = 0$. With the decompositions of $J^{\mu}_{(0)}$ and $J^{\mu\nu}_{(1)}$, we can write the equations of motion as

$$J^{\mu}_{\perp(1)} = \perp^{\mu}{}_{\nu}\nabla_a\left(p^{a\nu} + J^{\nu a}_{(1)}\right) \quad , \tag{1.11}$$

$$\nabla_a\left(\hat{J}^a + p^{b\mu}K^a{}_{b\mu}\right) = 0 \quad , \tag{1.12}$$

where $\hat{J}^a = J^a_{(0)} - u^a_{\mu}\nabla_b J^{\mu b}_{(1)}$. The second equation corresponds to world-volume current conservation. Note that in the case $J^{\mu\nu}_{(1)} = 0$ for which the brane is infinitely thin, the equation (1.12) reduces to that obtained previously in the literature using the same method [73]. The equations of motion must be supplemented by the boundary conditions

$$\left(p^{a\mu} + J^{\mu a}_{\perp(1)}\right)\eta_a|_{\partial\mathcal{W}_{p+1}} = 0 \quad , \quad J^{ab}_{(1)}\eta_a\eta_b|_{\partial\mathcal{W}_{p+1}} = 0 \quad , \tag{1.13}$$

$$\nabla_{\hat{a}}J^{\hat{a}}_{(1)} - \eta_a\left(\hat{J}^a + p^{b\mu}K^a{}_{b\mu}\right)\Big]\,|_{\partial\mathcal{W}_{p+1}} = 0 \quad , \tag{1.14}$$

where we have defined the boundary degrees of freedom $J^{\hat{a}}_{(1)} = J^{ab}_{(1)}v^{\hat{a}}_b\eta_a$ accounting for possible extra current sources on the brane boundary. Note that the structure $m^{\mu\nu}$ entering in the decomposition of $J^{\mu\nu}_{(1)}$ does not play a role in the equation of motion (1.12) neither in the boundary conditions (1.13)-(1.14) though it may be relevant when considering external couplings to background fields.

It should be mentioned that the expansion of the current 1.4 also enjoys the two symmetries as the stress-energy tensor 1.2 coined by the authors of [46] as 'extra symmetry 1' and 'extra symmetry 2'. The extra symmetries and invariance of the equations of motion are also discussed in appendix B. In particular, one finds that it is possible to gauge away one of the structures in the decomposition (1.10).[2] This is why the last term in (1.10) was left neither parallel nor orthogonal to the world-volume. Furthermore, for $q = 0$ invariance of the equations of motion requires that $J^{ab}_{(1)} = J^{(ab)}_{(1)}$.

The derivation of the equations of motion for branes carrying string charge ($q = 1$) or branes charged under higher-form fields ($q > 1$) follows a similar pattern as above for branes carrying Maxwell charge ($q = 0$). For further details we refer to the sections B.2 and B.3 of the appendix. Here we just note that the generalization leads to the two objects: $m^{a_1\dots a_q\mu_{q+1}\rho}$ satisfying the properties $m^{a_1\dots a_q\mu_{q+1}\rho} = m^{[a_1\dots a_q]\mu_{q+1}\rho} = m^{a_1\dots a_q[\mu_{q+1}\rho]}$ and $p^{a_1\dots a_{q+1}\rho}$ satisfying the property $p^{a_1\dots a_{q+1}\rho} = p^{[a_1\dots a_{q+1}]\rho}$.

1.2 Fluids on dynamical surfaces

In this section, we will consider infinitely thin branes, $d^{\mu\nu\rho} = 0$, characterized by $T^{ab}_{(0)}$ taking the form of the stress-energy tensor of a perfect fluid. The stress-energy tensor given by equation (1.2) is thus truncated at order $\mathcal{O}(1)$. Inserting it into equation (1.3) one finds that the stress-energy tensor has support on the $(p+1)$-dimensional world-volume $\mathcal{W}_{p+1}$, i.e. $\perp^{\rho}{}_{\mu} T^{\mu\nu}_{(0)} = 0$. It is therefore natural to write $T^{\mu\nu}_{(0)} = T^{ab}_{(0)} u^{\mu}_a u^{\nu}_b$ such that the equations of motion of the fluid branes can be written as the tangentially divergence of the stress-energy tensor [65]

$$\gamma^{\rho}{}_{\nu}\nabla_{\rho}T^{\mu\nu}_{(0)} = 0 \quad , \quad T^{\mu\nu}_{(0)}\eta_{\nu}|_{\partial\mathcal{W}_{p+1}} = 0 \quad . \tag{1.15}$$

With this particular form of the stress-energy tensor the equations can be separated into $D - p - 1$ equations in directions orthogonal to the world-volume governing the *extrinsic* dynamics and $p + 1$ equations parallel to the world-volume $\mathcal{W}_{p+1}$ governing the *intrinsic* dynamics. The equations of motion (1.15) can therefore be written as

$$T^{ab}_{(0)}K_{ab}{}^{\rho} = 0 \quad , \tag{1.16}$$

$$\nabla_a T^{ab}_{(0)} = 0 \quad , \tag{1.17}$$

subject to the boundary condition $T^{ab}_{(0)}u^{\mu}_a\eta_b|_{\partial\mathcal{W}_{p+1}} = 0$. The equations can also be obtained directly from (1.7)-(1.9) by setting $d^{\mu\nu\rho} = 0$. The intrinsic equations are interpreted as conservation of the world-volume stress-energy and encode the hydrodynamic equations in the case of $T^{ab}_{(0)}$ describing a fluid. The extrinsic equations determines the coupling of the stress-energy tensor to the extrinsic geometry. It is interesting to write out the extrinsic curvature in terms of the embedding functions,

$$T^{ab}_{(0)} \perp_{\sigma}{}^{\rho} \left(\partial_a\partial_b X^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}\right) = 0 \quad , \tag{1.18}$$

thus giving the interpretation as a generalized geodesic equation for p-branes or simply as "mass times acceleration equal to zero". In the case where the brane is carrying a q-brane

[2] Except on the boundary, see appendix B.

charge, the fluid is in addition to the stress-energy tensor also characterized by a totally antisymmetric current tensor $J^{\mu_1 \cdots \mu_{q+1}}$. Inserting the current (1.4) truncated to order $\mathcal{O}(1)$ in the equations of motion (1.5) implies that the current only takes tangential components, $J_{(0)}^{\mu_1 \ldots \mu_{q+1}} = J_{(0)}^{a_1 \ldots a_{q+1}} u_{a_1}^{\mu_1} \ldots u_{a_{q+1}}^{\mu_{q+1}}$ and leads to the world-volume current continuity equations

$$\nabla_{a_1} J_{(0)}^{a_1 \ldots a_{q+1}} = 0 \quad , \quad J_{(0)}^{a_1 \ldots a_{q+1}} \eta_{a_1} |_{\partial \mathcal{W}_{p+1}} \quad . \tag{1.19}$$

The equations can also be obtained directly from (1.11)-(1.14) by setting $J_{(1)}^{\mu\nu} = 0$.

In the following we will review the cases studied in [47, 60] for p-branes with q-brane charges dissolved in their world-volume, $0 \leq q \leq p$ under the assumption that dissipative effects are absent. In section 1.2.2, we will briefly consider the form of first-order dissipative corrections.

1.2.1 Fluid branes with q-brane charges

We consider a fluid configuration that supports several conserved q-brane currents on its world-volume. Let $\hat{V}_{q+1}$ denote the $(q+1)$-dimensional volume form of the world-sheet of the q-brane current, then the q-brane current can be expressed on differential form as

$$J_{(q+1)}^{(0)} = \mathcal{Q}_q \hat{V}_{q+1} \quad , \tag{1.20}$$

where $\mathcal{Q}_q$ is the local q-brane charge density. Here we assume, for each current, the existence of integral $(q+1)$-dimensional submanifolds $\mathcal{C}_{q+1} \subset \mathcal{W}_{p+1}$ parallel to the projector $h_{ab}^{(q)}$; that acts as the projector onto the space parallel to the world-sheet of the q-brane current.[3] The charge density $\mathcal{Q}_q$ is thus constant along the submanifold $\mathcal{C}_{q+1}$, but is allowed to vary along the $p-q$ directions transverse to the current.

In addition to the currents, the perfect fluid is also characterized by a stress-energy tensor. The presence of the currents causes the pressure densities in directions parallel and transverse to differ which in general makes the fluid anisotropic. Introducing the local fluid velocity u^a and the potential Φ_q conjugate to $\mathcal{Q}_q$, the stress-energy tensor takes the general form [60][4]

$$T_{(0)}^{ab} = \mathcal{T} s u^a u^b - \mathcal{G} \gamma^{ab} - \sum_{q=0}^{p} \Phi_q \mathcal{Q}_q h_{(q)}^{ab} \quad , \tag{1.21}$$

where $\mathcal{T}$ is the local temperature, s is the entropy density and the Gibbs free energy density is given by

$$\mathcal{G} = \epsilon - \mathcal{T} s - \sum_{q=0}^{p} \Phi_q \mathcal{Q}_q \quad , \tag{1.22}$$

with the energy density ϵ. Furthermore, the fluid satisfies (locally) the first law of thermodynamics

$$d\epsilon = \mathcal{T} ds + \sum_{q=0}^{p-1} \Phi_q d\mathcal{Q}_q \quad . \tag{1.23}$$

[3]It can be shown from the current continuity equations $d * J = 0$ using Frobenius theorem that the currents are required to be surface-forming [47, 60]. We will assume this in the following.

[4]The authors of [60] does not claim that this is the universal form for any charged brane. It should however be quite generic.

We now turn to explicit cases where the fluid configuration carries a single q-brane current. We begin with the simplest case where the world-volume carries p-brane charge [60]. In this case the charge can not redistribute itself and is therefore trivially conserved over the world-volume and in many aspects this configuration is similar to the neutral case [27]. We then turn to the case of a world-volume theory with smeared q-brane charges [47]. For these configurations the charges are allowed to redistribute which leads to non-trivial conservation equations. The simplest case is for a world-volume theory with Maxwell charges $q = 0$, since in this case the stress-energy tensor still takes the form of an isotropic perfect fluid while for higher brane charges $0 < q < p$ the stress-energy tensor is anisotropic as mentioned.

Fluids with p-brane charge

We consider the dynamics of a perfect fluid that lives on the world-volume $\mathcal{W}_{p+1}$ carrying a p-brane current studied in [60]. Let $\hat{V}_{p+1}$ denote the volume form on $\mathcal{W}_{p+1}$, then the current can be expressed as

$$J^{(0)}_{(p+1)} = \mathcal{Q}_p \hat{V}_{p+1} \quad , \tag{1.24}$$

where $\mathcal{Q}_p$ is the local charge density. The world-volume conservation equations (1.19) implies that $\partial_a \mathcal{Q}_p = 0$. The charge density is therefore not allowed to vary along the world-volume directions and equals the total integrated charge Q_p. This means that there is no local degree of freedom associated with the charge Q_p and the fluid variables reduce to those of a neutral fluid, that is, the local fluid velocity u^a and the energy density ϵ. The effective world-volume theory is therefore given in terms of an isotropic perfect fluid with stress-energy tensor

$$T^{ab}_{(0)} = \epsilon u^a u^b + P \Delta^{ab} \quad , \tag{1.25}$$

where we have introduced the orthogonal projector $\Delta^{ab} \equiv \gamma^{ab} + u^a u^b$ and P is the pressure density. The intrinsic dynamics is governed by the world-volume conservation equations given by (1.17). For the stress-energy tensor given by equation (1.25) they evaluate to the energy continuity equation (by projecting with u^a) and the Euler equation (by projecting with Δ^{ab}),

$$\dot{\epsilon} = -w\vartheta, \quad \dot{u}^a = -w^{-1}\Delta^{ab}\partial_b P \quad , \tag{1.26}$$

where $\vartheta \equiv \nabla_a u^a$ is the expansion of u^a and a dot denotes the directional derivative along u^a. Since the fluid satisfies (locally) the thermodynamic relations

$$d\epsilon = \mathcal{T} ds \quad , \quad w \equiv \epsilon + P = \mathcal{T} s \quad , \tag{1.27}$$

the energy continuity equation can also be expressed as the conservation of the entropy density while the Euler equation can be expressed as a relation between the fluid acceleration $\dot{u}$ and the temperature gradient. The equations of motion (1.26) are therefore equivalent to

$$\nabla_a (s u^a) = 0 \quad , \tag{1.28}$$

and

$$\Delta^{ab}(\dot{u}_b + \partial_b \log \mathcal{T}) = 0 \quad . \tag{1.29}$$

In addition to the intrinsic equations, the world-volume is also regarded as being dynamical. The dynamics of the world-volume is governed by the extrinsic equations given by (1.16). For the perfect isotropic fluid (1.25) these equations evaluate to

$$PK^{\rho} = -w \perp^{\rho}{}_{\mu} \dot{u}^{\mu} \quad , \tag{1.30}$$

where we have used that the vector u is tangent to the world-volume $\mathcal{W}_{p+1}$ and therefore satisfy $u^{\mu}u^{\nu}K_{\mu\nu}{}^{\rho} = \perp^{\rho}{}_{\mu} \dot{u}^{\mu}$. This reveals that the acceleration of the fluid element along the transverse directions is given by the mean extrinsic curvature $K^{\rho} \equiv \gamma^{ab}K_{ab}{}^{\rho}$. One can also use the thermodynamic relation (1.27) to write

$$PK^{\rho} = -s\mathcal{T} \perp^{\rho}{}_{\mu} \dot{u}^{\mu} \quad . \tag{1.31}$$

Finally, note that the above analysis is equivalent to considering a neutral fluid configuration, because Q_p does not appear in thermodynamical relations (1.27). However, the presence of the charge will of course show up in the equation of state of the fluid.

Fluids with Maxwell charge

Perfect fluids with Maxwell charge ($q = 0$) dissolved in their world-volume studied in [47,60] provide the simplest non-trivial example of fluid configurations with charge, since the stress-energy tensor remains on the form of a perfect isotropic fluid (1.25). The conservation of stress-energy is therefore given by the equations (1.26), but are now supplemented by a non-trivial current conservation equation (1.19).

The one-form current supported on the $(p+1)$-dimensional world-volume is proportional to the fluid velocity u which we can express as

$$J^{a}_{(0)} = \mathcal{Q}u^{a} \quad , \tag{1.32}$$

with $\mathcal{Q}$ denoting the Maxwell charge density.[5] The current conservation equation (1.19) evaluates to

$$\dot{\mathcal{Q}} = -\mathcal{Q}\vartheta \quad , \tag{1.33}$$

where $\vartheta = \nabla_a u^a$ is the expansion of the velocity field. The charge density can therefore vary along the world-volume adding an extra degree of freedom to the system. Again, thermodynamical equilibrium is satisfied locally thus the fluid satisfy the first law and thermodynamic relation given by

$$d\epsilon = \mathcal{T}ds + \Phi d\mathcal{Q} \quad , \quad w \equiv \epsilon + P = \mathcal{T}s + \Phi\mathcal{Q} \quad , \tag{1.34}$$

where Φ is the electric potential conjugate to $\mathcal{Q}$. Using the thermodynamic relations and the current conservation equation (1.33), the energy continuity equation (1.26) can again be shown to be equivalent to conservation of entropy density (1.28), while the Euler force equation takes the form

$$\Delta^{ab}\left[s\mathcal{T}(\dot{u}_b + \nabla_b \log \mathcal{T}) + \mathcal{Q}\Phi(\dot{u}_b + \nabla_b \log \Phi)\right] = 0 \quad . \tag{1.35}$$

[5] We will omit the index q on $\mathcal{Q} \equiv \mathcal{Q}_q$ and $\Phi \equiv \Phi_q$ when it is clear from the context what is referred to.

The extrinsic equations again evaluates to equation (1.30), however, using (1.34) one can write

$$PK^{\rho} = -(s\mathcal{T} + \Phi\mathcal{Q}) \perp^{\rho}{}_{\mu} \dot{u}^{\mu} \quad , \tag{1.36}$$

making the presence of the charge explicit.

Fluids with string charge

There is a natural way of writing the stress-energy tensor and current tensor for fluids carrying string currents ($q = 1$) in their world-volume [47]. In particular, the string current can be expressed as

$$J^{ab}_{(0)} = 2\mathcal{Q}u^{[a}v^{b]} \quad , \tag{1.37}$$

where v is a normalized spacelike eigenvector $v^2 = 1$ which satisfy $u \cdot v = 0$. This defines the string charge density $\mathcal{Q}$. The current conservation equation (1.19) evaluates to

$$\dot{\mathcal{Q}} = -\mathcal{Q}\vartheta - \mathcal{Q}u \cdot v' \quad , \quad \mathcal{Q}' = -\mathcal{Q}\hat{\vartheta} + \mathcal{Q}v \cdot \dot{u} \quad , \tag{1.38}$$

by projection along u and v, respectively. The prime denotes a directional derivative along v^a i.e. $v' \equiv v^a \nabla_a v$, and we have defined $\hat{\vartheta} \equiv \nabla_a v^a$. As mentioned above, the current is assumed to spans a two-dimensional submanifold $\mathcal{C}_2 \subset \mathcal{W}_{p+1}$ parallel to the projector $h_{ab} = -u_a u_b + v_a v_b$.

The presence of string charge breaks the isotropy of the world-volume. The stress-energy tensor therefore takes the form of an anisotropic perfect fluid characterized by two pressures; the pressure density $P_{\|}$ along the spatial direction in which the string charge lie and the pressure density $P_{\perp}$ transverse to that direction,

$$T^{ab}_{(0)} = \epsilon u^a u^b + P_{\|} v^a v^b + P_{\perp}\Delta^{ab} \quad , \tag{1.39}$$

now with $\Delta^{ab} = \gamma_{ab} - h_{ab}$. Evaluating the equations of motion (1.17) along u and v one finds

$$\dot{\epsilon} = -w\vartheta - (P_{\perp} - P_{\|})u \cdot v' \quad , \quad wv \cdot \dot{u} = (P_{\perp} - P_{\|})\hat{\vartheta} - P_{\|}' \quad , \tag{1.40}$$

with $w \equiv \epsilon + P_{\perp}$. Now using that the difference in pressures is given by the effective tension $\Phi\mathcal{Q}$ along the current i.e. $P_{\perp} - P_{\|} = \Phi\mathcal{Q}$ together with the thermodynamic relations (1.34) with $P = P_{\perp}$, the first set of the equations in (1.38) and (1.40), respectively, is equivalent to conservation of entropy density (1.28). Orthogonal projection of the stress-energy conservation equations leads to

$$\Delta^{ab}\left[w\dot{u}_b - (P_{\perp} - P_{\|})v_b' + \nabla_b P_{\perp}\right] = 0 \quad . \tag{1.41}$$

The Euler force equations (given by orthogonal projection and projection onto v) can be rewritten using the thermodynamic relations. The second set of the equations in (1.38) and (1.40) lead to

$$v^a v^b s\mathcal{T}(\dot{u}_b + \nabla_b \log \mathcal{T}) = 0 \quad , \tag{1.42}$$

while (1.41) gives

$$\Delta^{ab}\left[s\mathcal{T}(\dot{u}_b + \nabla_b \log \mathcal{T}) + \Phi\mathcal{Q}(\dot{u}_b - v_b' + \nabla_b \log \Phi)\right] = 0 \quad . \tag{1.43}$$

The extrinsic equations (1.16) evaluate to

$$P_\perp K^\rho = - \perp^\rho{}_\mu \left(s\mathcal{T}\dot{u}^\mu + \Phi\mathcal{Q}\hat{K}^\mu \right) \quad , \tag{1.44}$$

where $\hat{K}^\rho = h^{ab}K_{ab}{}^\rho$ is the mean curvature of the embedding of the world-sheet $\mathcal{C}_2$.

It is worth pointing out that the equations of motion in the different cases, as expected, show a similar structure. For example, the last form of the extrinsic equations given by (1.44) is written in very suggestive form. Indeed, one can identify the mean curvature $\hat{K}^\rho = h^{ab}_{(q)}K_{ab}{}^\rho$ of the embedding of the world-volume $\mathcal{C}_{q+1}$ for each of the different cases given by (1.31), (1.36) and (1.44). Similarly, one can for the Euler force equations (1.29), (1.35) and (1.43) in each case identify the pullback of the mean curvature $\hat{K}^a = u^a_\mu \hat{K}^\mu$. Furthermore, projecting the intrinsic equations along u in each case lead to conservation of entropy density.

1.2.2 Viscous stress-energy tensor and current

Finally, we will briefly discuss the general form of the first-order viscous corrections to the stress-energy tensor and current in the case of Maxwell charge ($q = 0$). We thus consider the corrections to a generic hydrodynamic flow to first order in a derivative expansion when the stress-energy tensor is localized on the surface described by $X^\mu(\sigma)$ and $T^{\mu\nu\rho}_{(1)} = 0, J^{\mu\nu}_{(1)} = 0$.

We write the stress-energy tensor and the current as

$$\hat{T}^{ab} = T^{ab}_{(0)} + \Pi^{ab}_{(1)} + \mathcal{O}(\partial^2) \quad , \quad \hat{J}^a = J^a_{(0)} + \Upsilon^a_{(1)} + \mathcal{O}(\partial^2) \quad , \tag{1.45}$$

where the derivative expansion is considered with respect to slowly varying world-volume fields in a controlled way such that the magnitude of the fluctuations are small along the world-volume (see section 2.2). The tensors $\Pi^{ab}_{(1)}$ and $\Upsilon^{ab}_{(1)}$ are the first-order dissipative derivative corrections to the perfect fluid stress-energy tensor and current, respectively. As is well-known, to any order in derivatives, it is in principle possible to write down all the terms that can contribute to the stress-energy tensor and current (see e.g. [74]). In this way the dissipative corrections to the stress-energy tensor and the current can be characterized in terms of a set of transport coefficients. It is possible to show that the most general form of $\Pi^{ab}_{(1)}$ is given by[6]

$$\Pi^{ab}_{(1)} = -2\eta\sigma^{ab} - \zeta\vartheta\Delta^{ab} \quad , \tag{1.46}$$

where σ^{ab} is the usual shear tensor and is given by

$$\sigma^{ab} = \Delta^{ac}\left(\partial_{(c}u_{d)} - \Delta_{cd}\frac{\vartheta}{p}\right)\Delta^{db} \quad \text{with} \quad \vartheta = \nabla_a u^a \quad . \tag{1.47}$$

The coefficients η and ζ are respectively the shear and bulk viscosity transport coefficients and were computed for the neutral brane in [32]. We will compute them in the case of charged branes in section 2. The viscosities η and ζ are required to be positive in order to ensure entropy creation in the fluid [75].

[6]Here we have imposed the Landau frame gauge on the stress-energy tensor $u_a\Pi^{ab}_{(1)} = 0$. Similarly one can impose a Landau frame condition on the current. It takes the form $u_a\Upsilon^a_{(1)} = 0$.

Using similar reasoning, it is possible to show that the most general form of $\Upsilon^a_{(1)}$ (in the Landau frame) is given by[7]

$$\Upsilon^a_{(1)} = -\mathfrak{D}\left(\frac{\mathcal{Q}\mathcal{T}}{w}\right)^2 \Delta^{ab}\partial_b\left(\frac{\Phi}{\mathcal{T}}\right) \quad . \tag{1.48}$$

Indeed, it is possible to derive that with $\mathfrak{D} > 0$, the term (1.48) is the only term which can be constructed from the fields and that is consistent with the 2nd law of thermodynamics [75].

1.3 Stationary fluid configurations

We will now focus on stationary perfect fluid configurations. In particular, we will solve the intrinsic equations following the analysis of [27, 47, 60]. Assuming a stationary fluid flow we shall see that the solution can be completely specified providing a set of vector fields, a global temperature and a global electric potential leaving only the embedding functions $X^\mu(\sigma^a)$ to be determined.

1.3.1 Intrinsic dynamics

In order to describe stationary fluid configurations one must require the existence of a timelike Killing vector field in the background space-time that naturally satisfies the Killing equation

$$\nabla_{(\mu}k_{\nu)} = 0 \quad , \tag{1.49}$$

and whose pullback $\mathbf{k}_a = u^\mu_a k_\mu$ form the world-volume timelike Killing vector field. Now, in order for the fluid to be stationary, it can not have dissipative terms in its stress-energy tensor. In general this means that the corrections to the stress-energy tensor must vanish i.e. the shear and expansion of the velocity field u given in equation (1.47) have to vanish. For general stationary fluid configurations [76], the velocity of the fluid u must be proportional to the world-volume timelike Killing vector field $\mathbf{k} = \mathbf{k}^a\partial_a$,

$$\mathbf{k} = |\mathbf{k}|u, \quad |\mathbf{k}| = \sqrt{-\gamma_{ab}\mathbf{k}^a\mathbf{k}^b} \quad . \tag{1.50}$$

From the world-volume Killing equation $\nabla_{(a}\mathbf{k}_{b)} = 0$, one has

$$\nabla_{(a}u_{b)} = -u_{(a}\nabla_{b)}\log|\mathbf{k}| \quad , \tag{1.51}$$

and $\mathbf{k}^a\partial_a|\mathbf{k}| = 0$ with $|\mathbf{k}| \neq 0$. The acceleration is therefore given by

$$\dot{u}_a = \nabla_a \log|\mathbf{k}| \quad . \tag{1.52}$$

This is sufficient to determine the solution of the intrinsic equations (1.29) for fluid configurations with unsmeared charge,

$$\mathcal{T}(\sigma^a) = \frac{T}{|\mathbf{k}|} \quad , \tag{1.53}$$

[7] It is possible to include a parity violating term as was found in [30]. However, since we will not consider Chern-Simons terms such a term is not relevant.

where T is an integration constant with the interpretation of the global temperature of the fluid. The local temperature is thus obtained from a redshift of the uniform global temperature.

In the case of smeared charges the intrinsic equations have the additional freedom from the potential Φ_q. As for the temperature, the potential can not depend on time, since it is incompatible with stationarity. Furthermore, due to the equipotential condition for electric equilibrium [60], the potential can not depend on directions transverse to the current [47]. It is therefore possible to form a constant by integrating over the directions along the current,

$$\Phi_H^{(q)} = \int_{\mathcal{C}_{q+1}} d^q\sigma \sqrt{|h^{(q)}(\sigma^a)|}\Phi_q(\sigma) \quad , \tag{1.54}$$

with the interpretation of the global q-brane potential. Here, $\sqrt{|h^{(q)}(\sigma^a)|}$ is the volume element on $\mathcal{C}_{q+1}$.

For stationary fluid branes carrying Maxwell charge ($q = 0$), the intrinsic equations can be solved by taking (1.53) while the global potential fixes the local potential on the world-volume by

$$\Phi_0(\sigma^a) = \frac{\Phi_H^{(0)}}{|\mathbf{k}|} \quad . \tag{1.55}$$

The solution of the intrinsic equations is therefore completely specified given the Killing vector field $\mathbf{k}$ and the constants T and $\Phi_H^{(0)}$.

For stationary fluid branes carrying string charge ($q = 1$), it is necessary to specify the solution for v. To proceed we assume the existence of a spacelike Killing vector field ψ, that commutes with $\mathbf{k}$, in which the string charge lie along. We then construct the component orthogonal to $\mathbf{k}$[8]

$$\boldsymbol{\zeta}^a = \psi^a - (\psi^b u_b)u^a \quad , \tag{1.56}$$

satisfying

$$\boldsymbol{\zeta}^a\nabla_{(a}\boldsymbol{\zeta}_{b)} = 0 \quad , \quad \mathbf{k}^a\nabla_{(a}\boldsymbol{\zeta}_{b)} \quad , \quad \nabla_a\boldsymbol{\zeta}^a = 0 \quad . \tag{1.57}$$

Now taking

$$\boldsymbol{\zeta} = |\boldsymbol{\zeta}|v, \quad |\boldsymbol{\zeta}| = \sqrt{-\gamma_{ab}\boldsymbol{\zeta}^a\boldsymbol{\zeta}^b} \quad , \tag{1.58}$$

we have $[u, v] = 0$, which is the necessary and sufficient condition for the vector fields to be surface-forming, that is, they form $\mathcal{C}_2$. The spatial vector v satisfies

$$\hat{\vartheta} \equiv \nabla_a v^a = 0 \quad , \quad v'_a = -\nabla_a \log|\boldsymbol{\zeta}| \quad , \quad v \cdot \dot{u} = 0 \quad . \tag{1.59}$$

Using these relations and equation (1.51), the current conservation equations (1.38) is therefore $\mathcal{Q}' = 0$. With equation (1.53) the electric potential Φ_1 is therefore constant along v due to equation (1.43). Hence, assuming that ζ^a has compact orbits of periodicity 2π the local potential is given by

$$\Phi_1(\sigma^a) = \frac{1}{2\pi}\frac{\Phi_H^{(1)}}{|\mathbf{k}||\boldsymbol{\zeta}|} \quad . \tag{1.60}$$

[8] Note that since the function $\psi^b u_b$ can vary in directions transverse to ψ and $\mathbf{k}$, the vector $\boldsymbol{\zeta}$ is in general not a Killing vector field.

The solution of the intrinsic equations are therefore completely specified given the timelike Killing vector field $\mathbf{k}$ and the spacelike vector field $\boldsymbol{\zeta}$ together with the constants T and $\Phi_H^{(1)}$.

This completes the analysis of the solutions to the intrinsic equations of the system. The remaining extrinsic equations constrain the embedding functions of the surface that the stationary fluid configuration lives on. Explicit solutions to these equations in the case of configurations carrying charge have been considered in [47, 60]. However, we will just note that in general under small deformations, it is possible to show that the surface, even at this order, behaves as an elastic brane [41]. Furthermore, it should be mentioned that the extrinsic equations for stationary configurations can also be obtained from an action formalism. Recently, it was shown how the effective action formalism for neutral fluid brane configurations can be related to the multipole expansion of the stress-energy tensor. This precise relation was put forward by Armas [42]. It is expected that an action formalism for stationary fluid configurations that carries charge would be very similar in the case of $q = 0$ and $q = p$, but new contributions would need to be added in the case of the anisotropic fluids with smeared charge $1 < q < p$.

1.3.2 Physical quantities

For stationary fluid configurations it is possible to form conserved surface quantities by assuming that the stationary Killing vector field $\mathbf{k}$ is given by a combination of linear independent world-volume Killing vector fields,

$$\mathbf{k} = \xi + \sum_i \Omega_{(i)} \chi_{(i)} \quad , \tag{1.61}$$

where $\xi^a \partial_a$ is the world-volume timelike Killing vector field assumed to be hypersurface orthogonal to $\mathcal{W}_{p+1}$ and whose norm is the redshift factor R_0 i.e. $\sqrt{-\xi^2} = R_0$. Furthermore, $\chi^a_{(i)} \partial_a$ is a set of rotational Killing vector fields with the associated constant angular velocities $\Omega_{(i)}$.

The mass and angular momenta can be obtained by integrals of the corresponding effective currents $\xi^b T^{(0)}_{ab}$ and $\chi^b_{(i)} T^{(0)}_{ab}$ over the spatial section of the world-volume $\mathcal{B}_p$ with coordinates σ^i,

$$M = \int_{\mathcal{B}_p} dV_{(p)} \xi^b T^{(0)}_{ab} n^a \quad , \quad J_{(i)} = -\int_{\mathcal{B}_p} dV_{(p)} \chi^b_{(i)} T^{(0)}_{ab} n^a \quad , \tag{1.62}$$

where $dV_{(p)} = d^p\sigma \sqrt{\gamma^{(p)}}$ is the measure on $\mathcal{B}_p$ with $\gamma^{(p)}_{ij}$ being the induced metric of the spatial section of the world-volume and $n^a = \xi^a / R_0$ the associated unit normal vector. The total charge passing through the spacelike hypersurface $\mathcal{B}_p$ can be obtained by the integral

$$Q_0 = -\int_{\mathcal{B}_p} dV_{(p)} J^{(0)}_a n^a \quad \text{for} \quad q = 0 \quad , \tag{1.63}$$

where the minus sign will ensure that a positive charge density and a future-pointing normal vector will give a positive total charge. To obtain the total q-brane charge we need to consider spatial sections of the submanifold $\mathcal{C}_{q+1}$ orthogonal to n^a. Let the associated volume q-form of these sections be given by

$$\omega_{(q)} = \frac{\hat{V}_{q+1} \cdot n}{\sqrt{-h^{(q)}_{ab} n^a n^b}} \quad , \tag{1.64}$$

then the total q-brane charge can be obtained as

$$Q_q = -\int_{\mathcal{B}_{p-q}} dV_{(p-q)} J^{(0)}_{q+1} \cdot (n \wedge \omega_q) \quad . \tag{1.65}$$

The total entropy can be obtained from the conserved entropy current su^a as

$$S = -\int_{\mathcal{B}_p} dV_{(p)} s_a n^a \quad . \tag{1.66}$$

Finally, the conserved quantities satisfy the Smarr relation [47, 60]

$$(D-3)M - (D-2)\left(\sum_i \Omega_i J_i + TS\right) - \sum_q (D-3-q)\Phi^{(q)}_H Q_q = \mathcal{T}_{\text{tot.}} \quad , \tag{1.67}$$

with the total tensional energy obtained by integrating the local tension,

$$\mathcal{T}_{\text{tot.}} = -\int_{\mathcal{B}_p} dV_{(p)} R_0 \left(\gamma^{ab} + n^a n^b\right) T^{(0)}_{ab} \quad . \tag{1.68}$$

For Minkowski backgrounds, where $R_0 = 1$, one has the zero tension condition $\mathcal{T}_{\text{tot.}} = 0$. In instances where the extrinsic equations (1.16) reduces to one equation, this equation will be equivalent to the zero tension condition. Finally, it is worth mentioning that in the presence of an external field in the background, the conserved quantities can be obtained using the prescription given in [73].

1.4 Brane electroelasticity

In this section, we discuss the physical interpretation of the structures entering the dipole contribution of the stress-energy tensor and the electric current. The physical interpretation of the structures $j^{b\mu\nu}$ and $d^{ab\rho}$ introduced in equation (1.6) was given in [37, 42] while the physical interpretation of the different structures appearing in the decompositions of the electric current (1.10) and (B.49) was given in [39, 40]. We will review these structures here. Furthermore, we will discuss the form of these general dipole contributions under the assumption that the charged fluid can be described by a covariantized linear response theory inspired by classical electroelasticity theory. This will involve the introduction of response coefficients corresponding to the Young modulus and piezoelectric moduli.

1.4.1 Bending moment and Young modulus

As mentioned in the beginning of section 1.1.1 the structure $d^{ab\rho}$ accounts for the bending moment of the brane [37, 42]. To see this note that we can compute the total bending moment from the stress-energy tensor (1.2) by

$$D^{ab\rho} = \int_{\Sigma} d^{D-1}x\sqrt{-g}T^{\mu\nu}u^a_\mu u^b_\nu x^\rho = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}d^{ab\rho} \quad , \tag{1.69}$$

where Σ is a constant timeslice in the bulk space time and we have ignored boundary terms (which we will continue to do so in the following). Hence we identify $d^{ab\rho}$ as the bending moment density on the brane. Note that for the case of a point particle ($p = 0$), $d^{ab\rho}$ has only one non-vanishing component, namely, $d^{\tau\tau\rho}$ where τ is the proper time coordinate of the world-line. Since the stress-energy tensor (1.2) also enjoys the 'extra symmetry 2' acting as $\delta_2 d^{ab\rho} = -T^{ab}_{(0)}\tilde{\varepsilon}^\rho$ one can, by an appropriate choice of $\tilde{\varepsilon}^\rho$, gauge away the component $d^{\tau\tau\rho}$ [37]. Thus point particles do not carry world-volume mass dipoles but in the case $p > 0$ these components cannot, in general, be gauged away.

The bending moment $d^{ab\rho}$ is a priori unconstrained but assuming that the brane will behave according to classical (Hookean) elasticity theory we consider it to be of the form

$$d^{ab\rho} = \tilde{Y}^{abcd}K_{cd}{}^\rho \quad , \tag{1.70}$$

which is the form of the bending moment expected for a thin elastic brane that has been subject to pure bending. Here, the extrinsic curvature $K_{cd}{}^\rho$ has the interpretation of the Lagrangian strain since it measures the variation of the induced metric on the brane along transverse directions to the world-volume while $\tilde{Y}^{abcd}$ is the Young modulus of brane.[9]

The linear response exhibited in equation (1.70) will be analyzed for the case of bending deformations of fluid branes which are stationary. In these situations the general structure of $\tilde{Y}^{abcd}$ has been classified for neutral isotropic fluids using an effective action approach [42]. For the isotropic cases studied here, making a slight generalization to the case of p-branes with world-volume Maxwell charge, it takes the form [42][10]

$$\begin{aligned}\tilde{Y}^{abcd} = -2\Big(&\lambda_1(\mathbf{k};T,\Phi_H)\gamma^{ab}\gamma^{cd} + \lambda_2(\mathbf{k};T,\Phi_H)\gamma^{a(c}\gamma^{d)b} + \lambda_3(\mathbf{k};T,\Phi_H)\mathbf{k}^{(a}\gamma^{b)(c}\mathbf{k}^{d)} \\ &+\lambda_4(\mathbf{k};T,\Phi_H)\frac{1}{2}(\mathbf{k}^a\mathbf{k}^b\gamma^{cd} + \gamma^{ab}\mathbf{k}^c\mathbf{k}^d) + \lambda_5(\mathbf{k};T,\Phi_H)\mathbf{k}^a\mathbf{k}^b\mathbf{k}^c\mathbf{k}^d\Big) \quad ,\end{aligned} \tag{1.71}$$

where $\mathbf{k}^a$ is the Killing vector field along which the fluid is moving given by equation (1.50). We have also indicated explicitly the dependence on the global temperature T, and the generalization compared to the neutral isotropic case of [42] is that there is now in addition a dependence on the global chemical potential Φ_H given by equation (1.55). The Young modulus $\tilde{Y}^{abcd}$ satisfies the expected properties of a classical elasticity tensor $\tilde{Y}^{abcd} = \tilde{Y}^{(ab)(cd)} = \tilde{Y}^{cdab}$.

We will find explicit realizations of (1.71) in section 3.3.2 when we consider the bending of black p-branes with Maxwell charge. For this we note that not all of the five terms in the expression in (1.71) are independent. In fact, due to the 'extra symmetry 2', these include gauge dependent terms of the form $k\left(T^{ab}_{(0)}\gamma^{cd} + T^{cd}_{(0)}\gamma^{ab}\right)$, where k is a gauge parameter

[9] We use the convention that $\tilde{Y} = YI$ (omitting tensor indices) where Y is the conventionally normalized Young modulus and I the moment inertia of the object with respect to the choice of world-volume surface.

[10] Note that the Young modulus $\tilde{Y}^{abcd}$ introduced here is related to the one introduced in [42] via the relation $\tilde{Y}^{abcd} = -\mathcal{Y}^{abcd}$.

corresponding to the choice of origin. In the end only three out of the five λ-coefficients are independent, when using also the equations of motion.

It is not the purpose of this work to construct the effective action for anisotropic fluid branes. However, as seen in section 1.2.1, we note that the simplest case of p-branes carrying string charge with $p > 1$, are characterized by the additional vector v satisfying $v \cdot u = 0$ and $v^2 = 1$, aligned in the direction of the smeared string charge along the brane world-volume. Following the analysis of [42], there are four further response coefficients that can in principle be added to the effective action and, in turn, to the Young modulus defined in (1.71), restricting to terms that contain only even powers of u^a and/or v^a. The expression should therefore be supplemented with a contribution of the form

$$\begin{aligned}\hat{Y}^{abcd} = -2 \Big(& \lambda_6(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H)\mathbf{k}^a\mathbf{k}^b\boldsymbol{\zeta}^c\boldsymbol{\zeta}^d + \lambda_7(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H)\boldsymbol{\zeta}^a\boldsymbol{\zeta}^b\mathbf{k}^c\mathbf{k}^d \\ & + \lambda_8(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H)\mathbf{k}^{(a}\boldsymbol{\zeta}^{b)}\mathbf{k}^{(c}\boldsymbol{\zeta}^{d)} + \lambda_9(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H)\boldsymbol{\zeta}^a\boldsymbol{\zeta}^b\boldsymbol{\zeta}^c\boldsymbol{\zeta}^d \Big) \quad .\end{aligned} \tag{1.72}$$

Note that for this case the λ-coefficients introduced in (1.71) are now also functions of the non-normalized space-like vector $\boldsymbol{\zeta}^a$ given by equation (1.58). More generally, for branes with smeared q-brane charge with $0 < q < p$ one can introduce a set of vectors $v^a_{(i)}$, $i = 1, ..., q$, such that $v^a_{(i)} v^{(j)}_a = \delta^j_i$ and furthermore $v^a_{(i)} u_a = 0$. As a consequence, one can have for every vector $v^a_{(i)}$, a contribution of the form (1.72), but clearly more complicated contributions can appear as well. The analysis of this is beyond the scope of this thesis.

We note that the introduction of the new terms in (1.72) does not a priori guarantee that the expected classical symmetries $\hat{Y}^{abcd} = \hat{Y}^{(ab)(cd)} = \hat{Y}^{cdab}$ are preserved. However, one should properly take into account that just as in (1.71), as a consequence of gauge freedom not all of the terms in (1.72) are independent. In fact, we will see that for the particular cases of charged black branes with $q > 0$, the terms appearing in (1.72) can be transformed away, such that the Young modulus is described by the expression (1.71). In section 3, we will give explicit examples of charged black branes exhibiting these properties. It would be interesting to examine whether more general bent charged black brane solutions can be constructed that necessitate the introduction of the terms in (1.72), and, moreover, whether in those cases there is an anomalous contribution violating the classical symmetries mentioned above.

1.4.2 Electric dipole moment and piezoelectric moduli

We now proceed to interpret the structures entering in the decomposition of the electric current for the different cases, focusing first on the quantity $p^{a\rho}$ relevant to the case of $q = 0$ for which we have the current defined in (1.4). In close parallel to the bending moment in (1.69), a charged brane can have an electric dipole moment $P^{a\rho}$ due to the finite thickness. This is obtained by evaluating

$$P^{a\rho} = \int_\Sigma d^{D-1}x\sqrt{-g}\, J^\mu u^a_\mu x^\rho = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}p^{a\rho} \quad , \tag{1.73}$$

and hence the structure $p^{a\rho}$ should be interpreted as a density of world-volume electric dipole moment. Note that in the case of a point particle the structure $p^{a\rho}$ appearing in the decomposition (1.10) can be gauged away due to the 'extra symmetry 2' since it only has one world-volume index component $p^{\tau\rho}$ where τ is the proper time direction of the world-line. Since by equation (B.15) we have that $\delta_2 p^{\tau\rho} = -J^{\tau}_{(0)}\tilde{\varepsilon}^{\rho}$, one can always choose $\tilde{\varepsilon}^{\rho}$ such that the component $p^{\tau\rho}$ vanishes. However, as discussed below equation (1.69), the same symmetry can be used to gauge away the bending moment of a point particle. Since this uses the same gauge parameter, it is generally not possible to gauge away both the bending moment and the electric dipole moment at the same time. For extended objects ($p \geq 1$) the electric dipole moment cannot be removed generically.

We now specialize to a class of branes for which the form of $p^{a\rho}$ is that expected from classical electroelasticity theory

$$p^{a\rho} = \tilde{\kappa}^{abc} K_{bc}{}^{\rho} \quad , \tag{1.74}$$

which is the covariant generalization of the usual relation for the electric dipole moment of classical piezoelectrics [59]. Here, $\tilde{\kappa}^{abc}$ is a set of piezoelectric moduli encoding the response to bending deformations. The structure of $\tilde{\kappa}^{abc}$ has not yet been classified from an effective action perspective as was the case for (1.71). However, based on covariance, it is easy to write down the expected form for the cases we consider such that $\tilde{\kappa}^{abc}$ obeys the symmetry property $\tilde{\kappa}^{abc} = \tilde{\kappa}^{a(bc)}$ and respects the gauge freedom set by the transformation rule (B.15). This leads to the form

$$\tilde{\kappa}^{abc} = -2\left(\kappa_1(\mathbf{k};T,\Phi_H)\gamma^{a(b}\mathbf{k}^{c)} + \kappa_2(\mathbf{k};T,\Phi_H)\mathbf{k}^a\mathbf{k}^b\mathbf{k}^c + \kappa_3(\mathbf{k};T,\Phi_H)\mathbf{k}^a\gamma^{bc}\right) \quad . \tag{1.75}$$

In parallel with (1.71) this contains gauge-dependent terms with respect to the 'extra symmetry 2', which have the form $k\,J^{a}_{(0)}\gamma^{bc}$, with k the gauge parameter. In all, there is only one independent κ-parameter when using also the equations of motion.

We now turn our attention to the case of general p-branes carrying smeared q-brane charge with $q > 0$. The generalization of (1.73) is the electric dipole moment $P^{a_1...a_{q+1}\rho}$ defined by

$$P^{a_1...a_{q+1}\rho} = \int_{\Sigma} d^{D-1}x\sqrt{-g}\,J^{\mu_1...\mu_{q+1}\rho}u^{a_1}_{\mu_1}...u^{a_{q+1}}_{\mu_{q+1}}x^{\rho} = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}p^{a_1...a_{q+1}\rho} \quad , \tag{1.76}$$

and hence $p^{a_1...a_{q+1}\rho}$ has the same interpretation as for the $q = 0$ case. Now according to the expectation from classical electro-elastodynamics we assume the following form for $p^{a_1...a_{q+1}\rho}$

$$p^{a_1...a_{q+1}\rho} = \tilde{\kappa}^{a_1...a_{q+1}bc}K_{bc}{}^{\rho} \quad , \tag{1.77}$$

where $\tilde{\kappa}^{a_1...a_{q+1}bc}$ inherits the symmetries of $p^{a_1...a_{q+1}\rho}$, that is, $\tilde{\kappa}^{a_1...a_{q+1}bc} = \tilde{\kappa}^{[a_1...a_{q+1}]bc}$ and also the property $\tilde{\kappa}^{a_1...a_{q+1}bc} = \tilde{\kappa}^{a_1...a_{q+1}(bc)}$. In particular, for $q = 1$, one expects a structure of the form

$$\begin{aligned}\tilde{\kappa}^{abcd} = -2\Big(&\kappa_1(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H)\boldsymbol{\zeta}^{[a}\gamma^{b](c}\mathbf{k}^{d)} + \kappa_2(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H)\boldsymbol{\zeta}^{[a}\mathbf{k}^{b]}\mathbf{k}^c\mathbf{k}^d \\ &+\kappa_3(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H)\boldsymbol{\zeta}^{[a}\mathbf{k}^{b]}\boldsymbol{\zeta}^c\boldsymbol{\zeta}^d + \kappa_4(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H)\boldsymbol{\zeta}^{[a}\mathbf{k}^{b]}\gamma^{cd}\Big) \quad .\end{aligned} \tag{1.78}$$

Again, this includes a gauge-dependent term of the form $k\,J^{ab}_{(0)}\gamma^{cd}$. The symmetry property of the piezoelectric moduli $\tilde{\kappa}^{abcd}$ for $q \geq 1$, namely the anti-symmetry in its first $q+1$ indices is not something that has a classical analogue and has not been previously considered in the literature of charged elastic solids. In section 3.3 we will give examples of $\tilde{\kappa}^{a_1 \dots a_{q+1} bc}$ obtained from charged black branes in gravity. In particular, we will find that for all the cases considered, there is only one independent contribution.

1.4.3 Spin current and magnetic dipole moment

As mentioned in the beginning of section 1.1.1 the structure $j^{b\mu\nu}$ accounts for the spinning degrees of freedom of the brane [37, 42]. This can be seen by using the stress-energy tensor in (1.2) and constructing the total angular momentum in a (μ,ν)-plane orthogonal to the brane as

$$J_{\perp}^{\mu\nu} = \int_{\Sigma} d^{D-1}x\sqrt{-g}\left(T^{\mu 0}x^{\nu} - T^{\nu 0}x^{\mu}\right) = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma} j^{0\mu\nu} \quad . \tag{1.79}$$

Hence we recognize $j^{b\mu\nu}$ as the angular momentum density on the brane. Angular momentum conservation follows because the brane world-volume spin current $j^{b\mu\nu}$ is conserved [37, 42]. As we will now see, this quantity is also expected to play a role in relation to a particular component of the dipole contribution to the electric current for branes with q-charge.

For this we first turn to the quantity $m^{\mu\nu}$ entering the decomposition of the electric current for $q = 0$. Here, it is instructive to furthermore start by considering the case of a point particle ($p = 0$) with point-like charge. This can have a magnetic dipole moment $M^{\mu\nu}$ obtained by evaluating

$$M^{\mu\nu} = \int_{\Sigma} d^{D-1}x\sqrt{-g}\left(J^{\mu}x^{\nu} - J^{\nu}x^{\mu}\right) = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma} m^{\mu\nu} \quad . \tag{1.80}$$

Therefore, $m^{\mu\nu}$ should be seen as a world-volume density of magnetic dipole moment. Since a magnetic dipole moment requires a moving charge, one would naturally expect $m^{\mu\nu}$ to be proportional to the spin current $j^{\tau\mu\nu}$ of the particle. This interpretation also holds for any p-brane with smeared Maxwell charge and generically one should expect

$$m^{\mu\nu} = \lambda(\sigma^b) u_a j^{a\mu\nu} \quad , \tag{1.81}$$

for some world-volume function $\lambda(\sigma^b)$. Turning to the case of general p-branes carrying a smeared q-brane charge with $q > 0$, we can evaluate the magnetic dipole moment

$$M^{a_1 \dots a_q \mu\nu} = \int_{\Sigma} d^{D-1}x\sqrt{-g}\left(J^{\mu_1 \dots \mu_q \mu}x^{\nu} - J^{\mu_1 \dots \mu_q \nu}x^{\mu}\right) u^{a_1}_{\mu_1} \dots u^{a_q}_{\mu_q} = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma} m^{a_1 \dots a_q \mu\nu} \quad , \tag{1.82}$$

and hence generically the structure $m^{a_1 \dots a_q \mu\nu}$ should be interpreted as a density of magnetic dipole moment. Moreover, in analogy with (1.81) we expect this to be related to the spin current via the generic form

$$m^{a_1 \dots a_q \mu\nu} = \Xi^{a_1 \dots a_q}{}_b j^{b\mu\nu} \quad , \tag{1.83}$$

where $\Xi^{a_1 \dots a_q}{}_b$ is totally anti-symmetric in its indices $a_1 \dots a_q$. We will not find explicit examples of these responses to the spin, since the black branes that we consider in section 3 are non-spinning.

1.5 Gibbons-Maeda black branes

The next step is to consider long-wavelength perturbations of specific charged black brane solutions in (super)gravity theories and show that their effective long-wavelength description can be captured by the general results for charged fluid branes considered in the previous sections. We will consider the family of Gibbons-Maeda black branes derived in [47] through a double uplifting procedure of the Gibbons-Maeda black hole solution [77]. The solution describes a black p-brane with horizon topology $S^{n+1}\times\mathbb{R}^p$ which has electric q-charge diluted on its world-volume. The general solution is given in terms of a metric, a dilaton and a $(q+1)$-form gauge field under which the black p-brane is charged. We will start by presenting the solutions in section 1.5.1.

The remaining part of this section will introduce the framework for extracting the effective stress-energy tensor and current from the black brane solutions. We compute in section 1.5.2 to monopole order the effective stress-energy tensor $\hat{T}^{ab}$ and current $\hat{J}^{a_1\cdots a_{q+1}}$ and find in particular that the zeroth order stress-energy tensor of the Gibbons-Maeda black brane takes the form of a perfect fluid. In section 1.5.3, we will give the procedure for extracting the dipole corrections of the fields and explain their relation to the dipole moments entering the effective stress-energy tensor $\hat{T}^{ab}$ and effective current $\hat{J}^{a_1\cdots q_{q+1}}$.

1.5.1 Black brane solutions

We consider charged dilatonic black brane solutions of the action

$$S=\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\left[R-2(\nabla\phi)^2-\frac{1}{2(q+2)!}e^{-2a\phi}H^2_{[q+2]}\right] \quad . \tag{1.84}$$

The field content consists of the metric $g_{\mu\nu}$, the dilaton ϕ, and the $(q+1)$-form gauge field $A_{[q+1]}$ with field strength $H_{[q+2]}=\mathrm{d}A_{[q+1]}$. The field strength is coupled to the dilaton with coupling constant a.

The Gibbons-Maeda black brane is characterized by p spatial directions $\{\vec{y},\vec{z}\}$, a time direction t, and a radial direction r along with the transverse sphere S^{n+1}. The space-time dimension D is related to p and n by $D=p+n+3$. The metric is given by

$$\mathrm{d}s^2=h^{-A}\left(-f\mathrm{d}t^2+\sum_{i=1}^{q}(\mathrm{d}y^i)^2\right)+h^B\left(f^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{(n+1)}+\sum_{i=1}^{p-q}\left(\mathrm{d}z^i\right)^2\right) \quad , \tag{1.85}$$

with the two harmonic functions $f\equiv f(r)$ and $h\equiv h(r)$ given by

$$f(r)=1-\left(\frac{r_0}{r}\right)^n,\quad h(r)=1+\left(\frac{r_0}{r}\right)^n\gamma_0 \quad . \tag{1.86}$$

The solution is parametrized by two parameters r_0 and γ_0.[11] The exponents of the harmonic functions are given by

$$A=\frac{4(n+p-q)}{2(q+1)(n+p-q)+a^2(n+p+1)} \quad , \tag{1.87}$$

[11] In the blackfold literature $\gamma_0\equiv\sinh^2\alpha$ and $\gamma_0>0$.

and

$$B = \frac{4(q+1)}{2(q+1)(n+p-q)+a^2(n+p+1)} \quad . \tag{1.88}$$

The gauge field only lies in the q directions $y^1, ..., y^q$ and is given by

$$A_{[q+1]} = -\frac{\sqrt{N}}{h}\left(\frac{r_0}{r}\right)^n \sqrt{\gamma_0(1+\gamma_0)}\, \mathrm{d}t \wedge \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^q \quad , \tag{1.89}$$

where we have defined $N \equiv A+B$. It is worth mentioning that a and N are interchangeable and in general can be real numbers, but for many interesting solutions N takes integer values e.g. for the supergravity description of D-branes and NS-branes in type II string theory as well as the M-branes in M-theory all have $N=1$. Finally the dilaton is given by

$$\phi = -\frac{1}{4} N a \log h \quad . \tag{1.90}$$

For future reference the field equations associated to the action (1.84) are

$$\begin{aligned} G_{\mu\nu} - 2\nabla_\mu\phi\nabla_\nu\phi - S_{\mu\nu} = 0 \quad , \\ \nabla_\mu\left(e^{-2a\phi}H^\mu{}_{\rho_1\ldots\rho_{q+1}}\right) = 0 \quad , \\ \Box\phi + \frac{a}{4(q+2)!}e^{-2a\phi}H^2 = 0 \quad , \end{aligned} \tag{1.91}$$

with

$$S_{\mu\nu} = \frac{1}{2(q+1)!}e^{-2a\phi}\left(H_{\mu\rho_1\ldots\rho_{q+1}}H_\nu{}^{\rho_1\ldots\rho_{q+1}} - \frac{1}{2(q+2)}H^2 g_{\mu\nu}\right) \quad . \tag{1.92}$$

It is worth noting that the neutral limit is given by taking $\gamma_0 \to 0$ which reduces the solution to a black p-branes and the field equations to the Einstein vacuum equations $R_{\mu\nu} = 0$.

As we shall see in the following, the associated effective stress-energy tensor of the Gibbon-Maeda solution takes the form of an anisotropic perfect fluid. However, there are two interesting cases where the stress-energy tensor is isotropic. The first is in the case of Einstein-Maxwell-Dilaton theory obtained by taking $q=0$. This also includes the Reissner-Nordström black branes where $a=0$. The second case is theories with a $(p+1)$-form gauge field obtained by taking $p=q$. In the context of type II string theory the gauge field corresponds to a R-R field. We have already considered the effective description for these two cases in section 1.2.1. In section 2, we will consider intrinsic perturbations of the black brane solutions of these two theories and compute the dissipative corrections to their effective stress-energy tensor and current. In section 3, we will construct bent black brane solutions in the context of theories with $0 \leq q < p$ and compute the finite thickness effects to the effective stress-energy tensor and current.

1.5.2 Effective fluid

In the blackfold approach one works with an effective stress-energy tensor that encodes the short-wavelength degrees of freedom of the near region. The long-wavelength degrees of freedom existing in the far region couple to the near region through this tensor. The stress-energy tensor is computed in the asymptotically flat region ($r \to \infty$) of the black brane and acts as a source for the gravitational field in the far region. The stress-energy tensor can either be obtained from an ADM-type prescription [78] or equivalent from the quasilocal stress-energy tensor $\tau_{\mu\nu}$ introduced by Brown-York [79] which we will consider in what follows.

In order to adopt this formalism to the black brane space-times that are considered here we consider timelike hypersurfaces Σ_r at large constant r in the black brane space-time. These surfaces form a boundary of the brane with geometry $\mathbb{R}^{1,p} \times S^{n+1}$. Let n^μ denote the outward pointing spacelike normal vector field to the boundary surface, then the metric on Σ_r is

$$\bar{\gamma}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad . \tag{1.93}$$

This is also known as the first fundamental form or projection tensor of the hypersurface. The hypersurfaces Σ_r are embeddings in the space-time and therefore one can associate an extrinsic curvature tensor to them. Given the metric and normal, the extrinsic curvature of the hypersurface is given by the Lie derivative of the metric along the normal vector field,

$$\bar{K}_{\mu\nu} = \frac{1}{2}\mathcal{L}_n \bar{\gamma}_{\mu\nu} = \bar{\gamma}^{\alpha}{}_{\mu}\bar{\gamma}^{\beta}{}_{\nu}\nabla_{(\alpha}n_{\beta)} \quad . \tag{1.94}$$

The extrinsic curvature tensor is therefore symmetric in its lower indices. Now, in order to ascribe a stress-energy tensor to the surface we form the pullback of the extrinsic curvature

$$\bar{K}_{ij} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\bar{K}_{\mu\nu} \quad . \tag{1.95}$$

Similar we form the intrinsic metric $\bar{\gamma}_{ij}$ and can write the quasilocal stress-energy tensor as [79]

$$\tau^{ij} = \frac{1}{8\pi G}\left(\bar{K}^{ij} - \bar{\gamma}^{ij}\bar{K}\right) - \tau_0^{ij} \quad , \tag{1.96}$$

where τ_0^{ij} is a background reference for the space-time that constitute terms computed on a reference space-time with the same intrinsic geometry.

For the current setup the stress-energy tensor is computed far from the horizon of the brane where the gravitational field is weak. In flat space this is the region where the deviations away from Minkowski space-time are small. Since we do not consider deformations of the intrinsic sphere, it is possible to integrate over the transverse sphere and the world-volume stress-energy tensor can be obtained by

$$\hat{T}^{ab} = \lim_{r\to\infty}\frac{\Omega_{(n+1)}}{2}r^{n+1}\tau^{ab} \quad , \tag{1.97}$$

under the assumption that the integrated tension of the S^{n+1} directions does not diverge. We have now obtained the stress-energy tensor with support on the $(p+1)$-dimensional world-volume $\mathcal{W}_{p+1}$. This stress-energy tensor is what is referred to as the stress-energy tensor of the blackfold fluid.

Effective current

If one deals with a charged source this will give rise to an effective current. It can be obtained from the large r-asymptotics of the gauge field components far from the black brane horizon where the gravitational field is weak and requiring the Lorentz gauge $\nabla_{\mu_1} A^{\mu_1\mu_2\cdots\mu_{q+1}} = 0$. In this limit, the linearized equation of motion (with an effective source) for asymptotically flat space-times is

$$\nabla_\perp^2 A^{\mu_1\cdots\mu_{q+1}} = -16\pi G J^{\mu_1\cdots\mu_{q+1}} \delta^{(n+2)}(r) \quad , \tag{1.98}$$

where the Laplacian operator is taken along transverse directions to the world-volume and the dilaton is assumed to vanish at infinity. The effective current can then simply be obtained from the field using Green's function,

$$\hat{J}^{a_1\dots a_{q+1}} = \lim_{r\to\infty} \frac{\Omega_{(n+1)}}{16\pi G} n r^n A^{a_1\dots a_{q+1}} \quad . \tag{1.99}$$

This is the effective current with support on the world-volume $\mathcal{W}_{p+1}$.

World-volume stress-energy tensor and thermodynamics

The asymptotic charges and horizon quantities of the black brane solutions in section 1.5.1 serve as the leading order input to the blackfold formalism. We compute the quasilocal stress-energy tensor given by equation (1.96) of the solutions given in section 1.5.1 and evaluate equation (1.97) in order to obtain the blackfold fluid stress-energy tensor. In this case the stress-energy tensor describes an anisotropic perfect fluid (in the rest frame) living on the $(p+1)$-dimensional world-volume. We note that the quasilocal tensor in the directions of the transverse sphere is $\tau^{\Omega\Omega} = 0$ and therefore does not diverge when going to infinity. From the stress-energy tensor we can identify the energy density ϵ, the pressure $P_\parallel$ along the q directions of the gauge field, and the pressure $P_\perp$ in the $p-q$ remaining directions of the world-volume

$$\epsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \big(n+1+nN\gamma_0\big), \quad P_\perp = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n, \quad P_\parallel = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(1+nN\gamma_0\right) \quad . \tag{1.100}$$

The effective current can be found by using equation (1.99) which determines the charge density

$$\mathcal{Q} = \frac{\Omega_{(n+1)}}{16\pi G} n r_0^n \sqrt{N\gamma_0(1+\gamma_0)} \quad . \tag{1.101}$$

The quantities associated to the horizon can be obtained in the usual way. In particular, the temperature $\mathcal{T}$ is determined from the surface gravity while the entropy density s is obtained from the horizon area. Finally, we have the electric potential at the horizon Φ conjugate to $\mathcal{Q}$.

$$\mathcal{T} = \frac{n}{4\pi r_0 \sqrt{(1+\gamma_0)^N}} \quad , \quad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} \sqrt{(1+\gamma_0)^N} \quad , \quad \Phi = \sqrt{\frac{N\gamma_0}{1+\gamma_0}} \quad . \tag{1.102}$$

The quantities are parameterized in terms of two parameters: the charge parameter γ_0 and the radius r_0. With reference to section 1.2 we note that the general Smarr-like relation [60]

$$\epsilon = \frac{n+1}{n} \mathcal{T} s + \Phi \mathcal{Q} \quad , \tag{1.103}$$

is satisfied for the static black branes discussed here. The relation can equivalently be expressed as $\mathcal{G} = \frac{1}{n}\mathcal{T}s$ with $\mathcal{G}$ given by equation (1.22) and we can therefore by introducing the boost velocities u^a write the stress-energy tensor (1.21) in the form

$$T^{ab}_{(0)} = \mathcal{T}s\left(u^a u^b - \frac{1}{n}\gamma^{ab}\right) - \Phi\mathcal{Q}h^{ab}_{(q)} \quad . \tag{1.104}$$

In this form, the stress-energy tensor (1.104) immediately allows us to identify the thermal and the electrostatic parts. Since $r_0^n \sim \mathcal{T}s$, it gives a measure of the thermal energy density. In a similar manner γ_0 is identified with the thermodynamic ratio,

$$\gamma_0 = \frac{1}{N}\frac{\Phi\mathcal{Q}}{\mathcal{T}s} \quad , \tag{1.105}$$

and γ_0 therefore measures the electrostatic energy relative to the thermal energy of the black brane.

Zeroth order effective blackfold description

As mentioned, the dynamics of the charged black brane is to leading order given in terms of an effective description of a perfect fluid living on a dynamical world-volume $\mathcal{W}_{p+1}$ of vanishing thickness whose dynamics is governed by the equations (1.16) and (1.17) [26, 27]. The fluid is characterized by the effective stress-energy tensor (1.104) and current (1.20) with the charge density (1.101), those dynamics we have considered in detail in section 1.2, in terms of stress-energy conservation (1.17) and current conservation equations (1.19), respectively. The dynamics of the world-volume embedded in the background space-time is governed by the coupling between the perfect fluid stress-energy tensor and the extrinsic curvature of the submanifold. More precisely, the embedding functions satisfies the extrinsic equations (1.16), which for the current setting we can write as [60],

$$K^{\rho} = n \perp^{\rho}{}_{\mu}\left(\dot{u}^{\mu} - \gamma_0 N K^{\mu}_{(q)}\right) \quad , \tag{1.106}$$

where we have used equation (1.105) and $K^{\mu}_{(q)} = h^{ab}_{(q)} K_{ab}{}^{\rho}$ is the mean curvature vector of the embedding of $\mathcal{C}_{q+1}$ in the background space-time. For the case of $q = 0$, where $K^{\mu}_{(0)} = - \perp^{\rho}{}_{\mu}\, \dot{u}^{\mu}$, the effect of the brane carrying Maxwell charge on its world-volume is thus to decrease the necessary acceleration needed to maintain a given mean curvature. On the other hand, for higher-form charges $q > 1$ with positive mean curvature the necessary acceleration required to maintain a given mean curvature increases [47].

Finally, it should also be mentioned, in connection with bending black brane solutions, that the equations of motion (1.16) and (1.17) can be shown to be a subset of the Einstein equations [38]. The equations can therefore also be used to exclude possible horizon topologies. Indeed, if a particular embedding does not solve the blackfold equations it implies that such horizon geometry can not be realized as a solution to the Einstein equations, at least not within the regime of validity of the effective description.

1.5.3 Measuring dipole corrections

We now outline the method used to compute the dipole moments for generic stationary black brane solutions in the theory described by (1.84). As mentioned in the beginning of section 1.1.1, bent branes acquire a bending moment which in turn implies a dipole correction $T^{\mu\nu\rho}_{(1)}$ to the stress-energy tensor and if the brane is charged an electric dipole moment $J^{\mu_1\dots\mu_{q+1}\rho}_{(1)}$ is also induced. In order to compute these from a gravitational solution we look at the large r-asymptotics where the geometry and gauge field, as seen from a distant observer, can be replaced by effective sources of stress-energy and current. The task is then to find the effective stress-energy tensor (1.2) and current (1.4) that source the charged brane solution. To this end, we note that the equations of motion that follow from the action (1.84) in the presence of sources are given by equation (1.91) with effective sources $T^{\mu\nu}$ and $J^{\mu_1\dots\mu_{q+1}}$ appearing in the r.h.s.

The bending moment (1.70) and the electric dipole moment (1.77) are then related, via equation (1.91) with effective sources, to the dipole corrections occurring in the different fields as one approaches spatial infinity, which by definition have the fall-off behaviour $\mathcal{O}\left(r^{-n-1}\right)$ [37]. In particular, the bending moment is related to the dipole contributions to the metric $g_{\mu\nu}$ far away from the brane horizon. It is therefore convenient to decompose the metric according to

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(M)}_{\mu\nu} + h^{(D)}_{\mu\nu} + \mathcal{O}(r^{-n-2}) \ \ , \tag{1.107}$$

where the coefficients $h^{(M)}_{\mu\nu}$ represent the monopole structure of the metric, generically of order $\mathcal{O}\left(r^{-n}\right)$, while the coefficients $h^{(D)}_{\mu\nu}$ represent the dipole deformation of the metric of order $\mathcal{O}\left(r^{-n-1}\right)$. Similarly, the electric dipole moment is related to the dipole contributions to the gauge field $A_{\mu_1\dots\mu_{q+1}}$, also of order $\mathcal{O}\left(r^{-n-1}\right)$. Therefore we decompose the gauge field such that

$$A_{\mu_1\dots\mu_{q+1}} = A^{(M)}_{\mu_1\dots\mu_{q+1}} + A^{(D)}_{\mu_1\dots\mu_{q+1}} + \mathcal{O}\left(r^{-n-2}\right) \ \ , \tag{1.108}$$

where again the labels (M) and (D) indicate the monopole and dipole contributions respectively to the gauge field $A_{\mu_1\dots\mu_{q+1}}$. We note that in the cases studied here, there are no response coefficients associated with the dilaton ϕ, a fact that renders the analysis of the dilaton equation of motion unnecessary. In the following, we will review how the bending and electric dipole moments as well as the corresponding response coefficients can be extracted from the linearized equations of motion. We should emphasize that the procedure that will be outlined here only works under the assumption that there are no background fields, namely, no background gauge field nor a non-zero background dilaton and that the background metric is asymptotically flat.

Measuring the Young modulus

Considering the first equation in (1.91) with an effective source on the r.h.s. given by the pole-dipole expansion given by equation (1.2) and using the decomposition (1.6), one finds that the dipole contribution to the metric should satisfy the linearized equation of motion

$$\nabla^2_{\perp}\bar{h}^{(D)}_{\mu\nu} = 16\pi G d_{\mu\nu}{}^{r_\perp}\partial_{r_\perp}\delta^{n+2}(r) \ \ , \ \ \nabla_\mu \bar{h}^{\mu}{}_{\nu} = 0 \ \ , \tag{1.109}$$

where we have defined

$$\bar{h}^{(D)}_{\mu\nu} = h^{(D)}_{\mu\nu} - \frac{h^{(D)}}{2}\eta_{\mu\nu} \ \ , \ \ h^{(D)} = \eta^{\mu\nu}h^{(D)}_{\mu\nu} \ \ , \tag{1.110}$$

and the Laplacian operator is taken along transverse directions to the world-volume. The direction cosine $r_\perp = r\cos\theta$ is transverse to the direction along which the brane is bent. It is convenient to exploit the explicit r and θ dependence of the asymptotic form of the dipole contributions, thus we define

$$h^{(D)}_{ab} = f^{(D)}_{ab}\cos\theta\frac{r_0^{n+2}}{r^{n+1}} \ \ , \ \ h^{(D)}_{rr} = f^{(D)}_{rr}\cos\theta\frac{r_0^{n+2}}{r^{n+1}} \ \ , \ \ h^{(D)}_{ij} = r^2 g_{ij} f^{(D)}_{\Omega\Omega}\cos\theta\frac{r_0^{n+2}}{r^{n+1}} \ \ , \tag{1.111}$$

where $f^{(D)}_{\mu\nu}$ are the asymptotic metric coefficients which do not depend on r neither on θ.[12] With this definition, the transverse gauge condition gives rise to the constraint

$$\eta^{ab}f^{(D)}_{ab} + f^{(D)}_{rr} + (n-1)f^{(D)}_{\Omega\Omega} = 0 \ \ , \tag{1.112}$$

and hence one obtains

$$h^{(D)} = 2f^{(D)}_{\Omega\Omega}\cos\theta\frac{r_0^{n+2}}{r^{n+1}} \ \ . \tag{1.113}$$

The dipole contributions to the metric are therefore given by[13]

$$\hat{d}_{ab} = \bar{f}^{(D)}_{ab} = f^{(D)}_{ab} - f^{(D)}_{\Omega\Omega}\eta_{ab} \ \ , \tag{1.114}$$

and hence the Young modulus $\tilde{Y}^{abcd}$ can then be obtained via equation (1.70).

Measuring the piezoelectric moduli

The procedure for obtaining the piezoelectric moduli follows a similar logic. Using the linearized version of the second equation in (1.91) with an effective pole-dipole source on the r.h.s given by the expansion (1.4) and corresponding decomposition (see equation (1.10) and equation (B.25)), one finds that the gauge field satisfies

$$\nabla^2_\perp A^{(D)}_{\mu_1\ldots\mu_{q+1}} = 16\pi G p_{\mu_1\ldots\mu_{q+1}}{}^{r_\perp}\partial_{r_\perp}\delta^{(n+2)}(r) \ \ , \ \ \nabla_\mu A^{\mu\nu_1\ldots\nu_q} = 0 \ \ , \tag{1.115}$$

where it has been assumed that the dilaton vanishes at infinity. Again, it is convenient to write the asymptotic gauge field coefficients as

$$A^{(D)}_{\mu_1\ldots\mu_{q+1}} = a^{(D)}_{\mu_1\ldots\mu_{q+1}}\cos\theta\frac{r_0^{n+2}}{r^{n+1}} \ \ . \tag{1.116}$$

The electric dipole moment (1.74) follows from the equations (1.115)-(1.116) leading to the simple relation[14]

$$\hat{p}_{a_1\ldots a_{q+1}} = a^{(D)}_{a_1\ldots a_{q+1}} \ \ . \tag{1.117}$$

[12] Here $f^{(D)}_{\Omega\Omega}$ is the same function for all transverse sphere indices.

[13] Note that here we have defined $d_{ab} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}r_0^2\hat{d}_{ab}$ and omitted the transverse index $r_\perp$ from $d_{ab}{}^{r_\perp}$ since, according to the analysis of [38], also valid for the case at hand, perturbations in each direction $r_\perp$ decouple from each other to first order in the derivative expansion.

[14] Note that here we have defined $p_{a_1\ldots a_{q+1}} = \frac{\Omega_{(n+1)}r_0^n}{16\pi G}r_0^2\hat{p}_{a_1\ldots a_{q+1}}$ and again omitted the transverse index $r_\perp$ from $p_{a_1\ldots a_{q+1}}{}^{r_\perp}$ in parallel with our definition of $d_{ab}{}^{r_\perp}$.

The piezoelectric moduli $\tilde{\kappa}^{a1...a_{q+1}bc}$ can then be extracted from (1.117) via equation (1.77). This concludes our review of how the response coefficients are obtained from the field content of bent black brane solutions far away from the brane horizon.

Chapter 2

Hydrodynamic Perturbations

In this chapter, we extend the analysis of ref. [33] where the hydrodynamic limit of the Reissner-Nordström black p-brane was considered. In particular, we extend the analysis by introducing a dilaton field in the theory and consider black p-branes in two different settings: first charged under a Maxwell gauge field and second charged under a $(p+1)$-form gauge field.

As we shall see, since the background space-time exhibits a SO(p) invariance, the perturbations decouple into the different sectors of SO(p). The presence of a dilaton field will introduce an additional scalar perturbation in both settings that otherwise leads to a very similar system of differential equations except for the vector perturbations. The vector perturbations differ because whereas the Maxwell charge is allowed to redistribute itself on the world-volume the system with p-brane charge is more constrained and can not redistribute itself. There is therefore no diffusion associated with the $(p+1)$-form current and the vector perturbations are in that sense much simpler to deal with. Although, the procedure of the analysis is similar for the two systems, the details differ in the two cases. We therefore choose to focus on the analysis for the black brane charged under a Maxwell gauge field and relay the specifics of the computation for the black brane charged under a $(p+1)$-form gauge field to appendix A.

In section 2.1 we start by considering the leading order solution of the black p-brane in Einstein-Maxwell-dilaton theory. In section 2.2 we discuss the perturbation procedure and explain how the boundary conditions are handled. In section 2.3 the first-order equations are solved. Finally, in section 2.4 we present the effective stress-energy tensor and current for the Maxwell black branes and the effective stress-energy tensor for the brane carrying p-brane charge. In section 2.5 the transport coefficients are used to analyze the dispersion relations, the dynamical stability, and the relation to the branes thermodynamical stability.

2.1 Maxwell black branes

As mentioned above we start by considering black brane solutions of Einstein-Maxwell-dilaton theory. The action is given in section 1.5 by setting $q = 0$,

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - \frac{1}{4}e^{-2a\phi}F_{\mu\nu}F^{\mu\nu}\right] \quad . \tag{2.1}$$

Here $F_{\mu\nu}$ is the field strength of the Maxwell gauge field A_μ, $F = \mathrm{d}A$, which is coupled to the dilaton ϕ through the coupling constant a.

The black brane is characterized by p spatial directions x^i, a time direction t, and a radial direction r along with the transverse sphere S^{n+1}. The space-time dimension D is related to p and n by $D = p + n + 3$. The metric takes the form

$$\mathrm{d}s^2 = -h^{-A} f \mathrm{d}t^2 + h^B \left(f^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(n+1)} + \sum_{i=1}^{p} \left(\mathrm{d}x^i\right)^2 \right) \quad , \tag{2.2}$$

with the two harmonic functions $f \equiv f(r)$ and $h \equiv h(r)$ given by

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^n , \quad h(r) = 1 + \left(\frac{r_0}{r}\right)^n \gamma_0 \quad . \tag{2.3}$$

The solution is parametrized by two parameters r_0 and γ_0 which interpretation has been discussed in section 1.5.2. The exponents of the harmonic functions are given by

$$A = \frac{4(n+p)}{2(n+p) + a^2(n+p+1)} \quad , \quad B = \frac{4}{2(n+p) + a^2(n+p+1)} \quad . \tag{2.4}$$

The gauge field is given by

$$A_{[1]} = -\frac{\sqrt{N}}{h} \left(\frac{r_0}{r}\right)^n \sqrt{\gamma_0(1+\gamma_0)}\, \mathrm{d}t \quad , \tag{2.5}$$

where $N \equiv A + B$, and finally the dilaton is given by

$$\phi = -\frac{1}{4} N a \log h \quad . \tag{2.6}$$

We note for future reference that N can not take arbitrary values, since we must require that $a^2 > 0$. It is therefore bounded by

$$N \in \left]0; \quad 2\left(1 + \frac{1}{n+p}\right)\right] \quad , \tag{2.7}$$

where the upper bound is set by $a = 0$.

One can apply a uniform boost u^a to the static solution in the world-volume directions thus extending it to a stationary one. The metric (2.2) of the dilatonic boosted black brane then takes the form

$$\mathrm{d}s^2 = h^B \left(-h^{-N} f\, u_a u_b \, \mathrm{d}x^a \mathrm{d}x^b + f^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(n+1)} + \Delta_{ab}\, \mathrm{d}x^a \mathrm{d}x^b \right) \quad , \tag{2.8}$$

where $\Delta^a{}_b \equiv \delta^a{}_b + u^a u_b$ is the orthogonal projector defined by the boost u^a. The gauge field is given by

$$A_{[1]} = \frac{\sqrt{N}}{h} \left(\frac{r_0}{r}\right)^n \sqrt{\gamma_0(\gamma_0+1)}\, u_a \mathrm{d}x^a \quad , \tag{2.9}$$

while the dilaton fields remains unaffected.

We note here that a particularly nice property of the family of Gibbons-Maeda black branes is that the dilaton coupling a can be treated as a free parameter. This in particular means that we are free to set $a = 0$ and thus the Reissner-Nordström black brane solution considered in [33] is a subset of the solutions considered here.

2.1.1 Effective fluid

The blackfold effective theory of p-branes supporting particle charge ($q = 0$) was reviewed in section 1.2.1. For a uniform boost u^a of the brane, the effective blackfold stress-energy tensor can be written in the form given by equation (1.104) as

$$T^{ab}_{(0)} = \mathcal{T} s \left(u^a u^b - \frac{1}{n}\gamma^{ab} \right) + \Phi \mathcal{Q}\, u^a u^b \quad , \tag{2.10}$$

where γ_{ab} is the induced metric on the world-volume. For our purposes (flat extrinsic geometry), we have $\gamma_{ab} = \eta_{ab}$. Moreover $\mathcal{T}$ is the local temperature, s is the entropy density, $\mathcal{Q}$ is the charge density and finally Φ is the electric potential conjugate to $\mathcal{Q}$. The various quantities are provided in section 1.5.2 parameterized in terms of a charge parameter γ_0 and the horizon thickness r_0. The stress-energy tensor can be expressed (in standard form) in terms of the energy density ϵ and pressure $P = P_\perp$,

$$T^{ab}_{(0)} = \epsilon u^a u^b + P \Delta^{ab} \quad , \tag{2.11}$$

and the 1-form current supported by the p-brane is given by

$$J^a_{(0)} = \mathcal{Q}\, u^a \quad . \tag{2.12}$$

To leading order, the intrinsic blackfold equations take the form of the world-volume conservation equations $\nabla_a T^{ab}_{(0)} = 0$ and $\nabla_a J^a_{(0)} = 0$. They are evaluated in section 1.2.1 and are given by equation (1.26) and (1.33), respectively. The conservation equations will be important in the perturbative analysis as they will show up as constraint equations when solving the Einstein-Maxwell-dilaton system perturbatively.

2.2 The perturbative expansion

Our aim is to solve the Einstein-Maxwell-dilaton system in a derivative expansion around the solution given in section 2.1. In this section, we will define the appropriate coordinates to handle this problem and explain how the perturbations are classified according to their transformation properties under SO(p).

2.2.1 Setting up the perturbation

Before perturbing the brane, we first need to introduce Eddington-Finkelstein (EF) coordinates. The reason is two-fold. First, it is essential for the computation that we can ensure regularity at the horizon and since the Schwarzschild description breaks down at the horizon, it is clearly more useful to use EF coordinates. Secondly, since a gravitational disturbance moves along null-lines, in order to control the perturbation, we want the lines of constant world-volume coordinates to be radial null-curves i.e. $g_{rr} = 0$. This is exactly the defining property of EF coordinates. For a general boost u^a, we define the EF coordinates σ^a by

$$\sigma^a = x^a + u^a r_\star, \quad r_\star(r) = r + \int_r^\infty \left(\frac{f - h^{N/2}}{f} \right) dr \quad . \tag{2.13}$$

Here $r_\star$ is chosen such that $r'_\star = h^{N/2}/f$ and $r_\star \to r$ for large r. The first condition ensures that $g_{rr} = 0$ while the latter is chosen such that the EF coordinates reduce to ordinary radial Schwarzschild light cone coordinates for large r. Notice that it is possible to write down a closed form expression for $r_\star$ in terms of the hypergeometric Appell function F_1

$$r_\star(r) = r F_1\left(-\frac{1}{n}; -\frac{N}{2}, 1; 1-\frac{1}{n}; 1-h, 1-f\right) \approx r\left(1 - \frac{1}{n-1}\frac{r_0^n}{r^n}\left(1+\frac{N\gamma_0}{2}\right)\right) \quad , \quad (2.14)$$

where the last equality applies for large r and is valid up to $\mathcal{O}\left(\frac{1}{r^{2n-1}}\right)$. It is nice to note that the hypergeometric Appell function F_1 reduces to the ordinary hypergeometric function ${}_2F_1$ in the neutral limit $\gamma_0 \to 0$. Indeed

$$\lim_{\gamma_0\to 0} r_\star(r) = r_\star(r)\Big|_{\gamma_0=0} \equiv r + \int_r^\infty \left(\frac{f-1}{f}\right) dr = r\, {}_2F_1\left(1; -\frac{1}{n}; 1-\frac{1}{n}; 1-f\right) \quad , \quad (2.15)$$

which is the $r_\star$ used in [32]. With this definition of $r_\star$ we will limit our analysis to the case for which $n \geq 2$. In EF coordinates, the metric (2.8) takes the form

$$\mathrm{d}s^2_{(0)} = h^B\left(-h^{-N} f\, u_a u_b\, \mathrm{d}\sigma^a \mathrm{d}\sigma^b - 2h^{-N/2}\, u_a\, \mathrm{d}\sigma^a \mathrm{d}r + \Delta_{ab}\, \mathrm{d}\sigma^a \mathrm{d}\sigma^b + r^2 \mathrm{d}\Omega^2_{(n+1)}\right) \quad . \quad (2.16)$$

Here the subscript indicates that the metric solves the Einstein-Maxwell-dilaton equations to zeroth order in the derivatives. Notice that in these coordinates the gauge field will acquire a non-zero A_r component . However, we shall work in a gauge where this component is zero. We therefore take

$$A^{(0)} = \frac{\sqrt{N}}{h}\left(\frac{r_0}{r}\right)^n \sqrt{\gamma_0(\gamma_0+1)}\, u_a \mathrm{d}\sigma^a \quad , \quad \text{and in particular} \quad A_r^{(0)} = 0 \quad . \quad (2.17)$$

Having determined the EF form of the metric and gauge field, we are now ready to set up the perturbative expansion.

Following the lines of [32], we promote the parameters u^a, r_0 and γ_0 to *slowly* varying world-volume fields:

$$u^a \to u^a(\sigma^a), \quad r_0 \to r_0(\sigma^a), \quad \gamma_0 \to \gamma_0(\sigma^a) \quad . \quad (2.18)$$

By slowly varying we mean that the derivatives of the world-volume fields are sufficiently small. In order to quantify this, we introduce a set of re-scaled coordinates $\sigma^a_\varepsilon = \varepsilon\sigma^a$, $\varepsilon \ll 1$, and consider the w.v. fields to be functions of σ^a_ε. In this way each derivative will produce a factor of ε. Moreover, two derivatives will be suppressed by a factor of ε compared to one derivative and so on. Effectively what we are doing is to consider arbitrary varying world-volume fields (no restrictions on the size of derivatives) and "stretching" them by a factor of $1/\varepsilon \gg 1$. In this way we will only consider slowly varying fields and the derivative expansion is controlled by the parameter ε.[1] The fields can now be expanded around a given point $\mathcal{P}$

$$\begin{gathered} u^a(\sigma) = u^a\big|_{\mathcal{P}} + \varepsilon\sigma^b\partial_b u^a|_{\mathcal{P}} + \mathcal{O}(\varepsilon^2) \quad , \quad r_0(\sigma) = r_0\big|_{\mathcal{P}} + \varepsilon\sigma^a\partial_a r_0|_{\mathcal{P}} + \mathcal{O}(\varepsilon^2) \quad , \\ \gamma_0(\sigma) = \gamma_0\big|_{\mathcal{P}} + \varepsilon\sigma^a\partial_a\gamma_0|_{\mathcal{P}} + \mathcal{O}(\varepsilon^2) \quad . \end{gathered} \quad (2.19)$$

[1] In the end of the computation, we of course set $\varepsilon = 1$ and keep in mind that the expressions only hold as a derivative expansion i.e. for sufficiently slowly varying configurations.

We now seek derivative corrections to the metric, gauge field, and dilaton denoted by $\mathrm{d}s^2_{(1)}$, $A_{(1)}$, and $\phi_{(1)}$, respectively, so that

$$\begin{gathered}\mathrm{d}s^2 = \mathrm{d}s^2_{(0)} + \varepsilon \mathrm{d}s^2_{(1)} + \mathcal{O}(\varepsilon^2) \quad , \quad A = A_{(0)} + \varepsilon A_{(1)} + \mathcal{O}(\varepsilon^2) \quad , \\ \phi = \phi_{(0)} + \varepsilon\phi_{(1)} + \mathcal{O}(\varepsilon^2) \quad ,\end{gathered} \tag{2.20}$$

solves the equations of motion to order ε. By a suitable choice of coordinates, we can take the point $\mathcal{P}$ to lie at the origin $\sigma^a = (0, \mathbf{0})$. Moreover, we can choose coordinates so that $u^v|_{(0,\mathbf{0})} = 1$, $u^i|_{(0,\mathbf{0})} = 0$, $i = 1, ..., p$ (the rest frame of the boost in the origin).[2] In these coordinates the 0$^{\text{th}}$ order metric $\mathrm{d}s^2_{(0)}$ takes the form

$$\begin{aligned}\mathrm{d}s^2_{(0)} = h^B \Bigg[& -2h^{-\frac{N}{2}}\mathrm{d}v\mathrm{d}r - \left(\frac{f}{h^N}\right)\mathrm{d}v^2 + \sum_{i=1}^{p}(\mathrm{d}\sigma^i)^2 + r^2\mathrm{d}\Omega^2_{(n+1)}\Bigg] \\ & + \varepsilon h^B \Bigg[\frac{1}{h^N}\frac{r_0^n}{r^n}\left(\frac{n}{r_0}\left(1 + A\frac{r_0^n}{r^n}\gamma_0\right)\sigma^a\partial_a r_0 + A\frac{f}{h}\sigma^a\partial_a\gamma_0\right)\mathrm{d}v^2 \\ & + \frac{B}{h}\frac{r_0^n}{r^n}\left(\frac{n\gamma_0}{r_0}\sigma^a\partial_a r_0 + \sigma^a\partial_a\gamma_0\right)\left(\sum_{i=1}^{p}(\mathrm{d}\sigma^i)^2 + r^2\mathrm{d}\Omega^2_{(n+1)}\right) \\ & + 2\left(\frac{f}{h^N} - 1\right)\sigma^a\partial_a u_i\,\mathrm{d}v\mathrm{d}\sigma^i - \frac{2}{h^{N/2}}\sigma^a\partial_a u_i\,\mathrm{d}\sigma^i\mathrm{d}r \\ & + \frac{B-A}{h^{N/2+1}}\frac{r_0^n}{r^n}\left(\frac{n\gamma_0}{r_0}\sigma^a\partial_a r_0 + \sigma^a\partial_a\gamma_0\right)\mathrm{d}v\mathrm{d}r\Bigg] \quad ,\end{aligned} \tag{2.21}$$

where we have denoted $r_0|_{(0,\mathbf{0})} \equiv r_0$ and $\gamma_0|_{(0,\mathbf{0})} \equiv \gamma_0$. The system has a certain amount of gauge freedom. Following the discussion of the definition of $r_\star$, we want the r coordinate to maintain its geometrical interpretation. We therefore choose

$$g^{(1)}_{rr} = 0 \quad , \tag{2.22}$$

and we moreover take

$$g^{(1)}_{\Omega\Omega} = 0 \quad \text{and} \quad A^{(1)}_r = 0 \quad . \tag{2.23}$$

The background $g_{(0)}$ exhibits a residual SO(p) invariance. We can use this to split the system up into sectors of SO(p). The scalar sector contains 5 scalars, $A^{(1)}_v$, $g^{(1)}_{vr}$, $g^{(1)}_{vv}$, $\mathrm{Tr}g^{(1)}_{ij}$, and $\phi^{(1)}$. The vector sector contains 3 vectors $A^{(1)}_i$, $g^{(1)}_{vi}$ and $g^{(1)}_{ri}$. Finally, the tensor sector contains 1 tensor $\overline{g}^{(1)}_{ij} \equiv g^{(1)}_{ij} - \frac{1}{p}(\mathrm{Tr}g^{(1)}_{kl})\delta_{ij}$ (the traceless part of $g^{(1)}_{ij}$). We parameterize the three SO(p) sectors according to

$$\begin{gathered}\textbf{Scalar: } A^{(1)}_v = -\sqrt{N\gamma_0(1+\gamma_0)}\,\frac{r_0^n}{r^n}h^{-1}a_v, \quad g^{(1)}_{vr} = h^{\frac{B-A}{2}}f_{vr}, \\ g^{(1)}_{vv} = h^{1-A}f_{vv}, \quad \mathrm{Tr}\,g^{(1)}_{ij} = h^B\mathrm{Tr}f_{ij}, \quad \phi^{(1)} = f_\phi \quad , \\ \textbf{Vector: } A^{(1)}_i = -\sqrt{N\gamma_0(1+\gamma_0)}\,a_i, \quad g^{(1)}_{vi} = h^B f_{vi}, \quad g^{(1)}_{ri} = h^{\frac{B-A}{2}}f_{ri} \quad , \\ \textbf{Tensor: } \overline{g}^{(1)}_{ij} = h^B\overline{f}_{ij} \quad ,\end{gathered} \tag{2.24}$$

where $\overline{f}_{ij} \equiv f_{ij} - \frac{1}{p}(\mathrm{Tr}f_{kl})\delta_{ij}$. The parameterization is chosen in such a way that the resulting EOMs only contain derivatives of f_{ab}, f_ϕ and a_a and will thus be directly integrable.

[2] In these coordinates $u^v = 1 + \mathcal{O}(\varepsilon^2)$.

2.3 First-order equations

In order to compute the effective stress-energy tensor and current and thereby extract the transport coefficients, we need the large r-asymptotics of the perturbation functions which are decomposed and parametrized according to equation (2.24). The equations of motion associated to the action (2.1) are given by the equations in (1.91) with $q = 0$. We denote the first-order Einstein, Maxwell, and dilaton equations by

$$\begin{aligned} R_{\mu\nu} - 2\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}e^{-2a\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(n+p+1)}F^2 g_{\mu\nu}\right) &\equiv \varepsilon\mathcal{E}_{\mu\nu} + \mathcal{O}(\varepsilon^2) = 0 \ , \\ \nabla_\rho\left(e^{-2a\phi}F^\rho{}_\mu\right) &\equiv \varepsilon\mathcal{M}_\mu + \mathcal{O}(\varepsilon^2) = 0 \ , \\ g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{a}{8}e^{-2a\phi}F^2 &\equiv \varepsilon\mathcal{E}_{(\phi)} + \mathcal{O}(\varepsilon^2) = 0 \ . \end{aligned} \tag{2.25}$$

In this section, we will find the solution to each SO(p) sector in turn and explain how the regularity on the horizon is ensured.

2.3.1 Scalars of $SO(p)$

The scalar sector consists of eight independent equations which correspond to the vanishing of the components: $\mathcal{E}_{vv}, \mathcal{E}_{rv}, \mathcal{E}_{rr}, \mathrm{Tr}\mathcal{E}_{ij}, \mathcal{E}_{\Omega\Omega}, \mathcal{E}_{(\phi)}, \mathcal{M}_v, \mathcal{M}_r$.

Constraint equations: There are two constraint equations; $\mathcal{E}^r{}_v = 0$ and $\mathcal{M}_r = 0$. The two equations are solved consistently by

$$\partial_v r_0 = -\frac{r_0(1-(N-2)\gamma_0)}{n+1+(2-n(N-2))\gamma_0}\partial_i u^i \ , \tag{2.26}$$

and

$$\partial_v \gamma_0 = -\frac{2\gamma_0(1+\gamma_0)}{n+1+(2-n(N-2))\gamma_0}\partial_i u^i \ . \tag{2.27}$$

The first equation corresponds to conservation of energy while the second equation can be interpreted as current conservation. These are equivalent to the scalar conservation equations given by (1.26) in the rest frame.

We now proceed to solve for the first-order correction to the scalar part of the metric, gauge field, and dilaton under the assumption that the fluid configuration satisfy the above constraints. By imposing the constraint equations one is left with six equations with five unknowns.

Dynamical equations: The coupled system constituted by the dynamical equations is quite intractable. One approach to obtaining the solution to the system is to decouple the trace function $\mathrm{Tr}f_{ij}$. Once $\mathrm{Tr}f_{ij}$ is known, it turns out, as will be presented below, all the other functions can be obtained while ensuring that they are regular on the horizon.

It is possible to obtain a 3rd order ODE for $\mathrm{Tr}f_{ij}$ by decoupling it through a number of steps. However, first it is useful to note that the particular combination of $\mathrm{Tr}\mathcal{E}_{ij}$ and $\mathcal{E}_{(\phi)}$

leads to the equation

$$\frac{d}{dr}\left[r^{n+1}f(r)T'(r)\right] = -(\partial_i u^i)r^n\left(2(n+1)+C\frac{r_0^n}{r^n}\gamma_0\right)h(r)^{\frac{N}{2}-1} \quad , \tag{2.28}$$

where we have defined the constant $C = 2(n+1) - nN$ and

$$T(r) = \mathrm{Tr}f_{ij}(r) + \frac{4p}{(n+p+1)a}f_\phi(r) \quad . \tag{2.29}$$

As we shall see this equation is very reminiscent of the equations for the tensor perturbations found in section 2.3.3 for which we know the solution to be

$$T(r) = c_T^{(1)} - 2(\partial_i u^i)\left(r_\star - \frac{r_0}{n}(1+\gamma_0)^{\frac{N}{2}}\log f(r)\right) \quad , \tag{2.30}$$

where horizon regularity has been imposed, since we know that $\mathrm{Tr}f_{ij}$ and f_ϕ are individually regular on the horizon. Once f_ϕ is known in terms of $\mathrm{Tr}f_{ij}$ we can use $\mathcal{E}_{rr}$ to eliminate f'_{rv} and then take linear combinations of the remaining equations. The resulting combinations can then be used to eliminate f'_{vv} and f''_{vv} such that one is left with two equations in terms of a_v and $\mathrm{Tr}f_{ij}$ which can then be decoupled by standard means. The resulting equation is schematically of the form

$$H_3^{(n,p)}(r)\left[\mathrm{Tr}f_{ij}\right]'''(r) + H_2^{(n,p)}(r)\left[\mathrm{Tr}f_{ij}\right]''(r) + H_1^{(n,p)}(r)\left[\mathrm{Tr}f_{ij}\right]'(r) = S_{\mathrm{Tr}}(r) \quad , \tag{2.31}$$

where H_1, H_2 and H_3 do not depend on the sources (world-volume derivatives) and the source term S_{Tr} only depends on the scalar $\partial_i u^i$. The expressions for these functions are however very long and have therefore been omitted. After some work, one finds that the equation is solved by

$$\mathrm{Tr}f_{ij}(r) = c_{\mathrm{Tr}}^{(1)} + \gamma_0 c_{\mathrm{Tr}}^{(2)}G(r) - 2(\partial_i u^i)\mathrm{Tr}f_{ij}^{(\mathrm{s})}(r) \quad , \tag{2.32}$$

where the terms containing the two integration constants $c_{\mathrm{Tr}}^{(1)}$ and $c_{\mathrm{Tr}}^{(2)}$ correspond to the homogeneous solution. The entire family of homogeneous solutions to equation (2.31) of course has an additional one-parameter freedom which has been absorbed in the particular solution $\mathrm{Tr}f_{ij}^{(\mathrm{s})}(r)$ and been used to ensure horizon regularity.[3] With the introduction of $c_{\mathrm{Tr}}^{(1)}$ we can safely take $c_T^{(1)} = 0$. The function G is given by

$$G(r) = B\frac{r_0^n}{r^n}\left(-2+(Bn-2)\frac{r_0^n}{r^n}\gamma_0\right)^{-1} \quad , \tag{2.33}$$

and has an intricate relationship to the gauge choice (2.23) as we shall see in section 2.3.4. The particular solution which is regular on the horizon is given by

$$\mathrm{Tr}f_{ij}^{(\mathrm{s})}(r) = \frac{r_0}{n}(1+\gamma_0)^{\frac{N}{2}}\alpha\gamma_0 G(r) + \left(r_\star - \frac{r_0}{n}(1+\gamma_0)^{\frac{N}{2}}\log f(r)\right)(1+\beta\gamma_0 G(r)) \quad , \tag{2.34}$$

with the coefficients

$$\alpha = 2p\left[\frac{2(n+1)+C\gamma_0}{(n+1)^2+C\gamma_0(2(n+1)+C\gamma_0)}\right] \quad \text{and} \quad \beta = p\left[\frac{n+2+C\gamma_0}{n+1+C\gamma_0}\right] \quad . \tag{2.35}$$

[3] Note that equation (2.31) has been derived under the assumption that $\partial_i u^i \neq 0$. This especially means that when there are no sources the one-parameter freedom disappears in accordance with (2.32).

With $\text{Tr}f_{ij}$ (and f_ϕ) given, the equation $\mathcal{E}_{rr} = 0$ will provide the derivative of f_{rv},

$$f'_{rv}(r) = \frac{r}{\left(2(n+1) + C\frac{r_0^n}{r^n}\gamma_0\right) h(r)^{\frac{N}{2}-1}} \left(\frac{d}{dr}\left[h(r)^{\frac{N}{2}} [\text{Tr}f_{ij}]'(r)\right] - 4a\frac{d}{dr}\left[h(r)^{\frac{N}{2}}\right] f'_\phi(r) \right) \quad . \tag{2.36}$$

Since the equation is a 1st order ODE, the regularity of the horizon is ensured by $\text{Tr}f_{ij}$. Note that it is possible to perform integration by parts and use that the derivative of $r_\star$ takes a simpler form. One can thereafter obtain an analytical expression for the resulting integral. This expression is however rather long and does not add much to the question we are addressing for which we are in principle only interested in the large r behavior given by

$$f_{rv}(r) \approx f_{rv}^{(\text{h})}(r) + (\partial_i u^i) \sum_{k=1}^{\infty} \frac{r_0^{nk}}{r^{nk}} \left[\alpha_{rv}^{(k)} r + \beta_{rv}^{(k)} r_0\right] \quad , \tag{2.37}$$

where the homogeneous solution is given by

$$f_{rv}^{(\text{h})}(r) = c_{rv} + \gamma_0 N c_{\text{Tr}}^{(2)} \frac{r_0^n}{r^n} \left[\frac{2p(n+p+1) + (n+p)(2p+C)\frac{r_0^n}{r^n}\gamma_0}{2p\left(2(n+p+1) + (2p+C)\frac{r_0^n}{r^n}\gamma_0\right)^2}\right] \quad , \tag{2.38}$$

and the particular solution is given in terms of the coefficients $\alpha_{rv}^{(k)}$ and $\beta_{rv}^{(k)}$ which depend on n, p, a, and γ_0. The coefficients are in general very long and does not provide much insight. We therefore choose to omit them.

Using the expression for f'_{rv} in terms of $\text{Tr}f_{ij}$, the Maxwell equation $\mathcal{M}_v = 0$ becomes a 2nd order ODE for the gauge field perturbation,

$$\frac{d}{dr}\left[\frac{1}{r^{n-1}} a'_v(r)\right] = \frac{nr^2}{\left(2(n+1) + C\frac{r_0^n}{r^n}\gamma_0\right)} \frac{d}{dr}\left[\frac{1}{r^{n+1}}[\text{Tr}f_{ij}]'(r) + 4a\frac{(n+1)}{r^{n+2}} f'_\phi(r)\right] \quad . \tag{2.39}$$

This equation is solved by a double integration. The inner integral is manifestly regular at the horizon, one can therefore work directly with the asymptotic behavior of the right-hand side before performing the integrations. The large r behavior of the perturbation function is thus found to be

$$a_v(r) \approx a_v^{(\text{h})}(r) + (\partial_i u^i)\left[-\frac{n}{n-1} r + \sum_{k=1}^{\infty} \frac{r_0^{nk}}{r^{nk}} \left[\alpha_v^{(k)} r + \beta_v^{(k)} r_0\right]\right] \quad , \tag{2.40}$$

where the first term constitute the homogeneous solution,

$$a_v^{(\text{h})}(r) = c_v^{(1)} r^n + c_v^{(2)} - \gamma_0 c_{\text{Tr}}^{(2)} \frac{r_0^n}{r^n}\left[\frac{2p+C}{2p\left(2(n+p+1) + (2p+C)\frac{r_0^n}{r^n}\gamma_0\right)}\right] \quad , \tag{2.41}$$

and the particular solution is given in terms of the coefficients $\alpha_v^{(k)}$ and $\beta_v^{(k)}$ depending on n, p, a, and γ_0.

The last perturbation function f_{vv} can be obtained from $\text{Tr}\mathcal{E}_{ij} = 0$ which provides a 1st order ODE for the perturbation. Horizon regularity is therefore ensured by the horizon regularity of $\mathrm{Tr} f_{ij}$. Using the expression for f'_{rv} in terms of $\mathrm{Tr} f_{ij}$ the equation is schematically of the form

$$f'_{vv}(r) = G_1\left[\mathrm{Tr} f_{ij}(r)\right] + G_2\left[a_v(r)\right] + G_3\left[f_\phi(r)\right] + S_{ii}(r) \quad , \tag{2.42}$$

where G_1, G_2, G_3 are differential operators and the source S_{ii} depends on $\partial_i u^i$. Again, the full expressions have been omitted and we only provide the large r behavior,

$$f_{vv}(r) \approx f^{(\mathrm{h})}_{vv}(r) + (\partial_i u^i) \sum_{k=1}^{\infty} \frac{r_0^{nk}}{r^{nk}} \left[\alpha^{(k)}_{vv} r + \beta^{(k)}_{vv} r_0\right] \quad , \tag{2.43}$$

with the homogeneous part given by

$$\begin{aligned} f^{(\mathrm{h})}_{vv}(r) =& c^{(1)}_{vv} + \frac{r_0^n}{r^n}\frac{1}{h(r)}\Bigg[-2(1+\gamma_0)(c^{(2)}_v - c^{(1)}_v r_0^n \gamma_0) + c^{(1)}_{\mathrm{Tr}} \frac{(n+p+1)(1+\gamma_0)a^2}{2p} \\ &+ \left[\frac{(n+p+1+(2p+C)\gamma_0)h(r) - p\gamma_0 N f(r)}{p\left(2(n+p+1) + (2p+C)\frac{r_0^n}{r^n}\gamma_0\right)}\right] c^{(2)}_{\mathrm{Tr}}\Bigg] \quad , \end{aligned} \tag{2.44}$$

and the coefficients $\alpha^{(k)}_{vv}$ and $\beta^{(k)}_{vv}$ again depend on n, p, a, and γ_0.

Finally, one must ensure that the remaining equations coming from $\mathcal{E}_{vv}$ and the angular directions ($\mathcal{E}_{\Omega\Omega} = 0$) are satisfied. This will require the following relation,

$$c^{(1)}_{vv} = -2c_{rv} \quad . \tag{2.45}$$

This completes the analysis of the scalar sector. The remaining undetermined integration constants are thus: $c^{(1)}_{\mathrm{Tr}}, c^{(2)}_{\mathrm{Tr}}, c_{rv}, c^{(1)}_v, c^{(2)}_v$ for which c_{rv} and $c^{(1)}_{\mathrm{Tr}}$ will be fixed by requiring the space-time to be asymptotically flat while the rest constitute the freedom of the homogeneous solution. Note that the above functions reproduce the neutral case as $\gamma_0(\sigma^a) \to 0$.

2.3.2 Vectors of $SO(p)$

The vector sector consists of $3p$ independent equations which correspond to the vanishing of the components: $\mathcal{E}_{ri}, \mathcal{E}_{vi}$ and $\mathcal{M}_i$.

Constraint equations: The constraint equations are given by the Einstein equations ${\mathcal{E}^r}_i = 0$ and are solved by

$$\partial_i r_0 = r_0(1 + N\gamma_0)\partial_v u_i \quad , \tag{2.46}$$

which are equivalent to conservation of stress-momentum. These are part of the conservation equations given by (1.26) in the rest frame. Similar to above we now proceed solving for the first-order corrections to the metric and gauge field under the assumption that the fluid profile satisfy the above constraint (2.46).

Dynamical equations: The remaining equations consist of p pairs consisting of one Einstein equation $\mathcal{E}_{vi}=0$ and one Maxwell equation $\mathcal{M}_i=0$. The structure of these equations is the same as in the scalar sector. The Einstein equation $\mathcal{E}_{vi}=0$ is schematically of the form,

$$L_3^{(n,p)}(r)f_{vi}''(r)+L_2^{(n,p)}(r)f_{vi}'(r)+L_1^{(n,p)}(r)a_i'(r)=S_{vi}(r) \quad , \tag{2.47}$$

while the Maxwell equation $\mathcal{M}_i=0$ is,

$$M_3^{(n,p)}(r)a_i''(r)+M_2^{(n,p)}(r)a_i'(r)+M_1^{(n,p)}(r)f_{vi}'(r)=S_i(r) \quad . \tag{2.48}$$

Again the functions L_k and M_k, $k=1,\dots,3$ have been omitted.

To decouple the system we differentiate $\mathcal{E}_{vi}$ once and eliminate all $a_i(r)$ terms in $\mathcal{M}_i$. Doing so, one obtains a 3rd order ODE for $f_{vi}(r)$ which can be written on the form

$$\frac{d}{dr}\left[\frac{r^{n+1}f(r)}{h^N}\left(1-c_1\frac{r_0^n}{r^n}\right)^2\frac{d}{dr}\left[\frac{r^{n+1}h^{N+1}}{\left(1-c_1\frac{r_0^n}{r^n}\right)}f_{vi}'(r)\right]\right]=S_{vi}(r) \quad , \tag{2.49}$$

with

$$c_1=\frac{N-1}{1+N\gamma_0}\gamma_0 \quad . \tag{2.50}$$

It is possible to perform the first two integrations analytically and ensure regularity at the horizon. The first integration is straightforward while the second involves several non-trivial functions. The large r behavior of the f_{vi} function is found to be

$$f_{vi}(r)\approx c_{vi}^{(1)}-\left(1-\frac{f(r)}{h(r)^N}\right)c_{vi}^{(2)}-(\partial_v u_i)r+\sum_{k=1}^{\infty}\frac{r_0^{nk}}{r^{nk}}\left[\alpha_{vi}^{(k)}r+\beta_{vi}^{(k)}r_0\right] \quad , \tag{2.51}$$

where the first two terms constitute the homogeneous solution and we find, in particular, that in order to ensure horizon regularity one must have

$$\beta_{vi}^{(2)}=-\frac{N}{4n}\left(\frac{2\gamma_0(1+\gamma_0)(\partial_v u_i)+(\partial_i\gamma_0)}{(1+\gamma_0)^{\frac{N}{2}-1}(1+N\gamma_0)}\right) \quad , \tag{2.52}$$

while the remaining set of coefficients $\alpha_{vi}^{(k)}$ and $\beta_{vi}^{(k)}$ are in general complicated expressions depending on the parameters in the problem. We therefore omit them as they provide no insight. Also, we notice that the sum in the function (2.51) vanishes in the neutral limit.

Once the solution of f_{vi} is given we can use $\mathcal{E}_{vi}$ to determine a_i,

$$a_i(r)\approx c_i^{(1)}+\frac{r_0^n}{r^n}\frac{1}{h(r)}c_{vi}^{(2)}+\sum_{k=1}^{\infty}\frac{r_0^{nk}}{r^{nk}}\left[\alpha_i^{(k)}r+\beta_i^{(k)}r_0\right] \quad , \tag{2.53}$$

where the first two terms correspond to the homogeneous solution. Again, we choose to omit the coefficients $\alpha_i^{(k)}$ and $\beta_i^{(k)}$.

The remaining undetermined integration constants are thus: $c_i^{(1)}, c_{vi}^{(1)}$, and $c_{vi}^{(2)}$. The constant $c_{vi}^{(2)}$ corresponds to an infinitesimal shift in the boost velocities along the spatial directions of the brane while $c_i^{(1)}$ is equivalent to an infinitesimal gauge transformation. The last constant $c_{vi}^{(1)}$ will be determined by imposing asymptotically flatness at infinity.

2.3.3 Tensors of $SO(p)$

There are no constraint equations in the tensor sector which consists of $p(p+1)/2-1$ dynamical equations given by

$$\mathcal{E}_{ij} - \frac{\delta_{ij}}{p}\mathrm{Tr}(\mathcal{E}_{ij}) = 0 \quad . \tag{2.54}$$

This gives an equation for each component of the traceless symmetric perturbation functions $\bar{f}_{ij}$,

$$\frac{d}{dr}\left[r^{n+1}f(r)\bar{f}'_{ij}(r)\right] = -\sigma_{ij}r^n\left(2(n+1) + C\frac{r_0^n}{r^n}\gamma_0\right)h(r)^{\frac{N}{2}-1} \quad , \tag{2.55}$$

where $C = 2(n+1) - nN$ and

$$\sigma_{ij} = \partial_{(i}u_{j)} - \frac{1}{p}\delta_{ij}\partial_k u^k \quad . \tag{2.56}$$

The solution is given by,

$$\bar{f}_{ij}(r) = \bar{c}_{ij} - 2\sigma_{ij}\left(r_\star - \frac{r_0}{n}\left(1+\gamma_0\right)^{\frac{N}{2}}\log f(r)\right) \quad , \tag{2.57}$$

where horizon regularity has been imposed and the constant $\bar{c}_{ij}$ is symmetric and traceless and will be determined by imposing asymptotically flatness.

2.3.4 Comment on the homogeneous solution

We have now obtained the solution to the Einstein-Maxwell-dilaton equations for any first-order fluid profile which fulfill the constraint equations. These have been provided in large r expansions and are ensured to have the right behavior at the horizon for any of the remaining integration constants. One remark that is worth mentioning is that f_{ri} did not appear in the analysis above and corresponds to a gauge freedom. This gauge freedom does not play a role for $n \geq 2$, but is expected to play a role for $n = 1$ to ensure asymptotically flatness.

We now want to provide some insight into the meaning of the remaining integration constants. One can separate the constants into two categories; the subset that are fixed by asymptotically flatness and the subset that corresponds to the ε-freedom of the parameters in the zeroth order fields. The latter corresponds exactly to the remaining freedom of the homogeneous solution. In the above the homogeneous part of the fields are given exact.

One finds that the homogeneous part of the scalar sector corresponds to shifts in $r_0 \to r_0 + \varepsilon\delta r_0$, $\gamma_0 \to \gamma_0 + \varepsilon\delta\gamma_0$, and the gauge freedom $a_v \to a_v + \varepsilon\delta a_v$ of the zeroth order metric given by equation (2.8). Indeed, by redefining the r coordinate,

$$r \to r\left(1 + \epsilon\gamma_0(n\delta\log r_0 + \delta\log\gamma_0)G(r)\right) \quad , \tag{2.58}$$

with $G(r)$ given by equation (2.33), the angular directions does not receive first-order contributions in accordance with the gauge choice (2.23), one can relate the integration constants to the two shifts and gauge transformation by,

$$\begin{aligned} c_{\mathrm{Tr}}^{(2)} &= -2p(n\delta\log r_0 + \delta\log\gamma_0) \quad , \\ c_v^{(2)} &= -n\delta\log r_0 - \frac{1+2\gamma_0}{2(1+\gamma_0)}\delta\log\gamma_0 - \frac{\gamma_0}{\sqrt{N\gamma_0(1+\gamma_0)}}\delta a_v \quad , \\ c_v^{(1)} &= -\frac{\delta a_v}{r_0^n\sqrt{N\gamma_0(1+\gamma_0)}} \quad . \end{aligned} \tag{2.59}$$

For the vector sector one finds that the homogeneous part corresponds to the shift of $u_i \to u_i + \varepsilon\delta u_i$ and the gauge transformation $a_i \to a_i + \varepsilon\delta a_i$. The first transformation corresponds to global shifts in the boost velocities. With the same choice of r-coordinate, one has

$$\begin{aligned} c_{vi}^{(2)} &= \delta u_i \ , \\ c_i^{(1)} &= -\frac{\delta a_i}{\sqrt{N\gamma_0(1+\gamma_0)}} \ . \end{aligned} \tag{2.60}$$

This accounts for all the ε-freedom in the full solution.

2.3.5 Imposing asymptotically flatness

We now turn to imposing the boundary condition at infinity, namely requiring the solution to be asymptotically flat. To impose this we must first change coordinates back to the Schwarzschild-like form. Moreover, we need the fields expressed in Schwarzschild coordinates for obtaining the effective stress-energy tensor and current. In order to change coordinates, we use the inverse transformation of the one stated in equation (2.13). The transformation can be worked out iteratively order by order. To first order the transformation from EF-like coordinates to Schwarzschild-like coordinates for a general $r_0(\sigma^a)$ and $\gamma_0(\sigma^a)$ is given by,

$$\begin{aligned} v &= t + r_\star + \varepsilon\Big[(t+r_\star)\left(\partial_{r_0} r_\star \partial_t r_0 + \partial_{\gamma_0} r_\star \partial_t \gamma_0\right) + x^i\left(\partial_{r_0} r_\star \partial_i r_0 + \partial_{\gamma_0} r_\star \partial_i \gamma_0\right)\Big] + \mathcal{O}(\varepsilon^2) \ , \\ \sigma^i &= x^i + \varepsilon\Big[(t+r_\star)\partial_t u^i + \sigma^j \partial_j u^i\Big] r_\star + \mathcal{O}(\varepsilon^2) \ . \end{aligned} \tag{2.61}$$

It is now possible to transform all the fields to Schwarzschild coordinates and impose asymptotically flatness. This leads to

$$c_{rv} = 0, \quad c_{vi}^{(1)} = 0, \quad c_{\mathrm{Tr}}^{(1)} = 0, \quad \bar{c}_{ij} = 0 \ . \tag{2.62}$$

We now have the complete first-order solution for the black brane metric and Maxwell gauge field that solves the Einstein-Maxwell-dilaton equations.

2.4 Viscous stress-energy tensor and current

In this section, we will compute the effective stress-energy tensor and current of the first-order solution obtained above. The general form of the first-order derivative corrections to the stress-energy tensor and current was discussed in section 1.2.2 and we shall briefly discuss the form in the current setting.

The perfect fluid terms were written down for our specific fluid in section 2.1.1 and we can now provide the specific form of $\Pi_{(1)}^{ab}$ and $\Upsilon_{(1)}^{ab}$ as they are encoded in the first-order corrected solution obtained in the previous section. The bulk and shear viscosities η and ζ are associated with the scalar and tensor fluctuations, respectively. Note that although the overall form of $\Pi_{(1)}^{ab}$ is the same as in the neutral case, the transport coefficients are now expected to depend on both the temperature and the charge i.e. on both r_0, γ_0 as well as the dilaton coupling a. Furthermore, the charge diffusion constant $\mathfrak{D}$ is associated with the vector fluctuations. Plugging in the values of Φ and $\mathcal{T}$ in terms of r_0 and γ_0 and using the vector constraint equation (2.46), we find that (in the rest frame)

$$\Upsilon^v_{(1)} = 0, \quad \Upsilon^i_{(1)} \sim \gamma_0(1+\gamma_0)\partial_v u^i + \frac{1}{2}\partial_i\gamma_0 \quad . \tag{2.63}$$

Since the derivatives appear in a very specific combination in this expression, this in fact provides us with a non-trivial check of the blackfold fluid description.

2.4.1 Effective stress-energy tensor and current: Maxwell charge

The quasi-local stress-energy tensor τ_{ij} is obtained by background subtraction as presented in section 1.5.2. We consider a surface at large r (spatial infinity) with induced metric $\bar{\gamma}_{ij}$ and compute the components of the quasi-local tensor by,

$$8\pi G\tau_{ij} = \bar{K}_{ij} - \bar{\gamma}_{ij}\bar{K} - \left(\hat{K}_{ij} - \bar{\gamma}_{ij}\hat{K}\right) \quad , \tag{2.64}$$

where $\bar{K}_{ij}$ is the extrinsic curvature of the timelike hypersurface and $\bar{K} = \bar{\gamma}^{ij}\bar{K}_{ij}$. The hatted quantities are the subtracted terms and constitute τ^0_{ij}. These are computed on flat space-time with the same intrinsic geometry as the boundary of the black brane space-time. Notice that the transverse space bear the structure $h^B d\Omega^2_{(n+1)}$. One finds that for the transverse directions $\tau_{\Omega\Omega} = 0$ while for the brane directions we use equation (1.97) in order to obtain the fluid stress-energy tensor $\hat{T}_{ab}$ with the components,

$$\begin{aligned}
\hat{T}_{tt} &= \frac{\Omega_{(n+1)}}{16\pi G}\left(n+1+nN(\gamma_0+\varepsilon(\delta\gamma_0+x^a\partial_a\gamma_0))\right)\left(r_0+\varepsilon(\delta r_0+x^a\partial_a r_0)\right)^n \quad , \\
\hat{T}_{ij} &= -\frac{\Omega_{(n+1)}}{16\pi G}\delta_{ij}\left(r_0+\varepsilon(\delta r_0+x^a\partial_a r_0)\right)^n - \varepsilon\eta\left[2\left(\partial_{(i}u_{j)} - \frac{1}{p}\delta_{ij}\partial_k u^k\right) + \frac{\zeta}{\eta}\delta_{ij}\partial_k u^k\right] \quad , \\
\hat{T}_{tj} &= -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n n(1+N\gamma_0)\varepsilon(\delta u_j + x^a\partial_a u_j) \quad ,
\end{aligned} \tag{2.65}$$

where the expressions are valid to order $\mathcal{O}(\varepsilon)$. In a similar manner the current is obtained from large r-asymptotics of the gauge fields. Ensuring that the Lorenz gauge condition $\nabla^\mu A_\mu = 0$ is satisfied, the current is obtained using equation (1.99) and leads to

$$\begin{aligned}
\hat{J}_t &= -\frac{\Omega_{(n+1)}}{16\pi G}n\sqrt{N}\left(r_0+\varepsilon(\delta r_0+x^a\partial_a r_0)\right)^n\sqrt{\gamma_0(1+\gamma_0)+\varepsilon(\delta\gamma_0+x^a\partial_a\gamma_0)(1+2\gamma_0)} \quad , \\
\hat{J}_i &= \frac{\Omega_{(n+1)}}{16\pi G}n\sqrt{N}r_0^n\sqrt{\gamma_0(1+\gamma_0)}\left(\varepsilon\left(\delta u_j + x^a\partial_a u_j\right) - \varepsilon r_0\frac{\gamma_0(1+\gamma_0)\partial_v\beta_i + \frac{1}{2}\partial_i\gamma_0}{n(1+N\gamma_0)\gamma_0(1+\gamma_0)^{\frac{N}{2}}}\right) \quad .
\end{aligned} \tag{2.66}$$

Again these expressions are valid to $\mathcal{O}(\varepsilon)$. It is now possible to read off the transport coefficients. Before doing this, we require that the Landau frame renormalization conditions $\Pi^{tt}_{(1)} = \Pi^{ti}_{(1)} = 0$ and $\Upsilon^t_{(1)} = 0$ are satisfied. Equivalently we require the shifts δr_0 and $\delta\gamma_0$ of the zeroth order solution to vanish. Notice that the stress-energy tensor and current do not depend on the gauge transformation δa_a as they should of course not do. Also recall that the shifts were related to the integrations constants by (2.59).

Setting $\delta r_0 = \delta\gamma_0 = 0$, the shear and bulk viscosities are determined using the form given by equation (1.47),

$$\eta(a) = \frac{\Omega_{(n+1)}}{16\pi G} r_0^{n+1} (1+\gamma_0)^{\frac{N}{2}} \quad , \quad \frac{\zeta(a)}{\eta(a)} = \frac{2}{p} + \frac{2}{C}\left(2 - N + \frac{(n+1)N}{(n+1+C\gamma_0)^2}\right) \quad , \tag{2.67}$$

where $C = 2 + n(2-N)$. The second term of $\hat{J}_i$ is seen to have the right proportionality according to (2.63) and hence using the form of equation (1.48) the diffusion constant can be determined,

$$\mathfrak{D}(a) = \frac{\Omega_{(n+1)}}{4G} \frac{1+\gamma_0}{nN\gamma_0} r_0^{n+2} \quad . \tag{2.68}$$

Notice that all the transport coefficients are found to be positive which is expected for a consistent effective fluid dynamic theory. We have now obtained the first-order derivative corrections to the effective stress-energy tensor and current. In the limit $a = 0$, we note that the transport coefficients reproduces the results found for the Reissner-Nordström black brane in [33] while for $\gamma_0 = 0$ the transport coefficients reduces to the results found for the neutral black branes [32].

AdS/flat solution map

We shall now apply the recently established connection between the fluid/gravity correspondence and the blackfold formulation [57]. The connection consists of a map between asymptotically AdS solutions compactified on a torus and a corresponding Ricci-flat solution obtained by replacing the torus by a sphere. We will here provide evidence that the map can be applied in more general cases between asymptotically AdS solutions and solutions that are asymptotically flat i.e in our case theories that includes a dilaton and a gauge field (thus to theories that are not Ricci-flat).

One check is provided by the study of [53], where the fluid description of an asymptotically AdS black brane solution with one scalar and Maxwell gauge field was considered. The solution was obtained from a reduction on Einstein gravity with a negative cosmological constant in $2\sigma+1$ dimensions on a torus $\mathbb{T}^{2\sigma-d}$ to a theory in $d+1$ dimensions with a Kaluza-Klein gauge field and a scalar. We thus consider the specific value of the Kaluza-Klein value of the dilaton coupling

$$a_{\mathrm{KK}} = \sqrt{\frac{2(D-1)}{D-2}} \quad , \tag{2.69}$$

for which one has $N = 1$. We will map the transport coefficients (2.67) and (2.68) to those of the AdS solution. The map provided by [57] instruct us to take $2\sigma \leftrightarrow -n$ with the relation $d = p + 1$.[4] This leads to the coefficients

$$\eta_{\mathrm{AdS}}(a_{\mathrm{KK}}) = \frac{\ell}{r_0^{2\sigma-1}} \sqrt{1+\gamma_0} \quad , \tag{2.70}$$

[4] Note, that to have equivalent normalization of the actions for the two theories one must require that $\ell = \frac{\Omega_{(n+1)}}{16\pi G}$, where ℓ is the AdS radius of the $(d+1)$-dimensional theory.

$$\frac{\zeta_{\mathrm{AdS}}(a_{\mathrm{KK}})}{\eta_{\mathrm{AdS}}(a_{\mathrm{KK}})} = \frac{2}{2\sigma-1}\left(\frac{2\sigma-d}{d-1} - \frac{2\gamma_0((\gamma_0+1)(\sigma-1)+\sigma)}{(2(\gamma_0+1)(\sigma-1)+1)^2}\right) \quad , \tag{2.71}$$

and

$$\frac{\mathfrak{D}_{\mathrm{AdS}}(a_{\mathrm{KK}})}{\eta_{\mathrm{AdS}}(a_{\mathrm{KK}})} = \frac{1}{\gamma_0 \mathcal{T}} \quad , \tag{2.72}$$

in agreement with the result of the AdS computation with the identification $r_0 = 1/m$ and $\cosh^2\omega = \gamma_0 + 1$.[5]

A second check is provided for the case of vanishing coupling constant $a = 0$. The hydrodynamic limit of such AdS-Maxwell branes has been considered in [30, 31] in the case of five dimensions. If we once again apply the map in the specific case of $p = 3$ we obtain $N = 0$ and the transport coefficients takes the form

$$\eta_{\mathrm{AdS}}(0) = \ell r_0^{-3} \quad , \quad \zeta_{\mathrm{AdS}}(0) = 0 \quad , \quad \frac{\kappa_{\mathrm{AdS}}(0)}{\eta_{\mathrm{AdS}}(0)} = \mathcal{T}(1+\gamma_0)^2 \quad , \tag{2.73}$$

in agreement with the result of the AdS computation with the identification $r_0 = 1/R$.[6]

With these checks we conjecture that the transport coefficients of AdS-Maxwell black brane solutions in any dimension and for any dilaton coupling, under the AdS/flat map, takes the form given by equation (2.67) and (2.68).

2.4.2 Effective stress-energy tensor: p-brane charge

In appendix A, we have considered the derivative expansion of dilatonic black brane solutions charged under a $(p+1)$-form gauge field. In this section, we present the resulting effective stress-energy tensor for which the perfect fluid terms was given in section A.1.1. Applying the same method of section 1.5.2 on the first-order corrected solution, the effective fluid stress-energy tensor is of the form

$$\begin{aligned}
\hat{T}_{tt} =& \frac{\Omega_{(n+1)}}{16\pi G}\left(n+1+nN(\gamma_0+\varepsilon(\delta\gamma_0+x^a\partial_a\gamma_0))\right)\left(r_0+\varepsilon(\delta r_0+x^a\partial_a r_0)\right)^n \quad , \\
\hat{T}_{ij} =& -\frac{\Omega_{(n+1)}}{16\pi G}\delta_{ij}\left(r_0+\varepsilon(\delta r_0+x^a\partial_a r_0)\right)^n\left(1+nN(\gamma_0+\varepsilon(\delta\gamma_0+x^a\partial_a\gamma_0))\right) \\
& -\varepsilon\eta\left[2\left(\partial_{(i}u_{j)}-\frac{1}{p}\delta_{ij}\partial_k u^k\right)+\frac{\zeta}{\eta}\delta_{ij}\partial_k u^k\right] \quad , \\
\hat{T}_{tj} =& -\frac{\Omega_{(n+1)}}{16\pi G}r_0^n n\varepsilon(\delta u_j + x^a\partial_a u_j) \quad ,
\end{aligned} \tag{2.74}$$

valid up to order $\mathcal{O}(\varepsilon)$. The shear viscosity given by

$$\eta(a) = \frac{\Omega_{(n+1)}}{16\pi G} r_0^{n+1}(1+\gamma_0)^{\frac{N}{2}} \quad , \tag{2.75}$$

and the bulk viscosity given by

$$\frac{\zeta(a)}{\eta(a)} = 2\left(\frac{1}{p} + \frac{(2-nN)\gamma_0}{n+1+C\gamma_0} + \frac{(n+1)(1+(2-nN)\gamma_0)}{(n+1+C\gamma_0)^2}\right) \quad , \tag{2.76}$$

[5] Note, that [53] uses a different normalization of the diffusion constant such that $\hat{\kappa} = \left(\frac{\mathcal{Q}\mathcal{T}}{w}\right)^2 \mathfrak{D}_{\mathrm{AdS}}$.

[6] Again, one must require that $\ell = \frac{\Omega_{(n+1)}}{16\pi G}$, where ℓ is the AdS radius of the five dimensional theory. Furthermore, since the diffusion constant $\mathfrak{D}$ given by (2.68) diverges for $N \to 0$, we use the normalization $\kappa_{\mathrm{AdS}} = \left(\frac{\mathcal{Q}\mathcal{T}}{w}\right)^2 \mathfrak{D}_{\mathrm{AdS}}$.

where $C = 2 + n(2 - N)$ and $N \equiv A + B$ given by equation (A.3). In the case of p-brane charge we do not have a diffusion constant coming from the vector sector, since the charge can not redistribute itself on the world-volume.

We can compare the viscosities (2.75) and (2.76) with the result obtained in [43] for the D3-brane by setting $p = 3$, $n = 4$, and $a = 0$ for which we find

$$\eta(0) = \frac{\Omega_{(5)}}{16\pi G} r_0^5 \sqrt{1+\gamma_0} \quad , \quad \frac{\zeta(0)}{\eta(0)} = \frac{80}{3(5+6\gamma_0)^2} \quad , \tag{2.77}$$

in agreement for $R_c = 1$ with the identification $r_-^4 = r_0^4 \gamma_0$ and $r_+^4 = r_0^4(1+\gamma_0)$ such that $\gamma_0 = (1-\delta_{\mathrm{e}})/\delta_{\mathrm{e}}$.

2.4.3 Hydrodynamic bounds

We will check the results of the shear viscosities against the expectation that the transport coefficient should satisfy the bound

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \quad . \tag{2.78}$$

Using the value for the entropy density given by (1.102) and the fact that both values for the shear viscosity (2.75) and (2.67) takes the same form, both systems are seen to saturate the bound as expected.

There are different proposals for bounds on the bulk to shear viscosity ratio in the literature. One such bound is proposed by ref. [52],

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{p} - c_s^2\right) \quad , \tag{2.79}$$

where c_s is the speed of sound.[7] For the system with Maxwell charge distributed on the world-volume one has

$$c_s^2 = \left(\frac{\partial P_\perp}{\partial \epsilon}\right)_{\frac{s}{\mathcal{Q}}} = -\frac{1+(2-N)\gamma_0}{(1+\gamma_0 N)(n+1+C\gamma_0)} \quad , \tag{2.80}$$

for which we find that the bound is satisfied for all values of γ_0 when a^2 is sufficiently large. Otherwise, the bound is only satisfied for $0 \leq \gamma_0 \leq \gamma_{0,c}$ for some critical value $\gamma_{0,c}$, while for larger values of γ_0 the bound is violated. If we instead of c_s in (2.79) use the proposed quantity [53, 54]

$$c_{\mathcal{Q}}^2 \equiv \left(\frac{\partial P_\perp}{\partial \epsilon}\right)_{\mathcal{Q}} = -\frac{1+2\gamma_0}{n+1+C\gamma_0} \quad , \tag{2.81}$$

computed for fixed charge density $\mathcal{Q}$, we find that the bound will always be violated (except for the neutral case where it is saturated i.e. $c_{\mathcal{Q}} = c_s$). Applying the map used in section 2.4.1 to $c_{\mathcal{Q}}$, one finds that the ratio actually satisfies the proposed bound. However, this is also an indication that this version of the bound is not universal.

In the case of p-brane charge one finds

$$c_s^2 = \left(\frac{\partial P_\parallel}{\partial \epsilon}\right)_{Q_p} = -\frac{1+(2-nN)\gamma_0}{n+1+C\gamma_0} \quad . \tag{2.82}$$

[7] One should keep in mind that the proposal of this bound relies heavily on holographic considerations.

For this value the bound (2.79) is satisfied for all values of γ_0 and a in the case of p-brane charge.

2.5 Stability and dispersion relations

In ref. [27] the Gregory-Laflamme instability was successfully identified with the unstable sound mode of the neutral black brane. This analysis was further refined in [32] and considered for branes charged under p-form gauge fields in [60]. In this section, we address the issue of stability and dispersion of long-wavelength perturbations of black branes charged under a Maxwell gauge field and furthermore refine the analysis of [60] for black branes charged under a p-form gauge field. In each case, we comment on the connection to thermodynamic (in)stability.

2.5.1 Dispersion relations: Maxwell charge

It is straightforward to show that the first-order fluid (conservation) equations take the form

$$\begin{gathered}\dot{\epsilon} = -(w-\zeta\vartheta)\vartheta - 2\eta\sigma_{ab}\sigma^{ab} \quad , \qquad \dot{u}^a = -\frac{\Delta^{ab}\partial_b(P-\zeta\vartheta) - 2\eta\Delta^a{}_b\partial_c\sigma^{bc}}{w-\zeta\vartheta} \quad , \\ \dot{\mathcal{Q}} = -\mathcal{Q}\vartheta + \mathfrak{D}\left(\frac{\mathcal{Q}\mathcal{T}}{w}\right)^2\left(\vartheta u^b + \dot{u}^b + \Delta^{ab}\partial_a\right)\partial_b\left(\frac{\Phi}{\mathcal{T}}\right) \quad ,\end{gathered} \tag{2.83}$$

where the transport coefficients and the factor associated to $\mathfrak{D}$ are coefficients in the derivative expansion and should be treated as constants. In order to find the speed of sound and dispersion relations, we consider *small* long-wavelength perturbations of the fluid

$$\Phi \to \Phi + \delta\Phi\, e^{i\left(\omega t + k_j x^j\right)}, \quad \mathcal{T} \to \mathcal{T} + \delta\mathcal{T}\, e^{i\left(\omega t + k_j x^j\right)}, \quad u^a = (1,0,\dots) \to (1, \delta u^i\, e^{i\left(\omega t + k_j x^j\right)}) \ . \tag{2.84}$$

The charge density $\mathcal{Q}$, energy density ϵ, and pressure P are perturbed according to

$$\mathcal{Q} \to \mathcal{Q} + \delta\mathcal{Q}\, e^{i\left(\omega t + k_j x^j\right)}, \quad \epsilon \to \epsilon + \delta\epsilon\, e^{i\left(\omega t + k_j x^j\right)}, \quad P \to P + \delta P\, e^{i\left(\omega t + k_j x^j\right)} \quad , \tag{2.85}$$

where the amplitudes can be expressed in terms of thermodynamic derivatives that depend on the specific equation of state. Note that $\delta p = \mathcal{Q}\delta\Phi + s\delta T$ as a consequence of the Gibbs-Duhem relation. Plugging the expressions into the first-order fluid equations (2.83) and linearizing in the amplitudes, we obtain the $p+2$ equations

$$\begin{aligned} i\omega\left(\left(\frac{\partial\epsilon}{\partial\Phi}\right)_{\mathcal{T}}\delta\Phi + \left(\frac{\partial\epsilon}{\partial\mathcal{T}}\right)_{\Phi}\delta\mathcal{T}\right) + iwk_i\delta u^i = 0 \quad , \\ iw\omega\delta u^j + ik^j\left(\mathcal{Q}\delta\Phi + s\delta\mathcal{T}\right) + k^j\left(\eta\left(1-\frac{2}{p}\right)+\zeta\right)k_i\delta u^i + \eta k^2\delta u^j = 0 \quad , \\ i\omega\left(\left(\frac{\partial\mathcal{Q}}{\partial\Phi}\right)_{\mathcal{T}}\delta\Phi + \left(\frac{\partial\mathcal{Q}}{\partial\mathcal{T}}\right)_{\Phi}\delta\mathcal{T}\right) + i\mathcal{Q}k_i\delta u^i + \mathfrak{D}\mathcal{T}\frac{\mathcal{Q}^2}{w^2}\left(\frac{\Phi}{\mathcal{T}}\delta\mathcal{T} - \delta\Phi\right)k^2 = 0 \quad . \end{aligned} \tag{2.86}$$

We stress that the thermodynamic derivatives are not dynamical and do only depend on the equation of state of the fluid in question. In our case they can be computed from (1.100) and (1.102) with $q = 0$. In order to find the ω that solves this system for a given wave vector k^i, we set the determinant of the system of linear equations in the amplitudes to zero. To linear order in k^i (i.e. at the perfect fluid level) the dispersion relation gives the speed of sound $c_s = \omega/k$. Using the equation of state (1.100) and solving the system to linear order, one finds the speed of sound given by equation (2.80). For zero charge $\gamma_0 = 0$ we recover the neutral result $c_s^2 = -1/(n+1)$. Since a negative speed of sound squared signifies an unstable sound mode, the neutral brane is unstable under long-wavelength perturbations. Indeed, this instability is exactly identified with the GL instability [32].

Note that due to the bound on N given by equation (2.7) we have $C > 0$ and hence the denominator of equation (2.80) is always positive. We therefore find that a positive speed of sound squared requires sufficiently small values of the dilaton coupling a such that $N > 2$. Indeed when this is satisfied, the speed of sound squared becomes less and less negative as we increase γ_0 and for

$$\gamma_0 > \frac{1}{N-2} \quad , \tag{2.87}$$

the $q = 0$ brane becomes stable under long-wavelength perturbations to leading order. The condition (2.87) can be satisfied for any non-zero charge density if the black brane temperature is low enough. Indeed, stability is obtained for $\mathcal{T} \sim (G\mathcal{Q})^{-1/n}$ (where the exact numerical factor depends on the number of transverse and brane dimensions).

In order to check stability to next to leading order, we now work out the dispersion relation for the fluid to quadratic order in k. We solve the system of equations to $\mathcal{O}(k^2)$. Solving for the longitudinal modes, we find the equation

$$\omega - c_s^2\frac{k^2}{\omega} - i\frac{k^2}{w}\left(2\left(1-\frac{1}{p}\right)\eta + \zeta\right) + \frac{ik^2}{w}\mathfrak{D}\left(\mathcal{R}_1\left(\frac{k}{\omega}\right)^2 + \frac{\mathcal{R}_2}{w}\right) + \mathcal{O}(k^3) = 0 \quad , \tag{2.88}$$

where the coefficients $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}$ (introduced below) are given in appendix A.4. Solving for the sound mode(s), we find the dispersion relation

$$\omega(k) = \pm c_s k + \frac{ik^2}{w}\left(\left(1-\frac{1}{p}\right)\eta + \frac{\zeta}{2}\right) + ik^2\mathcal{R}\mathfrak{D} \quad . \tag{2.89}$$

For a general fluid both the first-order term (c_s) and the second order term must be positive in order for it to be dynamically stable. In this case, the above equation describes dampening of the long-wavelength sound waves in the fluid.

For $N < 2$, we find that the (second order) attenuation term is positive for all values of γ_0, however as we already mentioned the speed of sound squared is negative for this region of parameter space. For $N > 2$, figure 2.1 shows the general behavior of c_s and the attenuation term in (2.89). Here, the conditions on γ_0 for dynamical stability are found to be partially complementary; when the sound mode is stable to leading order it is unstable next-to-leading order and vice versa. However, as the charge parameter γ_0 becomes sufficiently large this behavior ceases to exist and the sound mode becomes stable. The Maxwell branes therefore seems to have a regime, in which γ_0 is large and the dilaton coupling a is small, where they do not suffer from a GL instability.

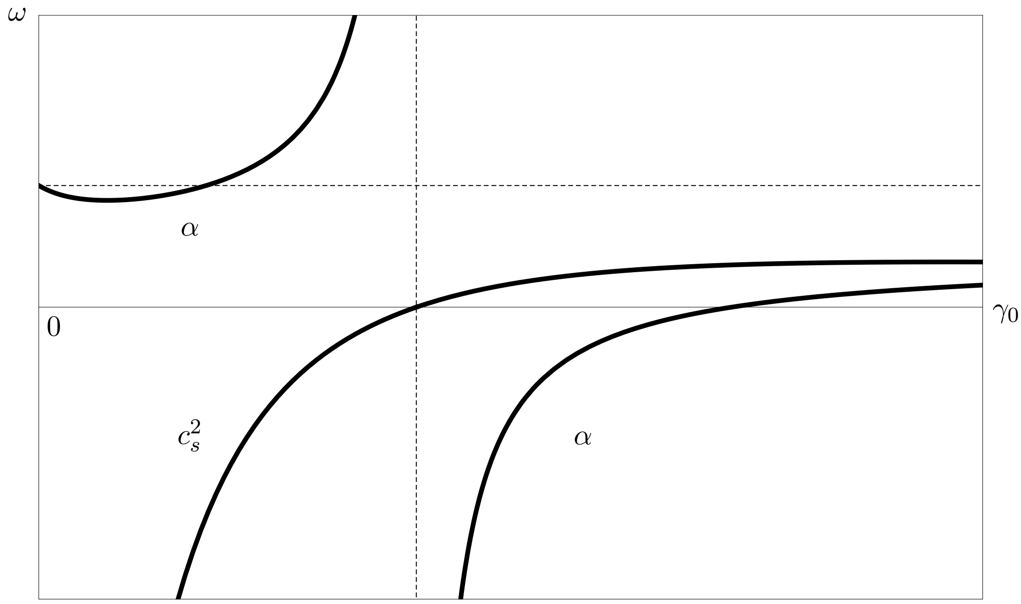

Figure 2.1: The qualitative behavior of the sound mode $\omega = c_s k + \alpha k^2 + \mathcal{O}(k^3)$ given by equation (2.89) as a function of γ_0 assuming $N > 2$. The first-order correction (speed of sound) is seen to be positive when the charge density passes the threshold $\gamma_0 = 1/(N-2)$ (indicated with the vertical dashed line) while the second-order correction diverges at the threshold and becomes negative. However, at sufficiently large γ_0 both the leading order and first-order correction becomes positive ensuring next-to-leading order stability of the system. The horizontal dashed line correspond to the first-order correction in the neutral case. For vanishing γ_0, the speed of sound reduces to the finite negative value obtained for the neutral system (not shown in the figure).

In addition to the sound mode we have a longitudinal diffusion mode given by

$$\omega(k) = \frac{i\mathfrak{D}\mathcal{R}_1}{c_s^2 w} k^2 = -ik^2 \frac{1}{4\pi\mathcal{T}} \left[\frac{(1+\gamma_0)^{1-N}}{1+(2-N)\gamma_0} \right] \quad . \tag{2.90}$$

We see that in general this mode is stable if and only if $\mathcal{R}_1/c_s^2 > 0$. In our case this amounts to the condition $\gamma_0 > \frac{1}{N-2}$ which is the same as (2.87). Finally, we also have a shear mode which takes the form

$$\omega(k) = \frac{i\eta}{w} k^2 \quad . \tag{2.91}$$

The fluctuations of the shear mode are very simple, they are transverse displacement of effective fluid with no variations in the charge and energy densities. Notice that this mode is always stable.

Thermodynamic stability

The conditions for thermodynamic stability of the Maxwell black brane are computed in the grand canonical ensemble since charge is allowed to redistribute itself in the directions of the brane. Using the thermodynamic quantities in equation (1.100)-(1.102), one finds the specific heat capacity $C_\mathcal{Q}$ and the (inverse) isothermal permittivity c to be,

$$\begin{aligned} C_\mathcal{Q} &= \left(\frac{\partial \epsilon}{\partial \mathcal{T}}\right)_\mathcal{Q} = \left(\frac{n+1+(2-n(N-2))\gamma_0}{(nN-2)\gamma_0-1}\right) s \ , \\ c &= \left(\frac{\partial \Phi}{\partial \mathcal{Q}}\right)_\mathcal{T} = \left(\frac{1}{(\gamma_0+1)(1-(nN-2)\gamma_0)}\right) \frac{1}{s\mathcal{T}} \ . \end{aligned} \tag{2.92}$$

Thermodynamical stability is obtained if the two quantities are positive. However, these two conditions are complementary and can never be satisfied. This is also what was found for the class of smeared Dp-branes considered in e.g. [80].

Although, this do not overlap in an exact manner, this complementary behavior is analogous to what was found for the dynamical analysis. It would be interesting to investigate higher order corrections or even perform a numerical analysis in order to further investigate how the behavior predicted by the dynamic analysis and the thermodynamic computation are related thus making a more precise connection to the correlated stability conjecture in the charged case [80].

2.5.2 Dispersion relations: p-brane charge

In order to find the speed of sound and dispersion relations in the case of a brane with unsmeared charge, we consider *small* long-wavelength perturbations of the fluid as before. Setting $\mathfrak{D} = 0$, the first-order equations takes the same form as given by (2.86).

Solving the system to linear order in k one finds the speed of sound given by (2.82) which recovers the neutral result for $\gamma_0 = 0$. From the speed of sound we find, as in ref. [60], that the p-brane with p-brane charge becomes stable under long-wavelength perturbations to linear order if

$$\gamma_0 > \frac{1}{nN-2} \ . \tag{2.93}$$

However, since N is bounded from above one can have instances where $nN < 2$ for which the speed of sound squared is negative for all values of γ_0. Also, note that due to the bound on N given by equation (A.6) we have for $n, p \geq 1$ that $C > 0$.

To quadratic order in k, the sound mode is given by equation (2.89) with $\mathfrak{D} = 0$,

$$\omega(k) = \pm c_s k + \frac{ik^2}{w}\left(\left(1-\frac{1}{p}\right)\eta + \frac{\zeta}{2}\right) \ . \tag{2.94}$$

The sound mode attenuation is therefore of the same form as found for the neutral brane [32], now with the shear and bulk viscosities given by (2.75) and (2.76), respectively. We find, for $nN > 2$, that the second order attenuation term is positive for all values of γ_0. In figure 2.2 the qualitative behavior is shown in the case where the threshold (2.93) exists. The black brane charged under $(p+1)$-form gauge field therefore seems to be dynamically stable for sufficiently large charge parameter γ_0 at least to next-to-leading order.

Finally, we also have the stable shear mode given by equation (2.91).

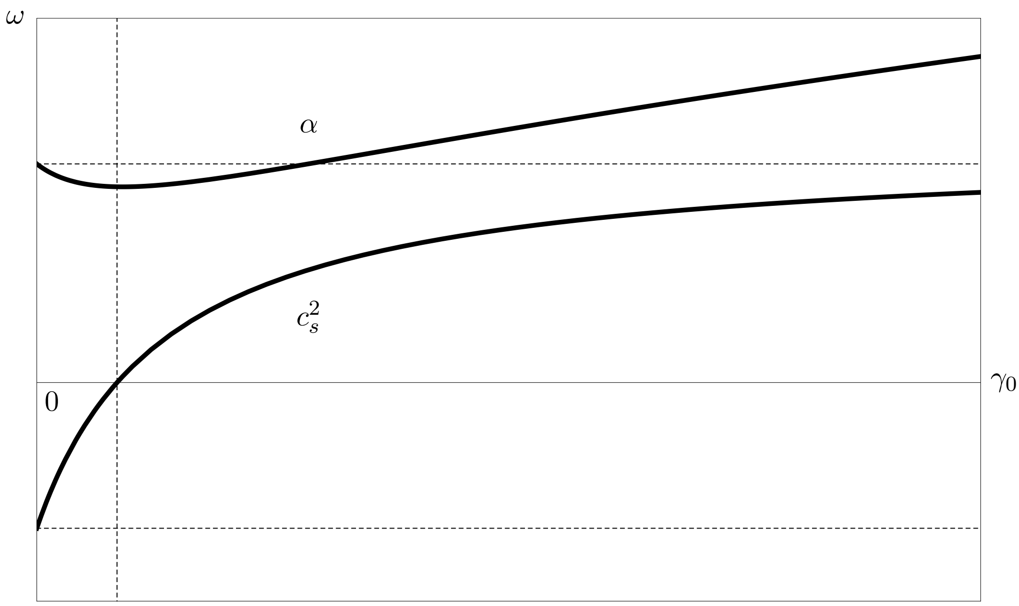

Figure 2.2: The qualitative behavior of the sound mode $\omega = c_s k + \alpha k^2 + \mathcal{O}(k^3)$ given by equation (2.94) as a function of γ_0. The leading order correction (speed of sound) is seen to be positive when the charge density passes the threshold $\gamma_0 = 1/(nN-2)$ (indicated with the vertical dashed line) assuming $nN > 2$. The two horizontal dashed lines show the neutral limit of the speed of sound squared and first-order correction, respectively.

Thermodynamic stability

For the brane with unsmeared charge, the conditions for thermodynamic stability are computed in the canonical ensemble and thus only requires positivity of the specific heat. The specific heat is given by the first expression in equation (2.92). Thus, the threshold value given by equation (2.93) overlaps precisely with the point where the specific heat becomes positive. In fact, as pointed out in ref. [60], there is a direct relation between the speed of sound and the specific heat c_{Q_p} at fixed charge given by

$$c_s^2 = \left(\frac{\partial P_\parallel}{\partial \epsilon}\right)_{Q_p} = s\left(\frac{\partial \mathcal{T}}{\partial \epsilon}\right)_{Q_p} = \frac{s}{c_{Q_p}} \quad . \tag{2.95}$$

For this system, we therefore find that the dynamical stability is in accordance with the correlated stability conjecture [80].

Chapter 3

Elastic Perturbations

The objective of this chapter is to provide explicit realizations in (super)gravity theories of the general results for charged fluid branes presented in section 1.1 and 1.4. We therefore construct slightly curved stationary charged black brane solutions and show that their dynamics can in their long-wavelength regime be described in terms of the effective theory of pole-dipole branes. This provides us with explicit examples of charged black branes that show electroelastic behavior. Furthermore, we obtain the associated response coefficients i.e. the Young modulus and piezoeletric moduli.

Originally, one of the purposes for constructing explicit approximate curved black brane solutions in the context of the blackfold approach was to show that perturbed geometries satisfying the extrinsic blackfold equations (1.16) also have regular horizons. This pursue was considered for black strings curved into black rings, black tori, and black rings in AdS space in refs. [34–36]. It was finally extended and proven for neutral black branes [38] where also a more suitable set of adapted coordinates was introduced to handle this problem. The analysis of the neutral black branes does however not readily apply to the case of charged black branes, since they are solutions to a different set of equations of motion. A priori, it is therefore not proven that the horizon will be regular for perturbed charged black brane solutions satisfying the extrinsic blackfold equations (1.16). We find that all the charged solutions that are constructed in this chapter satisfy horizon regularity. However, this information will be trivial, since we will utilize solution-generating techniques taking the neutral solution as a seed in order to obtain them. The solutions therefore trivially inherit the horizon regularity. Nevertheless, this is direct evidence that it might also hold for more general charged black brane solutions.

In section 3.1, we begin by discussing the setup of the suitable framework for handling extrinsic perturbations. In section 3.2 we review the perturbed neutral black brane and proceed with constructing two classes of bent charged black branes. We then compute the bending moment and electric dipole moment of the solutions using the procedure outlined in section 1.5.3. Finally, in section 3.3 we present the response coefficients of the charged bent black branes that characterize their effective world-volume theory.

3.1 The perturbation expansion

In this section, we discuss an appropriate framework for constructing approximate curved solutions of charged black branes. In order to setup the perturbative expansion we must identify two widely separated scales of the black brane horizon. We thus imagine the charged black brane wrapped on a submanifold with an extrinsic curvature set by a large length scale R that measures the size of the horizon along the world-volume of the submanifold. The size of the horizon transverse to the world-volume is measured by a small length scale that can, for the charged configurations we consider, come from two different scales parametrized by r_0 and γ_0. One can conveniently choose these scales to be the energy-density radius and the charge-density radius given by[1]

$$r_\epsilon = r_0(1+\gamma_0)^{\frac{1}{n}} \quad , \quad r_{\mathcal{Q}} = r_0\sqrt{\gamma_0(1+\gamma_0)}^{\frac{1}{n}} \quad . \tag{3.1}$$

However, since $r_{\mathcal{Q}} < r_\epsilon$, we can consistently take the energy-density radius r_ϵ to define the small length scale of the horizon. Assuming that the two scales are widely separated $r_\epsilon \ll R$, we can setup a consistent expansion in the parameter $\tilde{\varepsilon} = r_\epsilon/R$ using the method of matched asymptotic expansions. In this method, the full geometry is split into a near region ($r \ll R$) and a far region ($r_\epsilon \ll r$) that crucially share a common 'overlap' ($r_\epsilon \ll r \ll R$). One therefore works with two different coordinate patches in which the solution is expanded individually order-by-order. At each iteration of the construction, the regions are subject to boundary conditions from the other region. The full approximate solution to a given order is thus given by matching both of the expansions. The construction of the approximative solution using the matched asymptotic expansion is thus a iterative procedure where at each iteration k the solution receives corrections to order $\tilde{\varepsilon}^k$. The procedure is as follows:

- **0^{th} order:** At zeroth order, the near region solution is given by the Gibbons-Maeda black brane while the far region metric is given by the background metric (in our case flat space). To provide the input for the next step we relate the parameters of the solution r_0 and γ_0 to the asymptotic charges. These charges will constitute the distributional sources for the 1st order far region $T^{\mu\nu} = T^{\mu\nu}_{(0)}\delta^{(n+2)}(r)$ and $J^{\mu_1\cdots\mu_{q+1}} = J^{\mu_1\cdots\mu_{q+1}}_{(0)}\delta^{(n+2)}(r)$, where $T^{\mu\nu}_{(0)}$ and $J^{\mu_1\cdots\mu_{q+1}}_{(0)}$ are the zeroth order effective stress-energy tensor and current, respectively.

- **1st order far:** We now take the gravitational backreaction into account. The equations of motion are linearized around the background and solved to first order in r_ϵ/r for which the solution is completely determined by the zeroth order distributional sources and the boundary condition of asymptotically flatness. It is worth noting that the extrinsic blackfold equations (1.16) will here appear as constraint equations among the Einstein equations. They are necessary for ensuring the regularity of the horizon and can be interpreted as a balancing condition i.e. a centrifugal repulsion that compensates the tension. The far region solution will provide the asymptotic boundary conditions for the 1st order near region solution.

[1] The scales are based on estimates of the conserved mass and charge quantities given in section 1.3.2 (see e.g. [47]).

- **1st order near:** One continue by considering the perturbations of the near region solution, subject to the boundary conditions: regularity of the horizon and matching asymptotics with the far region solution. The solution will in principle determine the corrections to the entropy density, the temperature and the electrical potential. However, as we shall see these quantities do not receive corrections to first order. In order to provide corrected distributional sources for the 2nd order far region, one reads off the corrected solution in the overlap region. The first-order corrections will be dipole of nature and will correspond to the dipole contributions $T^{\mu\nu}_{(1)}$ and $J^{\mu_1\cdots\mu_{q+1}}_{(1)}$ to the effective stress-energy tensor and current, respectively. In section 3.2 we will show how they are obtained using the procedure of section 1.5.3.
- **Repeat:** Once the near and far region solution has been computed they provide the full approximate solution to order $\tilde{\varepsilon}$ and the construction now proceeds in a repeated iteration to compute the next order in the expansion.

We will take the construction through the step '1st order near', where we have first-order corrections to the charged black brane solution. Once the first-order corrected solution is obtained, the dipole terms in the large r-asymptotics of the fields can be extracted and we can read off the bending moment and electric dipole moment.

3.2 First-order solutions

We now turn to computing the first-order extrinsic perturbations of charged black brane solutions in the near region $r \ll R$. There are in principle no obstructions for applying the method of matched asymptotic expansions outlined in the previous section to the case of charged black branes. However, we will here exploit the fact that solutions of neutral black branes with extrinsic first-order corrections are known [38] and use solution-generating techniques in order to find the first-order corrections to a subset of the charged black brane solutions given by (1.85)-(1.90). In particular, using different techniques, we can construct slightly curved black branes with smeared q-brane charge coupled to a dilaton that are solutions to the field equations (1.91) of the action (1.84).

The first class of solutions that we consider consists of black dilatonic p-branes with a single Maxwell gauge field. This class is constructed by uplifting the corrected neutral black brane solution with $m+1$ additional flat directions. The resulting solution is then boosted along the time direction and one of the uplifted directions and followed by a Kaluza-Klein reduction along that particular uplifted and boosted direction. In this way we obtain p-brane solutions carrying Maxwell charge ($q = 0$). The brane directions of the produced solution all lie along the directions labeled by $\vec{z}$ which were introduced in (1.85). The extra m directions appearing as a byproduct of the uplift remain flat while the other world-volume directions now receive first-order corrections. We provide the details of this construction in section 3.2.2.

The second class of solutions we consider are supergravity solutions to type II string theory in $D = 10$ dimensions where we can use T-duality in order to generate higher-form gauge fields. The solution generating technique works in the following way. Starting with a p-brane carrying 0-brane charge, one can perform successive T-duality transformations on the m flat directions leading to a p-brane with q-brane charge, where $q = m$. The effect of this transformation is to introduce higher-form fields and to unsmear the m flat directions. In practice, this transforms the m directions originally included in $\vec{z}$ into m directions now included in $\vec{y}$. We thus end up with Dq-brane solutions smeared in $(p-q)$-directions constrained by the condition $n+p=7$ with $n \geq 1$. The details of this construction are presented in section 3.2.3.

We will start in section 3.2.1 by reviewing the neutral black brane solution in detail as it will act as the seed for the first class of solutions.

3.2.1 The neutral black brane: a review

In this section, we review the near region solution obtained in [38] and the adapted (Fermi normal) coordinates used to decouple the deformation along each direction orthogonal to the brane. Furthermore, we also review the details on the calculations of the Young modulus and the relation to the result obtained for the black string in [37].

Extrinsic perturbations

In order to study the extrinsic deformations of the world-volume of a flat black $\tilde{p}$-brane one introduces a suitable set of adapted coordinates. Due to linearity, perturbations in different orthogonal directions of black branes decouple in suitable coordinates. This decoupling can be achieved by using Fermi normal coordinates. Since the extrinsic curvature of the world-volume is the only first-order derivative correction which characterizes the bending of the brane [42], it is therefore possible to rewrite the induced metric on the brane in terms of the extrinsic curvature tensor ${K_{ab}}^{i}$.

Since in these coordinates the perturbations along each of the transverse directions y^i decouple from each other, one can consider the deformation in each normal direction separately. One can therefore limit the analysis to the study where ${K_{ab}}^{i}$ is non-zero along a single direction $i = \hat{i}$. Introducing a direction cosine, $y^{\hat{i}} = r\cos\theta$, the uniformly boosted flat black $\tilde{p}$-brane metric in the adapted coordinates is given by [38]

$$\begin{aligned} \mathrm{d}s_D^2 = &\left(\eta_{ab} - 2{K_{ab}}^{\hat{i}}\, r\cos\theta + \frac{r_0^n}{r^n}\tilde{u}_a\tilde{u}_b\right)\mathrm{d}\sigma^a\mathrm{d}\sigma^b + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\Omega^2_{(n)} \\ &+ h_{\mu\nu}(r,\theta)\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathcal{O}(r^2/R^2)\ , \end{aligned} \tag{3.2}$$

where $\tilde{u}^a$ are the boost velocities and $f(r) = 1 - \frac{r_0^n}{r^n}$. We can now drop the index on $\hat{i}$ without loss of generality. Since the corrections are of dipole nature one can parametrize the extrinsic perturbation functions according to $h_{\mu\nu}(r,\theta) = \cos\theta\,\hat{h}_{\mu\nu}(r)$ with

$$\begin{aligned} \hat{h}_{ab}(r) &= K_{ab}\mathsf{h}_1(r) + \tilde{u}^c\tilde{u}_{(a}K_{b)c}\mathsf{h}_2(r) + K\tilde{u}_a\tilde{u}_b h_\gamma(r)\ , \\ \hat{h}_{rr}(r) &= Kf(r)^{-1}h_r(r)\ , \\ \hat{h}_{\Omega\Omega}(r) &= Kr^2 h_\Omega(r)\ . \end{aligned} \tag{3.3}$$

We note again that the components of the metric corrections are linear in the extrinsic curvature and are therefore proportional to K_{ab}. The solution is invariant under the coordinate transformation

$$r \to r + K\cos\theta\gamma(r) \quad , \quad \theta \to \theta + K\sin\theta \int^{x} dx \frac{\gamma(x)}{x^2 f(x)} \quad , \tag{3.4}$$

where $\gamma(r)$ is an arbitrary function.[2] Under the transformation $\mathsf{h}_1(r)$ and $\mathsf{h}_2(r)$ are invariant, but the remaining functions transform according to

$$\begin{aligned} h_\gamma(r) &\to h_\gamma(r) - n\frac{r_0^n}{r^{n+1}}\gamma(r) \quad , \\ h_r(r) &\to h_r(r) + 2\gamma'(r) - n\frac{r_0^n}{r^{n+1}}\frac{\gamma(r)}{f(r)} \quad , \\ h'_\Omega(r) &\to h'_\Omega(r) + 2\frac{\gamma'(r)}{r} + 2\frac{r_0^n}{r^{n+2}}\frac{\gamma(r)}{f(r)} \quad . \end{aligned} \tag{3.5}$$

This coordinate-gauge freedom can be eliminated by forming invariant functions and taking combinations of the metric perturbations. For example one can take

$$\begin{aligned} \mathsf{h}_r(r) &= h_r(r) + \frac{2}{n}r_0\left(\frac{r^{n+1}}{r_0^{n+1}}h_\gamma(r)\right)' - \frac{h_\gamma(r)}{f(r)} \quad , \\ \mathsf{h}'_\Omega(r) &= h'_\Omega(r) + \frac{2}{n}\frac{r_0}{r}\left(\frac{r^{n+1}}{r_0^{n+1}}h_\gamma(r)\right)' + \frac{2}{nr}\frac{h_\gamma(r)}{f(r)} \quad . \end{aligned} \tag{3.6}$$

The perturbations can then be expressed in terms of four coordinate-gauge invariant functions for which the solution is

$$\begin{aligned} \mathsf{h}_1(r) &= 2r - AP_{1/n}\left(2\frac{r^n}{r_0^n} - 1\right) \quad , \\ \mathsf{h}_2(r) &= -A\frac{r_0^n}{r^n}\left[P_{1/n}\left(2\frac{r^n}{r_0^n} - 1\right) + P_{-1/n}\left(2\frac{r^n}{r_0^n} - 1\right)\right] \quad , \\ \mathsf{h}_r(r) &= \frac{n+1}{n^2 f(r)}\left[\left(\frac{n}{n+1} - 2\frac{r_0^n}{r^n}\right)(2r - \mathsf{h}_1) - \mathsf{h}_2\right] \quad , \\ \mathsf{h}'_\Omega(r) &= \frac{1}{nrf(r)}\left(2r - \mathsf{h}_1 + \frac{n+2}{2n}\mathsf{h}_2\right) \quad , \end{aligned} \tag{3.7}$$

with the dimensionful constant

$$A = 2r_0\frac{\Gamma\left[\frac{n+1}{n}\right]^2}{\Gamma\left[\frac{n+2}{n}\right]} \quad . \tag{3.8}$$

Here $P_{\pm 1/n}$ are Legendre polynomials. In order to impose horizon regularity one can change coordinates to Eddington-Finkelstein. In these coordinates, one can shows that it is necessary and sufficient to require

$$h_\gamma(r_0) = -\frac{A}{n} \quad . \tag{3.9}$$

With this, the solution is ensured to be regular on the horizon for any extrinsic perturbation that satisfy the leading order extrinsic blackfold equations $\nabla_a T^{ab}_{(0)} = 0$. Since the stress-energy tensor (3.61) of the neutral black $\tilde{p}$-brane is

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n\left(n\tilde{u}^a\tilde{u}^b - \eta^{ab}\right) \quad , \tag{3.10}$$

[2]The function $\gamma(r)$ has to satisfy $\gamma(r_0) = 0$ to ensure horizon regularity.

this is equivalent to the condition

$$n\tilde{u}^a\tilde{u}^b K_{ab} = K \quad . \tag{3.11}$$

It is worth pointing out that since the type of perturbations is purely dipolar, the horizon temperature, entropy-density, and horizon velocity does not receive any corrections.

Large r-asymptotics

We are in particular interested in finding the dipole corrections to the stress-energy tensor and the bending moment of the brane. We therefore focus on the large r-asymptotics of the solution. Below, we list these for the neutral $\tilde{p}$-branes with $n \geq 3$. Given the asymptotics of the Legendre polynomials $P_{\pm 1/n}$, one finds the asymptotics of the coordinate-gauge invariant functions to be

$$\begin{aligned}
\mathsf{h}_1(r) &= \frac{1}{n}\frac{r_0^n}{r^{n-1}} - \frac{\xi_2(n)}{n+2}\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}) \quad , \\
\mathsf{h}_2(r) &= -2\frac{r_0^n}{r^{n-1}} - 2\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}) \quad , \\
\mathsf{h}_r(r) &= \frac{2}{n}r - \frac{3}{n^2}\frac{r_0^n}{r^{n-1}} + \frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}) \quad , \\
\mathsf{h}_\Omega(r) &= \frac{2}{n}r - \frac{n-3}{n^2(n-1)}\frac{r_0^n}{r^{n-1}} - \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}) \quad ,
\end{aligned} \tag{3.12}$$

where[3]

$$\xi_2(n) = \frac{\Gamma\left[\frac{n-2}{n}\right]\Gamma\left[\frac{n+1}{n}\right]^2}{\Gamma\left[\frac{n+2}{n}\right]\Gamma\left[\frac{n-1}{n}\right]^2} = \frac{n\tan(\pi/n)}{4\pi r_0^2}A^2 \quad . \tag{3.13}$$

Choosing a suitable gauge

To find the actual large r-asymptotics of the metric one has to settle upon a gauge by choosing $\gamma(r)$. This choice is of course only for convenience, since it will not affect the actual response coefficients. The coordinate-gauge invariant functions are related to the metric perturbations via equation (3.6). Let us parameterize the asymptotics of the non-gauge invariant function h_γ by

$$h_\gamma(r) = b_0 r + b_1\frac{r_0^n}{r^{n-1}} + k_1\frac{r_0^{n+2}}{r^{n+1}} + b_4\frac{r_0^{2n}}{r^{2n-1}} + k_2\frac{r_0^{2n+2}}{r^{2n+1}} + \mathcal{O}\left(r^{-2n-2}\right) \quad , \tag{3.14}$$

were the coefficients b_0, b_1 and b_4 are in principle determined by matching the asymptotics with the boundary conditions given in the overlap region [34]. We are however free to choose a gauge where $b_0 = 0$ and $b_1 = \frac{1}{2}$. This leads to,

$$h_r(r) \approx \left[\frac{n^2-6+(4n^2-8n)b_4}{2n^2}\right]\frac{r_0^n}{r^{n-1}} + \left[k_1 + 2k_2 + \frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\right]\frac{r_0^{n+2}}{r^{n+1}} \quad , \tag{3.15}$$

$$h_\Omega(r) \approx \left[\frac{3-2b_4(n^2-2n)}{n^2(n-1)}\right]\frac{r_0^n}{r^{n-1}} + \left[\frac{2(k_1-nk_2)}{n(n+1)} + \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\right]\frac{r_0^{n+2}}{r^{n+1}} \quad , \tag{3.16}$$

[3]The function $\xi(n)$ given in [37] is related to $\xi_2(n)$ via $\xi(n) = \frac{n+1}{n^2(n+2)}\xi_2(n)$.

truncated at order $\mathcal{O}\left(r^{-n-2}\right)$. With this choice we eliminate some of the leading order terms in h_r and h_Ω. Note that we only need the dipole terms of these expansions in order to determine the response coefficients.

Measuring the Young modulus

In this section, we provide the details on how the Young modulus is obtained by applying the procedure outlined in section 1.5.3 to the case at hand. The dipole contributions can be read off from equation (3.12) and equation (3.15) such that

$$\begin{aligned} f^{(D)}_{ab} &= K_{ab}\left(-\frac{\xi_2(n)}{n+2}\right) + \tilde{u}^c\tilde{u}_{(a}K_{b)c}\left(-2\xi_2(n)\right) + K\tilde{u}_a\tilde{u}_b k_1 \ , \\ f^{(D)}_{rr} &= K\left[k_1 + 2k_2 + \frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\right] \ , \\ f^{(D)}_{\Omega\Omega} &= K\left[\frac{2(k_1 - nk_2)}{n(n+1)} + \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\right] \ , \end{aligned} \tag{3.17}$$

where the coefficients $f^{(D)}_{ab}$ where defined in equation (1.111). Since the Young modulus is obtained using equation (1.70), one is interested in the bending moment given by equation (1.114). With the transverse gauge condition given by equation (1.112) and a redefinition of k_2 such that

$$k_1 = \frac{2}{1-n}\left(\xi_2(n) + nk_2\right) \quad \text{and} \quad -\tilde{k}\xi_2(n) - \frac{(n+1)(n-4)}{n^2(n^2+n-2)}\xi_2(n) = k_2\left(\frac{2}{1-n}\right) \ , \tag{3.18}$$

one obtains the following form for the bending moment

$$\hat{d}_{ab} = -\xi_2(n)\left[\frac{1}{n+2}K_{ab} + 2\tilde{u}^c\tilde{u}_{(a}K_{b)c} + \frac{4+3n}{n(n+2)}K\tilde{u}_a\tilde{u}_b + \tilde{k}\left[K(n\tilde{u}_a\tilde{u}_b - \eta_{ab})\right]\right] \ . \tag{3.19}$$

Then using equation (3.11) together with (1.70) one finally obtains the Young modulus

$$\begin{aligned} \tilde{Y}_{ab}{}^{cd} = &-\frac{n\tan(\pi/n)}{4\pi}A^2\left[\frac{\Omega_{(n+1)}r_0^n}{16\pi G}\left(\frac{1}{n+2}\delta_{(a}{}^c\delta_{b)}{}^d + 2\tilde{u}_{(a}\delta_{b)}{}^{(c}\tilde{u}^{d)} + \frac{3n+4}{n+2}\tilde{u}_a\tilde{u}_b\tilde{u}^c\tilde{u}^d\right)\right. \\ &\left. + \left[T^{(0)}_{ab}\eta^{cd} + \eta_{ab}T^{cd}_{(0)}\right]\tilde{k}\right] \ , \end{aligned} \tag{3.20}$$

in agreement with what was found in [38].[4] Comparing the structure of (3.20) with equation (1.71) one can find the associated λ-coefficients (see [42]).

Relation to the black string

In this section, we provide the relation to the special case of $\tilde{p} = 1$ originally obtained in [37]. For this particular example we have $K_{ab} = \text{diag}\left(0, -1/R\right)$ and the leading order stress-energy tensor given by (3.10). Inserting the extrinsic curvature and boost in equation (3.11) we find

[4] Note that the Young modulus presented in [38] was defined with the opposite sign compared to the one presented here.

the condition for regularity at the horizon to be $s_\alpha^2 = 1/n$, where the boost is parameterized as

$$\tilde{u}_a = (c_\alpha, s_\alpha) = \left(\sqrt{\frac{n+1}{n}}, \frac{1}{\sqrt{n}} \right) \quad . \tag{3.21}$$

Given this parameterization we have that

$$\tilde{u}^c \tilde{u}_{(a} K_{b)c} = \frac{1}{R} \begin{bmatrix} 0 & \frac{1}{2} c_\alpha s_\alpha \\ \frac{1}{2} c_\alpha s_\alpha & s_\alpha^2 \end{bmatrix} \quad . \tag{3.22}$$

This can then be used together with equation (3.19) to find the coefficients,

$$\begin{aligned} \hat{d}_{tt} &= \frac{2k_2 n^2 (n+2)^2 + (n+1)(4+3n+2n^2)\xi_2(n)}{n^2(n^2+n-2)R} \quad , \\ \hat{d}_{tz} &= \frac{\sqrt{n+1}(2nk_2 + (n+1)\xi_2(n))}{n(n-1)R} \quad , \\ \hat{d}_{zz} &= \frac{(n+1)(4+3n)\xi_2(n)}{n^2(n+2)R} \quad . \end{aligned} \tag{3.23}$$

In order to compare to the original result for the black string [37] one can use the relation

$$\xi_2(n) = \frac{n^2(n+2)}{n+1} \xi(n) \quad , \tag{3.24}$$

and change the gauge by

$$k_2 = \frac{1}{2}((1-n)\bar{k}_2 - n(n+2)\xi(n)) \quad , \tag{3.25}$$

for which one finds the expected form

$$\hat{d}_{tt} = -\frac{1}{R}\left(\bar{k}_2(n+2) + (n^2+3n+4)\xi(n)\right) \quad , \quad \hat{d}_{tz} = -\frac{\bar{k}_2}{R}\sqrt{n+1} \quad , \quad \hat{d}_{zz} = \frac{1}{R}(3n+4)\xi(n) \quad . \tag{3.26}$$

Finally, we note that $\bar{k}_2 = 0$ corresponds, via equation (3.25), to

$$k_2 = -\frac{n(n+2)}{2}\xi(n) = -\frac{n+1}{2n}\xi_2(n) \quad , \tag{3.27}$$

which by equation (3.18) leads to

$$\tilde{k} = -\frac{(n+1)(n+4)}{n^2(n+2)} \quad . \tag{3.28}$$

This concludes our review of the first-order corrected neutral black brane solution and we now turn to generate extrinsically perturbed versions of the charged black branes.

3.2.2 Perturbed solutions with Maxwell charge

We generate charged solutions by using the solution generating technique consisting of applying an uplift-boost-reduce transformation to the neutral solution given in section 3.2.1. In this way we obtain the first-order extrinsic perturbations to charged solutions. The end configuration will consist of dilatonic black p-brane metrics charged under a Maxwell gauge field with Kaluza-Klein dilaton coupling.

The first step in this construction is to uplift the D-dimensional seed solution given by equation (3.2) with $m+1$ additional flat directions,

$$\mathrm{d}s^2_{d+1} = \mathrm{d}s^2_D + \sum_{i=1}^{m}(\mathrm{d}y_i)^2 + \mathrm{d}x^2 \quad , \tag{3.29}$$

where $d = \tilde{p}+m+n+3$. We denote the coordinates that span the original $\tilde{p}$-brane directions by $\sigma^a = (t, z^i)$ with $i = 1, \ldots, \tilde{p}$. The additional flat directions are labeled by y^i with $i = 1, \ldots, m$. Lastly, we have separated the flat x-direction from the rest as it will serve as the isometry direction which we will perform the reduction over.

The second step is to apply a uniform boost $[c_\kappa, s_\kappa]$, with rapidity κ along the t and x-direction such that

$$g^{(d+1)}_{tt} = g_{tt}c^2_\kappa + s^2_\kappa \quad , \quad g^{(d+1)}_{xx} = g_{tt}s^2_\kappa + c^2_\kappa \quad , \quad g^{(d+1)}_{tx} = s_\kappa c_\kappa(g_{tt}+1) \quad ,$$
$$g^{(d+1)}_{tz_i} = c_\kappa g_{tz_i} \quad , \quad g^{(d+1)}_{xz_i} = s_\kappa g_{tz_i} \quad , \tag{3.30}$$

where $g^{(d+1)}$ is the boosted metric and g is the metric given by (3.29).

Finally, we can perform a reduction along the x-direction. The Einstein frame decomposition is given by

$$\mathrm{d}s^2_{(d+1)} = e^{2\tilde{a}\phi}\mathrm{d}s^2_{(d)} + e^{2(2-d)\tilde{a}\phi}(\mathrm{d}x + A_\mu \mathrm{d}x^\mu)^2 \quad , \quad \tilde{a}^2 = \frac{1}{2(d-1)(d-2)} \quad , \tag{3.31}$$

where A_μ is the gauge field and ϕ the dilaton. The Lagrangian density is decomposed according to

$$\sqrt{-g_{(d+1)}}R_{(d+1)} = \sqrt{-g_{(d)}}\left(R_{(d)} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(d-1)\tilde{a}\phi}F^2_{[2]}\right) \quad , \tag{3.32}$$

where $F_{[2]} = \mathrm{d}A_{[1]}$. With this decomposition we find,

$$g^{(d)}_{\mu\nu} = e^{-2\tilde{a}\phi}\left(g^{(d+1)}_{\mu\nu} - \frac{g^{(d+1)}_{\mu x}g^{(d+1)}_{\nu x}}{g^{(d+1)}_{xx}}\right) \quad , \quad A_\mu = \frac{g^{(d+1)}_{x\mu}}{g^{(d+1)}_{xx}} \quad , \quad e^{2(2-d)\tilde{a}\phi} = g^{(d+1)}_{xx} \; . \tag{3.33}$$

Defining $p = \tilde{p} + m$ this provides us with the extrinsic perturbed solution of the black p-branes carrying $q = 0$ charge in the presence of a dilaton. We thus have access to the explicit $1/R$ corrections to the fields. Note that only $\tilde{p}$ directions are extrinsically perturbed while the remaining m directions remain flat. The rapidity κ now takes the interpretation of a charge parameter.

Large r-asymptotics

We are interested in the large r-asymptotics of the solution and we provide them in terms of the large r-asymptotics of the neutral solution. We denote the boost velocities of the neutral solution by $\tilde{u}^a$. It is convenient to define the object

$$\mathcal{K}_{\mu\nu} = \cos\theta\left[\hat{h}_{\mu\nu} - 2rK_{\mu\nu}\right] \quad , \tag{3.34}$$

with $\hat{h}_{\mu\nu}$ given by equation (3.3) together with (3.12) and (3.15). Here $K_{\mu\nu}$ is the extrinsic curvature tensor of the neutral brane solution which is equal to the extrinsic curvature tensor of the generated solution provided $K_{ta} = 0$ for all a, which we assume in the following. Recall, futhermore that the study is limited to deformations along a single direction $i = \hat{i}$ such that $K_{\mu\nu} \equiv K_{\mu\nu}{}^{\hat{i}}$. The large r-asymptotic behavior of the dilaton is

$$e^{-2\tilde{a}\phi} = 1 + \frac{s_\kappa^2 \tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n} + \frac{s_\kappa^2}{d-2}\mathcal{K}_{tt}\left(1 - \frac{d-3}{d-2}\frac{r_0^n}{r^n}s_\kappa^2 \tilde{u}^2\right) + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad . \tag{3.35}$$

The large r-asymptotics of the metric components are

$$\begin{aligned}
g_{tt} &= \eta_{tt} + \left(1 + \frac{d-3}{d-2}s_\kappa^2\right)\frac{r_0^n}{r^n}\tilde{u}_t^2 + \mathcal{K}_{tt}\left(1 + \frac{d-3}{d-2}s_\kappa^2 + \frac{r_0^n}{r^n}\mathcal{C}_{tt}\right) + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
g_{z_i z_j} &= \eta_{ij} + \left(\tilde{u}_i\tilde{u}_j + \eta_{ij}\frac{s_\kappa^2\tilde{u}_t^2}{d-2}\right)\frac{r_0^n}{r^n} + \left(\mathcal{K}_{ij} + \eta_{ij}\frac{s_\kappa^2\mathcal{K}_{tt}}{d-2}\right) + \frac{r_0^n}{r^n}\mathcal{C}_{z_i z_j} + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
g_{tz_i} &= c_\kappa\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_i + \mathcal{K}_{ti}c_\kappa - 2c_\kappa s_\kappa^2\tilde{u}_t\tilde{u}_{(t}\mathcal{K}_{i)t}\frac{d-3}{d-2}\frac{r_0^n}{r^n} + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
g_{y_i y_i} &= 1 + \frac{s_\kappa^2\tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n} + \frac{s_\kappa^2}{d-2}\mathcal{K}_{tt}\left(1 - \frac{d-3}{d-2}\frac{r_0^n}{r^n}s_\kappa^2\tilde{u}_t^2\right) + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
g_{rr} &= 1 + \left(1 + \frac{s_\kappa^2\tilde{u}_t^2}{d-2}\right)\frac{r_0^n}{r^n} + \mathcal{K}_{rr} + \frac{s_\kappa^2}{d-2}\mathcal{K}_{tt} + \frac{r_0^n}{r^n}\mathcal{C}_{rr} + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
g_{\Omega\Omega} &= g_{\xi_i\xi_j}\left(1 + \frac{s_\kappa^2\tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n}\right) + \mathcal{K}_{\Omega\Omega} + \frac{g_{\xi_i\xi_j}s_\kappa^2}{d-2}\mathcal{K}_{tt} + \frac{r_0^n}{r^n}\mathcal{C}_{\Omega\Omega} + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad ,
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
\mathcal{C}_{tt} &= s_\kappa^2\tilde{u}_t^2\frac{d-3}{d-2}\left(\frac{s_\kappa^2}{d-2} - 2c_\kappa^2\right) \quad , \\
\mathcal{C}_{z_i z_j} &= \frac{s_\kappa^2}{d-2}\left[\tilde{u}_t^2\mathcal{K}_{ij} + \left(\tilde{u}_i\tilde{u}_j - \eta_{ij}\frac{d-3}{d-2}s_\kappa^2\tilde{u}_t^2\right)\mathcal{K}_{tt}\right] - 2s_\kappa^2\tilde{u}_t\tilde{u}_{(i}\mathcal{K}_{j)t} \quad , \\
\mathcal{C}_{rr} &= \frac{s_\kappa^2}{d-2}\left[\tilde{u}_t^2\mathcal{K}_{rr} + \left(1 - \frac{d-3}{d-2}s_\kappa^2\tilde{u}_t^2\right)\mathcal{K}_{tt}\right] \quad , \\
\mathcal{C}_{\Omega\Omega} &= \frac{s_\kappa^2\tilde{u}_t^2}{d-2}\left(\mathcal{K}_{\Omega\Omega} - g_{\xi_i\xi_j}\mathcal{K}_{tt}\frac{d-3}{d-2}s_\kappa^2\right) \quad ,
\end{aligned} \tag{3.37}$$

with $\mathcal{K}_{ij} = \mathcal{K}_{z_i z_j}$, $\eta_{ij} = \eta_{z_i z_j}$ and $\tilde{u}_i = \tilde{u}_{z_i}$. The components $g_{\xi_i\xi_j}$ constitute the metric of a $(n+1)$-sphere of radius r. For the gauge field one finds in terms of the parameters of the neutral solution

$$\begin{aligned}
A_t &= s_\kappa c_\kappa\left[\frac{r_0^n}{r^n}\tilde{u}_t^2 + \mathcal{K}_{tt}\left(1 - 2s_\kappa^2\frac{r_0^n}{r^n}\tilde{u}_t^2\right)\right] + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad , \\
A_{z_i} &= s_\kappa\left[\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_i + \mathcal{K}_{ti} - 2s_\kappa^2\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_{(t}\mathcal{K}_{i)t}\right] + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad .
\end{aligned} \tag{3.38}$$

With the large r-asymptotics we can now read off the dipole contributions following the prescription given in section 1.5.3. For this matter it is convenient to use the relation between the metric obtained here and the asymptotically flat Gibbons-Maeda solutions.

Relation to the Gibbons-Maeda black brane solutions

The solutions given by (3.33) is a subset of the asymptotically flat Gibbons-Maeda family presented in section 1.5. The $\tilde{p}$-directions correspond to uniformly boosted directions with boost u^a while the m-directions remain unaffected. It is possible to connect to this family of solutions by a redefinition of the dilaton field $\phi \to 2\phi$ and the identification of the dilaton coupling

$$a^2 = 4(d-1)^2\tilde{a}^2 = \frac{2(\tilde{p}+m+n+2)}{\tilde{p}+m+n+1} \quad . \tag{3.39}$$

The leading order critical boost and charge parameter can be identified, e.g., from the components of the gauge field such that

$$\tilde{u}_t \cosh\kappa = u_t\sqrt{\gamma_0+1} \quad \text{and} \quad \tilde{u}^i = u^i\sqrt{\gamma_0+1} \quad \text{with} \quad \gamma_0 = \tilde{u}_t^2\sinh^2\kappa \quad , \tag{3.40}$$

where $\tilde{u}^a$ is the critical boost of the neutral solution and the normalization conditions $\tilde{u}^a\tilde{u}_a = u^a u_a = -1$ are satisfied. Finally, they correspond to solutions with

$$A = \frac{d-3}{d-2} \quad , \quad B = \frac{1}{d-2} \quad , \tag{3.41}$$

and therefore $N = 1$.

Dipole terms

The dipole contributions can be read off from the asymptotic expansion of the solution (3.36) and (3.38). In terms of the dipole contributions of the seed solution one finds

$$\hat{f}^{(D)}_{tt} = \left(1+\frac{d-3}{d-2}s_\kappa^2\right)f^{(D)}_{tt} \quad , \quad \hat{f}^{(D)}_{tz_i} = c_\kappa f^{(D)}_{tz_i} \quad , \quad \hat{f}^{(D)}_{z_iz_i} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{z_iz_i} \quad , \tag{3.42}$$

$$\hat{f}^{(D)}_{y_iy_i} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} \quad , \quad \hat{f}^{(D)}_{rr} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{rr} \quad , \quad \hat{f}^{(D)}_{\Omega\Omega} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{\Omega\Omega} \quad ,$$

where $\hat{f}_{\mu\nu}$ denote the dipole coefficients of the charged solution (see equation (1.111)). Notice that taking κ to zero reproduces the result of the neutral seed solution.

In order to obtain the dipole contribution to the stress-energy tensor (1.2) we use the method outlined in section 1.5.3. The transverse gauge condition given by equation (1.112) naturally takes the form

$$(\eta^{ab}\hat{f}_{ab} + m\hat{f}_{yy}) + \hat{f}_{rr} + (n-1)\hat{f}_{\Omega\Omega} = 0 \quad , \tag{3.43}$$

which ensures that the relation obtained in (3.18) stays the same, that is,

$$k_1 = \frac{2}{1-n}\left(\xi_2(n) + nk_2\right) \quad , \tag{3.44}$$

and similarly using equation (1.113) one finds $\hat{f}^{(D)} = 2\hat{f}_{\Omega\Omega}$. It is therefore possible to apply equation (1.114) to obtain the bending moment. In terms of the neutral dipole coefficients one finds

$$\hat{d}_{tt} = c_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{\Omega\Omega} \quad , \quad \hat{d}_{tz_i} = c_\kappa f^{(D)}_{tz_i} \quad , \quad \hat{d}_{z_iz_j} = f^{(D)}_{z_iz_j} - f^{(D)}_{\Omega\Omega} \quad , \quad \hat{d}_{y_iy_j} = -f^{(D)}_{\Omega\Omega} \quad . \tag{3.45}$$

Recall that the solution is not boosted along the y^i directions and that these directions are flat, i.e., $K_{y_i a} = 0$ for all a and i. Using the gauge choice given by equation (3.18) such that

$$k_1 = -\frac{3n+4}{n(n+2)}\xi_2(n) - n\tilde{k}\xi_2(n) \quad , \tag{3.46}$$

it is possible to see that in fact $f^{(D)}_{\Omega\Omega} = -\xi_2(n)\tilde{k}K$ and hence the dipole terms in those directions are indeed pure gauge.

Using the relations given in equation (3.40) one can express the coefficients of the bending moment in terms of the Gibbons-Maeda boost and charge parameters. Furthermore, it turns out that it is more natural to work in the gauge given by

$$\tilde{k} = \bar{k} - \frac{3n+4}{n^2(n+2)} \quad , \tag{3.47}$$

in the presence of charge, since the classical symmetries of the Young modulus will be manifestly apparent in this gauge. Suppressing the transverse index in the extrinsic curvature the bending moment is found to be

$$\begin{aligned} \hat{d}_{ab} = & -\xi_2(n)(\gamma_0+1)\left[\frac{K_{ab}}{(n+2)(\gamma_0+1)} + 2u^c u_{(a}K_{b)c} + \frac{3n+4}{n^2(n+2)}\eta_{ab}K\right] \\ & - \bar{k}\xi_2(n)\left[n(\gamma_0+1)\, u_a u_b - \eta_{ab}\right]K \quad , \end{aligned} \tag{3.48}$$

which, as mentioned in section 3.2, is only valid under the assumption that all time components of the extrinsic curvature are zero, that is, $K_{ta} = 0$ for all a.

Similarly, for the gauge field, the non-vanishing dipole terms can be read off from the asymptotic expansion given in equation (3.38)

$$a^{(D)}_t = c_\kappa s_\kappa f^{(D)}_{tt} \quad , \quad a^{(D)}_{z_i} = s_\kappa f^{(D)}_{tz_i} \quad . \tag{3.49}$$

Using equation (1.117) one can read off the electric dipole moment. Again, using the relations given in equation (3.40) and the above assumptions we have that

$$\hat{p}_a = -\xi_2(n)\sqrt{\gamma_0(1+\gamma_0)}\left[u^c K_{ca} + \bar{k}u_a K\right] \quad . \tag{3.50}$$

It is now possible to obtain the response coefficients using (3.48) and (3.50), which are presented in section 3.3.2.

3.2.3 T-duality transformation

With the solutions given in the previous section it is possible to use T-duality on the residual m isometries, if we impose $n + \tilde{p} + m = 7$ and start from a solution with $m \geq 1$. In order to make contact with type II string theory in $D = 10$ we consider the truncated effective action with zero NSNS 2-form B field and only one R-R field.[5] Thus the configurations are

[5] By using type IIB S-duality ($\phi \to -\phi$) it is not difficult to include the case of the NSNS 2-form field as well. In that way it is also possible to obtain bent versions of smeared F-strings and NS5-branes. Note that for the case of $q = 5$ we have $p = 6$ and $n = 1$, so while we can compute the bent solutions our results for the response coefficients are not valid since it requires $n \geq 3$. For $q = 1$ it is possible to have $n \geq 3$. In particular, we find that the response coefficients for the F1-string turn out to be the same as that of the D1-brane.

solutions of the equations of motion that follow from the action

$$S = \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(q+2)!}e^{\frac{3-q}{2}\phi}H^2_{[q+2]}\right] \quad . \tag{3.51}$$

Let z be an isometry direction, then in the Einstein frame, the T-duality transformation takes the form

$$\begin{aligned} g_{\mu\nu} &= e^{\frac{1}{8}\hat{\phi}}(\hat{g}_{zz})^{\frac{1}{4}}\left(\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu z}\hat{g}_{\nu z}}{\hat{g}_{zz}}\right) \quad , \quad g_{zz} = e^{-\frac{7}{8}\hat{\phi}}(\hat{g}_{zz})^{-\frac{3}{4}} \quad , \\ e^{2\phi} &= \frac{e^{\frac{3}{2}\hat{\phi}}}{\hat{g}_{zz}} \quad , \quad A_{[q+2]} = A_{[q+1]} \wedge dz \quad , \end{aligned} \tag{3.52}$$

where the hatted quantities denote the fields before the transformation. The first T-duality transformation is applied to the solution given by (3.33).[6] We apply the T-duality transformations in successive order to transform the m flat directions and gain an $(m+1)$-form gauge field. Performing a recursive bookkeeping one finds the relation between the m'th T-duality transformation and the starting configuration to be

$$g_{\mu\nu} = e^{\frac{m}{6}\hat{\phi}}\hat{g}_{\mu\nu} \quad , \quad g_{y_i y_j} = \delta_{ij}e^{\frac{m-7}{6}\hat{\phi}} \quad , \quad \phi = \frac{3-m}{3}\hat{\phi} \quad . \tag{3.53}$$

Here the y^i directions come from the m isometry directions and μ, ν labels the remaining directions. Note that one can take $m = 0$ and get the starting solution in which no y^i directions are present. The solution now overlaps with the metric given in (1.85) where the $q = m$ directions included in $\vec{y}$ remain flat while the $\tilde{p} = p - q$ directions included in $\vec{z}$ are extrinsically perturbed.

Large r-asymptotics

The large r-asymptotics of the solution (3.53) can be obtained from (3.36), (3.37) and (3.38) by noting that

$$A = \frac{1}{d-2} \to \frac{q+1}{8} \quad , \quad \text{and} \quad B = \frac{d-3}{d-2} \to \frac{7-q}{8} \quad . \tag{3.54}$$

The value $N = 1$ is therefore also preserved under the transformation. The isometry directions naturally differ from the rest and leads to

$$g_{y_i y_i} = 1 - \frac{7-q}{8}\frac{r_0^n}{r^n}s_\kappa^2\tilde{u}_t^2 - \frac{7-q}{8}s_\kappa^2\mathcal{K}_{tt}\left[1 - \left(1 + \frac{7-q}{8}\right)\frac{r_0^n}{r^n}s_\kappa^2\tilde{u}_t^2\right] + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right) \quad . \tag{3.55}$$

The form of the ten dimensional solutions obtained here are likewise a subset of the Gibbons-Maeda family and are related according to the identification given in section 3.2.2.

[6] To relate our constructions with the black branes of supergravity we take $\hat{\phi} = -\phi$ with ϕ being the dilaton given in equation (3.33).

Dipole terms

The dipole terms can be read off from the asymptotic expansion of the solution as before and read

$$\hat{f}_{tt}^{(D)} = \left(1 + \frac{7-q}{8}s_\kappa^2\right) f_{tt}^{(D)} \ \ , \ \ \hat{f}_{tz_i}^{(D)} = c_\kappa f_{tz_i}^{(D)} \ \ , \ \ \hat{f}_{z_iz_i}^{(D)} = \frac{q+1}{8}s_\kappa^2 f_{tt}^{(D)} + f_{z_iz_i}^{(D)} \ \ , \tag{3.56}$$

$$\hat{f}_{y_iy_i}^{(D)} = -\frac{7-q}{8}s_\kappa^2 f_{tt}^{(D)} \ \ , \ \ \hat{f}_{rr}^{(D)} = \frac{q+1}{8}s_\kappa^2 f_{tt}^{(D)} + f_{rr}^{(D)} \ \ , \ \ \hat{f}_{\Omega\Omega}^{(D)} = \frac{q+1}{8}s_\kappa^2 f_{tt}^{(D)} + f_{\Omega\Omega}^{(D)} \ \ ,$$

where $\hat{f}_{\mu\nu}$ denote the dipole coefficients of the charged solution.

The dipole contribution to the stress-energy tensor (1.2) can be extracted using the method outlined in section 3.2. The transverse gauge condition given by equation (1.112) takes the same form as equation (3.43) which ensures again that the relation given in equation (3.44) remains the same after the successive T-duality transformations. We can therefore use equation (1.114) to read off the bending moment and obtaining the only non-zero components

$$\begin{aligned} \hat{d}_{tt} &= c_\kappa^2 f_{tt}^{(D)} + f_{\Omega\Omega}^{(D)} \ \ , \ \ \hat{d}_{y_iy_j} = -f_{\Omega\Omega}^{(D)} - s_\kappa^2 f_{tt}^{(D)} \ \ , \\ \hat{d}_{z_iz_j} &= f_{z_iz_j}^{(D)} - f_{\Omega\Omega}^{(D)} \ \ , \ \ \hat{d}_{tz_i} = c_\kappa f_{tz_i}^{(D)} \ \ . \end{aligned} \tag{3.57}$$

The non-vanishing dipole terms appearing in the higher-form gauge field expansion read

$$a_{ty_1...y_q}^{(D)} = c_\kappa s_\kappa f_{tt}^{(D)} \ \ , \ \ a_{z_iy_1...y_q}^{(D)} = s_\kappa f_{tz_i}^{(D)} \ \ . \tag{3.58}$$

This class of solutions are special in the sense that the charge is always smeared along the directions along which the brane is bent. In other words, the directions in which the q-brane charge lies are always flat and therefore never critically boosted. It is therefore possible to introduce a set of vectors $v_a^{(i)}$, $i = 1...q$ describing the q directions in which the smeared q-charge is located (see section 1.4). The bending moment given in equation (3.57) can then be written in terms of the Gibbons-Maeda boost and charge parameters using the relations (3.40) and read

$$\begin{aligned} \hat{d}_{ab} =& -\xi_2(n)(\gamma_0+1)\left[\frac{K_{ab}}{(n+2)(\gamma_0+1)} + 2u^c u_{(a}K_{b)c} + \frac{3n+4}{n^2(n+2)}\eta_{ab}K\right] \\ & -\bar{k}\xi_2(n)\left[n((\gamma_0+1)\ u_a u_b - \gamma_0\sum_{i=1}^{q} v_a^{(i)}v_b^{(i)}) - \eta_{ab}\right] K \ \ , \end{aligned} \tag{3.59}$$

under the same assumptions on K_{ab} as before. Furthermore, the electric dipole moment is obtained using equation (1.117) and can be written as

$$\hat{p}_{ba_1...a_q} = -(q+1)!\xi_2(n)\sqrt{\gamma_0(1+\gamma_0)}\left[u^c v_{[a_1}^{(1)}...v_{a_q}^{(q)}K_{b]c} + \bar{k}u_{[b}v_{a_1}^{(1)}...v_{a_q]}^{(q)}K\right] \ \ . \tag{3.60}$$

The two dipole moments (3.59) and (3.60) have an identical form when compared to the dipole moments given by equation (3.48) and equation (3.50). This is perhaps not too surprising, since the extrinsic perturbations of all the solutions are always along the smeared directions, i.e., the type of bending is similar for all the solutions considered here. From the bending and electric dipole moments one can read off the corresponding response coefficients and these are presented in section 3.3.3.

3.2.4 Comments

We have now obtained the first-order corrected solution and read off the dipole terms in the large r-asymptotics of the fields. In this way we obtained the corresponding bending moment and electric dipole moment.

We should mention that we have not specified the solution in the far region. It was implicitly used in our starting point through the asymptotic boundary conditions on the near region solution of the neutral black brane. In principle, we could therefore read it off from the near region solutions that we have constructed. However, since we are only interested in the dipole corrections to the fields we do not need to construct the far region explicitly.

Both classes of generated solutions are valid for $n \geq 1$, but in order to compute the response coefficients one must require that $n \geq 3$ such that self-gravitational interactions are sub-leading with respect to the fine structure corrections [37]. Also, it is crucial to point out that we compute the response coefficients under the assumption that the extrinsic curvature components satisfy $K_{ta} = 0$ for all a. With this requirement the extrinsic curvature is invariant under the solution generating techniques used here. On the other hand, if this was not the case the solution generating technique would introduce a background gauge field and a non-zero background dilaton which is an interesting extension, but out of scope for this thesis.

Furthermore, it is worth noting that the solutions constructed here are ensured to be regular on the horizon, since they are generated from the neutral black brane. The corrections of the horizon quantities i.e the temperature, entropy density and electric potential are therefore also uniform. In particular, in case of charged solutions, it is interesting to check that the electric potential at the horizon Φ does not receive corrections. Computing Φ for the generated solutions one finds that it is sufficient that the condition (3.9) is satisfied. This is the same requirement for horizon regularity as found for the perturbed neutral black brane [38]. The quantities associated to the horizon therefore takes the values given by equation (1.102) up to second order with $N = 1$.

Finally, an important detail that we have left out of the discussion until now, is the residual freedom given by the parameter k_1 first encountered in equation (3.14). The parameter is not fixed by the asymptotic boundary conditions from the overlap region and also appeared in the generated solutions as shown. As it is presented, it simply correspond to the freedom in the choice of r-coordinate and it will go into the dipole corrections. From the effective pole-dipole effective theory in section 1.1 we know however that this freedom exactly corresponds to the freedom of displacing the location of the world-volume of the brane.

3.3 Measuring the response coefficients

In this section, we present the response coefficients of the bent charged black brane solutions obtained in section 3.2. We thereby show that these provide explicit realizations in (super)gravity theories of the general results for charged fluid branes presented in section 1.1 and 1.4. We begin by providing the thermodynamic quantities characterizing the world-volume stress-energy tensor and current. These provide the input for the effective world-volume theory for charged (infinitely thin) probe branes discussed in section 1.2. We then turn to the dipole corrections and provide the Young modulus (1.71) and the piezoelectric moduli (1.75) and (1.78).

3.3.1 World-volume stress-energy tensor and thermodynamics

Here we present the thermodynamic quantities characterizing the world-volume stress-energy tensor $T^{ab}_{(0)}$ and world-volume electric current $J^{a_1 \ldots a_{q+1}}_{(0)}$ of the generated solutions. The world-volume stress-energy tensor takes the form[7]

$$T^{ab}_{(0)} = \epsilon u^a u^b + P_\perp \left(\gamma^{ab} + u^a u^b - \sum_{i=1}^{q} v^a_{(i)} v^b_{(i)} \right) + P_\parallel \sum_{i=1}^{q} v^a_{(i)} v^b_{(i)} \quad , \tag{3.61}$$

while the world-volume electric current reads

$$J^{a_1 \ldots a_{q+1}}_{(0)} = (q+1)! \mathcal{Q} u^{[a_1} v^{a_2}_{(1)} \ldots v^{a_{q+1}]}_{(q)} \quad . \tag{3.62}$$

We note that the form presented here is the same as that obtained to leading order in the expansion considered in section 1.2.1 (generalized slightly). Indeed, as noted in [34, 38, 39] for elastically perturbed black branes there are no corrections to the world-volume stress-energy tensor $T^{ab}_{(0)}$ to order $\mathcal{O}(\tilde{\varepsilon})$. This fact is supported by a general analysis of the effective action for stationary black holes [42] which applies to the cases studied here. Therefore, the thermodynamic quantities entering (3.61)-(3.62) do not suffer corrections to this order and are given by (1.100)-(1.101).

3.3.2 Black branes carrying Maxwell charge

In this section, we present the response coefficients for the first class of solutions described in section 3.2.2 consisting of dilatonic black p-branes with a single Maxwell gauge field. For the $q = 0$ case the leading order world-volume stress-energy tensor (3.61) is isotropic

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n(1 + N\gamma_0) u^a u^b - \gamma^{ab} \right) \quad , \tag{3.63}$$

while the leading order electric current is simply

$$J^a_{(0)} = \mathcal{Q} u^a \quad . \tag{3.64}$$

To first order in the perturbative expansion, solving Einstein equations for the perturbed metric such that the horizon remains regular requires solving the leading order blackfold equations (1.16) and (1.17). For a stress-energy tensor of the form (3.63), this implies that the following equation of motion must be satisfied

$$n(1 + N\gamma_0) u^a u^b K_{ab}{}^i = K^i \quad , \tag{3.65}$$

[7] Notice that this form of the world-volume stress-energy tensor and current is certainly not the most general form for the stress-energy tensor and current of the Gibbons-Maeda family of solutions with $q < p$ charge, however, for the cases we considered in section 1.2.1 i.e. the cases $q = 0, 1$ the form presented here is the most general.

where $K^i \equiv \gamma^{ab} K_{ab}{}^i$ is the mean extrinsic curvature vector. For example, by only considering perturbations along a single direction $\hat{i}$, i.e., an extrinsic curvature tensor of the form $K_{ab}{}^{\hat{i}} = \text{diag}(0, -1/R, 0, \dots)$ one finds the leading order critical boost $u_a = [\cosh\beta, \sinh\beta, 0, \dots]$ where

$$\sinh^2\beta = \frac{1}{n(1+N\gamma_0)} \quad . \tag{3.66}$$

This condition is seen to be satisfied for the critical boost given by (3.40) in the case of $N = 1$.

As mentioned in section 1.3, the solution to the intrinsic equation (1.17), generically for $q = 0$, is obtained by requiring stationarity of the overall configuration given by equation (1.50) and setting the global horizon temperature T and global horizon chemical potential Φ_H according to equation (1.53) and (1.55), respectively. Using these relations we can express the solution parameters r_0 and γ_0 in terms of the global quantities using the thermodynamic quantities given in (1.102) such that

$$r_0 = \frac{n}{4\pi T}|\mathbf{k}|\left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \quad , \quad \gamma_0 = \frac{\Phi_H^2}{|\mathbf{k}|^2}\left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{-1} \quad . \tag{3.67}$$

The solutions constructed in section 3.2.2 are automatically stationary due to the stationarity of the neutral seed solution. The stress-energy tensor components of the solution to order $\mathcal{O}(\tilde{\varepsilon})$ are

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n(\gamma_0+1)u^a u^b - \eta^{ab}\right) \quad , \quad T^{y_i y_j}_{(0)} = P_\perp \delta^{y_i y_j} \quad , \tag{3.68}$$

with $a = (t, z_i)$. This result agrees with the form (3.63) by noting that $u^{y_i} = 0$. For $p = 1$ and $m = 1$ this would correspond to a charged tube.

Response coefficients

The components of the Young modulus can be obtained from the bending moment acquired by the bent metric which is given in equation (3.48) together with equation (1.70). It takes the covariant form

$$\begin{aligned} \tilde{Y}_{ab}{}^{cd} = P_\perp r_0^2 \xi_2(n)(\gamma_0+1)\left[\frac{3n+4}{n^2(n+2)}\eta_{ab}\eta^{cd} + \frac{1}{(n+2)(\gamma_0+1)}\delta_{(a}{}^{c}\delta_{b)}{}^{d} + 2u_{(a}\delta_{b)}{}^{(c}u^{d)}\right] \\ - \bar{k}\xi_2(n) r_0^2 \left[T^{(0)}_{ab}\eta^{cd} + \eta_{ab}T^{cd}_{(0)}\right] \quad , \end{aligned} \tag{3.69}$$

where $\bar{k}$ is a dimensionless gauge parameter and the function $\xi_2(n)$ is given by

$$\xi_2(n) = \frac{n\tan(\pi/n)}{\pi}\frac{\Gamma\left(\frac{n+1}{n}\right)^4}{\Gamma\left(\frac{n+2}{n}\right)^2} \quad , \quad n \geq 3 \quad . \tag{3.70}$$

From equation (1.71) together with (3.67) we can obtain the associated non-vanishing λ-coefficients, which read

$$\begin{aligned}\lambda_1(\mathbf{k};T,\Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n+2}\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{n}{2}} \\ &\times\left(\frac{3n+4}{2n^2(n+2)}-\bar{k}\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)\right) \quad , \\ \lambda_2(\mathbf{k};T,\Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n+2}\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{n}{2}+1}\frac{1}{2(n+2)} \quad , \\ \lambda_3(\mathbf{k};T,\Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n}\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{n}{2}} \quad , \\ \lambda_4(\mathbf{k};T,\Phi_H) =& \lambda_3(\mathbf{k};T,\Phi_H)n\bar{k} \quad . \end{aligned} \tag{3.71}$$

From (3.71) we therefore only have 3 independent transport coefficients. Similarly, the components of the piezoelectric moduli can be obtained from the electric dipole moment given in equation (3.50) together with equation (1.74). When written in a covariant form it reads

$$\tilde{\kappa}_a{}^{bc} = -\xi_2(n)r_0^2\left(\frac{\mathcal{Q}}{n}\delta_a{}^{(b}u^{c)}+\bar{k}J_a^{(0)}\eta^{bc}\right) \quad . \tag{3.72}$$

Again, we can obtain the associated non-vanishing κ-coefficients using equation (1.75) and the relations (3.67) yielding

$$\begin{aligned}\kappa_1(\mathbf{k};T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\frac{\xi_2(n)}{2}\left(\frac{n}{4\pi T}\right)^{n+2}\Phi_H|\mathbf{k}|^n\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{n}{2}} \quad , \\ \kappa_3(\mathbf{k};T,\Phi_H) &= \kappa_1(\mathbf{k};T,\Phi_H)n\bar{k} \quad , \end{aligned} \tag{3.73}$$

and therefore only one of the response coefficients is independent. Note that some of the coefficients presented in (3.71) and (3.73) are gauge dependent. The Young modulus (3.69) and the piezoelectric moduli (3.72) obtained here agree with the results of the case $p=1$ studied in reference [39] when using the map given in section 3.2.1. We conclude that the bent black branes carrying Maxwell charge constructed in this work are characterized by a total of 3+1=4 independent response coefficients.

3.3.3 Smeared black Dq-branes

In this section, we specialize to black branes in type II string theory in $D=10$ and present the corrections to the stress-energy tensor (1.2) and current (1.4) as well as the response coefficients. This class of solutions were constructed in section 3.2.3 and consists of black p-branes carrying Dq-charge. The components of the monopole source of stress-energy tensor can be obtained from the solution given in equation (3.53) and read

$$T_{(0)}^{ab} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n\left(n(\gamma_0+1)u^au^b-\eta^{ab}\right) \quad , \quad T_{(0)}^{y_iy_j} = P_{\parallel}\delta^{y_iy_j} \quad , \tag{3.74}$$

with $a=(t,z_i)$. We thus see that the effect of the T-duality transformation is to unsmear the $\vec{y}$ directions, which can be easily realized when comparing the above stress-energy tensor with (3.68). For this particular class of solutions the stress-energy tensor given in (3.74) can be put into the form (3.61) by taking $u^{y_i} = 0$ and noting that the $v^{(i)}_{y_i}$ vectors only take values in the y-directions, e.g., $v^{(2)}_{y_i} = (0, 1, 0, ..., 0)$. Similarly, the electric current can be put into the form (3.62).

Branes carrying string charge

In the case of $q = 1$ the world-volume stress-energy tensor (3.61) reduces to

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1}}{16\pi G} r_0^n \left(n u^a u^b - \gamma^{ab} - nN\gamma_0(-u^a u^b + v^a v^b) \right) \quad , \tag{3.75}$$

where we have omitted the index (1) from the vector $v^{(1)}_a$. The leading order equilibrium condition for configurations with $N = 1$ is obtained by solving equation (1.16) such that

$$n(u^a u^b(\gamma_0 + 1) - v^a v^b \gamma_0) K_{ab}{}^i = K^i \quad , \tag{3.76}$$

while the leading order solution to the intrinsic equation (1.17), in the case of $q = 1$, is again obtained by the requirement of stationarity given by equation (1.50) and with T and Φ_H given by equation (1.53) and (1.60), respectively. The solution parameters r_0 and γ_0 expressed in terms of global quantities using the thermodynamic quantities (1.102) read

$$r_0 = \frac{n}{4\pi T} |\mathbf{k}| \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right)^{\frac{1}{2}} \quad , \quad \gamma_0 = \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right)^{-1} \quad . \tag{3.77}$$

The components of the Young modulus can be obtained from the dipole contributions to the metric given in equation (3.59). This results in the same form as that obtained previously in (3.69) but now with $T^{(0)}_{ab}$ given by equation (3.75). The associated non-vanishing λ-coefficients now enjoy the intrinsic dynamics defined by the relations (3.77) and are given by

$$\begin{aligned}
\lambda_1(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n+2} \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right)^{\frac{n}{2}} \\
& \times \left(\frac{3n+4}{2n^2(n+2)} - \bar{k} \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right) \right) \quad , \\
\lambda_2(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n+2} \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right)^{\frac{n}{2}+1} \frac{1}{2(n+2)} \quad , \\
\lambda_3(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H) =& \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n} \left(1 - \frac{1}{(2\pi)^2} \frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2} \right)^{\frac{n}{2}} \quad , \\
\lambda_4(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H) =& \lambda_3(\mathbf{k}, \boldsymbol{\zeta}; T, \Phi_H) n\bar{k} \quad .
\end{aligned} \tag{3.78}$$

The piezoelectric moduli is a natural generalization of the $q = 0$ case given in (3.72). It can be obtained from the electric dipole moment given in equation (3.60) and takes the form

$$\tilde{\kappa}_{ab}{}^{cd} = -\xi_2(n) r_0^2 \left(2\frac{\mathcal{Q}}{n} \delta_{[a}{}^{(c} v_{b]} u^{d)} + \bar{k} J^{(0)}_{ab} \eta^{cd} \right) \quad , \tag{3.79}$$

where $\tilde{\kappa}_{ab}{}^{cd}$ satisfies the property $\tilde{\kappa}_{ab}{}^{cd} = \tilde{\kappa}_{[ab]}{}^{(cd)}$. Finally, the associated non-vanishing κ-coefficients are given by

$$\begin{aligned}
\kappa_1(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\frac{\xi_2(n)}{2}\left(\frac{n}{4\pi T}\right)^{n+2}\Phi_H|\mathbf{k}|^n\left(1-\frac{1}{(2\pi)^2}\frac{\Phi_H^2}{(|\mathbf{k}||\boldsymbol{\zeta}|)^2}\right)^{\frac{n}{2}} ~~, \\
\kappa_3(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H) &= \kappa_1(\mathbf{k},\boldsymbol{\zeta};T,\Phi_H)n\bar{k} ~~.
\end{aligned} \tag{3.80}$$

Again notice that a subset of the coefficients presented in (3.78) and (3.80) show gauge dependence. We conclude that these branes carrying string charge are characterized by a total of 3+1=4 independent response coefficients.

Branes charged under higher-form fields

For the case $1 < q < p$, one can again obtain the Young modulus from the bending moment given in equation (3.59). It will again lead to the expression written in equation (3.69) but now with $T^{ab}_{(0)}$ given by (3.61). For the piezoelectric moduli, by means of equation (3.60), we find the natural generalization

$$\tilde{\kappa}_{ba_1\ldots a_q}{}^{cd} = -\xi_2(n)r_0^2\left((q+1)!\frac{\mathcal{Q}}{n}\delta_{[b}{}^{(c}v^{(1)}_{a_1}\ldots v^{(q)}_{a_q]}u^{d)} + \bar{k}J^{(0)}_{ba_1\ldots a_q}\eta^{cd}\right) ~~, \tag{3.81}$$

which satisfies the property $\tilde{\kappa}_{ba_1\ldots a_q}{}^{cd} = \tilde{\kappa}_{[ba_1\ldots a_q]}{}^{(cd)}$.

We thus find that the Young modulus and the piezoelectric moduli of all the strained charged black brane solutions considered here can be parameterized by a total of four response coefficients. The fact that we find the same form for the response coefficients associated with the Young modulus is not surprising, since all the solutions obtained here are only perturbed along smeared directions.

Conclusion

We have considered long-wavelength perturbations of charged black branes of both hydrodynamic and electroelastic nature. This allowed us to extract a set of transport and response coefficients characterizing the corresponding effective theories. In order to put the results in a broader perspective we first reviewed the general form of the equations of motion to pole-dipole order of fluid branes carrying higher-form charge [40]. These equations generalize the results for neutral pole-dipole branes obtained in [37, 46] and are in particular important for understanding properties of charged branes in supergravity [47, 60, 72]. However, since these results are quite general they may also be useful in the study of charged extended objects in other settings.

Following [47, 60], we considered the details of the thermodynamics and equations of motion for infinitely thin perfect fluid branes carrying q-brane charge. We also discussed the general form of first-order hydrodynamic dissipative terms of fluid configurations carrying Maxwell charge i.e. shear and bulk viscosity as well as the transport coefficient associated with charge diffusion. Assuming stationarity of the fluid flow, we solved the intrinsic equations finding that the solutions for branes carrying Maxwell and string charge can be completely specified in terms of a set of vector fields, a global temperature and a global potential. In this connection, we note that it would be interesting to develop a formalism for p-branes carrying smeared charged that would enable a complete characterization of stationary solutions for $q > 1$. In the context of stationary solutions we also showed how to compute conserved quantities.

Applying linear response theory, we considered, following [37, 39, 40], the general form of the relevant response coefficients that characterize stationary bent charged (an)isotropic fluid branes i.e. the Young modulus and the piezoelectric moduli. It would be interesting to use these results as a starting point for the formulation of the general effective theory of thin elastic charged fluid branes in the same way as done for neutral branes in [40].

Finally, we introduced a large class of dilatonic black branes charged under a single higher-form gauge field in (super)gravity theories [47] and discussed the methods for extracting the transport and response coefficients that characterize their long-wavelength description. We then turned to study long-wavelength perturbations for which we will now outline the results.

Hydrodynamic perturbations

In chapter two, we investigated the nature of the hydrodynamic effective theory that governs the intrinsic long-wavelength fluctuations of black branes charged under two different settings; a one-form ($q = 0$) gauge field and a $(p + 1)$-form ($q = p$) gauge field. With the extraction of the effective stress-energy tensor and current, the analysis provided the generalizations of the known shear and bulk viscosities for neutral black branes [32] as well as the charge diffusion constant in the case of the Reissner-Nordström black brane [33]. In particular, we have found the dependence on the dilaton coupling of the transport coefficients for this large class of black branes.

It is worth pointing out that a subset of the black branes considered in this thesis contains the supergravity descriptions of D-branes and M-branes. As a comparison we find, in the appropriate limit, that the transport coefficients for the case of the D3-brane are in exact agreement with the results obtained in [43]. In this context, we expect the large class of solutions obtained here to give further input into the connection between the hydrodynamic sector of the blackfold effective theory and the hydrodynamic limit of the AdS/CFT correspondence.

We find that the shear viscosity for both black brane solutions receives the expected modification such that $\eta/s = 1/4\pi$. It is reasonable to expect that the result for η given by (2.67) and (2.75) holds for the entire black brane family presented in section 1.5.1, since the form is dictated by the entropy density. The bulk viscosity was found to be strictly positive for all values of the charge and dilaton coupling in both cases as expected for a non-conformal effective fluid. We also considered the proposed ζ/s bounds in [52–54], but they do not seem to hold for the complete class of solutions. For the Maxwell brane we computed the charge diffusion constant $\mathfrak{D}$. As with the value of η, the value of $\mathfrak{D}$ only show dependence on N which could be an indication that the result will hold for more general cases of smeared charge $q < p$. Furthermore, for the Maxwell black branes, we showed that the recently proposed AdS/Ricci flat correspondence [57] can potentially be extended to a larger class of asymptotically flat solutions e.g. to solutions including a dilaton and gauge field. In particular, we mapped the transport coefficients of the effective fluid to the corresponding results of known AdS solutions and found exact agreement with [31, 53, 54]. This extension of the map will be made more precise in [49].

For both the $q = 0$ and $q = p$ system, the dynamical stability was analyzed under small long-wavelength perturbations of the fluid. Both classes of black branes are found to be GL unstable for large values of the dilaton coupling due to an instability in the sound mode. For sufficiently small values of the dilaton coupling, the instability persists for the $q = p$ system, but as the charge density is increased above a certain threshold the system becomes stable. On the other hand, for sufficiently small values of the dilaton coupling, the $q = 0$ system also continues to be unstable, but now shows a partially complementary behavior in the following way: for small values of the charge density the speed of sound squared is negative while the attenuation of the sound mode is positive, but as the charge density exceeds a certain threshold the two terms exchange sign. The Maxwell black brane therefore seems to suffer from a GL instability. However, continuing to increase the charge density one finds that the sound mode eventually becomes stable. Thus, for sufficiently small values of the dilaton coupling and sufficiently large charge densities the Maxwell black brane is stable, at least to next-to-leading order.

The behavior of the $p = q$ system relates exactly to the expectations from thermodynamic stability. However, for the $q = 0$ system we find that even though the complementary behavior of the instability is reflected in the thermodynamic stability conditions where the specific heat capacity and isothermal permittivity show a similar behavior they do not overlap exactly in the same way and furthermore at large charge density this relation breaks down. It would therefore be interesting to investigate the relation between the two approaches in more detail, that is, establish a more precise connection to the correlated stability conjecture [80]. Also, it would be interesting for comparison to perform a numerical analysis of the long-wavelength perturbations in the current setting as was done in the case of the neutral brane, where excellent agreement was found [32].

A natural generalization of the above results would be to consider the derivative expansion to second order either by direct computation or by considering the corresponding, less technically demanding, charged AdS black brane solution and subsequently use the AdS/flat solution map. Furthermore, considering the hydrodynamic perturbations of spinning branes or multi-charged configurations would also be interesting extensions.

Elastic perturbations

In chapter three, we considered stationary extrinsic perturbations of charged black branes. One of the primary goal with constructing such solutions was to show their electroelastic behavior via the general results for charged fluid branes under the assumption of linear response theory. In particular, we constructed bent charged black branes by using a solution generating technique and the neutral bent black brane [38] as a seed solution as well as T-duality in ten dimensions. This resulted in two classes of solutions: the first class consisting of p-brane solutions carrying Maxwell charge ($q = 0$) coupled to a dilaton through a Kaluza-Klein dilaton coupling and the second class consisting of Dq-brane solutions smeared in $(p - q)$-directions constrained by the condition $n + p = 7$ with $n \geq 1$. The zeroth order effective fluid being isotropic and anisotropic in the first and second class, respectively. Although, perhaps trivial it is worth pointing out that these solutions remained regular under the extrinsic perturbations, a property, which is believed to be true if the leading order blackfold equations are satisfied. Thus, these examples gives evidence that this might hold for more general blackfold constructions carrying charge.

By computing the bending moment and electrical dipole moment of the solutions we explicitly showed that these quantities are captured by classical electroelastic theory. In particular, the Young modulus and piezoelectric moduli of the specific classes considered here of strained charged black brane solutions are parameterized by a total of $3 + 1 = 4$ response coefficients, both for the isotropic as well as anisotropic cases. While the black branes provide an interesting realization of electroelastic behavior of charged fluid branes, it is not surprising that they are characterized by just one more response coefficient, as compared to bent neutral black branes. This is a consequence of the fact that we obtain them by a solution generating technique, which causes the branes to be bent only in the smeared directions. It would therefore be very interesting to find more general bent black brane solutions, in which the bending also takes place in the directions in which the brane is charged. A particular special case of this would be to obtain, to first order, the solution of a bent D3-brane in type IIB string theory. This would allow to explore the physical interpretation of the response coefficients in the context of the AdS/CFT correspondence. Another important, but technically challenging, next step would be to obtain the metric of bent black branes to second order in the matched asymptotic expansion. This would provide further clues to a more formal development of electroelasticity of black branes. Even in the neutral case this would hold valuable information.

It is worth pointing out that the obtained forms of the Young modulus and piezoelectric moduli are results obtained from specific solutions where $N = 1$. However, it is reasonable to expect that due to the presence of $\mathcal{Q}$ and the structure of the large r-asymptotics, in particular the appearance of the coefficients A and B in equation (3.36), that the results also holds for general dimensions (and general N) as long as the bending is along smeared directions and gravitational backreaction is subleading.

Furthermore, it would be interesting to understand the coupling of fluid branes to external gauge fields. This would, in particular, allow us to compute electric (magnetic) susceptibilities, but more generally help us understand polarization effects. It would also be interesting to consider spinning charged branes as it would allow one to compute the magnetic moment. In a different direction it would also be interesting to consider multi-charged configurations. Finally, we also note that examining the elastic corrections for thermal string probes [72, 73, 81–85] is expected to shed further light on the physics of these finite temperature objects which where obtained using the blackfold method.

Open problems

We have already mentioned some interesting generalizations and extensions to the work presented in this thesis. We end with a set of interesting open problems and future directions of a more general nature:

- It would be interesting to include a Chern-Simon term in the theory. This was considered in AdS fluid/gravity in the papers [30, 31, 86]. Both examining the effect of Chern-Simons terms on the transport coefficients and the response coefficients would be relevant, in part due to the relation of these terms to the anomaly via the gauge/gravity correspondence. However, we note that black brane solutions analogous to the Gibbon-Maeda black brane solution with such a term in the action is to the knowledge of the author not known in the literature.

- A very interesting computation, that has not been investigated in the blackfold approach, is the computation of the entropy current à la [87]. Computing the entropy current could provide a consistency check of the transport coefficients and the framework in general.

- Studying time-dependent embeddings in order to address the damping of extrinsic oscillations of the world-volume. This would require a combination of hydrodynamic and elastic perturbations and would provide interesting insights into the stability of black branes.

- The blackfold approach is based on the property that large classes of known black holes look locally like flat black branes in certain limits. As mentioned in the introduction, certain black holes in AdS space-times even has this property [35, 61]. It could be interesting to explore whether the blackfold approach could be applied to black branes with different asymptotics e.g. asymptotically AdS black branes of higher co-dimension. In that way, one could bent branes with intrinsic curvature and thus gain access to new regimes with widely seperated scales. Even though the effective fluid would have to be obtained by numerically means (see e.g. [88, 89]), this would might give insights into obstacles like the existence of large black rings in AdS.

- Finally, it would be very interesting to examine whether the electroelastic behavior of black branes could provide clues towards the microscopics of black holes and branes. The AdS/CFT context is probably be the most natural starting point for this, but more generally for the asymptotically at black branes, this holds the potential of providing valuable insights towards flat space holography. One may wonder whether there is a microscopic way to derive the type of response coefficients that we have encountered in this thesis.

Appendix A

Hydrodynamics Perturbations

In this appendix we repeat the setup and calculation of section 2 for the case of a dilatonic black p-brane charged under a $(p+1)$-form gauge field [49]. In section A.1 we review this particular black brane solution. In section A.2, we apply the perturbation procedure and continue in section A.3 with solving the first-order equations. The resulting effective stress-energy tensor with dissipative corrections are presented in section 2.4.2.

A.1 Black branes charged under a $(p+1)$-form gauge field

We now consider black brane solutions with a $(p+1)$-form gauge field $A_{[p+1]}$. The action is given in section 1.5 by setting $p = q$,

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - 2(\nabla\phi)^2 - \frac{1}{2(p+2)!} e^{-2a\phi} H^2_{[p+2]} \right] \quad , \tag{A.1}$$

where $H_{[p+2]}$ is the field strength of the gauge field, $H = \mathrm{d}A$, and is coupled to the dilaton ϕ through the coupling constant a.

The world-volume of the black brane is characterized by p spatial directions denoted by x^i and a time direction t. The transverse space is characterized by a radial direction r along with the transverse sphere S^{n+1}. The space-time dimension D is related to p and n by $D = p + n + 3$. The metric takes the form

$$\mathrm{d}s^2 = h^{-A} \left(-f \mathrm{d}t^2 + \sum_{i=1}^{p} (\mathrm{d}x^i)^2 \right) + h^B \left(f^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(n+1)} \right) \quad , \tag{A.2}$$

with the harmonic functions f and h given by equation (1.86). The exponents of the harmonic functions are given by

$$A = \frac{4n}{2(p+1)n + (n+p+1)a^2} \quad , \quad B = \frac{4(p+1)}{2(p+1)n + (n+p+1)a^2} \quad . \tag{A.3}$$

The gauge field is given by

$$A_{[p+1]} = -\frac{\sqrt{N}}{h} \left(\frac{r_0}{r} \right)^n \sqrt{\gamma_0(1+\gamma_0)}\, \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^p \quad . \tag{A.4}$$

where $N \equiv A + B$, and finally the dilaton is given by

$$\phi = -\frac{1}{4} N a \log h \quad . \tag{A.5}$$

For future reference note that since $a^2 > 0$, the parameter N is bounded by

$$N \in \left]0; \quad 2\left(\frac{1}{n} + \frac{1}{p+1}\right)\right] \quad . \tag{A.6}$$

The static solution can be extended by applying a uniform boost u^a in the world-volume directions. The metric (A.2) of the dilatonic boosted black brane carrying p-charge then takes the form

$$\mathrm{d}s^2 = h^{-A}\left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) \mathrm{d}x^a \mathrm{d}x^b + h^B \left(f^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(n+1)}\right) \quad , \tag{A.7}$$

while the gauge field and dilaton field remains invariant under the boost.

We note that this class of solutions contain the black D/NS-branes of type II string theory. The solutions are obtained for $D = 10$ and $N = 1$ with $p = 0, ..., 6$ where the field strength takes the role of a RR field. For the particular case of $p = 3$, the hydrodynamic limit of the black D3-brane was considered in [43]. Furthermore, the M-branes of M-theory are obtained for $D = 11$ and $N = 1$ with $p = 2, 5$.

A.1.1 Effective fluid

The blackfold effective theory of p-branes with p-brane charge was reviewed in section 1.2.1. For a uniform boost u^a of the brane, the effective blackfold stress-energy tensor can be written in the form given by (1.104) as,

$$T^{ab}_{(0)} = \mathcal{T} s \left(u^a u^b - \frac{1}{n}\gamma^{ab}\right) - \Phi Q\, \gamma^{ab} \quad , \tag{A.8}$$

where γ_{ab} is the induced metric on the world-volume. For our purposes (flat extrinsic geometry), we have $\gamma_{ab} = \eta_{ab}$. Moreover $\mathcal{T}$ is the local temperature, s is the entropy density, $\mathcal{Q}$ is the charge density and finally Φ is the electric potential conjugate to $\mathcal{Q}$. The various quantities are are provided in section 1.5.2 parameterized in terms of a charge parameter γ_0 and the horizon thickness r_0. The stress-energy tensor can be expressed (in standard form) in terms of the energy density ϵ and pressure $P = P_{\|}$,

$$T^{ab}_{(0)} = \epsilon u^a u^b + P \Delta^{ab} \quad , \tag{A.9}$$

and the $(p+1)$-form current supported by the p-brane is given by

$$J^{(0)}_{(p+1)} = \mathcal{Q}_p \hat{V}_{p+1} \quad . \tag{A.10}$$

To leading order, the intrinsic blackfold equations take the form of the world-volume conservation equations $\nabla_a T^{ab}_{(0)} = 0$ and $d * J_{(0)} = 0$. The first is evaluated in section 1.2.1 and is given by equation (1.26) while the latter is trivially satisfied at leading order, since $\mathcal{Q}_p = Q_p$. However, both of the conservation equations will be important in the perturbative analysis as they will show up as constraint equations when solving the system perturbatively.

A.2 Setting up the perturbation

The setup follows the steps of the Einstein-Maxwell-dilaton system in section 2.2 now with the aim of solving the system in a derivative expansion around the solution given in section A.1. In order to control the perturbations and have a valid description at the horizon we will again use the EF coordinates defined by equation (2.13). In EF coordinates, the metric (A.7) takes the form

$$\mathrm{d}s^2_{(0)} = h^{-A}\left(-f u_a u_b \mathrm{d}\sigma^a \mathrm{d}\sigma^b - 2h^{\frac{N}{2}} u_a \mathrm{d}\sigma^a \mathrm{d}r + \Delta_{ab}\mathrm{d}\sigma^a \mathrm{d}\sigma^b\right) + h^B r^2 \mathrm{d}\Omega^2_{(n+1)} \quad . \tag{A.11}$$

Here the subscript indicates that the metric solves the system of equations to zeroth order in the derivatives. The gauge field will acquire the $p+1$ indpendent non-zero $A_{ra_1,\dots,a_p}$ components with $a \in \{v, \sigma^i\}$. However, we shall work in a gauge where these components are zero. We therefore take

$$A^{(0)} = -\frac{\sqrt{N}}{h}\left(\frac{r_0}{r}\right)^n \sqrt{\gamma_0(\gamma_0+1)}\, \mathrm{d}v \wedge \mathrm{d}\sigma^1 \wedge \dots \wedge \mathrm{d}\sigma^p \quad . \tag{A.12}$$

We promote the parameters u^a, r_0 and γ_0 to *slowly* varying world-volume fields,

$$u^a \to u^a(\sigma^a), \quad r_0 \to r_0(\sigma^a), \quad \gamma_0 \to \gamma_0(\sigma^a) \quad , \tag{A.13}$$

and expand the fields around a given point $\mathcal{P}$ as in equation (2.19). We then seek the derivative corrections to the metric, gauge field, and dilaton as prescribed in equation (2.20). Again, we want the r coordinate to maintain its geometrical interpretation and therefore we choose to work in the gauge where

$$g^{(1)}_{rr} = 0 \quad , \tag{A.14}$$

and moreover take

$$g^{(1)}_{\Omega\Omega} = 0 \quad \text{and} \quad A^{(1)}_{ra_1\dots a_p} = 0 \quad . \tag{A.15}$$

We use the SO(p) invariance of the background to classify the perturbations according to their transformation properties under SO(p). The scalar sector contains 5 scalars, $A^{(1)}_{vi_1\dots i_p}$, $g^{(1)}_{vr}$, $g^{(1)}_{vv}$, $\mathrm{Tr}g^{(1)}_{ij}$, and $\phi^{(1)}$. The vector sector contains 2 vector $g^{(1)}_{vi}$ and $g^{(1)}_{ri}$. Finally, the tensor sector contains 1 tensor $\overline{g}^{(1)}_{ij} \equiv g^{(1)}_{ij} - \frac{1}{p}(\mathrm{Tr}g^{(1)}_{kl})\delta_{ij}$ (the traceless part of $g^{(1)}_{ij}$). We parameterize the three SO(p) sectors according to

$$\begin{gathered}\textbf{Scalar: } A^{(1)}_{vi_1\dots i_p} = -\sqrt{N\gamma_0(1+\gamma_0)}\,\frac{r_0^n}{r^n} h^{-1} a_{vi_1\dots i_p}, \quad g^{(1)}_{vr} = h^{\frac{B-A}{2}} f_{vr} \quad , \\ g^{(1)}_{vv} = h^{1-A} f_{vv}, \quad \mathrm{Tr}\, g^{(1)}_{ij} = h^{-A}\mathrm{Tr} f_{ij}, \quad \phi^{(1)} = f_\phi \quad , \\ \textbf{Vector: } g^{(1)}_{vi} = h^{-A} f_{vi}, \quad g^{(1)}_{ri} = h^{\frac{B-A}{2}} f_{ri} \quad , \\ \textbf{Tensor: } \overline{g}^{(1)}_{ij} = h^{-A}\overline{f}_{ij} \quad ,\end{gathered} \tag{A.16}$$

where $\overline{f}_{ij} \equiv f_{ij} - \frac{1}{p}(\mathrm{Tr} f_{kl})\delta_{ij}$.

A.3 First-order equations

In order to compute the effective stress-energy tensor and current and thereby extract the transport coefficients, we need the large r asymptotics of the perturbation functions which are decomposed and parametrized according to equation (A.16). We denote the first-order Einstein, gauge field, and dilaton equations by

$$\begin{aligned} R_{\mu\nu} - 2\nabla_\mu\phi\nabla_\nu\phi - S_{\mu\nu} &\equiv \varepsilon\mathcal{E}_{\mu\nu} + \mathcal{O}(\varepsilon^2) = 0 \ , \\ \nabla_\mu \left(e^{-2a\phi} H^{\mu}{}_{\rho_1\ldots\rho_{q+1}} \right) &\equiv \varepsilon\mathcal{M}_{\rho_1\ldots\rho_{q+1}} + \mathcal{O}(\varepsilon^2) = 0 \ , \\ g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{a}{4(p+2)!}e^{-2a\phi}H^2 &\equiv \varepsilon\mathcal{E}_{(\phi)} + \mathcal{O}(\varepsilon^2) = 0 \ , \end{aligned} \tag{A.17}$$

with

$$S_{\mu\nu} = \frac{1}{2(p+1)!}e^{-2a\phi}\left(H_{\mu\rho_1\ldots\rho_{p+1}}H_\nu{}^{\rho_1\ldots\rho_{p+1}} - \frac{p+1}{(D-2)(p+2)}H^2 g_{\mu\nu} \right) \ . \tag{A.18}$$

They can be obtained from (1.91)-(1.92) by setting $q = p$. We will now provide some details on how the solution to each SO(p) sector can be obtained following closely the analysis of section 2.3.

A.3.1 Scalars of $SO(p)$

The scalar sector consists of eight independent equations which correspond to the vanishing of the components: $\mathcal{E}_{vv}, \mathcal{E}_{rv}, \mathcal{E}_{rr}, \mathrm{Tr}\mathcal{E}_{ij}, \mathcal{E}_{\Omega\Omega}, \mathcal{E}_{(\phi)}, \mathcal{M}_{a_1\ldots a_{p+1}}, \mathcal{M}_{ri_1\ldots i_p}$ with $i \in \{\sigma^i\}$.

Constraint equations: There are two constraint equations; $\mathcal{E}^r{}_v = 0$ and $\mathcal{M}_{ri_1\ldots i_p} = 0$ with $i \in \{\sigma^i\}$. The two equations are solved consistently by

$$\partial_v r_0 = -\frac{r_0(1+2\gamma_0)}{n+1+(2-n(N-2))\gamma_0}\partial_i u^i \ , \tag{A.19}$$

and

$$\partial_v \gamma_0 = -\frac{2n\gamma_0(1+\gamma_0)}{n+1+(2-n(N-2))\gamma_0}\partial_i u^i \ . \tag{A.20}$$

The first equation corresponds to conservation of energy while the second equation can be interpreted as the charge density being constant in time.

Dynamical equations: After the constraint equations (A.19) and (A.20) has been imposed one is left with a system very similar to the one obtained in the presence of Maxwell charge ($q = 0$). It consists of the six equations given by the components: $\mathcal{E}_{vv}$, $\mathcal{E}_{rr}$, $\mathrm{Tr}\mathcal{E}_{ij}$, $\mathcal{E}_{\Omega\Omega}$, $\mathcal{E}_{(\phi)}$, and $\mathcal{M}_{a_1\ldots a_{p+1}}$.

The particular combination of $\mathrm{Tr}\mathcal{E}_{ij}$ and $\mathcal{E}_{(\phi)}$ gives equation (2.28), but now with the relation

$$T(r) = \mathrm{Tr}f_{ij}(r) - \frac{4np}{(n+p+1)a}f_\phi(r) \ , \tag{A.21}$$

and is solved by the same expression given by (2.30). The equation for the trace $\mathrm{Tr} f_{ij}$ is again similar to the $q = 0$ case and the solution can be put on the form given by (2.32) with (2.33) and (2.34), but now with the coefficients

$$\alpha = \frac{2pn^2}{p+1}\left[\frac{2(n+1)+C\gamma_0}{(n+1)^2+C\gamma_0(2(n+1)+C\gamma_0)}\right] \quad \text{and} \quad \beta = \frac{p}{p+1}\left[\frac{n^2}{n+1+C\gamma_0}\right] \quad , \tag{A.22}$$

with $C = 2 + n(2 - N)$. The solution of $\mathrm{Tr} f_{ij}$ dictates the perturbation of the dilaton field through equation (A.21). With $\mathrm{Tr} f_{ij}$ determined, one can find the remaining perturbation functions as follows: f_{rv} from $\mathcal{E}_{rr}$ and f_{vv} from $\mathrm{Tr}\mathcal{E}_{ij}$ by a single integration while the gauge field perturbation $a_{vi_1\ldots i_p}$ can be obtained from $\mathcal{M}_{va_1\ldots a_p}$ by a double integration. Finally, we note that remaining undetermined integration constants are equivalent to the $q = 0$ case.

A.3.2 Vectors of $SO(p)$

The vector sector consists of $3p$ independent equations which correspond to the vanishing of the components: $\mathcal{E}_{ri}$, $\mathcal{E}_{vi}$ and $\mathcal{M}_{vrj_1\ldots j_{p-1}}$ with $j \neq i$.

Constraint equations: The constraint equations are given by the Einstein equations ${\mathcal{E}^r}_i = 0$ and $\mathcal{M}_{vrj_1\ldots j_{p-1}} = 0$ with $j \neq i$. For each spatial index i one has a pair of equations that are solved by

$$\partial_i r_0 = \frac{r_0(1+2\gamma_0)}{1-(nN-2)\gamma_0}\partial_v u_i \quad , \tag{A.23}$$

and

$$\partial_i \gamma_0 = -\frac{2n\gamma_0(1+\gamma_0)}{1-(nN-2)\gamma_0}\partial_v u_i \quad . \tag{A.24}$$

The first equation corresponds to conservation of stress-momentum while the second equation censures that the charge density does not have any spatial gradients over the world-volume. We see that the current is more constrained compared to the case with Maxwell charge which is tied to the fact that the p-brane charge is not able to redistribute itself.

Dynamical Equations: After the constraint equations (A.23) and (A.24) have been imposed the remaining p equations consist of 2nd order differential equations $\mathcal{E}_{vi} = 0$ of the form

$$\frac{d}{dr}\left[r^{n+1} f'_{vi}(r)\right] = S_{vi}(r) \quad . \tag{A.25}$$

Each equation can be integrated analytically in order to obtain the perturbation functions f_{vi}. Horizon regularity is ensured due to the form of the differential operator. The homogeneous solution gives rise to two integration constants: $c^{(1)}_{vi}$ and $c^{(2)}_{vi}$. The constant $c^{(2)}_{vi}$ corresponds to an infinitesimal shift in the boost velocities along the spatial directions of the brane while the constant $c^{(1)}_{vi}$ will be determined by imposing asymptotically flatness at infinity.

A.3.3 Tensors of $SO(p)$

It turns out that with the parametrization given by equation (A.16), the equations for the tensor perturbations take the exact same form as found for the Maxwell charge given by the form (2.55). The solution is therefore,

$$\bar{f}_{ij}(r) = \bar{c}_{ij} - 2\sigma_{ij}\left(r_\star - \frac{r_0}{n}\left(1+\gamma_0\right)^{\frac{N}{2}}\log f(r)\right) \quad , \tag{A.26}$$

with σ_{ij} given by equation (2.56) and regularity of the horizon has been imposed. The constant $\bar{c}_{ij}$ is symmetric and traceless and will be determined by imposing asymptotically flatness. Because of this closed-form expression, the shear viscosity will take the same form as found for the system with Maxwell charge.

A.3.4 Comments

We have now obtained the solution of the first-order perturbation functions for the system governing the dynamics of a dilatonic black p-brane charged under a $(p+1)$-form gauge field. It is a regular solution for any first-order fluid profile that fulfill the constraint equations. Although, the procedure of solving the individual sectors followed a very similar procedure, the specific solution is significantly different than the one obtained for the Einstein-Maxwell-dilaton system considered in section 2.3. The vector sector being a noteworthy exception due to the increasing number of constraint equations. Again, it is should be mentioned that f_{ri} did not appear in the analysis and corresponds to a gauge freedom. The gauge freedom does not play a role for $n \geq 2$, but is expected to play a role for $n = 1$ to ensure asymptotically flatness.

As in section 2.3.4, a subset of the remaining integration constants corresponds to the remaining freedom of the homogeneous solution, namely the ε-freedom of the parameters in the zeroth order fields, r_0, γ_0, u^i as well as the gauge transformation $A_{[p+1]} \to A_{[p+1]} + \mathrm{d}\Lambda_{[p]}$ for some p-form $\Lambda_{[p]}$. Indeed, one can again relate the integration constants to the shifts and gauge transformation by redefining the r coordinate according to equation (2.58). This way one finds that the integration constants $c_{\mathrm{Tr}}^{(2)}, c_v^{(2)}$ and $c_v^{(1)}$ appearing in the scalar sector correspond to shifts of r_0 and γ_0 and the gauge transformation, while in the vector sector one finds that $c_{vi}^{(2)}$ corresponds to global shifts in the boost velocities. This accounts for all the ε-freedom in the full solution.

The remaining integration constants are fixed by requiring asymptotically flatness. In order to impose asymptotically flatness at infinity we change coordinates back to Schwarzschild-like coordinates. These coordinates will also enables us to obtain the effective stress-energy tensor by the methods outlined in section 1.5.2. Using the inverse transformation of the EF-like coordinates given by equation (2.61) we find that asymptotically flatness requires

$$c_{rv} = 0, \quad c_{vi}^{(1)} = 0, \quad c_{\mathrm{Tr}}^{(1)} = 0, \quad \bar{c}_{ij} = 0 \quad . \tag{A.27}$$

This completes the computation of the first-order intrinsic derivative corrections for the dilatonic black brane charged under a $(p+1)$-form gauge field. The effective stress-energy tensor and current are presented in section 2.4.2.

A.4 Thermodynamic coefficients

In this subappendix we list a number of thermodynamic coefficients related to the analysis of section 2.5. The two coefficients $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by

$$\begin{aligned} \mathcal{R}_1 &= \mathcal{Q}^2 \left[\left(\frac{\partial \mathcal{Q}}{\partial \mathcal{T}} \right)_{\Phi} \left(\frac{\partial \epsilon}{\partial \Phi} \right)_{\mathcal{T}} - \left(\frac{\partial \mathcal{Q}}{\partial \Phi} \right)_{\mathcal{T}} \left(\frac{\partial \epsilon}{\partial \mathcal{T}} \right)_{\Phi} \right]^{-1} , \\ \mathcal{R}_2 &= -\mathcal{R}_1 \left[\mathcal{T} \left(\frac{\partial \epsilon}{\partial \mathcal{T}} \right)_{\Phi} + \Phi \left(\frac{\partial \epsilon}{\partial \Phi} \right)_{\mathcal{T}} \right] . \end{aligned} \tag{A.28}$$

The speed of sound obtained takes the form

$$c_s^2 = \frac{\mathcal{R}_1}{\mathcal{Q}^2 w} \left[w \left(\mathcal{Q} \left(\frac{\partial \mathcal{Q}}{\partial \mathcal{T}} \right)_{\Phi} - s \left(\frac{\partial \mathcal{Q}}{\partial \Phi} \right)_{\mathcal{T}} \right) - \mathcal{Q} \left(\mathcal{Q} \left(\frac{\partial \epsilon}{\partial \mathcal{T}} \right)_{\Phi} - s \left(\frac{\partial \epsilon}{\partial \Phi} \right)_{\mathcal{T}} \right) \right] . \tag{A.29}$$

It can be shown that it is equivalent to the expression given by equation (2.80) or (2.82). Finally the coefficient associated to the dispersion relation of the sound mode is given by

$$\mathcal{R} = \frac{1}{2} \frac{\mathcal{R}_1^2}{\mathcal{Q}^2 w^3 c_s^2} \left(\mathcal{Q} \left(\frac{\partial \epsilon}{\partial \mathcal{T}} \right)_{\Phi} - s \left(\frac{\partial \epsilon}{\partial \Phi} \right)_{\mathcal{T}} \right) \left(\mathcal{Q} \frac{\mathcal{R}_2}{\mathcal{R}_1} + w \left(\left(\frac{\partial \mathcal{Q}}{\partial \Phi} \right)_{\Phi} \Phi + \left(\frac{\partial \mathcal{Q}}{\partial \mathcal{T}} \right)_{\Phi} \mathcal{T} \right) \right) . \tag{A.30}$$

Appendix B

Derivation of the EOM

In this appendix, we provide the details on the derivation of the equations of motion for p-branes carrying q-brane charge given in [40]. In particular, we give the explicit details in the case of Maxwell charge ($q = 0$) and string charge ($q = 1$) as well as discuss their invariance under 'extra symmetry 1' and 'extra symmetry 2'. In the last section, we will present the results for generic p-branes carrying q-brane charge with $q > 1$.

B.1 Branes carrying Maxwell charge

Pole-dipole p-branes carrying Maxwell charge ($q = 0$) are characterized by a current J^μ of the form (1.4). In order to solve equation (1.5) we introduce an arbitrary scalar function $f(x^\alpha)$ of compact support and integrate (1.5) over the entire space-time following the method outlined in [46] applied to the stress-energy tensor (1.2)

$$\int d^D x \sqrt{-g} f(x^\alpha) \nabla_\mu J^\mu = 0 \quad . \tag{B.1}$$

In order to make further progress one decomposes the derivatives of $f(x^\alpha)$ in parallel and orthogonal components to the world-volume such that

$$\nabla_\mu f = f^\perp_\mu + u^a_\mu \nabla_a f \quad , \quad \nabla_\nu \nabla_\mu f = f^\perp_{\mu\nu} + 2 f^\perp_{(\mu a} u^a_{\nu)} + f_{ab} u^a_\mu u^b_\nu \quad . \tag{B.2}$$

Here the label $\perp$ on a tensor indicates that all of its space-time indices are transverse, for example, $u^\mu_a f^\perp_\mu = 0$. Explicit computation of the functions involved allows one to deduce

$$f^\perp_{\mu a} = {\perp^\lambda}_\mu \nabla_a f^\perp_\lambda + \left(\nabla_a u^b_\mu \right) \nabla_b f \quad , \quad f_{ab} = \nabla_{(a} \nabla_{b)} f - f^\perp_\mu \nabla_b u^\mu_a \quad . \tag{B.3}$$

This tells us that the only independent components on the world-volume surface $x^\alpha = X^\alpha(\sigma^a)$ are $f^\perp_{\mu\nu}$, $f^\perp_\mu$ and f. Using this and performing a series of partial integrations when introducing (1.4) into (B.1) leads to an equation with the following structure

$$\int_{\mathcal{W}_{p+1}} \sqrt{-\gamma} \left[Z^{\mu\nu} f^\perp_{\mu\nu} + Z^\mu f^\perp_\mu + Z f + \nabla_a \left(Z^{\mu a} f^\perp_\mu + Z^{ab} \nabla_b f + Z^a f \right) \right] = 0 \quad . \tag{B.4}$$

Requiring the above equation to vanish for each of the arbitrary independent components on the world-volume $f^{\perp}_{\mu\nu}$, $f^{\perp}_{\mu}$ and f results in the equations

$$\perp^{\lambda}{}_{\mu}\perp^{\rho}{}_{\nu}J^{(\mu\nu)}_{(1)}=0 \quad , \quad \perp^{\lambda}{}_{\mu}\left[J^{\mu}_{(0)}-\nabla_a\left(2\perp^{\mu}{}_{\nu}u^a_{\rho}J^{(\nu\rho)}_{(1)}+u^a_{\nu}u^b_{\rho}u^{\mu}_b J^{(\nu\rho)}_{(1)}\right)\right]=0 \quad , \tag{B.5}$$

$$\nabla_a\left(J^{\mu}_{(0)}u^a_{\mu}+2J^{(\mu\nu)}_{(1)}u^b_{\nu}\nabla_b u^a_{\mu}-\nabla_b\left(J^{(\mu\nu)}_{(1)}u^a_{\mu}u^b_{\nu}\right)\right)=0 \quad . \tag{B.6}$$

From equation (B.4), we are then left with a boundary term that vanishes by itself

$$\int_{\partial\mathcal{W}_{p+1}}\sqrt{-h}\eta_a\left(Z^{\mu a}f^{\perp}_{\mu}+Z^{ab}\nabla_b f+Z^a f\right)=0 \quad , \tag{B.7}$$

where h is the determinant of the induced metric on the boundary.

On the brane boundary, however, the components $\nabla_a f$ are not independent so we decompose them according to

$$\nabla_a f=\eta_a\nabla_{\perp}f+v^{\hat{a}}_a\nabla_{\hat{a}}f \quad , \tag{B.8}$$

where $\nabla_{\perp}\equiv\eta^a\nabla_a$, $v^{\hat{a}}_a$ are boundary coordinate vectors and $\nabla_{\hat{a}}$ is the boundary covariant derivative with the indices $\hat{a}$ labeling boundary directions. On the brane boundary the functions $f^{\perp}_{\mu}$, $\nabla_{\perp}f$ and f are mutually independent. Requiring the terms appearing in equation (B.7) proportional to these functions to vanish leads to the following boundary conditions:

$$\perp^{\lambda}{}_{\mu}J^{(\mu\nu)}_{(1)}u^a_{\nu}\eta_a|_{\partial\mathcal{W}_{p+1}}=0 \quad , \quad J^{(\mu\nu)}_{(1)}u^a_{\mu}u^b_{\nu}\eta_a\eta_b|_{\partial\mathcal{W}_{p+1}}=0 \quad , \tag{B.9}$$

$$\left[\nabla_{\hat{a}}\left(J^{(\mu\nu)}_{(1)}u^a_{\mu}u^b_{\nu}v^{\hat{a}}_b\eta_a\right)-\eta_b\left(J^{\mu}_{(0)}u^b_{\mu}+2J^{(\mu\nu)}_{(1)}u^a_{\nu}\nabla_a u^b_{\mu}-\nabla_a\left(J^{(\mu\nu)}_{(1)}u^a_{\mu}u^b_{\nu}\right)\right)\right]|_{\partial\mathcal{W}_{p+1}}=0 \quad . \tag{B.10}$$

We now wish to solve the equations of motion (B.5)-(B.6). To that end, we make the most general decomposition of $J^{\mu}_{(0)}$ and $J^{\mu\nu}_{(1)}$ in terms of tangential and orthogonal components given by equation (1.10). By introducing the decomposition of $J^{\mu\nu}_{(1)}$ into the first equation in (B.5) and obtain the constraint $m^{\mu\nu}=m^{[\mu\nu]}$. Introducing both decompositions (1.10) into the second equation in (B.5) leads to the relation (1.11). Furthermore, introducing the decompositions into the equation and (B.6) one finds the equation for world-volume current conservation given by equation (1.12). Turning to the boundary conditions (B.9)-(B.10) using (1.10) we obtain the equations given by (1.13) and (1.14).

Extra symmetries and invariance of the equations of motion

The current expansion (1.4) enjoys two symmetries as the stress-energy tensor (1.2) coined by the authors of [46] as 'extra symmetry 1' and 'extra symmetry 2'. Their transformation properties can be obtained by looking at the invariance of the functional

$$J[f]=\int d^Dx\sqrt{-g}J^{\mu}f_{\mu} \quad , \tag{B.11}$$

for an arbitrary tensor field $f_{\mu}(x^{\alpha})$ of compact support. The 'extra symmetry 1' is an exact symmetry to all orders in the expansion defined by the transformation

$$\delta_1 J^{\mu}_{(0)}=-\nabla_a\tilde{\varepsilon}^{\mu a} \quad , \quad \delta_1 J^{\mu\nu}_{(1)}=-\tilde{\varepsilon}^{\mu a}u^{\nu}_a \quad , \tag{B.12}$$

and leaves the functional (B.11) invariant as long as the parameters $\tilde{\varepsilon}^{\mu a}$ are required to obey $\tilde{\varepsilon}^{\mu a}\eta_a|_{\partial\mathcal{W}_{p+1}} = 0$. This means, for example, that it is possible to gauge away one of the structures in the decomposition (1.10) everywhere except on the boundary since $\delta_1(J^{\mu\nu}_{(1)}u^a_\nu) = -\tilde{\varepsilon}^{\mu a}$. This is why we have left the last term in (1.10) neither parallel nor orthogonal to the world-volume. Further, invariance of the equation of motion (B.6) under (B.12) requires that $J^{ab}_{(1)} = J^{(ab)}_{(1)}$.[1] Explicit use of (B.12) leads to the variations of the structures that characterize the charge current

$$\delta_1\hat{J}^a = 0 \quad , \quad \delta_1 p^{a\mu} = 0 \quad , \quad \delta_1 J^{\hat{a}}_{(1)} = 0 \quad , \tag{B.13}$$

and hence leave the equations of motion (1.11)-(1.14) invariant.

On the other hand, the 'extra symmetry 2' is a perturbative symmetry and defined as the transformation that leaves (B.11) invariant under the displacement of representative surface $X^\alpha(\sigma^a) \to X^\alpha(\sigma^a) + \tilde{\varepsilon}^\alpha(\sigma^a)$. This leads to the transformation rule

$$\delta_2 J^\mu_{(0)} = -J^\mu_{(0)}u^a_\rho\nabla_a\tilde{\varepsilon}^\rho - \Gamma^\mu{}_{\lambda\rho}J^\lambda_{(0)}\tilde{\varepsilon}^\rho \quad , \quad \delta_2 J^{\mu\rho}_{(1)} = -J^\mu_{(0)}\tilde{\varepsilon}^\rho \quad . \tag{B.14}$$

Explicit calculation using (B.14) leads to

$$\begin{aligned} \delta_2\hat{J}^a &= -J^a_{(0)}u^b_\rho\nabla_b\tilde{\varepsilon}^\rho - u^a_\rho J^b_{(0)}\nabla_b\tilde{\varepsilon}^\rho + \nabla_b\left(J^a_{(0)}\tilde{\varepsilon}^b\right) \quad , \\ \delta_2 p^{a\mu} &= -J^a_{(0)}\tilde{\varepsilon}^\mu \quad , \quad \delta_2 J^{\hat{a}}_{(1)} = -J^b_{(0)}v^{\hat{a}}_b\tilde{\varepsilon}^a\eta_a \quad , \end{aligned} \tag{B.15}$$

and renders the equations of motion (1.11)-(1.14) invariant. As we will be obtaining these tensor structures from specific black hole metrics in section 3.3, the existence of this symmetry implies the existence of a residual gauge freedom in this measurement procedure as seen before in the case of the stress-energy tensor [39].

B.2 Branes carrying string charge

Pole-dipole p-branes carrying string charge are characterized by an anti-symmetric current $J^{\mu\nu}$ of the form (1.4). In order to solve equation (1.5) we introduce an arbitrary tensor field $f_\mu(x^\alpha)$ of compact support and integrate (1.5) such that

$$\int d^Dx\sqrt{-g}f_\nu(x^\alpha)\nabla_\mu J^{\mu\nu} = 0 \quad . \tag{B.16}$$

We now decompose $f_\mu(x^\alpha)$ in parallel and orthogonal components to the world-volume such that [46]

$$\begin{aligned} \nabla_\lambda f_\mu &= f^\perp_{\mu\lambda} + u^a_\lambda\nabla_a f_\mu \quad , \\ \nabla_{(\rho}\nabla_{\lambda)}f_\mu = f^\perp_{\mu\lambda\rho} + 2f^\perp_{\mu(\lambda a}u^a_{\rho)} &+ f_{\mu ab}u^a_\lambda u^b_\rho \quad , \quad \nabla_{[\rho}\nabla_{\lambda]}f_\mu = \frac{1}{2}R^\sigma{}_{\mu\lambda\rho}f_\sigma \quad . \end{aligned} \tag{B.17}$$

[1] In the case $q > 0$ and also in the case of the stress-energy tensor [46], the equations of motion obtained by this procedure are invariant under both 'extra symmetries' without further requirements on the structures appearing in analogous decompositions to (1.10). The $q = 0$ case is a special case as it is characterized by a current J^μ with only one index. This requires an extra constraint such that $J^{ab}_{(1)} = J^{(ab)}_{(1)}$ for the equations of motion (B.6) to be invariant under both extra symmetry transformations.

Explicit calculation using (B.17) and the projectors u^a_λ and $\perp^\mu{}_\lambda$ leads to

$$\begin{aligned} f_{\mu ab} =& \nabla_{(a}\nabla_{b)}f_\mu - f^\perp_{\mu\nu}\nabla_a u^\nu_b \ , \\ f_{\mu\rho a} =& \perp^\nu{}_\rho \nabla_a f^\perp_{\mu\nu} + (\nabla_a u^b_\rho)\nabla_b f_\mu + \frac{1}{2}\perp^\lambda{}_\rho u^\nu_a R^\sigma{}_{\mu\nu\lambda} f_\sigma \ . \end{aligned} \tag{B.18}$$

The equations (B.17)-(B.18) indicate that on the world-volume only the components $f^\perp_{\mu\nu\rho}$, $f^\perp_{\mu\nu}$ and $f^\perp_\mu$ are mutually independent while on the boundary, as in section B.1, we need to decompose $\nabla_a f_\mu$ such that [46]

$$\nabla_a f_\mu = \eta_a \nabla_\perp f_\mu + v^{\hat{a}}_a \nabla_{\hat{a}} f_\mu \ . \tag{B.19}$$

On the brane boundary the components $f^\perp_{\mu\nu}$, $\nabla_\perp f_\mu$ and f_μ are mutually independent. Given this, solving equation B.16 results in an equation with the following structure

$$\int_{\mathcal{W}_{p+1}} \sqrt{-\gamma}\left[Z^{\mu\nu\rho}f^\perp_{\mu\nu\rho} + Z^{\mu\nu}f^\perp_{\mu\nu} + Z^\mu f_\mu + \nabla_a\left(Z^{\mu\nu a}f^\perp_{\mu\nu} + Z^{\mu ab}\nabla_b f_\mu + Z^{\mu a}f_\mu\right)\right] = 0 \ . \tag{B.20}$$

This equation has the same structure as that obtained for the stress-energy tensor (1.2). Requiring the first three terms to vanish independently leads to the equations of motion

$$\perp^\sigma{}_\nu \perp^\lambda{}_\rho J^{\mu(\nu\rho)}_{(1)} = 0 \ , \ \perp^\sigma{}_\nu\left[J^{\mu\nu}_{(0)} - \nabla_a\left(\perp^\nu{}_\lambda J^{\mu\lambda\rho}_{(1)} u^a_\rho + J^{\mu\rho\nu}_{(1)} u^a_\rho\right)\right] = 0 \ , \tag{B.21}$$

$$\nabla_a\left(J^{\mu\nu}_{(0)}u^a_\nu - 2J^{\mu(\nu\rho)}_{(1)}u^{[a}_\rho \nabla_b u^{b]}_\nu - \nabla_b\left(J^{\mu(\nu\rho)}_{(1)}\right)u^b_\nu u^a_\rho\right) - \left(\perp^\lambda{}_\nu J^{\sigma(\nu\rho)}_{(1)} R^\mu{}_{\sigma\rho\lambda} + \frac{1}{2}J^{\sigma\nu\rho}_{(1)} R^\mu{}_{\sigma\nu\rho}\right) = 0 \ . \tag{B.22}$$

These equations have exactly the same form as those obtained for the stress-energy tensor [46], the difference between the two is that now we are dealing with anti-symmetric tensors. Requiring the vanishing of the last three terms in equation (B.20) in terms of the mutually independent components leads to the boundary conditions

$$\perp^\nu{}_\lambda J^{\mu(\lambda\rho)}_{(1)} u^a_\rho \eta_a |_{\partial\mathcal{W}_{p+1}} = 0 \ , \ J^{\mu\lambda\rho}_{(1)} u^a_\lambda u^b_\rho \eta_a \eta_b |_{\partial\mathcal{W}_{p+1}} = 0 \ , \tag{B.23}$$

$$\left[\nabla_{\hat{a}}\left(J^{\mu(\lambda\rho)}_{(1)} u^a_\lambda u^b_\rho v^{\hat{a}}_a \eta_b\right) - \eta_a\left(J^{\mu\nu}_{(0)}u^a_\nu - 2J^{\mu(\nu\rho)}_{(1)} u^{[b}_\rho \nabla_b u^{a]}_\nu - \nabla_b\left(J^{\mu(\nu\rho)}_{(1)}\right)u^b_\nu u^a_\rho\right)\right]|_{\partial\mathcal{W}_{p+1}} = 0 \ . \tag{B.24}$$

We now want to recast the equation (B.21)-(B.24) into a more convenient form. First we note that the first constraint in equation (B.21) results in the decomposition of $J^{\mu\nu\rho}_{(1)}$ into the components

$$J^{\mu\nu\rho}_{(1)} = 2u^{[\mu}_a m^{a\nu]\rho} + u^\mu_a u^\nu_b p^{ab\rho} + J^{\mu\nu a}_{(1)} u^\rho_a \ , \tag{B.25}$$

where $m^{a\mu\nu}$ and $p^{ab\rho}$ are orthogonal in its space-time indices while $J^{\mu\nu a}_{(1)}$ is left neither parallel nor orthogonal to the world-volume due to the extra symmetry transformations that we will describe below. Now, as in the case of the stress-energy tensor [46] we introduce the analogous tensor structures

$$\mathcal{Q}^{\mu\nu a} = p^{ab[\mu}u^{\nu]}_b + J^{\mu\nu a}_{(1)} \ , \ \mathcal{M}^{\mu\nu a} = m^{a\mu\nu} - p^{ab[\mu}u^{\nu]}_b \ , \tag{B.26}$$

where $\mathcal{Q}^{\mu\nu a}$ and $\mathcal{M}^{\mu\nu a}$ are both anti-symmetric in their space-time indices and furthermore $\mathcal{M}^{\mu\nu a}$ has the property $\mathcal{M}^{\mu\nu(a}u_{\nu}^{b)}=0$. This means that the dipole correction to the current $J_{(1)}^{\mu\nu\rho}$ can be written as

$$J_{(1)}^{\mu\nu\rho}=2u_{a}^{[\mu}\mathcal{M}^{\nu]\rho a}+\mathcal{Q}^{\mu\nu a}u_{a}^{\rho} \quad . \tag{B.27}$$

Using these definitions in the second constraint given in (B.21) results in

$$\perp^{\sigma}{}_{\nu}\left[J_{(0)}^{\mu\nu}-\nabla_{a}\left(\mathcal{Q}^{\mu\nu a}-\mathcal{M}^{\mu\nu a}\right)\right]=0 \quad . \tag{B.28}$$

Now, making the most general decomposition of $J_{(0)}^{\mu\nu}$ results in

$$J_{(0)}^{\mu\nu}=u_{a}^{\mu}u_{b}^{\nu}J_{(0)}^{ab}+2u_{b}^{[\mu}J_{\perp(1)}^{\nu]b}+J_{\perp(1)}^{\mu\nu} \quad , \tag{B.29}$$

and taking all the possible projections of equation (B.28) leads to the relations

$$J_{\perp(1)}^{\rho b}=u_{\mu}^{b}\perp^{\rho}{}_{\nu}\nabla_{a}(\mathcal{Q}^{\mu\nu a}-\mathcal{M}^{\mu\nu a}) \quad , \quad J_{\perp(1)}^{\sigma\rho}=\perp^{\sigma}{}_{\mu}\perp^{\rho}{}_{\nu}\nabla_{a}\left(\mathcal{Q}^{\mu\nu a}-\mathcal{M}^{\mu\nu a}\right) \quad . \tag{B.30}$$

Note that, contrary to the equations of motion for the stress-energy tensor [46], we only have two constraints. The third one, which in the case of (1.2) lead to the conservation of the spin current $j^{a\mu\nu}$, is non-existent here because both tensors introduced in (B.26) are anti-symmetric. Finally, inserting the first relation in equation (B.30) into equation (B.22) we obtain the equation for current conservation

$$\nabla_{a}\left(\tilde{J}^{ab}u_{b}^{\mu}+u_{b}^{\mu}u_{\nu}^{a}u_{\rho}^{b}\nabla_{c}\mathcal{M}^{\nu\rho c}\right)=0 \quad , \tag{B.31}$$

where we have defined the effective world-volume current $\tilde{J}^{ab}=\tilde{J}^{[ab]}$ such that

$$\tilde{J}^{ab}=J_{(0)}^{ab}-u_{\mu}^{a}u_{\nu}^{b}\nabla_{c}\mathcal{Q}^{\mu\nu c} \,. \tag{B.32}$$

It is worth noticing that all the terms proportional to the Riemann tensor in equation (B.22) have dropped out of the equations of motion as a consequence of the anti-symmetry of (1.4). Moreover, at first sight, equation (B.30) seems to contain two sets of independent equations obtained by projecting tangentially and orthogonally to the world-volume. This is only apparent as the orthogonal projection of equation (B.30),

$$\left(\tilde{J}^{ab}+u_{\nu}^{a}u_{\rho}^{b}\nabla_{c}\mathcal{M}^{\nu\rho c}\right)K_{ab}{}^{\mu}=0 \quad , \tag{B.33}$$

trivially vanishes due to the anti-symmetry of $\tilde{J}^{ab}$ and $\mathcal{M}^{\nu\rho c}$ and the symmetry of $K_{ab}{}^{\mu}$ in its two world-volume indices. Now, we turn our attention to the boundary conditions (B.23)-(B.24). Introducing (B.26) leads to

$$\perp^{\nu}{}_{\rho}\,\mathcal{Q}^{\mu\rho a}\eta_{a}|_{\partial\mathcal{W}_{p+1}}=0 \quad , \quad \left(\mathcal{M}^{\mu\rho a}-\mathcal{Q}^{\mu\rho a}\right)u_{\rho}^{b}\eta_{a}\eta_{b}|_{\partial\mathcal{W}_{p+1}}=0 \quad , \tag{B.34}$$

$$\left[\nabla_{\hat{a}}\left(J_{(1)}^{\hat{a}\hat{b}}v_{\hat{b}}^{\mu}\right)-\eta_{a}\left(\tilde{J}^{ab}u_{b}^{\mu}+u_{b}^{\mu}u_{\nu}^{a}u_{\rho}^{b}\nabla_{c}\mathcal{M}^{\nu\rho c}\right)\right]_{\partial\mathcal{W}_{p+1}}=0 \quad . \tag{B.35}$$

Again, the orthogonal projection of equation (B.35) vanishes and we are left with

$$\left[v_{\hat{b}}^{b}\nabla_{\hat{a}}J_{(1)}^{\hat{a}\hat{b}}-\eta_{a}\left(\tilde{J}^{ab}+u_{\nu}^{a}u_{\rho}^{b}\nabla_{c}\mathcal{M}^{\nu\rho c}\right)\right]_{\partial\mathcal{W}_{p+1}}=0 \quad . \tag{B.36}$$

Reintroducing $m^{a\mu\nu}$ and $p^{ab\mu}$ the relations given by equation (B.30) gives

$$J^{\mu b}_{\perp(1)} = u^b_\rho {\perp^\mu}_\lambda \nabla_c (J^{\rho\lambda c}_{(1)} - m^{c\rho\lambda}) \ \ , \ \ J^{\mu\nu}_{\perp(1)} = {\perp^\mu}_\lambda {\perp^\nu}_\rho \nabla_c (J^{\rho\lambda c}_{(1)} - m^{c\rho\lambda}) \ \ , \tag{B.37}$$

while taking the parallel projection of equation (B.31) and using (B.26) leads to

$$\nabla_a \left(\hat{J}^{ab} - 2p^{c[a|\mu|} {K_{c\mu}}^{b]} \right) = 0 \ \ , \tag{B.38}$$

where the vertical bars on the index μ means that it is insensitive to the anti-symmetrization taking place only on the world-volume indices. In equation (B.38) we have introduced the effective world-volume current $\hat{J}^{ab} = \hat{J}^{[ab]}$ such that

$$\hat{J}^{ab} = J^{ab}_{(0)} - u^a_\mu u^b_\nu \nabla_c J^{\mu\nu c}_{(1)} \ \ . \tag{B.39}$$

Note again that, as in the case of branes carrying Maxwell charge, the components $m^{a\mu\nu}$ entering the decomposition (B.25) do not play a role in the equation for current conservation. Similar for the boundary conditions reintroducing $m^{a\mu\nu}$ and $p^{ab\mu}$ using (B.26) leads to

$$\left(p^{ba\mu} + 2{\perp^\mu}_\lambda J^{\lambda ab}_{(1)} \right) \eta_b|_{\partial\mathcal{W}_{p+1}} = 0 \ \ , \ \ J^{\mu ab}_{(1)} \eta_a \eta_b|_{\partial\mathcal{W}_{p+1}} = 0 \ \ , \tag{B.40}$$

$$\left[v^b_{\hat{b}} \nabla_{\hat{a}} J^{\hat{a}\hat{b}}_{(1)} - \eta_a \left(\hat{J}^{ab} - 2p^{c[a|\mu|} {K_{c\mu}}^{b]} \right) \right] |_{\partial\mathcal{W}_{p+1}} = 0 \ \ , \tag{B.41}$$

where we have defined the boundary degrees of freedom $J^{\hat{a}\hat{b}}_{(1)} = J^{\mu\nu c}_{(1)} u^\rho_c \eta_{(\rho} v^{\hat{a}}_{\nu)} v^{\hat{b}}_\mu$, with $v^{\hat{a}}_\mu = u^a_\mu v^{\hat{a}}_a$. Again, the components $m^{a\mu\nu}$ have dropped out of the boundary conditions. We see that branes carrying string charge are characterized by a world-volume effective current $\hat{J}^{ab}$, an electric dipole moment $p^{ab\mu}$ and a boundary current $J^{\hat{a}\hat{b}}_{(1)}$.

Extra symmetries and invariance of the equations of motion

The extra symmetries associated with the current (1.4) for branes carrying string charge are now deduced by looking at the functional

$$J[f] = \int d^D x \sqrt{-g} J^{\mu\nu} f_{\mu\nu} \ \ , \tag{B.42}$$

for an arbitrary tensor field $f_{\mu\nu}$ of compact support. The 'extra symmetry 1' acts on the current (1.4) such that

$$\delta_1 J^{\mu\nu}_{(0)} = -\nabla_a \tilde{\varepsilon}^{\mu\nu a} \ \ , \ \ \delta_1 J^{\mu\nu\rho}_{(1)} = -\tilde{\varepsilon}^{\mu\nu a} u^\rho_a \ \ , \tag{B.43}$$

where the parameters $\tilde{\varepsilon}^{\mu\nu a}$ satisfy the properties $\tilde{\varepsilon}^{\mu\nu a} = \tilde{\varepsilon}^{[\mu\nu]a}$ and $\tilde{\varepsilon}^{\mu\nu a}\eta_a|_{\partial\mathcal{W}_{p+1}} = 0$. This is turn implies that the components $J^{\mu\nu a}_{(1)}$ entering the decomposition (B.25) are pure gauge everywhere except on the boundary since $\delta_1 (J^{\mu\nu\rho}_{(1)} u^a_\rho) = J^{\mu\nu a}_{(1)} = -\tilde{\varepsilon}^{\mu\nu a}$. Evaluating (B.43) for the components that describe the charged pole dipole brane leads to the variations

$$\delta_1 \hat{J}^{ab}_{(0)} = 0 \ \ , \ \ \delta_1 p^{ab\rho} = 0 \ \ , \ \ \delta_1 J^{\hat{a}\hat{b}}_{(1)} = 0 \ \ , \tag{B.44}$$

and hence the equations of motion (B.37)-(B.38) together with the boundary conditions (B.40)-(B.41) are invariant under this transformation rule. Turning our attention to the 'extra symmetry 2', invariance of (B.42) requires

$$\delta_2 J^{\mu\nu}_{(0)} = -J^{\mu\nu}_{(0)} u^a_\rho \nabla_a \tilde{\varepsilon}^\rho - 2\Gamma^{[\mu}{}_{\rho\lambda} J^{\nu]\lambda}_{(0)} \tilde{\varepsilon}^\rho \quad , \quad \delta_2 J^{\mu\nu\rho}_{(1)} = -J^{\mu\nu}_{(0)} \tilde{\varepsilon}^\rho \quad , \tag{B.45}$$

which upon explicit calculation leads to

$$\begin{aligned} \delta_2 \hat{J}^{ab} &= -J^{ab}_{(0)} u^c_\rho \nabla_c \tilde{\varepsilon}^\rho - 2u^{[a}_\rho J^{b]c}_{(0)} \nabla_c \tilde{\varepsilon}^\rho + \nabla_c \left(J^{ab}_{(0)} \tilde{\varepsilon}^c \right) \quad , \\ \delta_2 p^{ab\mu} &= -J^{ab}_{(0)} \tilde{\varepsilon}^\mu \quad , \quad \delta_2 J^{\hat{a}\hat{b}}_{(1)} = -J^{ab}_{(0)} \tilde{\varepsilon}^c u^\rho_c \eta_{(\rho} v^{\hat{a}}_{\nu)} u^\nu_a v^{\hat{b}}_b \quad , \end{aligned} \tag{B.46}$$

and leaves the equations of motion (B.37)-(B.38) and boundary conditions (B.40)-(B.41) invariant. As a final comment, we note that the equations in terms of $\mathcal{Q}^{\mu\nu a}, \mathcal{M}^{\mu\nu a}$ and $\tilde{J}^{ab}$ are also invariant under both extra symmetries and their transformation follows from equation (B.43) and equation (B.45) yielding

$$\delta_1 \tilde{J}^{ab} = 0 \quad , \quad \delta_1 \mathcal{M}^{\mu\nu a} = 0 \quad , \tag{B.47}$$

$$\delta_2 \tilde{J}^{ab} = -J^{ab}_{(0)} u^c_\rho \nabla_c \tilde{\varepsilon}^\rho + J^{c[a}_{(0)} u^{b]}_\rho \nabla_c \tilde{\varepsilon}^\rho + \nabla_c \left(J^{ab}_{(0)} \tilde{\varepsilon}^c \right) \quad , \quad \delta_2 \mathcal{M}^{\mu\nu a} = J^{ab}_{(0)} \tilde{\varepsilon}^{[\mu}_\perp u^{\nu]}_b \quad . \tag{B.48}$$

B.3 Branes charged under higher-form fields

In this section, we conjecture the equations of motion for the cases $q > 1$. In these cases, the electric current of the p-brane are characterized by two structures $J^{\mu_1 \dots \mu_{q+1}}_{(0)}$ and $J^{\mu_1 \dots \mu_{q+1}\rho}_{(1)}$. It is straightforward to derive a similar constraint as in equation (B.5) by solving the conservation equation (1.5). This constraint allows us to make the general decomposition of $J^{\mu_1 \dots \mu_{q+1}\rho}_{(1)}$ such that

$$J^{\mu_1 \dots \mu_{q+1}\rho}_{(1)} = (q+1) u^{[\mu_1}_{a_1} \dots u^{\mu_q}_{a_q} m^{|a_1 \dots a_q|\mu_{q+1}]\rho} + u^{\mu_1}_{a_1} \dots u^{\mu_{q+1}}_{a_{q+1}} p^{a_1 \dots a_{q+1}\rho} + J^{\mu_1 \dots \mu_{q+1} a}_{(1)} u^\rho_a \quad , \tag{B.49}$$

where the vertical bars around the indices $a_1 \dots a_q$ indicates that the indices are insensitive to the anti-symmetrization which is done only over the space-time indices $\mu_1 \dots \mu_{q+1}$. Moreover, $m^{a_1 \dots a_q \mu_{q+1}\rho}$ satisfies the properties $m^{a_1 \dots a_q \mu_{q+1}\rho} = m^{[a_1 \dots a_q]\mu_{q+1}\rho} = m^{a_1 \dots a_q[\mu_{q+1}\rho]}$ while $p^{a_1 \dots a_{q+1}\rho}$ has the property $p^{a_1 \dots a_{q+1}\rho} = p^{[a_1 \dots a_{q+1}]\rho}$ and also $J^{\mu_1 \dots \mu_{q+1} a}_{(1)} = J^{[\mu_1 \dots \mu_{q+1}]a}_{(1)}$. A similar decomposition of $J^{\mu_1 \dots \mu_{q+1}}_{(0)}$ as in (B.29) is assumed and the final form of the equations of motion is conjectured to be

$$\nabla_{a_1} \left(\hat{J}^{a_1 \dots a_{q+1}} + (-1)^q (q+1)\, p^{c[a_1 \dots a_q|\mu|} K_{c\mu}{}^{a_{q+1}]} \right) = 0 \quad , \tag{B.50}$$

while the boundary conditions have an analogous form to the $q = 1$ case presented in the previous section

$$\left(p^{a_1 \dots a_{q+1}\mu} + (-1)^q (q+1)! \perp^\mu{}_\lambda J^{\lambda a_1 \dots a_{q+1}}_{(1)} \right) \eta_{a_{q+1}} |_{\partial \mathcal{W}_{p+1}} = 0 \quad , \tag{B.51}$$

$$J^{\mu_1 \dots \mu_q a_{q+1} b}_{(1)} \eta_{a_{q+1}} \eta_b |_{\partial \mathcal{W}_{p+1}} = 0 \quad , \tag{B.52}$$

$$\left[v^{a_2}_{\hat{a}_2}...v^{a_{q+1}}_{\hat{a}_{q+1}}\nabla_{\hat{a}_1}J^{\hat{a}_1...\hat{a}_{q+1}}_{(1)} - \eta_{a_1}\left(\hat{J}^{a_1...a_{q+1}} + (-1)^q(q+1)\,p^{c[a_1...a_q|\mu|}K_{c\mu}{}^{a_{q+1}]}\right)\right]|_{\partial\mathcal{W}_{p+1}} = 0 \quad . \tag{B.53}$$

Here we have introduced the effective world-volume current $\hat{J}^{a_1...a_{q+1}}$, as well as the boundary degrees of freedom $J^{\hat{a}_1...\hat{a}_{q+1}}_{(1)}$ given by

$$\hat{J}^{a_1...a_{q+1}} = J^{a_1...a_{q+1}}_{(0)} - u^{a_1}_{\mu_1}...u^{a_{q+1}}_{\mu_{q+1}}\nabla_c J^{\mu_1...\mu_{q+1}c}_{(1)} \quad , \tag{B.54}$$

$$J^{\hat{a}_1...\hat{a}_{q+1}}_{(1)} = J^{a_1...a_{q-1}\mu_q\mu_{q+1}c}_{(1)}u^\rho_c\eta_{(\rho}v^{\hat{a}_q}_{\mu_{q+1})}v^{\hat{a}_{q+1}}_{\mu_q}v^{\hat{a}_1}_{a_1}...v^{\hat{a}_{q-1}}_{a_{q-1}} \quad . \tag{B.55}$$

The conjectured form of the equations of motion (B.50)-(B.53) for any value of q is supported by their invariance under the extra symmetry transformations which we will now analyze.

The ‘extra symmetry 1’ acting on the generic form of the current (1.4) has the following transformation rule

$$\delta_1 J^{\mu_1...\mu_{q+1}}_{(0)} = -\nabla_a\tilde{\varepsilon}^{\mu_1...\mu_{q+1}a} \quad , \quad \delta_1 J^{\mu_1...\mu_{q+1}\rho}_{(1)} = -\tilde{\varepsilon}^{\mu_1...\mu_{q+1}a}u^\rho_a \quad , \tag{B.56}$$

where $\tilde{\varepsilon}^{\mu_1...\mu_{q+1}a}$ has the property $\tilde{\varepsilon}^{\mu_1...\mu_{q+1}a} = \tilde{\varepsilon}^{[\mu_1...\mu_{q+1}]a}$ and is constrained on the boundary such that $\tilde{\varepsilon}^{\mu_1...\mu_{q+1}a}\eta_a|_{\partial\mathcal{W}_{p+1}} = 0$. Correspondingly, this implies that the structures characterizing the charged pole-dipole brane transform as $\delta_1\hat{J}^{a_1...a_{q+1}} = \delta_1 p^{a_1...a_{q+1}\mu} = \delta_1 J^{\hat{a}_1...\hat{a}_{q+1}} = 0$, leaving the equations of motion invariant. As in the cases $q=0$ and $q=1$ analyzed previously, this symmetry implies that the last structure introduced in (B.49) can be gauged away everywhere on the world-volume. Furthermore, under the action of the ‘extra symmetry 2’ the structures entering (1.4) transform according to

$$\begin{aligned} \delta_2 J^{\mu_1...\mu_{q+1}}_{(0)} &= -J^{\mu_1...\mu_{q+1}}_{(0)}u^c_\rho\nabla_c\tilde{\varepsilon}^\rho - (q+1)\Gamma^{[\mu_1}{}_{\rho\lambda}J^{\mu_2...\mu_{q+1}]\lambda}_{(0)}\,\tilde{\varepsilon}^\rho \quad , \\ \delta_2 J^{\mu_1...\mu_{q+1}\rho}_{(1)} &= -J^{\mu_1...\mu_{q+1}}_{(0)}\tilde{\varepsilon}^\rho \quad , \end{aligned} \tag{B.57}$$

which upon explicit calculation lead to the ‘extra symmetry 2’ transformations

$$\begin{aligned} \delta_2\hat{J}^{a_1...a_{q+1}} &= -J^{a_1...a_{q+1}}_{(0)}u^c_\rho\nabla_c\tilde{\varepsilon}^\rho - (q+1)u^{[a_1}_\rho J^{a_2...a_{q+1}]c}_{(0)}\nabla_c\tilde{\varepsilon}^\rho + \nabla_c\left(J^{a_1...a_{q+1}}_{(0)}\tilde{\varepsilon}^c\right) \quad , \\ \delta_2 p^{a_1...a_{q+1}\mu} &= -J^{a_1...a_{q+1}}_{(0)}\tilde{\varepsilon}^\mu \quad , \quad \delta_2 J^{\hat{a}_1...\hat{a}_{q+1}}_{(1)} = -J^{a_1...a_{q+1}}_{(0)}\tilde{\varepsilon}^c u^\rho_c\eta_{(\rho}v^{\hat{a}_1}_{\nu)}u^\nu_{a_1}v^{\hat{a}_2}_{a_2}...v^{\hat{a}_{q+1}}_{a_{q+1}} \quad . \end{aligned} \tag{B.58}$$

The above transformation rules leave the equations of motion (B.50)-(B.53) invariant. In section 3.3 we will give examples of these structures measured for bent black branes.

References

[1] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," *Living Rev.Rel.* **11** (2008) 6, arXiv:0801.3471 [hep-th].

[2] R. Emparan and H. S. Reall, "Black Rings," *Class.Quant.Grav.* **23** (2006) R169, arXiv:hep-th/0608012 [hep-th].

[3] N. A. Obers, "Black Holes in Higher-Dimensional Gravity," *Lect.Notes Phys.* **769** (2009) 211–258, arXiv:0802.0519 [hep-th].

[4] V. Niarchos, "Phases of Higher Dimensional Black Holes," *Mod.Phys.Lett.* **A23** (2008) 2625–2643, arXiv:0808.2776 [hep-th].

[5] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Blackfold approach for higher-dimensional black holes," *Acta Phys.Polon.* **B40** (2009) 3459–3506.

[6] M. J. Rodriguez, "On the Black Hole Species (By Means of Natural Selection)," arXiv:1003.2411 [hep-th].

[7] R. Gregory and R. Laflamme, "The Instability of charged black strings and p-branes," *Nucl.Phys.* **B428** (1994) 399–434, arXiv:hep-th/9404071 [hep-th].

[8] R. Emparan and H. S. Reall, "A Rotating black ring solution in five-dimensions," *Phys.Rev.Lett.* **88** (2002) 101101, arXiv:hep-th/0110260 [hep-th].

[9] S. Hollands and A. Ishibashi, "Black hole uniqueness theorems in higher dimensional spacetimes," *Class.Quant.Grav.* **29** (2012) 163001, arXiv:1206.1164 [gr-qc].

[10] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys.Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029 [hep-th].

[11] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[12] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[13] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[14] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys.Rept.* **323** (2000) 183–386, arXiv:hep-th/9905111 [hep-th].

[15] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class.Quant.Grav.* **26** (2009) 224002, arXiv:0903.3246 [hep-th].

[16] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," *J.Phys.* **A42** (2009) 343001, arXiv:0904.1975 [hep-th].

[17] J. McGreevy, "Holographic duality with a view toward many-body physics," *Adv.High Energy Phys.* **2010** (2010) 723105, arXiv:0909.0518 [hep-th].

[18] S. Sachdev, "What can gauge-gravity duality teach us about condensed matter physics?," *Ann.Rev.Condensed Matter Phys.* **3** (2012) 9–33, arXiv:1108.1197 [cond-mat.str-el].

[19] L. Huijse, S. Sachdev, and B. Swingle, "Hidden Fermi surfaces in compressible states of gauge-gravity duality," *Phys.Rev.* **B85** (2012) 035121, arXiv:1112.0573 [cond-mat.str-el].

[20] N. Ogawa, T. Takayanagi, and T. Ugajin, "Holographic Fermi Surfaces and Entanglement Entropy," *JHEP* **1201** (2012) 125, arXiv:1111.1023 [hep-th].

[21] S. A. Hartnoll and E. Shaghoulian, "Spectral weight in holographic scaling geometries," *JHEP* **1207** (2012) 078, arXiv:1203.4236 [hep-th].

[22] C. Charmousis, B. Gouteraux, B. Kim, E. Kiritsis, and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," *JHEP* **1011** (2010) 151, arXiv:1005.4690 [hep-th].

[23] J. Gath, J. Hartong, R. Monteiro, and N. A. Obers, "Holographic Models for Theories with Hyperscaling Violation," *JHEP* **1304** (2013) 159, arXiv:1212.3263 [hep-th].

[24] R. C. Myers and M. J. Perry, "Black Holes in Higher Dimensional Space-Times," *Ann. Phys.* **172** (1986) 304.

[25] R. Emparan and R. C. Myers, "Instability of ultra-spinning black holes," *JHEP* **0309** (2003) 025, arXiv:hep-th/0308056 [hep-th].

[26] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "World-Volume Effective Theory for Higher-Dimensional Black Holes," *Phys.Rev.Lett.* **102** (2009) 191301, arXiv:0902.0427 [hep-th].

[27] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Essentials of Blackfold Dynamics," *JHEP* **1003** (2010) 063, arXiv:0910.1601 [hep-th].

[28] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," *JHEP* **0802** (2008) 045, arXiv:0712.2456 [hep-th].

[29] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," *JHEP* **0812** (2008) 116, arXiv:0809.4272 [hep-th].

[30] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, "Fluid dynamics of R-charged black holes," *JHEP* **0901** (2009) 055, arXiv:0809.2488 [hep-th].

[31] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, *et al.*, "Hydrodynamics from charged black branes," *JHEP* **1101** (2011) 094, arXiv:0809.2596 [hep-th].

[32] J. Camps, R. Emparan, and N. Haddad, "Black Brane Viscosity and the Gregory-Laflamme Instability," *JHEP* **1005** (2010) 042, arXiv:1003.3636 [hep-th].

[33] J. Gath and A. V. Pedersen, "Viscous Asymptotically Flat Reissner-Nordström Black Branes," arXiv:1302.5480 [hep-th].

[34] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers, and M. J. Rodriguez, "The Phase Structure of Higher-Dimensional Black Rings and Black Holes," *JHEP* **0710** (2007) 110, arXiv:0708.2181 [hep-th].

[35] M. M. Caldarelli, R. Emparan, and M. J. Rodriguez, "Black Rings in (Anti)-deSitter space," *JHEP* **0811** (2008) 011, arXiv:0806.1954 [hep-th].

[36] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "New Horizons for Black Holes and Branes," *JHEP* **1004** (2010) 046, arXiv:0912.2352 [hep-th].

[37] J. Armas, J. Camps, T. Harmark, and N. A. Obers, "The Young Modulus of Black Strings and the Fine Structure of Blackfolds," *JHEP* **1202** (2012) 110, arXiv:1110.4835 [hep-th].

[38] J. Camps and R. Emparan, "Derivation of the blackfold effective theory," *JHEP* **1203** (2012) 038, arXiv:1201.3506 [hep-th].

[39] J. Armas, J. Gath, and N. A. Obers, "Black Branes as Piezoelectrics," *Phys.Rev.Lett.* **109** (2012) 241101, arXiv:1209.2127 [hep-th].

[40] J. Armas, J. Gath, and N. A. Obers, "Electroelasticity of Charged Black Branes," arXiv:1307.0504 [hep-th].

[41] J. Armas and N. A. Obers, "Relativistic Elasticity of Stationary Fluid Branes," *Phys.Rev.* **D87** (2013) 044058, arXiv:1210.5197 [hep-th].

[42] J. Armas, "How Fluids Bend: the Elastic Expansion for Higher-Dimensional Black Holes," arXiv:1304.7773 [hep-th].

[43] R. Emparan, V. E. Hubeny, and M. Rangamani, "Effective hydrodynamics of black D3-branes," arXiv:1303.3563 [hep-th].

[44] D. Brattan, J. Camps, R. Loganayagam, and M. Rangamani, "CFT dual of the AdS Dirichlet problem : Fluid/Gravity on cut-off surfaces," *JHEP* **1112** (2011) 090, arXiv:1106.2577 [hep-th].

[45] M. Vasilic and M. Vojinovic, "Classical string in curved backgrounds," *Phys.Rev.* **D73** (2006) 124013, arXiv:gr-qc/0610014 [gr-qc].

[46] M. Vasilic and M. Vojinovic, "Classical spinning branes in curved backgrounds," *JHEP* **0707** (2007) 028, arXiv:0707.3395 [gr-qc].

[47] M. M. Caldarelli, R. Emparan, and B. Van Pol, "Higher-dimensional Rotating Charged Black Holes," *JHEP* **1104** (2011) 013, arXiv:1012.4517 [hep-th].

[48] A. Di Dato, "Kaluza-Klein reduction of relativistic fluids and their gravity duals," arXiv:1307.8365 [hep-th].

[49] A. Di Dato, J. Gath, and A. V. Pedersen, "Work in Progress," *Forthcoming* (2013) .

[50] G. Policastro, D. Son, and A. Starinets, "The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma," *Phys.Rev.Lett.* **87** (2001) 081601, arXiv:hep-th/0104066 [hep-th].

[51] A. Buchel and J. T. Liu, "Universality of the shear viscosity in supergravity," *Phys.Rev.Lett.* **93** (2004) 090602, arXiv:hep-th/0311175 [hep-th].

[52] A. Buchel, "Bulk viscosity of gauge theory plasma at strong coupling," *Phys.Lett.* **B663** (2008) 286–289, arXiv:0708.3459 [hep-th].

[53] B. Gouteraux, J. Smolic, M. Smolic, K. Skenderis, and M. Taylor, "Holography for Einstein-Maxwell-dilaton theories from generalized dimensional reduction," *JHEP* **1201** (2012) 089, arXiv:1110.2320 [hep-th].

[54] M. Smolic, "Holography and hydrodynamics for EMD theory with two Maxwell fields," arXiv:1301.6020 [hep-th].

[55] T. Harmark, V. Niarchos, and N. A. Obers, "Instabilities of black strings and branes," *Class.Quant.Grav.* **24** (2007) R1–R90, arXiv:hep-th/0701022 [hep-th].

[56] D. T. Son and A. O. Starinets, "Hydrodynamics of r-charged black holes," *JHEP* **0603** (2006) 052, arXiv:hep-th/0601157 [hep-th].

[57] M. M. Caldarelli, J. Camps, B. Gouteraux, and K. Skenderis, "AdS/Ricci-flat correspondence and the Gregory-Laflamme instability," arXiv:1211.2815 [hep-th].

[58] L. Landau and E. M. Lifshitz, "Theory of elasticity," *Course of Theoretical Physics* **Vol. 7** (1959) 134.

[59] D. Bardzokas, M. Filshtinsky, and L. Filshtinsky, *Mathematical Methods in Electro-Magneto-Elasticity.* Lecture Notes in Applied and Computational Mechanics. Springer, 2007.

[60] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Blackfolds in Supergravity and String Theory," *JHEP* **1108** (2011) 154, arXiv:1106.4428 [hep-th].

[61] J. Armas and N. A. Obers, "Blackfolds in (Anti)-de Sitter Backgrounds," *Phys.Rev.* **D83** (2011) 084039, arXiv:1012.5081 [hep-th].

[62] J. J. Blanco-Pillado, R. Emparan, and A. Iglesias, "Fundamental Plasmid Strings and Black Rings," *JHEP* **0801** (2008) 014, arXiv:0712.0611 [hep-th].

[63] J. Camps, R. Emparan, P. Figueras, S. Giusto, and A. Saxena, "Black Rings in Taub-NUT and D0-D6 interactions," *JHEP* **0902** (2009) 021, arXiv:0811.2088 [hep-th].

[64] J. Le Witt and S. F. Ross, "Black holes and black strings in plane waves," *JHEP* **1001** (2010) 101, arXiv:0910.4332 [hep-th].

[65] B. Carter, "Essentials of classical brane dynamics," *Int.J.Theor.Phys.* **40** (2001) 2099–2130, arXiv:gr-qc/0012036 [gr-qc].

[66] A. Polyakov, "Fine structure of strings," *Nuclear Physics B* **268** (1986) no. 2, 406 – 412.

[67] H. Kleinert, "The Membrane Properties of Condensing Strings," *Phys.Lett.* **B174** (1986) 335–338.

[68] W. Dixon, "Dynamics of extended bodies in general relativity: Equations of Motion," *Trans. R. Soc.* **A277** (1974) 59.

[69] J. Souriou, "Modele de particule spin dans le champ electromagnetique et gravitationnel," *Ann. Inst. H. Poincare* **20** (1974) 315.

[70] L. F. O. Costa, J. Natario, and M. Zilhao, "Spacetime dynamics of spinning particles - exact gravito-electromagnetic analogies," arXiv:1207.0470 [gr-qc].

[71] A. Papapetrou, "Spinning test particles in general relativity. 1.," *Proc.Roy.Soc.Lond.* **A209** (1951) 248–258.

[72] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Heating up the BIon," *JHEP* **1106** (2011) 058, arXiv:1012.1494 [hep-th].

[73] J. Armas, T. Harmark, N. A. Obers, M. Orselli, and A. V. Pedersen, "Thermal Giant Gravitons," *JHEP* **1211** (2012) 123, arXiv:1207.2789 [hep-th].

[74] P. Romatschke, "Relativistic Viscous Fluid Dynamics and Non-Equilibrium Entropy," *Class.Quant.Grav.* **27** (2010) 025006, arXiv:0906.4787 [hep-th].

[75] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*. Course of Theoretical Physics, Vol. 6. Pergamon Press, London, 2nd ed., 1987. Translated from the Russian by J. B. Sykes and W. H. Reid.

[76] M. M. Caldarelli, O. J. Dias, R. Emparan, and D. Klemm, "Black Holes as Lumps of Fluid," *JHEP* **0904** (2009) 024, arXiv:0811.2381 [hep-th].

[77] G. Gibbons and K.-i. Maeda, "Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields," *Nucl.Phys.* **B298** (1988) 741.

[78] R. C. Myers, "Stress tensors and Casimir energies in the AdS / CFT correspondence," *Phys.Rev.* **D60** (1999) 046002, arXiv:hep-th/9903203 [hep-th].

[79] J. D. Brown and J. York, James W., "Quasilocal energy and conserved charges derived from the gravitational action," *Phys.Rev.* **D47** (1993) 1407–1419, arXiv:gr-qc/9209012 [gr-qc].

[80] T. Harmark, V. Niarchos, and N. A. Obers, "Instabilities of near-extremal smeared branes and the correlated stability conjecture," *JHEP* **0510** (2005) 045, arXiv:hep-th/0509011 [hep-th].

[81] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Thermodynamics of the hot BIon," *Nucl.Phys.* **B851** (2011) 462–480, arXiv:1101.1297 [hep-th].

[82] G. Grignani, T. Harmark, A. Marini, N. A. Obers, and M. Orselli, "Thermal string probes in AdS and finite temperature Wilson loops," *JHEP* **1206** (2012) 144, arXiv:1201.4862 [hep-th].

[83] V. Niarchos and K. Siampos, "M2-M5 blackfold funnels," *JHEP* **1206** (2012) 175, arXiv:1205.1535 [hep-th].

[84] V. Niarchos and K. Siampos, "Entropy of the self-dual string soliton," *JHEP* **1207** (2012) 134, arXiv:1206.2935 [hep-th].

[85] V. Niarchos and K. Siampos, "The black M2-M5 ring intersection spins," arXiv:1302.0854 [hep-th].

[86] D. T. Son and P. Surowka, "Hydrodynamics with Triangle Anomalies," *Phys.Rev.Lett.* **103** (2009) 191601, arXiv:0906.5044 [hep-th].

[87] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, *et al.*, "Local Fluid Dynamical Entropy from Gravity," *JHEP* **0806** (2008) 055, arXiv:0803.2526 [hep-th].

[88] R. B. Mann, E. Radu, and C. Stelea, "Black string solutions with negative cosmological constant," *JHEP* **0609** (2006) 073, arXiv:hep-th/0604205 [hep-th].

[89] B. Kleihaus, J. Kunz, and E. Radu, "New AdS solitons and brane worlds with compact extra-dimensions," *JHEP* **1009** (2010) 047, arXiv:1006.3290 [hep-th].