# D3-Brane Descriptions of Wilson Loops

In the AdS/CFT Correspondence

## Masters thesis


Johannes Weirich


26. April 2007


Department of physics
University of Copenhagen


Academic supervisor
Dr. Niels Obers

# Contents

# Chapter 1

# Introduction

One of the most fundamental concepts in theoretical physics is unification. In the history of physics famous examples of unification are the simultaneous description of electricity and magnetism in the Maxwell equations or the embedding of electrodynamics and weak interactions into the theory of electroweak interactions.
The theories that describe interactions of fundamental particles are quantum field theories, and the theory summarizing the present theoretical knowledge of experimentally verified particles is the standard model. Even though it reproduces many experimental result quite exactly it has two significant shortcomings, namely that it does not include gravity and that it contains a lot of parameters that cannot be calculated from first principles.
String theory is a promising candidate for a theory of all fundamental forces. It naturally incorporates gravity and it contains only one dimensionful parameter, the string length $l_s$.
Another important aspect is that it contains interacting gauge theories on the worldvolume of D-branes. Since these are the kind of theories that are used in the standard model there is hope that the results of the standard model could be somehow reproduced by string theory.
The first steps towards describing a gauge theory in terms of strings were already taken in the middle of the 1970s. 't Hooft made a suggestion based on similarities in the diagrammatic expansions of $U(N)$ gauge theories and string theory [1]. The diagrams in the gauge theory can be categorized either as planar or as non-planar diagrams where the former can be drawn in a plane while the latter only fit on tori or Riemann surfaces of higher genus. These objects are the basic objects in a diagrammatic expansion of string theory and the suggestion was to describe the gauge theory in terms of string theory in the large $N$ limit. The full partition function for a theory where the basic field is a hermitian matrix, for example an $U(N)$ gauge theory with matter fields in the adjoint representation, has the form [2]

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(g^2 N) \tag{1.1}$$

where $f_h$ summarizes all diagrams of genus $h$ in a $g^2N$ expansion. The t' Hooft limit is given for large $N$ by keeping the effective coupling constant $\lambda = g^2 N$ ('t Hooft coupling) constant. In this limit higher genus contributions are suppressed and the partition function looks very similar to the loop expansion in string theory if we identify the string

coupling $g_s$ with $1/N$.
After the work of t'Hooft it was long suspected that large $N$ gauge theories should be described by string theories but it was not until 1997 that Maldacena made a concrete suggestion of a duality between a conformal gauge theory in four dimensions and string theory on a certain gravitational background, namely 5 dimensional Anti-de Sitter space times a 5-sphere [3]. He conjectured the duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensional spacetime and full quantum type IIB string theory on $AdS_5 \times S^5$, which is known as the AdS/CFT correspondence. The parameters on the gauge theory side are $g_{YM}$ and $N$ while the string theory is characterized by the radius of $AdS_5$ in string units $L/l_s = L/\sqrt{\alpha'}$[1] and the string coupling $g_s$. In the AdS/CFT correspondence these quantities are related as

$$L^4 = 4\pi g_s N \alpha'^2 = \lambda \alpha'^2. \tag{1.2}$$

It is not known how to solve quantized string theory on $AdS_5 \times S^5$, and therefore one mostly considers a weaker form of the correspondence. In the 't Hooft limit the duality becomes the identity between $\mathcal{N} = 4$ SYM in the large $N$ limit (with constant $\lambda$) and non interacting string theory on $AdS$. If in addition we take the large $\lambda$ limit the duality is formulated for the low energy limit of type IIB string theory, type IIB supergravity. In this form of the correspondence strongly coupled $\mathcal{N} = 4$ SYM is dual to classical type IIB supergravity on $AdS$. Most of the calculations on the AdS side can only be done in this limit. We are therefore in particular interested in operators whose expectation values when calculated in $\mathcal{N} = 4$ SYM can be extrapolated to strong coupling[2].

In this thesis we investigate a special class of operators, Wilson loops, both in $\mathcal{N} = 4$ SYM and AdS[3]. The Wilson loop operator in $\mathcal{N} = 4$ SYM measures the holonomy of a very heavy W-boson associated with the propagation along a given path $C$. It is[4]

$$W(C) = \exp\left(\int_C d^4x (iA_\mu \dot{x}^\mu + \Phi_i \dot{y}^i)\right) \tag{1.3}$$

with $A_\mu$ a non Abelian gauge field and $\Phi_i$ six scalar fields, both in the adjoint representation of the gauge group. For some special contours $C$ the Wilson loop is one out of few operators whose expectation value in the large $N$ limit is thought to be known to all orders in perturbation theory. In some cases we can even extrapolate to strong 't Hooft coupling. Wilson loops are therefore objects we can use to compare results from $\mathcal{N} = 4$ SYM with the ones from supergravity, and they thus provide non trivial tests of the AdS/CFT correspondence.
The thesis is organized as follows: In chapter 2 we start by describing some general aspects of supersymmetry and continue by focussing on some properties of $\mathcal{N} = 4$ SYM.
In chapter 3 we first give a short review of open and closed string theory and show how the field content is obtained from the quantization of relativistic strings. Then we use

[1] Here we used the relation between the string length and the Regge slope $l_s = \sqrt{\alpha'}$.

[2] The strong coupling region is not accessible in perturbation theory, therefore we do calculations in the 't Hooft limit and extrapolate to strong coupling a posteriori.

[3] In the following we will often use the term AdS with different meanings depending on the context. We use it as abbreviation for string theory on $AdS_5 \times S^5$ as well as for Anti-de Sitter space.

[4] We use Euclidean signature.

T-duality to uncover D-branes, which play an important role in connection with supersymmetric gauge theories. In another section we describe some aspects of AdS spaces and how to relate its boundary to Minkowski space. In the last section of this chapter we show how $AdS_5 \times S^5$ is obtained as a D3-brane solution of type IIB supergravity, and thereby we also find a solution for the dilaton and the R-R 5-form $F_5$.
In chapter 4 we give a more detailed description of the AdS/CFT correspondence. We describe the idea behind the Maldacena conjecture and show how symmetries on both sides can be mapped. We also show that each operator in $\mathcal{N} = 4$ SYM corresponds to exactly one field in type IIB string theory.
In chapter 6 we calculate vacuum expectation values of Wilson loop operators (1.3) with different geometries using perturbation theory in the large $N$ limit. If $C$ is a straight line or a circle the result can be obtained to all orders in a perturbative expansion in $\lambda$.
In the first section of chapter 6 we find expectation values for Wilson loops in the large $N$ and large $\lambda$ limit as minimal surfaces traced by a fundamental macroscopic string ending on the loop $C$ at the boundary. For multiply wound Wilson loops or many coincident loops it is actually more convenient to use D-branes instead of fundamental strings. In section 6.2 we show these calculations and in the case of a circular loop we find that the solution not only reproduces the planar limit result but also includes all higher genus contributions corresponding to string loop diagrams.
In chapter 8 we summarize the obtained results and discuss the implications.
In the appendix we give a summary on differential form notation and discuss magnetic and electric duality in type II supergravity.

# Chapter 2

# Supersymmetric Gauge Theories

In this chapter we review some important aspects of supersymmetry. We start with the supersymmetry algebra and its representations in four spacetime dimensions. After deriving the important $BPS$ condition we focus on $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory which we derive from maximally supersymmetric ($\mathcal{N} = 1$ SYM) theory in ten dimensions. We end this chapter by giving an overview of the symmetries of $\mathcal{N} = 4$ SYM.

## 2.1 Supersymmetry Algebra and BPS-bound

We can extend the symmetries of a given field theory by introducing supersymmetry transformations, which transform bosonic into fermionic fields and vice-versa. Schematically they act as

$$\delta_\epsilon \phi \sim \epsilon \psi, \quad \delta_\epsilon \psi \sim \bar{\epsilon} \phi \tag{2.1}$$

with $\epsilon$ a spinorial supersymmetry parameter and where $\phi$ represents the bosonic fields and $\psi$ the fermionic ones. The supersymmetry transformations are generated by $\mathcal{N}$ spinorial supercharges

$$\begin{gathered} Q^a_\alpha, \quad \bar{Q}_{\dot{\alpha} a} = (Q^\alpha_a)^\dagger \\ a = 1 \ldots \mathcal{N}, \quad \alpha = 1, 2 \end{gathered}$$

with $Q^a_\alpha$ are left and $(Q^a_\alpha)^\dagger$ right Weyl spinors. They act on the fields as

$$\delta_\epsilon \varphi = \bar{\epsilon} Q \ \varphi \tag{2.2}$$

for any field $\varphi$. The supersymmetry algebra extends the Poincaré algebra[1] and the additional anti-commutators are given as

$$\begin{aligned} \{Q^a_\alpha, \bar{Q}_{\dot{\beta} b}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^a_b && (2.3) \\ \{Q^a_\alpha, Q^b_\beta\} &= 2\epsilon_{\alpha\beta} Z^{ab} \\ \{\bar{Q}^a_{\dot{\alpha}}, \bar{Q}^b_{\dot{\beta}}\} &= 2\epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{ab}, && (2.4) \end{aligned}$$

with antisymmetric central charges $Z^{ab}$, that commute with all generators of the supersymmetry (and Poincaré) algebra. For the case $\mathcal{N} \geq 1$ the supersymmetry algebra is

[1] a good summary of the Poincaré algebra can for example be found [4]

invariant under a $U(1)_R$ phase rotation of the supercharges. In additional, for $\mathcal{N} > 1$ we can rotate the supercharges into one another leading to an $SU(\mathcal{N})$ R-symmetry.
It is clear from equation (2.3) that the representation of the supersymmetry algebra is different for the massive and the massless case.
In the massive case we can choose the rest frame $P_\mu = (M, \vec{0})$ and obtain the algebra for a fermionic oscillator from equation (2.3),

$$\{a^a_\alpha, (a^b_\beta)^\dagger\} = \delta_{\alpha\beta}\delta^{ab}, \tag{2.5}$$

where we have defined

$$a^a_\alpha = \frac{1}{\sqrt{2M}} Q^a_\alpha \tag{2.6}$$

and analogously for the Hermitian conjugate.
We can construct the state space [5] by acting with creation operators on a vacuum state $|0\rangle$ defined by

$$a^a_\alpha |0\rangle = 0. \tag{2.7}$$

The state space is then obtained by acting with creation operators on the vacuum,

$$(a^{a_1}_{\alpha_1})^\dagger \dots (a^{a_n}_{\alpha_n})^\dagger |0\rangle \quad n = 0, 1, \dots, 2\mathcal{N} \tag{2.8}$$

and for each $n$ we obtain $\begin{pmatrix} 2\mathcal{N} \\ n \end{pmatrix}$ states. Summing over all $n$ we obtain

$$\sum_{n=0}^{2\mathcal{N}} \begin{pmatrix} 2\mathcal{N} \\ n \end{pmatrix} = 2^{2\mathcal{N}} \tag{2.9}$$

states in the representation for the massive supersymmetry algebra. For $\mathcal{N} = 1$ the central charges vanish due to their antisymmetry property. For $\mathcal{N} > 1$, central charges are present and we use the R-symmetry to rotate them into the form of a block diagonal matrix:

$$Z = \begin{pmatrix} \epsilon Z_1 & & & \\ & \ddots & & \\ & & \epsilon Z_n & \\ & & & (Z_{n+1} = 0) \end{pmatrix} \tag{2.10}$$

with $\epsilon = \epsilon_{\alpha\beta}$ the total antisymmetric tensor. We split the index $a$ of the supercharges $Q^a$ as $a = (\hat{a}, \tilde{a})$, where $\hat{a} = 1, 2$ and $\tilde{a} = 1, \dots, \frac{\mathcal{N}}{2}$ label the $\tilde{a}$'th central charge $Z_{\tilde{a}}$[2].
We can now build the state space by considering the operators

$$\mathcal{Q}^{\tilde{a}}_{\alpha\pm} = \frac{1}{2}\left(Q^{1\tilde{a}}_\alpha \pm \sigma^0_{\alpha\dot{\beta}} (Q^{2\tilde{a}}_\beta)^\dagger\right) \tag{2.11}$$

which satisfy the anti-commutating rule

$$\{\mathcal{Q}^{\tilde{a}}_{\alpha\pm}, (\mathcal{Q}^{\tilde{b}}_{\beta\pm})^\dagger\} = \delta^{\tilde{a}}_{\tilde{b}} \delta^\beta_\alpha \left(M \pm Z_{\tilde{b}}\right). \tag{2.12}$$

[2]in equation (2.10) the last entry ($Z_{n+1} = 0$) is only there in the case $\mathcal{N}$ odd, and then we also have $\tilde{a} = 1 \dots \frac{\mathcal{N}+1}{2}$

We are only considering unitary representations, and by taking the expectation value of (2.12) with respect to an arbitrary state $|\psi\rangle$ we obtain (with $a = b$ and $\alpha = \beta$)

$$\langle\psi|\{\mathcal{Q}^{\tilde{a}}_{\alpha\pm}, (\mathcal{Q}^{\tilde{a}}_{\alpha\pm})^\dagger\}|\psi\rangle = \|(\mathcal{Q}^{\tilde{a}}_{\alpha\pm})^\dagger|\psi\rangle\|^2 + \|\mathcal{Q}^{\tilde{a}}_{\alpha\pm}|\psi\rangle\|^2 \geq 0. \tag{2.13}$$

Comparing this with equation (2.12) we obtain a very important relation between the mass and the central charges, the so called $BPS$ bound,

$$M \geq |Z_{\tilde{a}}| \quad \tilde{a} = 1, \ldots, \mathcal{N}/2. \tag{2.14}$$

What happens if $M = Z_{\tilde{a}}$ ? Each time the bound is saturated for the $\mathcal{Q}_-$ equation (2.13) becomes zero

$$\|(\mathcal{Q}^b_{\alpha-})^\dagger|\psi\rangle\|^2 + \|\mathcal{Q}^b_{\alpha-}|\psi\rangle\|^2 = 0, \tag{2.15}$$

and it follows that the operators $Q^{\tilde{a}}_{\alpha-}$ and its Hermitian conjugate vanish. We can build the state space like in equation (2.8) and if for some $Z_{\tilde{a}}$ the $BPS$ bound is saturated $M = |Z_{\tilde{a}}|$ ($\tilde{a} = 1, \ldots, n_0$) and for all other $\tilde{a} > n_0$ the mass is greater than the central charge $M > Z_{\tilde{a}}$ we reduce the dimension of the state space to $2^{2\mathcal{N}-2n_0}$. We are speaking of an $(\frac{1}{2})^{n_0}$ $BPS$ representation if (2.14) is satisfied for $n_0$ among the central charges in (2.12).

To obtain the massless respresentation we can choose $P_\mu = (E, 0, 0, -E)$ and the supersymmetry algebra (2.3) becomes

$$\{Q^a_\alpha, (Q^b_\beta)^\dagger\} = 2(\sigma^\mu p_\mu \delta^b_a)_{\alpha\dot{\beta}} = 4E\delta_{1\dot{\beta}}\delta^b_a. \tag{2.16}$$

For $\dot{\beta} = 2$ the anti commutator vanishes

$$\{Q^a_2, (Q^b_2)^\dagger\} = 0 \tag{2.17}$$

and for a unitary representation the supercharges with $\alpha, \beta = 2$ become zero

$$Q^a_2 = \bar{Q}^b_{\dot{2}} = 0. \tag{2.18}$$

Using this in equation (2.4) it follows that the central charges become zero as well. Analogous to the massive case we construct the state space by acting with creation operators on a vacuum

$$(a^{a_1})^\dagger \ldots (a^{a_n})^\dagger|0\rangle \quad n = 0, 1, \ldots, \mathcal{N} \tag{2.19}$$

with

$$a^a = \frac{1}{2\sqrt{E}} Q^a_1, \quad (a^b)^\dagger = \frac{1}{2\sqrt{E}} (Q^a_1)^\dagger, \tag{2.20}$$

For each $n$ we obtain

$$\binom{\mathcal{N}}{n} \tag{2.21}$$

massless states and as the total number of states in the multiplet for a given $\mathcal{N}$ we obtain

$$\sum_{n=0}^{\mathcal{N}} \binom{\mathcal{N}}{n} = 2^{\mathcal{N}}. \tag{2.22}$$

In the case $\mathcal{N} = 4$ we obtain the states and their helicity $h$ as follows.

$$\begin{aligned}
|0\rangle,\ h &= -1\text{(one state)} \\
(\alpha^a)^\dagger|0\rangle,\ h &= -\frac{1}{2}\text{ (four states)}, \\
(\alpha^a)^\dagger(\alpha^b)^\dagger|0\rangle,\ h &= 0\text{ (six states)}, \\
(\alpha^a)^\dagger(\alpha^b)^\dagger(\alpha^b)^\dagger|0\rangle,\ h &= \frac{1}{2}\text{ (four states)}, \\
(\alpha^a)^\dagger(\alpha^b)(\alpha^c)^\dagger(\alpha^d)|0\rangle,\ h &= 1\text{ (one state)}
\end{aligned} \tag{2.23}$$

where we used (2.21) to calculate the number of states. These states correspond to a massless vector field, six scalars and four Majorana spinors. The on-shell degrees of freedom are 8 bosonic and 8 fermionic ones, and the matching of the physical bosonic and fermionic degrees of freedom has to hold for any supersymmetric multiplet. A complete list of the multiplets in four dimensions is for example given in [6]. We are mostly interested in the $\mathcal{N} = 4$ massless multiplet, which we will explicitly consider now.

## 2.2 $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

The AdS/CFT correspondence conjectures the duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensional spacetime. We derive the action by dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions and we will describe the symmetries of the theory.

### 2.2.1 Dimensional reduction

In this section we derive the Lagrangian and field content of $\mathcal{N} = 4$ SYM by dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. In chapter 3 we will show that a stack of $N$ D3-branes in superstring theory carries a supersymmetric gauge theory in four dimensions with gauge group $U(N)$. The D3-branes are uncovered by doing 6 T-duality transformations on a D9-brane, which are done by compactifying 6 spatial directions and fixing the brane at the compact direction. The gauge theory on the D3-brane is derived by dimensional reduction of the gauge theory on the D9-brane whose worldvolume is just ten dimensional spacetime. It is therefore natural to make a Kaluza-Klein compactification of maximally supersymmetric Yang-Mills theory in ten dimension to obtain the maximally supersymmetric theory in four dimensions.

A gauge field in $D$ dimensions has $D - 2$ physical degrees of freedom, whereas an unconstrained Dirac spinor in $D$ dimensions has $2^{\frac{D}{2}}$ real degrees of freedom. The bosonic and fermionic degrees of freedom have to be equal, and in ten dimensions we have to restrict the fermions with a Weyl and a Majorana condition, to reduce their degrees of freedom to eight.

In the following we are using a mostly negative metric $\eta_{AB}$ $(A, B \in 0, \dots, 9)$, $32 \times 32$ gamma matrices that satisfy the Clifford algebra $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}\mathbb{1}$ and $\Gamma_{11} = \Gamma_0\Gamma_1 \dots \Gamma_9$, $\Gamma_{11}^2 = 1$. The action of $\mathcal{N} = 1$ SYM in ten dimensions is given by

$$S = \frac{1}{g_{YM}^2} \int d^{10}x \left( -\frac{1}{4} G^{AB} G_{AB} + \frac{i}{2} \bar{\Psi} \Gamma^A D_A \Psi \right) \tag{2.24}$$

with field strength and covariant derivative

$$G_{AB} = \partial_A W_B - \partial_B W_A - g_{YM}[W_A, W_B], \quad D_A\Psi = \partial_A\Psi - g_{YM}[W_A, \Psi] \tag{2.25}$$

The Majorana and Weyl conditions are

$$\bar{\Psi} = \Psi^\dagger \Gamma_0 = \Psi^* \mathcal{C}, \quad \Gamma_{11}\Psi = -\Psi \tag{2.26}$$

where $\mathcal{C} = -\mathcal{C}^t$ is the charge conjugation operator. The Lagrangian is obviously invariant under a $SU(N)$ gauge transformation, and the fields are all in the adjoint representation of $SU(N)$ with generators $T^a$. The action is further invariant under $\mathcal{N} = 1$ supersymmetry transformations

$$\begin{aligned}
\delta W_A &= i\bar{\epsilon}\Gamma_A\Psi = -i\bar{\Psi}\Gamma_A\epsilon \\
\delta\Psi &= \frac{1}{2}G^{AB}\Gamma_{AB}\epsilon, \\
\delta\bar{\Psi} &= -\frac{1}{2}\bar{\epsilon}\Gamma_{AB}G^{AB}, \quad \Gamma_{AB} = \frac{1}{2}[\Gamma_A, \Gamma_B]
\end{aligned}$$

with $\epsilon$ a constant anticommuting Majorana-Weyl spinor.

To obtain the dimensionally reduced theory we split the ten dimensional coordinates in the following way

$$x^A = (x^\mu, x^{3+I}, x^{6+J}), \quad I, J = 1, 2, 3 \tag{2.27}$$

and decompose the $32 \times 32$ gamma matrices as

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes \mathbb{1}_4 \otimes \sigma_3 \qquad \Gamma^{3+I} = \mathbb{1}_4 \otimes \alpha^I \otimes \sigma_1 \\
\Gamma^{6+J} &= i\gamma^5 \otimes \beta^J \otimes \sigma_3 \qquad \mathcal{C} = C \otimes \mathbb{1}_4 \otimes \mathbb{1}_2 \\
\Gamma^{11} &= -\mathbb{1}_4 \otimes \mathbb{1}_4 \otimes \sigma_2
\end{aligned} \tag{2.28}$$

where the subscripts $_2$ and $_4$ are indicating $2 \times 2$ and $4 \times 4$ matrices. $\{\alpha^I\}$ and $\{\beta^J\}$ are $4 \times 4$ real antisymmetric matrices, satisfying the following (anti-) commutation rules

$$\begin{aligned}
[\alpha^I, \alpha^J] &= -2\epsilon^{IJK}\alpha^K \qquad \{\alpha^I, \alpha^J\} = -2\delta^{IJ}\mathbb{1}_4 \\
[\beta^I, \beta^J] &= -2\epsilon^{IJK}\beta^K \qquad \{\beta^I, \beta^J\} = -2\delta^{IJ}\mathbb{1}_4 \\
[\alpha^I, \beta^J] &= 0.
\end{aligned} \tag{2.29}$$

The antisymmetric matrices $\alpha^I$ and $\beta^J$ belong to $SO(4)$ and from equation (2.29) we see that they form independent SO(3) subalgebras. An explicit representation can be found by decomposing the $4 \times 4$ matrices into a self dual and an anti self dual part. We can further define 15 matrices

$$A^{IJ} = \epsilon^{IJK}\alpha^K, \quad B^{IJ} = \epsilon^{IJK}\beta^K, \quad C^{IJ} = i\{\alpha^I, \alpha^J\} \tag{2.30}$$

which generate an $SU(4)$-Lie algebra in the fundamental representation. They are the generators of the $SU(4)_R$ symmetry.
We have decomposed the gamma matrices in ten dimensions into a four dimensional spinor component, an internal $SU(4)$ component and another internal $SU(2)$ component,

the latter two in the fundamental representation.
We write the Majorana-Weyl spinor as

$$\Psi = \psi \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{2.31}$$

where $\psi$ carries a spinor index $\alpha$ and an $SU(4)$ index $a$, $\psi = \psi_\alpha^a$. One can show that a Weyl spinor in ten dimensions becomes an unconstrained Dirac spinor in four dimensions and the Majorana condition in ten dimensions becomes a Majorana condition in four dimensions.

$$C\psi = -\gamma^0 \psi^*. \tag{2.32}$$

Thus a Majorana-Weyl spinor in ten dimensions becomes a quartet of Majorana spinors in four dimensions or in other words, a Majorana spinor that carries an additional $SU(4)$ index in the fundamental representation.
We now define scalar and pseudo-scalar fields $S_I = W_{3+I}$ and $P_J = W_{6+J}$. With this we obtain the field content of $\mathcal{N}=4$ SYM: A spin-1 gluon field $A_\mu = W_\mu$, three scalars $S^I$, three pseudo scalars $P^J$ and the four spin-$\frac{1}{2}$ Majorana gluinos $\psi_\alpha^a$.
We dimensionally reduce the theory by postulating that the derivatives in the directions $A > 3$ have to vanish

$$\partial_{3+I} = \partial_{6+J} = 0, \quad I, J = 1, 2, 3 \tag{2.33}$$

i.e. the fields are independent of the compact directions.
The field tensor $G_{AB}$ is then decomposed as follows

$$\begin{aligned} F_{\mu\nu} &= G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g_{YM}[A_\mu, A_\nu] \\ G_{\mu I} &= \partial_\mu S_I - g_{YM}[A_\mu, S_I] = D_\mu S_I \\ G_{\mu J} &= \partial_\mu P_J - g_{YM}[A_\mu, P_J] = D_\mu P_J \\ G_{IJ} &= -g_{YM}[S_I, P_J], \quad G_{II} = -g_{YM}[S_I, S_I], \quad G_{JJ} = -g_{YM}[P_J, P_J] \end{aligned} \tag{2.34}$$

and the bosonic part of the Lagrangian (2.24) becomes

$$\begin{aligned} \mathcal{L}_{bos} = &-\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} D_\mu S_I \cdot D^\mu S_I + \frac{1}{2} D_\mu P_J \cdot D^\mu P_J \\ &-\frac{1}{4} g_{YM}^2 \left( \|[S_I, S_J]\|^2 + \|[P_I, P_J]\|^2 + 2\|[S_I, P_J]\|^2 \right). \end{aligned} \tag{2.35}$$

Using the decomposition for the $\Gamma$-matrices and the spinor in ten dimensions, equations (2.28) and (2.33), the fermionic part in (2.24) becomes

$$\mathcal{L}_{fer} = \frac{i}{2} \bar{\psi} \cdot \gamma^\mu D_\mu \psi + \frac{g_{YM}}{2} \bar{\psi} \cdot [(\alpha^I S_I + \beta^J P_J)\gamma_5, \psi]. \tag{2.36}$$

The action for $\mathcal{N}=4$ SYM is then given by

$$S = \frac{1}{g_{YM}^2} \int d^4x \left( \mathcal{L}_{bos} + \mathcal{L}_{fer} \right). \tag{2.37}$$

with $\mathcal{L}_{bos}$ and $\mathcal{L}_{fer}$ given in (2.35) and (2.36) respectively.

**Supersymmetry**

The supersymmetry parameter $\epsilon$ in ten dimensions is a Majorana-Weyl spinor and after dimensional reduction we obtain a quartet of Majorana spinors in four dimensions. Hence the four dimensional theory will be $\mathcal{N}=4$ supersymmetric, and the action (2.37) is invariant under the following supersymmetry transformations

$$\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}\gamma_\mu\psi \\
\delta S^I &= \bar{\epsilon}\alpha^I\psi \\
\delta P^J &= \bar{\epsilon}\gamma_5\beta^J\psi \\
\delta\psi &= \frac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}\epsilon + iD_\mu S_I\gamma^\mu\alpha^I\epsilon + iD_\mu P_j\gamma^\mu\gamma_5\beta^J\epsilon \\
&\quad + g_{YM}\left(-[S_I,P_J]\gamma_5\alpha^I\beta^J\epsilon + \frac{1}{2}\epsilon_{IJK}[S_I,S_J]\alpha^K\epsilon + \frac{1}{2}\epsilon_{IJK}[P_I,P_J]\beta^K\epsilon\right)
\end{aligned} \tag{2.38}$$

It is often more convenient to use an action with only the bosonic fields in (2.24) dimensionally reduced. In this case we use the original $\Gamma$ matrices, $\Gamma^A = (\Gamma^\mu, \Gamma^i)$ and the original spinor $\Psi$. We further collect the scalar and pseudoscalar fields as

$$\Phi = \begin{pmatrix} S \\ P \end{pmatrix}. \tag{2.39}$$

We then obtain the action

$$S = \frac{1}{g_{YM}^2}\int d^4x Tr\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_I D^\mu\Phi^I + \frac{1}{4}g_{YM}^2[\Phi_I,\Phi_J]^2 \tag{2.40}$$

$$+\frac{1}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + g_{YM}\frac{1}{2}\bar{\Psi}\ \Gamma^I[\Phi_I,\Psi]\Big). \tag{2.41}$$

where we shifted to Euclidean signature.

### 2.2.2 Symmetries

A very important property of the AdS/CFT correspondence is the matching of the symmetries on both sides. The full symmetry group of $\mathcal{N}=4$ SYM is the supergroup $SU(2,2|4)$, which consists of conformal invariance, Poincaré supersymmetry, R-Symmetry and conformal supersymmetry. The Poincaré supersymmetry transformations are given in (2.38) and in the following we review the other symmetries of the theory.

**R-symmetry**

In the previous section we have obtained the Lagrangian for $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions by dimensionally reducing $\mathcal{N}=1$ supersymmetric Yang-Mills in ten dimensions, and we have seen that the spinor obtains an additional $SU(4)$ index, which gives rise to a global internal symmetry, called R-symmetry. It stems from the Lorentz invariance in ten dimensions $SO(1,9)$ which after dimensional reduction becomes $SO(1,3)\times SO(6)$. $SO(6)\cong SU(4)$ is the bosonic part of the $R$-symmetry and the scalars $\Phi^I$ form a vector representation of $SO(6)$. The spinors in (2.31) are in the fundamental representation of $SU(4)_R$ and the universally covering group is given by $Spin(6)\cong SU(4)$.

### Conformal symmetry

The conformal transformations contain Lorentz transformations $L_{\mu\nu}$, translations $P_\mu$, dilatations [3] $D$ and special conformal transformations $K_\mu$. In four dimensions the conformal algebra is isomorphic to the algebra of $SO(4,2)$.
A conformal transformation leaves the metric invariant up to a coordinate dependent scale factor. Using this we can derive the action of the conformal transformation on a coordinate $x^\mu$. For a general coordinate transformation $x^\mu \to x'^\mu$ the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} \tag{2.42}$$

whereas a conformal transformation is given by

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.43}$$

We now consider an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. Inserting this in equation (2.42) we obtain to first order in $\epsilon(x)$

$$g'_{\mu\nu}(x') = g_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu = (1+\omega(x)) g_{\mu\nu} \tag{2.44}$$

with $\nabla_\mu \epsilon_\nu = \partial_\mu \epsilon_\nu - \Gamma^\rho_{\mu\nu}\epsilon_\rho$ the covariant derivative and in the last step we have used the infinitesimal version of the conformal transformation in (2.43). In the special case of a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ we obtain from equation (2.44)

$$\omega(x) = -\frac{2}{D}\partial_\mu \epsilon^\mu. \tag{2.45}$$

Using this in equation (2.44) we obtain

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{D}\partial_\rho \epsilon^\rho(x) \eta_{\mu\nu} \tag{2.46}$$

which has the general solution in dimensions $D > 2$ [7]

$$\epsilon^\mu(x) = \alpha^\mu + a x^\mu + \omega^\mu_\nu x^\nu + (\beta^\mu x^2 - 2x^\mu \beta_\nu x^\nu). \tag{2.47}$$

In this equation the infinitesimal conformal transformations become manifest as translations $\delta x^\mu = \alpha^\mu$, dilatations $\delta x^\mu = a x^\mu$, Lorentz transformations $\delta x^\mu = \omega^\mu_\nu x^\nu$ and special conformal transformations $\delta x^\mu = \beta^\mu x^2 - 2x^\mu \beta_\nu x^\nu$. The finite conformal transformations and the corresponding generators are given by

$$\begin{array}{ccc}
\text{Translations} & x'^\mu = x^\mu + \alpha^\mu & P_\mu = -i\partial_\mu \\
\text{Lorentz transformations} & x'^\mu = M^\mu_\nu x^\nu & L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
\text{Dilatations} & x'^\mu = \lambda x^\mu & D = -i x^\mu \partial_\mu \\
\text{Special conformal transf.} & x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x - b^2 x^2} & K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)
\end{array} \tag{2.48}$$

How do conformal transformations act on fields? We are interested in fields with a definite conformal dimension $\Delta$ which is just the mass dimension of the classical fields. The action

[3] In the literature one sometimes reads the term dilation instead of dilatation. We will use the latter throughout this thesis.

of the conformal generators on such fields is

$$\begin{aligned} P_\mu \varphi_\Delta(x) &= -i\partial_\mu \varphi_\Delta(x) \\ M_{\mu\nu}\varphi_\Delta(x) &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\varphi_\Delta(x) + \Sigma_{\mu\nu}\varphi_\Delta(x) \\ D\varphi_\Delta(x) &= -i(\Delta + x^\mu\partial_\mu)\varphi_\Delta(x) \\ K_\mu\varphi_\Delta(x) &= -i(2\Delta x_\mu + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\varphi_\Delta(x) - x^\nu\Sigma_{\mu\nu}\varphi_\Delta(x) \end{aligned} \tag{2.49}$$

Where $\varphi_\Delta$ stands for any of the fields in the theory and the subscript $\Delta$ indicates its conformal dimension. The fields in $\mathcal{N} = 4$ SYM have the conformal dimensions $\Delta_\Phi = \Delta_A = 1$ and $\Delta_\Psi = \frac{3}{2}$.
We can now show the invariance of the action under the transformations (2.49). We restrict ourselves to show the action of the dilatations and special conformal transformations on the kinetic term of the gauge field. Doing an infinitesimal dilatation we obtain

$$F_{\mu\nu}F^{\mu\nu} \to (1 - i\sigma D)F_{\mu\nu}(1 - i\sigma D)F^{\mu\nu} \tag{2.50}$$

with an infinitesimal parameter $\sigma$. The conformal dimension of the gauge potential is one, hence the conformal dimension of the field strength is equal to two. Using this and (2.49) the dilatation in (2.50) becomes to first order in $\sigma$

$$\begin{aligned} (1 - i\sigma D)F_{\mu\nu}(1 - i\sigma D)F^{\mu\nu} &= F_{\mu\nu}F^{\mu\nu} - i\sigma DF_{\mu\nu}F^{\mu\nu} - i\sigma F_{\mu\nu}DF^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - 2\sigma x^\rho\partial_\rho F_{\mu\nu}F^{\mu\nu} - 4\sigma F_{\mu\nu}F^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - \sigma\partial_\rho(x^\rho F_{\mu\nu}F^{\mu\nu}). \end{aligned} \tag{2.51}$$

The change in the Lagrangian is just a surface term which can be neglected assuming the fields are vanishing at infinity.
The special conformal transformation of the kinetic term of the gauge field is given by

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} \quad &\to (1 - i\alpha^\rho K_\rho)F_{\mu\nu}F^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - \epsilon^\rho F_{\mu\nu}(8x_\rho + x^\rho x^\sigma\partial_\sigma - 2x^2\partial_\rho)F^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - 2\epsilon^\rho\partial_\sigma(x_\rho x^\sigma F_{\mu\nu}F^{\mu\nu}) + \epsilon^\rho\partial_\rho(F_{\mu\nu}F^{\mu\nu}x^2) \end{aligned} \tag{2.52}$$

and again the change is just a surface term which vanishes at infinity. In a similar way, one can show the invariance of the remaining terms in the action (2.40) under conformal transformations.

We have seen that the symmetries of $\mathcal{N} = 4$ SYM are R-symmetry, forming the group $SU(4) \cong SO(6)_R$, the conformal symmetry with the group $SO(2,4) \cong SU(2,2)$ and Poincaré supersymmetry, generated by the supercharges $Q^a_\alpha$. The bosonic subalgebras for the R-symmetry and conformal symmetry, $SO(2,4)$ and $SO(6)_R$ commute, but the Poincaré supersymmetries and the special conformal transformations $K_\mu$ do not. Both are symmetries of the theory, thus we expect their commutator $[Q^\alpha_a, K_\mu]$ to be a symmetry of the theory as well. These are the generators $S^\alpha_a$ of the so called conformal supersymmetry, and the full symmetry group of $\mathcal{N} = 4$ SYM is therefore $SU(2,2|4)$.

# Chapter 3

# String Theory, Supergravity and D-branes

## 3.1 Superstring Theory

In the following we review open and closed superstring theory. We start with the action for supersymmetric strings and then turn to the equations of motion. We present the solutions with respect to the boundary conditions for open and closed strings, and quantize the theory in both sectors. We finally present the field content of the theory.

The propagation of a supersymmetric string in the superconformal gauge is described by the action

$$S = -\frac{T}{2}\int_M d\tau d\sigma \left(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{3.1}$$

with $T = (2\pi\alpha')^{-1}$ the string tension and M the world sheet parametrized by the coordinates $\tau \in (-\infty, \infty)$ and $\sigma \in [0, \beta\pi]$, with $\beta = 1$ for open strings and $\beta = 2$ for closed strings. The coordinates $X^\mu(\tau, \sigma)$ describe the embedding of the string into the given (in our case 10 dimensional) spacetime[1], in general described by an arbitrary metric $g_{\mu\nu}$, but in the following we will choose the metric to be flat, $g_{\mu\nu} = \eta_{\mu\nu}$. $\psi^\mu(\tau, \sigma)$ is a world-sheet Majorana spinor

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{3.2}$$

with $\psi^*_\pm = \psi_\pm$. The matrices $\rho^\alpha$ form a Clifford algebra $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ in two dimensions and we take $\eta^{\alpha\beta} = diag(-+)$. We can then choose

$$\rho^0 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right), \quad \rho^1 = \left(\begin{array}{cc} 0 & i \\ i & 0 \end{array}\right). \tag{3.3}$$

The action (3.1) is the Polyakov action with a supersymmetric extension in the superconformal gauge. It is invariant under the following supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta_\epsilon \psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon \tag{3.4}$$

[1]The 10-dimensional spacetime is a necessary consequence if we demand Lorentz invariance on quantum level.

with $\epsilon$ a constant, anticommuting two-component spinor. Varying the action (3.1) with respect to $X^\mu$ we obtain the equations of motion and boundary conditions for the string coordinates

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0 \tag{3.5}$$
$$\partial_\sigma X \cdot \delta X|_{\sigma=\beta\pi} - \partial_\sigma X \cdot \delta X|_{\sigma=0} = 0. \tag{3.6}$$

The boundary conditions (3.6) can be satisfied in two ways. If the string ends coincide we obtain the periodicity condition

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \tag{3.7}$$

which describes closed strings.
For open strings we obtain the boundary conditions

$$\partial_\sigma X \cdot \delta X|_{\sigma=0,\pi} = 0, \tag{3.8}$$

which correspond to Dirichlet ($\delta X^\mu|_{\sigma=0,\pi} = 0$) or Neumann ($\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$) boundary conditions.
The equations of motion (3.5) with respect to the closed string boundary conditions in (3.7) are solved by a sum of left and right moving coordinates

$$\begin{aligned} X^\mu(\tau,\sigma) &= X_L^\mu + X_R^\mu \\ X_L^\mu &= \frac{1}{2}x_0^\mu + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \\ X_R^\mu &= \frac{1}{2}x_0^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)} \end{aligned} \tag{3.9}$$

and using equation (3.7) we obtain

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu. \tag{3.10}$$

The center of mass momentum is given by

$$p^\mu = \frac{\delta S}{\delta\partial_\tau X^\mu} = -\frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma(-2\partial_\tau X^\mu) = \sqrt{\frac{1}{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu. \tag{3.11}$$

The equations of motion with respect to the open string boundary conditions in (3.8) lead to

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma), \tag{3.12}$$

and the relation between the zero modes and the center of mass momentum is now given by

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}\alpha_0^\mu. \tag{3.13}$$

In (3.12) we have imposed full Neumann boundary conditions which corresponds to freely moving strings. Dirichlet boundary conditions will become important in connection with

D-branes and we will describe them in the next section.
Because the embedding coordinates $X^\mu$ are real the center of mass momentum and position $p^\mu$ and $x_0^\mu$ should be real as well and the oscillator modes $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ satisfy

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu, \qquad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu. \tag{3.14}$$

Varying the fermionic part in equation (3.1) we obtain the equations of motion and boundary conditions

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0 \tag{3.15}$$
$$[\psi_+ \delta\psi_+ - \psi_- \delta\psi_-]_{\sigma=\beta\pi} - [\psi_+ \delta\psi_+ - \psi_- \delta\psi_-]_{\sigma=0} = 0, \tag{3.16}$$

where we introduced $\xi_\pm = \tau \pm \sigma$. Again the boundary conditions can be satisfied in two ways, either by matching the string ends or by letting both parentheses in (3.16) vanish.

We obtain the open string boundary conditions by demanding that the string ends do not coincide and (3.16) becomes

$$\begin{aligned} \psi_+ \delta\psi_+|_{\sigma=0} &= \psi_- \delta\psi_-|_{\sigma=0} \\ \psi_+ \delta\psi_+|_{\sigma=\pi} &= \psi_- \delta\psi_-|_{\sigma=\pi}. \end{aligned} \tag{3.17}$$

These boundary conditions can be satisfied by demanding that $\psi_+ = \pm\psi_-$ at the string ends. The overall relative sign between $\psi_+$ and $\psi_-$ has no influence on the physics, and we set

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0). \tag{3.18}$$

We are then left with two possibilities for the endpoint at $\sigma = \pi$, which are called Ramond (R) and Neveu-Schwarz (NS) boundary conditions

$$\psi_+^\mu(\tau, \pi) = \psi_-^\mu(\tau, \pi) \qquad (R) \tag{3.19}$$
$$\psi_+^\mu(\tau, \pi) = -\psi_-^\mu(\tau, \pi) \qquad (NS) \tag{3.20}$$

In the closed string case $\psi_\pm$ are independent and (3.16) leads to the periodicity conditions

$$\psi_-^\mu(\tau, \sigma) = \beta_1 \psi_-^\mu(\tau, \sigma + 2\pi), \qquad \psi_+^\mu(\tau, 0) = \beta_2 \psi_+^\mu(\tau, \sigma + 2\pi). \tag{3.21}$$

We obtain four different sectors

$$\begin{aligned} \beta_1 &= \beta_2 = 1 \qquad (R - R) \\ \beta_1 &= \beta_2 = -1 \qquad (NS - NS) \\ \beta_1 &= -\beta_2 = 1 \qquad (R - NS) \\ \beta_1 &= -\beta_2 = -1 \qquad (NS - R). \end{aligned} \tag{3.22}$$

The solution for the equations of motion with respect to the open string boundary conditions in (3.20) is

$$\psi_\pm^\mu = \frac{1}{\sqrt{2}} \sum_r \psi_r^\mu e^{-ir(\tau\pm\sigma)} \quad \text{for} \begin{cases} r \in \mathbb{Z} + \frac{1}{2} \rightarrow \text{NS sector} \\ r \in \mathbb{Z} \rightarrow \text{R sector} \end{cases}$$

and the one satisfying the closed string boundary conditions (3.22) is

$$
\begin{aligned}
\psi_-^\mu &= \frac{1}{\sqrt{2}}\sum_r \psi_r^\mu e^{-ir(\tau-\sigma)} \quad \text{for} \begin{cases} r\in\mathbb{Z}+\frac{1}{2}\rightarrow \text{NS sector}\\ r\in\mathbb{Z}\rightarrow \text{R sector}\end{cases}\\
\psi_+^\mu &= \frac{1}{\sqrt{2}}\sum_r \tilde{\psi}_r^\mu e^{-ir(\tau+\sigma)} \quad \text{for} \begin{cases} r\in\mathbb{Z}+\frac{1}{2}\rightarrow \tilde{\text{NS}}\text{ sector}\\ r\in\mathbb{Z}\rightarrow \tilde{\text{R}}\text{ sector}\end{cases}
\end{aligned}
\tag{3.23}
$$

with left and right moving modes $\psi_r^\mu$ and $\tilde{\psi}_r^\mu$ where the Majorana-condition for $\psi^\mu$ leads to a constraint for the modes

$$
\psi_{-r}^\mu = (\psi_r^\mu)^*, \qquad \tilde{\psi}_{-r}^\mu = (\tilde{\psi}_r^\mu)^*. \tag{3.24}
$$

We are now ready to quantize the theory to obtain the state space and the corresponding fields. Canonical quantization leads to the (anti-)commutation rules for the bosonic and fermionic oscillators as well as the center of mass position and momentum

$$
[\alpha_m^\mu,\alpha_n^\nu] = [\tilde{\alpha}_m^\mu,\tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{3.25}
$$

$$
[x_0^\mu,p^\nu] = i\,\eta^{\mu\nu} \tag{3.26}
$$

$$
\{\psi_r^\mu,\psi_s^\nu\} = \delta_{rs}\eta^{\mu\nu} \tag{3.27}
$$

with all other commutators vanishing.
Before we can construct the state space we have to introduce the operators

$$
L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{n-m}\cdot\alpha_m + \frac{1}{4}\sum_r (2r-n)\psi_{n-r}\cdot\psi_r \tag{3.28}
$$

$$
G_r = \sum_{m=-\infty}^{\infty}\alpha_m\cdot\psi_{r-m} \tag{3.29}
$$

with analogous expressions $\tilde{L}_m$ and $\tilde{G}_r$ in the closed string case. $L_n$ are called Virasoro generators which are the Fourier components of the energy momentum tensor, which has two non vanishing components

$$
T_{++} = \partial_+X\cdot\partial_+X + \frac{i}{2}\psi_+\cdot\partial_+\psi_+, \qquad T_{--} = \partial_-X\cdot\partial_-X + \frac{i}{2}\psi_-\cdot\partial_-\psi_-, \tag{3.30}
$$

and $G_r$ are the Fourier components of the Noether current associated with the supersymmetry transformations (3.4).
The state space for the closed string can be constructed by defining a vacuum $|0\rangle|\tilde{0}\rangle|p\rangle$ which is an eigenstates of the center of mass momenta $p^\mu, \tilde{p}^\mu$ and satisfies the conditions

$$
\begin{aligned}
\alpha_n^\mu|0\rangle|\tilde{0}\rangle|p\rangle &= \tilde{\alpha}_n^\mu|0\rangle|\tilde{0}\rangle|p\rangle = 0, \qquad (n>0)\\
\psi_r^\mu|0\rangle|\tilde{0}\rangle|p\rangle &= \tilde{\psi}_r^\mu|0\rangle|\tilde{0}\rangle|p\rangle = 0 \qquad \begin{cases} r\in\mathbb{N}+\frac{1}{2}\rightarrow \tilde{\text{NS}}\text{ sector}\\ r\in\mathbb{N}_+\rightarrow \tilde{\text{R}}\text{ sector}\end{cases}
\end{aligned}
\tag{3.31}
$$

The open string the vacuum state is just $|0\rangle|p\rangle$ and we only consider one set of operators. Physical states should have positive norm. The states constructed in (3.31) can actually obtain a negative norm due to the Lorentz metric in the (anti)commutators (3.25). It can be shown [8] that physical states satisfy the conditions

$$\begin{aligned}(L_0 - a_0)|\psi_{phys}\rangle &= 0 \\ L_n|\psi_{phys}\rangle &= 0, \qquad n > 0 \\ G_r|\psi_{phys}\rangle &= 0, \qquad r \geq 0\end{aligned} \tag{3.32}$$

with $a_0 = 0$ in the R sector and $a_0 = \frac{1}{2}$ in the NS sector and analogous relations with $\tilde{L}_n$ and $\tilde{G}_r$ for closed strings.

## Open string state space and field content

The normal orderer Virarsoro generator $L_0$ is given by

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{m=0}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r>0} r\psi_{-r} \cdot \psi_r = \alpha' p^2 + N_b + N_f \tag{3.33}$$

where we used $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ and $N_b$ and $N_f$ are the number operators counting the bosonic and fermionic modes

$$N_b = \sum_{m=0}^{\infty} \alpha_{-m} \cdot \alpha_m, \qquad N_f = \sum_{r>0} r\psi_{-r} \cdot \psi_r \tag{3.34}$$

From (3.32) we obtain the mass formula for the open string

$$m^2 = -p_\mu p^\mu = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r>0} r\psi_{-r} \cdot \psi_r - a_0\right). \tag{3.35}$$

We can now construct the states by acting with creation operators $\psi_{-r}^\mu$ on the vacuum $|0, p\rangle = |0\rangle|p\rangle$ and we are mainly interested in the massless states of the theory. In the NS sector we obtain

$$A_\mu(p)\psi_{-1/2}^\mu|0, p\rangle \tag{3.36}$$

which corresponds to a vector field with transverse polarization, $p \cdot A = 0$.
In the R sector $a_0 = 0$ and the states $\psi_0^\mu|0, p\rangle$ are also vacuum states according to (3.31). The fermionic zero modes $\psi_0^\mu$ satisfy the Dirac algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}, \tag{3.37}$$

and can be represented by Γ-matrices. Therefore we label the R sector ground state by a spinor index $|0, p\rangle_R = |A, p\rangle$. The massless state in the R sector is then given by

$$u_A(p)|A, p\rangle, \qquad A = 1, \ldots, 16 \tag{3.38}$$

which corresponds to a Majorana spinor in $d = 10$ dimensions.
To obtain spacetime supersymmetry the physical bosonic degrees of freedom have to match with the fermionic ones. The vector field from the NS sector has 8 physical degrees

of freedom (superstring theory is only well defined in $D = 10$ dimensions [9]), but the Majorana spinors in ten dimensions have 16 physical degrees of freedom. To preserve supersymmetry we have to project out some fermionic states. This is done by the so called GSO projection, where we replace physical states according to

$$|\psi_{phys}\rangle \rightarrow P_{GSO}|\psi_{phys}\rangle. \tag{3.39}$$

The projection operator $P_{GSO}$ is defined as

$$P_{GSO} = \frac{1}{2}\left(1 - (-1)^F\right). \tag{3.40}$$

In the R sector the GSO projection is done to get rid of the superfluous fermionic degrees of freedom while in the NS sector it projects out the states with half integer mass (in units of $\alpha'$). The operator $(-1)^F$ is defined as

$$(-1)^F = \begin{cases} (-1)^{\sum\limits_{r=1/2}^{\infty}\psi_{-r}\cdot\psi_r} & \text{NS sector} \\ \pm\Gamma^{11}(-1)^{\sum\limits_{r=1}^{\infty}\psi_{-r}\cdot\psi_r} & \text{R sector} \end{cases} \tag{3.41}$$

Equation (3.39) leads to the required supersymmetry as well as it projects out the tachyon $|0, p\rangle$ with negative mass from the NS sector.

The massless fields from the open string sector are therefore a gauge field with eight physical degrees of freedom and a Majorana-Weyl spinor with the same amount of physical degrees of freedom. We should also mention that in (3.41) we either choose the positive or the negative sign in the R sector which corresponds to choosing fermions with positive or negative chirality.

## Closed string state space and field content

Physical states in the closed string sector satisfy the conditions in equation (3.32) as well as the analogous ones with operators $\tilde{L}$ and $\tilde{G}$. We also have to include the level matching condition

$$(L_0 - \tilde{L}_0 - a_0 + \tilde{a}_0)|\psi_{phys}\rangle = 0. \tag{3.42}$$

The spectrum is then obtained as

$$m^2 = \frac{2}{\alpha'}\left(N_b + N_f + \tilde{N}_b + \tilde{N}_f - a_0 - \tilde{a}_0\right) \tag{3.43}$$

with the number operators $N$ given in equation (3.34) and the $\tilde{N}$'s are defined in an analogous way.

The states space is built by acting with creation operators $\psi^{\mu}_{-n}$ on the vacuum $|0\rangle|\tilde{0}\rangle|p\rangle$ and like in the open string case we have to do a GSO projection, in each of the four closed string sectors. This is done by using the projection operators from equations (3.40) and (3.41) acting on the left moving states and equivalent ones acting on the right moving states $|\tilde{0}\rangle$, defined by

$$\begin{aligned} \tilde{P}_{GSO} &= \frac{1}{2}\left(1 - (-1)^{\tilde{F}}\right) \\ (-1)^{\tilde{F}} &= \begin{cases} (-1)^{\sum\limits_{r=1/2}^{\infty}\tilde{\psi}_{-r}\cdot\tilde{\psi}_r} & \tilde{\text{NS}} \text{ sector} \\ \pm\Gamma^{11}(-1)^{\sum\limits_{r=1}^{\infty}\tilde{\psi}_{-r}\cdot\tilde{\psi}_r} & \tilde{\text{R}} \text{ sector} \end{cases} \end{aligned} \tag{3.44}$$

In the R-R sector we can choose the projection operators $P_R$ and $\tilde{P_R}$ either with the same sign or with different sign in the definition of $(-1)^F$ and $(-1)^{\tilde{F}}$. For equal signs the spinors $u_A$ and $\tilde{u}_A$ have the same chirality and we obtain a chiral theory which is called type IIB string theory. In the case of different signs we cannot assign a defined chirality, and the theory is called type IIA string theory.

Again we want to find the field content in the massless case. The massless states in the NS-NS sector are

$$\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|0\rangle|\tilde{0}\rangle|p\rangle. \tag{3.45}$$

The corresponding polarization tensor $\alpha_{\mu\nu}$ can be decomposed as

$$\alpha_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu} + \eta_{\mu\nu}\phi \tag{3.46}$$

with $S_{\mu\nu}$ a symmetric traceless tensor corresponding to the graviton, $A_{\mu\nu}$ an antisymmetric tensor corresponding to the NS-NS 2-form field and $\phi$ a scalar corresponding to the Dilaton.
In the R-NS sector the massless state is

$$u_A(p)\epsilon_\mu(p)\psi^\mu_{-1/2}|A\rangle|\tilde{0}\rangle|p\rangle \tag{3.47}$$

and the vector spinor $\chi_\mu = u_A(p)\epsilon_\mu(p)$ can be decomposed as

$$\chi_\mu = \frac{1}{10}\Gamma_\mu(\Gamma^\nu\chi_\nu) + \left(\chi_\mu - \frac{1}{10}\Gamma_\mu(\Gamma^\nu\chi_\nu)\right). \tag{3.48}$$

where the two terms corresponds to the superpartner of the dilaton and the graviton respectively, the dilatino with spin 1/2 and the gravitino with spin 3/2. We can make the same considerations in the NS-R sector and we obtain another gravitino and dilatino.

Finally, the massless state in the R-R sector is

$$u_A(p)\tilde{u}_B(p)|A\rangle|B\rangle|p\rangle \tag{3.49}$$

The direct product of the spinors $u_A(p)\tilde{u}_B(p) = u_A(p)\otimes\tilde{u}_B(p)$ can be decomposed into antisymmetric tensors [8]. This is done in the following way: The R-R vacuum state $|A\rangle|\tilde{B}\rangle$ can be written in terms of the NS-NS vacuum by introducing spin fields $S^A(z), \tilde{S}^B(\bar{z})$ defined by

$$\lim_{z\to 0} S^A(z)\tilde{S}^B(\bar{z})|0\rangle|\tilde{0}\rangle = |A\rangle|B\rangle \tag{3.50}$$

where we used the conformal formulation of the theory with coordinates

$$z = e^{\beta i(\tau-\sigma)} \tag{3.51}$$

with $\beta = 2$ in the closed string case (and $\beta = 1$ for open strings, for a more detailed description see for example [7]). We then expand (3.49) as

$$\begin{aligned} u_A(p)\tilde{u}_B(p)|A\rangle|B\rangle|p\rangle &= \frac{1}{2^5}\sum_{n=0}^{9}\frac{(-1)^{n+1}}{n!}u_A(p)(\Gamma_{[\mu_1}\ldots\Gamma_{\mu_n]}C^{-1})^{AB}\tilde{u}_B(p)\times \\ &\times\lim_{z\to 0} S^C(z)(C\Gamma^{[\mu_1}\ldots\Gamma^{\mu_n}])_{CD}\tilde{S}^D(\bar{z})|0\rangle|\tilde{0}\rangle|p\rangle \end{aligned} \tag{3.52}$$

with $\Gamma_{[\mu_1} \dots \Gamma_{\mu_n]}$ the totally antisymmetric product of the involved $\Gamma$ matrices and $C$ the charge conjugation operator.
Defining

$$F_{\mu_1 \dots \mu_n} = u \Gamma_{[\mu_1} \dots \Gamma_{\mu_n]} C^{-1} \tilde{u} \tag{3.53}$$

the state (3.52) can be written in terms of the antisymmetric tensors $F_{\mu_1 \dots \mu_n}$. The sum in (3.52) suggests that we should consider $F_n$ with $n = 0 \dots 10$. But in type IIA non chiral string theory only terms with even $n$, while in the IIB chiral case only terms with odd $n$ contribute. Further, the terms for $p \geq 5$ are obtained by taking the Hodge dual of the corresponding $n$-form with, thus we only consider the case for $n \leq 5$. Summarizing we obtain the following field strength's

$$\text{Type II A} \qquad F_2,\ F_4 \tag{3.54}$$

$$\text{Type II B} \qquad F_1,\ F_3,\ F_5 \tag{3.55}$$

where the 5-form $F_5$ satisfies the self-duality condition $F_5 = *F_5$.
The massless field content of the closed string sector of type II superstring theory can be summarized as foolows. From the NS-NS sector we obtain the graviton $g_{\mu\nu}$ the anti-symmetric 2-form (Kalb-Ramond-field) $B_{\mu\nu}$ and the dilaton $\phi$. From the R-NS and NS-R sectors we obtain obtain each a gravitino and a dilatino, and from the R-R sector we obtain the $n$-form field strengths $F_n$ which are related to the $(n-1)$-form R-R potentials by $F_n = dC_{n-1}$. In the case $n$ even we obtain type IIA and for $n$ odd type IIB superstring theory.

## 3.2 D-Branes and T-Duality

In addition to fundamental strings, type II superstring theory contains other objects, the Dp-branes, which are $p$- dimensional hypersurfaces where open strings can end on. Their $(p+1)$-dimensional world volume is embedded into the 10-dimensional spacetime. These objects are necessary for the consistency of the theory because elementary string states cannot carry R-R charges [9], thus we need other objects in the theory which are charged under the (p+1)-form potential $C^{p+1}$. A minimal coupling is given by the pullback of $C^{(p+1)}$ to the world volume of the brane

$$\int d^{p+1}\xi \epsilon^{\alpha_0 \dots \alpha_p} \frac{\partial x^{\mu_0}}{\partial \xi^{\alpha_0}} \dots \frac{\partial x^{\mu_p}}{\partial \xi^{\alpha_p}} C^{(p+1)}_{\mu_0 \dots \mu_p}. \tag{3.56}$$

where $\xi^{\alpha_i}$ parametrize the worldvolume of the Dp-brane.
A natural way to see that D-branes are part of open string theory is to impose Dirichlet boundary conditions for the string endpoints in some directions $x^a$

$$\begin{aligned} X^a(\tau, 0) &= x_1^a(\tau), \\ X^a(\tau, \pi) &= x_2^a(\tau), \qquad a = p+1, \dots, 9. \end{aligned} \tag{3.57}$$

The Dp-branes are then the hypersurfaces defined by (3.57) with tangential coordinates $x^\mu, (\mu = 0, 1, \dots p)$ and transverse coordinates $x^a$.
A symmetry of string theory that is closely related to D-branes is T-Duality. In the

following we study T-duality for open and closed (bosonic) strings[2]. We will see how D-branes are a natural part of string theory and that they carry a gauge theory on their world volume, which we will introduce in the end of this section.

### 3.2.1 T-Duality for Closed Strings

In closed string theory the mode expansion for the string embedding fields was given in (3.9) and we rewrite it in terms of left and right moving coordinates[3] using (3.10)

$$\begin{aligned} x^m &= x_L^m + x_R^m \\ &= x_0^m + \sqrt{\alpha'/2}(\alpha_0^m + \tilde{\alpha}_0^m)\tau + \sqrt{\alpha'/2}(\alpha_0^m - \tilde{\alpha}_0^m)\sigma + \ldots \end{aligned} \tag{3.58}$$

with $m = 0, \ldots, 9$ and the dots in (3.58) stand for the oscillators. Considering that the embedding coordinates have to be periodic

$$x^m(\tau, \sigma) = x^m(\tau, \sigma + 2\pi), \tag{3.59}$$

we obtain, by inserting $\sigma = \sigma' + 2\pi$ into equation (3.58),

$$x^m(\tau, \sigma' + 2\pi) = x^m(\tau, \sigma') + 2\pi\sqrt{\alpha'/2}\,(\alpha_0^m - \tilde{\alpha}_0^m)\,. \tag{3.60}$$

Comparing this with equation (3.59) leads to

$$\alpha_0^m = \tilde{\alpha}_0^m. \tag{3.61}$$

The results in the previous equations hold as long as all directions are flat and infinitely extended. Compactifying one dimension on a circle with radius R, say

$$x^9 \sim x^9 + 2\pi R, \tag{3.62}$$

there will be some significant changes in the physics. Considering that the translation operator in the 9-direction should be single valued under (3.62)

$$e^{ip^9 x^9} = e^{ip^9(x^9 + 2\pi R)} \tag{3.63}$$

leads to the quantization of the momentum in the compactified direction

$$p^9 = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^9 + \tilde{\alpha}_0^9) = \frac{n}{R}, \qquad n = 1, 2, \ldots \tag{3.64}$$

where we used equation (3.11). It is the same effect that we would obtain for a point like particle, but for strings there is another phenomenon. Because of the identification in

[2]In connection with D-branes it is natural to consider the bosonic part of strings. Nevertheless T-Duality can be extended to closed superstrings, and in this case it interchanges type IIA and IIB string theory

[3]In this section we use the index $m$ for all ten spacetime directions and $\mu, \nu$ for the directions along the brane. We also do not use different notation for the spacetime coordinates and the string embedding fields, as opposed to the notation in the last section.

equation (3.62), the embedding coordinates for the string ends may differ by the winding around the compact dimension

$$x^9(\tau,\sigma+2\pi)-x^9(\tau,\sigma)=2\pi wR \tag{3.65}$$

with $w$ an integer number describing how often the string winds around the circle. Using (3.58) in (3.65) we obtain for the difference between the left and the right moving zero modes

$$\alpha_0^9-\tilde{\alpha}_0^9=\sqrt{\frac{2}{\alpha'}}wR. \tag{3.66}$$

By combing equations (3.64) and (3.66) we find

$$\begin{aligned}\alpha_0^9&=\sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)\\ \tilde{\alpha}_0^9&=\sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)\end{aligned} \tag{3.67}$$

The mass spectrum for the remaining 1+8 uncompactified dimensions can be found using the constraints for physical states in equation (3.32). The normal ordered Virasoro zero modes become [4]

$$L_0=\frac{1}{2}(\alpha_0^9)^2+\frac{\alpha'}{4}p^\mu p_\mu+N_b \tag{3.68}$$

$$\tilde{L}_0=\frac{1}{2}(\tilde{\alpha}_0^9)^2+\frac{\alpha'}{4}p^\mu p_\mu+\tilde{N}_b \tag{3.69}$$

$$\mu=0\ldots 8$$

where we have used that $\sqrt{\frac{\alpha'}{2}}p^\mu=\alpha_0^\mu=\tilde{\alpha}_0^\mu$ still holds for the uncompactified directions. Using the conditions for physical states (3.32) we obtain

$$m^2=-p_\mu p^\mu=\frac{n^2}{R^2}+\frac{w^2R^2}{\alpha'^2}+\frac{2}{\alpha'}\left(N_b+\tilde{N}_b-a_0-\tilde{a}_0\right). \tag{3.70}$$

The mass formula is changed for closed strings winding around a compact dimension and we obtain a contribution from both, the Kaluza-Klein and winding states, where the latter is a purely stringy phenomenon.
Having a closer look at the mass formula (3.70) we can see that it is symmetric under the T-duality transformations

$$n\leftrightarrow w,\quad R\leftrightarrow R'=\frac{\alpha'}{R}. \tag{3.71}$$

The theory with one dimension compactified on a circle with radius $R$, momentum $n$ and winding $w$ is therefore equivalent to a theory with radius $R'=\alpha'/R$, winding $n$ and momentum $w$.
The whole symmetry can be realized by a transformation of the embedding coordinate in the compactified direction

$$x'^9=x_L^9-x_R^9. \tag{3.72}$$

[4]Here and in the following we only consider the bosonic oscillators.

We then obtain

$$\sqrt{2\alpha'}p'^9 = \left(\alpha_0^9 - \tilde{\alpha}_0^9\right) = 2wR = 2\frac{n'\alpha'}{R'} \tag{3.73}$$

where we have used equation (3.71). We conclude that the momentum in the dual theory is the winding in the original theory and the radii are interchanged according to (3.71).

### 3.2.2 T-Duality for open Strings

In the open string case we consider a string world sheet given by $\tau \in (-\infty, \infty)$, $\sigma \in [0, \pi]$. Let us again start by compactifying one spatial dimension on a circle with radius $R$

$$x^9 \sim x^9 + 2\pi R, \tag{3.74}$$

leading to the quantization of the momentum like in equation (3.64)

$$p^9 = \frac{n}{R}. \tag{3.75}$$

We assume that the string ends obey Neumann boundary conditions for all ten spacetime dimensions

$$\partial_\sigma x^m(\tau, \sigma)|_{\sigma=0,\pi} = 0. \tag{3.76}$$

The endpoints of an open string can thus move freely in space, in particular along the compact coordinate. Closed strings obtain winding if they wrap around a compactified dimension, but open strings are not fixed in this direction so in general we can still continuously deform them to a point hence we cannot assign winding. The T-dual picture is obtained by transforming the radius of the compactified dimension in the same way as we did in the closed string case,

$$R' = \frac{\alpha'}{R} \tag{3.77}$$

but it is only a symmetry of the theory, if we also change the boundary conditions along the compactified direction from Neumann to Dirichlet

$$\partial_\tau x'^9(\tau, \sigma)|_{\sigma=0,\pi} = 0. \tag{3.78}$$

The string ends are now fixed on a point on the compactified dimension, they are restricted to move on a $(9-1)$-dimensional hypersurface, a D8- brane.
By doing a T-duality transformation we have therefore naturally uncovered Dp-branes, which will play an important role in the following chapters.
We fixed the string coordinates at a point on the compactified dimension, and by using the identification (3.74), we obtain

$$x'^9|_{\sigma=\pi} - x'^9|_{\sigma=0} = 2\pi w R' \tag{3.79}$$

showing that in the dual picture we can actually assign winding.
For a more detailed discussion we proceed as we did in the closed string case. The embedding coordinates and boundary conditions for open strings propagating freely in

space are given by

$$
\begin{aligned}
x^m &= x^m(\tau+\sigma) + x^m(\tau-\sigma) && (3.80)\\
&= x_0^m + \alpha' p^m \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^m}{n}e^{-in\tau}\cos(n\sigma)\\
x^m(\tau\pm\sigma) &= \frac{1}{2}(x_0^m \pm q_0^m) + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau\pm\sigma) && (3.81)\\
&+ i\frac{\sqrt{\alpha'}}{2}\sum_{n\neq 0}\frac{\alpha_n^m}{n}e^{-in(\tau\pm\sigma)}\\
\partial_\sigma x^m|_{\sigma=0,\pi} &= 0 && (3.82)
\end{aligned}
$$

and we have used that the center of mass momentum is given by

$$
p^m = \int d\sigma \partial_\tau x^m = \frac{1}{\sqrt{2\alpha'}}\alpha_0^m. \tag{3.83}
$$

Compactifying one direction we obtain the quantization of the physical momentum like in equation (3.64). As in the closed string case the T-dual picture is given by subtracting the left and right movers in the compact direction

$$
x'^9 = x^9(\tau+\sigma) - x^9(\tau-\sigma) = q_0^9 + 2\alpha'\frac{n}{R}\sigma + \frac{1}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^9 e^{-in\tau}\sin n\sigma \tag{3.84}
$$

where we have used equations (3.81) and (3.64). The Neumann and Dirichlet boundary conditions are automatically interchanged

$$
\begin{aligned}
\partial_\tau x'^9|_{\sigma=0,\pi} &= 0 && (3.85)\\
x'^9(\tau,\pi) - x'^9(\tau,0) &= 2\pi\alpha'\frac{n}{R} = 2\pi n R' && (3.86)
\end{aligned}
$$

and we see that the string obtains winding $w' = n$ around the compact direction with radius $R'$ in the dual picture. By doing a T-duality transformation we imposed Dirichlet boundary conditions on the $x^9$ coordinate, fixing the string at a point on the compactified direction. The strings can still move freely along the 1+8 remaining directions of the 10-dimensional spacetime, and we have found a D8-brane with a nine dimensional world volume. A Dp-brane wrapping a compact dimension becomes after a T-duality transformation a D$(p-1)$-brane that is fixed on a point at the compact dimension. T-Duality therefore relates Dp-branes to D(p±1)-branes. We can generalize this picture by compactifying more dimensions and for $n$ T-duality transformations we obtain a Dp-brane with $p = 9 - n$.

### 3.2.3 D-Branes in type II string theory

In the last section we have seen that Dp-branes are p-dimensional hypersurfaces with a $(p+1)$-dimensional world volume. Open strings start and end on them and we uncover D-branes by imposing Dirichlet rather than Neumann boundary conditions on the transverse directions. The boundary conditions corresponding to a Dp-brane are

$$
\begin{aligned}
\partial_\sigma x^\mu(\tau,\sigma)|_{\sigma=0,\pi} &= 0,\ \mu = 0,1,\ldots,p,\ \text{Neumann b.c.} && (3.87)\\
\delta x^i(\tau,\sigma)|_{\sigma=0,\pi} &= 0,\ i = p+1\ldots 9,\ \text{Dirichlet b.c.} && (3.88)
\end{aligned}
$$

The strings can move freely along the directions with Neumann boundary conditions, while they are fixed to the Dp-brane in the transverse directions.
Dp-branes in type II superstring theory couple to R-R potentials $C^{(p+1)}$ and recalling the different potentials we obtained from the closed string sector in type IIA and IIB string theory, section 3 , we can identify which Dp-branes will appear in the theory.
In type IIA the R-R potentials and the corresponding Dp-branes are given as

$$\begin{array}{c} C^1, C^3, C^5, C^7, C^9 \\ p = 0 \;\; 2 \;\; 4 \;\; 6 \;\; 8 \end{array} \tag{3.89}$$

where the D0-brane is a particle and the D0- and D6-brane as well as the D2- and D4-brane are connected via a duality transformation, see appendix A.
In type IIB theory we find the R-R potentials and corresponding Dp-branes

$$\begin{array}{c} C^0, C^2, C^4, C^6, C^8 \\ p = -1 \;\; 1 \;\; 3 \;\; 5 \;\; 7 \;\; 9 \end{array} \tag{3.90}$$

where the case $p = -1$ corresponds to a "D-instanton" and the case $p = 1$ to a "D-string". Again we have an electric-magnetic duality between the D1- and D5-brane, the D-instanton and the D7-brane whereas the D3-brane is self dual. The D9-brane is a space filling one and there is no R-R- potential it can couple to.

We now turn to a non perturbative description of the D-branes. The massless modes of open bosonic string theory are the gauge fields $A_m$ and we can split them into components along and transverse to the worldvolume of the Dp-brane

$$A_m = (A_\mu, A_a)\,,\;\; \mu = 0, 1, \ldots, p;\;\; a = p+1, \ldots, 9. \tag{3.91}$$

We will find that the gauge field components along the D-brane describe a gauge theory living on the D-brane. The components perpendicular can be interpreted as scalar fields,

$$\Phi^a = A^a,\;\; a = p+1, \ldots, 9, \tag{3.92}$$

which describe the transverse displacement of the brane. The remaining massless modes are fermions coming from the R-NS and NS-R sector.
We saw that a T-duality transformation goes along with a compactification of one or more spatial directions, and the low energy description of the gauge theory on the brane is described by a supersymmetric Yang-Mills theory, which can be obtained by dimensional reduction from the maximally supersymmetric theory in ten dimensions [9].

### 3.2.4 T-Duality and Gauge Fields

String ends carry charge with respect to the gauge field $A_\mu$ living on the branes, a fact leading us to add a coupling term of the form $S_A = \int d\tau A_\mu \dot{x}^\mu$ to the string action. The string ends carry opposite charge and the action becomes

$$\begin{aligned} S &= S_p + S_A \\ &= T \int d\tau d\sigma\; \eta^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu \\ &+ \int d\tau A_\mu \dot{x}^\mu(\tau,\sigma)|_{\sigma=\pi} - \int d\tau A_\mu \dot{x}^\mu(\tau,\sigma)|_{\sigma=0} \end{aligned} \tag{3.93}$$

where $S_p$ is the Polyakov action in the conformal gauge and we have chosen the string end at $\sigma = \pi$ to be charged with $q = +1$ , the one at $\sigma = 0$ with $q = -1$. The two terms in $S_A$ cancel each other if the string ends are coupling to the same gauge field, equivalent to ending on the same brane, but in the more general case with the string attached to two different branes this leads to non trivial results.

We first consider an $U(1)$ gauge field and just one string end coupling to it, and we keep in mind that this term might be canceled if the other string end is attached to the same brane as well. The coupling term in a path integral formalism is obtained as

$$W = e^{iq\int_C A_\mu dx^\mu} \tag{3.94}$$

with the appropriate charge $q$. This defines a Wilson line along the path $C$. We now compactify one direction like in equation (3.74) and make a gauge choice for the $U(1)$-field to point along this direction $A_\mu = \delta_{\mu,9}A_9$. A field that trivially satisfies the Maxwell equations and the gauge parameter $U(x)$ are respectively given by

$$A_9 = \frac{\theta}{2\pi R} = \frac{-i}{q}\partial_9 U U^{-1}, \quad U(x) = \exp\big(\frac{iq\theta x^9}{2\pi R}\big). \tag{3.95}$$

Locally this is pure gauge and we can transform it away. But $U(x^9)$ is not single valued under $x^9 \to x^9 + 2\pi R$ and the gauge transformation leads to a phase factor $e^{iq\theta}$ for the transformed states. The effect is measured by the Wilson line

$$W = e^{iq\oint dx^9 A_9} = e^{iq\theta}, \tag{3.96}$$

which is the phase factor associated with the gauge field along the compact $x^9$ direction.

The canonical momentum along this direction obtains an additional contribution and considering the string end with $q = -1$ we obtain

$$p_c^9 = \frac{\partial L}{\partial \dot{x}^9} = \int d\sigma \frac{\dot{x}^9}{2\pi\alpha'} - \frac{\theta}{2\pi R} = p^9_{\text{phys}} - \frac{\theta}{2\pi R} = \frac{n}{R} \tag{3.97}$$

where we used (3.93). The physical momentum is therefore given by

$$p^9_{\text{phys}} = \frac{n}{R} + \frac{\theta}{2\pi R} \tag{3.98}$$

and the mass formula becomes

$$m^2 = -(p^9_{\text{phys}})^2 - p_\mu p^\mu = \left(\frac{2\pi n + \theta}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N-1)\,. \tag{3.99}$$

with $\mu = 0, \ldots, 8$. Up to now we have only considered one string end. If the other string end is attached to the same brane we obtain another contribution to the physical momentum in equation (3.98) with a value $-\frac{\theta}{2\pi R}$, which comes from the positively charged string end. Hence the two contributions to mass and momentum cancel each other out due to the oppositely charged string ends. We are left with our usual result with the standard mass formula plus a contribution from the quantization of the momentum along the compact dimension.

This changes if we consider $N$ coincident $D$-branes labeled by Chan-Paton factors, equivalent to an U(N) gauge theory. The gauge parameter is given by

$$U(x) = e^{i\alpha^a(x)T^a} \tag{3.100}$$

where $T^a$ are the generators of the $U(N)$ gauge group which together with the brackets $[T^a, T^b] = i\epsilon^{abc}T^c$ form a Lie algebra. We can make a gauge choice for the potential to point along the compact direction

$$A_\mu = \delta_{9,\mu} \begin{pmatrix} \frac{\theta_1}{2\pi R} & & 0 \\ & \ddots & \\ 0 & & \frac{\theta_N}{2\pi R} \end{pmatrix}. \tag{3.101}$$

This breaks the Chan-Paton gauge symmetry $U(N) \rightarrow U(1)^N$ and the gauge parameter is given by

$$U(x) = \begin{pmatrix} \frac{i\theta_1 x^9}{2\pi R} & & 0 \\ & \ddots & \\ 0 & & \frac{i\theta_N x^9}{2\pi R} \end{pmatrix} \tag{3.102}$$

In this configuration $A_\mu = -iU^{-1}\partial_9 U$ is pure gauge and the configuration trivially satisfies the source free Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. We could now make a gauge transformation

$$A_9^{'} = UA_9U^{-1} + iU^{-1}\partial_9 U \tag{3.103}$$

to gauge $A^9$ away. But again the gauge parameter $U(x)$ is not single valued under a transformations $x^9 \rightarrow x^9 + 2\pi R$ and the states pick up a phase factor associated with the winding around the compact direction. The effect is measured by the Wilson line

$$W = \mathrm{diag}(e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_N}) \tag{3.104}$$

Using Chan-Paton degrees of freedom $(i, j)$ [9], a string in the state $|p, ij>$ is attached with one end (which we choose to be the one for $\sigma = \pi$) to the i'th and the other to the j'th brane. The gauge field couples to the Chan-Paton indices and the i'th string end has opposite charge of the j'th [10]. According to equation (3.98) we now obtain the momentum

$$p^9_{\text{phys}} = \frac{2\pi n + \theta_i - \theta_j}{2\pi R}, n \in \mathbb{N} \tag{3.105}$$

and the mass formula becomes

$$m^2 = \frac{(2\pi n + \theta_i - \theta_j)^2}{(2\pi R)^2} + \frac{1}{\alpha'}(N-1). \tag{3.106}$$

Ignoring the contribution from the winding states, we see that the gauge bosons obtain a mass if $\theta_i \neq \theta_j$. This corresponds to placing the two branes at different positions, thus braking the gauge symmetry to an $U(1)$ theory on each brane.
In the T-dual picture the coordinates are given in (3.84) and subtracting the coordinates for $\sigma = \pi$ from the one at $\sigma = 0$ we obtain

$$x^{'9}(\tau, \pi) - x^{'9}(\tau, 0) = 2\pi\alpha' p^9 = \frac{\alpha'}{R}(2\pi n + \theta_i - \theta j) = R'(2\pi n + \theta_i - \theta j). \tag{3.107}$$

Up to an overall constant the position of the endpoint on the i'th brane is given by

$$x'^9(\tau,\pi) = \theta_i R' = 2\pi\alpha' A^9_{ii}, \tag{3.108}$$

where we used equation (3.101). This shows that the position of the brane in the T-dual picture is described by the transverse components of the gauge field in the original picture. If we visualize the compact dimension as a circle we can interpret $\theta_i$ as the angle characterizing the position of the i'th D-brane at the compact direction.
Again we can generalize the discussion by compactifying more than one direction. We take the coordinates $\{x^\mu\}$ along and $\{x^a\}$ transversal to the brane. The string ends move on the (p+1)-dimensional world volume of the Dp-branes which are dynamical objects and can fluctuate in shape and position [7]. This follows from equation (3.108).
In the T-dual picture we now have gauge fields along the directions of the D-brane and scalar fields in the transverse directions, corresponding to the states

$$\alpha^\mu_{-1}|p,ii\rangle, \quad \alpha^a_{-1}|p;ii\rangle. \tag{3.109}$$

In the original theory $A_a$ are the gauge field along the compact directions, whereas $A_\mu$ are the ones along the other directions. After performing the T-duality, transformation $A_\mu$ is the gauge field living on the brane while the $A_a = \Phi_a$ are interpreted as a set of scalars living on the world volume and describing the transverse position of the brane. They therefore describe the embedding of the brane into a given spacetime.

### 3.2.5 The Dirac-Born-Infeld action

In the previous sections we have discussed D-branes in string theory and argued that a gauge theory is living on their worldvolume. We also found that their embedding into a given spacetime is described by some scalar fields. The gauge theory living on the D-brane is not described by a Lagrangian with a kinetic term of the form $F_{\mu\nu}F^{\mu\nu}$, which would be the one we find in the usual gauge theories. But this term can be found in a small field approximation taking the limit $\alpha' \to 0$.
To derive the action we consider for definiteness a D2-brane extended in the $x^1$ and $x^2$ directions[5] as well as a constant gauge field $F_{12}$. Making a T-duality transformation along the $x^2$ direction, the displacement of the brane in the transverse direction (3.108) is

$$x'^2 = 2\pi\alpha' A_2 = 2\pi\alpha' x^1 F_{12} \tag{3.110}$$

where in the last step we chose the gauge $A_2 = F_{12}x^1$. The D1-brane worldvolume action then becomes

$$\begin{aligned} S &= T_1 \int d\tau \int ds = T_1 \int \sqrt{(dx'^1)^2 + (dx'^2)^2} \\ &= T_1 \int d\tau dx^1 \sqrt{1 + \left(\frac{dx'^2}{dx'^1}\right)^2} = T_1 \int d\tau dx^1 \sqrt{1 + (2\pi\alpha' F_{12})^2}. \end{aligned} \tag{3.111}$$

[5]We leave the possibility for the brane to extend in other directions $x^m$, $m > 2$ as well, but for our discussion it is enough to consider these two directions.

In the case of a space filling brane we can boost the directions of the brane to be parallel to the coordinate axes and rotate the gauge field $F_{mn}$ into block diagonal form. Then the action becomes a product of factors like in (3.111) and we obtain the Born-Infeld action

$$S = T_9 \int d^{10}x \sqrt{\det\left(\eta_{mn} + 2\pi\alpha' F_{mn}\right)}, \tag{3.112}$$

which describes the gauge theory living on a space filling brane in a flat spacetime.
To obtain the action for a Dp-brane we dimensionally reduce (3.112). We take the coordinates $x^\mu$, $(\mu = 0 \dots p)$ along the brane and $x^a$, $(a = p+1 \dots D-1)$ transverse to the brane corresponding to compactifying the directions $x^a$. By taking the compact directions very small, the derivatives with respect to $x^a$ can then be neglected and we obtain

$$S \sim \int d^{p+1}x \sqrt{\det\left(\eta_{\mu\nu} + \partial_\mu x^a \partial_\nu x_b + 2\pi\alpha' F_{\mu\nu}\right)}. \tag{3.113}$$

where we used equation (3.108). This is not yet the final version of an action for a Dp-brane. We also have to consider the case of non trivial metric, Kalb-Ramond field and dilaton. Demanding spacetime gauge invariance and considering that we are at tree level, we finally obtain the Dirac-Born-Infeld action as [7]

$$S_{DBI} = -T_p \int d^{p+1}x e^{-\Phi} \sqrt{\det(G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \tag{3.114}$$

with $G_{\mu\nu}$ and $B_{\mu\nu}$ the pullback of the spacetime metric and 2-form field to the world volume and the Dp-brane tension given by

$$T_p = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_s}. \tag{3.115}$$

The action (3.113) describes the dynamics of the gauge field living on the Dp-brane worldvolume in an arbitrary background. It will be the starting point for the calculation of expectation values of Wilson loops using D-branes in chapter 6.

## 3.3 Anti-de Sitter Spaces

The Maldacena conjecture states the duality between $\mathcal{N} = 4$ SYM on four dimensional Minkowski space and type IIB superstring theory on $AdS_5 \times S^5$. In the following we give the derivation of the metric of AdS in different coordinates. Further we discuss the boundary of AdS which is in particular important because, given a Wilson loop parametrization on the SYM side and thus on the boundary of AdS we have to find the transcription to the interior of AdS.

#### General properties

De Sitter and Anti de Sitter spaces are solutions of the Einstein Hilbert action with a cosmological term [11]. The Einstein Hilbert action in $D$ dimensions is given by

$$S = \frac{-s}{16\pi G_D} \int d^D x \sqrt{|g|}(R + \Lambda) \tag{3.116}$$

with $s=-1$ for Minkowski and $s=+1$ for Euclidean signature. From the action follows the Einstein equation with a cosmological term

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{1}{2}\Lambda g_{\mu\nu} \tag{3.117}$$

Contracting with the inverse metric $g^{\mu\nu}$ we find

$$(1-\frac{D}{2})R=\frac{D}{2}\Lambda \tag{3.118}$$

and we see that the Ricci scalar is a constant $R=\frac{D}{2-D}\Lambda$ and the Ricci tensor is proportional to the metric. Spaces with these properties are called Einstein spaces. De Sitter and Anti de Sitter spaces are Einstein spaces with maximal symmetry

$$R_{\mu\nu\rho\sigma}=\frac{R}{D(D-1)}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}). \tag{3.119}$$

With these properties Anti de Sitter spaces are obtained for $\Lambda>0$, De Sitter spaces for $\Lambda<0$.

**Anti de Sitter space by embedding**

We can define Anti de Sitter space in $(n+1)$ dimensions by considering an $(n+2)$ dimensional pseudo Euclidean space with metric

$$ds^2=-dx_0^2-dx_{n+1}^2+d\vec{x}^2 \tag{3.120}$$

where $\vec{x}$ includes the components $\{x_i\},(i=1\dots n)$.
The isometries are given by matrices $\Lambda$ that preserve the metric such that

$$ds'^2=\eta_{\mu\nu}dx'^\mu dx'^\nu=\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma dx^\rho dx^\sigma=\eta_{\rho\sigma}dx^\rho dx^\sigma \tag{3.121}$$

with $\eta=diag(+,+,-,\dots,-)$, and we obtain

$$\mathbf{\Lambda}^T\eta\mathbf{\Lambda}=\eta \tag{3.122}$$

hence $\mathbf{\Lambda}\in SO(2,n)$.
We can now define $AdS_{n+1}$ as the locus of all points $x^\mu$, $(\mu=0\dots n+1)$, with

$$x_0^2+x_{n+1}^2-\vec{x}^2=b^2. \tag{3.123}$$

This equation is solved by setting

$$\begin{aligned} x_0 &= b\cosh\rho\cos\tau, \quad x_{n+1}=b\cosh\rho\sin\tau, \\ x_i &= b\sinh\rho\;\Omega_i \end{aligned} \tag{3.124}$$

where $\Omega_i$ is a unit vector. Using these coordinates in (3.120) we obtain the metric

$$ds^2=b^2\left(-\cosh^2\rho\;d\tau^2+d\rho^2+\sinh^2\rho\;d\Omega_{n-1}^2\right) \tag{3.125}$$

where $\rho\in[0,\infty)$ and $\tau\in[0,2\pi]$. The coordinates (3.124) cover the hyperbolic space once and are called global AdS coordinates. To obtain a causal spacetime structure we have

to take $\tau \in (-\infty, \infty)$ and we obtain the universal covering of the hyperboloid. For $\rho \to 0$ the metric behaves like $ds^2 \sim b^2 \left(-d\tau^2 + d\rho^2 + \rho^2 \; d\Omega_{n-1}^2\right)$, hence the hyperboloid has the topology $S^1 \times \mathrm{R}^n$.
We can conformally compactify (3.125) by introducing the coordinate $\theta \in [0, \pi/2]$ defined as $\tan\theta = \sin\rho$. We then obtain

$$ds^2 = \frac{b^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{n-1}^2). \tag{3.126}$$

Leaving out the factor $\frac{b^2}{\cos^2\theta}$ in front of the brackets does not change the causal structure of the space-time. In this case the metric has the topology $\mathbb{R} \times S^n$ and it is equal to the metric of the Einstein static universe, apart from the fact that the coordinate $\theta$ only takes values $0 \leq \theta \leq \pi/2$ which only covers half of the Einstein static universe.
The boundary in these coordinates is given for $\theta = \pi/2$ and the topology of the metric becomes $\mathbb{R} \times S^{n-1}$. We will later see that the boundary of conformally compactified AdS is identical to the conformal compactification of Minkowski space, which is a very important fact for the AdS/CFT correspondence.

Apart from the global coordinates we will also use the so called Poincaré coordinates of AdS. These are obtained by setting

$$\begin{aligned} x_0 &= \frac{1}{2}\left(\frac{1}{u} + u(b^2 + \vec{X}^2 - t^2)\right) \\ x^n &= \frac{1}{2}\left(\frac{1}{u} + u(b^2 - \vec{X}^2 + t^2)\right) \\ x_{n+1} &= but \\ x^i &= buX^i \quad i = 1, \ldots, n-1 \end{aligned} \tag{3.127}$$

Using this in (3.120) we obtain the metric

$$ds^2 = \frac{b^2}{u^2} du^2 + b^2 u^2 \left(-dt^2 + d\vec{X}^2\right) \tag{3.128}$$

and another convenient form is obtained by substituting $u = 1/Y,\; du/u = -dY/Y$,

$$ds^2 = \frac{b^2}{Y^2}\left(dY^2 - dt^2 + d\vec{X}^2\right). \tag{3.129}$$

In terms of the embedding coordinates we can describe the boundary of $AdS_{n+1}$ by introducing a scaling factor $R$

$$x^\mu = R\tilde{x}^\mu. \tag{3.130}$$

The boundary of AdS is then defined by letting $R \to \infty$ and we find a point with the coordinates $(\tilde{x_0}, \tilde{x_{n+1}}, \tilde{x}^1 \ldots, \tilde{x}^n)$ at the boundary given by

$$\tilde{x_0}\tilde{x_{n+1}} - \tilde{\vec{x}}^2 = 0, \tag{3.131}$$

where we have used the definition of AdS space, equation (3.123). It is important to see that every multiple $t\tilde{x}^\mu$ of a point at the boundary satisfies equation (3.131) for a real parameter $t > 0$. Thus the boundary is defined as the projective equivalence class

$$\begin{gathered} x_0^2 + x_{n+1}^2 - \vec{x}^2 = 0 \\ (x_0, x_{n+1}, \vec{x}) \sim t(x_0, x_{n+1}, \vec{x}). \end{gathered} \tag{3.132}$$

This gives us the possibility to rescale the coordinates such that

$$(x^0)^2 + (x^{n+1})^2 = 1 = \vec{x}^2 \tag{3.133}$$

and the boundary therefore has the topology $S^1 \times S^{n-1}$, which we can also obtain by a conformal compactification of Minkowski space with a compactified Euclidean time [11], which we will show in the next section. In terms of Poincaré AdS coordinates in equation (3.128), $(u, X^i)$ the boundary is given by

$$u \to \infty,\ X^i = \frac{x^i}{bu} = \frac{\tilde{x}^i}{b\tilde{u}} = \tilde{X}^i. \tag{3.134}$$

The coordinates $X^i$ are the same in the interior and at the boundary. This gives us the possibility to transcribe the boundary coordinates from Minkowski space to the AdS coordinates $X^i$, which will become important when we choose a parametrization of the Wilson loop on the boundary of AdS.

## Euclidean signature

It is often convenient to use Euclidean signature, which we obtain by rotating the time like coordinate of the global metric as

$$\tau \to \tau_E = -i\tau \tag{3.135}$$

The metric (3.125) then becomes

$$ds_E^2 = b^2 \left(\cosh^2 \rho d\tau_E^2 + d\rho^2 + \sinh^2 \rho d\Omega_{n-1}^2\right). \tag{3.136}$$

The Wick rotation of $\tau$ is equivalent to rotate $t \to t_E = -it$ in Poincaré coordinates (3.129). We then obtain

$$ds_E^2 = \frac{b^2}{u^2} du^2 + b^2 u^2 \left(dt^2 + d\vec{X}^2\right) \tag{3.137}$$

and the second version, equation (3.129), is changed to

$$\frac{b^2}{Y^2}\left(dY^2 + dt^2 + d\vec{X}^2\right) \tag{3.138}$$

### A closer look at the symmetries

We have already seen that the isometries of $AdS_{n+1}$ are given by $SO(2, n)$. For $AdS_5$ we thus have the symmetry group $SO(2, 4)$, and because there are spinors involved, we have to consider the covering group $SU(2, 2)$. The isometries of the 5-sphere are given by $SO(6)$ and thus we have the symmetry $SU(2, 2) \times SU(4)$. If we further consider supersymmetry and consider the 32 Majorana spinor supercharges of the IIB theory the full invariance is given by $SU(2, 2|4)$.

## Conformal compactification

We have seen that the boundary of $AdS_5$ has the topology $S^1 \times S^3$ (or in the case of the covering space $\mathbb{R} \times S^3$). The Maldacena conjecture states the equivalence between type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM on Minkowski space. We should therefore be able to identify Minkowski space with $S^1 \times S^3$ (or $\mathbb{R} \times S^3$). For this we should have a look of the conformal structure of flat space. Starting with the flat Minkowski-metric

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{3.139}$$

and introducing 'light cone' coordinates

$$u = t + r \tag{3.140}$$
$$v = t - r \tag{3.141}$$

equation (3.139) becomes

$$ds^2 = -du dv + (\frac{u-v}{2})^2 d\Omega_2^2. \tag{3.142}$$

Changing coordinates once more

$$\begin{aligned} u &= \tan X \\ v &= \tan Y \\ X, Y &\in [-\pi/2, \pi/2] \end{aligned} \tag{3.143}$$

the metric becomes[6]

$$\begin{aligned} ds^2 &= \frac{1}{(\cos X \cos Y)^2}(-dXdY + (\frac{1}{2}(\sin X \cos Y + \cos X \sin Y))^2 d\Omega_2^2 \\ &= \frac{1}{4\cos^2\frac{(\tau+\Theta)}{2}\cos^2\frac{(\tau-\Theta)}{2}}(-d\tau^2 + d\Theta^2 + \sin^2\Theta d\Omega_2^2) \end{aligned} \tag{3.144}$$

where for the second version we have once more introduced new coordinates

$$\begin{aligned} \tau = X - Y \qquad & \Theta = X + Y \\ \tau \in [-\pi/2, \pi/2], \qquad & \Theta \in [0, \pi] \end{aligned} \tag{3.145}$$

For a fixed point on the $S^2$ we have mapped the half plane given by $t$ and $r$ into a triangular region in the $\tau$ and $\Theta$ plane, see Figure 3.1. Now we see the structure of $S^1 \times S^3$ multiplied by a conformal factor and we can therefore identify the boundary of $AdS_5$ with four dimensional Minkowski space. As before, the conformally scaled metric can be analytically continued such that $-\infty < \tau < \infty$ and in this way we obtain the topology of $R \times S^3$ [12]. In chapter 2 we have seen that the global conformal symmetry of $\mathbb{R}^{1,3}$ is $SO(2,4)$ which is generated by Lorentz transformations $M_{\mu\nu}$, translations $P_\mu$, Dilatations $D$ and special conformal transformations $K_\mu$. The global time translation generator in terms of the coordinates in (3.144) can be written as

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\frac{\partial}{\partial X} + \frac{1}{2}\frac{\partial}{\partial Y} = \frac{1}{2}(1+u^2)\frac{\partial}{\partial u} + \frac{1}{2}(1+v^2)\frac{\partial}{\partial v} \tag{3.146}$$

[6] We have to pay attention to the fact that the range of $X$ and $Y$ is between $[-\pi/2, \pi/2]$ but, due to $r \geq 0$ the sum $X + Y$ has to be zero.

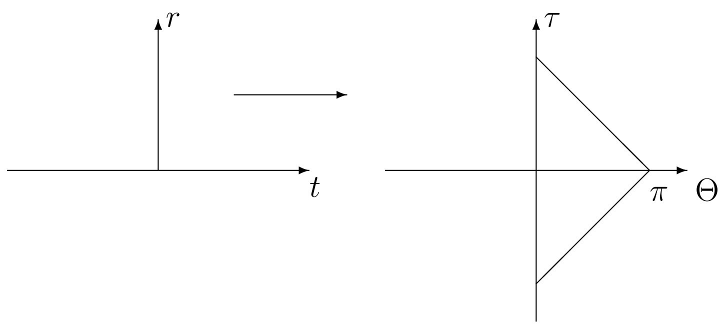


Figure 3.1: The $(t,r)$ plane of four dimensional Minkowski space is mapped into a triangular region in the $(\tau,\Theta)$ plane

which is the linear combination

$$\frac{\partial}{\partial\tau} = \frac{1}{2}(P_0 + K_0) \tag{3.147}$$

with $P_0$ and $K_0$ given by

$$P_0 = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right), \qquad K_0 = \frac{1}{2}\left(u^2\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v}\right) \tag{3.148}$$

The maximally compact subgroup of $SO(2,4)$ is $SO(2)\times SO(4)$ and the generator $\partial_\tau$ in equation (3.146) belongs to the $SO(2)$ part. The isometries of $\mathbb{R}\times S^3$ are thus given by $SO(2)\times SO(4)$ and the generator $\partial_\tau$ guarantees that we we can analytically extend correlation functions of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on $\mathbb{R}^{1,3}$ to $\mathbb{R}\times S^3$.

## 3.4 P-Brane Solutions of Type IIB Supergravity

In section 3.2 we have seen that D-branes are part of open string theory and that a gauge theory is living on their world volume. But also closed string theories contain, apart from strings, other object called $p$-branes, which are solitonic solutions for the semiclassical low energy action of closed string theory, type IIA or IIB supergravity.

Solitons are solutions of the non-linear field equations and they are non perturbative objects, i.e. we cannot find them using perturbation theory around the linear solution. They carry a conserved charge which ensures their stability and their mass density is usually inversely proportional to some power of a coupling constant,

$$m \sim \frac{1}{g^r}, \tag{3.149}$$

hence they become very massive at weak coupling and quantum effects due to soliton exchange will be suppressed to all orders in perturbation theory. On the other hand, according to the classical mass formula, they will become very light at strong coupling and therefore dominate the strong coupling low energy dynamics. But the question arizes when we can actually trust the classical mass formula. Here supersymmetry comes into play. If we are dealing with a half-BPS solitons we have the mass-charge relationship

$$M = |Z| \tag{3.150}$$

which is protected from quantum corrections and ensures the strong coupling properties mentioned before.
The solitons we will consider are $p$-branes, hypersurfaces embedded in the 10-dimensional space, acting as sources for the massless Ramond-Ramond (R-R) closed string fields [11]. A $p$-brane has a $(p+1)$- dimensional world volume and couples naturally to a $(p+1)$-form potential, which is associated with a $(p+2)$-form field strength.
In the following we will mostly be interested in the type IIB case, as it is the one that is used in the AdS/CFT correspondence. The type II supergravity action is the effective low energy description of type II superstring theory, and its complete form is given for example in [7]. For our purpose we can neglect the Fermionic part and make an ansatz with the NS-NS 3-form set equal to zero, hence we obtain the following action in the string frame[7]

$$S_{II} = -\frac{s}{16\pi G_{10}} \int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}\sum_n \frac{1}{n!}F_n^2 + \ldots\right). \tag{3.151}$$

The fields of interest are the metric $g^{\mu\nu}$, the dilaton $\phi$ and the R-R $n$-forms $F_n$. The factor $s$ in front of the action is one or minus one, for Euclidean or Minkowski signature respectively. The dots in (3.151) are standing for the fermionic terms as well as the NS-NS 3-form, which we set to zero in the following. It is often more convenient to use another frame, the Einstein frame, which we obtain by rescaling the metric as

$$g_{\mu\nu} \rightarrow e^{-\phi/2}g_{\mu\nu}. \tag{3.152}$$

The action becomes

$$-\frac{s}{16\pi G_{10}} \int d^{10}x\sqrt{-g}\big(R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sum_n \frac{1}{n!}e^{a_n\phi}F_n^2 + \ldots\big), \tag{3.153}$$

with $a_n = -\frac{n-5}{2}$. We are interested in a $p$-brane solution for this action. A $p$-brane is charged under a R-R $(p+1)$-form and we thus consider just one $n = (p+2)$-form field strength $F_n$. With this in mind, varying the action in (3.153) leads to the following equations of motion

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2n!}e^{a_n\phi}\big(nF_{\mu\xi_2\ldots\xi_n}F_\nu{}^{\xi_2,\ldots,\xi_n} - \frac{n-1}{D-2}g_{\mu\nu}F_n^2\big) \tag{3.154}$$

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial_\nu g^{\mu\nu}\phi) = \frac{a_n}{2n!}e^{a_n}F_n^2 \tag{3.155}$$

$$\partial_\mu(\sqrt{g}e^{a_n\phi}F^{\mu\xi_1\ldots\xi_n}) = 0 \tag{3.156}$$

We are looking for 'electric' solutions for the equations of motion (compare appendix A) and as in classical electrodynamics we expect them to have a singularity. One should

[7]See appendix A for a short summary on form notation

therefore in principle add a source term to the truncated action, which has the form

$$S_s = S_{SBI} + S_{WZ} = T_p \left( \int_{V_{p+1}} e^{-\phi} \sqrt{g + 2\pi\alpha' F} + \int_{V_{p+1}} P[C_{p+1}] \right) \tag{3.157}$$

with $S_{DBI}$ the Dirac Born Infeld action known from section 3.2 which describes the $p$-brane dynamics and $S_{WZ}$ the Wess Zumino term which reflects the coupling of the brane to the $C_{p+1}$ form. Here we used differential form notation and $P[C_{p+1}]$ indicates that we take the pullback of the $(p+1)$-form to the world volume of the Dp-brane.

## Ansatz for a p-brane solution

We want to find a solution with $N$ p-branes embedded in the 10 dimensional spacetime of type IIB supergravity. For the metric we demand the highest degree of symmetry, i.e. $R^{p+1} \times SO(p+1)$ Poincaré symmetry for the coordinates along the $(p+1)$-dimensional world volume of the brane and $SO(9-p)$ invariance in the transverse directions. This suggests that we split the coordinates into $(t, x^i)$ along and $y^a$ transverse to the p-brane worldvolume.

$$x^\mu = (t, x^i, y^a), \; i = 1 \dots 3, \; a = 4 \dots 9 \tag{3.158}$$

This suggests to make an ansatz for the metric of the form

$$ds^2 = sB^2 dt^2 + C^2 \sum_{i=1}^{p} (dx^i)^2 + D^2 dr^2 + Er^2 d\Omega^2_{(8-p)} \tag{3.159}$$

with $d\Omega_{(8-p)}$ being the metric of the unit $(8-p)$-sphere on the transverse space. For the dilaton we make the ansatz

$$\phi = \phi(r) \tag{3.160}$$

and an ansatz for an electric solution for the R-R field strength $F_n$, respecting the symmetries, is given by

$$F_{ti_1 \dots i_p r}(r) = \epsilon_{i_1 \dots i_p} k(r) \tag{3.161}$$

with all other components set to zero. By defining

$$\tilde{F}_{10-n} = e^{a_n \phi} * F_n \tag{3.162}$$

we can actually see that the equations of motion (3.154),(3.155),(3.156) are invariant under the duality transformations

$$n \to 10 - n, \quad F_n \to \tilde{F}_{10-n}, \quad a\phi \to -a\phi, \tag{3.163}$$

meaning that we will obtain magnetic solutions as well.

Inserting our ansatz for $F_n$ (3.161) into the equation of motion (3.156) leads to

$$\partial_r \left( \sqrt{g} e^{a_n \phi} g^{tt} g^{i_1 \alpha_1} \dots g^{i_p \alpha_p} g^{rr} \epsilon_{\alpha_1 \dots \alpha_p} k(r) \right) = 0 \tag{3.164}$$

and using the ansatz for the metric (3.159) we obtain

$$\partial_r \left( \frac{(Gr)^{d-1}}{BC^P F} e^{a_n \phi} k(r) \right) = 0. \tag{3.165}$$

We find a solution by taking the term in the brackets in equation (3.165) to be a constant

$$F_{ti_1\ldots i_p r} = \epsilon_{i_1\ldots i_p} k(r) = \epsilon_{i_1\ldots i_p} e^{-a_n\phi} BC^p F \frac{Q}{(Gr)^{(d-1)}}, \tag{3.166}$$

with $Q$ a constant of integration. Using the definition in equation (3.162), we calculate the dual field strength to be

$$\tilde{F}_{\alpha_1\ldots\alpha_{(10-n)}} = \sqrt{g_{(10-n)}}\epsilon_{\alpha_1\ldots\alpha_{(10-n)}} Q, \tag{3.167}$$

with $g_{10-n}$ is the determinant of the metric for the $(10-n)$-sphere on the transverse space. We obtain the 'electric charge density' $\mu_p$, which is the electric flux through the world volume of the brane as

$$\mu_p = \frac{1}{\sqrt{16\pi G_{10}}} \int_{S^{8-p}} \tilde{F}_{8-p} = \frac{\Omega_{8-p} Q}{\sqrt{16\pi G_{10}}} \tag{3.168}$$

with $\Omega_{10-n}$ being the volume of the unit sphere in the transverse space.

According to the section about AdS-spaces we know that a solution for the metric that respects all the symmetries (namely Poincare symmetry $R^{p+1} \times SO(1,p)$ for the worldvolume coordinates and $SO(9-p)$ for the transverse space) could be $AdS_p \times M^{(8-p)}$, with $M^{(8-p)}$ a $(8-p)$ dimensional manifold, with isometries $SO(9-p)$. It can actually be shown [13], that in 10 dimensional type IIB supergravity the case $n = p+2 = 5$ is the only possibility for obtaining a metric in the form $AdS_n \times S^n$. In this case the dilaton has to decouple and it will be a constant, which can be set to zero. We therefore use the 5-form field strength, which corresponds to a 3-brane solution. Our ansatz for the metric, equation (3.159) then becomes

$$ds^2 = sB^2 dt^2 + C^2 \sum_{i=1}^{3} (dx^i)^2 + D^2 dr^2 + E^2 r^2 d\Omega_5^2 \tag{3.169}$$

Using this in the equation of motion (3.154) we obtain after some calculations [13]

$$ds^2 = H^{-\frac{1}{2}}(r) \left( sf(r)dt^2 + \sum_{i=1}^{3} (dx^i)^2 \right) + H^{\frac{1}{2}}(r) \left( \frac{dr^2}{f(r)} + r^2 d\Omega_5^2 \right) \tag{3.170}$$

with

$$\begin{aligned} H(r) &= 1 + \left(\frac{h}{r}\right)^4 \\ h^4 &= \frac{1}{4}\left(-r_0^4 + \sqrt{r_0^{16} + Q^2}\right) \\ f(r) &= 1 - \left(\frac{r_0}{r}\right)^4. \end{aligned} \tag{3.171}$$

For $r \to \infty$ the factors $H(r)$ and $f(r)$ tend to one and the metric becomes flat in the region far away from the branes. The parameter $r_0$ is zero if the branes are in the ground state of a quantum description. For $r_0 \neq 0$ the metric describes a black D3-brane with

horizon at $r = r_0$
Using the result for the metric (3.170) in equation (3.166) we obtain

$$F_{ti_1\ldots i_p r} = \epsilon_{i_1\ldots i_p} H^{-2}(r)\frac{Q}{r^5} \tag{3.172}$$

for the 5-form field strength.

### The Extremal Solution

An extremal configuration is given by the brane(s) being in the ground state, corresponding to $r_0 = 0$. Using this in equation (3.171) we obtain

$$H = 1 + \frac{Q}{4r^4}, \quad f(r) = 1. \tag{3.173}$$

and we obtain the metric

$$ds^2 = H^{-1/2}(r)\left(sdt^2 + \sum_{i=1}^{3}(dx^i)^2\right) + H^{1/2}(r)\left(dr^2 + r^2 d\Omega_5^2\right). \tag{3.174}$$

For a brane in the ground state the relation between the flux $\mu_p$ in equation (3.168) and the brane tension (3.115) is given by [7],[10]

$$\mu_p = T_p\sqrt{16\pi G_{10}} \tag{3.175}$$

and by comparing with equation (3.168) this leads to the result $Q = \frac{16\pi G_{10} T_3}{\Omega_5}$. The flux scales with the numbers of coincident branes $N$ [6] and we obtain from equation (3.168)

$$\frac{\Omega_5}{\sqrt{16\pi G_{10}}}Q = N \rightarrow Q = 16\pi g_s N \alpha^{'2} = 4L^4 \tag{3.176}$$

where we have used

$$16\pi G_{10} = (2\pi l_s)^4 g_s^2. \tag{3.177}$$

With this we obtain

$$H(r) = 1 + \frac{4\pi g_s N\alpha'^2}{r^4} \tag{3.178}$$

which determines the 3-brane solution for the truncated IIB supergravity action in the extremal limit.

### The extremal solution in the near horizon limit

We now describe the extremal near horizon solution, taking the limit very close to the branes, given by $r \rightarrow 0$. Introducing the variable $u = r/\alpha'$ we also take $\alpha' \rightarrow 0$ such that $u$ is a final variable. In this limit we obtain

$$H(r) \simeq \frac{Q}{4r^4} = \frac{4\pi g_s N}{u^4\alpha'^2} = \frac{L^4}{u^4\alpha'^4} \tag{3.179}$$

and the metric becomes

$$\begin{aligned} ds^2 &= \frac{u^2\alpha'}{\sqrt{4\pi g_s N}}\left(dt + \sum_{i=1}^{3}(dx^i)^2\right) + \sqrt{4\pi g_s N}\alpha'\left(\frac{du^2}{u^2} + d\Omega_5^2\right) \\ &= \frac{L^2}{y^2}\left(dt + \sum_{i=1}^{3}(dx^i)^2 + dy^2\right) + L^2 d\Omega_5^2 \end{aligned} \tag{3.180}$$

where in the last step we have made the coordinate transformation $Y = 1/u$ and rescaled the metric as we did in (3.129). The metric thus describes $AdS_5 \times S^5$ with common radius $L$.

We obtain the 5-form field strength by inserting (3.179) into (3.172), leading to

$$F_{ti_1\ldots i_3 r} = \epsilon_{i_1\ldots i_p}\frac{r^3}{L^4} \tag{3.181}$$

where we have used the t'Hooft coupling $\lambda = 4\pi g_s N = g_{YM}^2 N$.

We have obtained the important result that $AdS_5 \times S^5$ is a 3-brane solution of type IIB supergravity in the extremal and near horizon limit. We also found that the dilaton can be set to zero $\phi = 0$ and we obtained a result for the 5-form field strength $F_5$.

### P-branes and symmetries

A very important property of the AdS/CFT correspondence is that the global unbroken symmetries of the two theories are identical. In 3.3 we showed that the isometrie group of $AdS_5$ is $SO(2,4)$, and the 3-brane solution we obtained in this section has therefore the isometry group $SO(2,4) \times SO(6)$, where the $SO(6)$ symmetry is the isometry group of the 5-sphere. Here we recognize the maximal bosonic subgroup of the superconformal group $SU(2,2|4)$, which is the global symmetry group of $\mathcal{N} = 4$ SYM.

The 3-brane solution also preserves 16 of the 32 Poincaré supersymmetries, and the way to show that can be outlined as follows: Because we set all fermionic degrees of freedom to zero, the supersymmetric variation of the bosonic fields, which vary into the fermionic ones, do automatically vanish. To find the number of supersymmetries preserved by the 3-brane solution we have to find the number of linearly independent supersymmetry parameters for which the supersymmetry variations of the gravitino and dilatino become zero. In the string frame these are given by[8]

$$\begin{aligned} \delta\psi_M &= D_M\epsilon - i\frac{1}{960}e^{\phi}F_{M_1\ldots M_5}\Gamma^{M_1\ldots M_5}\Gamma_M\epsilon \\ \delta\lambda &= \partial_M\phi\Gamma^M\epsilon, \end{aligned} \tag{3.182}$$

with $D_M\psi = \partial_M\psi + \frac{i}{2}\Gamma_{ab}\omega_\mu^{ab}\psi$ the covariant derivative with spin connection $\omega_\mu^{ab}$. The variation of the dilatino is identically zero, since we consider a configuration with vanishing dilaton. We therefore only have to consider how many linearly independent supersymmetry parameters $\epsilon$ satisfy the Killing spinor equation in.

$$\delta\psi_M = D_M\epsilon - i\frac{1}{960}e^{\phi}F_{M_1\ldots M_5}\Gamma^{M_1\ldots M_5}\Gamma_M\epsilon = 0. \tag{3.183}$$

[8]In our case we are only considering the 5-form field strength and the NS-NS two form $B_{\mu\nu}$ set to zero.

Using our solution for the 5-form $F_{M_1 \dots M_5}$, equation (3.181) we find that there are 16 independent solutions, and therefore the 3-brane solution preserves half of the 32 Poincaré supersymmetries. In the near horizon limit these are supplemented by 16 conformal supersymmetries [6] and therefore the global symmetries on the AdS-side are given by $SU(2,2|4)$.

**Dp-branes and $p$-branes**

In this section we showed that $p$-branes are solitonic solutions for the equations of motion of type II supergravity, and by construction they are charged under the corresponding $C^{p+1}$ form potentials. On the other hand, in section 3.2 we considered Dp-branes with gauge theories living on their worldvolume. The supergravity $p$-branes we constructed in this section preserve half of the supersymmetries of the theory, and it can be shown [14] that we can use T-duality to relate $p$-branes to $(p-1)$ branes. It is therefore most likely that $p$-brane solitons and D-branes describe the same underlying object.

# Chapter 4

# The AdS/CFT Correspondence

The AdS/CFT correspondence is the duality between type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on four dimensional Minkowski space. In the following we will describe the considerations behind this duality using the theory we have developed in the previous sections. We will then formulate the three different forms of the correspondence and show how the symmetries and operators can be identified on both sides.

## 4.1 Gauge Theory on D3-Branes

In chapter 3 we have seen how D-branes are a part of string theory and how they can be uncovered by using T-duality. We are interested in type IIB superstring theory with a stack of $N$ D3-branes. The theory will then contain open and closed strings with the branes acting as topological defects.

Open strings end on D-branes and the massless open string states correspond to gauge fields that live on the world volume of the D-branes. For a single D3-brane we obtain a gauge theory in four dimensional spacetime and since the brane breaks half of the total number of supersymmetries [7], [14] in ten dimensional superstring theory we expect the gauge theory to be $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory.

In the case of $N$ coincident D3-branes we obtain an $U(N)$ gauge theory. We have seen that for spatially separated branes the lowest lying string states obtain a mass if the open string ends lie on two different branes. Each brane we separate from the stack of D-branes corresponds to breaking the $U(N)$ gauge theory to $U(N-1) \times U(1)$ and if all D3-branes are separated we obtain $U(1)^N$, see figure 4.1. In the low energy limit, for $E \ll 1/\sqrt{\alpha'}$, only massless open string states can be excited and the dynamics of the corresponding field theory is described by $\mathcal{N} = 4$ SYM theory on the four dimensional world volume of the branes. On the other hand, the low energy dynamics of the fields arising from the closed string states are described by type IIB supergravity in ten dimensional flat space. In general we should consider the complete effective action of the massless modes as a sum of the bulk action, the D3-brane action and an interaction term. If we take the limit $\alpha' \to 0$ with the energy and the dimensionless parameters fixed, the interaction term vanishes [12] and we obtain two decoupled theories: $\mathcal{N} = 4$ SYM in $3+1$ dimensions and free supergravity in flat ten dimensional spacetime.

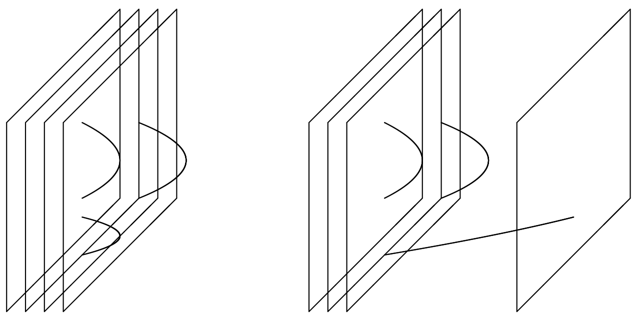


Figure 4.1: A stack of $N$ coincident D-branes corresponding to a $U(N)$ gauge theory. By separating one D-brane the symmetry is broken to $U(N-1)\times U(1)$.

## 4.2 D3-Brane Solution of Supergravity

D-branes are massive charged objects and do therefore influence the geometry of space. As we have seen in section 3.4 the metric for $AdS_5 \times S^5$ is obtained from the D3-brane solution of IIB supergravity in the near horizon limit. The metric corresponding to the p-branes in the ground state is given by

$$ds^2 = \left(1+\frac{L^4}{u^4}\right)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + \left(1+\frac{L^4}{u^4}\right)^{1/2}(du^2+u^2d\Omega_5^2) \tag{4.1}$$

with $L^4 = 4\pi g_s N\alpha'^2$. We can now consider two limits. The first one is the near horizon limit $u \ll L$ in which we obtain $AdS_5 \times S^5$ as was discussed in detail in section 3.4. On the other hand, in the limit far away from the branes, $u \gg L$, we obtain a flat ten dimensional spacetime.

For the metric (4.1) the energy $E_0$ of an object as measured by an observer at a constant position $u = u_0$ is related to the energy $E_\infty$ measured by an observer far away from the branes by

$$E_\infty = (1+\frac{L^4}{u_0^4})^{-1/4}E_0. \tag{4.2}$$

For a region close to the branes $u_0 \ll L$ we obtain $E_\infty = \frac{u_0}{L}E_0$. The observer at infinity will then observe all energies to be small and he will find two kinds of low energy excitations. Massless particles propagating in the asymptotically flat region and all other particles close to the branes. In the low energy limit he will not observe any particles in the region between the two limits because the excitations close to $u = 0$ have no chance to escape the gravitational potential and reach the asymptotically flat area. The theories in the two regions therefore decouple and we obtain free type IIB supergravity in the bulk and type IIB superstring theory in the near horizon region, which has the geometry of $AdS_5 \times S^5$ .

We can now compare this result with the one from the last section. There the low energy limit of IIB string theory with a stack of $N$ D3-branes was described by two theories, which decouple if we take the limit $\alpha' \to 0$ (and the energy and all dimensionless parameters fixed): $\mathcal{N} = 4$ SYM living on the (flat) four dimensional world volume of the D3-branes and type IIB supergravity on a flat ten dimensional spacetime, describing the massless excitations from the closed string sector. The two parts of the theory we obtain from the supergravity description are IIB supergravity in the asymptotically flat ten dimensional spacetime while in the near horizon area we find IIB string theory on $AdS_5 \times S^5$. From both sides we obtain type IIB supergravity in flat ten dimensional spacetime. Maldacena identified the remaining parts and predicted the identity of $\mathcal{N} = 4$ SYM theory on four dimensional Minkowski space and type IIB string theory on $AdS_5 \times S^5$, which is the famous AdS/CFT correspondence.

## 4.3 The Three Forms of AdS/CFT

In its strongest form the AdS/CFT correspondence predicts the duality between $\mathcal{N} = 4$ conformal SYM theory with gauge group $U(N)$ and gauge coupling $g_{YM}$ and full quantum type IIB string theory in $AdS_5 \times S^5$. The string coupling is identified as $g_s = g_{YM}^2$ and the radius of $L$ is given in terms of the string coupling $g_s$, the gauge group $N$ and the Regge slope $\alpha'$ as

$$L^4 = 4\pi g_s N \alpha'^2. \tag{4.3}$$

The second version asserts the duality of $\mathcal{N} = 4$ SYM in the 't Hooft limit, where we take $N \to \infty$ and the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed, and type IIB superstring theory on $AdS_5 \times S^5$ in the limit $g_s \to 0$. Corrections to classical supergravity from stringy effects are of order $\alpha'$ and they would agree with corrections to the large $\lambda$ limit on the SYM side, which are of order $\lambda^{-1/2}$. Higher order corrections in $g_s$ on the supergravity side could disagree with the corrections coming from non-planar diagrams on the SYM side.
The weakest version of the AdS/CFT correspondence is given by $\mathcal{N} = 4$ SYM in the large $N$, large $\lambda$ limit and classical type IIB supergravity on $AdS_5 \times S^5$. On the gauge theory side we expect order $\lambda^{-1/2}$ and order $1/N^2$ corrections, but they might not agree with the corrections of order $\alpha'$ and $g_s$ on the supergravity side. In this version the AdS/CFT correspondence is a duality between a strongly coupled Yang-Mills theory and the low energy limit of type IIB string theory. While we usually cannot access the former (with some exceptions) a lot of calculations can be done on the string theory side giving us the possibility to make predictions for the behavior of the gauge theory at strong 't Hooft coupling.

## 4.4 Comparison of Global Symmetries

A very important ingredient for the AdS/CFT correspondence is that the global symmetries on both sides are identical. In chapter 2 we have shown that the global symmetry group of $\mathcal{N} = 4$ SYM is the superconformal group $SU(2, 2|4)$ with maximal bosonic subgroup $SU(2, 2) \times SU(4)_R \simeq SO(2, 4) \times SO(6)_R$.
On the string theory side we can find these symmetries as the isometries of $AdS_5 \times S^5$ which we showed in section 3.3. One can actually show [13] that the isometries of $AdS_5$ induce the conformal symmetry on the boundary of AdS space.
The completion to the full supergroup is due to $\mathcal{N} = 4$ supersymmetry, and Poincaré and conformal supersymmetry give rise to a total of 32 supersymmetries on the SYM side. On the AdS side we would naively expect 32 Poincaré supersymmetries, but only half of

them are preserved due to the existence of the stack of D3-branes. The other 16 supersymmetries are conformal supersymmetries which are obtained in the near horizon-limit. Therefore we obtain the same global symmetry $SU(2,2|4)$ for IIB superstring theory on $AdS_5 \times S^5$.
There is an additional (conjectured) Montonen-Olive or S-duality symmetry SL(2,$\mathbb{Z}$) on both the SYM and the AdS side. The symmetry transformation is given as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z} \tag{4.4}$$

Using the relation between the Yang-Mills coupling and the string coupling we obtain $\tau$ on both sides as

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g_{YM}^2} = \frac{i}{g_s} + \frac{\chi}{2\pi} \tag{4.5}$$

with $\theta_I$ the instanton angle and $\chi$ the expectation value of the R-R scalar.

## 4.5 Mapping $\mathcal{N} = 4$ SYM Operators and Type IIB fields

With the symmetries on both sides matching we also have to show that the operators on the SYM side can be identified with the fields on the string theory side. Let us first consider a change in the value of the coupling constant on the SYM side which leads to a change of the string coupling according to (4.5). The string coupling is related to the expectation value of the dilaton by [7]

$$g_s = e^{\langle \phi \rangle}, \tag{4.6}$$

and the expectation value of the dilaton is determined by its boundary conditions at infinity. From this we can see that a change in the gauge coupling results in a change in the boundary value of the dilaton. By adding a source term $\int d^4x \phi_0(x) \mathcal{O}(x)$ to the action of $\mathcal{N} = 4$ SYM, where $\mathcal{O}$ is the operator responsible for the change of the coupling, we change the boundary condition for the dilaton as

$$\phi(\vec{x}, y)|_{y=0} = \phi_0(\vec{x}) \tag{4.7}$$

with $\vec{x}, y$ Poincaré coordinates on $AdS_5$, see equation (3.129). It is then proposed [15] that we can relate the partition functions on both sides

$$\langle e^{\int d^4x \phi_0(x) \mathcal{O}(x)} \rangle = Z_{string}|_{\phi(\vec{x}, y=0) = \phi_0(\vec{x})} \tag{4.8}$$

with the left hand side the generating functional for correlation functions on the SYM side and $Z_{string}$ the partition function of string theory restricted by the boundary condition on the dilaton $\phi$. The relation (4.8) is not restricted to the dilaton, and similar equations can be found for any field on AdS, including the graviton and the gauge field. They all have in common that the boundary values of the string theory fields are sources for the operators on the theory on the boundary. Each operator on the SYM side corresponds to exactly one field on the string theory side, and to preserve conformal invariance an operator with

a conformal dimension $\Delta$ corresponds to a field $\phi$ with conformal dimension $4 - \Delta$. In general all quantum numbers of the operator should be supplemented by the conjugate quantum numbers of the corresponding field on AdS. For any given operator in $\mathcal{N} = 4$ SYM we can find the corresponding field in string theory on $AdS_5 \times S^5$, and we can then obtain correlation functions by differentiation of the partition function in (4.8) with respect to the sources $\phi_0$ and setting $\phi_0$ to zero afterwards.
In the large $N$ limit we can make a saddle-point approximation of the AdS partition function and (4.8) becomes

$$\langle \exp\left(\int d^4x \phi_{0,i}(x) \mathcal{O}^i(x)\right)\rangle = e^{S_{IIB}(\phi_i)} \tag{4.9}$$

with $S_{IIB}$ the classical string or (for large $\lambda$) supergravity action and $\phi_i$ the fields on the string theory side, restricted by the condition that they take the values $\phi_{0,i}$ at the boundary. Equation (4.9) is a useful tool to relate expectation values of operators on the SYM side to the classical action on the AdS side and it will be the starting point for the calculations of the expectation value of the Wilson loop.

# Chapter 5

# Wilson Loops in $\mathcal{N} = 4$ SYM

In this chapter we will use perturbation theory to calculate expectation values of the Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. All calculations will be done in the large $N$ limit, while keeping $\lambda = g_{YM}^2 N$ fixed. We will also give a short description of the supersymmetry that Wilson loops with different geometries preserve.
The Wilson loop operator in $\mathcal{N} = 4$ SYM measures the holonomy of a very heavy W-boson which obtains its mass through symmetry breaking of the $U(N)$ gauge symmetry to $U(N-1) \times U(1)$. It is[1]

$$W(C) = \frac{1}{N} Tr P \exp \int_C d\tau \left( i A_\mu(x) \dot{x}^\mu + \Phi_i(x) \theta^i |\dot{x}| \right) \tag{5.1}$$

with the field $A_\mu$, coupling to $x^\mu(\tau)$, which parametrizes the loop $C$ and the six scalars $\Phi^i$, coupling to the unit vector $\theta^i(\tau)$ which can be regarded as coordinates on the sphere $S^5$. We have $\dot{x}^\mu = \partial_\tau x^\mu(\tau)$ and we take the trace over the gauge indices and evaluate the path ordered expression. In general $\theta$ depends on the parameter $\tau$ but in the perturbative calculations we will choose it to be a constant. In the following we will be interested in the large $N$ limit, where only planar diagrams contribute.
We can consider a more general class of Wilson loops

$$W(C) = \frac{1}{N} Tr P \exp \int_C d\tau \left( i A_\mu(x) \dot{x}^\mu + \Phi_i(x) \dot{y}^i \right) \tag{5.2}$$

where $y^i(\tau)$ is an arbitrary six dimensional vector. This is the Wilson loop one obtains by dimensional reduction from $\mathcal{N} = 1$ SYM theory in ten dimensions [16]. Comparing this with (5.1) we obtain the constraint

$$\dot{x}^2 = \dot{y}^2, \tag{5.3}$$

which will become important in the D-brane calculations in the next chapter where we will regard the coordinates $y^i$ as T-dual coordinates transverse to the brane.
In the following we will only consider smooth loops. The case with cusps is for example described in [16].

[1] In this chapter we will consistently use Euclidean signature

## 5.1 Supersymmetric Wilson loops

A very important fact in the AdS/CFT correspondence is that the symmetries on both sides match. The amount of supersymmetry the Wilson loop operator preserves depends on its geometry. We will show that the straight line preserves half of the supersymmetries in the theory and the more complex the geometry becomes the less supersymmetry will be preserved. The supersymmetry transformations of the bosonic fields are

$$\begin{aligned} \delta_\epsilon A_\mu &= \bar{\Psi}\Gamma_\mu \epsilon \\ \delta_\epsilon \Phi_i &= \bar{\Psi}\Gamma_i \epsilon, \end{aligned} \tag{5.4}$$

with $\epsilon$ an infinitesimal anticommuting Majorana spinor in ten dimensions. Using this in (5.1) the variation of an arbitrary Wilson loop operator under a supersymmetry transformation to first order in $\epsilon$ is

$$\delta W(C) = \frac{1}{N} Tr\mathcal{P}\left(\int d\tau \bar{\Psi}(i\Gamma^\mu \dot{x}_\mu + \Gamma^i \theta_i |\dot{x}|)\epsilon\right) W(C). \tag{5.5}$$

Some of the 16 supersymmetries of $\mathcal{N}=4$ SYM will be preserved if

$$(i\Gamma^\mu \dot{x}_\mu + \Gamma^i \theta_i |\dot{x}|)\epsilon = 0. \tag{5.6}$$

One can show that $M^2 \equiv (i\Gamma^\mu \dot{x}_\mu + \Gamma^i \theta_i |\dot{x}|)^2 = 0$, hence the 16 dimensional matrix $M$ has eight linearly independent eigenvectors. The solution of equation (5.6) will depend on the parametrization of the loop, so we only obtain local supersymmetry which is not a symmetry of the action.
For constant $\theta_i$ and parameter independent $\epsilon$ equation (5.6) has only a solution if the contour of the Wilson loop is a straight line. This can easily be seen if we differentiate (5.6) with respect to $\tau$ and parametrize $C$ such that $|\dot{x}| = 1$. We then obtain

$$i\Gamma^\mu \ddot{x}^\mu \epsilon = 0 \tag{5.7}$$

from which we conclude that $x^\mu(\tau)$ describes a straight line. The straight Wilson line thus commutes with eight of the sixteen supercharges of the theory. It is an $\frac{1}{2}$-BPS object and is protected from quantum corrections.
In [17] Zarembo has shown how to construct Wilson loops that preserve a certain amount of supersymmetry. The ansatz is a restriction on the position of the loop on the $S^5$, requiring that it follows the tangent vector $\frac{\dot{x}^\mu}{|\dot{x}|}$ of the spacetime contour $C$

$$\theta^i = M^i_\mu \frac{\dot{x}^\mu}{|\dot{x}|} \tag{5.8}$$

One can then show that the amount of supersymmetry preserved is correlated to the spacetime dimensionality of the Wilson loop. The straight line is a one dimensional object and preserves half of the supersymmetry of the theory. If we construct the Wilson loops as in (5.8), then for each additional dimension the Wilson loop extends to, supersymmetry is reduced by a factor 1/2. Thus, a 2D Wilson loop commutes with four supercharges and is $\frac{1}{4}$ BPS, whereas the 3D and 4D Wilson loops are 1/8 and 1/16 BPS respectively.

## 5.2 Wilson Loops in Perturbation Theory

We can calculate the expectation value $\langle W(C)\rangle$ in perturbation theory in the 't Hooft limit as an expansion in $\lambda = g_{YM}^2 N$. In the following we will describe the calculations for a Wilson loop with an arbitrary shape up to second order in $\lambda$. We then describe the particular geometries of a straight line, a circle and two antiparallel straight lines.
In the following we will use the notation

$$\begin{aligned} x_i &= x^{(i)} = x_\mu(\tau_i) \\ A(\tau_i) &= A_\mu^a(x)\dot{x}^\mu(\tau_i)T^a \\ \Phi(\tau_i) &= \Phi^a(x)|\dot{x}(\tau_i)|T^a \\ g &= g_{YM} \end{aligned} \tag{5.9}$$

and work in Feynman gauge[2].

**First order contribution**

We can expand the Wilson loop in $\lambda$ as

$$\begin{aligned} \langle W(C)\rangle &= \sum_{n=0}^{\infty} a_n \lambda^n \\ a_0 &= 1 \\ a_1 &= \frac{1}{2N}\oint ds_1 ds_2 Tr\left(-\dot{x}_1^\mu \dot{x}_2^\nu \langle A_\mu(x_1)A_\nu(x_2)\rangle + |\dot{x}_1|\theta_1^i\ |\dot{x}_2|\theta_2^j \langle \Phi_i(x_1)\Phi_j(x_2)\rangle\right) \\ a_2 &= (\ldots). \end{aligned} \tag{5.10}$$

The free gluon and scalar propagators in $2\omega$ dimensions are

$$\begin{aligned} G_{\mu\nu}^{ab} &= \langle A_\mu^a(x_1)A_\nu^b(x_2)\rangle_0 = g^2\frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{\delta^{ab}\delta_{\mu\nu}}{(x_1-x_2)^{2(\omega-1)}} \\ G_{ij}^{ab} &= \langle \Phi_i^a(x_1)\Phi_j^b(x_2)\rangle_0 = g^2\frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{\delta^{ab}\delta_{ij}}{(x_1-x_2)^{2(\omega-1)}}, \end{aligned} \tag{5.11}$$

with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Inserting these expressions (in four dimensions) into (5.11) we obtain the contribution to the expectation value of the Wilson loop to first order in $\lambda$

$$a_1\lambda = \frac{\lambda}{4\pi^2}\oint_C d\tau_1 d\tau_2 \frac{|\dot{x}(\tau_1)||\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{|x(\tau_1)-x(\tau_2)|^2}. \tag{5.12}$$

where we used $T^aT^a = \frac{N}{2}\mathbb{1}$ for the generators of the $U(N)$ Lie algebra and the factor $N^2$ in (5.12) is obtained by taking the trace of the unit matrix.

[2]We will only calculate a subset of diagrams to all orders, namely planar diagrams without internal vertices. Therefore, the choice of a different gauge could lead to different results. In the cases we consider, diagrams with internal vertices cancel to second order in $\lambda$ and assuming that this result holds to all orders in $\lambda$ the gauge choice does not lead to different results

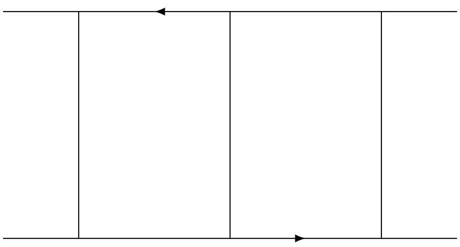

Figure 5.1: A ladder diagram

For $\tau_1 = \tau_2$ (5.12) is divergent. If we regularize the propagators (5.11) in the denominator as $(x_1 - x_2)^2 + \epsilon^2$ (5.12) becomes

$$\frac{\lambda}{4\pi^2} \oint d\tau_1 d\tau_2 \frac{|\dot{x}(\tau_1)||\dot{x}(\tau_1)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_1)}{\epsilon^2} \tag{5.13}$$

from which we conclude that the divergent contributions from the gauge field and the scalars mutually cancel.

### Second order contributions

To obtain the second order in $\lambda$ we have to consider the fourth order term in the expansion of the Wilson loop

$$\Sigma_2 = \frac{1}{4!N} \oint d\tau_1 d\tau_2 d\tau_3 d\tau_4 Tr\mathcal{P}\Big\langle \big(iA(\tau_1) + \Phi(\tau_1)\big) \\ \times \big(iA(\tau_2) + \Phi(\tau_2)\big)\big(iA(\tau_3) + \Phi(\tau_3)\big)\big(iA(\tau_4) + \Phi(\tau_4)\big)\Big\rangle. \tag{5.14}$$

Inserting the free propagators (5.11) and performing a series of Wick contractions (5.14) becomes

$$\Sigma_2 = \frac{g^4 N^2}{6} \oint_{\tau_1 > \tau_2 > \tau_3 > \tau_4} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \frac{|\dot{x}^{(1)}||\dot{x}^{(2)}| - \dot{x}^{(1)} \cdot \dot{x}^{(2)}}{|\dot{x}^{(1)} - \dot{x}^{(2)}|^2} \\ \times \frac{|\dot{x}^{(3)}||\dot{x}^{(4)}| - \dot{x}^{(3)} \cdot \dot{x}^{(4)}}{|\dot{x}^{(3)} - \dot{x}^{(4)}|^2}. \tag{5.15}$$

This describes the contribution of the ladder diagrams, which obtain their name due to their particular form, see figure 5.2. The expression (5.15) is divergent for $\tau_1 = \tau_2$ and $\tau_3 = \tau_4$, but as in equation (5.13) the contributions from the gauge field cancel the ones from the scalars.

### One loop self energy contributions

To second order in $\lambda$ we also have to consider one loop corrections to the scalar and vector propagators. We use the spacetime dimensionality $D = 2\omega$ because in four dimensions the result will be divergent. We obtain the one loop corrections for the gluon propagator from the diagrams in figure 5.2. The calculations are elementary and for example done in [18]. The corrected propagator in momentum space then becomes

$$\Delta^{ab}_{\mu\nu} = g^2 \frac{\delta^{ab}\delta_{\mu\nu}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot \frac{(8\omega - 4)\delta^{ab}(\delta_{\mu\nu} - p_\mu p_\nu / p^2)}{p^{6-2\omega}} \tag{5.16}$$

Figure 5.2: One loop diagrams for the gauge propagator: The vector and ghost loop, the scalar loop and the fermion loop

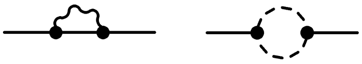

Figure 5.3: The one loop contributions to the scalar propagator: The scalar-vector intermediate state and the fermion loop

The one loop corrections to the scalar propagator are diagrammatically shown in figure 5.3. The scalar propagator then becomes

$$D_{ij}^{ab} = g^2 \delta^{ab} \frac{\delta_{ij}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^{\omega}\Gamma(2\omega)} \cdot \frac{(8\omega-4)\delta_{ij}\delta^{ab}}{p^{6-2\omega}}. \tag{5.17}$$

We can see that there is a singularity at $\omega = 2$ which is due to an ultraviolet divergence. In general, for computing expectation values of local operators, we should add a counterterm to the action but this is not necessary for computing the (non-local) Wilson loop.
To compute the corrected Wilson loop we have to Fourier transform (5.16) and (5.17) and insert the result into the first order term for the expectation value of the Wilson loop, equation (5.12). We then obtain the following correction

$$\delta_{loop} = -g^4 N^2 \frac{\Gamma^2(\omega-1)}{2^7 \pi^{2\omega}(2-\omega)(2\omega-3)} \oint d\tau_1 d\tau_2 \frac{|\dot{x}(\tau_1)||\dot{x}(\tau_2)| - \dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{(|x(\tau_1)-x(\tau_2)|^2)^{2\omega-3}} \tag{5.18}$$

and we see that the integral is the same as in (5.12) just with a different coefficient.

**Diagrams with one internal vertex**

By expanding the Wilson loop to third order and contracting with the relevant vertices we obtain further contributions to the Wilson loop of order $\lambda^2$. This becomes clear if one considers that a vertex contributes a factor $1/g^2$ and each propagator a factor $g^2$, hence diagrams with internal vertices are of order $\lambda^2$. The relevant quantities are

$$\begin{aligned} \delta_V \;=\; & \frac{-i}{3!} \oint d\tau_1 \tau_2 \tau_3 \big\langle Tr\mathcal{P}[\Phi(\tau_1)A(\tau_2)\Phi(\tau_3)] \left( - \int d^{2\omega}y f^{abc} \partial_\mu \Phi_i^a(y) A_\mu^b(y) \Phi_i^c(y) \right) \big\rangle \\ & + \frac{i}{2!} \oint d\tau_1 \tau_2 \tau_3 \big\langle Tr\mathcal{P}[A(\tau_1)A(\tau_2)A(\tau_3)] \left( - \int d^{2\omega}y f^{abc} \partial_\mu A_\nu^a(y) A_\mu^b(y) A_\nu^c(y) \right) \big\rangle \end{aligned} \tag{5.19}$$

By using Wick's theorem we obtain

$$\delta V = -\frac{g^4N^2}{4}\oint d\tau_1 d\tau_2 d\tau_3 \epsilon(\tau_1,\tau_2,\tau_3)(|\dot{x}^{(1)}||\dot{x}^3| - \dot{x}^{(1)}\dot{x}^3)$$
$$\times \dot{x}_2^\mu \frac{\partial}{\partial x_1^\mu}\int d^{2\omega}y \Delta(x^{(1)}-y)\Delta(x^{(2)}-y)\Delta(x^{(3)}-y) \tag{5.20}$$

where we introduced the antisymmetric path ordering symbol defined as $\epsilon(\tau_1,\tau_2,\tau_3) = 1$ for $\tau_1 > \tau_2 > \tau_3$ and antisymmetric under transpositions of $\tau_i$. We further used the Greens functions $\Delta(x-y)$ defined as

$$\partial_x^2 \Delta(x-y) = -\delta^{2\omega}(x-y) \tag{5.21}$$

and this equation is solved by

$$\Delta(x-y) = \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{e^{ip\cdot x}}{p^2} = \frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{1}{(x-y)^{2(\omega-1)}}, \tag{5.22}$$

and we see that, apart from the Dirac-Delta functions, this is equal to the free gauge and scalar field propagators.
To get rid of the factors $(x^{(i)}-y)^{2(\omega-1)}$ in the denominator one introduces Feynman parameters $\alpha,\beta,\gamma$. We can then do the $y$-integral and (5.20) becomes

$$\delta_V = g^4N^2\frac{\Gamma(2\omega-2)}{2^7\pi^{2\omega}}\int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2}\delta(1-\alpha-\beta-\gamma)\oint d\tau_1 d\tau_2 d\tau_3 \epsilon(\tau_1,\tau_2,\tau_3)$$
$$\times \frac{(|\dot{x}^{(1)}||\dot{x}^{(3)}| - \dot{x}^{(1)}\cdot\dot{x}^{(3)})\left(\alpha(1-\alpha)\dot{x}^{(1)}\cdot\dot{x}^{(2)} - \alpha\gamma\dot{x}^{(2)}\dot{x}^{(3)} - \alpha\beta(\dot{x}^{(2)})^2\right)}{\left(\alpha\beta|x^{(1)}-x^{(1)}|^2 + \alpha\gamma|x^{(1)}-x^{(3)}|^2 + \beta\gamma|x^{(2)}-x^{(3)}|^2\right)^{2\omega-2}}. \tag{5.23}$$

For $\tau_2 \to \tau_1$ there is a logarithmic divergence in the integral in equation (5.20). It comes from the integration over $y$ close to $x^{(1)}$ and is cancelled by the divergence from the one loop correction (5.18) in $D=4$ dimensions. This can be shown as follows: The $y$-integral in 5.23 in four dimensions can be written as [19]

$$\int d^4y \Delta(y-x^{(1)})\Delta(y-x^{(2)})\Delta(y-x^{(3)}) \sim -\frac{1}{64\pi^2}\frac{\log|x^{(1)}-x^{(2)}|^2/\delta^2}{|x^{(1)}-x^{(2)}|^2} \tag{5.24}$$

where we take the limit $\tau_1 \to \tau_2$ and $\delta$ is the upper integration limit, going to infinity for $\tau_1$ very close to $\tau_2$. We introduce $\tau = \tau_1 - \tau_2$ and Taylor expand $x^{(1)} = x(\tau_1) = x(\tau_2+\tau) = x^{(2)} + \dot{x}^{(2)}\tau + \ldots$. Using this and equation (5.24) we obtain the divergent part of (5.20) as

$$-\frac{g^4N^2}{128\pi^4}\oint d\tau_2 d\tau_3 \frac{|\dot{x}^{(2)}||\dot{x}^{(3)}| - \dot{x}^{(2)}\cdot\dot{x}^{(3)}}{|x^{(2)}-x^{(3)}|^2}\int d\tau \text{sign } \tau \frac{\dot{x}^{(2)}\cdot(x^{(1)}-x^{(2)})}{|x^{(1)}-x^{(2)}|^2}$$
$$= -\frac{g^4N^2}{64\pi^4}\oint d\tau_2 d\tau_3 \frac{|\dot{x}^{(2)}||\dot{x}^{(3)}| - \dot{x}^{(2)}\cdot\dot{x}^{(3)}}{|x^{(2)}-x^{(3)}|^2}\log\epsilon. \tag{5.25}$$

with $\epsilon = |x^{(1)}-x^{(2)}|^2 \to 0$. We obtain the divergent contribution from the one loop calculations by replacing $\frac{1}{2-\omega}$ with $-2\log\epsilon$ for $\omega \to 2$ in equation (5.18) and we see that this cancels the divergence from equation (5.25).

Having done the more general calculations we can now turn on the particular loop geometries. We will consider the straight line, two antiparallel lines and a circular loop.

### 5.2.1 Straight line

The straight infinite Wilson line is a BPS-object and preserves half of the supersymmetries of the theory. It thus commutes with eight of the 16 supercharges and this residual symmetry protects it from quantum corrections.

We can parametrize it by $x^\mu = (\tau, 0, 0, 0)$ and obtain to second order in $\lambda$

$$\langle W(C) \rangle = 1 \tag{5.26}$$

where we used the expansion (5.12) and (5.14) as well as the correction terms (5.18) and (5.23). Due to supersymmetry we expect this result to hold to all orders in perturbation theory.

### 5.2.2 Circular loop

The circular Wilson loop can be parametrized by

$$x^\mu(\tau) = (\cos\tau, \sin\tau, 0, 0). \tag{5.27}$$

We can always choose the radius to be one due to conformal invariance. For the circular loop we can calculate the sum of all planar diagrams without internal vertices. Taylor expanding the Wilson loop to $2n$-th order we obtain[3]

$$\frac{1}{N}\int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{2n-1}} d\tau_{2n} Tr \left\langle (iA(\tau_1) + \Phi(\tau_1)) \ldots (iA(\tau_{2n}) + \Phi(\tau_{2n})) \right\rangle \tag{5.28}$$

We now want to perform all Wick contractions that represent planar diagrams without internal vertices. Then (5.28) is the integral over $n$ factors of the form

$$\left\langle (iA^a(\tau_1) + \Phi^a(\tau_1)) \left(iA^b(\tau_2) + \Phi^b(\tau_2)\right) \right\rangle_0 = \frac{g^2\delta^{ab}}{8\pi} \tag{5.29}$$

where we have used (5.12) and the parametrization of the loop (5.27). We see that the possible Wick contractions for planar diagrams without internal vertices always give the same result, and the contribution of a planar diagram without internal vertices to order $\lambda^n$ is

$$\frac{(g^2N/4)^n}{(2n)!} \tag{5.30}$$

where we evaluated the integral and took the trace in (5.28). To find the contribution of all planar diagrams we have to multiply (5.30) with the number of diagrams containing $n$ propagators and sum over all $n$. We then obtain [19]

$$\langle W(C) \rangle = \frac{2}{\sqrt{g^2N}} I_1(\sqrt{g^2N}) \tag{5.31}$$

with $I_1(x)$ a Bessel function given by

$$I_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(x^2/4)^n}{(n+1)!\; n!}. \tag{5.32}$$

[3]The factor $1/2n!$ from the Taylor expansion is canceled by our choice of a particular time ordering.

In [19] it is shown to second order in $\lambda$ that the loop contribution to the free propagators cancels with the contribution from the diagrams with internal vertices and it is suggested that this holds to all orders.
In the large $\lambda = g^2 N$ limit equation (5.31) becomes

$$\langle W(C)\rangle = \frac{\exp\sqrt{\lambda}}{(\pi/2)^{1/2}(\lambda)^{3/4}} \tag{5.33}$$

and we will find this exponential behavior in the string calculations in chapter 6.
We can also obtain (5.31) by mapping the straight line to the circle with a conformal transformation. If the expectation value for a Wilson loop was invariant under a conformal transformation the results for the straight line and the circle should be identical. This is not the case due to a conformal anomaly which can be calculated exactly to all orders in the coupling.

The line is mapped to a circle by an inversion about the origin

$$x^\mu \to \frac{x^\mu}{x^2}. \tag{5.34}$$

By using the conformal transformation from chapter 2 the scalar and gauge propagators in equation (5.11) transform to

$$\begin{aligned}\tilde{G}^{ab}_{ij} &= \frac{g^2}{4\pi^2} x_1^2 x_2^2 \frac{\delta_{ij}\delta^{ab}}{(x_1-x_2)^2}\\ \tilde{G}^{ab}_{\mu\nu} &= \frac{g^2}{4\pi^2} x_1^2 x_2^2 I_{\mu\rho}(x_1) I_{\nu\sigma}(x_2)\frac{\delta_{\rho\sigma}\delta^{ab}}{(x_1-x_2)^2}\end{aligned} \tag{5.35}$$

with

$$I_{\mu\nu} = g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{5.36}$$

The transformation of the scalar propagator leads to no additional contribution, which can be shown as follows. After expanding the Wilson loop and doing Wick contractions terms involving the scalar fields look like

$$|\dot{x}_1|\theta_1^i|\dot{x}_2|\theta_2^j G^{ab}_{ij}, \tag{5.37}$$

which is due to the coupling of $\Phi_i$ to $\dot{y}^i = |\dot{x}|\theta^i$ in equation (5.1). Taking into account that $\dot{y}^i$ transforms like $|\dot{x}|\theta^i \to (|\dot{x}|/|x^2|)\theta^i$, we see that (5.37) remains unchanged under an inversion.
There is a divergence in (5.37) when $x_1 \to x_2$ but we have already shown that this is canceled by a divergent term coming from the gluon propagator. Because the transformation of the scalar propagator does not give an extra contribution, the conformal anomaly must arise from transforming the gauge field. The terms involving the gluon propagator are of the form

$$\dot{x}_1^\mu \dot{x}_2^\nu G^{ab}_{\mu\nu} \tag{5.38}$$

which change under an inversion by a total derivative

$$\dot{x}_1^\mu \dot{x}_2^\nu G^{ab}_{\mu\nu}(x_1,x_2) \to \dot{x}_1^\mu \dot{x}_2^\nu \left[G^{ab}_{\mu\nu} + \frac{g^2\delta^{ab}}{4\pi^2}\partial^1_\mu\left(\log\frac{(x_1-x_2)^2}{|x_1|}\partial^2_\nu \log x_2^2\right)\right]. \tag{5.39}$$

A total derivative can usually be neglected, and one could conclude that an inversion is indeed a symmetry of the Wilson loop. But there is a possible singularity in the surface term that actually gives a contribution.
Let us start with a straight line parametrized by

$$x^\mu(\tau) = \frac{1}{2}(1, \tan(\tau/2), 0, 0) \tag{5.40}$$

which leads to the expectation value $\langle W(C)\rangle = 1$. By an inversion the line is mapped to the circle passing trough the origin

$$\tilde{x}^\mu(\tau) = \frac{x^\mu}{x^2} = (1 + \cos\tau, \sin\tau, 0, 0). \tag{5.41}$$

We now obtain the Wilson loop along the circle in terms of the diagrams for the straight line modified in the way described in equation (5.39). Because the modification is a surface term, we only obtain a contribution if (5.39) is singular. In [20] it is shown that this is in fact the case if we map the straight line to the circle. The gauge propagator for the circular loops behaves like a delta function and contributes a constant factor in each diagram. If one assumes that diagrams with internal vertices vanish, one simply has to add all graphs with noninteracting modified gauge propagators, as we did in (5.30). To leading order in $1/N$ one obtains the result (5.31).
The fact that we obtain a nonvanishing contribution to the gluon propagator only from a singular point, and that this contribution is just a constant leads to the suggestion that the calculations can be done in a 0-dimensional field theory, a matrix model. The expectation value is then calculated as

$$\langle W(C)\rangle = \langle \frac{1}{N} Tr \exp(M)\rangle = \frac{1}{Z}\int \mathcal{D}M \frac{1}{N} Tr \exp(M) \exp(-\frac{2N}{\lambda} Tr M^2). \tag{5.42}$$

The calculations are shown in detail in [20], and the result is

$$\langle W(C)\rangle = \frac{1}{N} L^1_{N-1}(-\lambda/4N) \exp(\lambda/8N) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} + \dots \tag{5.43}$$

with $I_n$ the modified Bessel functions and $L^m_n = \frac{1}{n!}\exp(x) x^{-m} (d/dx)^n (\exp(-x) x^{n+m})$ the Laguerre polynomials. To leading order in $\frac{1}{N}$ this simplifies to (5.31) which in the large $\lambda$ limit agrees with the string theory calculations. It is worth mentioning that (5.43) agrees with the result we obtain from the D-brane calculations, which we will describe in chapter 6.
The circular Wilson loop does not preserve any Poincaré supersymmetries, but it can be shown that it commutes with eight linear combinations of Poincaré and conformal supercharges [21]. The cancellation of diagrams with internal vertices is expected to be due to this invariance [17].

### 5.2.3 Rectangular loop

For the rectangular loop we consider two anti-parallel lines of lengths $S$ and $T$ separated by a distance L in the limit $S, T \gg L$. From the expectation value of the anti-parallel lines we can read off the quark-antiquark energy via the relation $\langle W(C)\rangle \sim e^{-TE(L)}$. We obtain a recursive relation for the sum of all ladder-like diagrams without internal vertices [19]

$$\Sigma(S,T) = 1 + \int_0^S ds \int_0^T dt \frac{g^2N}{4\pi^2}[(s-t)^2 + L^2]^{-1}\Sigma(s,t). \tag{5.44}$$

We should obtain the result for the straight line in the limit $S = 0$ or $T = 0$ which leads to the boundary conditions

$$\Sigma(S,0) = \Sigma(0,T) = 1. \tag{5.45}$$

From (5.44) we obtain a differential equation for $\Sigma$ by taking derivatives with respect to $S$ and $T$

$$\frac{\partial^2\Sigma(S,T)}{\partial S\partial T} = \frac{g^2N}{4\pi^2}[(s-t)^2 + L^2]^{-1}\Sigma(s,t) \tag{5.46}$$

We use the usual ansatz and write $\Sigma(S,T)$ as a product with factors depending only on a single variable

$$\Sigma(x,y) = A(x)B(y) \tag{5.47}$$

where we defined $x = (S-T)/L$, and $y = (S+T)/L$. Inserting this into (5.46) we obtain

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{g^2N}{4\pi^2(x^2+1)}\right)A(x)B(y) + \frac{\partial^2}{\partial y^2}B(y)A(x) = 0 \tag{5.48}$$

We can easily find that $B(y) \sim \exp(\Omega y/2)$ and our ansatz for $\Sigma$ becomes

$$\Sigma(x,y) = \sum_n c_n\psi_n(x)e^{\Omega_n y/2}. \tag{5.49}$$

Inserting this in (5.48) we obtain

$$\left(\frac{d^2}{dx^2} + \frac{g^2N}{4\pi^2(x^2+1)} - \frac{\Omega_n^2}{4}\right)\psi_n(x) = 0 \tag{5.50}$$

In the limit $S \to T$ corresponding to $x \to 0$ we can Taylor expand $1/(1+x^2)$ and (5.50) becomes

$$\left(\frac{d^2}{dx^2} + \frac{g^2N}{4\pi^2}(1-x^2) - \frac{\Omega_n^2}{4}\right)\psi_n(x) = 0 \tag{5.51}$$

which is the Schroedinger equation with a harmonic oscillator potential. In [19] it is illustrated that the major contribution to $\Sigma(x,y)$ comes from the ground state $\psi_0$. The ground state of the harmonic oscillator has a solution of the form $\psi_0(x) \sim e^{\alpha x^2}$. Using this in (5.50) we obtain

$$\psi_0 \sim e^{-\frac{\sqrt{\lambda}}{4\pi}x^2}, \quad \Omega_0^2 = \frac{\lambda}{\pi^2} - 2\frac{\sqrt{\lambda}}{\pi}. \tag{5.52}$$

By inserting this into (5.44) we finally obtain the sum of all ladder-like diagrams without internal vertices in the large $\lambda$ limit

$$\Sigma(T,T) = \exp\left[\left(\frac{\sqrt{\lambda}}{\pi} - 1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\right)\frac{T}{L}\right] \tag{5.53}$$

This is the also the final result we obtain to second order perturbation theory in $\lambda$

$$\langle W(C) \rangle = \Sigma(T, T), \tag{5.54}$$

because the contribution from diagrams with internal vertices cancel with the one loop corrections to the propagators [19].

# Chapter 6

# Wilson Loops in String Theory

## 6.1 Area Law Calculations

In the last chapter we described how to use perturbation theory to calculate expectation values of Wilson loops with different geometries in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In this section we will calculate the expectation values of Wilson loops in string theory using the classical string action.

The Wilson loop describes the phase factor associated with the propagation of a heavy W-boson in the fundamental representation of the gauge group. In $\mathcal{N} = 4$ SYM all fields are in the adjoint representation of the gauge group, and the W-boson arises when the $SU(N)$ gauge symmetry is broken to $SU(N-1) \times U(1)$ by giving an expectation value to some scalar field $\Phi$ and taking this expectation value to infinity.

In the language of string theory we construct a heavy W-boson by separating one of the $N$ coincident D-branes, and stretching a string between the single and the bunch of branes, thus breaking the $SU(N)$ symmetry on the worldvolume and giving a mass to the W-boson. We want to find the expectation value of the Wilson loop operator $W(C)$ in the large $N$ and large $\lambda$ limit. Here, the standard prescription for the calculation on the AdS side is to take the exponent of the proper area given by a fundamental string ending on the loop at the boundary of AdS. We will consider different geometries for the loop, an infinite straight line, a rectangle and a circle.

In $\mathcal{N} = 4$ SYM the Wilson loop operator is given by

$$W(C) = \frac{1}{N} TrP \exp\left(\oint_C d\tau \left(iA_\mu \dot{x}^\mu + \Phi_i \dot{y}^i\right)\right), \tag{6.1}$$

where we used Euclidean signature. It turns out [16] that only Wilson loops satisfying

$$\dot{x}^2 = \dot{y}^2 \tag{6.2}$$

can be described by minimal surfaces ending on the boundary[1].

The amplitude for a W-boson with mass $M$ moving along a curve $C$ with length $L(C)$ is given by the expectation value of the Wilson loop times a mass dependent factor [22]

$$\mathcal{A} = \langle W(C) \rangle e^{-ML(C)}. \tag{6.3}$$

[1]This can be shown by use of the Hamilton-Jacobi equation for a minimal surface. It can only be satisfied if the string ends at the boundary and this leads to (6.2)

According to the Maldacena conjecture the prescription for calculating this amplitude on the AdS side is

$$\mathcal{A} = \int_{\delta X = C} \mathcal{D}X^{\mu}\mathcal{D}\gamma_{\alpha\beta}\mathcal{D}\Theta^{i} \exp\left(-S\right) \tag{6.4}$$

where $S$ is the Polyakov action (3.1) on $AdS_5 \times S^5$ with supersymmetric extension

$$S = \frac{1}{4\pi\alpha'}\int_D d^2\sigma \left(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN} + \text{Fermions}\right) \qquad M, N = 0\ldots 10 \tag{6.5}$$

with $\Theta^i$ the anticommuting superspace coordinates whose bosonic part is $AdS_5 \times S^5$. We integrate over the string world sheet $D$ and the string should end on the curve $C$ at the boundary. It is thus restricted to some particular boundary conditions, which we will consider later in this chapter. Equation (6.4) describes a two dimensional Sigma model that cannot be solved exactly. In the large $\lambda = g_{YM}^2 N$ limit string fluctuations are suppressed and the bosonic part of the action dominates. In this case we can obtain the value of the path integral (6.4) using the steepest descent method [16] where we expand the action around its minimal value

$$S[X] = S[X_0] + \left.\frac{\delta S}{\delta X^M}\right|_{X_0}(X^M - X_0^M) + \mathcal{O}(X^2). \tag{6.6}$$

By definition the first order term vanishes, and considering that the $AdS_5 \times S^5$ metric $G_{MN}$ contains a factor $\sqrt{\lambda}$ we see that the second order term (and all higher power terms in X) becomes very large and equation (6.4) becomes the minimal area law

$$\langle W(C)\rangle = e^{-A(C)+ML(C)}. \tag{6.7}$$

$A(C)$ is the minimal surface ending on the loop $C$ at the boundary, and given by the minimal Polyakov action (6.5) or, classically equivalent, the Nambu-Goto (NG) action

$$S_{NG} = A(C) = \mathcal{T}\int d\tau d\sigma \sqrt{g} \tag{6.8}$$

with string tension $\mathcal{T} = \frac{1}{2\pi\alpha'}$ and $g$ the determinant of the pullback of the metric to the worldsheet, given by

$$g = \det g_{\alpha\beta} = \det\left(G_{MN}\partial_\alpha X^M \partial_\beta X^N\right). \tag{6.9}$$

### 6.1.1 Equations of motion and boundary conditions

To minimize the $NG$ action we have to find the equations of motion for the fields $X^M = X^M(\tau,\sigma)$ subject to some boundary conditions. Varying (6.8) with respect to $X^M$ leads to

$$\delta S_{NG} = \int d\tau d\sigma \left\{\left(\frac{\partial\mathcal{L}}{\partial X^M} - \partial_\alpha \frac{\partial\mathcal{L}}{\partial\partial_\alpha X^M}\right)\delta X^M + \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^M}\delta X^M\right)\right\}. \tag{6.10}$$

The term in the first parentheses corresponds to the equations of motion while the on in the second is a surface term that should vanish. Using

$$\delta(\sqrt{g}) = \frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta} \tag{6.11}$$

the surface term becomes

$$\begin{aligned}&\mathcal{T}\int d\tau d\sigma \partial_\alpha \left(\sqrt{g} g^{\alpha\beta}\partial_\beta X^M \delta X^N G_{MN}\right)\\ &= \mathcal{T}\int d\tau \left[\sqrt{g} g^{2\beta}\partial_\beta X^M \delta X^N G_{MN}\right]_{\sigma=0}^{\sigma=2\pi} = 0 \end{aligned}\tag{6.12}$$

where we used that the fields vanish at $\tau = \pm\infty$. The boundary term (6.12) vanishes by either imposing Neumann or Dirichlet boundary conditions. To get physical information on the configuration we have to focus on the string end at the boundary of $AdS_5$. In the following we will use one of the following two forms for the Euclidean metric of $AdS_5 \times S^5$

$$\begin{aligned} ds^2 &= L^2 \left( U^2 dX^\mu dX^\mu + \frac{dU^2}{U^2} + L^2 d\Omega_5^2 \right) &(6.13)\\ &= \frac{L^2}{Y^2}\left(dX^\mu dX^\mu + dY^i dY^i\right), \ (\mu = 1\dots 4,\ i = 1\dots 6) &(6.14)\end{aligned}$$

where for the latter we introduced

$$Y^i = \frac{\Omega^i}{U} \rightarrow Y = \frac{1}{U}. \tag{6.15}$$

In these coordinates the boundary is $Y = \frac{1}{U} = 0$.
We have to impose Dirichlet Boundary conditions for the four coordinates $X^\mu$ and Neumann coordinates for $Y^i$ [16]. Heuristically this is clear because the string is attached to the Wilson loop at the boundary, and following our discussion about $AdS$ spaces in chapter 3 we can identify $X^\mu$ with the boundary coordinates. Another argument comes from T-Duality. We start by considering a string ending on a space filling brane in ten dimension which corresponds to Neumann boundary conditions for all directions $X^\mu$. On the other hand, demanding that the string should end on a Wilson loop in ten dimensions leads to complementary boundary conditions, i.e. Dirichlet boundary conditions for all directions. We obtained $\mathcal{N} = 4$ SYM in four dimensions by dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. Analogously in the case of a string moving on a space filling brane we can perform 6 T-duality transformations, leading to 4 Neumann boundary conditions tangential and 6 Dirichlet BC transverse to the D3-brane worldvolume. The string ending on the Wilson loop is then subjected to complementary boundary conditions, von Neumann boundary conditions for the transverse coordinates and Dirichlet boundary conditions for the coordinates along the Brane.
Taking the string end with $\sigma = 0$ connected to the Wilson loop at the boundary the four Dirichlet boundary conditions are

$$X^\mu(\tau, 0) = x^\mu(\tau), \tag{6.16}$$

with $x^\mu(\tau)$ the parametrization of the Wilson loop at the boundary. The Neumann boundary conditions on the tangential coordinates are

$$\sqrt{g} g^{2\beta}\partial_\beta Y^i(\tau, 0) = \partial_\tau y^i(\tau). \tag{6.17}$$

with $y^i(\tau)$ the coordinate on the boundary coupling to the scalar fields in (6.1).
It is not immediately clear why we choose the boundary conditions (6.17), but the identification of the SO(6) symmetry on the $S^5$ with the $R$-symmetry in $\mathcal{N} = 4$ SYM and

reparametrization invariance of the boundary conditions demands that we set

$$\sqrt{g}g^{2\beta}\partial_\beta Y^i(\tau,0) = c\dot{y}^i(\tau). \tag{6.18}$$

with $c$ a normalization constant. The Neumann boundary conditions for $Y^i$ do not ensure that the string actually ends at the boundary $Y = 0$. Demanding that $Y(\tau, \sigma = 0) = 0$ together with $\dot{x}^2 = \dot{y}^2$ ensures that the normalization constant in (6.18) is one and we obtain the boundary conditions (6.17).

If the constraint $\dot{x}^2 = \dot{y}^2$ holds we can actually reinterpret the Neumann boundary conditions for $Y^i$ as Dirichlet boundary conditions for $\Omega^i$ defined in (6.15). From equation (6.15) it follows that $Y^i = Y\Omega^i$ and equation (6.17) turns into

$$\sqrt{g}g^{2\beta}\partial_\beta(Y\Omega^i)|_{\sigma=0} = \sqrt{g}g^{2\beta}\frac{\partial_\beta Y^j Y^j}{Y}\Omega^i|_{\sigma=0} = \dot{y}^j\Omega^j\Omega^i|_{\sigma=0} \tag{6.19}$$

where we used that at the boundary a smooth loop is given by[2] $\partial_\alpha(Y\Omega^i) = \partial_\alpha Y\Omega^i$. Inserting this into (6.17) and using $\dot{y}^2 = \dot{x}^2$ we obtain

$$|\dot{x}|\theta^j\Omega^j\Omega^i = |\dot{x}|\theta^i \rightarrow \Omega^i = \theta^i = \frac{\dot{y}^i}{|\dot{y}|}. \tag{6.20}$$

The Neumann boundary conditions for $Y^i$ in equation (6.17) therefore turn into Dirichlet BC for the coordinates $\Omega^i$ on the sphere $S^5$.

The full set of boundary conditions for the coordinates $X^\mu, Y, \Omega^i$ is thus given by

$$X^\mu(\tau,0) = x^\mu(\tau),\ Y(\tau,0) = 0,\ \Omega^i(\tau,0) = \frac{\dot{y}^i}{|\dot{y}|}. \tag{6.21}$$

We will see, that the minimal area bound by the curve $C$ at the boundary is divergent. It can be shown that the divergency is proportional to the length of $C$ and it is canceled by the factor $e^{-ML(C)}$ in (6.3), if we let the mass $M$ go to infinity [22].

In [16] they introduced a method for removing this divergence by taking the Legendre transformation of the Nambu Goto action. The coordinates $Y^i$ satisfy Neumann boundary conditions which are actually a restriction for the conjugate momenta $P_i$, defined as

$$P_i = \frac{\delta A(C)}{\delta \partial_2 Y^i} = \frac{1}{2\pi\alpha'}G_{ij}\sqrt{g}g^{2\alpha}\partial_\alpha Y^j. \tag{6.22}$$

Inserting this into (6.17) the boundary conditions (6.17) become

$$2\pi\alpha' P^i = \frac{2\pi}{\sqrt{\lambda}}Y^2 P_i = \dot{y}^i. \tag{6.23}$$

It is now natural to choose the variables $X^\mu$ and $P_i$ for the area. This is achieved by adding a boundary term to the action[3]

$$\tilde{A} = A(C) - \int d\tau P_i Y^i|_{\sigma=\sigma_0}^{\sigma=\sigma_1} = A(C) - \frac{\sqrt{\lambda}}{2\pi}\int d\tau \frac{\dot{y}^i}{Y^2}Y^i|_{\sigma=\sigma_0}^{\sigma=\sigma_1} = A(C) - \frac{\sqrt{\lambda}}{2\pi}\int d\tau \frac{|\dot{y}^i|}{Y}|_{\sigma=\sigma_0}^{\sigma=\sigma_1}. \tag{6.24}$$

[2]This follows from the classical equations of motion.

[3]Here we consider the fact that we are dealing with macroscopic strings and that the parametrization can differ from $\sigma \in [0, \pi]$.

$\tilde{A}$ is the Legendre transformation of the Nambu Goto action. It still has the same equations of motion but the minimal value of $\tilde{A}$ is generically different from $A$. The divergence is proportional to the length of the loop $L(C)$ which we see by using $\dot{x}^2 = \dot{y}^2$ in equation (6.24)

$$\tilde{A} = A(C) - \frac{\sqrt{\lambda}}{2\pi\epsilon} \oint d\tau |\dot{x}| = A(C) - \frac{\sqrt{\lambda}}{2\pi\epsilon} L(C), \tag{6.25}$$

where we introduced a cutoff $\epsilon$ for $Y \to 0$ at the boundary.

### 6.1.2 Minimal area calculations

After a lot of preliminary considerations we are now ready to calculate the expectation value of the Wilson loop on the AdS side using the minimal area law (6.7). We will consider the geometry of an infinite straight line, a circle and two anti-parallel lines, and from the latter we obtain the energy of a quark-antiquark pair.

**Infinite straight line**

On the boundary we parametrize the line by

$$x^\mu = (\tau, 0, 0, 0). \tag{6.26}$$

We have to find a string that ends on that line at the boundary and extends to the interior of AdS. The Nambu Goto action was given in (6.8) and using the metric (6.14) we obtain

$$A(C) = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \frac{1}{Y^2} \sqrt{\det\left(\partial_\alpha X^\mu \partial_\beta X^\mu + \partial_\alpha Y \partial_\beta Y\right)}, \tag{6.27}$$

where we considered that $\Omega^i(\tau,\sigma) = \Omega^i_0$ as explained before. The parametrization for the string world sheet should be compatible with the boundary conditions and the equations of motion [4] and a convenient choice is

$$X^\mu = (\tau, 0, 0, 0),\ \ Y = \sigma, \quad \tau \in (-\infty, \infty);\ \sigma \in (0, \infty). \tag{6.28}$$

Using this in equation (6.27) we obtain

$$A(C) = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} d\sigma \frac{1}{\sigma^2} = \frac{\sqrt{\lambda}}{2\pi\epsilon} \int |\dot{x}| d\tau, \tag{6.29}$$

where we introduced a cutoff $\epsilon$ and we used that $|\dot{x}| = 1$. We find that the divergence is proportional to the length of the loop and as discussed before we have to add the boundary term (6.24). Our final result for the infinite straight line is

$$\tilde{A} = \frac{\sqrt{\lambda}}{2\pi\epsilon} \int d\tau |\dot{x}| - \frac{\sqrt{\lambda}}{2\pi\epsilon} \int d\tau |\dot{y}^i| = \frac{\sqrt{\lambda}}{2\pi\epsilon} \int d\tau (|\dot{x}| - |\dot{y}|) = 0 \tag{6.30}$$

where we once more used the constraint equation(6.2). We obtain the value of the Wilson loop for the infinite straight line $\langle W(C) \rangle = 1$ as expected from the SYM calculations.

[4] In general we should make an ansatz $Y = Y(\sigma)$, and it can then be shown that this ansatz satisfies the equations of motion independent of $\sigma$.

**Circular loop**

We can obtain the result for the circular loop by mapping the straight line to the circle with a conformal transformation. We first shift the line on the boundary of AdS away from the origin and parametrize it by

$$x^\mu = \left(\tau, \frac{1}{2}, 0, 0\right). \tag{6.31}$$

The conformal transformation

$$x^{\mu'} = \frac{x^\mu - \delta^{1\mu} x^2}{1 - 2x_1 + x^2} \tag{6.32}$$

maps the line to a circle with unit radius,

$$(x'^\mu)^2 = \frac{x^2 - 2\frac{1}{2}x^2 - x^4}{(1 - 2\frac{1}{2} + x^2)^2} = 1. \tag{6.33}$$

We can extend the conformal symmetry to an isometry of $AdS_5$

$$\begin{aligned} X^\mu \to X^{\mu'} &= \frac{X^\mu - \delta^{1\mu}(X^2 + Y^2)}{1 - 2X_1 + X^2 + Y^2} \\ Y \to Y' &= \frac{Y}{1 - 2X_1 + X^2 + Y^2}. \end{aligned} \tag{6.34}$$

where we used the AdS part of the metric (6.14). The circle on the boundary is defined by $x^2 = 1$ and using (6.34) this equation becomes

$$X^2 + Y^2 = 1 \to Y = \sqrt{1 - X^2}. \tag{6.35}$$

A parametrization for the Wilson loop on AdS that respects the symmetries is

$$\begin{aligned} X^\mu &= (r\cos\phi, r\sin\phi, 0, 0), \\ Y &= \sqrt{1 - r^2} \end{aligned} \tag{6.36}$$

and at the boundary $Y = 0$ we obtain $r = 1$ and $X^\mu|_{Y=0} = (\cos\phi, \sin\phi, 0, 0) = x^\mu$ which is exactly the definition of the circle with unit radius on the SYM side.
Inserting this into the action (6.27) we obtain

$$\begin{aligned} A(C) &= \frac{\sqrt{\lambda}}{2\pi} \int dr d\phi \frac{r}{Y^2(r)} \sqrt{1 + (\partial_r Y(r))^2} \\ &= \sqrt{\lambda} \int_0^1 dr \frac{r}{(1 - r^2)^{3/2}} = \sqrt{\lambda} \left[(1 - r^2)^{-1/2}\right]_0^1 = \frac{\sqrt{\lambda}}{\epsilon} - \sqrt{\lambda}. \end{aligned} \tag{6.37}$$

Where we introduced a cutoff $\epsilon \to 0$. The divergence is removed by adding the boundary term (6.24),

$$\tilde{A} = \frac{\sqrt{\lambda}}{\epsilon} - \sqrt{\lambda} - \frac{\sqrt{\lambda}}{2\pi\epsilon} \int_{\delta D} d\tau |\dot{x}| = -\sqrt{\lambda} \tag{6.38}$$

and the expectation value of the Wilson loop becomes

$$\langle W(C)\rangle = e^{-\tilde{A}} = e^{\sqrt{\lambda}}. \tag{6.39}$$

The circle is obtained from the straight line by a conformal transformation. Due to conformal invariance of $\mathcal{N}=4$ SYM we expect the result to be the same as the one for the infinite line, in contradiction to our results in equations (6.30) and (6.39). This can be explained by the fact that we have regularized the action, which explicitly breaks conformal invariance.

**Rectangular loop**

We consider a rectangular loop with sides of length $T$ and $L$. We can use it to determine the energy $E(L)$ of a quark-antiquark pair via the relation [23]

$$\langle W(C)\rangle = A(L)\exp\left(-TE(L)\right) \tag{6.40}$$

where we take the limit $T\to\infty$ which gives the rectangle the shape of two anti-parallel lines and the problem becomes translational invariant along the $\hat{T}$ direction.
We choose the lines to point in $x^0$ direction and we put the quark in $x^1$ direction at $L/2$, the antiquark at $-L/2$. The two lines are then parametrized as

$$\begin{aligned} x_1^\mu &= (\tau, L/2, 0, 0), \qquad \tau\in[-T/2,T/2], \\ x_2^\mu &= (-\tau, -L/2, 0, 0), \qquad \tau\in[-T/2,T/2]. \end{aligned} \tag{6.41}$$

We are interested in the minimal area of a string world sheet closing off on the lines at the boundary and fixed at a point on the $S^5$. We use the metric (6.13) for which the boundary is at $U\to\infty$. The boundary conditions (6.21) and the translational invariance of the problem lead us to parametrize the world sheet as

$$X^\mu = (\tau,\sigma,0,0),\ U=U(\sigma),\ \tau\in[-T/2,T/2],\ \sigma\in[-L/2,L/2], \tag{6.42}$$

$$\theta^i = \theta_0^i. \tag{6.43}$$

At the boundary $U\to\infty$ the string should end on the anti parallel lines, leading to the boundary conditions

$$U(\pm L/2)\to\infty \tag{6.44}$$

which automatically make sure that $X^\mu|_{U(\pm 1/2)} = x^\mu_{1/2}$. Inserting (6.42) into the Nambu Goto action (6.8) we obtain

$$A(C) = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{(\partial_\sigma U)^2+U^4} = \frac{T\sqrt{\lambda}}{2\pi}\int d\sigma\sqrt{(\partial_\sigma U)^2+U^4}. \tag{6.45}$$

To minimize the area we should find the equation of motion for $U=U(\sigma)$. The Lagrangian $L\equiv\frac{T\sqrt{\lambda}}{2\pi}\sqrt{(\partial_\sigma U)^2+U^4}$ does not explicitly depend on $\sigma$ and therefore the Hamiltonian

$$H \equiv \frac{\partial L}{\partial\partial_\sigma U}\partial_\sigma U - L = \frac{T\sqrt{\lambda}}{2\pi}\frac{U^4}{\sqrt{(\partial_\sigma U(\sigma))^2-U^4}} \tag{6.46}$$

is a constant of motion[5]. The string stretches between the two lines at $\sigma = \pm L/2$ on the boundary $U \to \infty$. Due to the symmetry of the problem $U$ assumes its minimal value $U_0$ at $\sigma = 0$. Inserting $U_0$ into (6.46) we obtain

$$H = \frac{U^4}{\sqrt{(\partial_\sigma U(\sigma))^2 - U^4}} = U_0^2. \tag{6.47}$$

Separation of variables leads to

$$d\sigma = U_0^2 \frac{dU}{U^2\sqrt{U^4 - U_0^4}} \tag{6.48}$$

Using $U(L/2) = \infty$ and the $U_0 = U(\sigma = 0)$ we can determine the value of $U_0$ by integrating equation (6.48)

$$\frac{L}{2} = U_0^2 \int_{U_0}^{\infty} \frac{dU}{U^2\sqrt{U^4 - U_0^2}} = \frac{1}{U_0}\int_1^{\infty} \frac{dt}{t^2\sqrt{t^4-1}} = \frac{1}{U_0}\frac{\sqrt{2}\pi^{3/2}}{\Gamma^2(\frac{1}{4})}, \tag{6.49}$$

where we transformed to the variable $t = U/U_0$. Inserting equations (6.48),(6.49) into the action (6.45) we obtain

$$A(C) = 2\frac{T\sqrt{\lambda}}{2\pi}\int_{U_0}^{\infty} dU \frac{U^2}{\sqrt{U^4 - U_0^4}} = \frac{T\sqrt{\lambda}}{\pi} U_0 \int_1^{1/U_0\epsilon} dt \frac{t^2}{\sqrt{t^4-1}} \tag{6.50}$$

where the factor 2 accounts for the fact that we are only integrating over half the $\sigma$ interval and we have introduced a cutoff $\epsilon \to 0$. According to equation (6.40) we can read off the energy of an quark-antiquark pair from the expectation value of the Wilson Loop. After evaluating the integral in (6.50) and Taylor expanding the resulting hypergeometric function in $U_0\epsilon$, we obtain

$$A(C) = 2\frac{T\sqrt{\lambda}}{2\pi\epsilon} - \frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)}\frac{T}{L} + \mathcal{O}((U_0\epsilon)^3) \tag{6.51}$$

Again the divergent part is proportional to the length of the loop and, following our previous discussion, it is removed by adding the boundary term in (6.25) to the action. We finally obtain the result

$$\langle W(C)\rangle = \exp\left(-TE(L)\right) = \exp\left(\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)}\frac{T}{L}\right). \tag{6.52}$$

and we can read off the energy

$$E(L) = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)}\frac{1}{L} \tag{6.53}$$

We can see that this result is rather different from the one we obtained with the perturbative calculations in $\mathcal{N} = 4$ SYM. This is not surprising though, the anti-parallel lines are not a BPS-object and therefore we should not expect that the sum of ladder diagrams will reproduce the correct strong coupling behavior.

[5]In the following we will drop the factor $\frac{T\sqrt{\lambda}}{2\pi}$, in the definition of $H$

## 6.2 D-Brane Solutions

In the last section we saw that the standard way of evaluating expectation values of Wilson loops in the large-$N$, large $\lambda$ limit on the AdS side is the area law, given by $\langle W(C)\rangle \sim \exp(-A)$, with $A$ being the minimal area of a string world sheet ending on the loop $C$ at the boundary. If we have multiply wound loops or many coinciding loops it is better to use a description in terms of D-branes. The action should describe a brane in the $AdS_5 \times S^5$ spacetime, pinching off the Wilson loop at the boundary. We have to consider the gauge field on the world volume as well as the coupling to the Ramond-Ramond background. The action thus contains a Born Infeld and a Wess Zumino term restricted by some boundary conditions.

### 6.2.1 The Action

The Dirac- Born Infeld action describes the dynamics of a $Dp$ brane in a gravitational background. We derived it in section 3.2 and it is

$$S_{DBI} = T_p \int_{M_{p+1}} d^{p+1}\xi e^{-\phi} \sqrt{\det(g + 2\pi\alpha' F)} \tag{6.54}$$

where we consider a configuration with vanishing 2-form field strength $B_{ij}$. The brane tension was given by

$$T_p = (2\pi)^{-p}(\alpha')^{-\frac{p+1}{2}}, \tag{6.55}$$

and $g$ is the pullback of the metric to the world volume

$$g_{ij} = \partial_i X^m \partial_j X^n G_{mn}. \tag{6.56}$$

$F$ is the two-form field strength of a $U(1)$ gauge field living on the world volume of the brane.
We consider a brane in a certain background, given by the fields from the closed string sector. A D-$p$ brane is naturally charged under a R-R $(p+1)$- form potential, hence we have to include a Wess Zumino (WZ) term in the action, which for a D3-brane is given by

$$S_{WZ} = -T_3 \int_{M_4} P[C^{(4)}] \tag{6.57}$$

where we again take the pullback of the $C^{(4)}$ form to the brane worldvolume. Equation (6.57) is not the most general expression for the coupling of a Dp-brane to the background fields. In general the WZ term is given by $T_p \int \sum_{r=0}^{10} P[C^r e^F]$ with $C^r$ the $r$-form field strength and $F$ the gauge field on the D-brane and it is implied that after expanding the exponential only the $(p+1)$-form term is taken [24], since it is the one that couples to the Dp-brane. In the background we consider, there is only the 4-form R-R potential, and therefore we use (6.57).
The action we are interested in thus describes a D-brane in a gravitational background with a gauge field living on it, coupling to a R-R 4-form potential. It is the dual description of a string ending on the brane, considering that the string ends are charged under a U(1)

gauge potential and that the brane couples to the R-R background.
To obtain the expectation value of the Wilson loop we now use the action with a DBI and WZ term instead of the Nambu Goto action. This action is reparametrization invariant but we should find a parametrization that respects the symmetries given by the loop at the boundary.

### 6.2.2 D3-brane calculation for the infinite straight line

We use the Euclidean metric for $AdS_5 \times S^5$

$$ds^2 = \frac{L^2}{Y^2}(dY^2 + d\vec{X}^2) + L^2 d\Omega_5^2 \tag{6.58}$$

with $L^2 = \sqrt{\lambda}\alpha'$ as before. On the boundary we parametrize the Wilson loop by

$$x^\mu = (t, 0, 0, 0) \tag{6.59}$$

with the Euclidean time $t \in ]-\infty, \infty[$. A closer look at equations (6.54) and (6.57) shows that we can restrict our considerations to the AdS-part of the metric if we do not extend the brane on the $S^5$. In this case we are at a fixed point on the sphere $\theta^i(s) = \theta^i_0$ and, because we take the pullback of the metric and the four form to the world volume, terms including coordinates on the $S^5$ vanish.
The boundary of $AdS_5$ is given by $Y \to 0$ and we have to take care that the brane pinches off at the straight line and respects the symmetries given by the Wilson line on the boundary. The four dimensional world volume of the brane requires one restriction of the AdS-coordinates, and we parametrize the brane world volume by

$$(\xi^0, \dots, \xi^3) = (x^0, r, \theta, \phi), \tag{6.60}$$

and take $Y = Y(r)$. We have introduced spherical coordinates perpendicular to the Wilson line on the boundary such that $d\vec{X}^2 = (dX^0)^2 + dr^2 + r^2 d\Omega_2^2$ in equation (6.58), and $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. This ansatz preserves the same symmetries as the Wilson line extending in the $x^0$ direction on the boundary. To find an ansatz for the gauge field we consider the entire picture including the string ending on the single D3-brane that pinches off on Wilson loop at the boundary. We therefore describe a brane with a gauge field on the world volume induced by the charged string end. We take the gauge potential $A_\mu$ as

$$A_0 = A_0(r); \quad A_j = 0; \quad j = 1, 2, 3 \tag{6.61}$$

and the only non vanishing component of the field strength is therefore $F_{0r}(r)$. This choice is also consistent with the symmetries.

In equation (6.54) we have to take the determinant of the matrix

$$g + 2\pi\alpha' F = \frac{L^2}{Y^2}\begin{pmatrix} 1 & 2\pi\alpha' F_{0r}\frac{Y^2}{L^2} & 0 & 0 \\ 2\pi\alpha' F_{0r}\frac{Y^2}{L^2} & 1 + (\partial_r Y)^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{6.62}$$

where we have taken the pullback of the metric in (6.58). The DBI part of the action thus becomes

$$S_{DBI} = T_3 L^4 \int dx_1 dr\ r^2 d\Omega_2 \frac{1}{Y^4} \sqrt{1 + (\partial_r Y)^2 + (2\pi\alpha' F_{0r})^2 \frac{Y^4}{L^4}}, \tag{6.63}$$

and from our previous consideration we know that we can take the 4-form potential $C^{(4)}$ proportional to the determinant of the part of the metric that is parallel to the D3-brane: [25],[26]

$$C^{(4)}_{0r\theta\phi} dX^0 \wedge dr \wedge d\theta \wedge d\phi = \frac{L^4 r^2 \sin\theta}{Y^4} dX^0 \wedge dr \wedge d\theta \wedge d\phi. \tag{6.64}$$

Taking the pullback we obtain

$$\begin{aligned} P[C^{(4)}] &= \frac{1}{4!} \epsilon^{i_0 \ldots i_3} \partial_{i_0} X^{M_0} \ldots \partial_{i_3} X^{M_3} C^{(4)}_{M_0 \ldots M_3} d\xi^0 \wedge \ldots \wedge d\xi^3 \\ &= \frac{L^4 r^2 \sin\theta}{Y^4} dx^0 \wedge dr \wedge d\theta \wedge d\phi. \end{aligned} \tag{6.65}$$

The fact that $P[C^{(4)}] = C^{(4)}$ is due to our choice of parametrization. The full action is then calculated to be

$$\begin{aligned} S &= S_{DBI} + S_{WZ} \\ &= T_3 L^4 \int dx^0 dr\ r^2 d\Omega_2 \frac{1}{Y^4} \left( \sqrt{1 + (\partial_r Y)^2 + (2\pi\alpha' F_{0r})^2 \frac{Y^4}{L^4}} - 1 \right) \\ &= \frac{2N}{\pi} \int dx^0 dr \frac{r^2}{Y^4} \left( \sqrt{1 + (\partial_r Y)^2 + (2\pi\alpha' F_{0r})^2 \frac{Y^4}{L^4}} - 1 \right). \end{aligned} \tag{6.66}$$

The fields in the Lagrangian are independent of the angular coordinates and in the last step in equation (6.66) we have integrated over $d\Omega_2$ and used that

$$T_3 = \frac{N}{2\pi^2 \lambda (\alpha')^2} = \frac{N}{2\pi^2 L^4}. \tag{6.67}$$

To minimize the action we have to solve the equations of motion. We can use the Euler-Lagrange equations or, more conveniently, use the fact that the conjugate momentum to the gauge potential is a first integral, noting that the Lagrangian does not depend explicitly on $A_0(r)$. The conjugate momentum is

$$\Pi_A = i \frac{\delta S}{\delta \partial_r A_0} = \frac{-4iNr^2}{\lambda} \frac{2\pi F_{0r}}{\sqrt{1 + (\partial_r Y)^2 + 4\pi^2 F_{0r}^2 \frac{y^4}{\lambda}}}. \tag{6.68}$$

The factor $i$ is there because with Euclidean signature the electrical field is imaginary and we want to produce a real quantity. Having already integrated over the angular variables, $\Pi_A$ is the conserved charge corresponding to the fundamental string density [25] and thus an integer number which corresponds to the number of coincident Wilson loops.

Varying the action with respect to $Y$ we find the equation of motion which is a complicated expression in terms of first and second derivatives of $Y(r)$ and $A_0(r)$

$$Y'' = f(Y, Y', F_{0r}, F'_{0r}). \tag{6.69}$$

From (6.68) we obtain $F_{0r}$ in terms of $Y(r)$ and $Y'(r)$ and inserting this in the differential equation (6.69) we are left with a second order differential equation for $Y$.
Considering a D3-brane in flat space we know from the T-Duality considerations in section 3.2 that after the T-duality transformations the transverse coordinates in the T-dual picture can be considered as the gauge field components in the original picture. If we for example compactify the $X^9$ direction in a 10 dimensional spacetime we obtain $X'^9 = 2\pi\alpha' A^9$. For a point charge we expect $A^9 \sim \frac{1}{r}$ [27] where $r$ is the distance to the charge. The excitation of a transverse coordinate is interpreted as a fundamental string attached to the brane and stretching to infinity. The charged end on the brane induces the gauge field whereas the other end is connected to the transverse displacement. In our case the relevant transverse coordinate is $Y = Y(r)$. For $Y \to 0$ we are far away from the charge while for $Y \to \infty$ we are at the charged endpoint. This leads us to the ansatz

$$Y(r) = \frac{r}{\kappa}. \tag{6.70}$$

The equation of motion for $Y$ is then solved for

$$\kappa = \frac{\Pi_A}{\sqrt{\lambda}4N}. \tag{6.71}$$

This shows that the brane pinches off at the line $C$ on the boundary since for $Y = 0$ we have $r = 0$, leaving us with the boundary coordinate $X^\mu|_{\delta AdS_5} = x^\mu = (x^0, 0, 0, 0)$.

The electric field on the brane becomes

$$E_r(r) = F_{0r}(r) = i\frac{k\lambda}{8\pi N r^2}. \tag{6.72}$$

Using the result for $Y$ (6.70) with the induced metric we obtain

$$g_{ij}d\xi^i d\xi^j = \frac{L^2\kappa^2}{r^2}\left(dx_1^2 + (1 + \frac{1}{\kappa}^2)dr^2 + r^2 d\Omega_2^2\right) \tag{6.73}$$

which has the topological structure of $AdS_2 \times S^2$ with a constant curvature radius $L\kappa$ of the sphere. Now we can evaluate the action with the fields $Y$ and $F$ a solution to the equations of motion (on shell values) and obtain

$$S = S_{DBI} + S_{WZ} = 0 \tag{6.74}$$

as expected from the minimal area calculations.

### Regularization with boundary terms

In the area law calculations we added boundary terms to the action in order to work with the proper canonical variables and to counteract divergences. The addition of these terms corresponds to taking the Legendre transformation with respect to the dynamical variables.
The conjugate momentum $\Pi_Y$ to the transverse coordinate $Y$ is given by

$$\Pi_Y = \frac{\partial \mathrm{L}}{\partial \partial_r Y} = \frac{2N}{\pi}\frac{\partial_r Y\ r^2}{Y^4\sqrt{1 + (\partial_r Y)^2 + (2\pi\alpha' F_{0r})^2\frac{Y^4}{L^4}}}. \tag{6.75}$$

Varying our action with respect to $Y$ and using the equations of motion we obtain

$$\delta S = \int dX^0 dr \left( \frac{\partial \mathrm{L}}{\partial Y} \delta Y - \frac{\partial \mathrm{L}}{\partial \partial_r Y} \delta \partial_r Y \right) = \int dx^0 dr \partial_r \left( \frac{\partial \mathrm{L}}{\partial \partial_r Y} \delta Y \right) = \int dx_0 \Pi_Y \; \delta Y|_{r=0}^{r=\infty} \tag{6.76}$$

which shows that the action is a functional of $Y$. Adding the boundary term

$$S_Y = -\int_C dX^0 Y \Pi_Y|_{r=0}^{r=\infty} \tag{6.77}$$

the variation of the Legendre transformed action becomes

$$\delta(S + S_Y) = -\int_C dX^0 Y \delta \Pi_Y|_{r=0}^{r=\infty} \tag{6.78}$$

and it becomes evident that $S+S_Y$ is a functional of the conjugate momentum. Evaluating (6.75) on shell we obtain

$$\Pi_Y = \frac{2N\kappa^3}{\pi r^2} = \frac{2N\kappa}{\pi Y^2} \tag{6.79}$$

and inserting this into (6.77) we obtain

$$S_Y = -\int_C dX^0 \Pi_Y Y|_{r=0}^{r=\infty} = -\frac{2NX^0\kappa}{\pi Y}|_{r=0}^{r=\infty} = \frac{2NX^0\kappa}{\pi Y_0} \tag{6.80}$$

where we used $r = \kappa Y$ and we introduced a cutoff $Y_0 \to 0$ at the boundary. The action evaluated on shell is already zero. Adding the boundary term actually changes this result. We will obtain the correct result by including the boundary term corresponding to the conjugate momentum of the electrical field $\Pi_A$ given by

$$S_A = -i \int dX^0 \Pi_A A_0(r)|_{r=0}^{r=\infty} = -i \int dX^0 dr \Pi_A \partial_r A_0 = i \int_0^\infty dX^0 dr \Pi_A F_{0r}. \tag{6.81}$$

Equation (6.68) tells us that $\Pi_A$ is a first integral and we argued that it should be an integer number $k$. Using the result for $F_{0r}$ in (6.72) we obtain

$$S_A = -\frac{2NX^0\kappa}{\pi Y_0} \tag{6.82}$$

which exactly cancels the boundary term corresponding to $\Pi_Y$. Including both boundary terms we therefore obtain the expected result for the action

$$S = S_{DBI} + S_{WZ} + S_Y + S_A = 0 \tag{6.83}$$

and the expectation value for the Wilson loop $\langle W \rangle = e^{-S} = 1$ agrees with the calculations using fundamental strings as well as with the perturbative calculations on the SYM side.

**Supersymmetry**

In (3.4) we found that the 3-brane solution in type IIB supergravity preserves 16 Poincaré and another 16 conformal supersymmetries. On the SYM side the straight Wilson line preserved half of the Poincaré supersymmetries, and the same amount of supersymmetry should also be preserved by the D3-brane solution.
The type IIB background with the stack of 3-branes has 16 linearly independent solutions of the Killing spinor equation. Since our D3-brane solution has no dependence on the $S^5$ part of the metric, we only need to consider the form of the Killing spinors of $AdS_5$, which are for example given in [28].
The D3-brane solution preserves the supersymmetries generated by Killing spinors which also satisfy the equation

$$(\Gamma - 1)\epsilon = 0 \tag{6.84}$$

where $\Gamma$ is the projector associated with the D3-brane [25]

$$\Gamma = \frac{1}{\sqrt{-\det(g + 2\pi\alpha' F)}}\left(-i(Y'(r)\gamma_{ty\theta\phi} + \gamma_{tr\theta\phi}) + i2\pi\alpha' F_{tr}\gamma_{\theta\phi}K\right). \tag{6.85}$$

The matrices $\gamma_\mu$ are defined in terms of the vielbeins $e^a_\mu$ and the original $\Gamma$- matrices by the relation

$$\gamma_\mu = e^a_\mu \Gamma_a \tag{6.86}$$

and the operator $K$ acts on the spinors by complex conjugation. It can then be shown [25] that the D3-brane solution for the straight line preserves half the supersymmetries of the theory and is therefore a BPS-object. This is in perfect agreement with the straight Wilson line on the SYM side.

### 6.2.3 D3-brane solution for a circular loop

We can find a solution for the circular loop by using the result for the straight line and performing a conformal transformation. As we did in the minimal area calculations we map the line to the circle and extend the conformal transformation to an isometry of $AdS_5$ space. We start with the metric for $AdS_5$

$$ds^2 = \frac{L^2}{Y^2}\left(dY^2 + d\vec{X}^2\right) = \frac{L^2}{Y^2}\left(dY^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2 + r_2^2 d\xi^2\right) \tag{6.87}$$

where we introduced polar coordinates in the $X^0, X^1$ plane as well as in the $X^2, X^3$ plane

$$X^0 = r_1 \cos\psi, \qquad X^1 = r_1 \sin\psi, \qquad X^2 = r_2 \cos\xi, \qquad X^3 = r_2 \sin\xi. \tag{6.88}$$

We shift the line on the boundary away from the origin and parametrize it by $X^\mu = (X^0, 1/2, 0, 0)$. The conformal transformation

$$X^{\mu'} = \frac{X^\mu - \delta^{1\mu}(X^2 + Y^2)}{1 - 2X_2 + X^2 + Y^2} \tag{6.89}$$

maps the line to

$$X^\mu = \left(\frac{X^0}{(X^0)^2 + 1/4}, \frac{-(X^0)^2 + 1/4}{(X^0)^2 + 1/4}, 0, 0\right) \tag{6.90}$$

which defines a circle with unit radius . By extending the conformal transformation (6.89) to an isometry of $AdS_5$

$$X^{\mu'} = \frac{X^\mu - \delta^{1\mu}(X^2+Y^2)}{1-2X_2+X^2+Y^2}, \qquad Y' = \frac{Y}{1-2X_2+X^2+Y^2}, \tag{6.91}$$

the solution of the equations of motion in the straight line case,

$$Y(r) = r\kappa^{-1} = (\sqrt{r_1^2\sin^2\psi + r_2^2})\kappa^{-1} \tag{6.92}$$

is transformed into

$$(2\kappa)^2Y^2 = (2r_2)^2 + (Y^2 + r_1^2 + r_2^2 - 1)^2. \tag{6.93}$$

At the boundary $Y = 0$ equation (6.93) is satisfied by $r_1 = 1, r_2 = 0$ which is exactly the definition of a circle with unit radius placed in the $X^0, X^1$ plane. By doing a conformal transformation of the straight line on the boundary and extending it to the isometries of AdS we have found a solution for the for the transverse coordinate $Y(r) = Y(r_1, r_2)$.
We can now write down the action which is analogous to the straight line case. It is more convenient to work with the variables [25].

$$\begin{aligned} r_1 &= \frac{\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}, &(6.94)\\ r_2 &= \frac{\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta}, &(6.95)\\ Y &= \frac{\sin\eta}{\cosh\rho - \sinh\rho\cos\theta}. &(6.96)\end{aligned}$$

and equation (6.93) simplifies to

$$\sin\eta = \frac{1}{\kappa}\sinh\rho. \tag{6.97}$$

The range of the original coordinates was $Y \in (0,\infty)$, $r_1, r_2 \in [0,\infty)$, so we take $\theta \in [0,\pi]$, $\rho \in [0,\infty)$ and $\eta \in [0, \frac{\pi}{2}]$. The metric in terms of the coordinates $\eta, \rho, \theta, \psi, \xi$ becomes

$$ds^2 = \frac{L^2}{\sin^2\eta}\left(d\eta^2 + \cos^2\eta d\psi^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta d\xi)\right) \tag{6.98}$$

and we reach the boundary for $\eta = 0$ as well as for $\rho \to \infty$.

The location of the $D3$-brane is defined by the solution (6.97) which we found with the conformal transformation, and we parametrize it by

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (\psi, \rho, \theta, \xi). \tag{6.99}$$

Analogously to the straight line case we take the four-form potential $C^{(4)}$ as

$$C^{(4)} = \frac{L^4 r_1 r_2}{Y^4} dr_1 \wedge d\psi \wedge dr_2 \wedge d\xi$$

which in terms of the new coordinates becomes

$$\begin{aligned} C^{(4)} &= L^4\frac{\cos^2\eta\sin\theta\sinh^2\rho}{\sin^4\eta}d\rho\wedge d\psi\wedge d\theta\wedge d\xi \\ &+ L^4\frac{\cos\eta\sin\theta\sinh^2\rho(\sinh\rho-\cosh\rho\cos\theta)}{\sin^3\eta(\cosh\rho-\sinh\rho\cos\theta)}d\eta\wedge d\psi\wedge d\theta\wedge d\xi \\ &- L^4\frac{\cos\eta\sin^2\theta\sinh\rho}{\sin^3\eta(\cosh\rho-\sinh\rho\cos\theta)}d\eta\wedge d\psi\wedge d\rho\wedge d\xi. \end{aligned} \tag{6.100}$$

The three terms come from the coordinate transformation and the corresponding sum (the indices $\mu$ and $\nu$ run over $\{\eta,\rho,\theta\}$ and there is no summation over $\psi$ and $\xi$)

$$\begin{aligned} & C_{r_1\psi r_2\xi}dr_1\wedge d\psi\wedge dr_2\wedge d\xi \\ =\; & C_{r_1\psi r_2\xi}\frac{\partial r_1}{\partial x^\mu}\frac{\partial r_2}{\partial x^\nu}dx^\mu\wedge d\psi\wedge dx^\nu\wedge d\xi \end{aligned} \tag{6.101}$$

$$\tag{6.102}$$

Due to symmetry arguments we consider the gauge field $F_{\psi\rho}(\rho)$ and the action becomes

$$\begin{aligned} S &= S_{DBI}+S_{WZ} \\ &= 2N\int d\rho d\theta\frac{\sin\theta\sinh^2\rho}{\sin^4\eta}\sqrt{\cos^2\eta(1+(\partial_\rho\eta)^2)+(2\pi\alpha' F_{\psi\rho})^2\frac{\sin^4\eta}{L^4}} \\ &- 2N\int d\rho d\theta\frac{\cos\eta\sin\theta\sinh^2\rho}{\sin^4\eta}\left(\cos\eta+(\partial_\rho\eta)\sin\eta\frac{\sinh\rho-\cosh\rho\cos\theta}{\cosh\rho-\sinh\rho\cos\theta}\right), \end{aligned} \tag{6.103}$$

where we again integrated over the angular variables $\psi$ and $\xi$ and the derivative of $\eta$ in the WZ term is due to the pullback of the four form potential.
The Lagrangian density does not explicitly depend on the gauge potential $A_\psi$ and therefore its conjugate momentum

$$\begin{aligned} \Pi_A &= -i\frac{\partial \mathrm{L}}{\partial\partial_\rho A_\psi} \\ &= i\frac{2(2\pi\alpha')^2N\sin\theta\sin^2\rho F_{\psi\rho}}{L^4\sqrt{\cos^2\eta(1+(\partial_\rho\eta)^2)+(2\pi\alpha' F_{\psi\rho})^2\frac{\sin^4\eta}{L^4}}} \end{aligned} \tag{6.104}$$

is a conserved quantity. We will actually work with $\Pi_A$ integrated over $\theta$ and $\xi$, but not over $\psi$, therefore it is equal to the number of coincident loops and given by an integer number $k$. Inserting the solution we found via the conformal transformation, equation (6.97) into equation (6.104) and doing the $\xi$ and $\psi$ integration we obtain

$$F_{\psi\rho}=\frac{ik\lambda}{8\pi N\sinh^2\rho}. \tag{6.105}$$

Using this solution for the induced metric leads to

$$ds^2=\frac{L^2}{\sinh^2\eta}\left(\frac{1+\kappa^2}{1+\kappa^2\sinh^2\eta}d\eta^2+\cosh^2\eta d\psi^2\right)+L^2\kappa^2(d\theta^2+\sin^2\theta d\xi^2), \tag{6.106}$$

which is again the geometry of $AdS_2 \times S^2$, and the curvature radii of the AdS part and the sphere are $L\sqrt{1+\kappa^2}$ and $L\kappa$ respectively, meaning that the sphere never shrinks. The difference in comparison to the straight line is that the structure of the AdS part is the Poincaré disk for the circular loop while in the former case it was the upper half plane.
By inserting the solutions for $F_{\psi\rho}$ and $\eta$ into the action (6.103) we obtain

$$\begin{aligned} S_{DBI} + S_{WZ} &= 2N\kappa^2 \int d\rho d\theta \frac{\sin\theta\cos\theta}{\sinh\rho(\cosh\rho - \sinh\rho\cos\theta)} \\ &= 2N\kappa^2 \left[\coth\rho - \frac{\rho}{\sinh^2\rho}\right]_{\rho=0}^{\sinh\rho=\kappa}. \end{aligned} \tag{6.107}$$

where the upper limit is due to the solution $\sinh\rho = \kappa\sin\eta$. This is the minimized action but still not our final result, because we have to include the boundary terms.

### Including the boundary terms

Again we have to regularize the action with boundary terms. Like for the straight line we make a Legendre transformation with respect to $\eta$ and $A_\psi$.
The variation of the action with respect to $\eta$ is

$$\begin{aligned} \delta S &= \int d\rho d\theta d\psi d\xi \left(\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\eta'}\delta\eta'\right) \\ &= \int\limits_{\rho=0}^{\sinh\rho=\kappa} d\rho \int d\theta d\psi d\xi \partial_\rho \left(\frac{\partial\mathcal{L}}{\partial\eta'}\delta\eta\right) = \int d\theta d\psi d\xi \Pi_\eta \ \eta|_{\rho=0}^{\sinh\rho=\kappa} \end{aligned} \tag{6.108}$$

where the upper limit is again due to the solution of the equations of motion. To make the action a functional of $\Pi_\eta$ we add the boundary term $S_\eta = 0\int d\psi d\theta d\xi \Pi_\eta \ \eta|_{\rho=0}^{\sinh\rho=\kappa}$.
The conjugate momentum to the field $\eta$ is given by

$$\Pi_\eta = \frac{\partial \mathrm{L}}{\partial\partial_\rho\eta} = \frac{N \ \sin\theta\sinh^2\rho\cos^2\eta}{2\pi^2\sin^4\eta}\left(\partial_\rho\eta - \tan\eta\frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta}\right), \tag{6.109}$$

and this time we do not take the expression that is integrated over the $S^2$ because $\Pi_\eta$ has a non trivial $\theta$ dependence. The boundary is at $\eta \to 0$ which according to the solution (6.97) corresponds to $\rho \to 0$. We evaluate (6.109) at this solution and obtain

$$\Pi_\eta = \frac{\kappa^2 N}{2\pi^2}\frac{\sin\theta\sqrt{\kappa^2 - \sinh^2\rho}}{\sinh^2\rho}\left(\cosh\rho - \sinh\rho\frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta}\right). \tag{6.110}$$

At the boundary $\rho \to 0$ this behaves like

$$\Pi_\eta|_{\rho\to 0} = \frac{\kappa^3 N}{2\pi^2}\frac{\sin\theta}{\rho^2} \tag{6.111}$$

and the boundary term becomes

$$S_\eta = -\int d\psi d\theta d\xi \Pi_\eta \eta|_{\rho=0}^{\sinh\rho=\kappa} = -4N\frac{\kappa}{\eta_0} + \mathcal{O}(\eta_0). \tag{6.112}$$

The upper limit does not contribute and for the lower limit we have introduced a cutoff $\eta_0 = \kappa^{-1}\rho \to 0$. The divergence from this boundary term is removed by the one from the Legendre transformation with respect to $A_\psi$. It is

$$S_A = -i \int d\psi A_\psi \Pi_A|_{\rho=0}^{\sinh\rho=\kappa} = -i\ k \int d\psi\ A_\psi|_{\rho=0}^{\sinh\rho=\kappa}. \tag{6.113}$$

We can rewrite the gauge potential $A_\psi$ evaluated at the boundary in terms of the field strength tensor

$$A_\psi|_{\rho=0}^{\sinh\eta=\kappa} = \int_0^{\sinh\eta=\kappa} d\rho\partial_\rho A_\psi = -\int_0^{\sinh\eta=\kappa} d\rho\ F_{\psi\rho}. \tag{6.114}$$

Inserting $F_{\psi\rho}$ from (6.105) into this equation we obtain

$$\begin{aligned} S_A &= i\ k \int_0^{\sinh\eta=\kappa} d\psi d\rho F_{\psi\rho} = -\frac{k^2\lambda}{8\pi N}\int_0^{\sinh\eta=\kappa} d\psi d\rho \frac{1}{\sinh^2\rho} \\ &= -4N\kappa^2\left[\coth\rho\right]_0^{\sinh\rho=\kappa} \\ &= -4N\kappa\sqrt{1+\kappa^2} + 4N\kappa\frac{1}{\eta_0}, \end{aligned} \tag{6.115}$$

where we again introduced a cutoff $\eta_0 \to 0$. The divergent term exactly cancels the divergence from $S_\eta$ and we obtain the final result as a sum of the minimized action and the boundary terms

$$S = S_{DBI} + S_{WZ} + S_{\Pi_\eta} + S_{\Pi_A} = -2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right) \tag{6.116}$$

At first view our result looks different from the one we know from the minimal area calculations, $S = -\sqrt{\lambda}$. We can expand it for large $N$ and constant $\lambda$ where $\kappa = \frac{k\sqrt{\lambda}}{4N}$ is small. We obtain

$$S = -2N\left(2\kappa + \frac{\kappa^3}{3} - \frac{\kappa^5}{20} + \dots\right) = -k\sqrt{\lambda} - \frac{k^3\lambda^{3/2}}{96N^2} + \frac{k^5\lambda^{5/2}}{10240N^4} + \dots \tag{6.117}$$

This is a remarkable result, found by Drukker and Fiol in [25]. It not only reproduces the result from the fundamental string calculations $S = -\sqrt{\lambda}$ but also all higher genus corrections to the Wilson loop operator as expected from bulk calculations including string loop diagrams. This correspond to including non planar diagrams on the gauge theory side. The string loop corrections are given as an expansion in $1/N^2$ as we have seen in (6.117).

This result is actually in agreement with the one we obtained from the Matrix model calculations in Section 5.2. There we obtained the result

$$\langle W(C)\rangle = \left\langle \frac{1}{N} Tr\ \exp M \right\rangle = \frac{1}{N} L_{N-1}^1(-\lambda/4N)\exp(\lambda/8N) \tag{6.118}$$

where $L_n^m$ are the Laguerre polynomials and $C$ is the path that describes just one loop. For a loop that winds the circle $k$ times we have to consider the expectation value of $Tr \exp kM$ which correspond to $\lambda \to k^2\lambda$. From (6.71) we obtain

$$\lambda = \frac{(4N\kappa)^2}{k^2} \tag{6.119}$$

for $k$ coincident loops. Inserting this into (6.118) we obtain

$$\langle W(C)\rangle = L^1_{N-1}(-\frac{4N\kappa^2}{\exp}(2N\kappa^2). \tag{6.120}$$

The Laguerre polynomials satisfy the differential equation

$$\left(x\frac{d^2}{dx^2} + (m+1-x)\frac{d}{dx} + n\right) L^m_n(x) = 0 \tag{6.121}$$

from which we obtain the following differential equation for the Wilson loop

$$[\kappa\partial^2_\kappa + 3\partial_\kappa - 16N^2\kappa(1+\kappa^2)]\langle W(C)\rangle = 0 \tag{6.122}$$

In the D3-brane calculations we found the minimal action $S$ with boundary terms, and we obtained the expectation value as $W(C) = e^{-S}$. We therefore rewrite the expectation value obtained from the matrix model as the exponent of an effective action $S_{eff}$

$$\langle W(S)\rangle = \exp\left(-NS_{eff}(\kappa)\right) \tag{6.123}$$

Inserting this into (6.122) we obtain

$$(S'_{eff})^2 - \frac{1}{N\kappa}(\kappa S''_{eff} + 3S'_{eff}) = 16(1+\kappa^2) \tag{6.124}$$

For $\lambda \gg 1$ the factor $\frac{1}{N\kappa} = \frac{4}{k\sqrt{\lambda}} \to 0$ and (6.124) becomes

$$\frac{dS_{eff}}{d\kappa} = \pm 4\sqrt{1+\kappa^2} \tag{6.125}$$

which has the solution

$$S_{eff} = \pm 2\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right) + \text{const.} \tag{6.126}$$

By setting the constant to zero and choosing the negative sign we obtain perfect agreement with the D3-brane calculation by identifying $S = NS_{eff}$.

# Chapter 7

# Polyakov-Maldacena loop at finite temperature

The Phase structure of $\mathcal{N}=4$ SYM is well known at weak and at strong coupling, but the interpolation between the two regimes remains to be determined. In [32] it was argued that for all values of the coupling the low energy and high energy regimes are separated by at least one phase transition. The order parameter for these transitions is the eigenvalue distribution of the Polyakov loop, which is the Wilson loop winding the compactified Euclidean time direction

$$U = P\exp\left(i\oint A_0 d\tau\right). \tag{7.1}$$

The eigenvalue distribution is calculated by taking the trace of some powers of this operator

$$\langle TrP\exp\left(ik\oint A_0 d\tau\right)\rangle \qquad k=1,2,\ldots \tag{7.2}$$

which is equivalent to calculating the expectation value of the trace of a multiply wound Polyakov loop.
In the AdS/CFT correspondence we cannot access the Polyakov loop since we have to consider the coupling of the $S^5$ coordinates to the six scalar fields $\Phi_i$. The operator of interest is therefore the Polyakov-Maldacena loop, which is the Wilson loop winding the compact Euclidean time direction, and a parametrization is given by $x^\mu=(\tau,0,0,0)$. The eigenvalue distribution is obtained from the expectation value

$$\langle U\rangle = \langle TrP\exp\left(k\oint\left(iA_0+\Phi_i\theta^i\right)d\tau\right)\rangle, \tag{7.3}$$

and we can calculate it at strong coupling in type IIB supergravity, using either a macroscopic string ending on the circle at the boundary or a D-brane ansatz, as we did in section 6.2. Since we are considering a Polyakov-Maldacena loop winding the compact time direction $k$-times, it is more convenient to use a D-brane ansatz.
The dual geometry for the finite temperature case is the Schwarzschild AdS- blackhole times an $S^5$, which is obtained as a non extremal D3-brane solution of type IIB supergravity in the near horizon limit. We found this solution in section 3.4 and the metric

with Euclidean signature was given by

$$ds^2 = L^2 \left( u^2 \left( f(u)dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \frac{du^2}{f(u)u^2} + d\Omega_5^2 \right),$$
$$L^2 = \sqrt{\lambda}\alpha', \qquad f(u) = 1 - \frac{u_0^4}{u^4}, \qquad u_0 = \pi T, \tag{7.4}$$

with periodically identified time

$$t \sim t + \frac{1}{T}. \tag{7.5}$$

In [32] it was argued that there is no D3-brane solution that describes the eigenvalue distribution for the Polyakov-Maldacena loop. They used arguments based on numerical analysis and they did the calculations in global AdS coordinates. In principle there is no physical argument that contradicts a D3-brane solution and in the following we make a first approach towards a D3-brane solution in Poincaré coordinates.

## 7.1 Action and Equations of Motion

Analogous to equation (6.54) the action is given by a DBI and WZ term

$$S = T_3 \int d^4\xi \sqrt{\det(g + 2\pi\alpha' F)} - T_3 \int P[C_4], \tag{7.6}$$

with $g = g_{ij}$ and $P[C_4]$ the pullback of the metric and the 4-form potential to the world-volume and $F = F_{ij}$ the gauge field living on the brane. We write the metric in (7.4) as

$$ds^2 = \frac{L^2}{y^2} \left( f(y)dt^2 + dr^2 + r^2 d\Omega_2^2 + \frac{dy^2}{f(y)} \right),$$
$$f(y) = 1 - \frac{y^4}{y_0^4}, \qquad y_0 = \frac{1}{u_0}, \tag{7.7}$$

where we have substituted $y = 1/u$ and introduced spherical coordinates for the directions transverse to the time direction. We also ignored the $S^5$ part of the metric, since we do not extend the brane on the $S^5$ and the pullback of the metric and the 4-form potential ensures that all dependence of the $S^5$ coordinates in (7.6)vanishes.
The Polyakov-Maldacena loop is wrapping the compact time direction, and our parametrization is therefore analogous to the on for the straight line in section 6.2,

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (t, r, \theta, \phi), \qquad y = y(r). \tag{7.8}$$

Using the metric (7.7) and the ansatz for the field strength $F = F_{tr}(r)$ the action (7.6) becomes

$$S = \frac{2N}{\pi} \int dtdr \frac{r^2}{y^4(r)} \left( \sqrt{1 + (y'(r))^2 + (\frac{4\pi^2}{\lambda} F_{tr}^2(r) - \epsilon) y^4(r)} - 1 \right) \tag{7.9}$$

where we performed the integration over $\theta$ and $\phi$ and introduced $\epsilon = 1/y_0^4 = \pi^4 T^4$, which is the quantity containing the temperature dependence.

The Lagrangian does not explicitly depend on the gauge potential $A_t(r)$, therefore its conjugate momentum

$$\Pi_A \equiv \frac{\partial \mathcal{L}}{\partial \partial_r A_t} = -\frac{\partial \mathcal{L}}{\partial F_{tr}} = -\frac{8\pi N}{\lambda} \frac{r^2 F_{tr}(r)}{\sqrt{1 + (y'(r))^2 + (\frac{4\pi^2}{\lambda} F_{tr}^2(r) - \epsilon) y^4(r)}} \tag{7.10}$$

is a constant of motion. We use Euclidean signature and the gauge field is therefore imaginary, hence the quantity $i\Pi$ is real and we identify it with the winding number of the Polyakov-Maldacena loop around the compact time dimension

$$i\Pi_A = k, \qquad k = 1, 2, \ldots \tag{7.11}$$

Using this, equation (7.10) can be solved for $F_{tr}(r)$

$$F_{rt}(r) = i\frac{\sqrt{\lambda}\kappa}{2\pi} \sqrt{\frac{1 - (y'(r))^2 - \epsilon y^4(r)}{r^4 + \kappa^2 y^4(r)}} \tag{7.12}$$

where we used $\kappa = \frac{k\sqrt{\lambda}}{4N}$ which was already defined in equation (6.71).
The equations of motion for $y(r)$ are obtained from the Euler-Lagrange equations and we obtain a second order differential equation of the form

$$y''(r) = f(y'(r), y(r), F'_{tr}(r), F_{tr}(r), r). \tag{7.13}$$

Using (7.12) we can eliminate the dependence on the gauge field. Since we could not solve this equation analytically we will approach this problem with a perturbative calculation around the zero temperature solution. From the straight line case in section 6.2 we know the solution for $y(r)$ for $T = 0$

$$y(r)|_{T=0} = \frac{r}{\kappa}. \tag{7.14}$$

To make a perturbative expansion around the zero-temperature solution we rewrite the field $y(r)$ as

$$y(r) = \frac{r}{\kappa} + \epsilon y^1(r) + \mathcal{O}(\epsilon^2). \tag{7.15}$$

Inserting this into the Euler-Lagrange equations of motion and doing a Taylor expansion in $\epsilon$ we can compare the terms order by order, and from the first order term we obtain the following differential equation for $y^1(r)$

$$\frac{d^2}{dr^2} y^1(r) = \frac{2}{r^2} \left( \frac{-1024 N^5 r^5}{n^5 \lambda^{5/2}} - r(y^1(r))' - y^1(r) \right). \tag{7.16}$$

The general solution is

$$y^1(r) = -\frac{r^5}{6\kappa^5} + C_1 r^2 + C_2 r \tag{7.17}$$

with some constants $C_1, C_2$. The boundary condition is that the brane pinches off at the loop on the boundary, and the parametrization for the loop was given by $x^\mu(\tau) = (\tau, r = 0, 0, 0)$. The boundary is at $y = 0$, and $y(r) = r/\kappa + \epsilon y^1(r) = 0$ clearly has the solution $r = 0$ . But for non vanishing constants $C_1, C_2$ the equation $y(r) = 0$ has further solutions

with $r \neq 0$, and therefore they should be set to zero.
To first order in $\epsilon$ the solution for $y(r)$ is then given by

$$y(r) = \frac{r}{\kappa} - \epsilon \frac{r^5}{6\kappa^5}. \tag{7.18}$$

Inserting this equation into the Lagrangian we obtain the on shell value

$$\mathcal{L} = -\epsilon \frac{Nr^2}{\pi} + \mathcal{O}(\epsilon^2). \tag{7.19}$$

To obtain the value of the action we have to integrate the Lagrangian over $t$ and $r$. For a $k$-times wound Polyakov-Maldacena loop the k-dependence is automatically captured by the D-brane solution via the conjugate momentum for the gauge field and therefore we take $t \in [0, 1/T]$. The variable $r$ ranges in principle from zero to infinity. But this is only valid in the zero temperature limit, in our case we have to ensure that the maximum value for $y(r)$ is given at the horizon $y_{max} = y_0$. Using (7.18) we obtain the value for $r$ at the horizon

$$r_0 = a y_0 \kappa, \tag{7.20}$$

where $a$ satisfies the equation $a - a^5 = 1$. This can only be satisfied for a negative value of $a$, and therefore $r_0$ would be negative, too. This means that we can not find a value of $a$, such that the brane reaches the horizon. We therefore have to find the value of $r$ for which $y(r)$ assumes its maximal value. Using the first and second derivative of (7.18) we find

$$r_{max} = (\frac{5}{6})^{1/4} \kappa y_0. \tag{7.21}$$

Using this and (7.19) in the action (7.9) we obtain

$$S = -\frac{N}{\pi} \int_0^{1/T} dt \int_0^{(\frac{5}{6})^{1/4} y_0 \kappa} dr \, \epsilon r^2 = -\frac{(\frac{5}{6})^{3/4} \lambda^{3/2} k^3}{192 N^2} \tag{7.22}$$

## 7.2 Boundary Terms

As discussed in section 6.2 the minimal action is not the final result, we rather have to take the Legendre transformation with respect to the dynamical fields in the action. This was obtained by adding boundary terms. The boundary term corresponding to the field $y(r)$ is given by

$$S_y = -\int dt \Pi_y y(r)|_{r=0}^{r=r_{max}} \tag{7.23}$$

with $\Pi_y = \frac{\partial \mathcal{L}}{\partial y'(r)}$ the conjugate momentum to the field $y(r)$. Using the equations of motion and Taylor expanding in $\epsilon$ equation (7.23) becomes

$$S_y = -\int dt \left( \frac{n^2 \lambda}{8\pi N r} + \epsilon \frac{16 N^3 r^3}{3\pi n^2 \lambda} + \mathcal{O}(\epsilon^2) \right) |_{r=0}^{r=r_{max}}. \tag{7.24}$$

The boundary term corresponding to the gauge potential is

$$S_A = -\int dt \Pi_A A_t(r)|_{r=0}^{r=r_{max}} = \int dt dr \Pi_A F_{tr}|_{r=0}^{r=r_{max}}. \tag{7.25}$$

with $\Pi_A$ given in (7.10) and $F_{tr}$ in (7.12). Evaluated on shell we obtain to first order in $\epsilon$

$$S_A = \int dtdr \left( -\frac{n^2\lambda}{8\pi N r^2} + \epsilon \frac{16N^3 r^2}{\pi n^2 \lambda} + \mathcal{O}(\epsilon^2) \right). \tag{7.26}$$

Evaluating the $r$ integral we see that the boundary term corresponding to the gauge field and the boundary term corresponding to $y(r)$ are identical up to a minus sign, and the sum of both therefore vanishes.
The final result is then the one we obtain from equation (7.22),

$$S = -\frac{(\frac{5}{6})^{3/4}\lambda^{3/2}k^3}{192N^2} + \mathcal{O}(\epsilon^2) = -N\frac{(\frac{5}{6})^{3/4}\kappa^3}{3} \tag{7.27}$$

which is independent of the temperature.

## 7.3 Discussion

We have found the correction term to the expectation value of a zero temperature Wilson loop wrapping the compact time direction to first order in $\epsilon = \pi^4 T^4$. We solved the equation of motion for the first order correction to the transverse coordinate $y(r)$, which characterizes the embedding of the D3-brane into $AdS_5$. In [32] was argued, that the only D3-brane solution for the Polyakov-Maldacena loop that reaches the horizon is the collapsed solution, which is the solution one finds using a fundamental string (the D-brane worldvolume is collapsed to a world-sheet). This is supported by our calculations, since our (expanded) D3-brane does not reach the horizon. On the other hand, our solution is temperature independent and hence not reliable. A possible explanation is that the expansion parameter $\epsilon$ is contained in the upper integration limit for the $r$-integral, and this should be considered in a consistent expansion.
A general solution for a Wilson loop at zero temperature should be of the form [25](compare equation (6.117))

$$S = Nf(\kappa) \tag{7.28}$$

with $f$ an arbitrary function depending only on $\kappa$. At finite temperature (7.28) is modified as

$$S = \frac{N}{T} f(\kappa, y_0) \tag{7.29}$$

capturing the dependence on the temperature and the horizon. Our solution (7.22) shows the behavior of (7.28), and has no dependence on the temperature. This might be due to the fact that we used the Poincaré AdS-coordinates, which only cover half of the AdS space.
Summarizing we can say that our first order result for $y(r)$ shows a behavior that supports the argumentation in [32]. Nevertheless this does not rule out the possibility for a D3-brane solution for the Polyakov-Maldacena loop. To obtain an understanding of the temperature dependence and to investigate the behavior of the transverse coordinate at the horizon one could do a similar perturbative computation in global AdS. If in this way one could show that the transverse coordinate actually reaches the horizon, it would be most likely that there is also a D3-brane solution for the Polyakov-Maldacena loop.

# Chapter 8

# Summary and Outlook

In this master's thesis we investigated Wilson loops in the framework of the AdS/CFT correspondence and in particular the description in terms of D3-branes on the string theory side. To obtain a better understanding of the field we reviewed some aspects of $\mathcal{N} = 4$ SYM and string theory. We also considered the geometry of AdS space and showed how to relate its boundary to Minkowski space.

## Summary

We used perturbation theory to calculate expectation values of Wilson loops in $\mathcal{N} = 4$ SYM. All calculations were done in the 't Hooft limit, where we take the limit $N \to \infty$ and keep $\lambda$ fixed. In this limit, only planar diagrams contribute, and we studied the case of a straight infinite line, a circular and a rectangular loop, where from the latter we obtained the quark anti-quark potential.
On the string theory side we calculated expectation values for these geometries in the large $N$ and large $\lambda$ limit using minimal surfaces defined by a string ending on the Wilson loop at the boundary.
In the case of a straight Wilson line and a circular Wilson loop we computed the expectation values by using a D3-brane coupling to the background R-R potential. This setup is described by an action consisting of a DBI and WZ -term. Using D-branes to calculate the expectation values of Wilson loops is in particular useful if we are dealing with a multiply wound loop or many coincident loops, since the solution naturally includes the correct scaling behavior for a given number of coincident Wilson loops.

For the straight line in SYM we calculated planar diagrams to second order in $\lambda$ and found the result

$$\langle W(C) \rangle = 1. \tag{8.1}$$

To this order, the diagrams with one internal vertex as well as one loop corrections to the free propagators vanish. The straight line is a BPS-object and the residual supersymmetry protects it from quantum corrections. Therefore, this result holds to all orders in perturbation theory. Because $\langle W(C) \rangle = 1$ is independent of $\lambda$ the extrapolation to strong coupling is trivial. We find perfect agreement with the calculations on the string theory side, where we obtained this result with both, the minimum area law and the D-brane calculations.

The expectation value of a circular Wilson loop in SYM can be found by using perturbation theory, and summing all planar diagrams without internal vertices gives the result

$$\langle W(C)\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \tag{8.2}$$

where $I_1$ is the Bessel function. In four dimensions, contributions from diagrams with internal vertices cancel exactly to second order, and it was predicted that this result holds to all orders in perturbation theory. This prediction is supported by the fact that the Wilson loop operator commutes with eight linear combinations of conformal and Poincaré supercharges.
Another method to calculate the expectation value for a circular loop is to transform the straight line to the circle with a conformal transformation. The expectation value of the Wilson loop is then completely determined by the anomaly which we calculated to all orders in perturbation theory. We then obtain the exact result wich agrees with (8.2).
The fact that the result obtains a contribution associated to just a single point, which is the point where the surface term coming from the conformal transformation diverges, leads to the suggestion to calculate the expectation value with a matrix model. The result is then exact to all orders in $\lambda$ and in addition to all orders in $\frac{1}{N^2}$. It is

$$\langle W(C)\rangle = \frac{1}{N} L^1_{N-1}(-\lambda/4N)\exp(\lambda/8N) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} + \ldots \tag{8.3}$$

where $L^n_m$ are the generalized Laguerre polynomials and we see that it reproduces (8.2) in the limit $N \to \infty$. In the large $N$ and large $\lambda$ limit (8.3) becomes

$$\langle W(C)\rangle = \frac{\exp\sqrt{\lambda}}{(\pi/2)^{1/2}(\lambda)^{3/4}} \tag{8.4}$$

which shows the same exponential behavior that we obtained from the minimal area calculations.
The result we found with a D3-brane calculation for the circular Wilson loop is remarkable because it not only shows the exponential behavior (8.4) in the large $\lambda$ limit but it also captures all corrections in $\lambda$ and $\frac{1}{N^2}$ expansions. It is in perfect agreement with the result from the matrix model calculations, equation (8.3).

For the case of two anti-parallel Wilson lines in $\mathcal{N} = 4$ SYM we can sum all diagrams without internal vertices leading to the result

$$\langle W(C)\rangle = \exp\left[\left(\frac{\sqrt{\lambda}}{\pi} - 1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})\right)\frac{T}{L}\right] \tag{8.5}$$

where we extrapolated to the large $\lambda$ limit. Although quantum corrections cancel to second order in $\lambda$, the calculations on the SYM side lead to a different result than the one we obtained using the minimal area law, which is

$$\langle W(C)\rangle = \exp(-TE(L)) = \exp(\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)}\frac{T}{L}). \tag{8.6}$$

This is actually not surprising. Since the anti-parallel lines are not a BPS object, we do not expect that the sum of all planar diagrams without internal vertices reflect the correct large coupling behavior.
It is worth mentioning that up to now there has not been found a solution for the two anti-parallel lines in terms of D-branes.

In chapter 7 we made a first approach to calculate the eigenvalue distribution for the Polyakov-Maldacena loop using a D3-brane. We found a perturbative correction to the zero temperature Wilson-loop wrapping the compact time direction. The value of the minimized action we obtained to first order is independent of the temperature, and hence not reliable. But we found that the D3-brane solution does not reach the horizon, a result that agrees with the behavior that was found in [32].

# Outlook

Due to the successful descriptions of Wilson loops in terms of D-branes there are many different areas people are applying this technique. Some examples are descriptions of Wilson loops in terms of D5-branes [29] corresponding to Wilson loops in antisymmetric representations, investigations of correlation functions of Wilson loops [30] and Wilson loop operators with insertions [31].
Most of the investigations were done at zero temperature. For the finite temperature case a remaining problem is the interpolation for the phase structure of $\mathcal{N} = 4$ SYM between weak and strong coupling. The corresponding order parameter is the eigenvalue distribution of the Polyakov-Maldacena loop and we made a first approach with a perturbative D3-brane solution. In [32] they argued that there is no D3-brane solution that reaches the horizon. They further argued that the correct description for the Polyakov-Maldacena loop should be in terms of a D5-brane. Although they found order $N^{-2/3}$ corrections to the result one obtains if one just takes $k$-times the result for a fundamental string, concrete results concerning the phase structure of finite temperature $\mathcal{N} = 4$ SYM were not found.
To obtain a better understanding of this matter one could first investigate the perturbative behavior of a D3-brane in global AdS. Another possibility is to make a perturbative expansion around the zero-temperatur solution for a D5-brane. The hope is to obtain further insights into the behavior of the Polyakov-Maldacena loop, and in not too far future be able to exactly describe the phase structure of $\mathcal{N} = 4$ SYM for all values of the 't Hooft coupling.

# Acknowledgments

I would like to thank Niels Obers for giving me an introduction to this exciting field and for the many productive and inspiring discussions that greatly helped me writing this thesis. I am also very grateful to Nadav Drukker and Shinji Hirano for many useful ideas and constructive advises. Last but not least I would like to thank Mads Toudal Frandsen, Rafal Mostowy and Mikkel Bøje Nielsen for many enlightening conversations helping to improve this project.

# Appendix A

# Differential Forms and Duality

## A.1 Differential Forms

In superstring theory and its low energy limit supergravity we find tensors of different ranks and it is often convenient to use differential form notation to makes the presentation clearer by avoiding to use all the indices. Let us start with the definition of a $p$-form $F_p$. It is associated with an antisymmetric tensor $F_{\mu_1 \dots \mu_p}$ and given by the volume element

$$F_p \equiv \frac{1}{p!} F_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{A.1}$$

It is important to note that the sum over the indices $\mu_i$ reduces a lot due to anti-symmetry of both, the tensor $F$ and the wedge product $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$.
The wedge product of a $p$ and a $q$-form is defined as the anti-symmetrized product

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} \equiv A_{[\mu_1 \dots \mu_p} B_{\nu_1 \dots \nu_q]}. \tag{A.2}$$

The exterior derivative of a $p$-form is a $(p+1)$-form defined as

$$dF_p \equiv \partial_{[\mu} F_{\mu_1 \dots \mu_p]} dx^{\mu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \tag{A.3}$$

and the Hodge dual in D-dimensional spacetime is an $(D-p)$-form, defined as

$$\tilde{F}_{D-p} = *F_p \equiv \frac{1}{n!} \sqrt{g} \epsilon_{\mu_1 \dots \mu_{(D-p)} \nu_1 \dots \nu_p} F^{\nu_1 \dots \nu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{(D-p)}} \tag{A.4}$$

with $s = \pm 1$ for the Euclidean or Minkowski signature respectively and $g = \det g_{\mu\nu}$. In (A.4) we used the tensorial quantity

$$\omega_{\rho_1 \dots \rho_D} \equiv \sqrt{g} \epsilon_{\rho_1 \dots \rho_D}, \tag{A.5}$$

hence we can rise and lower indices with the metric as usual

$$\omega^{\mu_1 \dots \mu_D} = g^{\mu_1 \rho_1} \ldots g^{\mu_D \rho_D} \omega_{\rho_1 \dots \rho_D} = \frac{1}{\sqrt{g}} \epsilon^{\mu_1 \dots \mu_D}, \tag{A.6}$$

where we have used a property for the determinant of a general matrix, $M = \det M_{\mu\nu}$

$$M \epsilon_{\mu_1 \dots \mu_D} = \epsilon^{\rho_1 \dots \rho_D} M_{\mu_1 \rho_1} \ldots M_{\mu_D \rho_D} \tag{A.7}$$

## A.2 Electric and Magnetic Solitons

In section 3.4 we consider $p$-brane solutions for the type II supergravity action. Apart from electrically charged branes, we also have to consider magnetic solutions. Let us first consider standard Maxwell electrodynamics. An electrically charged point like particle couples to the gauge field via

$$\int d^4x\ J_E^\mu(x)A_\mu(x) = \int d\tau A_\mu(x(\tau))\dot{x}^\mu(\tau) = \int_C A_\mu dx^\mu = \int_C A \tag{A.8}$$

where in the last step we have used differential form notation and we consider the particle to move along a curve $C$. The point like particle corresponds to a delta function source term

$$d * F = J_E^\mu \sim \delta(C) \tag{A.9}$$

and its 1-dimensional "world volume" couples naturally to a one form $A_\mu dx^\mu$. By using the definitions (A.3) and (A.4) we can write the covariant Maxwell equations in differential form notation as

$$d * F = J \qquad \leftrightarrow \qquad \partial_{[\alpha}\left(\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu]\rho\sigma}F^{\rho\sigma}\right) = J_{\alpha\mu\nu} \tag{A.10}$$

$$*d * F = *J \qquad \leftrightarrow \qquad \nabla^\mu F_{\mu\nu} = J_\nu, \tag{A.11}$$

with $\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu}V^\nu$ the covariant derivative, and we note that the dual field strength $\tilde{F}_{\mu\nu} = 1/2\sqrt{g}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ corresponds to the Hodge dual of the Field strength tensor. Using Gauss law, we obtain the electric charge $Q_E$ by

$$Q_E = \int_V d^3x\rho_E = \int_{S=\delta V}\vec{E}d\vec{S} = \int_{S=\delta V}\tilde{F}_{\mu\nu}dx^\mu dx^\nu = \int_{S=\delta V} *F = \int_V d * F \tag{A.12}$$

with $V$ the volume we consider and $S$ its boundary, and in the last step we have used the Maxwell equations (A.10) and Stokes theorem for differential forms

$$\int_{\mathcal{M}} dA_p = \int_{\partial\mathcal{M}} A_p, \tag{A.13}$$

where A is an arbitrary $p$-form and M a $(p+1)$-dimensional manifold. If we now consider electric-magnetic duality in the Maxwell equations, corresponding to $F \to *F$, the dual magnetic soliton carries a magnetic charge $Q_M$,

$$Q_M = \int_{\partial V} F \tag{A.14}$$

It is a topological quantum number, which is associated to the Bianchi identity,

$$dF = 0. \tag{A.15}$$

It is important to note that $Q_M$ is not a Noether charge, which would arise due to the invariance of the action under a field transformation. Having a look at the definition of the hodge dual of the field strength, equation (A.4), we see, that our magnetic dual field strength is again a 2-form. So the soliton should couple to a 1-form potential, hence we

expect it to be a point like particle.
We can now easily generalize the whole discussion to arbitrary dimensions and theories. We are in particular interested in $p$- brane solutions of type II supergravity, where we consider various R-R field strengths $F_{p+2}$ and their magnetic duals. We can thus generalize our Maxwell theory as follows. The field strengths are given by the R-R fields

$$F_{p+2} = dC_{p+1} \tag{A.16}$$

with $C_{p+1}$ being the $(p+1)$-form potentials coupling to objects with $(p+1)$-dimensional world volume, which turn out to be the $p$-branes. The corresponding term in the action is

$$\int_V C_{p+1}. \tag{A.17}$$

and the $p$ branes carry an "electric" R-R charge

$$Q_E = \int_{\partial V} *F_{p+2}. \tag{A.18}$$

Their magnetic duals couple to the $(D-p-3)$-form potential $\tilde{C}_{D-p-3}$ associated with the Hodge dual of the field strength as

$$*F_{p+2} = \tilde{F}_{D-p-2} = d\tilde{C}_{D-p-3} \tag{A.19}$$

and carry a magnetic charge

$$Q_M = \int_{\partial V} F_{p+2}. \tag{A.20}$$

An important case is the 5-form $F_5$ from type IIB supergravity in 10-dimensional space-time. The associated potential is $C_4$ which couples to a 3-brane. The dual magnetic field strength $*F_5 = \tilde{F}_{10-5}$ is a 5-form as well and thus the dual magnetic object to the 3-brane is again a 3-brane. In this case the theory is self dual.
It is important to notice that there is actually nothing that tells us to prefer a R-R $p$-form from its $(10-p)$ dual, so in general we could also take the dual field strength for the electric solution. The duality relation (A.4) tells us how to describe a $(D-p)$-form in terms of its $p$-dimensional dual. We can thus restrict ourselves to the cases $p = 0, \ldots, 5$ and the electric solutions have a description in terms of D-branes.

# Bibliography


[1] G. 't Hooft, "A planar diagram theory for strong interactions," *Nucl. Phys.* **B72** (1974) 461.

[2] J. M. Maldacena, "Lectures on AdS/CFT," `hep-th/0309246`.

[3] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`.

[4] J. M. Figueroa-O'Farrill, "BUSSTEPP lectures on supersymmetry," `hep-th/0109172`.

[5] P. Di Vecchia, "Duality in N = 2,4 supersymmetric gauge theories," `hep-th/9803026`.

[6] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," `hep-th/0201253`.

[7] C. V. Johnson, "D-branes,". Cambridge, USA: Univ. Pr. (2003) 548 p.

[8] P. Di Vecchia and A. Liccardo, "Gauge theories from D branes," `hep-th/0307104`.

[9] R. J. Szabo, "BUSSTEPP lectures on string theory: An introduction to string theory and D-brane dynamics," `hep-th/0207142`.

[10] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,". Cambridge, UK: Univ. Pr. (1998) 402 p.

[11] P. Di Vecchia, "Large N gauge theories and ADS/CFT correspondence," `hep-th/9908148`.

[12] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `hep-th/9905111`.

[13] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," *Int. J. Mod. Phys.* **A14** (1999) 3597–3672, `hep-th/9902131`.

[14] M. Blau, "Supergravity solitons," *www.unine.ch/phys/string/lecturesSUGRA2.ps.gz* (2002).

[15] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[16] N. Drukker, D. J. Gross, and H. Ooguri, "Wilson loops and minimal surfaces," *Phys. Rev.* **D60** (1999) 125006, `hep-th/9904191`.

[17] K. Zarembo, "Supersymmetric Wilson loops," *Nucl. Phys.* **B643** (2002) 157–171, `hep-th/0205160`.

[18] P. Ramond, "FIELD THEORY: A modern primer," *Front. Phys.* **74** (1989) 1–329.

[19] J. K. Erickson, G. W. Semenoff, and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B582** (2000) 155–175, `hep-th/0003055`.

[20] N. Drukker and D. J. Gross, "An exact prediction of N = 4 SUSYM theory for string theory," *J. Math. Phys.* **42** (2001) 2896–2914, `hep-th/0010274`.

[21] M. Bianchi, M. B. Green, and S. Kovacs, "Instanton corrections to circular Wilson loops in N = 4 supersymmetric Yang-Mills," *JHEP* **04** (2002) 040, `hep-th/0202003`.

[22] G. W. Semenoff and K. Zarembo, "Wilson loops in SYM theory: From weak to strong coupling," *Nucl. Phys. Proc. Suppl.* **108** (2002) 106–112, `hep-th/0202156`.

[23] J. M. Maldacena, "Wilson loops in large N field theories," *Phys. Rev. Lett.* **80** (1998) 4859–4862, `hep-th/9803002`.

[24] E. Bergshoeff, R. Kallosh, T. Ortin, and G. Papadopoulos, "Kappa-symmetry, supersymmetry and intersecting branes," *Nucl. Phys.* **B502** (1997) 149–169, `hep-th/9705040`.

[25] N. Drukker and B. Fiol, "All-genus calculation of Wilson loops using D-branes," *JHEP* **02** (2005) 010, `hep-th/0501109`.

[26] M. T. Grisaru, R. C. Myers, and O. Tafjord, "SUSY and Goliath," *JHEP* **08** (2000) 040, `hep-th/0008015`.

[27] J. Callan, Curtis G. and J. M. Maldacena, "Brane dynamics from the Born-Infeld action," *Nucl. Phys.* **B513** (1998) 198–212, `hep-th/9708147`.

[28] K. Skenderis and M. Taylor, "Branes in ads and pp-wave spacetimes," *JHEP* **06** (2002) 025, `hep-th/0204054`.

[29] S. Yamaguchi, "Wilson loops of anti-symmetric representation and D5- branes," *JHEP* **05** (2006) 037, `hep-th/0603208`.

[30] T.-S. Tai and S. Yamaguchi, "Correlator of fundamental and anti-symmetric Wilson loops in ads/cft correspondence," *JHEP* **02** (2007) 035, `hep-th/0610275`.

[31] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, "On the D3-brane description of some 1/4 BPS Wilson loops," `hep-th/0612168`.

[32] S. A. Hartnoll and S. Prem Kumar, "Multiply wound Polyakov loops at strong coupling," *Phys. Rev.* **D74** (2006) 026001, `hep-th/0603190`.

# List of Figures