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Master Thesis

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Hyperscaling Violation in $\mathcal{N} = 2$ Gauged Supergravity

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Submitted: September 21, 2012

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Abstract

In recent years there has been much interest in the application of holography to condensed matter physics. In particular, gravity duals with non-relativistic scaling symmetries have been constructed for this purpose. These are known as Lifshitz and Schrödinger spacetimes. Recently, metrics with hyperscaling violation have also become important. These are not invariant under the non-relativistic scaling, but instead scale covariantly.

In this thesis, the equations of motion of four-dimensional $\mathcal{N} = 2$ gauged supergravity are derived in order to find Lifshitz solutions with non-zero hyperscaling violation. Both vector multiplets and gauged hypermultiplets are included in this general analysis. Focussing then on the $F = -iX^0X^1$ model, which contains a single vector multiplet coupled to gravity, explicit solutions are found. Further, it is shown that one of these solutions solves first-order flow equations. This implies that the solution is supersymmetric.

Along the way, relevant topics are introduced, such as special Kähler manifolds, quaternionic Kähler manifolds, gauging of $\mathcal{N} = 2$ supergravity, and also some aspects of non-relativistic holography and the formalism of first-order flow equations.

Contents

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Chapter 1

Introduction

The twentieth century saw huge leaps forward in theoretical physics. Albert Einstein's theory of relativity revolutionized the classical understanding of gravity, space, and time. Quantum mechanics offered a description of small scale physics, introducing concepts such as the uncertainty principle and wave-particle duality. The unification of special relativity and quantum physics led to quantum field theory, the language of the Standard Model of particle physics. However, Einstein gravity cannot be formulated consistently as a quantum field theory, due to non-renormalizability. A consistent quantum theory of gravity is not yet well established, however, among the leading candidates for such a theory is string theory.

This thesis may be considered as part of the wide area of modern string theory research, even though we will not address the question of whether string theory is a good quantum theory of gravity. The main topics of this thesis are classical supergravity and solutions relevant for the gauge/gravity correspondence. The aim of this Introduction is to place this work into the context of modern research.

String theory

String theory [1, 2] was first introduced in the late 1960's and early 1970's, initially as an attempt to describe the strong interactions [3–6]. The theory essentially postulates that the elementary constituents of Nature are not point-particles, but rather one-dimensional strings. These may be closed loops or open ended strings. The quantization of a string gives rise to a discrete, infinite spectrum of vibrational modes, corresponding to different types of particles. Among the initial problems was that the ground state was tachyonic, and that the spectrum included a massless spin-2 particle, which was unwanted in the context of the strong interactions. Further, the theory included only bosons, and consistent quantization required precisely 26 spacetime dimensions, rather than our familiar four.

Later, it was realized that the massless spin-2 particle could be interpreted as the graviton [7, 8]. Thus, string theory is actually a quantum theory of gravity. Inclusion of world-sheet supersymmetry led to the introduction of fermions and got rid of the tachyon state [9]. It was further shown, that this implies also spacetime supersymmetry, and that consistent quantization of the superstring requires spacetime to be ten-dimensional.

Supergravity

Supergravity was originally discovered independently of string theory [10]. Inclusion of supersymmetry in quantum field theories improves the UV behavior and delay divergences to higher loop orders. Originally, it was hoped that supersymmetry would render gravity finite, but this was realized not to be the case. Four-dimensional $\mathcal{N} = 8$ maximal supergravity, however, is still not ruled out as a finite quantum theory of gravity [11].

Supergravity does however find an important role in (super)string theory, as ten-dimensional supergravity emerges in the classical limit. By compactifying some dimensions, effective lower-dimensional theories may be obtained. These include gauged supergravities.

In this thesis we will take a more bottom-up approach to gauged supergravity. We will discuss how supergravity arises as local supersymmetry, and gauged supergravity can be regarded as a deformation of the ungauged theory, by promoting a global internal symmetry to be local.

The gauge/gravity correspondence

Among the most notable recent discoveries from string theory is the gauge/gravity correspondence, or holography. This states that a quantum gravity theory, in particular string theory, is dual to a lower-dimensional quantum field theory without gravity. For a quantum field theory at strong coupling and large N, the string theory dual reduces to the limit of classical gravity. On the one hand, this potentially allows new insight into quantum field theories from calculations in a classical gravity theory, while on the other hand we might learn more about string theory by studying field theories in flat spacetime. The first example was conjectured in 1997 by Maldacena [12–14]. This is a duality between type IIB supergravity on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills, which is a conformal field theory (CFT). Hence, the duality is known as AdS/CFT.

There are obviously interesting non-conformal field theories, e.g. quantum chromodynamics (QCD), to which one might want to extend the conjecture. In recent years, there has been much attention on the application of holography to condensed matter physics [15–25]. In particular, in many condensed matter systems one finds phase transitions governed by fixed points exhibiting non-relativistic scaling invariance. The gravity duals are known as Lifshitz and Schrödinger spacetimes [16–19].

Recently, aspects of hyperscaling violation have also been addressed in the context of holography [25, 26]. Roughly speaking, a d-dimensional condensed matter system with hyperscaling violation exponent θ has the thermodynamic behaviour of a $(d-\theta)$-dimensional system [27].

Embedding in string theory

Gravity duals are often first constructed without any relationship to string theory. Naturally, it is simpler to construct a specific solution if one is free to choose the matter content. Such models may yield phenomenological results and improve the understanding of the gauge/gravity correspondence itself. However, the duality actually involves a quantum gravity theory. It is therefore of interest to embed gravity duals into string theory as a UV complete theory.

Embeddings in string theory from constructions of Dp-branes in ten-dimensional supergravity can be considered a top-down approach. As a bottom-up approach one can work directly in the lower-dimensional supergravity theories, which results from compactifications. As compactifications in general increase the number of fields, it is convenient to work with consistent truncations of the compactified theory. A solution of a consistent truncation is then a guaranteed solution of the higher dimensional theory.

String/M-theory embeddings of Lifshitz solutions with hyperscaling violation have recently been found [25, 28–32], and also some solutions with preserved supersymmetry [29, 33].

This thesis

This thesis investigates Lifshitz spacetimes with non-zero hyperscaling violation, as solution of four-dimensional $\mathcal{N} = 2$ gauged supergravity. Such solutions with non-zero gaugings have not yet been found. We find indeed explicit solutions in the $F = -iX^0X^1$ model, however, these are restricted to vanishing gaugings. One of these solutions solve first-order flow equations, implying it is supersymmetric. Solutions with preserved supersymmetry have only previously been constructed differently, from near-horizon geometries of black holes or branes.

This thesis is organized as follows. In Chapter 2, we review supersymmetry and aspects of supergravity. Chapter 3 gives more details on four-dimensional $\mathcal{N} = 2$ gauged supergravity. Chapter 4 reviews the useful formalism of supersymmetric first-order flow equations. In Chapter 5, we discuss briefly the AdS/CFT correspondence and some aspects of non-relativistic generalizations thereof. In Chapter 6, we investigate the equations of motions of $\mathcal{N} = 2$ gauged supergravity in a Lifshitz-like background with non-zero hyperscaling violation. In Chapter 7, we consider the equations of motion of the $F = -iX^0X^1$ model with Fayet-Iliopoulos gaugings, as an explicit example. We find that we can solve these equations, but only for vanishing gauging. We then show that one of these solutions solves the first-order flow equations, implying it is supersymmetric. Finally, we conclude in Chapter 8 with a discussion of the results.

Chapter 2

Supergravity

In this section, we discuss the basics of supergravity. As supergravity is essentially the supersymmetric extension of General Relativity, we start by reviewing rigid supersymmetry. We then move on to different aspects of supergravity theories.

2.1 Rigid Supersymmetry

Supersymmetry has interesting properties desirable in particle physics, e.g. improved UV behaviour and stabilization of the Higgs mass (the hierarchy problem) [34, 35]. However, supersymmetry has not yet been observed in experiments. Hence, if it exists, it must be broken at low energies. In the context of this thesis, supersymmetry is important since it is required for consistency of string theory, in the sense mentioned in the Introduction. For further details or proofs, see e.g. [34–38]. We take the Minkowski metric to be $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The Poincaré algebra

Symmetries play a great role in particle theories. For example, relativistic theories are formulated to be invariant under the Poincaré group, the group of all isometries of Minkowski spacetime. Particles belong to representations of the Lorentz group with integer or half-integer spin, i.e. scalars, spinors, vectors, etc. The Poincaré algebra contains the generators of the Lorentz group, $\mathcal{J}_{\mu\nu}$, generating rotations and boosts, and the momentum P_μ generating spacetime translations. The algebra is

$$\begin{aligned} [\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}), \\ [P_\mu, \mathcal{J}_{\nu\rho}] &= i\eta_{\mu\nu}P_\rho - i\eta_{\mu\rho}P_\nu, \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{2.1}$$

From the elements of the Poincaré algebra, one can construct two Casimir operators, which commute with all elements of the algebra. The first is $P_\mu P^\mu$. To consider the action of this operator on massive particle states, we can choose the rest frame in which the particle has four-momentum $k^\mu = (m, 0, 0, 0)$. The eigenvalue of the momentum operator is the four-momentum. Thus, applying the Casimir operator yields

$$P_\nu P^\nu |k^\mu\rangle = k_\nu k^\nu |k^\mu\rangle = m^2 |k^\mu\rangle, \tag{2.2}$$

since $k_\mu k^\mu = m^2$. For a massless particle, one can choose the frame $k^\mu = (E, 0, 0, E)$ to find $P_\nu P^\nu |k^\mu\rangle = 0$. The other Casimir operator is $W_\mu W^\mu$, where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{J}^{\nu\rho}P^\sigma$ is the so-called Pauli-Lubanski pseudovector. In a somewhat similar fashion, one can show that $W_\mu W^\mu$ yields the spin (helicity) for massive (massless) particles.

Since $P_\mu P^\mu$ and $W_\mu W^\mu$ are Casimir operators, mass and spin/helicity are thus Poincaré invariant labels for particles.

The super-Poincaré algebra

In 1967, Coleman and Mandula considered the possible symmetries of the S-matrix under very general assumptions, such as unitarity [39]. They concluded that the Lie algebra of the largest possible symmetry must be a direct product of the Poincaré algebra and the algebra of an internal symmetry. In 1975, Haag, Łopuszański and Sohnius proved that the Coleman-Mandula theorem can be generalized, if the algebra of the symmetry is a superalgebra [40]. Thus, the Poincaré algebra can be extended to the super-Poincaré algebra by adding anti-commuting generators Q^A, where $A = 1, ..., \mathcal{N}$. These are the generators of supersymmetry. They extend the Poincaré algebra with the relations

$$\begin{array}{rclrcl} [Q^A_\alpha, P_\mu] & = & 0, & \{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} & = & 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu\delta^A_B, \\ [Q^A_\alpha, \mathcal{J}_{\mu\nu}] & = & (\sigma_{\mu\nu})_\alpha^{\ \beta}Q^A_\beta, & \{Q^A_\alpha, Q^B_\beta\} & = & \epsilon_{\alpha\beta}Z^{AB}, \end{array} \tag{2.3}$$

along with also the conjugate relations. Here, Q^A_α are Weyl spinors, and $\bar{Q}_{\dot{\alpha}A} = (Q^A_\alpha)^\dagger$. The components of Q^A are known as supercharges. When $\mathcal{N} > 1$, the theory has extended supersymmetry. The complex central charge $Z^{AB} = -Z^{BA}$ commutes with the whole algebra. Due to its antisymmetry there is no central charge for $\mathcal{N} = 1$. Further, for extended supersymmetry we can introduce another bosonic symmetry. The R-symmetry group is defined to be the largest subgroup of the automorphism group of the supersymmetry algebra that commutes with the Lorentz group. It corresponds to rotations between the supercharges,

$$\begin{array}{rclrcl} [P_\mu, R^a] & = & 0, & [\mathcal{J}_{\mu\nu}, R^a] & = & 0, \\ [Q^A_\alpha, R^a] & = & (U^a)^A_{\ B}Q^B_\alpha, & [R^a, R^b] & = & if^{ab}_{\ \ c}R^c. \end{array} \tag{2.4}$$

In four-dimensional spacetime in the absence of central charges the R-symmetry group is $U(\mathcal{N})$, while in the presence of central charges it is restricted to $USp(\mathcal{N}) = Sp(\mathcal{N}, \mathbb{C}) \cap U(\mathcal{N})$ [36].

Particles related by supersymmetry are called superpartners, and irreducible representations of the supersymmetry algebra form supermultiplets. Since the supercharges commute with P_μ, the operator $P_\mu P^\mu$ is still a Casimir. Hence, particles belonging to the same supermultiplet must have the same mass. The operator $W_\mu W^\mu$, however, turns out not to commute with the supercharges, which indicates that spin is not conserved under supersymmetry transformations. Since the supercharges are fermionic, their action on particle states will change the spin statistics of the state,

$$Q|\text{boson}\rangle = |\text{fermion}\rangle, \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle, \tag{2.5}$$

schematically.

Massless supermultiplets

Consider a massless particle with momentum k^μ, for which we may choose the frame $k_\mu = (E, 0, 0, E)$. The anti-commutator of the supercharges becomes

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^A_B = 2E\delta^A_B \left(\mathbb{1} + \sigma^3\right)_{\alpha\dot{\alpha}} = 4E\delta^A_B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}}. \tag{2.6}$$

Note in particular $\{Q^A_2, \bar{Q}_{\dot{2}B}\} = 0$. This implies Q^A_2 for all A lead to zero-norm states,

$$0 = \langle\psi|\{Q^A_2, \bar{Q}_{\dot{2}B}\}|\psi\rangle = ||Q^A_2|\psi\rangle||^2 + ||\bar{Q}_{\dot{2}B}|\psi\rangle||^2 \quad \Rightarrow \quad Q^A_2 = \bar{Q}_{\dot{2}B} = 0 \ \ \forall A, B. \tag{2.7}$$

Plugging $Q^A_2 = \bar{Q}_{\dot{2}A} = 0$ back into the super-Poincaré algebra (2.3) further implies vanishing central charges, $Z^{AB} = 0$, for the massless representations. On the other hand, defining $q_A \equiv (1/\sqrt{4E})Q^A_1$ and $q^\dagger_A \equiv (1/\sqrt{4E})\bar{Q}_{\dot{1}A}$ leads to cannonical fermionic creation and annihilation operators,

$$\{q_A, q^\dagger_B\} = \delta_{AB} \quad \{q_A, q_B\} = \{q^\dagger_A, q^\dagger_B\} = 0. \tag{2.8}$$

Recall above, that for a massless state with energy E and helicity λ, applying W_μ yields the helicity. In the chosen frame,

$$W_0|E,\lambda\rangle = \lambda P_0|E,\lambda\rangle = \lambda E|E,\lambda\rangle. \tag{2.9}$$

To find the helicity of the state $q^\dagger_A|E,\lambda\rangle$, one can apply W_0 and subsequently use the commutators of $\bar{Q}_{\dot{\alpha}A}$ with P_μ and $\mathcal{J}^{\mu\nu}$ to find

$$W_0 q^\dagger_A|E,\lambda\rangle = E(\lambda + \tfrac{1}{2})q^\dagger_A|E,\lambda\rangle. \tag{2.10}$$

Thus, $q^\dagger_A$ raises the helicity by $\frac{1}{2}$. To build a supermultiplet, one now chooses a vacuum state of helicity λ_0 satisfying

$$q_A|E_0,\lambda_0\rangle = 0 \quad \forall A, \tag{2.11}$$

which is always possible due to the anticommutativity of the supercharges. By applying the $\mathcal{N}$ creation operators one can construct $2^{\mathcal{N}}$ states with helicities from λ_0 to $\lambda_0 + \mathcal{N}/2$. However, helicity changes sign under CPT conjugation, so unless the multiplet is self-conjugate, the states of opposite helicity must be added to ensure CPT invariance.

As an example, consider a massless $\mathcal{N}=1$ multiplet. This contains the helicity states

$$\{\lambda_0 + \tfrac{1}{2},\ \ \lambda_0,\ \ -\lambda_0,\ \ -\lambda_0 - \tfrac{1}{2}\}. \tag{2.12}$$

Choosing $\lambda_0 = 0$ the field content corresponds to a complex scalar field (or two real scalar fields) and a spin-$\frac{1}{2}$ Weyl fermion. This is called the chiral multiplet. The vector multiplet corresponds to $\lambda_0 = \frac{1}{2}$, and contains a spin-$\frac{1}{2}$ Weyl fermion and a spin-1 gauge field. For rigid supersymmetry without gravity, we do not choose $\lambda_0 > \frac{1}{2}$, due to renormalizability. However, in supergravity the $\mathcal{N}=1$ gravity multiplet is build from $\lambda_0 = \frac{3}{2}$ and contains the spin-2 graviton and one spin-$\frac{3}{2}$ gravitino.

Massive supermultiplets

Now, consider a massive particle. In the rest frame, $k^\mu = (m,0,0,0)$, we have

$$\{Q^A_\alpha, \bar{Q}_{\dot\alpha B}\} = 2m\,\sigma^0_{\alpha\dot\alpha}\delta^A_B = 2m\,\delta_{\alpha\dot\alpha}\delta^A_B. \tag{2.13}$$

Consider first the super-Poincaré algebra (2.3) with vanishing central charges, $Z^{AB}=0$. Defining $q_{\alpha A} \equiv (1/\sqrt{2m})Q^A_\alpha$, the relevant part of the super-Poincaré algebra (2.3) becomes

$$\{q_{\alpha A}, q^\dagger_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}, \quad \{q_{\alpha A}, q_{\beta B}\} = \{q^\dagger_{\alpha A}, q^\dagger_{\beta B}\} = 0. \tag{2.14}$$

Choosing again a vacuum state and applying the creation and annihilation operators yields $2^{2\mathcal{N}}$ states, instead of just $2^{\mathcal{N}}$ as in the massless case. Such multiplets are called long multiplets.

Massive multiplets may be shortened due to non-vanishing central charges. For $Z^{AB} \neq 0$ the definition of the creation and annihilation operators is more subtle, since $\{Q^A_\alpha, Q^B_\beta\} \neq 0$. By appropriate symmetry transformations, the antisymmetric central charge can be brought to the form:

$$Z^{AB} = \begin{pmatrix} 0 & Z_1 & & & & & & \\ -Z_1 & 0 & & & & & & \\ & & 0 & Z_2 & & & & \\ & & -Z_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & Z_{\mathcal{N}/2} \\ & & & & & & -Z_{\mathcal{N}/2} & 0 \end{pmatrix}, \tag{2.15}$$

where we have assumed that $\mathcal{N}$ is even. The entries Z_L can be chosen to be real and non-negative. Splitting the indices $A=(a,L)$, where $L,M = 1,...,\mathcal{N}/2$ and $a,b=1,2$, the central charges can be written $Z^{AB} = Z^{(a,L)(b,M)} = -\epsilon^{ab}\delta^{LM}Z_M$, with no sum over M and $\epsilon^{12}=-1$. The anticommutators of the super-Poincaré algebra can then be written

$$\{Q^{(a,L)}_\alpha, \bar{Q}_{\dot\alpha(b,M)}\} = 2m\delta^L_M\delta^a_b\delta_{\alpha\dot\alpha}, \tag{2.16}$$

$$\{Q^{(a,L)}_\alpha, Q^{(b,M)}_\beta\} = -2\epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LM}Z_M, \tag{2.17}$$

$$\{\bar{Q}_{\dot\alpha(a,L)}, \bar{Q}_{\dot\beta(b,M)}\} = -2\epsilon_{\alpha\beta}\epsilon_{ab}\delta_{LM}Z_M. \tag{2.18}$$

The creation and annihilation operators can then be defined as

$$q^{\pm}_{\alpha L} = \tfrac{1}{2}\left(Q^{1L}_{\alpha} \pm \epsilon_{\alpha\beta}\delta^{\beta\dot{\gamma}}\bar{Q}_{\dot{\gamma}2L}\right), \tag{2.19}$$

$$q^{\pm\,\dagger}_{\dot{\alpha}L} = \tfrac{1}{2}\left(\bar{Q}_{\dot{\alpha}1L} \pm \epsilon_{\dot{\alpha}\dot{\beta}}\delta^{\dot{\beta}\gamma}Q^{2L}_{\gamma}\right), \tag{2.20}$$

and the algebra can be worked out to be

$$\{q^{\pm}_{\alpha L}, q^{\pm\,\dagger}_{\dot{\alpha}M}\} = (m \pm Z_M)\delta_{\alpha\dot{\alpha}}\delta_{LM}, \tag{2.21}$$

$$\{q^{\pm}_{\alpha L}, q^{\pm}_{\beta M}\} = \{q^{\pm\,\dagger}_{\dot{\alpha}L}, q^{\pm\,\dagger}_{\dot{\beta}M}\} = \{q^{\mp}_{\alpha L}, q^{\pm\,\dagger}_{\alpha M}\} = ... = 0. \tag{2.22}$$

Now, the $q^{\pm}_{\alpha L}$ act as annihilation operators, while the $q^{\pm\,\dagger}_{\dot{\alpha}L}$ act as creation operators. Thus, one chooses again a vacuum state, annihilated by all $q^{\pm}_{\alpha L}$, and then applies the $q^{\pm\,\dagger}_{\dot{\alpha}L}$. One can show that $q^{\pm\,\dagger}_{\dot{1}L}$ raises spin by $\frac{1}{2}$, while $q^{\pm\,\dagger}_{\dot{2}L}$ lowers spin by $\frac{1}{2}$. An important observation in eqn. (2.21) is that one must require

$$m \geq Z_L, \tag{2.23}$$

in order to avoid unphysical negative norm states. This is known as the Bogomolnyi-Prasad-Sommerfeld bound, or BPS bound for short. If none of the central charges Z_L saturate the BPS bound, the multiplet is constructed as in the case with vanishing central charges. Hence it is a long multiplet with $2^{2\mathcal{N}}$ states. If $n \leq \mathcal{N}/2$ of the central charges saturate the bound, according to (2.21) the corresponding creation operators $q^{-\,\dagger}_{\dot{\alpha}L}$ yield zero-norm states and do not contribute to the state spectrum. The multiplet will then contain $2^{2\mathcal{N}-2n}$ states, and is called a short multiplet.

Supersymmetric Lagrangians

One of the simplest examples of a supersymmetric Lagrangian is the Wess-Zumino model [41], containing a single chiral multiplet and a complex auxiliary field F. In the free, massless case, the Lagrangian is

$$\mathcal{L}_{WZ} = \partial_\mu\phi\partial^\mu\bar{\phi} + i\partial_\mu\chi\,\sigma^\mu\bar{\chi} + F\bar{F}. \tag{2.24}$$

The auxiliary field has the trivial equation of motion $F = 0$, which means it has no physical on-shell degrees of freedom. The field is needed, however, to close the supersymmetry algebra off-shell, and to match the bosonic and fermionic off-shell degrees of freedom. The off-shell closure and matching of degrees of freedom are important for the quantum theory. However, since this thesis is primarily concerned with classical solutions of supergravity, on-shell closure will do.

The Lagrangian (2.24) is invariant, up to total derivatives, under the infinitesimal supersymmetry variations:

$$\begin{aligned} \delta\phi &= \epsilon\chi, \\ \delta\chi &= \epsilon F + i\sigma^\mu\bar{\epsilon}\partial_\mu\phi, \\ \delta F &= -i\partial_\mu\chi\sigma^\mu\bar{\epsilon}. \end{aligned} \tag{2.25}$$

Here, ϵ is a spinorial parameter, with no spacetime dependence. Masses and interactions can consistently be added to the Lagrangian (2.24), see e.g. [34, 37]. Note that in the case at hand, when adding masses, the field content remains the same, even though the massive representation is a long multiplet (recall, there is no central charge for $\mathcal{N} = 1$). This is due to the added CPT conjugates in the massless representation, doubling the number of states. As a final remark, let us mention that a powerful formalism for constructing manifestly supersymmetric Lagrangians is the superspace formalism, see [37].

2.2 Local supersymmetry: Supergravity

In the previous section rigid supersymmetry was discussed. In the super-Poincaré algebra (2.3) the supercharges anticommutate to a global translation,

$$\{Q_\alpha^A, \bar{Q}_{\dot{\alpha}B}\} = 2\delta^A_B \sigma^\mu_{\alpha\dot{\alpha}} P_\mu. \tag{2.26}$$

Promoting supersymmetry to be a local symmetry, the above indicates that one must allow for spacetime dependent translations, i.e. general coordinate transformations. Indeed, local supersymmetry will necessarily require the inclusion of gravity. We will now review how to turn a global symmetry into a local one, and then apply this to supersymmetry.

Gauging a global symmetry

The procedure of turning a global symmetry into a local one is sometimes called the Noether method [42–44]. See also [45] for further details and generalizations. It is in general an iterative process, where terms must be added to the Lagrangian in order to end up with an invariant theory.

As an example, consider a spin-$\frac{1}{2}$ Dirac spinor ψ in Minkowski spacetime with the Lagrangian,

$$\mathcal{L} = i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi. \tag{2.27}$$

The above Lagrangian is invariant under global $U(1)$ transformations, $\psi \to e^{i\alpha}\psi$. This symmetry is gauged by promoting α to be spacetime dependent. While the mass term in (2.27) is still invariant, the kinetic term is not, due to the derivative,

$$\delta\mathcal{L} = j^\mu \partial_\mu \alpha(x), \qquad j^\mu = -\bar{\psi}\gamma^\mu\psi. \tag{2.28}$$

j^μ is the Noether current. Invariance can be restored by introducing a vector gauge field A_μ transforming as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x), \tag{2.29}$$

with coupling constant e. Adding to the Lagrangian the term

$$\mathcal{L}_N = e j^\mu A_\mu, \tag{2.30}$$

gauge invariance is restored. The gauge field A_μ is in fact a connection, and we can introduce covariant derivative, $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$, transforming as

$$D_\mu\psi \to e^{i\alpha(x)} D_\mu\psi. \tag{2.31}$$

Adding now also the gauge invariant kinetic terms for the gauge field, the action becomes

$$\mathcal{L} = i\bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.32}$$

We have then obtained the familiar gauge invariant Lagrangian of quantum electrodynamics (QED).

To conclude, the gauging of the global $U(1)$ symmetry of (2.27) introduces a covariant derivative, as well as a spin-1 gauge field as a connection.

Local supersymmetry

Now, consider promoting global supersymmetry to be local. For definiteness, consider the Wess-Zumino model, eqn. (2.24), consisting of a single $\mathcal{N} = 1$ chiral multiplet,

$$\mathcal{L}_{WZ} = \partial_\mu\phi\partial^\mu\bar{\phi} + i\partial_\mu\chi\sigma^\mu\bar{\chi}. \tag{2.33}$$

We consider here only the on-shell form, since we will be interested in classical solutions in the later chapters. The supersymmetry variations with global parameter ϵ are

$$\delta\phi = \epsilon\chi, \qquad \delta\chi = i\sigma^\mu\bar{\epsilon}\partial_\mu\phi\ . \tag{2.34}$$

Promoting now $\epsilon \to \epsilon(x)$, the Lagrangian is no longer invariant. Including now spinor indices for clarity, the variation and Noether current is

$$\delta\mathcal{L} = \partial_\mu\epsilon^\alpha\mathcal{J}^\mu_\alpha + \text{h.c.}, \qquad \mathcal{J}^\mu_\alpha = \chi^\beta\sigma^\mu_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\bar{\phi} \tag{2.35}$$

Analogous to eqn. (2.30), a gauge field must be added. Notice, however, that the gauge field ψ^α_μ must carry both a Lorentz index and a spinor index,

$$\mathcal{L}_N = \kappa\,\psi^\alpha_\mu\mathcal{J}^\mu_\alpha + \text{h.c.}. \tag{2.36}$$

It must transform as

$$\psi^\alpha_\mu \to \psi^\alpha_\mu - \frac{1}{\kappa}\partial_\mu\epsilon^\alpha + ..., \tag{2.37}$$

in order to cancel the variation (2.35). This field is the spin-$\frac{3}{2}$ representation of the Lorentz group, called the gravitino. The kinetic term for a spin-$\frac{3}{2}$ field is the Rarita-Schwinger Lagrangian [46],

$$\mathcal{L}_\psi = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma. \tag{2.38}$$

The term (2.38) is in fact invariant under gauge transformations of the form (2.37). Working out the local supersymmetry variations of the Lagrangian $\mathcal{L}_{WZ}+\mathcal{L}_N+\mathcal{L}_\psi$, the theory is still not invariant, however. Among other terms, one finds a term proportional to the energy-momentum tensor of the scalar field, $T_{\mu\nu}(\phi)$. Yet another field must therefore be added to the theory, a bosonic symmetric rank 2 tensor field, $h_{\mu\nu}$, along with a coupling,

$$\mathcal{L}_N^{(2)}=\frac{1}{2}\kappa h_{\mu\nu}T^{\mu\nu}. \tag{2.39}$$

The bosonic, symmetric rank-2 tensor field which couples to the energy-momentum tensor is identified with the graviton. The metric is then introduced as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. Also, one can work out that the coupling constant κ is dimensionful, unlike the QED case above. In fact, it is identified with the Newton constant, $\kappa^2=8\pi G$. Continuing the iteration process, the end result is a Lagrangian of matter-coupled gravity, invariant under appropriate supersymmetry variations. Hence the name: supergravity.

2.3 Four-dimensional $\mathcal{N}=1$ supergravities

We now discuss the structure of supergravity actions. We consider here only ungauged actions, postponing the gauged case specifically to Chapter 3.

A supergravity action always contains the gravity multiplet. This is constructed by choosing the graviton as the state of highest helicity, and then lowering helicity with the supercharges. For example, an $\mathcal{N}=1$ gravity multiplet contains just the graviton and one gravitino. Theories containing only the gravity multiplet are called minimal supergravities.

Minimal $\mathcal{N}=1$ supergravity

The four-dimensional $\mathcal{N}=1$ minimal supergravity action was first constructed in [10, 47], see reviews [43, 48]. In order to couple spinors to gravity, we need to introduce the vielbein $e_\mu{}^a$, such that $g_{\mu\nu}=e_\mu{}^a e_\nu{}^b\eta_{ab}$, and the spin connection $\omega_\mu{}^{ab}$. Appendix B includes some details on this.

The action contains the Einstein-Hilbert term, the Rarita-Schwinger term, and also higher order terms in the gravitino. We will use units in which $\kappa^2=8\pi G=1$, so the action has the form

$$S=\int e\,\mathrm{d}^4x\left(\frac{R}{2}+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5 D_\rho\psi_\sigma+\mathcal{L}_{\psi^4}\right). \tag{2.40}$$

Here $e\equiv\det e_\mu{}^a=\sqrt{-g}$, and $D_\mu=\left(\partial_\mu+\frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)$ contains the torsionless spin connection. The fact that the connection is torsionless implies $D_{[\mu}e_{\nu]}{}^a=0$. The terms in (2.40) with higher orders of fermions are typical to supergravity models. Such terms are generically non-renormalizable. However, since Einstein gravity is non-renormalizable anyways, this will not concern us.

The action (2.40) may be written simpler, by introducing a connection with torsion,

$$\hat{\omega}_\mu^{ab}=\omega_\mu{}^{ab}+K^a{}_\mu{}^b,\qquad K^a{}_\mu{}^b=-i\left(\bar{\psi}^{[a}\gamma^{b]}\psi_\mu+\tfrac{1}{2}\bar{\psi}^a\gamma_\mu\psi^b\right). \tag{2.41}$$

Then, $\hat{D}_{[\mu}e_{\nu]}{}^a = -\frac{i}{2}\bar{\psi}_\mu\gamma^a\psi_\nu$, where the covariant derivative is $\hat{D}_\mu = \left(\partial_\mu + \frac{1}{4}\hat{\omega}_\mu{}^{ab}\gamma_{ab}\right)$. The action can then be written as

$$S = \int e\,\mathrm{d}^4x \left(\frac{\hat{R}}{2} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma\right), \tag{2.42}$$

where $\hat{R}$ and $\hat{D}_\mu$ are functions of $\hat{\omega}_\mu{}^{ab}$, rather than $\omega_\mu{}^{ab}$. The action (2.42) is invariant under local supersymmetry variations,

$$\delta e_\mu{}^a = -i\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta\psi_\mu = \hat{D}_\mu\epsilon. \tag{2.43}$$

Matter-coupled $\mathcal{N} = 1$ supergravity

As discussed in section 2.1, the massless multiplets of $\mathcal{N} = 1$ supersymmetry are the chiral multiplet, the vector multiplet, and in supergravity also the gravity multiplet. An arbitrary number of matter-multiplets may be coupled to the minimal action above. We take n_C chiral multiplets and n_V vector multiplets. Thus the action contains n_C complex scalar fields and n_V vector fields, which we take to be Abelian. Since we are interested in solutions of supergravity where all fermions are truncated, we will focus on the bosonic part of the action. This has the form

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4}I_{\Lambda\Sigma}(\phi,\bar{\phi})\,F^\Lambda_{\mu\nu}F^{\Sigma|\mu\nu} + \frac{1}{4}R_{\Lambda\Sigma}(\phi,\bar{\phi})\,F^\Lambda_{\mu\nu}{}^\star F^{\Sigma|\mu\nu} + g_{i\bar{\jmath}}(\phi,\bar{\phi})\,\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}}\right). \tag{2.44}$$

The action contains a topological term $\mathcal{L}_{top} = \frac{1}{4}R_{\Lambda\Sigma}F^\Lambda_{\mu\nu}{}^\star F^{\Sigma\mu\nu}$. This is a generalization of the theta-angle of quantum chromodynamics [49]. The couplings of the vector kinetic terms and the topological term are given by the so-called period matrix or kinetic matrix, $\mathcal{N}_{\Lambda\Sigma}$, such that $I_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$ and $R_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$.

The kinetic terms of the scalar fields are not canonical. Rather, they are described by a non-linear sigma model, where the scalar fields may be interpreted as coordinates on a manifold called the target space or the scalar manifold, $\mathcal{M}_{scalar}$. The scalar fields define a set of maps from spacetime $\mathcal{M}_4$ to the target manifold,

$$\phi^i(x): \quad \mathcal{M}_4 \longrightarrow \mathcal{M}_{scalar}. \tag{2.45}$$

The scalar kinetic terms in the action are given by the metric $g_{i\bar{\jmath}}$ on $\mathcal{M}_{scalar}$. For $\mathcal{N} = 1$, supersymmetry dictates that the target space is a Kähler manifold (see Section 3.2).

2.4 Extended supergravity

In four dimensions, supergravity can be extended with $\mathcal{N} \leq 8$. Theories with $\mathcal{N} > 8$ must necessarily contains fields with helicity $\lambda > 2$. A finite number of fields with $\lambda > 2$ cannot consistently couple to other fields, at least in Minkowski space. However, recently there has been much interest in higher-spin theories and their role in holography. For a review see e.g. [50]. For further details on extended supergravity than given below, see e.g. [43,51].

Field content

The possible matter couplings depend on the value of $\mathcal{N}$. The question is if any multiplets can be constructed with helicities $-1 \leq \lambda \leq 1$, for consistency.

For $\mathcal{N} = 2$, there are two types of matter-multiplets, vector multiplets and hypermultiplets. We will discuss this in more detail in Chapter 3. For $\mathcal{N} = 4$, the only matter-multiplets are vector multiplets, while for $\mathcal{N} > 4$ the field content is completely fixed. Due to the large supersymmetry, such theories must contain many fields. For example, maximal $\mathcal{N} = 8$ supergravity contains the graviton, 8 gravitinos, 28 vectors, 56 spin-$\frac{1}{2}$ fermions, and 70 real scalar fields. Vectors belonging to the gravity multiplet are known as graviphotons.

Action

The bosonic part of the action has the same overall form as in the $\mathcal{N} = 1$ case,

$$S = \int \sqrt{-g}\, \mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma}(\phi, \bar{\phi})\, F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma}(\phi, \bar{\phi})\, F^{\Lambda}_{\mu\nu} {}^{\star}F^{\Sigma|\mu\nu} + g_{i\bar{\jmath}}(\phi, \bar{\phi})\, \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{\jmath}} \right). \tag{2.46}$$

The vector couplings are again described by the period matrix $\mathcal{N}_{\Lambda\Sigma} = R_{\Lambda\Sigma} + i I_{\Lambda\Sigma}$, and the scalar fields are described by a non-linear sigma model. For $\mathcal{N} > 2$ (but also some theories with $\mathcal{N} \leq 2$) the target space $\mathcal{M}_{scalar}$ is a coset space G/H. This means the target space has a global isometry group G, and an isotropy group (also called the little group or stabilizer) $H \subset G$. The isotropy group H associated to a chosen point $x \in \mathcal{M}_{scalar}$ is the largest subgroup of G, which leaves x fixed, i.e. $hx = x$ for all $h \in H$. A simple example is the 2-sphere S^2, which is the coset $SO(3)/SO(2)$.

As an example, the scalar manifold for $\mathcal{N} = 4$ supergravity with n_V vector multiplets is

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6, n_V)}{SO(6) \times SO(n_V)}. \tag{2.47}$$

2.5 Electric-magnetic duality

Electric-magnetic duality is well-known for simpler theories, such as classical Maxwell-theory. Maxwell's equations in vacuum,

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}, \tag{2.48}$$

are invariant under a transformation $(\mathbf{E}, \mathbf{B}) \to (\mathbf{B}, -\mathbf{E})$. Supergravity theories also exhibit electric-magnetic duality, though a little more involved, due to the scalar fields in the same supermultiplets as the vectors [52–54].

Assume the action (2.46) contains n vector fields A^{Λ}_{μ}. The n equations of motion for these fields are

$$\sqrt{-g} \frac{\partial \mathcal{L}}{\partial A^{\Lambda}_{\nu}} - \partial_\mu \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{\Lambda}_{\nu})} \right) = 0 \quad \Leftrightarrow \quad \nabla_\mu \frac{\partial \mathcal{L}}{\partial F^{\Lambda}_{\mu\nu}} = 0. \tag{2.49}$$

Defining

$$^\star G^{\mu\nu}_\Lambda \equiv 2\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = I_{\Lambda\Sigma}F^{\Sigma|\mu\nu} + R_{\Lambda\Sigma}{}^\star F^{\Sigma|\mu\nu}, \tag{2.50}$$

sometimes called the magnetic dual field strength, we can write the equations of motion as

$$\nabla_\mu {}^\star G^{\mu\nu}_\Lambda = 0. \tag{2.51}$$

We can also introduce the Bianchi identities

$$\nabla_\mu {}^\star F^{\Lambda|\mu\nu} = 0. \tag{2.52}$$

Gauge theories are naturally formulated in differential geometry. A theory with a (possibly non-Abelian) gauge group G has an associated G-principal bundle. Gauge transformations correspond to different choices of local sections. Let $\mathfrak{g}$ denote the Lie-algebra of G. The gauge potential $A = A_\mu \mathrm{d}x^\mu$ is a $\mathfrak{g}$-valued 1-form, and can be regarded as a local expression for the connection. The field strength $F = \mathrm{d}A + A \wedge A$ is a $\mathfrak{g}$-valued 2-form. It is a local form of the curvature associated with the connection. The Bianchi identity is a geometrical constraint on such a curvature: $\mathrm{d}F + [A, F] = 0$. In the Abelian case, this becomes just $\mathrm{d}F = 0$, which can be brought to the form of eqn. (2.52) (using such identities as found in Appendix C). Alternatively, since the Abelian field strength is an exact 2-form, $F = \mathrm{d}A$, it must be closed, $\mathrm{d}F = \mathrm{d}^2A = 0$, since the exterior derivative satisfies $\mathrm{d}^2 = 0$. The Bianchi identity can be violated, however, by magnetic monopoles or magnetic gaugings, introducing a current on the r.h.s. of eqn. (2.52).

Consider linear combinations of $^\star F$ and $^\star G$,

$$\begin{pmatrix} ^\star\tilde{F} \\ ^\star\tilde{G} \end{pmatrix} = \mathcal{S}\begin{pmatrix} ^\star F \\ ^\star G \end{pmatrix}, \tag{2.53}$$

where $\mathcal{S} \in GL(2n_V, \mathbb{R})$ is a constant matrix. This is a symmetry of the combined Bianchi identities (2.52) and equations of motion (2.51),

$$\nabla_\mu {}^\star\tilde{F}^{\Lambda|\mu\nu} = 0, \qquad \nabla_\mu {}^\star\tilde{G}^{\mu\nu}_\Lambda = 0. \tag{2.54}$$

Such rotations are called electric-magnetic duality rotations.

Now, one can think of the duality rotation (2.53) as also inducing a diffeomorphism on the scalar manifold, as well as a suitable transformation of the period matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$. Thus under a transformation,

$$\begin{array}{ccc} (^\star F^\Lambda, {}^\star G_\Lambda) & \to & \mathcal{S}(^\star F^\Lambda, {}^\star G_\Lambda), \\ \phi & \to & \phi'(\phi), \\ \mathcal{N}_{\Lambda\Sigma}(\phi) & \to & \mathcal{N}'_{\Lambda\Sigma}(\phi') . \end{array} \tag{2.55}$$

The transformation of $\mathcal{N}_{\Lambda\Sigma}$ is fixed by demanding $^\star G_\Lambda$ to still be defined as a variation of the action, eqn. (2.50), leading to [51, 55]

$$\mathcal{N}_{\Lambda\Sigma}(\phi) \quad \to \mathcal{N}'_{\Lambda\Sigma}(\phi') = \left[(C + D\mathcal{N}) \cdot (A + B\mathcal{N})^{-1}\right]_{\Lambda\Sigma}, \tag{2.56}$$

where

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.57}$$

Symmetry of $\mathcal{N}'_{\Lambda\Sigma}$ implies the contraint $\mathcal{S} \in Sp(2n, \mathbb{R}) \subset GL(2n_V, \mathbb{R})$. The symplectic group $Sp(2n, \mathbb{R})$ is the group of all real $2n \times 2n$ matrices satisfying

$$\mathcal{S}^T \mathbb{C} \mathcal{S} \ = \ \mathbb{C}, \tag{2.58}$$

where

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{2.59}$$

is the symplectic invariant matrix. This implies the homomorphism,

$$\mathrm{Diff}(\mathcal{M}_{scalar}) \ \rightarrow \ Sp(2n, \mathbb{R}), \tag{2.60}$$

such that all diffeomorphisms on the scalar manifold, $\mathrm{Diff}(\mathcal{M}_{scalar})$, have an image in the symplectic group.

Though (2.55) is a symmetry of the equations of motion and the Bianchi identities, the Lagrangian is not invariant in general. While this may be important for quantum calculations, classical solutions are on-shell. Thus, one may rotate one classical solution to other solutions. For example, a black hole may have magnetic charges p^Λ and electric charges q_Λ, given by

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda, \qquad\qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda, \tag{2.61}$$

where S^2 is a two-sphere enclosing the black hole. This will perhaps be more clear from Chapter 4.2. However, the point to make here is that duality rotations exchange electric and magnetic charges, without changing the spacetime metric. Hence the name, electric-magnetic duality.

Chapter 3

Four-dimensional $\mathcal{N} = 2$ gauged supergravity

The calculations in the later chapters of this thesis are performed in four-dimensional $\mathcal{N} = 2$ supergravity, which we will study in this section. In the last two decades, $\mathcal{N} = 2$ supergravity has been chosen for many studies of black holes, e.g. [56–61]. One reason for studying $\mathcal{N} = 2$ is that it has enough symmetry to make calculations tractable, yet not so much symmetry as to only yield very restricted solutions. E.g. one can couple an arbitrary number of supermultiplets to the theory, unlike $\mathcal{N} > 4$ where the matter content is completely fixed.

The action is comprised of the gravity multiplet coupled to n_V vector multiplets and n_H hypermultiplets. The full action with electric gaugings can be found in [55]. However, in the solutions we will study, fermions will always be put to zero and therefore we will primarily study the bosonic part of the action [62]. Also, we assume Abelian gauge groups (though, a few details of non-Abelian gauging will be included below in Section 3.4). The bosonic action then has form[1]

$$S = \int \sqrt{-g}\mathrm{d}^4x \left(\frac{R}{2}+\frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu}+\frac{1}{4}R_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}{}^{\star}F^{\Sigma\mu\nu}+g_{i\bar{\jmath}}\nabla_\mu z^i\nabla^\mu\bar{z}^{\bar{\jmath}}+h_{uv}\nabla_\mu q^u\nabla^\mu q^v-g^2V\right), \tag{3.1}$$

where $\nabla_\mu z^i = \partial_\mu z^i + gk^i_\Lambda A^\Lambda_\mu$ and $\nabla_\mu q^u = \partial_\mu q^u + gk^u_\Lambda A^\Lambda_\mu$ are the covariant derivative arising when gauging the supergravity. The potential depends on general on both vector scalars and hyperscalars $V = V(z, \bar{z}, q)$. Taking $g = 0$ corresponds to ungauged supergravity.

As described in the previous Chapter, all couplings in the action are fixed by supersymmetry. The couplings of the vector kinetic terms and the topological terms are determined by the period matrix $\mathcal{N}_{\Lambda\Sigma}$, such that $I_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$ and $R_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$. The scalar kinetic terms are in general nonlinear sigma-models, $g_{i\bar{\jmath}} = g_{i\bar{\jmath}}(z, \bar{z})$ and $h_{uv} = h_{uv}(q)$, displaying a fascinating interplay between supersymmetry and geometry. The vector multiplets are described by special Kähler geometry, while the hypermultiplets are described by qauternionic Kähler geometry, see below.

[1] In the gauging, we will only be concerned with the "classical symmetries" of ref. [55]. An additional term containing the $c_{\Lambda,\Sigma\Gamma}$-tensor can be added in some cases, which is included in [62, 63]. See also [64].

When discussing multiplets and gaugings below, the supersymmetry variations of the fermionic fields will be given. The variations of the bosons are all linear combinations of fermions. For example for the vielbein:

$$\delta e_\mu{}^a = -i\left(\bar{\psi}_{A\mu}\gamma^a\epsilon^A + \bar{\psi}^A_\mu\gamma^a\epsilon_A\right), \tag{3.2}$$

where ψ^A_μ is the gravitino. Since fermions are put to zero in solutions, the variations of the bosonic fields vanish. Therefore, only the variations of the fermions will be relevant.

3.1 The gravity multiplet

The gravity multiplet consists of the vielbein, a doublet of spin-$\frac{3}{2}$ gravitinos with opposite chirality, and the graviphoton A^0_μ:

$$(e_\mu{}^a, \psi^A_\mu, \psi_{\mu A}, A^0_\mu).$$

Thus, $A = 0, 1$ labels the gravitinos and is the index acted on by R-symmetry. The position of the index A denotes the chirality.

Minimal $\mathcal{N} = 2$ supergravity contains only the gravity multiplet. The bosonic action is just that of Einstein-Maxwell theory,

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} - \frac{1}{4}F^0_{\mu\nu}F^{0|\mu\nu}\right). \tag{3.3}$$

Hence, Einstein-Maxwell theory can be embedded into $\mathcal{N} = 2$ supergravity by adding the two gravitinos.

In ungauged supergravity, the $U(1) \times SU(2)$ R-symmetry acts as a global symmetry and rotates between the gravitinos. A $U(1)$ subgroup of the R-symmetry can be promoted to a local symmetry [65]. The gravitinos then become charged under the gauge field via a covariant derivative with coupling constant g. Due to supersymmetry, the bosonic action further gets a negative cosmological constant, $\Lambda = -3g^2$. Hence, the bosonic action becomes

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} - \frac{1}{4}F^0_{\mu\nu}F^{0|\mu\nu} + 3g^2\right). \tag{3.4}$$

This action admits an AdS_4 vacuum with negative curvature, $R = -12g^2$. In the context of e.g. black hole solutions, this is important since the gauged case allows asymptotic AdS_4 black holes, whereas the ungauged case does not. Supersymmetric AdS_4 black holes in gauged minimal $\mathcal{N} = 2$ supergravity were found in [66, 67].

The supersymmetry variation of the gravitino in minimal supergravity is

$$\delta\psi_{\mu A} = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon_A + F^0_{\mu\nu}\gamma^\nu\epsilon_{AB}\epsilon^B - \frac{1}{2}g\sigma^x_{AB}\epsilon^B, \tag{3.5}$$

where ϵ^A are two Weyl spinors. Also, $x = 1, 2$ or 3 depending on how the $U(1)$ gauge group is embedded in the R-symmetry.

3.2 Vector multiplets & special Kähler manifolds

Each of the n_V vector multiplets consists of a vector field A^i_μ, a doublet of spinors with opposite chirality called gauginos, $\lambda^{iA}, \lambda^i_A$, and a complex scalar field z^i,

$$(A^i_\mu, \lambda^{iA}, \lambda^i_A, z^i).$$

where $i = 1..., n_V$. When n_V vector multiplets are added, the action contains $n_V + 1$ vector fields, due to the graviphoton. These are collectively labelled by $\Lambda = 0, ..., n_V$. The kinetic terms of the vector scalars are described by a non-linear sigma-model,

$$\mathcal{L}_{vec.sc.} = g_{i\bar{\jmath}}(z, \bar{z})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}}. \tag{3.6}$$

Thus, $g_{i\bar{\jmath}}$ is the metric on the target space. Supersymmetry dictates the target space to be a so-called special Kähler manifold, or just special manifold for short [68].

Kähler manifolds

In the following we will review some geometry. For further proofs and details, see e.g. [69]. Consider a complex manifold $\mathcal{M}$ of dimension $\dim_{\mathbb{C}} \mathcal{M} = n_V$ (since there are n_V complex scalar fields). The tangent space $T_p\mathcal{M}$, with $p \in \mathcal{M}$, is spanned by $2n_V$ vectors

$$\left\{ \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}; \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^m} \right\}, \tag{3.7}$$

where $z^i = x^i + iy^i$ are the coordinates of p in a chart. The dual tangent space $T^*_p\mathcal{M}$ is then spanned by the one forms

$$\left\{ \mathrm{d}x^1, ..., \mathrm{d}x^m; \mathrm{d}y^1, ..., \mathrm{d}y^m \right\}. \tag{3.8}$$

Defining instead $2n_V$ complex vectors,

$$\frac{\partial}{\partial z^i} \equiv \frac{1}{2}\left\{ \frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i} \right\}, \qquad \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} \equiv \frac{1}{2}\left\{ \frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i} \right\}, \tag{3.9}$$

these form the complex tangent space $T_p\mathcal{M}^{\mathbb{C}}$. Likewise for $T^*_p\mathcal{M}^{\mathbb{C}}$,

$$\mathrm{d}z^i \equiv \mathrm{d}x^i + i\,\mathrm{d}y^i, \qquad \mathrm{d}\bar{z}^{\bar{\imath}} \equiv \mathrm{d}x^i - i\,\mathrm{d}y^i. \tag{3.10}$$

A complex manifold admits a globally defined almost complex structure J. This is a linear map $J : T_p\mathcal{M} \to T_p\mathcal{M}$ defined by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}. \tag{3.11}$$

Note that

$$J^2 = -\mathrm{id}_{T_p\mathcal{M}}, \tag{3.12}$$

familiar from the imaginary unit $i^2 = -1$. The almost complex structure J may also be defined on $T_p\mathcal{M}^{\mathbb{C}}$,

$$J(X + iY) \equiv JX + iJY, \qquad X, Y \in T_p\mathcal{M}. \tag{3.13}$$

From (3.11) it follows that

$$J\left(\frac{\partial}{\partial z^i}\right) = i\frac{\partial}{\partial z^i}, \qquad J\left(\frac{\partial}{\partial \bar{z}^{\bar{\imath}}}\right) = -i\frac{\partial}{\partial \bar{z}^{\bar{\imath}}}. \tag{3.14}$$

A Riemannian metric g of a complex manifold $\mathcal{M}$ is said to be Hermitian if it satisfies

$$g_p(JX, JY) = g_p(X, Y) \tag{3.15}$$

for any point $p \in \mathcal{M}$ and for any $X, Y \in T_p\mathcal{M}$. The pair $(\mathcal{M}, g)$ is called a Hermitian manifold. A complex manifold always admits a Hermitian metric. From the definition (3.15) it follows for a Hermitian metric,

$$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}\right) = i^2 g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = -g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = -g_{ij}. \tag{3.16}$$

Therefore, $g_{ij} = 0$ and likewise $g_{\bar{\imath}\bar{\jmath}} = 0$. A Hermitian metric must take the form[2]:

$$g = g_{i\bar{\jmath}}\, \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{\jmath}} + g_{\bar{\imath}j}\, \mathrm{d}\bar{z}^{\bar{\imath}} \otimes \mathrm{d}z^j. \tag{3.17}$$

Define now the tensor field K by its action on $X, Y \in T_p\mathcal{M}$,

$$K_p(X, Y) = g_p(JX, Y). \tag{3.18}$$

This implies that K is anti-symmetric and hence a two-form, called the Kähler form of the metric g. Extending the domain of K from $T_p\mathcal{M}$ to $T_p\mathcal{M}^{\mathbb{C}}$ leads to the form

$$K = i g_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}. \tag{3.19}$$

Now, a Kähler manifold is a Hermitian manifold $(\mathcal{M}, g)$ whose Kähler form K is closed, i.e. $\mathrm{d}K = 0$. The metric g is then called the Kähler metric of $\mathcal{M}$. One can show that this implies

$$\partial_k g_{i\bar{\jmath}} = \partial_i g_{k\bar{\jmath}}, \qquad \partial_{\bar{k}} g_{i\bar{\jmath}} = \partial_{\bar{\jmath}} g_{i\bar{k}}. \tag{3.20}$$

The closure of the Kähler form also implies that the Kähler metric may locally be expressed as

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \mathcal{K}, \tag{3.21}$$

where the function $\mathcal{K}$ is called the Kähler potential of the Kähler metric. The Kähler potential of the metric g is not unique. Under Kähler transformations

$$\mathcal{K} = \mathcal{K} + f(z) + \bar{f}(\bar{z}), \tag{3.22}$$

the Kähler metric is unchanged by eqn. (3.21).

[2]Of course we ought to also include the tensor product $\otimes$ when writing metrics on spacetime. It is convention in physics, however, to not do so. We include them here for clarity.

Special Kähler manifolds

Now, a special Kähler manifold is a Kähler manifold with further restrictions. Also, one may distinguish the rigid and local case, corresponding to rigid supersymmetry and supergravity, respectively. We focus here on the local case, introducing details needed for calculations in later chapters. For a complete definition of special Kähler manifolds, see e.g. [51, 55, 68, 70–73].

A special Kähler manifold $\mathcal{M}_{SK}$ of dimension n_V, requires the existence of a $(2n_V+2)$-dimensional symplectic bundle with holomorphic sections $\Omega = (X^\Lambda, F_\Lambda)$, $\Lambda = 0, ..., n_V$,

$$\partial_{\bar{\imath}}\Omega = 0. \tag{3.23}$$

In terms of these, the Kähler potential may be written as

$$\mathcal{K} = -\log\left[i\langle\Omega,\bar{\Omega}\rangle\right] = -\log\left[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)\right]. \tag{3.24}$$

Here, the symplectic product is defined by $\langle A, B\rangle \equiv A^t\mathbb{C}B$, where $\mathbb{C}$ is the symplectic invariant matrix

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.25}$$

We may also introduce the sections $\mathcal{V} = (L^\Lambda, M_\Lambda) = e^{K/2}(X^\Lambda, F_\Lambda)$, which satisfy

$$i\langle\mathcal{V},\bar{\mathcal{V}}\rangle = 1. \tag{3.26}$$

Under Kähler transformations, (3.22), a covariant derivative is needed. For a generic section Φ with weight p, the covariant derivative is

$$D_i\Phi \equiv \left(\partial_i + \tfrac{1}{2}p\partial_i\mathcal{K}\right)\Phi, \qquad D_{\bar{\imath}}\Phi \equiv \left(\partial_{\bar{\imath}} - \tfrac{1}{2}p\partial_{\bar{\imath}}\mathcal{K}\right)\Phi. \tag{3.27}$$

The sections $\mathcal{V}$ have weight $p = 1$, while $\bar{\mathcal{V}}$ have weight $p = -1$. A simple calculation then shows that the sections $\mathcal{V}$ are covariantly holomorphic,

$$D_{\bar{\imath}}\mathcal{V} = 0. \tag{3.28}$$

We further define to quantities

$$\begin{aligned} f_i^\Lambda &\equiv D_i L^\Lambda, \qquad & h_{\Lambda i} &\equiv D_i M_\Lambda, \\ \bar{f}_{\bar{\imath}}^\Lambda &\equiv D_{\bar{\imath}}\bar{L}^\Lambda, \qquad & \bar{h}_{\Lambda\bar{\imath}} &\equiv D_{\bar{\imath}}\bar{M}_\Lambda, \end{aligned} \tag{3.29}$$

with the appropriate weights given above. Using these quantities, the symmetric period matrix is defined by the relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma, \qquad \bar{h}_{\Lambda\bar{\imath}} = \mathcal{N}_{\Lambda\Sigma}\bar{f}_{\bar{\imath}}^\Sigma. \tag{3.30}$$

This can be solved to yield

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \bar{h}_{\Lambda\bar{\imath}} \\ M_\Lambda \end{pmatrix} \begin{pmatrix} \bar{f}_{\bar{\imath}}^\Sigma \\ L^\Sigma \end{pmatrix}^{-1}. \tag{3.31}$$

The matrices

$$I_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}, \qquad R_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} \tag{3.32}$$

appear in the action (3.1) as the scalar dependent couplings of the gauge field strengths. In particular, $I_{\Lambda\Sigma}$ is negative definite such that the gauge fields have positive kinetic energy. Thus, if the holomorphic sections $\Omega = (X^\Lambda, F_\Lambda)$ are specified, the couplings in the action, $g_{i\bar{\jmath}}$, $I_{\Lambda\Sigma}$ and $R_{\Lambda\Sigma}$, can be derived from it.

There are a number of identities on the special Kähler manifold [70], which may be useful for calculations, such as

$$I_{\Lambda\Sigma}L^\Lambda \bar{L}^\Sigma = -\frac{1}{2}, \tag{3.33}$$

$$g^{i\bar{\jmath}} f_i^\Lambda \bar{f}_{\bar{\jmath}}^\Sigma = -\frac{1}{2} I^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma. \tag{3.34}$$

We defined here $g^{i\bar{\jmath}}$ as the inverse of the Kähler metric, and $I^{\Lambda\Sigma} \equiv (\mathrm{Im}\mathcal{N})^{-1\,\Lambda\Sigma}$.

Prepotential

In some cases, the geometric quantities on the special Kähler manifold can all be derived from a holomorphic function called the prepotential $F(X)$.

When the prepotential exists, the holomorphic sections can be derived as

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda}. \tag{3.35}$$

Also, we define

$$F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}, \qquad F_{\Lambda\Sigma\Gamma} \equiv \frac{\partial^3 F}{\partial X^\Lambda \partial X^\Sigma \partial X^\Gamma}, \qquad \text{etc.} \tag{3.36}$$

Due to supersymmetry, the prepotential must be homogeneous of degree two, i.e.

$$F(\lambda X) = \lambda^2 F(X). \tag{3.37}$$

This implies some identities, e.g.

$$2F = X^\Lambda F_\Lambda, \qquad F_\Lambda = F_{\Lambda\Sigma}X^\Sigma, \qquad F_{\Lambda\Sigma\Gamma}X^\Gamma = 0. \tag{3.38}$$

The physical scalar fields may be chosen as so-called special coordinates, $z^i = X^i/X^0$. By eqns. (3.21) and (3.24), the prepotential thus gives the Kähler metric. Further, when a prepotential exists, the period matrix is determined as

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\mathrm{Im}(F_{\Lambda\Gamma})X^\Gamma \, \mathrm{Im}(F_{\Sigma\Delta})X^\Delta}{X^\Omega \mathrm{Im}(F_{\Omega\Upsilon})X^\Upsilon}. \tag{3.39}$$

The prepotential thus specifies all couplings in the (ungauged) action coupled to vector multiplets.

Two examples of prepotentials are,

$$F = -iX^0X^1 \qquad \text{and} \qquad F = -\frac{X^1X^2X^3}{X^0}. \tag{3.40}$$

Both are clearly homogeneous of degree two. The first contains just a single vector multiplet, and is among the simplest models due to the linearity. We will derive the action from this prepotenial is Chapter 7. The second contains three vector multiplets. In this case, the physical scalar fields are often called S, T and U, and therefore this model is commonly known as the STU model.

Electric-magnetic duality

As discussed in Section 2.5, diffeomorphisms on the scalar manifold are embedded in the symplectic group, $Sp(2n_V+2,\mathbb{R})$ for n_V+1 vector fields. Under such transformations, the sections of the symplectic bundle transform as

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix} = \mathcal{S}\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \qquad \mathcal{S} \in Sp(2n_V+2,\mathbb{R}). \tag{3.41}$$

For the ungauged theory this is a symmetry of the equations of motion, but not necessarily of the Lagrangian. Allowing for all possible symplectic matrices $\mathcal{S}$, the duality rotations thus generate an orbit of Lagrangians with the same solutions. In every orbit at least one Lagrangian will have a prepotential. This illustrates the convenience of prepotentials. These have the advantage of the nicely compact notation, while representing the whole duality orbit.

As an example, the STU model in eqn. (3.40) is related to the prepotential $\tilde{F} = -2i\sqrt{X^0X^1X^2X^3}$ by a symplectic transformation [60],

$$S = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ \hline & & & & 1 & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{array}\right), \tag{3.42}$$

such that $(\tilde{X}^\Lambda, \tilde{F}_\Lambda) = \mathcal{S}(X^\Lambda, F_\Lambda)$, where $(\tilde{X}^\Lambda, \tilde{F}_\Lambda)$ is derived from $\tilde{F}$.

Action and supersymmetry variations

The bosonic action with ungauged vector multiplets is given by

$$S = \int \sqrt{-g}\, \mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^\Lambda_{\mu\nu} {}^\star F^{\Sigma\mu\nu} + g_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \right). \tag{3.43}$$

The variations of the gravitinos and gauginos are

$$\delta\psi_{\mu A} = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab}\right)\epsilon_A + \frac{i}{2}\tilde{A}_\mu\epsilon_A + \frac{1}{2}T^-_{\mu\nu}\gamma^\nu\epsilon_{AB}\epsilon^B, \tag{3.44}$$

$$\delta\lambda^{iA} = i\partial_\mu z^i\gamma^\mu\epsilon^A + \frac{1}{2}G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\epsilon_B, \tag{3.45}$$

where

$$\tilde{A}_\mu = -\frac{i}{2}\left(\partial_i\mathcal{K}\partial_\mu z^i - \partial_{\bar{\imath}}\mathcal{K}\partial_\mu\bar{z}^{\bar{\imath}}\right) \tag{3.46}$$

is a $U(1)$ connection on the special Kähler manifold. $T^-_{\mu\nu}$ and $G^-_{\mu\nu}$ are the field strengths dressed with the scalars,

$$T^-_{\mu\nu} = 2iL^\Lambda I_{\Lambda\Sigma}F^{\Sigma-}_{\mu\nu}, \tag{3.47}$$

$$G^{i-}_{\mu\nu} = -g^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}I_{\Lambda\Sigma}F^{\Sigma-}_{\mu\nu}, \tag{3.48}$$

where $F^{\Lambda\pm}_{\mu\nu}$ are the (anti-)selfdual field strengths,

$$F^{\Lambda\pm}_{\mu\nu} = \frac{1}{2}\left(F^\Lambda_{\mu\nu} \pm i\,{}^\star F^\Lambda_{\mu\nu}\right). \tag{3.49}$$

3.3 Hypermultiplets & Quaternionic Kähler Manifolds

One may couple n_H hypermultiplets to the theory. Each multiplet consists of a doublet of spinors called hyperinos ζ^α and four real scalars q^u called the hyperscalars or simply the hypers,

$$(\zeta^\alpha, q^u),$$

where $\alpha = 1, ..., 2n_H$ and $u = 1, ..., 4n_H$. The kinetic terms of the $4n_H$ scalars are described by a non-linear sigma model,

$$\mathcal{L}_{hyp.scal.} = h_{uv}(q)\partial_\mu q^u\partial^\mu q^v, \tag{3.50}$$

with the target space being a quaternionic Kähler manifold [74]. Despite the name, a quaternionic Kähler manifold is not necessarily Kähler [75]. For $\mathcal{N} = 2$ rigid sypersymmetry, the target space is a instead a hyperKähler manifold, which is closely related.

Quaternionic Kähler manifolds

Quaternions can be thought of as an extension of complex numbers. A complex number has the form $z = a + ib$, where a,b are real numbers and $i^2 = -1$. Quaternions instead have the form $a + ib + jc + kd$, where a,b,c,d are real numbers, and

$$i^2 = j^2 = k^2 = ijk = -1. \tag{3.51}$$

Following here refs. [55,73,75], both a quaternionic Kähler manifold $\mathcal{M}_Q$ and a hyperKähler manifold $\mathcal{M}_{HK}$ is a $4n_H$-dimensional real manifold endowed with a metric h,

$$\mathrm{d}s^2 = h_{uv}(q)\mathrm{d}q^u \otimes \mathrm{d}q^v, \qquad u, v = 1, ..., 4n_H. \tag{3.52}$$

The manifold admits three almost complex structures,

$$J^x \;:\; T_p\mathcal{M}_Q \longrightarrow T_p\mathcal{M}_Q, \qquad x = 1, 2, 3, \tag{3.53}$$

(likewise for $\mathcal{M}_{HK}$), satisfying the quaternionic algebra,

$$J^x J^y = -\delta^{xy}\mathbb{1} + \epsilon^{xyz}J^z. \tag{3.54}$$

This has the form of eqn. (3.51), and extends the complex structures, eqn. (3.12). The metric must be hermitian with respect to the three complex structures,

$$h_p(J^x X, J^x Y) = h_p(X, Y), \quad (x = 1, 2, 3), \tag{3.55}$$

where X, Y are generic tagent vectors. A triplet of two-forms may be associated to the complex structures

$$K^x = K^x_{uv}\mathrm{d}q^u \wedge \mathrm{d}q^v, \qquad K^x_{uv} = h_{uw}(J^x)^w_v, \tag{3.56}$$

called the hyperKähler forms. Recall, for a Kähler manifold, the Kähler form is closed. In the quaternionic and hyperKähler case, the hyperKähler forms must be covariantly closed,

$$DK^x \equiv \mathrm{d}K^x - \epsilon^{xyz}\omega^y \wedge K^z = 0, \tag{3.57}$$

with respect to a connection ω^x on a principal $SU(2)$-bundle. Supersymmetry requires the existence of such a principal bundle, being essentially a fiber bundle with an associated group structure. The curvature of the bundle is given by

$$\Omega^x = \mathrm{d}\omega^x - \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z. \tag{3.58}$$

For a hyperKähler manifold $\mathcal{M}_{HK}$ of $\mathcal{N} = 2$ rigid supersymmetry, the principal bundle must be flat,

$$\Omega^x = 0. \tag{3.59}$$

A quaternionic manifold $\mathcal{M}_Q$ has instead curvature proportional to the hyperKähler two-form, i.e.

$$\Omega^x = \lambda K^x, \tag{3.60}$$

where λ is real and non-vanishing. In fact, $\lambda = -1$ (in natural units) in order to have correct normalization of the kinetic terms of the scalars.

We can define a vielbein one-form,

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q)\,\mathrm{d}q^u, \tag{3.61}$$

such that the metric can be decomposed as

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta}\epsilon_{AB}. \tag{3.62}$$

The antisymmetric matrices $\mathbb{C}_{\alpha\beta}$ and ϵ_{AB} are the flat $Sp(2n_H)$ and $SU(2)$ invariant metrics, respectively (see eqn. (3.25)).

Action and supersymmetry variations

When the hypermultiplets are ungauged, the bosonic part of the action is just the non-linear sigma model,

$$\mathcal{L}_{hyp.scal.} = h_{uv}(q)\partial_\mu q^u \partial^\mu q^v. \tag{3.63}$$

The supersymmetry variation of the hyperinos is given by

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \epsilon^A \epsilon_{AB}\mathbb{C}_{\alpha\beta}. \tag{3.64}$$

Note that in the ungauged theory, the hyperscalars do not couple to the vector multiplets in the action, and neither through the supersymmetry variations. The hypermultiplets do therefore not participate in the electric/magnetic duality. In fact, the hyperscalars can consistently be set to constant values and truncated.

The universal hypermultiplet

In compactifications of ten-dimensional type II supergravity, a particular hypermultiplet is always present. It is therefore called the universal hypermultiplet. In real coordinates (R, D, u, v), the metric on the quaternionic manifold can be written as (leaving out again the tensor product $\otimes$)

$$\mathrm{d}s^2 = \frac{1}{R^2}\left(\mathrm{d}R^2 + R\left(\mathrm{d}u^2 + \mathrm{d}v^2\right) + \left(\mathrm{d}D + \frac{u\mathrm{d}v}{2} - \frac{v\mathrm{d}u}{2}\right)^2\right), \qquad R > 0. \tag{3.65}$$

This metric describes in fact the coset space $SU(2,1)/U(2)$. For details, such as Killing vectors and Killing prepotentials (see below), see [62, 76, 77].

3.4 Gauging isometries on the scalar manifolds

The target space of the scalar non-linear sigma models is the product space

$$\mathcal{M} = \mathcal{M}_{SK} \otimes \mathcal{M}_{Q}. \tag{3.66}$$

Assume now the target space has isometries generated by Killing vectors $k_\Lambda = k_\Lambda^i(z)\partial_i$ on $\mathcal{M}_{SK}$ and $\tilde{k}_\Lambda = k_\Lambda^u(q)\partial_u$ on $\mathcal{M}_Q$. The kinetic terms of the scalars are then invariant under the transformations

$$\delta z^i = -\,\alpha^\Lambda k_\Lambda^i, \qquad \delta q^u = -\,\beta^\Lambda k_\Lambda^u, \tag{3.67}$$

with global transformation parameters $\alpha^\Lambda, \beta^\Lambda$. Gauging the supergravity theory means to promote a subgroup G of the group of isometries to a local symmetry. Since the theory contains $n_V + 1$ vector fields, the dimension of the gauge group can be at most $n_V + 1$. This is the reason for using the index $\Lambda = 0, ..., n_V$ in eqn. (3.67).

Vector scalar isometries

For gauging of isometries on the special Kähler manifold, the gauged Killing vectors must span the Lie-algebra,

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma k_\Gamma, \tag{3.68}$$

of the gauge group G with structure constants $f_{\Lambda\Sigma}{}^\Gamma$. Not all Killing vectors can be gauged on the special Kähler manifold, see e.g. [55, 62, 73]. The vector scalars become charged under the gauge fields via a covariant derivative with coupling constant g,

$$\nabla_\mu z^i = \partial_\mu z^i + g k^i_\Lambda A^\Lambda_\mu. \tag{3.69}$$

The gauge fields transform as $\delta A^\Lambda_\mu = \partial_\mu \alpha^\Lambda$, such that the kinetic terms are gauge invariant. For non-Abelian gauging, the field strengths must take the proper form,

$$F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu + g f_{\Sigma\Gamma}{}^\Lambda A^\Sigma_\mu A^\Gamma_\nu. \tag{3.70}$$

As was noted, it was already assumed that the gauge group is Abelian in the supergravity action (3.1).

To preserve supersymmetry when gauging the special Kähler isometries, a scalar potential must be added to the Lagrangian:

$$V(z, \bar{z}) = g_{i\bar{\jmath}} k^i_\Lambda \bar{k}^{\bar{\jmath}}_\Sigma \bar{L}^\Lambda L^\Sigma. \tag{3.71}$$

The supersymmetry variations must also be modified, see below.

Hyperscalar isometries

For gauging of the hyperscalars, the Killing vectors $\tilde{k}_\Lambda = k^u_\Lambda(q)\partial_u$ on the quaternionic manifold must again span the Lie-algebra of the gauge group G,

$$\left[\tilde{k}_\Lambda, \tilde{k}_\Sigma\right] = f_{\Lambda\Sigma}{}^\Gamma \tilde{k}_\Gamma. \tag{3.72}$$

The hyperscalars become charged via a covariant derivative,

$$\nabla_\mu q^u = \partial_\mu q^u + g k^u_\Lambda A^\Lambda_\mu. \tag{3.73}$$

This breaks again electric-magnetic duality. Due to supersymmetry, a potential must be introduced, which depends on both the vector scalars and the hypers,

$$V(z, \bar{z}, q) = 4 h_{uv} k^u_\Lambda k^v_\Sigma \bar{L}^\Lambda L^\Sigma + \left(g^{i\bar{\jmath}} f^\Lambda_i \bar{f}^\Sigma_{\bar{\jmath}} - 3\bar{L}^\Lambda L^\Sigma\right) \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma. \tag{3.74}$$

Here, $\mathcal{P}^x_\Lambda = \mathcal{P}^x_\Lambda(q)$ is a triplet of real zero-forms, called Killing prepotentials or moment maps. They are related to the Killing vectors by

$$K^x_{uv} k^v_\Lambda = D_v \mathcal{P}^x_\Lambda = \partial_v \mathcal{P}^x_\Lambda - \epsilon^{xyz} \omega^y_v \mathcal{P}^z_\Lambda \tag{3.75}$$

(using $\lambda = -1$ from eqn. (3.60)). When gauging both vector scalars and hyperscalars, the total scalar potential is the sum of (3.71) and (3.74).

For non-Abelian gauge groups, the gauging must include isometries on the special manifold. For Abelian gauge groups, however, the hyperscalars can be gauged without involving the vector scalars. We will be dealing with this case in Chapter 6. Let us write the action here for clarity,

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} {}^{\star}F^{\Sigma|\mu\nu} + g_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} + h_{uv} \nabla_\mu q^u \nabla^\mu q^v - g^2 V \right), \tag{3.76}$$

with V given by (3.74).

Such models have been studied in recent years in the context of black holes [62, 76] and e.g. Lifshitz solutions [77, 78]. However, supersymmetric solutions of theories with both gauged vector scalars and hypers have also been studied [63, 79], and the attractor mechanism has recently been extended to such theories [61].

Supersymmetry variations

For both gauged vector scalars and hyperscalars, the supersymmetry variations of the fermions are:

$$\begin{aligned} \delta\psi_{A\mu} &= \left(\partial_\mu + \frac{1}{4} \omega_\mu^{\ ab} \gamma_{ab} \right) \epsilon_A + \frac{i}{2} \tilde{A}_\mu \epsilon_A + \omega_{\mu A}{}^B \epsilon_B \\ &\quad + \frac{1}{2} T^-_{\mu\nu} \gamma^\nu \epsilon_{AB} \epsilon^B + i g S_{AB} \gamma_\mu \epsilon^B, \end{aligned} \tag{3.77}$$

$$\delta\lambda^{iA} = i \nabla_\mu z^i \gamma^\mu \epsilon^A + \frac{1}{2} G^{-i}_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \epsilon_B + g W^{iAB} \epsilon_B, \tag{3.78}$$

$$\delta\zeta_\alpha = i \mathcal{U}^{B\beta}_u \nabla_\mu q^u \gamma^\mu \epsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta} + g N^A_\alpha \epsilon_A. \tag{3.79}$$

Here,

$$\omega_{\mu\,A}{}^B = \partial_\mu q^u \omega_{u\,A}{}^B + \frac{i}{2} g \sigma^x{}_A{}^B \mathcal{P}^x_\Lambda A^\Lambda_\mu \tag{3.80}$$

involves the $SU(2)$ connection on the quaternionic manifold. The so-called gravitino, gaugino and hyperino mass matrices are given by

$$S_{AB} = \frac{i}{2} \mathcal{P}^x_\Lambda L^\Lambda (\sigma^x)_A{}^C \epsilon_{BC}, \tag{3.81}$$

$$W^{iAB} = k^i_\Lambda \bar{L}^\Lambda \epsilon^{AB} + i g^{i\bar{\jmath}} \bar{f}^\Lambda_{\bar{\jmath}} \mathcal{P}^x_\Lambda (\sigma^x)_C{}^B \epsilon^{CA}, \tag{3.82}$$

$$N^A_\alpha = 2 \mathcal{U}^A_{\alpha u} k^u_\Lambda \bar{L}^A. \tag{3.83}$$

As the names suggest, the fermions develop masses when scalar isometries are gauged. The matrices (3.81)-(3.83) appear in those mass terms.

The dressed field strengths $T^-_{\mu\nu}$ and $G^-_{\mu\nu}$ were introduced in eqns. (3.47)-(3.48). The $U(1)$ Kähler connection (3.46) is however modified to

$$\tilde{A}_\mu = -\frac{i}{2} \left(\partial_i \mathcal{K} \nabla_\mu z^i - \partial_{\bar{\imath}} \mathcal{K} \nabla_\mu \bar{z}^{\bar{\imath}} \right) - \frac{i}{2} g A^\Lambda_\mu (r_\Lambda - \bar{r}_\Lambda), \tag{3.84}$$

when the vector scalars are gauged. Here, $r_\Lambda(z)$ is a holomorphic function, see e.g. [62].

The supersymmetry variations in the case of Abelian gauged hypers and ungauged vector scalars is a special case of the above. They correspond to $k^i_\Lambda = r_\Lambda = 0$ and $\nabla_\mu z^i = \partial_\mu z^i$.

Electric-magnetic duality

The gaugings discussed above are called electric gaugings. They introduce a current $j^\mu_\Lambda = \partial\mathcal{L}/\partial A^\Lambda_\mu$ into the equations of motion

$$\nabla_\mu {}^\star F^{\Lambda|\mu\nu} = 0, \qquad \nabla_\mu {}^\star G^{\mu\nu}_\Lambda = j^\nu_\Lambda. \tag{3.85}$$

The electric-magnetic duality discussed in Sections 2.5 and 3.2 is then broken (at least to a smaller subgroup of $Sp(2n_V+2,\mathbb{R})$). Symplectic rotations lead to new field dynamics. To restore the invariance, magnetic gaugings must be introduced. The most general Lagrangian with both electric and magnetic gaugings is not yet known. As shown in [80, 81], massive tensor multiplets must be introduced. More recent progress was reported in [82].

3.5 Fayet-Iliopoulos gauging

Instead of gauging an isometry on the scalar manifold, one may gauge a subgroup of the $U(1)\times SU(2)$ R-symmetry, analogous to the gauging of minimal supergravity. This is known as Fayet-Iliopoulos (FI) gauging. In this case, the gravitinos become charged by a linear combination $\xi_\Lambda A^\Lambda_\mu$ of the graviphoton and the gauge fields from the vector multiplets. The constants ξ_Λ are known as FI parameters.

In FI gauging, the vector multiplet scalars remain uncharged. If hyperscalars are present, however, they will be charged. We will consider the case where only vector multiplets are coupled to the theory.

Recall, the gauging of minimal supergravity gave rise to a cosmological constant. In FI gauging, this cosmological constant is replaced by a scalar potential of the form

$$V(z,\bar z) = \left(g^{i\bar\jmath} f^\Lambda_i \bar f^\Sigma_{\bar\jmath} - 3\bar L^\Lambda L^\Sigma\right)\xi_\Lambda\xi_\Sigma. \tag{3.86}$$

Using the identity (3.34), this may also be written as

$$V(z,\bar z) = -\left(\frac{1}{2} I^{\Lambda\Sigma} + 4\bar L^\Lambda L^\Sigma\right)\xi_\Lambda\xi_\Sigma. \tag{3.87}$$

The action is thus a simpler case of eqn. (3.1):

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^\Lambda_{\mu\nu} {}^\star F^{\Sigma\mu\nu} + g_{i\bar\jmath}\partial_\mu z^i \partial^\mu \bar z^{\bar\jmath} - V(z,\bar z)\right). \tag{3.88}$$

Both FI gauging and the gauging of scalar isometries lead to a scalar potential admitting AdS_4 vacua. However, with FI gauging one can consider the simpler case of uncharged vector scalars. Such models have been used to study e.g. AdS_4 black holes [83–86], attractors [58–60], and more general supersymmetric solutions [87].

There is an alternative view on the constant FI parameters ξ_Λ. They can be understood as the quaternionic Killing prepotentials $\mathcal{P}^x_\Lambda(q)$ in the absence of hypermultiplets [85]. The potential (3.86) then arises from the potential (3.74) with no quaternionic Killing vectors, $k^u_\Lambda = 0$, and with the Killing prepotenials replaced by constants, $\mathcal{P}^x_\Lambda(q) \to \xi^x_\Lambda$,

$$V(z,\bar{z}) = \left(g^{i\bar{\jmath}} f_i^\Lambda \bar{f}_{\bar{\jmath}}^\Sigma - 3\bar{L}^\Lambda L^\Sigma\right)\xi^x_\Lambda \xi^x_\Sigma, \qquad x = 1,2,3. \tag{3.89}$$

The FI parameters must however satisfy $\epsilon^{xyz}\xi^y_\Lambda\xi^z_\Sigma = 0$ [55,73]. Using the local R-symmetry they can be brought to the form $\xi^x_\Lambda = (\xi_\Lambda, 0, 0)$, such that $\xi^x_\Lambda\xi^x_\Sigma = \xi_\Lambda\xi_\Sigma$. This recovers the form of the potential, eqn. (3.86).

Magnetic gaugings

The FI gaugings discussed above are again electric and break electric-magnetic duality. Though the full Lagrangian is not yet known, magnetic FI gaugings are sometimes included, e.g. [59,60]. In this case, we define

$$\mathcal{G} = (\xi^\Lambda, \xi_\Lambda), \qquad \mathcal{W} = \langle \mathcal{G}, \mathcal{V}\rangle = L^\Lambda \xi_\Lambda - F_\Lambda \xi^\Lambda. \tag{3.90}$$

where ξ_Λ are the electric and ξ^Λ are magnetic gaugings, while the symplectic product and the sections $\mathcal{V}$ were defined in Section 3.2. The potential is then

$$V(z,\bar{z}) = g^{i\bar{\jmath}}\mathcal{W}_i \bar{\mathcal{W}}_{\bar{\jmath}} - 3\mathcal{W}\bar{\mathcal{W}}, \tag{3.91}$$

where

$$\mathcal{W}_i = \left(\partial_i + \tfrac{1}{2}\partial_i\mathcal{K}\right)\mathcal{W}, \qquad \bar{\mathcal{W}}_{\bar{\imath}} = \left(\partial_{\bar{\imath}} + \tfrac{1}{2}\partial_{\bar{\imath}}\mathcal{K}\right)\bar{\mathcal{W}}. \tag{3.92}$$

As one can verify, this reduces to eqn. (3.86) if the magnetic gaugings vanish $\xi^\Lambda = 0$.

Supersymmetry variations

With electric FI gauging, the supersymmetry variations of the gravitinos and gauginos are

$$\begin{aligned}
\delta\psi_{\mu A} &= \Big(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\Big)\epsilon_A + \frac{i}{2}\tilde{A}_\mu\epsilon_A + \xi^x_\Lambda A^\Lambda_\mu \sigma^x{}_A{}^B\epsilon_B + \frac{1}{2}T^-_{\mu\nu}\gamma^\nu\epsilon_{AB}\epsilon^B - \frac{1}{2}g\xi^x_\Lambda L^\Lambda \sigma^x_{AB}\gamma_\mu\epsilon^B, \\
\delta\lambda^{iA} &= i\partial_\mu z^i\gamma^\mu\epsilon^A + \frac{1}{2}G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\epsilon_B + ig^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}\xi^x_\Lambda\sigma^{x,AB}\epsilon_B,
\end{aligned} \tag{3.93}$$

where $\tilde{A}_\mu$, $T^-_{\mu\nu}$ and $G^{-i}_{\mu\nu}$ are introduced in eqns. (3.46)-(3.48). As discussed, one can choose e.g. $\xi^x_\Lambda = (\xi_\Lambda, 0, 0)$.

Chapter 4

First-order flow equations

Using the structure of $\mathcal{N} = 2$ supergravity, one can set up the formalism of first-order flow equations. For supersymmetric solutions, these can be thought of as the combination of the second-order equations of motion with the Killing spinor equations, which guarantee preserved supersymmetry. Since the flow equations are first-order differential equations, they provide in some cases a simpler route to solutions than the equations of motion. The formalism can also in some cases be extended to non-supersymmetric solutions. In this Chapter, we review aspects of supersymmetric solutions and the first-order flow equations. We will apply these equations in Chapter 7 to the solutions found therein.

4.1 Black holes in supergravity

Supergravity naturally embeds General Relativity. Therefore, classical solutions of supergravity include black holes, or more generally p-branes in $4 \leq D \leq 11$ dimensions. As the spacetime curvature may become large at black holes, quantum gravity effects must become important. However, when the spacetime curvature is much smaller than the string scale, supergravity provides a reliable description of black holes. At the classical level, string theory further induces higher-derivative corrections in the action. However, we will not consider such higher derivatives here.

Supergravity generically contains vector fields and scalar fields. From the Reissner-Nordström black hole of Einstein-Maxwell theory, some essential features can be understood.

The Reissner-Nordström metric

Perhaps the simplest example of a supersymmetric four-dimensional black hole is the extremal Reissner-Nordström (RN) metric. This is a solution of Einstein-Maxwell theory,

$$S = \int \sqrt{-g} d^4x \left(\frac{R}{2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{4.1}$$

The (non-extremal) Reissner-Nordström metric with mass M, electric charge q, and magnetic charge p is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{4.2}$$

where $\mathrm{d}\Omega^2$ is the line element on a two-sphere and $Q^2 = q^2 + p^2$. This metric is asymptotically flat for $r \to \infty$, while it reduces to the Schwarzschild metric for $Q = 0$. For $M > |Q|$, this metric has two horizons at $r_\pm$, where the norm of the Killing vector ∂_t changes sign,

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{4.3}$$

For $M < |Q|$, the horizons disappear, leaving a naked singularity. Due to the weak cosmic censorship, this is considered an unphysical solution, hence introducing the bound

$$M \geq Q. \tag{4.4}$$

In the context of this Chapter, the extremal limit $M = |Q|$ is the most interesting. Defining also $\rho = r - M$, the extremal metric can be written in isotropic coordinates (where the spatial coordinates are conformally flat),

$$\mathrm{d}s^2 = \left(1 + \frac{Q}{\rho}\right)^{-2}\mathrm{d}t^2 - \left(1 + \frac{Q}{\rho}\right)^{2}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2\right). \tag{4.5}$$

Taking the near-horizon limit, $\rho \to 0$, the metric becomes

$$\mathrm{d}s^2 = \frac{\rho^2}{M_{BR}^2}\mathrm{d}t^2 - \frac{M_{BR}^2}{\rho^2}\mathrm{d}\rho^2 - M_{BR}^2\,\mathrm{d}\Omega^2, \tag{4.6}$$

with $M_{BR} = Q = M$. Clearly, the metric factorizes as $AdS_2 \times S^2$, both with radius M_{BR}. This is known as the Bertotti-Robinson metric.

Supersymmetric solutions

As discussed earlier, an action is supersymmetric if it is invariant under supersymmetry transformations with fermionic parameters, ϵ^A. A solution of supergravity may or may not preserve supersymmetry. What we mean by this is whether the supersymmetry variations vanish,

$$\begin{aligned} \delta_\epsilon \psi_{A\mu} &= 0, \\ \delta_\epsilon \lambda^{iA} &= 0, \\ \delta_\epsilon \zeta_\alpha &= 0, \end{aligned} \tag{4.7}$$

implying that the solution is invariant under the transformations. Eqns. (4.7) are known as the Killing spinor equations and ϵ^A are the Killing spinors. Note that we only included the variations of the fermions above, the gravitinos, gauginos, and hyperinos of $\mathcal{N} = 2$

supergravity. As already noted in Section 3, it is consistent to truncate all fermions from the supersymmetry action and construct only bosonic solutions. Also, the variations of the bosons are linear combinations of the fermions. Hence, the Killing spinor equations of the bosons are trivially satisfied when truncating the fermions.

Since Einstein-Maxwell theory is easily embedded in $\mathcal{N} = 2$ minimal supergravity, it follows that the extremal RN metric is also a supergravity solution. Furthermore, it can be shown that the solution preserves half the supersymmetries. The extremal RN black hole can in fact be interpreted as a supersymmetric soliton [51, 88]. By a soliton, we mean a stationary, regular and stable solution of the equations of motion with finite energy (mass). The metric (4.5) is not only stationary, but in fact static, and it is regular in the sense that it does not exhibit a naked singularity. It is stable both as a classical solution, as well as thermodynamically, since extremal black holes do not emit semiclassical Hawking radiation. The interpretation as a supersymmetric soliton implies that the charge Q should be replaced by the central charge of the supersymmetry algebra, $|Z|$, given below in eqn. (4.28)[3]. The bound (4.4), which was necessary to avoid naked singularities, now follows from the BPS bound, eqn. (2.23),

$$M \geq |Z|. \tag{4.8}$$

The extremal RN solution saturates the BPS bound, $M = |Z|$. Non-renormalization theorems of supersymmetric theories guarantee that the BPS bound must hold beyond the perturbative regime. In this way extremal black holes provide information about non-perturbative string theory, where the supergravity approximation to string theory must break down. In Section 2.1 we discussed massive supermultiplets. In particular, for an $\mathcal{N} = 2$ multiplet saturating the BPS bound, half the supercharges act trivially. Likewise, the extremal RN black hole preserves half of the supersymmetries, as mentioned above. At the horizon, however, the Bertotti-Robinson geometry, $AdS_2 \times S^2$, preserves all the supercharges, as does Minkowski space at spatial infinity. Such spaces preserving the full supersymmetry can be considered as supersymmetric vacua. The extremal Reissner-Nordström solution is therefore said to interpolate between two vacua, which is typical for solitonic solutions.

4.2 First-order flow equations in $\mathcal{N} = 2$ supergravity

Supersymmetric flow equations were first derived for supersymmetric black holes in ungauged supergravity [57,89,90]. We first review this case and then move on to more general cases.

Ungauged supergravity

Consider the action of $\mathcal{N} = 2$ ungauged supergravity coupled to n_V vector multiplets,

$$S = \int \sqrt{-g}\, \mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} {}^{\star}F^{\Sigma\mu\nu} + g_{i\bar{\jmath}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{\jmath}} \right). \tag{4.9}$$

[3] In Section 2.1 we took the central charges to be real. We do not assumes this here.

We will make an ansatz for the metric of a static and spherically symmetric black hole. Since the action is ungauged, we will take the solution to be asymptotically flat. Such an ansatz is

$$\mathrm{d}s^2 = e^{2U(r)}\mathrm{d}t^2 - e^{-2U(r)}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right). \tag{4.10}$$

The metric has four Killing vectors,

$$\begin{aligned} K_1 &= \partial_t, \qquad & K_3 &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi, \\ K_2 &= \partial_\phi, \qquad & K_4 &= -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi, \end{aligned} \tag{4.11}$$

corresponding to stationarity and the $SO(3)$ isometry. This implies that the Lie derivative of the metric along any of the Killing vectors (4.11) vanishes,

$$\mathcal{L}_{K_n} g_{\mu\nu} = 0, \qquad n = 1, ..., 4. \tag{4.12}$$

We can impose these symmetries on the matter fields by demanding the Lie derivatives along the Killing vectors K_n to vanish. A calculation yields,

$$\mathcal{L}_{K_n} z^i = 0 \qquad \Rightarrow \qquad z^i = z^i(r), \tag{4.13}$$

$$\mathcal{L}_{K_n} F^\Lambda = 0 \qquad \Rightarrow \qquad F^\Lambda = F^\Lambda_{tr}(r)\,\mathrm{d}t\wedge\mathrm{d}r + \tilde{F}^\Lambda_{\theta\phi}(r)\,\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi. \tag{4.14}$$

Imposing the Bianchi identity, $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F^\Lambda_{\rho\sigma} = 0$, further restricts $\tilde{F}^\Lambda_{\theta\phi}$ to be constants, which are the magnetic charges, p^Λ. The fields strengths must then be of the form

$$F^\Lambda = F^\Lambda_{tr}(r)\,\mathrm{d}t\wedge\mathrm{d}r + p^\Lambda\,\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi. \tag{4.15}$$

Now, the equations of motion for the field strengths are

$$0 = \partial_\mu\left(\sqrt{-g}I_{\Lambda\Sigma}F^{\Sigma|\mu\nu} + \frac{1}{2}R_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Sigma_{\rho\sigma}\right). \tag{4.16}$$

Inserting (4.15) and taking the scalars to be functions of only r, the only non-trivial equation in (4.16) is for $\nu = t$,

$$0 = \partial_r\left(-e^{-2U}r^2 I_{\Lambda\Sigma}F_{rt}(r) + R_{\Lambda\Sigma}p^\Sigma\right). \tag{4.17}$$

Since the derivative of the parenthesis vanishes, the content of parenthesis must be equal to a constant. This is the electric charge q_Λ. By re-arranging, we can eliminate the function $F^\Lambda_{rt}(r)$ in terms of the charges, the scalar couplings, and the metric components,

$$F^\Lambda_{rt}(r) = \frac{e^{2U}}{r^2}I^{\Lambda\Sigma}\left(R_{\Sigma\Gamma}p^\Gamma - q_\Sigma\right). \tag{4.18}$$

We can now make the statements of eqn. (2.61) more precise. As in Section 2.5, we define

$${}^\star G^{\mu\nu}_\Lambda \equiv 2\frac{\partial\mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = I_{\Lambda\Sigma}F^{\Sigma|\mu\nu} + R_{\Lambda\Sigma}{}^\star F^{\Sigma|\mu\nu}. \tag{4.19}$$

The equations of motion for the gauge fields of the action (4.9), along with the Bianchi identities can be written

$$\nabla_\mu {}^\star G_\Lambda^{\mu\nu} = 0, \qquad \nabla_\mu {}^\star F^{\Lambda|\mu\nu} = 0. \tag{4.20}$$

It is now straightforward to calculate

$$p^\Lambda = \frac{1}{4\pi} \int_{S^2} F^\Lambda, \qquad\qquad q_\Lambda = \frac{1}{4\pi} \int_{S^2} G_\Lambda, \tag{4.21}$$

which we stated without further explanation in eqn. (2.61). While p^Λ and q_Λ appeared above as integration constants, eqn. (4.21) identifies them as the black hole charges.

The supersymmetry variations of the action (4.9) were given in eqns. (3.44)-(3.45). As described in Section 4.1, we obtain the Killing spinor equations by demanding the variations to vanish,

$$0 = \delta_\epsilon \psi_{\mu A} = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{\ ab} \gamma_{ab} \right) \epsilon_A + \frac{i}{2} \tilde{A}_\mu \epsilon_A + \frac{1}{2} T^-_{\mu\nu} \gamma^\nu \epsilon_{AB} \epsilon^B, \tag{4.22}$$

$$0 = \delta_\epsilon \lambda^{iA} = i \partial_\mu z^i \gamma^\mu \epsilon^A + \frac{1}{2} G^{-i}_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \epsilon_B. \tag{4.23}$$

If these equations are satisfied, the solution preserves supersymmetry. To derive the first-order flow equations, the solutions of the equations of motion for the gauge fields, eqns. (4.15) and (4.18), are inserted into the Killing spinor equations, (4.22)-(4.23). We take the Killing spinors to only depend on the radial coordinate,

$$\epsilon_A(r) = e^{f(r)} \chi_A, \qquad \chi_A = \text{constant}, \tag{4.24}$$

and we impose also a suitable projector on the constant spinor

$$\chi_A = i \frac{Z}{|Z|} \epsilon_{AB} \gamma_0 \chi^B. \tag{4.25}$$

Notice that the last equation implies that only half of the supercharges are independent. This corresponds to the earlier statement, that extremal black holes may preserve half the supersymmetry. Using also identities (3.30), the Clifford algebra, $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, and the chiral projector $\gamma^5 \chi = \pm \chi$, the Killing spinor equations can be brought to the form

$$U' = -\frac{e^U}{r^2} |Z|, \tag{4.26}$$

$$z'^i = -\frac{2e^U}{r^2} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} |Z|. \tag{4.27}$$

Primes denote derivatives w.r.t. r, and the central charge Z is defined by

$$Z = \langle Q, \mathcal{V} \rangle = L^\Lambda q_\Lambda - M_\Lambda p^\Lambda. \tag{4.28}$$

From this definition and from eqn. (3.28), it follows that Z is covariantly holomorphic, $D_{\bar{\imath}} Z = 0$. This was also used to bring the equations to the form (4.26)-(4.27).

Eqns. (4.26)-(4.27) are known as the first-order flow equations. Their usefulness resides in the fact that they are relatively simple first-order equations, whose solutions are guaranteed to be supersymmetric and to satisfy the gauge field equation of motion. The flow equations therefore provide a simpler way of obtaining supersymmetric solutions than the full equations of motion.

Flow equations in gauged supergravity

It is of interest to consider a generalization of the first-order flow equations to include gauged supergravity. This allows a broader class of solutions, such as black holes and black branes with AdS asymptotics. Indeed, such work has been done [59, 60]. We take here the approach of ref. [60]. For consistency with refs. [59,60], we take in this subsection the metric signature to be $(-+++)$[4]. Appendix A.1 includes some details for such a change of notation. (Also, we take the convention $(+++)$ as described in Appendix A.1).

To allow for solutions without asymptotic flatness, the metric contains the function $e^{2\psi(r)}$, rather than just r^2 as in eqn. (4.10),

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\left(\mathrm{d}r^2 + e^{2\psi(r)}\mathrm{d}\Omega^2\right). \tag{4.29}$$

Consider the action of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and with Fayet-Iliopoulos gauging,

$$S = \int \sqrt{-g}\,\mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{4}R_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}{}^{\star}F^{\Sigma\mu\nu} - g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu \bar{z}^{\bar{\jmath}} - V(z,\bar{z})\right). \tag{4.30}$$

Following [60], we allow for both electric, ξ_Λ, and magnetic gaugings, ξ^Λ. The action (4.30) can be reduced to an effective one-dimensional action. As above, the spacetime symmetries restrict the form of the matter-fields, eqns. (4.13)-(4.14), and the gauge field equation of motion can be solved in terms of the charges, p^Λ and q_Λ. This is inserted into the action (4.30). However, when inserting the equations of motion for the gauge fields into the action in order to use the charges q_Λ and p^Λ rather than the functions F^Λ_{rt}, one must perform a Legendre transformation for consistency. This implies adding a term $-q_\Lambda F^\Lambda_{rt}$, as shown in [59]. The resulting action is independent of t and ϕ, while the θ-dependence factorizes out. This yields the effective one-dimensional action,

$$\begin{aligned} S_{1d} &= \int \mathrm{d}r\Big\{e^{2\psi}\big[-U'' + 2\psi'' + (U'-\psi')^2 + 2\psi'^2 + g_{i\bar{\jmath}}z'^i\bar{z}'^{\bar{\jmath}} + e^{2U-4\psi}V_{BH} + e^{-2U}V\big] - 1\Big\} \\ &= \int \mathrm{d}r\Big\{e^{2\psi}\big[U'^2 - \psi'^2 + g_{i\bar{\jmath}}z'^i\bar{z}'^{\bar{\jmath}} + e^{2U-4\psi}V_{BH} + e^{-2U}V\big] - 1\Big\} + \int \mathrm{d}r\big[e^{2\psi}(2\psi' - U')\big]'. \end{aligned} \tag{4.31}$$

The so-called black hole potential is given by

$$V_{BH} \equiv -\tfrac{1}{2}Q^t\mathcal{M}Q, \tag{4.32}$$

[4] In particular, we perform this change of convention in order to avoid a sign issue later in Section 7.4.

where Q is a symplectic $(2n_V+2)$-vector of the charges, while $\mathcal{M}$ is constructed from the period matrix,

$$Q \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} (I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda{}^\Sigma \\ -(I^{-1}R)^\Lambda{}_\Sigma & I^{-1\,\Lambda\Sigma} \end{pmatrix}. \tag{4.33}$$

The black hole potential can also be written in terms of the central charge, $Z = \langle Q, \mathcal{V}\rangle$, such that

$$V_{BH} = |Z|^2 + g^{i\bar{\jmath}} D_i Z D_{\bar{\jmath}} \bar{Z}, \qquad \text{where} \quad D_i Z = \big(\partial_i + \tfrac{1}{2}\partial_i \mathcal{K}\big) Z. \tag{4.34}$$

The approach taken in [60], building on [86, 87, 91], is again to solve the Killing spinor equations for generic configurations preserving half the supersymmetries. Up to total derivatives, and a constraint given below, the one-dimensional action can then be written as a sum of squares, each of which is first order in derivatives of r. By demanding every such squared term to vanish, we ensure that a variation of the action also vanishes. Hence, this procedure yields first-order differential equations, whose solutions are guaranteed to solve both the equations of motion and the Killing spinor equations. Such a rewritting is sometimes called a BPS squaring.

To write the first-order flow equations in a compact form, we define first the metric component $A = \psi - U$. We also define the superpotential, $\mathcal{B}$, as

$$\mathcal{B} \equiv e^U |Z - i e^{2A} \mathcal{W}|, \tag{4.35}$$

where Z is the central charge, and

$$\mathcal{G} = (\xi^\Lambda, \xi_\Lambda), \qquad \mathcal{W} = \langle \mathcal{G}, \mathcal{V}\rangle = L^\Lambda \xi_\Lambda - F_\Lambda \xi^\Lambda. \tag{4.36}$$

Eqn. (4.36) was also given in eqn. (3.90), and is part of the scalar potential. Using these definitions, the flow equations can be written,

$$U' = -e^{-2(A+U)}(\mathcal{B} - \partial_A \mathcal{B}), \tag{4.37}$$
$$A' = e^{-2(A+U)}\mathcal{B}, \tag{4.38}$$
$$z'^i = -2e^{-2(A+U)} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} \mathcal{B}. \tag{4.39}$$

The constraint mentioned above, which is necessary in order to complete the BPS squaring is

$$\langle \mathcal{G}, Q\rangle = -1. \tag{4.40}$$

It is straightforward to verify that the flow equations (4.37) and (4.39) reduce to (4.26) and (4.27), if we take the gaugings to vanish, $\mathcal{G} = 0$, and take $e^{2(A+U)} = e^{2\psi} = r^2$. However, this is inconsistent with the constraint (4.40). As noted in [60], one can indeed take $\mathcal{G} = 0$ consistently, by rewriting the BPS squares, from which $e^{2\psi} = r^2$ then follows.

The BPS squaring can be performed in a similar way for a metric of the form

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}r^2 + e^{2A(r)}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big). \tag{4.41}$$

In this case, as noted in [59, 60], the symplectic constraint is instead

$$\langle \mathcal{G}, Q \rangle = 0, \tag{4.42}$$

in order to complete the squares. We note, that there is no inconsistency in taking $\mathcal{G} = 0$ in this case. Rather, this exactly solves the constraint (4.42). The flow equations then simplify,

$$U' = -e^{-2A-U}|Z|, \tag{4.43}$$

$$A' = e^{-2A-U}|Z|, \tag{4.44}$$

$$z'^i = -2e^{-2A-U} g^{i\bar{j}} \partial_{\bar{j}} |Z|. \tag{4.45}$$

In Chapter 7, we indeed apply eqns. (4.43)-(4.45).

Chapter 5

The gauge/gravity correspondence and Lifshitz holography

The gauge/gravity correspondence is the conjecture that a theory of quantum gravity theory on a spacetime is equivalent to a quantum field on the conformal boundary of the spacetime. Below, we will sketch the basics of AdS/CFT correspondence and discuss aspects of non-relativistic generalizations thereof.

5.1 The AdS/CFT correspondence

The AdS/CFT correspondence was the first example of a gauge/gravity duality, and it is the best studied case. It was conjectured in 1997 by Maldacena [12], based on ideas about the holographic principle by 't Hooft [92] and Susskind [93].

The Maldacena Conjecture

The conjecture arises by considering a stack of N $D3$-branes in string theory. In string theory, Dp-branes are p-dimensional extended objects, upon which the end points of open strings with Dirichlet boundary conditions are restricted to. It can be shown that the massless modes of open strings on N Dp-branes stacked at the same position in spacetime will give rise to a $U(N)$ gauge theory on the branes. The gauge theory further inherits supersymmetry from the superstrings. In the case of $D3$-branes, the gauge theory is four-dimensional $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills (SYM). In the infrared, the $U(1)$ subgroup of the gauge group $U(N) = SU(N) \times U(1)$ decouples, leaving $\mathcal{N} = 4$ $SU(N)$ SYM.

Now, 3-branes also arise as charged black brane solutions of ten-dimensional type IIB supergravity, the low-energy limit of type IIB string theory. The brane is charged under the four-form Ramond-Ramond-field or RR-field[5]. When the brane is extremal, it is sypersymmetric, preserving 16 of the 32 supercharges. The geometry in the near-horizon limit

[5] A q-form gauge field can in general be sourced by a $p = (q-1)$-brane, which is called an electric brane. In D-dimensional spacetime, one can also have magnetic $(D-q-3)$-brane solutions. In four-dimensional black holes (0-branes) may thus be charged both electrically and magnetically under the one-form gauge field, A_μ. Eleven-dimensional supergravity (M-theory) contains 2-brane and 5-brane solutions, charged

is then $AdS_5 \times S^5$. In 1995, Polchinski showed that Dp-branes, on which open strings end, are in fact charged under RR-fields, and also preserve 16 supercharges [95]. This indicated that the metric of an extremal p-brane in fact describes a Dirichlet-brane.

Maldacena then conjectured the duality, that type IIB string theory on a $AdS_5 \times S^5$ spacetime is dual to four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM on the conformal boundary of the AdS_5 space. The isometry group of AdS_5 is $SO(2,4)$. On the four-dimensional boundary, $SO(2,4)$ acts as the conformal group, and hence, the field theory on the boundary must be a conformal field theory (CFT). This is indeed the case for four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM. This theory is both classically and quantum mechanically conformal, with a vanishing β-function. Hence, this duality is known as the AdS/CFT correspondence.

Weak-strong duality

Now, the $AdS_5 \times S^5$ metric can be written in so-called Poincaré coordinates,

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}t^2 - \mathrm{d}x_i^2 - \mathrm{d}r^2}{r^2} \right) - L^2 \mathrm{d}\Omega_5^2, \qquad i = 1, 2, 3 \; , \tag{5.1}$$

where $\mathrm{d}\Omega_5^2$ is the metric on a 5-sphere. Thus, both the AdS_5 and the S^5 have radius L. In these coordinates, the conformal boundary is located at $r \to 0$. Since this metric was derived as a classical solution to (supersymmetric) Einstein gravity, we must take L to be much larger than the Planck and string scales.

Using the interpretation that the metric is the near-horizon geometry of an extremal $D3$-brane, the geometric quantity L can be expressed in terms of string theory parameters,

$$\frac{L^4}{\ell_s^4} = 4\pi g_s N = g_{YM}^2 N = \lambda. \tag{5.2}$$

Here, g_s is the string coupling constant, $g_{YM} = \sqrt{4\pi g_s}$ is the coupling constant of the SYM theory on the brane, and $\ell_s^2 = \alpha'$ sets the string tension $T = 1/(2\pi \ell_s^2)$. The parameter $\lambda = g_{YM}^2 N$ is known as the 't Hooft coupling. 't Hooft showed that in the limit $N \to \infty$ with λ fixed, one can simplify computations and do a perturbative expansion in $1/N$, known as a large-N expansion [96].

From (5.2) we see that in the 't Hooft limit of fixed λ with $N \to \infty$, the string coupling must go to zero, $g_s \to 0$. Hence, we can describe the field theory at large N using only tree diagrams on the string theory side of the duality.

Further, consider the limit of $N \to \infty$ with also $\lambda \to \infty$. Then also the coupling constant g_{YM} must be large, so the field theory is strongly coupled. Eqn. (5.2) then demands we take $\ell_s \to 0$, the classical limit of string theory. Thus, we find a duality between classical supergravity and a quantum field theory at strong coupling and large N. This is indeed a remarkable duality.

electrically and magnetically, respectively, under the 3-form gauge field. A nice review is [94].

The dictionary

Further details on how to map the two theories onto each other were subsequently given in [13, 14]. This has become known as the dictionary. We give here a rough overview, for more details see e.g. [97–100].

In a conformal field theory, primary fields or operators are classified by their transformation properties under dilatations,

$$x^\mu \to \lambda x^\mu, \qquad \mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(x), \tag{5.3}$$

where Δ is the conformal weight of the operator $\mathcal{O}(x)$ (and the parameter λ is not related to the 't Hooft coupling above). We can define the partition function for $\mathcal{O}(x)$ with sources $\phi_0(x)$,

$$Z_{CFT}[\phi_0] = \left\langle e^{\int \mathrm{d}^4x\, \phi_0(x)\mathcal{O}(x)} \right\rangle. \tag{5.4}$$

An interesting physical object is the n-point correlation function for $\mathcal{O}(x)$. This is obtained by functional derivatives,

$$\left\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \right\rangle = \frac{\delta}{\delta\phi_0(x_1)} \cdots \frac{\delta}{\delta\phi_0(x_n)} Z_{CFT}[\phi_0]\bigg|_{\phi_0=0}. \tag{5.5}$$

Let now $\phi(x,r)$ be a field in the five-dimensional AdS_5 spacetime, known as the bulk. Here, r is the radial coordinate of the metric (5.1), while x^μ with $\mu = 0, ..., 3$ correspond to the coordinates on the boundary when $r \to 0$. The prescription in AdS/CFT is to identity the boundary values of the bulk fields with the sources of the CFT,

$$\phi(x,r) \xrightarrow[r\to 0]{} f(r)\phi_0(x). \tag{5.6}$$

The function $f(r)$ is needed to ensure regularity of the boundary values. A bulk field, characterized by its mass and spin, is in this way dual to some boundary field, characterized by its spin and conformal weight. Further, the partition functions of the bulk and the boundary are identified,

$$Z_{CFT}[\phi_0] = \left\langle e^{\int \mathrm{d}^4x \phi_0(x)\mathcal{O}(x)} \right\rangle = Z_{AdS}[\phi(x,r)]\big|_{\phi(x,r)\to f(r)\phi_0(x)}. \tag{5.7}$$

In particular, in the strong coupling, large-N limit of the boundary theory, we can use classical supergravity in the bulk, as described above. The partition function in the bulk is then just the classical action,

$$Z_{AdS}[\phi(x,r)] = e^{iS_{AdS}[\phi(x,r)]}. \tag{5.8}$$

Thus, by combining the above equations, correlation functions in the boundary theory can be computed from the purely classical bulk action.

5.2 Non-relativistic holography

While the AdS/CFT case is a beautiful example, it is also a very restricted one. The boundary field theory has both $\mathcal{N} = 4$ supersymmetry and conformal invariance. Since the conjecture, there has been much research devoted to generalizing the concepts of AdS/CFT.

In recent years, there has been much research in the application of holography to condensed matter physics ([15–24] and refs. therein). In many condensed matter systems, one finds phase transitions governed by fixed points exhibiting anisotropic, non-relativistic scaling invariance. The gravity duals of such systems are known as Lifshitz and Schrödinger spacetimes. In the following, we will focus on Lifshitz spacetimes.

Lifshitz holography

In the AdS/CFT case, the field theory is conformal. It it thus invariant under dilatations,

$$t \to \lambda t, \qquad x^i \to \lambda x^i. \tag{5.9}$$

Following conventions, we take $i = 1, ..., d$, such that the boundary theory is $(d + 1)$-dimensional. The gravity dual is the AdS_{d+2} metric,

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}t^2 - \mathrm{d}x^i \mathrm{d}x^i - \mathrm{d}r^2}{r^2} \right), \qquad i = 1, ..., d. \tag{5.10}$$

This metric is also scale invariant,

$$t \to \lambda t, \qquad x^i \to \lambda x^i, \qquad r \to \lambda r. \tag{5.11}$$

In many condensed matter systems, however, there are phase transitions governed by fixed points exhibiting anisotropic scaling invariance,

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad z \neq 1. \tag{5.12}$$

This is known as a Lifshitz scaling. Since time and space are scaled differently, Lorentz invariance is broken. The parameter z is known as the dynamical critical exponent. An example of a toy model exhibiting such scale invariance with $d = 2$ and $z = 2$ is

$$S = \int \mathrm{d}^2 x \mathrm{d}t \left((\partial_t \phi)^2 - k(\nabla^2 \phi)^2 \right). \tag{5.13}$$

Such a theory arises at finite temperature critical points in the phase diagrams of known metals [101, 102] and strongly correlated electron systems [103–106]. Holography for such systems was first considered in [16].

Since Lorentz invariance is broken, the theory enjoys fewer symmetries than a conformal theory. The $(d + 2)$-dimensional gravity dual must exhibit the same symmetries, i.e. scale invariance, translational invariance in (t, x^i), spatial rotations in x^i, and P and T symmetry. The metric has the form

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}t^2}{r^{2z}} - \frac{\mathrm{d}x^i \mathrm{d}x^i + \mathrm{d}r^2}{r^2} \right), \quad i = 1, ..., d, \tag{5.14}$$

with $0 < r < \infty$. It is invariant under the scaling

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad r \to \lambda r. \tag{5.15}$$

The metric is often written in another form. Performing a simple coordinate transformation, $\tilde{r} = 1/r$, and then renaming $\tilde{r} \to r$, the Lifshitz metric takes the form,

$$\mathrm{d}s^2 = L^2 \left(r^{2z} \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{r^2} - r^2\, \mathrm{d}x^i \mathrm{d}x^i \right), \qquad i = 1, ..., d. \tag{5.16}$$

The Lifshitz scaling (5.15) becomes

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad r \to \frac{r}{\lambda}. \tag{5.17}$$

It was shown in [107] that bulk-matter violating the null energy condition (NEC) leads to causality violation in the boundary theory[6]. The NEC is the constraint on the energy-momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} n^\mu n^\nu \geq 0, \tag{5.18}$$

where n^μ is a generic null vector, i.e. $n^\mu n_\mu = 0$. Since on-shell $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$, we easily get an expression for the energy-momentum tensor in terms of the parameters in the metric (5.16). The NEC is satisfied for

$$z \geq 1. \tag{5.19}$$

Lifshitz solutions

The Ricci scalar associated to the metric is

$$R = -2\, \frac{z^2 + 2z + 3}{L^2}. \tag{5.20}$$

This is constant and negative for all values of z. Of course, this is familiar from the AdS case with $z = 1$, and indicates the need for a negative cosmological constant or a scalar potential with a negative minimum. However, in order to support the Lifshitz metric (5.16), a cosmological constant alone is not sufficient.

In [16], four-dimensional Lifshitz spacetimes were constructed in Einstein gravity coupled to a negative cosmological constant, a one-form A_1 and a two-form B_2 with a topological term:

$$S = \int \sqrt{-g}\, \mathrm{d}^4 x \left(\frac{R}{2} - \Lambda \right) - \frac{1}{2} \int \left(\frac{1}{e^2} F_2 \wedge {}^\star F_2 + F_3 \wedge {}^\star F_3 + c\, B_2 \wedge F_2 \right), \tag{5.21}$$

[6] Recall, without any restrictions, the Einstein equations allow solutions for metrics with unphysical matter or energy. The energy conditions are coordinate-invariant restrictions on the energy-momentum tensor. [108]

where $F_2 = \mathrm{d}A_1$ and $F_3 = \mathrm{d}B_2$. This action allows for Lifshitz solutions with $z \geq 1$. In [109], it was shown that Lifshitz spacetimes can be engineered with a cosmological constant and a timelike massive vector field, rather than a two-form.

While these models support Lifshitz spacetimes, they are phenomenological in the sense that they were constructed without any relationship to a UV finite theory, such as string theory. Attempts to embed Lifshitz solutions into string theory first led to no-go theorems [110,111]. String theory embedding, however, was first achieved for $z = 2$ in [112], then followed by [113–115], [77, 78]. All values of $z \geq 1$ were embedded in string theory in [116].

For example, in [78] it was demonstrated that a $(D + 1)$-dimensional gravity theory with quite general constraints on the matter content can be dimensionally reduced on a circle S^1 and will admit a D-dimensional $z = 2$ Lifshitz solution. The $(D+1)$-dimensional theory must contain a scalar potential $V(\phi)$ with a negative minimum $\partial V/\partial\phi^u(\phi_{ex}) = 0$, $V(\phi_{ex}) < 0$, such that it allows for an AdS_{D+1} vacuum. The theory must also contain a scalar axion. By this, we mean that after a suitable field redefinition of the scalar fields ϕ^u, the metric on the scalar target space is independent of the axion, ξ. Thus, a vector $n\,\partial/\partial\xi$ on the target space with constant n is a Killing vector. While the other scalar fields are assumed to be independent of the coordinate $\hat{x}$ of the compact S^1, the axion is taken to have the form

$$\xi(x, \hat{x}) = \xi(x) + n\hat{x}, \tag{5.22}$$

where x are the coordinates of the D-dimensional space. Such a field dependence in the compact dimension is known as a flux. In the effective D-dimensional action, the axion gets coupled to the Kaluza-Klein vector $\mathcal{A}_\mu$ via a covariant derivative,

$$\nabla_\mu\xi\nabla^\mu\xi = (\partial_\mu\xi - n\mathcal{A}_\mu)(\partial^\mu\xi - n\mathcal{A}^\mu)\,, \tag{5.23}$$

assuming the axion was uncharged in the $(D + 1)$-dimensional action. A coupling of the form (5.23) is known as a Stückelberg coupling, and effectively provides a mass term for the vector, $n^2\mathcal{A}_\mu\mathcal{A}^\mu$. In [78], a five-dimensional consistent truncation from type IIB supergravity was used for this procedure, thus embedding $z = 2$ Lifshitz in string theory.

In [77] $z = 2$ Lifshitz solutions which preserve supersymmetry were found in four-dimensional $\mathcal{N} = 2$ gauged supergravity. They were also embedded in string theory via consistent truncations. As will be noted, part of Chapter 6 in this thesis can be viewed as a generalization of part of [77], since hyperscaling violation is included. However, we will not attempt to solve the Killing spinor equations as in [77], rather we will investigate the full equations of motion.

Hyperscaling violation in Lifshitz holography

Gravity duals for theories with hyperscaling violation has received much attention recently [25, 26, 28–32, 117–126]. Roughly speaking, condensed matter systems in d spatial dimensions with hyperscaling violation exponent θ exhibit the thermodynamic scaling behaviour of a theory living in $d - \theta$ spatial dimensions [27].

In the bulk, hyperscaling violation is a violation of the scaling invariance, which otherwise characterizes AdS, Lifshitz and Schrödinger spacetimes. The Lifshitz metric with hyperscaling violation exponent θ can be written as

$$\mathrm{d}s^2 = L^2 r^{-2\theta/d}\left(r^{2z}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{r^2} - r^2\mathrm{d}x^i\mathrm{d}x^i\right), \quad i = 1, ..., d. \tag{5.24}$$

It is conformally equivalent to the Lifshitz metric (5.16), which is in fact a special case for $\theta = 0$. The metric (5.24) is hence also known as a Lifshitz-like metric. Under a scaling,

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad r \to \frac{r}{\lambda}, \tag{5.25}$$

the metric (5.24) is not invariant. Rather, it scales covariantly,

$$\mathrm{d}s^2 \;\to\; \lambda^{2\theta/d}\mathrm{d}s^2. \tag{5.26}$$

In the gauge/gravity correspondence, one associates an energy scale with the radial coordinate in the gravity dual. The metric (5.24) is not expected to be a good description of the boundary theory for all values of r as discussed in [25]. Therefore, the dual theory lives on a finite r slice, and there could be important corrections for $r \to \infty$ or for very large r. As in the Lifshitz case, the gravity theory should satisfy the null energy condition (NEC), eqn. (5.18), to yield a physically sensible dual field theory [25]. Using the metric (5.24), the constraints coming from the NEC become

$$(d-\theta)(d(z-1)-\theta) \geq 0, \tag{5.27}$$
$$(z-1)(d+z-\theta) \geq 0. \tag{5.28}$$

For example, in the scale invariant case, $\theta = 0$, the constraint is $z \geq 1$, familiar from eqn. (5.19). The range $z < 1$ is allowed by the NEC, however, if $\theta \neq 0$. For $z = 1$, the NEC implies $\theta \leq 0$ or $\theta \geq d$, which both have string theory realizations in [25]. Other possible solutions are e.g. $0 < z < 1$ with $\theta \geq d + z$. It is argued in [25], however, that $\theta > d$ leads to instabilities in the gravity side, even though this range is allowed by the NEC.
The case $\theta = d - 1$ is emphasized in [25, 26] to be dual to particularly interesting field theories. In this case, the NEC requires

$$z \geq 2 - 1/d. \tag{5.29}$$

Metrics of the form (5.24) are solutions to Einstein-Maxwell-dilaton actions [117, 121, 122, 127],

$$S = \int \sqrt{-g}\,\mathrm{d}^{d+2}x\left(\frac{R}{2} - \frac{e^{\alpha\phi}}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_0 e^{\beta\phi}\right). \tag{5.30}$$

There are also top-down constructions from the near-horizon geometry of D-branes in type II supergravity, which hence are embedded into string theory. Solutions with $z \neq 1$ and $\theta \neq 0$ have been found in [25, 28–30, 32, 128], and e.g. a configuration with $z = 2$ and $\theta = 1$ for $d = 2$ was found in [31].

The calculations in Chapters 6 and 7 of this thesis are thus to be seen in this context. In this thesis, we investigate solutions of the form (5.24) in four-dimensional $\mathcal{N} = 2$ gauged supergravity. While this theory has a rich structure, the form of e.g. the potential is however restricted, whereas such models as (5.30) are constructed exactly for the purpose of finding such solutions.

Chapter 6

Four-dimensional Lifshitz spacetimes with hyperscaling violation in $\mathcal{N}=2$ supergravity

In this Chapter we investigate the equations of motion of four-dimensional $\mathcal{N}=2$ gauged supergravity in Lifshitz-like backgrounds with dynamical exponent z and hyperscaling violation exponent θ. As discussed in Chapter 5, the spacetime metric can be written as

$$\mathrm{d}s^2 = L^2 r^{-\theta}\left(r^{2z}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{r^2} - r^2(\mathrm{d}x^2+\mathrm{d}y^2)\right). \tag{6.1}$$

The structure of $\mathcal{N}=2$ gauged supergravity was discussed in Chapter 3. We consider here an action of the form

$$S = \int\sqrt{-g}\mathrm{d}^4x\left(\frac{R}{2}+\frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma|\mu\nu}+\frac{1}{4}R_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}{}^{\star}F^{\Sigma|\mu\nu}+g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu\bar{z}^{\bar{\jmath}}+h_{uv}\nabla_\mu q^u\nabla^\mu q^v-g^2V\right), \tag{6.2}$$

where $\nabla_\mu q^u = \partial_\mu q^u + g k^u_\Lambda A^\Lambda_\mu$. This action includes n_V vector multiplets and n_H Abelian gauged hypermultiplets[7]. The potential is given by eqn. (3.74),

$$V(z,\bar{z},q) = 4h_{uv}k^u_\Lambda k^v_\Sigma \bar{L}^\Lambda L^\Sigma + \left(g^{i\bar{\jmath}}f^\Lambda_i \bar{f}^\Sigma_{\bar{\jmath}} - 3\bar{L}^\Lambda L^\Sigma\right)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma. \tag{6.3}$$

Recall, that $k^u_\Lambda(q)$ are Killing vectors on the quaternionic target space, and $\mathcal{P}^x_\Lambda(q)$ are the Killing prepotentials. $L^\Lambda(z)$ are sections on the special manifold, and $f^\Lambda_i(z) \equiv (\partial_i + \frac{1}{2}\partial_i\mathcal{K})L^\Lambda$.

We keep in mind that the action (6.2) can be reduced to the case of only vector multiplets with Fayet-Iliopoulos gaugings. This corresponds to $k^u_\Lambda \to 0$, $q^u \to 0$, with $\mathcal{P}^x_\Lambda \to \xi_\Lambda$. We will indeed consider this case later on.

[7] We will use z for the dynamical exponent and z^i with $i=1,...,n_V$ for the complex vector scalars. The difference should be clear from the context. E.g. for the scalar potential $V(z,\bar{z})$, z refers to the scalar dependence, not the dynamical exponent.

Vacuum solutions

In order to consider the equations of motion, we need of course to calculate the Ricci tensor from the metric (6.1). The non-vanishing components of the Ricci tensor are

$$\begin{aligned} R_{tt} &= r^{2z}\,\frac{(2z-\theta)(\theta-z-2)}{2}, \\ R_{rr} &= \frac{2z^2-\theta z-2\theta+4}{2r^2}, \\ R_{xx} = R_{yy} &= r^2\,\frac{(2-\theta)(2+z-\theta)}{2}. \end{aligned} \tag{6.4}$$

From these, the Ricci scalar also follows,

$$R = g^{\mu\nu}R_{\mu\nu} = -\frac{r^\theta}{L^2}\Big(2z^2+z(4-3\theta)+\frac{3}{2}(\theta-2)^2\Big). \tag{6.5}$$

Before looking for more complicated solutions supported by matter-fields, one may consider the vacuum Einstein equations,

$$R_{\mu\nu} = 0. \tag{6.6}$$

These equations have three solutions for particular values of z and θ,

$$(z,\theta) = \big\{(0,2),(1,2),(4,6)\big\}. \tag{6.7}$$

For the sake of interest, these can be found in Appendix D for arbitrary spacetime dimension. The solutions $(z,\theta) = (0,2),(1,2)$, have vanishing Riemann tensor, $R^\mu{}_{\nu\rho\sigma} = 0$, and hence correspond to just flat spacetime. For example, for $z = 1$ and $\theta = 2$ the metric (6.1) becomes

$$\mathrm{d}s^2 = L^2\left(\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{r^4} - \mathrm{d}x^2 - \mathrm{d}y^2\right). \tag{6.8}$$

By a coordinate transformation $\tilde{r} = 1/r$ and a rescaling of the coordinates to absorb L, this metric becomes just the canonical Minkowski metric. For $z = 0$ and $\theta = 2$, one can likewise recover the Minkowski metric with $\tilde{t} = r^{-1}\sinh t$ and $\tilde{r} = r^{-1}\cosh t$. The solution with $z = 4$ and $\theta = 6$, however, has non-vanishing Riemann tensor,

$$R^t{}_{xtx} = R^t{}_{yty} = -2r^2, \qquad R^r{}_{xrx} = R^r{}_{yry} = -2r^2, \qquad R^t{}_{rtr} = \frac{4}{r^2}, \qquad R^x{}_{yxy} = 4r^2\,. \tag{6.9}$$

Thus, this is a curved spacetime. Such a solution does not seem to have been noted in the literature on Lifshitz-like solutions until ref. [33] appeared while this thesis was in the final stages of preparation.

6.1 Field ansätze

In order to simplify the later calculations, it is worthwhile to consider the symmetries of the metric (6.1).

Killing vectors

The metric has four Killing vectors,

$$\partial_t, \qquad \partial_x, \qquad \partial_y, \qquad y\partial_x - x\partial_y. \tag{6.10}$$

These correspond to translations in t, x, y, as well as $SO(2)$ rotations in the x, y plane. Recall that the Lie derivative of the metric vanishes along the Killing vectors,

$$\mathcal{L}_K\, g_{\mu\nu} = 0, \tag{6.11}$$

where K is any of the Killing vectors (6.10).

Under the anisotropic scaling,

$$t \to \lambda^z t, \qquad r \to \lambda^{-1} r, \qquad x_i \to \lambda x_i, \tag{6.12}$$

the metric (6.1) scales covariantly,

$$\mathrm{d}s^2 \to \lambda^\theta \mathrm{d}s^2. \tag{6.13}$$

The scaling (6.12) is generated by the conformal Killing vector C,

$$C = zt\,\partial_t - r\,\partial_r + x\,\partial_x + y\,\partial_y. \tag{6.14}$$

In general, the Lie derivative of the metric along a conformal Killing vector yields the metric itself multiplied by a constant or a spacetime-dependent function. In our case, one finds the Lie derivative of the metric along C to be

$$\mathcal{L}_C\, g_{\mu\nu} = \theta g_{\mu\nu}, \tag{6.15}$$

i.e. the metric multiplied by a constant.

Gauge fields

In order to simplify the equations of motion, we will impose the spacetime symmetries on the matter fields. We therefore demand the Lie derivative of the field strengths along the Killing vectors to vanish,

$$\mathcal{L}_K F^\Lambda_{\mu\nu} = 0. \tag{6.16}$$

After a short calculation, one finds that the most general solution for F^Λ is

$$F^\Lambda = F^\Lambda_{tr}(r)\,\mathrm{d}t \wedge \mathrm{d}r + F^\Lambda_{xy}(r)\,\mathrm{d}x \wedge \mathrm{d}y. \tag{6.17}$$

Imposing further the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F^\Lambda_{\rho\sigma} = 0$ restricts F^Λ_{xy} to be constants. As in Section 4.2, we identify these as the magnetic charges and denote them as p^Λ. Also, we denote $F_{rt}(r) = E'^\Lambda(r)$, hence

$$F^\Lambda = E'^\Lambda(r)\,\mathrm{d}r \wedge \mathrm{d}t + p^\Lambda\,\mathrm{d}x \wedge \mathrm{d}y. \tag{6.18}$$

Under the anisotropic scaling (6.12), the components of the field strengths scale as

$$E'^\Lambda(r)\,\mathrm{d}r \wedge \mathrm{d}t \quad\longrightarrow\quad \lambda^{z-1}E'^\Lambda(\lambda^{-1}r)\,\mathrm{d}t \wedge \mathrm{d}r, \tag{6.19}$$

$$p^\Lambda\,\mathrm{d}x \wedge \mathrm{d}y \quad\longrightarrow\quad \lambda^2 p^\Lambda\,\mathrm{d}x \wedge \mathrm{d}y. \tag{6.20}$$

We will demand that the field strengths scale covariantly,

$$F^\Lambda \quad\longrightarrow\quad \lambda^{\alpha_\Lambda}F^\Lambda \qquad \text{(no sum over }\Lambda), \tag{6.21}$$

where α_Λ are constants, associated to each field. Since the xy-component scales with λ^2 if $p^\Lambda \neq 0$, so must the rt-component. From (6.19),

$$\lambda^{z-1}E'^\Lambda(\lambda^{-1}r) = \lambda^2 E'(r). \tag{6.22}$$

To satisfy this, we take $E'^\Lambda(r) = e^\Lambda\, r^{z-3}$, where e^Λ are constants. However, if $p^\Lambda = 0$ for some of the field strengths, then $E'^\Lambda(r) = e^\Lambda\, r^{\beta_\Lambda}$ will scale covariantly for any β_Λ. We will therefore take the ansatz (no sum over Λ)

$$F^\Lambda = e^\Lambda r^{\beta_\Lambda}\,\mathrm{d}r \wedge \mathrm{d}t + p^\Lambda\,\mathrm{d}x \wedge \mathrm{d}y, \qquad\qquad \beta_\Lambda = z-3 \quad \text{if} \quad p^\Lambda \neq 0. \tag{6.23}$$

For the sake of overview, we will just write $E'^\Lambda(r)$ below and then keep in mind that $E'^\Lambda(r) = e^\Lambda r^{\beta_\Lambda}$.

Let us mention that a similar ansatz can be obtained by considering the Lie derivative of F^Λ along the conformal Killing vector (6.14). This is essentially the same argument as above, but on an infinitesimal form. Analogous to eqn. (6.15), we demand

$$\mathcal{L}_C F^\Lambda = \tilde{\alpha}^\Lambda F^\Lambda, \tag{6.24}$$

for some constants $\tilde{\alpha}^\Lambda$. If $F^\Lambda_{xy} = p^\Lambda$ is non-zero, this fixes again $\tilde{\alpha}^\Lambda$, since a calculation yields

$$\tilde{\alpha}^\Lambda F^\Lambda_{xy} \;=\; \mathcal{L}_C F^\Lambda_{xy} \;=\; 2F^\Lambda_{xy}. \tag{6.25}$$

For the rt-component, a calculations yields

$$\tilde{\alpha}^\Lambda F^\Lambda_{rt} \;=\; \mathcal{L}_C F^\Lambda_{rt} \;=\; (z-1)F^\Lambda_{rt} - r\partial_r F^\Lambda_{rt} \qquad\Rightarrow\qquad F^\Lambda_{rt} = e^\Lambda r^{z-1-\tilde{\alpha}_\Lambda}. \tag{6.26}$$

Hence, we find again $E'(r)^\Lambda = e^\Lambda r^{z-3}$ if $p \neq 0$, while for $p^\Lambda = 0$ there is no constraint on $\tilde{\alpha}_\Lambda$. This coincides with (6.23).

Another observation is that some gauge fields, A^Λ_μ, may appear in the action not only through their field strengths, but also in the covariant derivative, $\nabla_\mu q^u = \partial_\mu q^u + k^u_\Lambda A^\Lambda_\mu$. Up to gauge transformations, a gauge field corresponding to (6.23) is

$$A^\Lambda = E(r)\,\mathrm{d}t \,+\, xp^\Lambda\,\mathrm{d}y. \tag{6.27}$$

The component $A^\Lambda_y = xp^\Lambda$ violates the symmetries of the problem if $p^\Lambda \neq 0$, in the sense that $\mathcal{L}_{\partial_x} A^\Lambda_y = p^\Lambda \neq 0$. Thus, for the gauge fields that are gauging a scalar isometry, p^Λ must vanish. We will enforce this by taking $k^u_\Lambda\, p^\Lambda = 0$, and then remember that such p^Λ must also vanish elsewhere in the equations of motion.

Scalar fields

Demanding vanishing Lie derivatives of the scalar fields along the Killing vectors (6.10) restricts the fields to

$$z^i = z^i(r), \qquad q^u = q^u(r). \tag{6.28}$$

Using the conformal Killing vector, we could now try to restrict the form even further, analogous to (6.24). For example,

$$\alpha^i z^i = \mathcal{L}_C z^i(r) = -r\partial_r z^i \quad \Rightarrow \quad z^i(r) = z_0^i\, r^{-\alpha^i}, \tag{6.29}$$

where α^i and z_0^i are constants. However, for the scalar fields, we can do field redefinitions which are diffeomorphisms of the scalar target space. For example, a non-linear sigma model with a single complex scalar $\tau = \phi + i\chi$ could be

$$g_{\tau\bar\tau}\partial_\mu\tau\partial^\mu\bar\tau = \frac{\partial_\mu\phi\partial^\mu\phi + \partial_\mu\chi\partial^\mu\chi}{\phi^2}, \qquad \text{with } \phi > 0. \tag{6.30}$$

By a field redefinition $\phi = \tilde\phi\sin\tilde\chi$ and $\chi = \tilde\phi\cos\tilde\chi$, the non-linear sigma model becomes

$$\frac{1}{\sin^2\tilde\chi}\left(\frac{\partial_\mu\tilde\phi\partial^\mu\tilde\phi}{\tilde\phi^2} + \partial_\mu\tilde\chi\partial^\mu\tilde\chi\right), \qquad \text{with } 0 < \tilde\chi < \pi \ \text{ and } \ \tilde\phi > 0. \tag{6.31}$$

We can choose either set of coordinates on the target space. Thus, if we choose e.g.

$$\phi(r) = \phi_0\, r^\alpha, \qquad \chi(r) = \chi_0\, r^\beta, \tag{6.32}$$

the same form of r-dependence does not hold in the $(\tilde\phi, \tilde\chi)$-coordinates. We will therefore use the ansätze (6.28).

6.2 Equations of motion

A variation of the action w.r.t. the inverse metric yields the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \quad \Leftrightarrow \quad R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T^\rho{}_\rho\, g_{\mu\nu}. \tag{6.33}$$

For the energy momentum tensor, one finds

$$\begin{aligned} T_{\mu\nu} =& g_{\mu\nu}\left(\frac{1}{4}I_{\Lambda\Sigma}F^\Lambda_{\rho\sigma}F^{\Sigma|\rho\sigma} + g_{i\bar\jmath}\partial_\rho z^i\partial^\rho\bar z^{\bar\jmath} + h_{uv}\nabla_\rho q^u\nabla^\rho q^v - g^2 V\right) \\ &- 2\left(\frac{1}{2}I_{\Lambda\Sigma}F^\Lambda_{\mu\rho}F^{\Sigma\rho}_\nu + g_{i\bar\jmath}\partial_\mu z^i\partial_\nu\bar z^{\bar\jmath} + h_{uv}\nabla_\mu q^u\nabla_\nu q^v\right). \end{aligned} \tag{6.34}$$

Variations w.r.t z^i, q^u, and A^Λ yield,

$$0 = \frac{1}{4}\partial_i I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{4}\partial_i R_{\Lambda\Sigma} F^\Lambda_{\mu\nu} {}^\star F^{\Sigma|\mu\nu} + \partial_i g_{j\bar{k}} \partial_\mu z^j \partial^\mu \bar{z}^{\bar{k}} - g^2 \partial_i V - \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g_{i\bar{\jmath}} \partial^\mu \bar{z}^{\bar{\jmath}}\right), \tag{6.35}$$

$$0 = \partial_u \left(h_{vw} \nabla_\mu q^v \nabla^\mu q^w\right) - g^2 \partial_u V - \frac{2}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} h_{uv} \nabla^\mu q^v\right), \tag{6.36}$$

$$0 = 2h_{uv} k^v_\Lambda \partial^\mu q^u + 2h_{uv} k^u_\Lambda k^v_\Sigma A^{\Sigma\mu} - \frac{1}{\sqrt{-g}} \partial_\nu \left[\sqrt{-g}\left(I_{\Lambda\Sigma} F^{\Sigma|\nu\mu} + R_{\Lambda\Sigma} {}^\star F^{\Sigma|\nu\mu}\right)\right], \tag{6.37}$$

respectively, where $\partial_i \equiv \partial/\partial z^i$ and $\partial_u \equiv \partial/\partial q^u$.

Inserting now the ansätze (6.23), (6.28), and the metric, the equations become more concrete, yet still involved. The Einstein equations become

$$(\theta - 2z)(\theta - z - 2) = -\mathcal{E} - 2g^2 L^2 V r^{-\theta} + 4g^2 h_{uv} k^u_\Lambda k^v_\Sigma E^\Lambda E^\Sigma r^{-2z}, \tag{6.38}$$

$$2z^2 - \theta z - 2\theta + 4 = -\mathcal{E} - 2g^2 L^2 V r^{-\theta} - 4g_{i\bar{\jmath}} z'^i \bar{z}'^{\bar{\jmath}} r^2 - 4h_{uv} q'^u q'^v r^2, \tag{6.39}$$

$$(\theta - 2)(2 + z - \theta) = -\mathcal{E} + 2g^2 L^2 V r^{-\theta}, \tag{6.40}$$

$$0 = g h_{uv} q'^u k^v_\Lambda E^\Lambda. \tag{6.41}$$

Primes denote derivatives w.r.t. r, and we defined also

$$\mathcal{E} \equiv \frac{1}{L^2} I_{\Lambda\Sigma} E'^\Lambda E'^\Sigma r^{2-2z+\theta} + \frac{1}{L^2} I_{\Lambda\Sigma} p^\Lambda p^\Sigma r^{\theta-4}. \tag{6.42}$$

The equation of motion for the vector scalars becomes

$$\begin{aligned} 0 = {} & \partial_i I_{\Lambda\Sigma} E'^\Lambda E'^\Sigma r^{2-2z} - \partial_i I_{\Lambda\Sigma} p^\Lambda p^\Sigma r^{-4} - 2\partial_i R_{\Lambda\Sigma} E'^\Lambda p^\Sigma r^{-z-1} \\ & + 2g^2 L^4 \partial_i V r^{-2\theta} + 2L^2 \partial_i g_{j\bar{k}} z'^j \bar{z}'^{\bar{k}} r^{2-\theta} - 2L^2 r^{-z-1} \partial_r (r^{z+3-\theta} g_{i\bar{\jmath}} \bar{z}'^{\bar{\jmath}}). \end{aligned} \tag{6.43}$$

The equation for the hyperscalars becomes

$$\begin{aligned} 0 = {} & g^2 L^2 \partial_u V r^{-\theta} - g^2 \partial_u (h_{vw} k^v_\Lambda k^w_\Sigma) E^\Lambda E^\Sigma r^{-2z} \\ & + \partial_u (h_{vw}) q'^v q'^w r^2 - 2r^{\theta-z-1} \partial_r (h_{uv} q'^v r^{z+3-\theta}). \end{aligned} \tag{6.44}$$

Finally, the gauge field equations become

$$0 = \partial_r \left(I_{\Lambda\Sigma} E'^\Sigma r^{3-z} - R_{\Lambda\Sigma} p^\Sigma\right) + 2g^2 L^2 h_{uv} k^u_\Lambda k^v_\Sigma E^\Sigma r^{1-z-\theta}, \tag{6.45}$$

$$0 = g h_{uv} q'^u k^v_\Lambda. \tag{6.46}$$

Eqn. (6.46) clearly implies the Einstein equation (6.41). The above equations are obviously quite involved. Recall that not only E'^Λ is a function of r. The period matrix, the potential, the Killing vectors, and the scalar metrics all depend on the scalar fields, and hence in general have implicit r-dependence. An issue is that an equation may split into more equations, depending on the r-dependence, since we are interested in solutions valid for all values of r. A simple example of this is

$$0 = a + b r^c, \tag{6.47}$$

where a, b, c are constants. If $c = 0$, then $a = -b$. However, when $c \neq 0$, one must take $a = 0$ and $b = 0$ separately.

6.3 Constant scalar fields

As a simplifying case, we can consider the scalars as constants. In this case, the scalar-dependent quantities are also constants w.r.t. r. The equations of motion, eqns. (6.38)-(6.46), then simplify somewhat, and after re-arranging the Einstein equations, we have

$$\tfrac{1}{2}\theta^2 - \theta - z^2 + z = \frac{1}{L^2} I_{\Lambda\Sigma} E'^{\Lambda} E'^{\Sigma} r^{2-2z+\theta} + \frac{1}{L^2} I_{\Lambda\Sigma} p^{\Lambda} p^{\Sigma} r^{\theta-4}, \tag{6.48}$$

$$-\tfrac{1}{2}\theta^2 + 3\theta - z^2 - z + \theta z - 4 = 2g^2 L^2 V r^{-\theta}, \tag{6.49}$$

$$\theta^2 - 2\theta z + 4z - 4 = 4g^2 h_{uv} k^u_{\Lambda} k^v_{\Sigma} E^{\Lambda} E^{\Sigma} r^{-2z}, \tag{6.50}$$

$$\begin{aligned} 0 &= \partial_i I_{\Lambda\Sigma} E'^{\Lambda} E'^{\Sigma} r^{2-2z} - \partial_i I_{\Lambda\Sigma} p^{\Lambda} p^{\Sigma} r^{-4} \\ &\quad - 2\partial_i R_{\Lambda\Sigma} E'^{\Lambda} p^{\Sigma} r^{-z-1} + 2g^2 L^4 \partial_i V r^{-2\theta}, \end{aligned} \tag{6.51}$$

$$0 = g^2 L^2 \partial_u V r^{-\theta} - g^2 \partial_u (h_{vw} k^v_{\Lambda} k^w_{\Sigma}) E^{\Lambda} E^{\Sigma} r^{-2z}, \tag{6.52}$$

$$0 = \partial_r \left(I_{\Lambda\Sigma} E'^{\Sigma} r^{3-z} - R_{\Lambda\Sigma} p^{\Sigma} \right) + 2g^2 L^2 h_{uv} k^u_{\Lambda} k^v_{\Sigma} E^{\Sigma} r^{1-z-\theta}. \tag{6.53}$$

Lifshitz metrics, $\theta = 0$

Consider the special case of the Lifshitz metric with $\theta = 0$. In this case, the anisotropic scaling (6.12) is an isometry of the metric. The vector C, eqn. (6.14), is then a true Killing vector, rather than a conformal Killing vector. Imposing the scaling invariance on the fields, the scalars must be constants. For the field strengths, it is evident from the finite scaling, eqn. (6.20), that the xy-component is not scale invariant. Hence, imposing the symmetry restricts $p^{\Lambda} = 0$. From eqn. (6.19), it follows that we must take $E^{\Lambda} = e^{\Lambda} r^z$.

Again, same conclusions follow from the Lie derivatives of the fields, in fact more elegantly. Since C is a true Killing vector, the Lie derivatives along C must vanish. A short calculation yields

$$\begin{aligned} 0 &= \mathcal{L}_C z^i(r) \quad \Rightarrow \quad \partial_r z^i(r) = 0, \\ 0 &= \mathcal{L}_C F^{\Lambda}_{\mu\nu}(r) \quad \Rightarrow \quad A^{\Lambda} = E^{\Lambda}\,\mathrm{d}t = e^{\Lambda} r^z\,\mathrm{d}t, \qquad p^{\Lambda} = 0. \end{aligned} \tag{6.54}$$

With these ansätze and $\theta = 0$, the equations of motion simplify. The Einstein equations (6.48)-(6.50) take the form

$$I_{\Lambda\Sigma} e^{\Lambda} e^{\Sigma} = \frac{1-z}{z} L^2, \tag{6.55}$$

$$g^2 V = -\frac{z^2 + z + 4}{2L^2}, \tag{6.56}$$

$$g^2 h_{uv} k^u_{\Lambda} k^v_{\Sigma} e^{\Lambda} e^{\Sigma} = z - 1. \tag{6.57}$$

The matter-field equations become

$$0 = z^2 \partial_i I_{\Lambda\Sigma} e^{\Lambda} e^{\Sigma} + 2g^2 L^4 \partial_i V, \tag{6.58}$$

$$0 = g^2 L^2 \partial_u V - g^2 \partial_u (h_{vw} k^v_{\Lambda} k^w_{\Sigma}) e^{\Lambda} e^{\Sigma}, \tag{6.59}$$

$$0 = z I_{\Lambda\Sigma} e^{\Sigma} + g^2 L^2 h_{uv} k^u_{\Lambda} k^v_{\Sigma} e^{\Sigma}. \tag{6.60}$$

Eqns. (6.55)-(6.60) exactly coincides with ref. [77][8], where Lifshitz spacetimes with dynamical exponent $z = 2$ and preserved supersymmetry were found in $\mathcal{N} = 2$ gauged supergravity. Note that the r-dependence completely drops out in eqns. (6.55)-(6.60). This will not be the case in general, when hyperscaling violation is included below.

Several lessons can be learned about pure Lifshitz solutions. The r.h.s. of eqn. (6.55) is negative for all $z > 1$. This is consistent with $I_{\Lambda\Sigma}$ being negative definite, as noted in Section 3.2. The r.h.s. of eqn. (6.56) is negative for all values of z. This requires a negative scalar potential, $V(z, \bar{z}, q) < 0$. Thus, ungauged supergravity does not admit Lifshitz solutions. Eqn. (6.57) requires gauged hyperscalars for interesting Lifshitz solutions with $z \neq 1$. For example, consider supergravity with Fayet-Iliopoulos gaugings. In this case, there is a scalar potential but no hyperscalars, $h_{uv}k^u_\Lambda k^v_\Sigma \to 0$. Taking $g \to 1$, for convenience, eqns. (6.55)-(6.57) easily yield:

$$z = 1, \qquad V = -\frac{3}{L^2}, \qquad I_{\Lambda\Sigma}e^\Lambda e^\Sigma = 0. \tag{6.61}$$

Of course, a Lifshitz spacetime with $z = 1$ is just familar AdS_4. Taking $e^\Lambda = 0$, i.e. no gauge fields, the action effectively consists only of the Einstein-Hilbert term and the scalar potential, which plays the role of a cosmological constant. However, since the potential is actually scalar dependent, eqn. (6.58) requires that the scalar fields extremize the potential,

$$\partial_i V(z, \bar{z}, q) = 0. \tag{6.62}$$

In Section 5.2 it was noted that Lifshitz solutions can be constructed with a massive vector field and a cosmological constant. From the above discussion, we see that $\mathcal{N} = 2$ supergravity with gauged scalars may also admit Lifshitz solutions. The potential plays the role of the cosmological constant, while the gauging of the constant scalars may effectively give mass to the gauge fields via a Stückelberg coupling.

Lifshitz-like metrics, $\theta \neq 0$

Consider again the Einstein equations with constant scalar fields,

$$\tfrac{1}{2}\theta^2 - \theta - z^2 + z = \frac{1}{L^2}I_{\Lambda\Sigma}E'^\Lambda E'^\Sigma r^{2-2z+\theta} + \frac{1}{L^2}I_{\Lambda\Sigma}p^\Lambda p^\Sigma r^{\theta-4}, \tag{6.63}$$

$$-\tfrac{1}{2}\theta^2 + 3\theta - z^2 - z + \theta z - 4 = 2g^2L^2Vr^{-\theta}, \tag{6.64}$$

$$\theta^2 - 2\theta z + 4z - 4 = 4g^2h_{uv}k^u_\Lambda k^v_\Sigma E^\Lambda E^\Sigma r^{-2z}. \tag{6.65}$$

The l.h.s. of eqn. (6.64) is a constant on spacetime, while the r.h.s. scales as $r^{-\theta}$. For $\theta \neq 0$, both sides must vanish independently, since we are interested in solutions valid for all values of r. Thus, in a gauged theory, the constant scalars must take values, z_0, q_0, such that the potential vanishes,

$$V(z_0, \bar{z}_0, q_0) = 0. \tag{6.66}$$

[8] Except for typos in the "effective potential" in [77], to which Nick Halmagyi has agreed in a private correspondence.

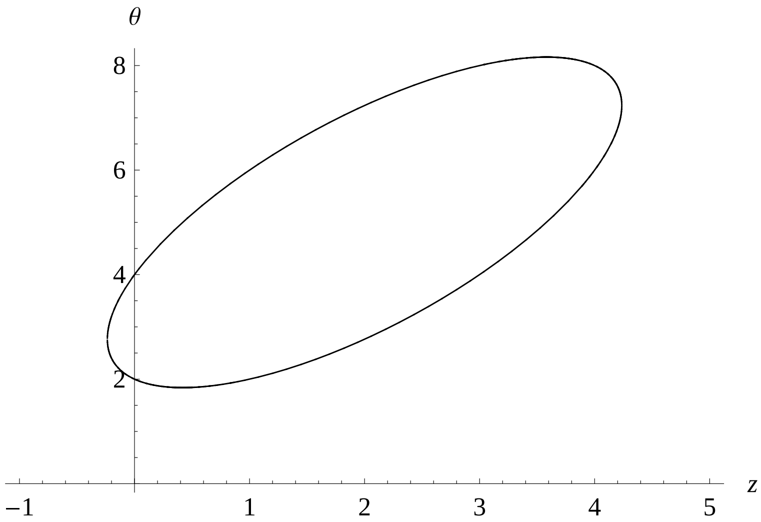

Figure 6.1: Values of z and $\theta \neq 0$ admitted by eqn. (6.67).

The l.h.s. of eqn. (6.64) vanishes for

$$\theta = 3 + z \pm \sqrt{1 + 4z - z^2}. \tag{6.67}$$

This constrains z and θ,

$$2 - \sqrt{5} \ \leq \ z \ \leq \ 2 + \sqrt{5} \qquad \text{and} \qquad 5 - \sqrt{10} \ \leq \ \theta \ \leq \ 5 + \sqrt{10}. \tag{6.68}$$

The values of (z, θ) for which (6.67) is solved are shown in Fig. 6.1. This followed from the analysis of just one equation of motion. A solution must further satisfy the remain equations. From eqns. (6.63) and (6.65) it follows that in order to obtain the full orbit of solutions of Fig. 6.1, the theory must contain gauged hyperscalars. The subtlety, however, is the r-dependence of the vectors, specifically $E^\Lambda(r)$. Non-constant terms must vanish, and do not support the solutions of Fig. 6.1. While it is simple to choose the vectors such that the r.h.s. of either (6.63) or (6.65) is constant, the r.h.s. of both equations must be constant and non-vanishing. Without vectors, $E^\Lambda = p^\Lambda = 0$, the only solutions are

$$(z, \theta) = \big\{(0, 2)\,,\,(1, 2)\,,\,(4, 6)\big\}. \tag{6.69}$$

These are clearly just the solutions of the vacuum Einstein equations, eqn. (6.7). If instead we consider the case without gauged hypers, $g^2 h_{uv} k^v_\Lambda k^u_\Sigma E^\Lambda E^\Sigma = 0$, but with $E^\Lambda = e^\Lambda r^{z-\theta/2}$, eqns. (6.63)-(6.65) have only one solution,

$$z = 3, \qquad \theta = 4, \qquad z^2 I_{\Lambda\Sigma} e^\Lambda e^\Sigma + I_{\Lambda\Sigma} p^\Lambda p^\Sigma = -2L^2, \qquad g^2 V = 0, \qquad g^2 h_{uv} k^u_\Lambda k^v_\Sigma e^\Lambda e^\Sigma = 0, \tag{6.70}$$

along with the three vacuum solutions (6.69). Note that $\theta = 4$ happens to be the value of θ where eqn. (6.63) allows non-zero magnetic charges, p^Λ. In fact, $z = 3$, $\theta = 4$ implies that the field strengths are constants, $E'^\Lambda = (z - \theta/2) e^\Lambda r^{z-1-\theta/2} = e^\Lambda$. Such constant field strengths are known as fluxes. Notice also, that this precisely respects the covariant scaling of the field strengths for $p^\Lambda \neq 0$, from eqn. (6.21), $F^\Lambda \to \lambda^2 F^\Lambda$.

Let us sum up what we found above. $\mathcal{N} = 2$ supergravity with constant scalar fields is well suited to obtain Lifshitz solutions with $\theta = 0$. The negative scalar potential and the gauged scalars are crucial ingredients in such solutions. For hyperscaling violation, $\theta \neq 0$, we found however that only solutions with very restricted values of z and θ are possible for constant scalar fields. The potential, which is essentially a cosmological constant for constant scalars, must in fact vanish. We may note at this point, that in the Einstein-Maxwell-Dilaton model, eqn. (5.30), which has been used to construct solutions with $\theta \neq 0$, there is a scalar potential rather than a cosmological constant. Clearly, this indicates that one should allow for non-trivial scalar profiles to find more general solutions with $\theta \neq 0$.

6.4 Vector multiplets with Fayet-Iliopoulos gauging

In this subsection, we consider supergravity coupled only to vector multiplets. We will allow for FI terms supporting a scalar potential. As noted in the beginning of this Chapter, this may be considered a special case of the calculations above. For constant scalar fields, this case was essentially included in the previous section, but yields a nicer result below. In the following, we absorb g into the FI parameters ξ_Λ.

After some rearranging, the Einstein equations (6.38)-(6.40) become

$$(z-1)(\theta - z - 2) = \frac{1}{L^2} I_{\Lambda\Sigma} E'^\Lambda E'^\Sigma r^{2-2z+\theta} + \frac{1}{L^2} I_{\Lambda\Sigma} p^\Lambda p^\Sigma r^{\theta-4}, \tag{6.71}$$

$$(\theta - z)^2 - z^2 + 4z - 4 = 4 g_{i\bar{\jmath}} z'^i \bar{z}'^{\bar{\jmath}} r^2, \tag{6.72}$$

$$-\left(z + \tfrac{3}{2} - \theta\right)^2 + \tfrac{1}{4} = 2L^2 V r^{-\theta}. \tag{6.73}$$

The vector scalar equation of motion looks the same as (6.43),

$$\begin{aligned} 0 = {} & \partial_i I_{\Lambda\Sigma} E'^\Lambda E'^\Sigma r^{2-2z+\theta} - \partial_i I_{\Lambda\Sigma} p^\Lambda p^\Sigma r^{\theta-4} - 2\partial_i R_{\Lambda\Sigma} E'^\Lambda p^\Sigma r^{-z-1+\theta} \\ & + 2L^4 \partial_i V r^{-\theta} + 2L^2 \partial_i g_{j\bar{k}} z'^j \bar{z}'^{\bar{k}} r^2 - 2L^2 r^{\theta-z-1} \partial_r \left(r^{z+3-\theta} g_{i\bar{\jmath}} \bar{z}'^{\bar{\jmath}} \right). \end{aligned} \tag{6.74}$$

The equation of motion for the gauge fields, simplifies without gauged scalar fields,

$$0 = \partial_r \left(I_{\Lambda\Sigma} E'^\Sigma r^{3-z} - R_{\Lambda\Sigma} p^\Sigma \right). \tag{6.75}$$

The gauge field equation (6.75) can be solved analogously to Section 4.2. Since the derivative vanishes, the terms in the parenthesis must be equal to a constant, the electric charge, $-q_\Lambda$,

$$-q_\Lambda = I_{\Lambda\Sigma} E'^\Sigma r^{3-z} - R_{\Lambda\Sigma} p^\Sigma. \tag{6.76}$$

Solving for $E'^\Lambda(r)$, we may eliminate this function in terms of the charges,

$$E'^\Lambda(r) = r^{z-3} I^{\Lambda\Sigma} \left(R_{\Sigma\Gamma} p^\Gamma - q_\Sigma \right). \tag{6.77}$$

Inserting this into eqn. (6.71), the Einstein equation becomes

$$-\frac{2V_{BH}}{L^2}r^{\theta-4} = (z-1)(\theta-z-2). \tag{6.78}$$

The black hole potential,

$$V_{BH} \equiv -\tfrac{1}{2}Q^t\mathcal{M}Q, \tag{6.79}$$

where

$$Q \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} (I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda{}^\Sigma \\ -(I^{-1}R)^\Lambda{}_\Sigma & I^{-1\,\Lambda\Sigma} \end{pmatrix}, \tag{6.80}$$

was introduced in Chapter 4. Inserting (6.77) into eqn. (6.74) yields

$$0 = \hat{V}_i - L^4\partial_i V r^{4-2\theta} - L^2\partial_i g_{j\bar{k}}z'^j\bar{z}'^{\bar{k}}r^{6-\theta} + L^2r^{3-z}\partial_r(r^{z+3-\theta}g_{i\bar{j}}\bar{z}'^{\bar{j}}), \tag{6.81}$$

where

$$\hat{V}_i \equiv -\frac{1}{2}\Big((pR-q)I^{-1}\partial_i II^{-1}(Rp-q) - p\partial_i Ip - 2(pR-q)I^{-1}\partial_i Rp\Big). \tag{6.82}$$

Eqns. (6.81)-(6.82) have the advantage that we have replaced the function $E^\Lambda(r)$ by the constant charges q_Λ and p^Λ. Note that V_{BH} and $\hat{V}_i$ only depend on r through the scalar fields.

For a theory coupled only to vector multiplets, the equations of motion to solve are hence (6.72), (6.73), (6.78) and (6.81). We will indeed solve these in Chapter 7.

Constant scalar fields

Consider again the special case of constant scalar fields, z^i. The r.h.s. of the Einstein equation (6.72) vanishes. For $\theta \neq 0$, the Einstein equation (6.73) implies again that the potential must vanish, since the l.h.s. is a constant, whereas the r.h.s. scales as $r^{-\theta}$. The last Einstein equation (6.78) requires $V_{BH} = 0$, unless $\theta = 4$. After these considerations, the eqns. (6.72), (6.73), and (6.78), have four solutions:

$$\big(z,\theta,V_{BH}\big) = \Big\{(0,2,0),(1,2,0),(4,6,0),(3,4,L^2)\Big\} \tag{6.83}$$

along with the constraint

$$V = 0. \tag{6.84}$$

The three first solutions in (6.83) are clearly the solutions of the vacuum Einstein equations, which we will not deal with further. The fourth is the non-vacuum solution also found in (6.70). Since $z = 3$, the gauge field strengths are constant by eqn. (6.77). The scalar field equation of motion (6.81) becomes

$$0 = \hat{V}_i - L^4\partial_i V r^{-4}. \tag{6.85}$$

Due to the r-dependence, the two terms must vanish separately,

$$\hat{V}_i = 0, \tag{6.86}$$

$$\partial_i V = 0. \tag{6.87}$$

Recall that the difference in the bosonic part of the action, between an ungauged theory and a theory with FI gaugings, is the existence of the potential, V. It is clear from eqns. (6.84) and (6.87), that for constant scalar fields the gauging does not contribute constructively to the equations of motion.

Nevertheless, for the solution with $z = 3$, $\theta = 4$, and constant scalar fields, we have found that the equations of motion reduce to the constraints (6.84), (6.86), (6.87), and $V_{BH} = L^2$. We will solve these explicitly in the next Chapter.

Chapter 7

Solutions in the $F = -iX^0X^1$ model

In this Chapter, we derive the relevant components for the equations of motion from the prepotential $F = -iX^0X^1$ with Fayet-Iliopoulos gauging. We then use this model as an explicit example, following the model-independent analysis of Chapter 6.

7.1 The $F = -iX^0X^1$ model with FI gauging

In Section 3.2, we discussed the target space of the vector multiplet scalars, namely special Kähler manifolds. Recall that for the ungauged theory coupled to vector multiplets, the Lagrangian is specified simply by the prepotential, when a such exists. The prepotential $F = -iX^0X^1$ is among the simplest, since it is linear in X^Λ and contains just one vector multiplet. The scalar target space manifold is in fact the coset space $SL(2,\mathbb{R})/SO(2)$ [129].

Now, differentiating the prepotential yields

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} = -i\left(X^1, X^0\right). \tag{7.1}$$

Choosing the physical scalar to be $\tau = X^1/X^0$ and the gauge $X^0 = 1$, the holomorphic section Ω becomes

$$\Omega = \left(X^\Lambda, F_\Lambda\right) = (1, \tau, -i\tau, -i). \tag{7.2}$$

From this, the Kähler potential follows,

$$\mathcal{K} = -\log\left[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)\right] = -\log\left[2(\tau + \bar{\tau})\right]. \tag{7.3}$$

It is clear that we must restrict $(\tau + \bar{\tau}) > 0$, or

$$\mathrm{Re}\tau > 0. \tag{7.4}$$

This is the so-called positivity domain [62]. Now, since the model has just one complex scalar field τ, the only non-zero component of the Kähler metric is

$$g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} \mathcal{K} = \frac{1}{(\tau + \bar{\tau})^2}. \tag{7.5}$$

From the Kähler potential we can also find the sections

$$\mathcal{V} = (L^\Lambda, M_\Lambda) = e^{K/2}(X^\Lambda, F_\Lambda) = \frac{1}{\sqrt{2(\tau+\bar\tau)}}\,(1, \tau, -i\tau, -i)\,. \tag{7.6}$$

These are needed to construct the scalar potential. It is also easily verified that they satisfy eqn. (3.26),

$$i\langle \mathcal{V}, \bar{\mathcal{V}}\rangle = 1. \tag{7.7}$$

Of course, we also need the period matrix, $\mathcal{N}_{\Lambda\Sigma}$. This determines the vector couplings, and also appears in the scalar potential. We first need to calculate

$$F_{\Lambda\Sigma} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.8}$$

From this, the period matrix may be obtained,

$$\mathcal{N}_{\Lambda\Sigma} = \bar F_{\Lambda\Sigma} + 2i\frac{\mathrm{Im}(F_{\Lambda\Gamma})X^\Gamma\,\mathrm{Im}(F_{\Sigma\Delta})X^\Delta}{X^\Upsilon \mathrm{Im}(F_{\Upsilon\Omega})X^\Omega} = -i\begin{pmatrix} \tau & 0 \\ 0 & 1/\tau \end{pmatrix}. \tag{7.9}$$

Thus, the couplings of the vector kinetic terms and the topological terms are,

$$I_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} = -\frac{1}{2}(\tau+\bar\tau)\begin{pmatrix} 1 & 0 \\ 0 & 1/\tau\bar\tau \end{pmatrix}, \tag{7.10}$$

$$R_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} = \frac{\tau-\bar\tau}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1/\tau\bar\tau \end{pmatrix}. \tag{7.11}$$

The above expressions specify the bosonic action of the ungauged theory. As described in Section 3.5, under gauging with Fayet-Iliopoulos terms, the two gravitinos will be charged under the two gauge fields. In the bosonic action, this gives rise to a scalar potential given by eqn. (3.87). Using the quantities above, we find the potential to be

$$\begin{aligned} V(\tau,\bar\tau) &= -\left(\frac{1}{2}I^{\Lambda\Sigma} + 4\bar L^\Lambda L^\Sigma\right)\xi_\Lambda\xi_\Sigma \\ &= 2\xi_0\xi_1 - \frac{\xi_0^2 + \xi_1^2\tau\bar\tau}{\tau+\bar\tau}. \end{aligned} \tag{7.12}$$

Below, we shall also need the derivatives

$$\partial_\tau g_{\tau\bar\tau} = -2\frac{1}{(\tau+\bar\tau)^3}, \tag{7.13}$$

$$\partial_\tau I_{\Lambda\Sigma} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1/\tau^2 \end{pmatrix}, \tag{7.14}$$

$$\partial_\tau R_{\Lambda\Sigma} = \frac{i}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1/\tau^2 \end{pmatrix}, \tag{7.15}$$

$$\partial_\tau V = \frac{\xi_0^2 - \bar\tau^2\xi_1^2}{(\tau+\bar\tau)^2}. \tag{7.16}$$

For simplicity, we define

$$\phi \equiv \mathrm{Re}\tau, \qquad \chi \equiv \mathrm{Im}\tau, \tag{7.17}$$

with $\phi > 0$. The above expressions are then generally neater,

$$g_{\tau\bar{\tau}} = \frac{1}{4\phi^2}, \tag{7.18}$$

$$I_{\Lambda\Sigma} = \begin{pmatrix} -\phi & 0 \\ 0 & \frac{-\phi}{\phi^2+\chi^2} \end{pmatrix}, \tag{7.19}$$

$$R_{\Lambda\Sigma} = \begin{pmatrix} \chi & 0 \\ 0 & -\frac{\chi}{\phi^2+\chi^2} \end{pmatrix}, \tag{7.20}$$

$$V(\phi,\chi) = 2\xi_0\xi_1 - \frac{\xi_0^2 + \xi_1^2(\phi^2+\chi^2)}{2\phi}. \tag{7.21}$$

Also, let us write the needed derivatives,

$$\partial_\tau g_{\tau\bar{\tau}} = -\frac{1}{4\phi^3}, \tag{7.22}$$

$$\partial_\tau I_{\Lambda\Sigma} = -i\,\partial_\tau R_{\Lambda\Sigma} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & \frac{\phi^2-\chi^2-2i\phi\chi}{(\phi^2+\chi^2)^2} \end{pmatrix}, \tag{7.23}$$

$$\partial_\tau V = \frac{\xi_0^2 - \xi_1^2(\phi^2-\chi^2-2i\phi\chi)}{4\phi^2}. \tag{7.24}$$

7.2 Constant scalar fields

In Section 6.4, we studied the equations of motion for theories coupled only to vector multiplets with constant scalar fields. We found that the only possible solution was $z = 3$ and $\theta = 4$. Further, the solution must satisfy the constraints

$$V = 0, \tag{7.25}$$

$$\partial_\tau V = 0, \tag{7.26}$$

$$\hat{V}_i = 0, \tag{7.27}$$

$$V_{BH} = L^2. \tag{7.28}$$

Clearly, the potential does not contribute to the solution. However, we can check if any configuration of τ, $\bar{\tau}$ and ξ_Λ solves eqns. (7.25)-(7.26) with non-zero gaugings, $\xi_\Lambda \neq 0$. For $\xi_\Lambda \neq 0$, the gravitinos are charged and the FI gauging also has effects on the supersymmetry transformations.

The scalar potential (7.21) vanishes for

$$\xi_0 = \xi_1\left(2\phi \pm \sqrt{3\phi^2-\chi^2}\right). \tag{7.29}$$

However, to make the derivative (7.24) vanish as well, we must take $\xi_0 = \xi_1 = 0$. Hence, the solutions is restricted to ungauged supergravity.

Consider next eqns. (7.27) and (7.28). Using the definition of $\hat{V}_i$, eqn. (6.82), along with the quantities (7.19), (7.20) and (7.23) from the $F = -iX^0X^1$ model, we find

$$\begin{aligned} 0 &= \hat{V}_i \\ &= -\frac{1}{2}\Big[(pR-q)I^{-1}\partial_i I I^{-1}(Rp-q) - p\partial_i Ip - 2(pR-q)I^{-1}\partial_i Rp\Big] \\ &= -\frac{1}{2\phi}\left[\frac{(p_0^2+q_1^2)\left(\phi^2-\chi^2\right)+2\chi(p_0q_0-p_1q_1)-p_1^2-q_0^2}{2\phi} + i\,\left(p_0q_0-p_1q_1-\chi(p_0^2+q_1^2)\right)\right]. \end{aligned} \tag{7.30}$$

We have introduced here the notation for the charges $p^\Lambda = (p_0, p_1)$ and $q_\Lambda = (q_0, q_1)$. From the definition of the black hole potential, eqn. (6.79), we find

$$V_{BH} = \frac{q_0^2 + p_1^2 + (\phi^2+\chi^2)(p_0^2+q_1^2) + 2\chi(q_1p_1 - q_0p_0)}{2\phi}. \tag{7.31}$$

Demanding the real and imaginary part of (7.30) to vanish independently, and inserting the black hole potential into (7.28), we thus have three equations to solve. Indeed, this system has the solution,

$$L^2 = \phi\left(p_0^2+q_1^2\right), \qquad q_0 = \epsilon\phi q_1 + \chi p_0, \qquad p_1 = \epsilon\phi p_0 - \chi q_1, \qquad \text{with } \epsilon = \pm 1. \tag{7.32}$$

This solution contains both electric and magnetic charges, but also allows for only electric (p^Λ=0) or only magnetic charges ($q_\Lambda = 0$). We have thus found that the $F = -iX^0X^1$ model with constant scalar fields admits the $z = 3$, $\theta = 4$ solution with constant field strengths (fluxes), but only in the ungauged theory where the FI terms vanish, $\xi_\Lambda = 0$.

7.3 Non-constant scalar fields

Consider now the equations of motion with non-constant scalar fields, which we found to be (6.72), (6.73), (6.78) and (6.81). Inserting the special Kähler metric (7.18) into the Einstein equation (6.72), this equation becomes

$$(\theta - z)^2 - z^2 + 4z - 4 = \frac{\phi'^2(r) + \chi'^2(r)}{\phi^2} r^2. \tag{7.33}$$

This differential equation is easily solved, if either ϕ or χ is set to a constant value. Consider here the case of non-trivial $\phi(r)$. Rather than taking χ to be constant, we will take it to be

$$\chi(r) = \beta\phi(r) + \chi_0, \tag{7.34}$$

where χ_0 is a real constant. Setting $\beta = 0$ corresponds to setting χ to a constant. However, allowing for arbitrary β gives a bit more freedom to solve the system. Then eqn. (7.33) becomes

$$(\theta - z)^2 - z^2 + 4z - 4 = \left(1+\beta^2\right)\frac{\phi'^2(r)}{\phi^2} r^2. \tag{7.35}$$

This can be easily solved for $\phi(r)$,

$$\phi(r) = \phi_0 r^a, \qquad \text{where} \quad a^2 = \frac{\theta^2 + 4z - 2\theta z - 4}{(1+\beta^2)^2}. \tag{7.36}$$

Plugging now the potential (7.21) into the Einstein equation (6.73) yields

$$-\left(z + \frac{3}{2} - \theta\right)^2 + \frac{1}{4} = 2L^2 \left(2\xi_0\xi_1 - \frac{\xi_0^2 + \xi_1^2(\phi^2+\chi^2)}{2\phi}\right) r^{-\theta}. \tag{7.37}$$

Using the explicit form of the scalars, (7.34) and (7.36), this can be written as

$$\frac{\phi_0}{L^2}\left(\frac{1}{4} - \left(z + \frac{3}{2} - \theta\right)^2\right) = 2\phi_0\xi_1\left(2\xi_0 + \xi_1\alpha\chi_0\right) r^{-\theta} + \left(\xi_1^2\chi_0^2 - \xi_0^2\right) r^{-a-\theta} + \left(1+\alpha^2\right)\xi_1^2\phi_0^2 r^{a-\theta}. \tag{7.38}$$

Now, the l.h.s. of this equation is constant, while the r.h.s. contains three terms with different r-dependences. We are of course interested in solutions valid for all values of r. Hence, the equation splits into more than one equation, depending on how the exponents are chosen. We are focusing on hyperscaling solutions with $\theta \neq 0$. Also, we take $a \neq 0$ since we already treated constant scalar fields. In this case, the term scaling as $r^{-\theta}$ must vanish,

$$\xi_1\left(2\xi_0 + \xi_1\beta\chi_0\right) = 0. \tag{7.39}$$

The remaining terms of (7.38) may split up in three different ways,
Case I:

$$a = \theta, \qquad \xi_1^2\chi_0^2 - \xi_0^2 = 0, \qquad (1+\beta^2)L^2\xi_1^2\phi_0 = -z^2 - \theta^2 + 3\theta - 3z + 2\theta z - 2. \tag{7.40}$$

Case II:

$$a = -\theta, \qquad (1+\beta^2)\xi_1^2 = 0, \qquad L^2\frac{\xi_1^2\chi_0^2 - \xi_0^2}{\phi_0} = -z^2 - \theta^2 + 3\theta - 3z + 2\theta z - 2. \tag{7.41}$$

Case III:

$$a \neq \pm\theta, \qquad \xi_1^2\chi_0^2 - \xi_0^2 = 0, \qquad (1+\beta^2)\xi_1^2 = 0, \qquad \text{with} \quad \theta = z+1 \quad \text{or} \quad \theta = z+2. \tag{7.42}$$

All three cases also include the constraint (7.39), and a is given by (7.36). These equations are hence solutions to two of the Einstein equations, and also the gauge field equation of motion. The remaining Einstein equation is (6.78).

Consider Case III, eqn. (7.42). To satisfy these equations, we must take $\xi_0 = \xi_1 = 0$, which restricts the solution ungauged supergravity. Using the black hole potential (7.31), the remaining Einstein equation (6.78) becomes

$$z^2 + \theta + z - \theta z - 2 = \frac{q_0^2 + p_1^2 + (\phi^2+\chi^2)(p_0^2+q_1^2) + 2\chi(q_1p_1 - q_0p_0)}{L^2\phi} r^{\theta-4}. \tag{7.43}$$

Inserting now $\phi = \phi_0 r^a$ and $\chi = \beta\phi_0 r^a + \chi_0$, and collecting terms according to their r-dependence, we find

$$\begin{aligned} L^2\phi_0\left(z^2+\theta+z-\theta z-2\right) &= \beta\phi_0\left[\chi_0(p_0^2+q_1^2)+2(q_1p_1-q_0p_0)\right]r^{\theta-4} \\ &+\left[q_0^2+p_1^2+\chi_0^2(p_0^2+q_1^2)+2\chi_0(q_1p_1-q_0p_0)\right]r^{\theta-4-a} \\ &+(1+\beta^2)\phi_0^2(p_0^2+q_1^2)r^{\theta-4+a}. \end{aligned} \tag{7.44}$$

Choosing now $\theta = z+2$ from (7.42), the l.h.s. of (7.44) vanishes. For $a \neq 0$, the three lines on the r.h.s. of (7.44) must vanish separately. This can only happen for $p^\Lambda = q_\Lambda = 0$. We have thus solved all Einstein equations by setting all charges and gaugings to zero. For the scalar fields we find for a, from (7.36) with $\theta = z+2$,

$$a^2 = \frac{4z-z^2}{(1+\beta^2)^2}. \tag{7.45}$$

Since a must be real, (7.45) restricts z to take values $0 \leq z \leq 4$. Having solved now the Einstein equations and the gauge field equation, only the equation of motion for the scalar fields remains, eqn. (6.81). With $p^\Lambda = q_\Lambda = \xi_\Lambda = 0$ and the scalar metric (7.20), eqn. (6.81) becomes

$$0 = -\frac{L^2}{2}(1+\beta^2)\frac{\phi'^2}{\phi^3}r^2 - L^2(1-i\beta)r^{\theta-z-1}\partial_r\left(r^{z+3-\theta}\frac{\phi'}{2\phi^2}\right). \tag{7.46}$$

We separate this equation into two equations, demanding both the real and imaginary part to vanish. Inserting $\phi = \phi_0 r^a$ and $\theta = z+2$, both equations reduce to

$$0 = a\beta. \tag{7.47}$$

Thus, taking $\beta = 0$, we have solved all the equations of motion.

Summarizing, the solution is

$$0 \leq z \leq 4, \qquad \theta = z+2, \qquad \phi = \phi_0 r^{\pm\sqrt{4z-z^2}}, \qquad \chi = \text{constant}, \qquad F^\Lambda = 0, \qquad \xi_\Lambda = 0. \tag{7.48}$$

Actually, to reach this solution we excluded $z = 0$ and $z = 4$, since these imply constant scalars, $a = 0$. However, these two special cases are in fact just two of the vacuum solutions of the Einstein equations, and the solution (7.48) recovers these solutions in the limits $z \to 0$ and $z \to 4$. We therefore consider these to be part of the solution found here.

Now, to find more solutions, one could obviously investigate Cases I and II above, as well as the choice $\theta = z+1$ for Case III. Indeed, during this work we carried out this analysis. However, we did not find that one could solve the remaining equations of motion for any of these cases. Also, one may go further back and consider eqn. (7.33). By choosing ϕ to be constant, $\phi' = 0$, the equation is solved by

$$\chi(r) = a\log r + \chi_0, \qquad \text{with } a^2 = \phi^2\left((\theta-z)^2 - z^2 + 4z - 4\right), \tag{7.49}$$

and where χ_0 is a constant. We also carried out this analysis, but again found that all equations of motion could not be satisfied.

Finally, one could consider solutions where both $\phi(r)$ and $\chi(r)$ are non-trivial functions of r. In this case, it is less simple to solve the Einstein equation (7.33) and the scalar equation of motion, (6.81). These are first order and second order differential equations, respectively. We have not proven, however, whether such solutions are possible.

Consistent truncation of the action

The solution (7.48) was found by studying the equations of motion of the supergravity action

$$S = \int \sqrt{-g}\mathrm{d}^4x \left(\frac{R}{2} + \frac{1}{4} I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{4} R_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} {}^{\star}F^{\Sigma|\mu\nu} + \frac{\partial_\mu \phi \partial^\mu \phi + \partial_\mu \chi \partial^\mu \chi}{4\phi^2} - V(z,\bar{z}) \right), \tag{7.50}$$

where $I_{\Lambda\Sigma}$, $R_{\Lambda\Sigma}$ and $V(z,\bar{z})$ are given by eqns. (7.19)-(7.21). However, the solution requires both vanishing charges and gaugings, as well as constant χ. We note here, that the solution can actually be obtained from a much simpler action. Setting $F^\Lambda = \xi_\Lambda = \partial_\mu \chi = 0$ in the action (7.50) yields

$$S = \int \sqrt{-g}\mathrm{d}^4x \left(\frac{R}{2} + \frac{\partial_\mu \phi \partial^\mu \phi}{4\phi^2} \right). \tag{7.51}$$

Deriving the full set of equations of motion from the action (7.50) and then setting $F^\Lambda = \xi_\Lambda = \partial_\mu \chi = 0$, one finds precisely the same set of equations as the equations of motion derived from (7.51). Therefore, any solution of (7.51) will also be a solution of (7.50). This is known as a consistent truncation. In general, one cannot simply plug e.g. constant scalar fields into an action and obtain a consistent truncation. Non-trivial constraints may result from the equations of motion when setting the scalar fields to constants, and these are not in general captured by the naively truncated action.

7.4 First-order flow equations

Having found explicit solutions in the $F = -iX^0X^1$ model, we will now plug these into the first-order flow equations discussed in Section 4.2. Recall, a solution of these flow equations must preserve supersymmetry.

As we found no solutions with non-zero gaugings, we take these to vanish, $\mathcal{G} = 0 \Rightarrow \mathcal{W} = 0$. The relevant flow equations are thus (4.43)-(4.45),

$$U' = -e^{-2A-U}|Z|, \tag{7.52}$$

$$A' = e^{-2A-U}|Z|, \tag{7.53}$$

$$z'^i = -2e^{-2A-U} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} |Z|. \tag{7.54}$$

The central charge is given

$$Z = \langle Q, \mathcal{V} \rangle = L^\Lambda q_\Lambda - M_\Lambda p^\Lambda = e^{\mathcal{K}/2} \left(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda \right), \tag{7.55}$$

and the warp factors U and A are the metric components,

$$\mathrm{d}s^2 = -e^{2U(\rho)}\mathrm{d}t^2 + e^{-2U(\rho)}\mathrm{d}\rho^2 + e^{2A(\rho)}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{7.56}$$

As in Section 4.2, we take in this section the metric signature $(-+++)$, for consistency with [60].

The metric

In the usual coordinates such as eqn. (6.1), the Lifshitz metric with hyperscaling violation is

$$\mathrm{d}s^2 = L^2 r^{-\theta}\left(-r^{2z}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2(\mathrm{d}x^2 + \mathrm{d}y^2)\right). \tag{7.57}$$

This does not have the form of (7.56). We can bring it to such a form by a coordinate transformation,[9]

$$\rho \equiv \pm\frac{L^2}{z-\theta} r^{z-\theta}. \tag{7.58}$$

The metric (7.57) then becomes (7.56) with

$$e^{2U} = L^{\frac{-2z}{z-\theta}}(z-\theta)^{\frac{2z-\theta}{z-\theta}}\rho^{\frac{2z-\theta}{z-\theta}}, \qquad e^{2A} = L^{\frac{2z-4}{z-\theta}}(z-\theta)^{\frac{2-\theta}{z-\theta}}\rho^{\frac{2-\theta}{z-\theta}}. \tag{7.59}$$

By differentiating this, and re-arranging, we further obtain

$$U' = \frac{2z-\theta}{z-\theta}\frac{1}{2\rho}, \qquad A' = \frac{2-\theta}{z-\theta}\frac{1}{2\rho}. \tag{7.60}$$

Solutions of the $F = -iX^0X^1$ model

We found two solutions in this Chapter, eqns. (7.32) and (7.48). We focus first on the latter with non-constant scalar fields, τ, but vanishing charges. This implies that the central charge vanishes, $Z = 0$. The flow equations (7.52)-(7.54) are then trivial,

$$U' = 0, \tag{7.61}$$

$$A' = 0, \tag{7.62}$$

$$\tau' = 0. \tag{7.63}$$

From eqn. (7.60), this implies

$$z = 1, \qquad \theta = 2. \tag{7.64}$$

Constant warp factors, U and A, correspond of course to the Minkowski metric. Indeed, it was shown in Chapter 6 that the metric with $z = 1$, $\theta = 2$ can be brought to the canonical form of the Minkowski metric, and this is also easily seen from (7.56) and (7.59).

[9] Clearly, this is not a good choice if $z = \theta$. In this case the coordinate transformation is $\rho = L^2 \log r$. However, we did not encounter such a solution.

The solution (7.32) with $z = 3$ and $\theta = 4$ is non-trivial, however. This solution has constant scalar fields, but non-zero charges. Consider first the coordinate transformation (7.58), which for $z = 3$ and $\theta = 4$ becomes

$$\rho = \mp \frac{L^2}{r}. \tag{7.65}$$

In order to have $\rho > 0$ for $r > 0$, we choose the lower sign in (7.58) and (7.65).

Now, for the $F = -iX^0X^1$ model, the symplectic sections, L^Λ and M_Λ, are given by (7.6). Inserting these into the definition of Z, eqn. (7.55), yields

$$Z = \frac{1}{\sqrt{2(\tau+\bar{\tau})}}(q_0 + \tau q_1 + \tau p_0 + ip_1). \tag{7.66}$$

By multiplying by the complex conjugate, $\bar{Z}$, we obtain

$$|Z|^2 = \frac{1}{2(\tau+\bar{\tau})}\left(|\tau|^2(p_0^2+q_1^2) + i(\tau-\bar{\tau})(p_0q_0 - p_1q_1) + (\tau+\bar{\tau})(p_0p_1 + q_0q_1) + p_1^2 + q_0^2\right). \tag{7.67}$$

As earlier, we define $\tau = \phi + i\chi$. Eqn. (7.67) can then be written as

$$|Z|^2 = \frac{1}{4\phi}\left((\phi^2+\chi^2)(p_0^2+q_1^2) - 2\chi(p_0q_0 - p_1q_1) + 2\phi(p_0p_1 + q_0q_1) + p_1^2 + q_0^2\right). \tag{7.68}$$

Taking the square root of this and inserting the solution (7.32), this reduces simply to

$$|Z| = \frac{1+\epsilon}{2}L. \tag{7.69}$$

For the flow equation (7.54), we need also to determine $\partial_{\bar{\tau}}|Z|$. By taking the square root of eqn. (7.67) and differentiating, we obtain

$$\partial_{\bar{\tau}}|Z| = -\frac{(p_1 + iq_0 - \tau(p_0 + iq_1))\sqrt{p_1 - iq_0 + \tau(p_0 - iq_1)}}{2\sqrt{2}\sqrt{\tau+\bar{\tau}}\sqrt{p_1 + iq_0 + \bar{\tau}(p_0 + iq_1)}}. \tag{7.70}$$

Inserting again $\tau = \phi + i\chi$ and the solution (7.32), eqn. (7.70) simplifies to

$$\partial_{\bar{\tau}}|Z| = -\frac{(\epsilon - 1)}{8\phi}L. \tag{7.71}$$

Consider now the flow equation (7.54) for the scalar fields. For $\tau' = 0$, eqn. (7.54) becomes just

$$0 = \partial_{\bar{\tau}}|Z|. \tag{7.72}$$

Clearly, this is solved using (7.71) by choosing $\epsilon = +1$. Consider next eqns. (7.52) and (7.53). For $z = 3$ and $\theta = 4$, we find from (7.60),

$$U' = -\frac{1}{\rho}, \qquad A' = \frac{1}{\rho}, \tag{7.73}$$

and from (7.59), we find

$$e^{2U} = \frac{L^6}{\rho^2}, \qquad e^{2A} = \frac{\rho^2}{L^2}. \tag{7.74}$$

From this, it follows

$$e^{-2A-U} = e^{-2A}\sqrt{e^{-2U}} = \frac{1}{L\rho}. \tag{7.75}$$

Using (7.69) with $\epsilon = +1$, we then finally arrive at

$$e^{-2A-U}|Z| = \frac{1}{\rho}. \tag{7.76}$$

Plugging (7.73) and (7.76) into the flow equations (7.52) and (7.53), one sees that these equations are solved. As mentioned, this is a non-trivial result implying that the solution preserves supersymmetry.

Supersymmetric Lifshitz solutions with non-zero hyperscaling violation have only previously been constructed from near-horizon geometries of black holes or black branes. E.g. solutions in [33] include a supersymmetric four-dimensional solution with $z = 3$ and $\theta = 4$, such as the solution above. In fact, [33] appeared only when this thesis was in the final stages of preparation.

Chapter 8

Conclusions

Let us summarize and discuss some of the results of Chapters 6 and 7.

We noted a Ricci-flat solution with $z = 4$ and $\theta = 6$ with non-zero Riemann tensor. This simple solution was only mentioned in the literature [33] when this thesis was in the final stages of preparation.

For constant scalar fields in gauged supergravity, we found that only a restricted set of solutions are possible. In theories with gauged hyperscalars, the allowed solutions are those of eqn. (6.67), or equivalently Fig. 6.1. Without gauged scalars, only a single non-trivial solution is possible, for which $z = 3$ and $\theta = 4$. In either case, the scalar potential must vanish and therefore it does not contribute to the support of the solution for constant scalar fields. This is very different from the Lifshitz ($\theta = 0$) case.

For more general solutions, it is clearly necessary to allow for non-constant scalar fields. The equations of motion with gauged hyperscalars, eqns. (6.38)-(6.46), are quite complicated if one allows for non-trivial r-dependence, and in general contain many terms which scale differently with r. However, there are no immediate restrictions to the possible values of z and θ in this case.

In the $F = -iX^0X^1$ model, we found an explicit realization of the solution with constant scalar fields, (7.32), and also a single solution with a non-constant real scalar field, (7.48). As was mentioned, it is not excluded that allowing for non-trivial spacetime-dependence of both real scalars can lead to more solutions.

We further showed that the solution (7.32) is supersymmetric, since it solves the first-order flow equations. Supersymmetric Lifshitz-like solutions have only recently been constructed, and only from near-horizon geometries of black holes and branes. Our solution is novel in this sense.

In Section 5.2 it was noted that the value $\theta = d - 1$ is particularly interesting for holography. Since our calculations are for $d = 2$, this implies $\theta = 1$. Unfortunately, we did not find such a solution.

The null energy condition is not an issue for our solutions. However, all our solutions have $\theta > d = 2$, except the trivial Minkowski space. It was pointed out in Section 5.2, based on [25], that such gravity solutions may not be consistent. However, from our analysis, it is unclear where this inconsistency or instability arises from.

Recall from Section 5.2 that solutions with more general z and θ may be found from the Einstein-Maxwell-Dilaton (EMD) action, eqn. (5.30). It is interesting to ask what the key differences are between the EMD action and the $F = -iX^0X^1$ action, since the first allows more general solutions. The $F = -iX^0X^1$ action is essentially a doubling of the field content of the EMD action. The difference may reside in the fact that the latter is allowed free parameters in the dilatonic coupling of the scalar to the vector, as well as in the scalar potential. A comparison of the equations of motion could resolve this issue.

Finally, we did not succeed in finding explicit solutions with non-zero gauging. To find such solutions in $\mathcal{N} = 2$ supergravity seems to be an open problem. A possible way to address this problem could be the flow equations. Rather than plugging in a known solution, as we did, one could use these equations oppositely. Plugging in the metric components (7.59) and (7.60), and allowing for both non-zero charges and gaugings, this may shed light on such solutions.

Appendix A

Conventions

The metric convention on spacetime is mostly negative, $(+---)$ (except Section 7.4). On quaternionic and special Kähler manifolds, which are Riemannian, we take Euclidean signature.

The Riemann tensor is

$$R^{\mu}{}_{\nu\rho\sigma} = \epsilon \left(\partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \right), \tag{A.1}$$

while the Ricci tensor and Ricci scalar are

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{A.2}$$

On spacetime we take $\epsilon = -1$. Then AdS has negative curvature despite the mostly negative metric. For the special Kähler and quaternionic Kähler manifolds we take $\epsilon = 1$.

We use natural units where Newton's constant is $\kappa^2 = 1$. The Einstein-Hilbert action is then

$$S = \int \sqrt{g} d^D x \frac{R}{2\kappa^2} = \int \sqrt{g} d^D x \frac{R}{2}. \tag{A.3}$$

p-forms are defined with a factor $p!$ in front of the components,

$$\omega_p = \frac{1}{p!} (\omega_p)_{\mu\nu} \, dx^\mu \wedge dx^\nu. \tag{A.4}$$

In particular, this implies for the 2-form field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{A.5}$$

On a D-dimensional manifold endowed with a metric $g_{\mu\nu}$, the Hodge dual of a p-form is a $(D-p)$-form. In particular, in four-dimensions the Hodge dual of a two-form field strenths is again a two-form,

$$^\star F^\Sigma_{\mu\nu} = \frac{\sqrt{-g}}{2} \epsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma}, \tag{A.6}$$

$$^\star F^{\Sigma\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\rho\sigma}. \tag{A.7}$$

The totally antisymmetric Levi-Civita symbol on spacetimes with Lorentzian signature is

$$\epsilon_{0123} = -\epsilon^{0123} = 1, \tag{A.8}$$

while on Riemannian manifolds (the target spaces of the non-linear sigma models),

$$\epsilon_{123\dots} = \epsilon^{123\dots} = 1. \tag{A.9}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.10}$$

We then define $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$, and $\sigma^{\mu\nu} = \frac{i}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}$.

A.1 Comparison of conventions

Without an overview of conventions, it can be confusing to compare papers in supergravity, etc. Inspired by [130,131], we give here a short overview, which hopefully makes comparison between references easier.

Sign classification

First, there is a sign choice for the Lorentzian metric. Take the metric $\tilde{g}_{\mu\nu}$ to be "mostly positive", i.e. $(- + + \cdots +)$. When comparing this metric to a metric $g_{\mu\nu}$ in another convention, the choice of signature for the latter is then encoded in a factor $s_1 = \pm 1$, such that $g_{\mu\nu} = s_1 \tilde{g}_{\mu\nu}$.

In either case, the Christoffel symbols of the torsion-free Levi-Civita connection are defined as

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho} \right). \tag{A.11}$$

The definition of the Riemann tensor comes with another choice of sign, $s_2 = \pm 1$,

$$R^\mu{}_{\nu\rho\sigma} = s_2 \left(\partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \right). \tag{A.12}$$

Likewise, the Ricci tensor can be defined with a choice of sign, $s_3 = \pm 1$,

$$R_{\mu\nu} = s_3 R^\rho{}_{\mu\rho\nu}, \tag{A.13}$$

while the Ricci scalar is defined as

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.14}$$

Now, take $\tilde{g}_{\mu\nu}$, $\tilde{\Gamma}^\mu_{\nu\rho}$, $\tilde{R}^\mu{}_{\nu\rho\sigma}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ to be defined with $s_1 = s_2 = s_3 = +1$. In order to compare to other conventions, these quantities will pick up the signs,

$$g_{\mu\nu} = s_1 \tilde{g}_{\mu\nu}, \qquad \Gamma^\mu_{\nu\rho} = \tilde{\Gamma}^\mu_{\nu\rho}, \qquad R^\mu{}_{\nu\rho\sigma} = s_2 \tilde{R}^\mu{}_{\nu\rho\sigma}, \qquad R_{\mu\nu} = s_3 \tilde{R}_{\mu\nu}, \qquad R = s_1 s_2 s_3 \tilde{R}. \tag{A.15}$$

Considering the action of a gravity theory. In order to have positive kinetic energy, the signs for the Einstein-Hilbert term, a vector field, and a scalar field with a potential must be

$$S = \int d^D x \sqrt{-g} \left(s_1 s_2 s_3 \frac{\tilde{R}}{2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - s_1 \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right). \tag{A.16}$$

From the scalar kinetic term, one can read off the metric signature. The vector kinetic term, however, is independent of metric convention, since it has two upper indices, and thus gets a factor of $s_1^2 = +1$.

We can now characterize the conventions of many papers by their choice of $(s_1\, s_2\, s_3)$. In this thesis, we use $(- - +)$, like e.g. [62, 76]. Some papers use $(- - -)$, such as [55, 73], while e.g. [69] uses $(+ + +)$. All the signs are not always clear, nor necessary. For example in [25, 59], where $s_1 = +1$ and $s_2 s_3 = +1$.

Normalization

There are also different conventions on the normalization of the Einstein-Hilbert action,

$$S = \int d^D x \sqrt{-g} \left(\frac{R}{b} + \mathcal{L}_M \right). \tag{A.17}$$

The normalization shows up when doing a variation w.r.t. the metric, i.e. deriving the Einstein equations. Absorbing b into the energy-momentum tensor yields

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}, \tag{A.18}$$

with

$$\begin{aligned} T_{\mu\nu} &= \frac{-b}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} \mathcal{L}_M \right)}{\partial g^{\mu\nu}} \\ &= b \left(\frac{1}{2} g_{\mu\nu} \mathcal{L}_M - \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \right). \end{aligned} \tag{A.19}$$

In this thesis we take $b = 2$.

p-forms

In this thesis and in e.g. [59, 69, 132], a p-form is defined as,

$$\omega_p = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{A.20}$$

On the other hand, in e.g [55, 62, 73, 76, 77], the factor of $1/(p!)$ is absorbed into the components $\omega_{\mu_1 \cdots \mu_p}$, thus

$$\omega_p = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{A.21}$$

In particular, for the two-form field strength of vector fields, this means

$$F_{\mu\nu,\mathrm{here}} = 2F_{\mu\nu,\mathrm{there}}. \tag{A.22}$$

This shows up for example in the action in the kinetic terms for vectors. In our convention,

$$S = \int \sqrt{-g}d^4x \, \frac{1}{4} I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu}, \tag{A.23}$$

while in the other convention

$$S = \int \sqrt{-g}d^4x \; I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu}. \tag{A.24}$$

Appendix B

Vielbein formalism

In order to consistently couple spinors to gravity, we need to introduce the vielbein [43, 69, 108]. In the following we consider a pseudo-Riemannian manifold with Lorentzian signature, but the Riemannian analogue is straightforward.

Standard formalism

Recall, on D-dimensional spacetime $\mathcal{M}$ with metric g, the usual basis for vectors in $T_p\mathcal{M}$ is $\{\partial_\mu\}$, while for one-forms in $T_p^*\mathcal{M}$ the basis is $\{\mathrm{d}x^\mu\}$.

A tensor is independent of the basis chosen. For instance, for two choices of coordinates $\{\partial_\mu\}$ and $\{\partial_{\mu'}\}$, a vector has components

$$V = V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'}. \tag{B.1}$$

This is the reason for the tensorial transformation properties of its components,

$$V^{\mu'} \frac{\partial}{\partial x^{\mu'}} = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial}{\partial x^{\mu'}}, \tag{B.2}$$

from which it follows

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu. \tag{B.3}$$

The covariant derivative D_μ contains the Christoffel symbol,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho} \right), \tag{B.4}$$

which is torsionless $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ and metric compatible $D_\mu g_{\nu\rho} = 0$.

Non-coordinate frames

Consider now the linear combination of basis vectors

$$e_a = e_a{}^\mu \partial_\mu, \qquad \{e_a{}^\mu\} \in GL(D, \mathbb{R}), \tag{B.5}$$

where $\det e_a{}^\mu > 0$. Thus $\{e_a\}$ is a frame of basis vectors obtained by $GL(D,\mathbb{R})$ rotations of the basis $\{\partial_\mu\}$, preserving the orientation. The basis $\{e_a\}$ is called a non-coordinate bases. The coefficients $e_a{}^\mu$ are called vielbeins. The word viel is German for many. One may be more specific, e.g. *vier*bein for $D = 4$ or *zehn*bein for $D = 10$. However, we take the sloppy approach of always writting vielbein.

We demand the frame $\{e_a\}$ to be of unit length and orthogonal with respect to the metric,

$$g(e_a, e_b) = e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}. \tag{B.6}$$

We define the inverse of $e_a{}^\mu$ as $e_\mu{}^a$, such that

$$e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu, \qquad e_\mu{}^a e_b{}^\mu = \delta^a{}_b. \tag{B.7}$$

These form a dual basis: $e^a = e_\mu{}^a \mathrm{d}x^\mu$, such that the inner product of the basis and dual basis is

$$\langle e^a, e_b\rangle = e_\mu{}^a e_b{}^\nu \langle \mathrm{d}x^\mu, \partial_\nu\rangle = e_\mu{}^a e_b{}^\nu \delta^\mu{}_\nu = \delta^a{}_b. \tag{B.8}$$

Inverting (B.6) then yields

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}. \tag{B.9}$$

Thus, we can always express the metric in terms of the vielbein.

A tensor is independent of whether we can choose the coordinate basis $\{\partial_\mu\}$ or the non-coordinate basis $\{e_a\}$, e.g. a vector,

$$V = V^\mu \partial_\mu = V^a e_a = V^a e_a{}^\mu \partial_\mu. \tag{B.10}$$

It follows that,

$$V^\mu = V^a e_a{}^\mu, \qquad V^a = V^\mu e_\mu{}^a. \tag{B.11}$$

Local Lorentz transformations

For a given metric the vielbein are not uniquely determined by (B.9). Transformations of the form

$$e'_\mu{}^a = e_\mu{}^b \Lambda_b{}^a(x) \tag{B.12}$$

leave (B.9) invariant if

$$\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab} \qquad \Rightarrow \qquad \Lambda(x) \in SO(1, D-1). \tag{B.13}$$

Thus gravitational theories formulated using the vielbein are invariant under both general coordinate transformations and also the transformations above, called local Lorentz transformations. The indices $\mu, \nu, ...$ are sometimes called curved indices, as they are raised and lowered by $g_{\mu\nu}$. The indices $a, b, ...$ are known as local Lorentz indices or flat indices, and they are raised and lowered by η_{ab}.

Spin connection

Spinors transform as scalars under general coordinate transformations. Under local Lorentz transformations, however, a connection is needed. This is the so-called spin connection ${\omega_\mu}^{ab}$. The torsionless spin connection is

$$ {\omega_\mu}^{ab} = \frac{1}{2} e_{c\mu} \left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab} \right), \tag{B.14} $$

where

$$ \Omega^{abc} = \left(e^{a\mu} e^{b\nu} - e^{b\mu} e^{a\nu} \right) \partial_\mu {e_\nu}^c \tag{B.15} $$

are called the anholonomy coefficients. The torsionless spin connection is antisymmetric in a, b, i.e. ${\omega_\mu}^{ab} = -{\omega_\mu}^{ba}$. It is related to the Christoffel symbols as

$$ {\omega_\mu}^{ab}(e) = {e_\nu}^a \partial_\mu e^{b\nu} + {e_\nu}^a e^{b\sigma} \Gamma^\nu_{\sigma\mu} \tag{B.16} $$

The covariant derivative of a spinor χ is then given by

$$ D_\mu \chi = \left(\partial_\mu + \frac{1}{4} {\omega_\mu}^{ab} \gamma_{ab} \right) \chi, \tag{B.17} $$

with $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$. The gamma matrices with flat indices γ^a may be taken to be the gamma matrices of flat spacetime. As described in Section 2.3, it may be convenient in some instances to introduce a connection with torsion.
From the spin connection, we can define the Riemann tensor

$$ {R_{\mu\nu}}^{ab} = 2\partial_{[\mu} {\omega_{\nu]}}^{ab} + 2{\omega_{[\mu}}^{ac} {\omega_{\nu]c}}^b. \tag{B.18} $$

Contracting the indices yields the Ricci scalar. Also, denoting by e the determinant of the vielbein, one finds $e \equiv \det {e_\mu}^a = \sqrt{-g}$. The Einstein-Hilbert action can thus be written in terms of the vielbein,

$$ S = \int d^D x \, \frac{1}{2} \, e \, {e_a}^\mu {e_b}^\nu {R_{\mu\nu}}^{ab}. \tag{B.19} $$

Appendix C

Useful geometric identities

The following is a collection of some geometric identities and associated facts. These may seem randomly chosen, however, as pointed out in Section 2.5, these identities are useful for calculations therein. For further details, see e.g. [69]. We assume below in all cases that the manifold M is endowed with a metric g.

Contraction of Levi-Civita symbols

On a D dimensional manifold M, the contraction of p indices of two Levi-Civita symbols yields an antisymmetrized product of Kronecker deltas:

$$\epsilon_{\mu_1\mu_2\cdots\mu_p\alpha_1\cdots\alpha_{D-p}}\epsilon^{\mu_1\mu_2\cdots\mu_p\beta_1\cdots\beta_{D-p}} = s\,p!(D-p)!\delta^{[\beta_1}_{\alpha_1}\cdots\delta^{\beta_{D-p}]}_{\alpha_{D-p}}, \tag{C.1}$$

where $s=-1$ for Lorentzian signature and $s=1$ for Euclidean. Recall also the antisymmetrization is

$$T_{[\mu_1\mu_2\cdots\mu_n]} = \frac{1}{n!}(T_{\mu_1\mu_2\cdots\mu_n} + \text{alternating sum over permutations of indices}). \tag{C.2}$$

In particular, the contraction of two indices of the Levi-Civitas on a four-dimensional manifold with Lorentzian signature is then

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\lambda\kappa} = -2(\delta^\lambda_\rho\delta^\kappa_\sigma - \delta^\kappa_\rho\delta^\lambda_\sigma). \tag{C.3}$$

The Hodge dual

On a D-dimensional manifold M endowed with a metric g, denote the vector space of p-forms by $\Omega^p(M)$. Then the Hodge * operator defines an isomorphism between $\Omega^p(M)$ and $\Omega^{D-p}(M)$. Applying the Hodge * operator twice to a p-form ω_p thus yields a p-form, in fact the same form up to a sign,

$$^{\star\star}\omega_p = (-1)^{\tilde{s}+p(D-p)}\omega_p. \tag{C.4}$$

Here $\tilde{s}=1$ for Lorentzian signature and $\tilde{s}=0$ for Euclidean. In particular, for the two-form field strength on a four-dimensional manifold with Lorentzian signature,

$$^{\star\star}F = -F. \tag{C.5}$$

The variation of the determinant of the metric is

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu} = +\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}. \tag{C.6}$$

Using this, and the antisymmetry of the field strength, one can prove

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu}). \tag{C.7}$$

More generally this relation holds for antisymmetric tensors of rank one or higher.

Appendix D

Vacuum solutions in general spacetime dimension

For the sake of interest, this Appendix provides the vacuum solutions of the hyperscaling Lifshitz metric in arbitrary spacetime spacetime dimension $D = (d+2)$. The metric has the form,

$$ds^2_{d+2} = L^2 r^{-2\theta/d}\left(r^{2z}dt^2 - \frac{dr^2}{r^2} - r^2 dx^i dx^i\right), \quad i = 1, ..., d. \tag{D.1}$$

The Ricci tensor has components

$$R_{tt} = r^{2z}\,\frac{(\theta - z - d)(zd - \theta)}{d}, \tag{D.2}$$

$$R_{rr} = \frac{(d+z^2)d - (d+z)\theta}{r^2 d}, \tag{D.3}$$

$$R_{ij} = \delta_{ij} r^2\,\frac{(d-\theta)(d+z-\theta)}{d}, \tag{D.4}$$

and the Ricci scalar is

$$R = -r^{\frac{2\theta}{d}}\,\frac{d^3 + d^2(-2\theta + 2z + 1) + d\left((\theta-2)\theta + 2z^2 - 2\theta z\right) + \theta(\theta - 2z)}{L^2 d}. \tag{D.5}$$

The vacuum Einstein equations $R_{\mu\nu} = 0$ then has four solutions,

$$\begin{array}{ll} \theta = d, & z = 0, \\ \theta = d, & z = 1, \\ \theta = \frac{d(d+1)}{d-1}, & z = \frac{2d}{d-1}. \end{array} \tag{D.6}$$

The two first, however, have vanishing Riemann tensor.

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