Phases of Thermal $\mathcal{N} = 2$ Quiver Gauge Theories

Cand. scient. thesis

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Abstract

In this master thesis we consider large N $U(N)^M$ thermal $\mathcal{N}=2$ quiver gauge theories on $S^1 \times S^3$. We obtain a phase diagram of the theory with R-symmetry chemical potentials, separating a low-temperature/high-chemical potential region from a high-temperature/low-chemical potential region. In close analogy with the $\mathcal{N}=4$ SYM case, the free energy is of order $\mathcal{O}(1)$ in the low-temperature region and of order $\mathcal{O}(N^2M)$ in the high-temperature phase. We conclude that the $\mathcal{N}=2$ theory undergoes a first order Hagedorn phase transition at the curve in the phase diagram separating these two regions. We observe that in the region of zero temperature and critical chemical potential the Hilbert space of gauge invariant operators truncates to smaller subsectors. We compute a 1-loop effective potential with non-zero VEV's for the scalar fields in a sector where the VEV's are homogeneous and mutually commuting. At low temperatures the eigenvalues of these VEV's are distributed uniformly over an $S^5/\mathbb{Z}_M$ which we interpret as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual geometry $AdS_5 \times S^5/\mathbb{Z}_M$. Above the Hagedorn transition the eigenvalue distribution of the Polyakov loop opens a gap, resulting in the collapse of the joint eigenvalue distribution from $S^5/\mathbb{Z}_M \times S^1$ into $S^6/\mathbb{Z}_M$. We finally give a detailed computation of the 1-loop anomalous dimension matrix of the $SO(6)$ sector of single-trace scalar operators of $\mathcal{N}=4$ $SU(N)$ SYM theory in the planar limit.

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Introduction

The phase structure of large N $U(N)$ gauge theories at finite temperature is in itself a very rich and interesting subject that may provide qualitative insight into the phase structure of QCD. Even more so, the AdS/CFT correspondence ([1], [2], [3]) has provided a general framework for translating results obtained in weakly coupled thermal gauge theory into results about the finite temperature behavior of the physics of black holes and stringy geometry at strong coupling. One such connection was suggested by Witten [4] who argued that the Hawking-Page phase transition [5] between thermal AdS_5 and the large AdS_5 Schwarzschild black hole should have a holographic dual description as a confinement/deconfinement transition in the dual thermal field theory defined on the conformal boundary $S^1 \times S^3$ of thermal AdS_5.

A general framework for studying large N $U(N)$ gauge theories on S^3 at finite temperature was given in [6] and, independently, in [7]. Despite the fact that these theories are defined in a finite volume, they can still have phase transitions, since the $N \to \infty$ limit acts as a thermodynamic limit. Indeed it was found that the theories exhibit phase transitions even in the limit of zero gauge coupling. The origin of this is the Gauss law constraint that all states must be gauge invariant. The order parameters of the phase transition are the expectation value of the Polyakov loop[1] and the scaling of the free energy F with N. In the low-temperature phase, the Polyakov loop has zero expectation value (thus, the $\mathbb{Z}_N$ center symmetry is unbroken), and the free energy is of order 1 with respect to N. The N dependence of the free energy naturally suggests that the physical system in this phase is a non-interacting gas of color singlet states, and the phase is therefore called "confining" (consistent with the unbroken center symmetry). When the temperature is raised above a critical temperature T_H the theory enters a new phase in which the expectation value of the Polyakov loop is non-zero (so that center symmetry is spontaneously broken), and the free energy scales as N^2 as $N \to \infty$. Here the N dependence of the free energy suggests that the gauge theory in this phase describes a non-interacting plasma of color non-singlet states, and the phase is accordingly called "deconfined" (consistent with the spontaneously broken center symmetry). This phase transition is of first order[2] and is identified with a

[1] We use the term Polyakov loop loosely since we are actually referring to the holonomy matrix of the gauge field around the thermal circle and not just its trace.

[2] As pointed out in [6], the order of the phase transition may change for non-zero gauge coupling. However, settling this issue requires a 3-loop computation since the leading (1-loop) result for the phase

Hagedorn phase transition.

Specializing to $\mathcal{N} = 4$ $U(N)$ SYM theory, Yamada and Yaffe [9] and, independently, Harmark and Orselli [10] extended the analysis of the phase structure by including chemical potentials conjugate to the R-charges, and thus obtained a phase diagram of the theory as a function of both temperature and chemical potentials. As noted in [9], the phase diagram obtained from the weakly coupled gauge theory is in qualitative agreement with the phase diagram of R-charged gravitational solutions of 5-dimensional $\mathcal{N} = 2$ gauged supergravity, which are believed to be related to the strong-coupling limit of $\mathcal{N} = 4$ SYM theory [11], [12] through AdS/CFT.

As another application of the phase diagram, in [10] the observation was made that in regions of small temperature and critical chemical potential $\mathcal{N} = 4$ SYM theory reduces to a quantum mechanical theory. More specifically, the gauge invariant partition function reduces to that of an $SU(2)$ spin chain when two of the chemical potentials are turned on and are equal, and an $SU(2|3)$ spin chain when all three chemical potentials are turned on and are equal. These were identified with, respectively, the $SU(2)$ and $SU(2|3)$ closed subsectors of the conjectured complete $SU(2,2|4)$ spin chain of $\mathcal{N} = 4$ SYM theory that have been investigated in the study of integrability [13], [14], [15], [16], [17], [18].

Again for $\mathcal{N} = 4$ SYM theory, the framework of [6], [7] was generalized in a different direction in [19] by allowing non-zero VEV's for the scalar fields. The authors computed a one-loop effective potential for the theory at finite temperature on S^3 at weak 't Hooft coupling under the assumption that the VEV's of the scalar fields are constant[3] and diagonal matrices.[4] Above a critical temperature $T_c \gg T_H$ the effective potential was observed to develop new unstable directions along the scalar directions accompanied by new saddle points which only preserve an $SO(5)$ subgroup of the global $SO(6)$ isometry group. This phenomenon was identified as the weak coupling version of the Gregory-Laflamme localization instability[5] of the small AdS_5 black hole in the gravity dual of the strongly coupled gauge theory.

The solutions to the equations of motion obtained from the effective potential of [19] were given in terms of eigenvalue distribution functions in [21]. The authors considered the joint eigenvalue distribution of the Polyakov loop and the scalar VEV's and found that the topology of the joint distribution was tied to the Hagedorn phase transition: below the Hagedorn temperature T_H the eigenvalues of the scalar VEV's are distributed uniformly over an S^5 and the eigenvalues of the Polyakov loop are distributed uniformly over an S^1. Thus, the joint eigenvalue distribution is an S^5 fibered trivially over S^1; i.e., it is $S^1 \times S^5$.

transition is precisely on the borderline between a first order and a second order transition. This has so far only been carried out for pure $U(N)$ Yang-Mills theory [8] where it was found that the transition is indeed of first order.

[3] The VEV's are assumed homogeneous on the S^3 to preserve the $SO(4)$ isometry. Otherwise the vacuum would spontaneously break rotational invariance, and such an exotic phase is very different from what is observed at strong coupling.

[4] This potential was computed earlier in [9] for the special case of zero Polyakov loop eigenvalues.

[5] See [20] for a recent review of Gregory-Laflamme instability.

Above T_H the eigenvalue distribution of the Polyakov loop opens a gap and thus becomes an interval. The scalar VEV's are distributed uniformly over an S^5 fibered over this interval, with the radius of the S^5 at any point in the interval proportional to the density of Polyakov loop eigenvalues at that point (for fixed TR). The S^5 thus shrinks to zero radius at the endpoints of the interval: the topology of the joint eigenvalue distribution is an S^6. The authors interpreted the S^5 eigenvalue distribution of the scalar VEV's as the emergence of the S^5 factor of the holographic dual thermal $AdS_5 \times S^5$ geometry, whereas the S^6 was identified with a non-contractible S^6 appearing in the near-horizon geometry of a configuration of smeared D2-branes, arising from T-dualizing along the thermal circle of thermal AdS_5.

This thesis is organized as follows. In Chapter 1 we give an introduction to $\mathcal{N} = 2$ $U(N)^M$ quiver gauge theory on $S^1 \times S^3$ with chemical potentials conjugate to the R-charges. In particular, we note that some of the details given here we have not found elsewhere in the literature. In particular, we write the full Lagrangian density in terms of $SU(2)_R \times U(1)_R$ invariants.

In Chapter 2 we evaluate the quantum effective action of $\mathcal{N} = 2$ quiver gauge theory with non-zero R-symmetry chemical potentials and zero scalar VEV's in the $g_{\text{YM}} \to 0$ limit and express it in terms of single-particle partition functions. We use the effective action to construct a matrix model for $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$. The model turns out to be an M-matrix model with adjoint and bifundamental potentials. Before turning to the phase transition in this matrix model we discuss order parameters by which to identify the phase transition. We then move on to study the saddle points of the matrix model as functions of temperature and chemical potential and thereby examine the phase structure of the model. In the low-temperature phase we find a saddle point corresponding to a uniform distribution of the eigenvalues of the Polyakov loop[6]. In this phase the free energy is $\mathcal{O}(1)$ with respect to N. This behavior of the free energy suggests that the model in this phase describes a non-interacting gas of color singlet states, and the phase is therefore labelled "confining". This saddle point is observed to become unstable when the temperature is raised above a certain threshold temperature (which depends on the chemical potential). The model then enters a new phase in which the free energy scales as N^2M as $N \to \infty$. This phase is thus interpreted as describing a non-interacting plasma of color non-singlet states and is labelled "deconfined". The "deconfinement" transition is of first order and is identified with a Hagedorn phase transition. The condition of stability of the low-temperature saddle point is translated into a phase diagram of the gauge theory as a function of both temperature and chemical potentials. We subsequently study the phase diagram in regions of small temperature and critical chemical potential. We observe that the Hilbert space of gauge invariant operators truncates to the $SU(2)$ subsector when the

[6]We are using a somewhat sloppy terminology here: by 'Polyakov loop' we really mean the holonomy matrix of a closed curve winding about the thermal circle and not just its trace. Throughout this thesis we will use the word to describe both and leave the precise meaning to be determined from the context.

chemical potential corresponding to the $SU(2)_R$ factor of the R-symmetry group $SU(2)_R \times U(1)_R$ is turned on, whereas when both chemical potentials are turned on and set equal, it truncates to a larger subsector that corresponds to an orbifolded version of the $SU(2|3)$ sector found in $\mathcal{N} = 4$ SYM theory.

In Chapter 3 we develop a matrix model for $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ with non-zero VEV's for the scalar fields and zero R-symmetry chemical potentials. We carry out this computation in the special case where the background fields are assumed to be "commuting" in a sense that conforms to the quiver structure. Furthermore the background fields will be taken to be static and spatially homogeneous in order to preserve the $SO(4)$ isometry of the spatial S^3 manifold. The method employed for computing the effective potential will be the standard background field formalism. That is, we expand the quantum fields about classical background fields and path integrate over the fluctuations, discarding terms of cubic or higher order in the fluctuations. The resulting fluctuation operators turn out to have a particular tridiagonal structure in their quiver indices. By exploiting the vacuum structure of the theory we find that the determinants factorize, leading to an expression for the quantum effective action of $\mathcal{N} = 2$ $U(N)^M$ quiver gauge theory that explicitly displays the $\mathbb{Z}_M$ structure of the theory. Finally we generalize our results to a specific class of field theories that can be obtained as $\mathbb{Z}_M$ projections of $\mathcal{N} = 4$ SYM theory, of which $\mathcal{N} = 2$ quiver gauge theory is a special case.

In Chapter 4 we find the saddle points of the matrix model of Chapter 3 in the large N limit in a coarse grained approximation. We consider the joint eigenvalue distribution of the scalar VEV's and the Polyakov loop and find that the topology of the eigenvalue distribution is tied to the Hagedorn phase transition. Below the Hagedorn temperature the eigenvalues of the scalar VEV's are distributed uniformly over an $S^5/\mathbb{Z}_M$ and the eigenvalues of the Polyakov loop are distributed uniformly over an S^1. Thus, the joint eigenvalue distribution is an $S^5/\mathbb{Z}_M$ fibered trivially over S^1. We interpret this $S^5/\mathbb{Z}_M$ as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ geometry. Above the Hagedorn temperature the eigenvalue distribution of the Polyakov loop becomes gapped and is thus an interval. The scalar VEV's are now distributed uniformly over an $S^5/\mathbb{Z}_M$ fibered over this interval, with the radius of the $S^5/\mathbb{Z}_M$ at any point in the interval proportional to the density of Polyakov loop eigenvalues at that point (for fixed TR). The $S^5/\mathbb{Z}_M$ thus shrinks to zero radius at the endpoints of the interval: the topology of the joint eigenvalue distribution is an $S^6/\mathbb{Z}_M$ where the $\mathbb{Z}_M$ is understood to act on the S^5 transverse to an S^1 diameter. Finally we generalize our results to the $\mathbb{Z}_M$ orbifold field theories discussed at the end of Section 5. In particular we find that the geometry of the dual AdS spacetime is mirrored in the structure of the quantum effective action in a precise way within this class of orbifold field theories.

In the conclusion we discuss the results we have obtained in this thesis and suggest directions for future study.

Chapter 1

$\mathcal{N} = 2$ quiver gauge theory

In this chapter we review $\mathcal{N} = 2$ $U(N)^M$ quiver gauge theories on $S^1 \times S^3$ with R-symmetry chemical potentials.[1] A brief introductory review of $\mathcal{N} = 2$ quiver gauge theories on $S^1 \times S^3$ is given in Section 1.1. In the section 1.2 we show explicitly how $\mathcal{N} = 2$ quiver gauge theory can be obtained from $\mathcal{N} = 4$ SYM theory by orbifold projection. In 1.3 we display the $SU(2)_R \times U(1)_R$ R-symmetry of the theory by writing the Lagrangian density in terms of invariants. We note that many of the technical details given here are not available elsewhere in the literature. In Section 1.4 we review how to implement the chemical potentials into the Lagrangian density. In Section 1.5 we then write up the complete Lagrangian density including R-symmetry chemical potentials.

1.1 Review of $\mathcal{N} = 2$ quiver gauge theory

$\mathcal{N} = 2$ quiver gauge theory with gauge group $U(N)^M$ arises as the world-volume theory of open strings ending on a stack of N D3-branes placed on the orbifold $\mathbb{C}^3/\mathbb{Z}_M$. The gauge theory is thus superconformal [26] with 16 supercharges. It can be obtained as a $\mathbb{Z}_M$ projection of $\mathcal{N} = 4$ $U(NM)$ SYM theory as explained in detail in Section 1.2. The resulting gauge group is $U(N)^M$ where all the $U(N)$ factors of the gauge group have the same gauge coupling constant g_{YM} associated with them. Letting $i = 1, \dots, M$ and identifying $i \simeq i + M$, the field content can be summarized as follows. There are M vector multiplets[2] $(A_{\mu i}, \Phi_i, \psi_{\Phi,i}, \psi_i)$ where $A_{\mu i}$ is the gauge field, ψ_i is the gaugino, Φ_i is a complex scalar field, and $\psi_{\Phi,i}$ is the superpartner of Φ_i. We take ψ_i and $\psi_{\Phi,i}$ to be 2-component Weyl spinors. Furthermore there are M hypermultiplets $(A_{i,(i+1)}, B_{(i+1),i}, \chi_{A,i}, \chi_{B,i})$ where $A_{i,(i+1)}$ and $B_{(i+1),i}$ are complex scalar fields and $\chi_{A,i}$ and $\chi_{B,i}$ are their respective superpartners which we will take as 2-component Weyl spinors. The fields in the i'th vector multiplet all transform in the adjoint representation of the i'th $U(N)$ factor of the gauge group. The fields in the i'th hypermultiplet transform in a bifundamental representation of the i'th and

[1] It is assumed that the reader is already familiar with supersymmetric field theory. For reviews, see for example [22, 23, 24, 25].

[2] We will use an $\mathcal{N} = 1$ notation throughout this thesis since this proves convenient.

$(i+1)$'th factors. More specifically, letting $\mathbf{N}_i$ denote the fundamental representation of the i'th $U(N)$ factor and $\overline{\mathbf{N}}_i$ the corresponding antifundamental representation, $A_{i,(i+1)}$ and its superpartner $\chi_{A,i}$ transform in the $\mathbf{N}_i \otimes \overline{\mathbf{N}}_{i+1}$ representation, whereas $B_{(i+1),i}$ and its superpartner $\chi_{B,i}$ transform in the $\overline{\mathbf{N}}_i \otimes \mathbf{N}_{i+1}$ representation.

The field content is conveniently summarized in the quiver diagram in Figure 1.1. The diagram consists of M nodes, labelled by $i = 1, \ldots, M$ with the identification $i \simeq i + M$. The i'th node represents the i'th $U(N)$ gauge group factor. Fields belonging to the i'th vector multiplet are drawn as arrows that start and end on the i'th node. For the i'th hypermultiplet, the fields transforming in the $\mathbf{N}_i \otimes \overline{\mathbf{N}}_{i+1}$ representation are drawn as arrows that start at the i'th node and end at the $(i+1)$'th node; the fields transforming in the $\overline{\mathbf{N}}_i \otimes \mathbf{N}_{i+1}$ are depicted as arrows going from the $(i+1)$'th to the i'th node.

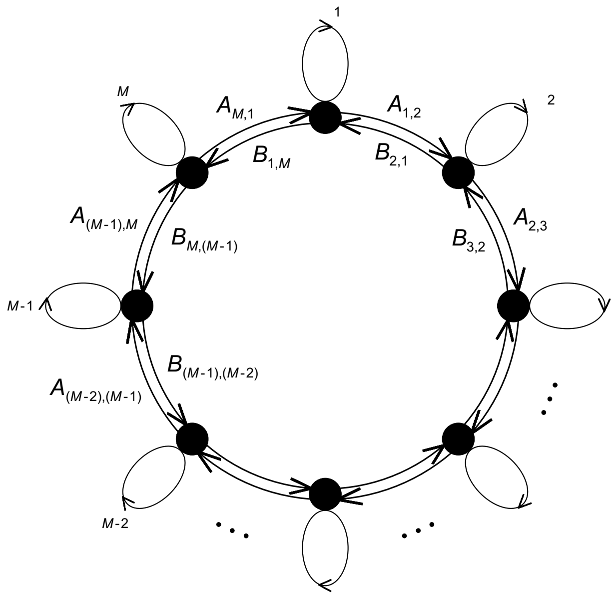


Figure 1.1: Quiver diagram summarizing the field content of $\mathcal{N} = 2\, U(N)^M$ quiver gauge theory. Each of the black dots (called nodes) represents a $U(N)$ gauge group factor. The nodes are labelled by $i = 1, \ldots, M$ with the identification $i \simeq i + M$. Arrows go from fundamental to antifundamental representations of the corresponding gauge group factors. The scalar fields $A_{i,(i+1)}, B_{(i+1),i}$ and Φ_i are shown in the figure, whereas the gauge fields and the superpartners of each field have been left implicit.

The holographic dual of $\mathcal{N} = 2$ quiver gauge theory was found in [26] to be Type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_M$.[3] The quotient $S^5/\mathbb{Z}_M$ is obtained by embedding S^5 in $\mathbb{C}^3$ where the action of the orbifold group $\mathbb{Z}_M$ is as defined in (1.2.1). Note that this statement implicitly contains the information that the $\mathcal{N} = 2$ quiver gauge theory is conformally invariant. Namely, since the AdS_5 part of the 10-dimensional geometry is not acted on by

[3] We will assume that the reader has some familiarity with the AdS/CFT-correspondence. For an excellent introductory review see [27].

the orbifold group, it preserves the isometry group $SO(2,4)$, and so the 4-dimensional field theory defined on the conformal boundary of AdS_5 will therefore be invariant under the full conformal group $SO(2,4)$.

The AdS_5 space has a radius given by

$$R^2_{AdS} = \sqrt{4\pi g_s(\alpha')^2 NM} \tag{1.1.1}$$

where g_s is the Type IIB string coupling. There are also NM units of 5-form RR-flux through the AdS_5. Due to the orbifold action the volume of the quotient $S^5/\mathbb{Z}_M$ equals the volume of the covering space S^5 divided by a factor M where the radius of the S^5 is the same as that of AdS_5 given in (1.1.1). Similarly, there are N units of 5-form RR-flux through the $S^5/\mathbb{Z}_M$ factor which originate from NM units of flux in the covering space. Finally, we note that the Yang-Mills coupling for each $U(N)$ gauge group factor $g_{\rm YM}$ is related to the Type IIB coupling by $g^2_{\rm YM} = 4\pi g_s M$. This means that the 't Hooft coupling relevant for each factor is $\lambda = g^2_{\rm YM}N = 4\pi g_s NM$. This is the same as the 't Hooft coupling on the original NM D3-branes before orbifolding, for which the Yang-Mills coupling was equal to $4\pi g_s$. In the following we will often denote the Yang-Mills coupling simply by g.

Before giving the action of $\mathcal{N}=2$ $U(N)^M$ quiver gauge theory on $S^1\times S^3$ we will fix our conventions. We set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + ig[A_\mu, \cdot\,]$. We denote the circumference of the circle S^1 with β and the radius of the spatial S^3 with R. The bosonic fields are required to satisfy periodic boundary conditions around the thermal circle, $\phi(\tau+\beta,\boldsymbol{x}) = \phi(\tau,\boldsymbol{x})$, whereas the fermionic fields have antiperiodic boundary conditions $\psi(\tau+\beta,\boldsymbol{x}) = -\psi(\tau,\boldsymbol{x})$. The fact that the boundary conditions are different for bosonic and fermionic fields explicitly breaks supersymmetry.[4] The temperature is defined to be the inverse circumference of S^1; that is, $T=\beta^{-1}$. The circle S^1 will be referred to throughout this thesis as the *thermal circle.*

The Euclidean action of $\mathcal{N}=2$ quiver gauge theory on $S^1\times S^3$ at the temperature β^{-1} is then

$$S = \int_{S^1\times S^3} d^4x\sqrt{|g|}\,\left(\mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm ferm}\right) \tag{1.1.2}$$

where the gauge boson, scalar field and spinor field Lagrangian densities are given by, respectively[5]

$$\begin{aligned}
\mathcal{L}_{\rm gauge} &= \frac{1}{4}\,{\rm Tr}\,F_{\mu\nu}F_{\mu\nu} && (1.1.3)\\
\mathcal{L}_{\rm scalar} &= {\rm Tr}\Big[\Big(D_\mu A\, D_\mu\overline{A} + D_\mu B\, D_\mu\overline{B} + D_\mu\Phi\, D_\mu\overline{\Phi}\Big)\\
&\quad + R^{-2}\Big(A\overline{A} + B\overline{B} + \Phi\overline{\Phi}\Big) + \frac{1}{2}g^2\Big([A,\overline{A}] + [B,\overline{B}] + [\Phi,\overline{\Phi}]\Big)^2\\
&\quad - 2g^2\Big(\big|[A,B]\big|^2 + \big|[A,\Phi]\big|^2 + \big|[B,\Phi]\big|^2\Big)\Big] && (1.1.4)
\end{aligned}$$

[4]We will moreover see a clear manifestation of the SUSY breaking in subsection 2.5.3.

[5]Note that for all fields, including the Weyl spinors $\chi_A, \chi_B, \psi_\Phi, \psi$, the bars denote the *Hermitian* conjugate, not the complex or Weyl conjugate. E.g., $(\overline{\chi_A})_{\alpha\beta} = (\chi_A)^*_{\beta\alpha}$ where α,β are gauge group indices and the * denotes complex conjugation. Furthermore, in the third line of Eq. (1.1.4), the notation means, e.g., $|[A,B]|^2 \equiv [A,B][\overline{A},\overline{B}]$.

$$\begin{aligned}\mathcal{L}_{\text{ferm}} &= i\,\mathrm{Tr}\Big(\overline{\chi_A}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overleftrightarrow{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overleftrightarrow{D}_\mu \overline{\psi_\Phi}\Big) \\ &\quad + \frac{g}{\sqrt{2}}\,\mathrm{Tr}\Big(\overline{\chi_A}\big([A,\psi_\Phi] - [\overline{B},\overline{\psi}]\big) + \overline{\chi_B}\big([\overline{A},\overline{\psi}] + [B,\psi_\Phi]\big) \\ &\qquad - \overline{\psi}\big([\overline{A},\overline{\chi_B}] - [\overline{B},\overline{\chi_A}]\big) - \psi_\Phi\big([A,\overline{\chi_A}] + [B,\overline{\chi_B}]\big) \\ &\qquad + \chi_A\big([\overline{A},\overline{\psi_\Phi}] - [B,\psi]\big) + \chi_B\big([A,\psi] + [\overline{B},\overline{\psi_\Phi}]\big) \\ &\qquad - \psi\big([A,\chi_B] - [B,\chi_A]\big) - \overline{\psi_\Phi}\big([\overline{A},\chi_A] + [\overline{B},\chi_B]\big) \\ &\qquad + \overline{\chi_A}\,[\overline{\Phi},\overline{\chi_B}] - \overline{\chi_B}\,[\overline{\Phi},\overline{\chi_A}] + \overline{\psi}\,[\Phi,\psi_\Phi] - \psi_\Phi\,[\Phi,\overline{\psi}] \\ &\qquad + \chi_A\,[\Phi,\chi_B] - \chi_B\,[\Phi,\chi_A] + \psi\,[\overline{\Phi},\overline{\psi_\Phi}] - \overline{\psi_\Phi}\,[\overline{\Phi},\psi]\Big). \qquad (1.1.5)\end{aligned}$$

The traces are taken over the $NM \times NM$ matrices. The spinor fields $\chi_A, \chi_B, \psi_\Phi, \psi$ are undotted 2-component Weyl spinors. We define $\tau_\mu = (1, i\boldsymbol{\sigma})$. The operator $\overleftrightarrow{D}_\mu$ is defined by $\psi_1 \overleftrightarrow{D}_\mu \psi_2 \equiv \frac{1}{2}\big(\psi_1 D_\mu \psi_2 - (D_\mu \psi_1)\,\psi_2\big)$. It is implied that the fields A, B, Φ, A_μ etc. take the orbifold projection invariant forms given in Eqs. (1.2.15)-(1.2.16) and (1.2.37)-(1.2.38). Note that the scalar fields are conformally coupled to the curvature of the spatial manifold S^3 through the term $R^{-2}\,\mathrm{Tr}\big(A\overline{A} + B\overline{B} + \Phi\overline{\Phi}\big)$ in (1.1.4). This effectively induces a mass for the scalar fields. Note here that, strictly speaking, the curvature coupling term should be expressed through the Ricci scalar $\mathcal{R} = 6R^{-2}$ of S^3 rather than the radius R. The reason for coupling the scalar fields to the curvature of the spatial manifold is to preserve the invariance of S_{scalar} under conformal transformations of the metric $g_{\mu\nu}(x) \longrightarrow \Omega^2(x)\, g_{\mu\nu}(x)$. In order for the kinetic part of the action to be invariant under a transformation with a constant Ω, any scalar field ϕ of the theory must accordingly transform as $\phi(x) \longrightarrow \Omega^{-1}(x)\phi(x)$ (this reflects the fact that ϕ has the dimension of $[\text{length}]^{-1}$ in a 4-dimensional spacetime). Allowing $\Omega(x)$ to vary over the spacetime, the variation of the kinetic part is cancelled by the variation of the curvature coupling term due to the non-trivial transformation of the Ricci scalar under the conformal transformation. This is explained in further detail in [28], below Eq. (3.27).

Finally we note that the R-symmetry group of $\mathcal{N} = 2$ $U(N)^M$ quiver gauge theory is $SU(2)_R \times U(1)_R$, resulting from breaking the $SU(4)_R$ R-symmetry group of $\mathcal{N} = 4$ theory via the action of the orbifold group $\mathbb{Z}_M$ as explained in the next section. Indeed, much of the renewed interest in orbifold quiver gauge theories during the past decade is motivated by the desire to extend the AdS/CFT correspondence between Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory to include boundary field theories with lower supersymmetry than $\mathcal{N} = 4$. Field theories obtained from $\mathcal{N} = 4$ SYM theory by orbifold projection with a discrete orbifold group $\Gamma \subseteq SU(4)_R$ present an elegant way of doing this since their dual holographic string theory description can often be deduced geometrically. Depending on whether Γ is embedded entirely in an $SU(2)$, $SU(3)$ or $SU(4)$ subgroup of the $SU(4)_R$ R-symmetry group, the resulting orbifold field theory will have $\mathcal{N} = 2, 1, 0$ supersymmetry, respectively [29, 30].

1.2 Relation to $\mathcal{N}=4$ SYM theory

In this section we give a detailed description of how $\mathcal{N}=2$ $U(N)^M$ quiver gauge theory can be obtained by applying a $\mathbb{Z}_M$ projection to $\mathcal{N}=4$ $U(NM)$ SYM theory. We include here details which we have not found elsewhere in the literature.

Consider Type IIB string theory and introduce a stack of NM coincident D3-branes into the 10-dimensional (initially flat) spacetime. It is well known that the low-energy effective field theory of open strings with endpoints attached to the D3-branes is 4-dimensional $\mathcal{N}=4$ SYM theory with gauge group $U(NM)$. The space transverse to the world volume of the D3-branes is $\mathbb{R}^6 \cong \mathbb{C}^3$ which has the isometry group $SO(6)$. Now we consider the action of the subgroup $\mathbb{Z}_M$ on $\mathbb{C}^3$ given by

$$(z_1, z_2, z_3) \longrightarrow (z_1, \omega^{-1} z_2, \omega z_3)\,, \qquad \omega \equiv e^{2\pi i/M}\,. \tag{1.2.1}$$

The group $\mathbb{Z}_M$ is called the *orbifold group*. We will denote the resulting quotient of $\mathbb{C}^3$ by $\mathbb{C}^3/\mathbb{Z}_M$ where it is implied that the action of $\mathbb{Z}_M$ on $\mathbb{C}^3$ is always that given in Eq. (1.2.1).

Consider now open strings living on the stack of D3-branes where the transverse space is $\mathbb{C}^3/\mathbb{Z}_M$. The low-energy effective field theory is no longer $\mathcal{N}=4$ $U(NM)$ SYM theory. This is because associated with the orbifold group action (1.2.1) on the coordinates of $\mathbb{C}^3$ there is an orbifold group action on the scalar fields and their superpartners (to be defined below), and we must require that all quantum fields of $\mathcal{N}=4$ SYM theory be invariant under this action. The gauge theory obtained from $\mathcal{N}=4$ SYM theory by truncating the Hilbert spaces of quantum fields to $\mathbb{Z}_M$-invariant fields is called $\mathcal{N}=2$ *quiver gauge theory*.

The orbifolding breaks the R-symmetry group $SU(4)$ of $\mathcal{N}=4$ SYM theory into $SU(2)_R \times U(1)_R$.[6] This is shown explicitly in Section 1.3 where the Lagrangian density of the quiver gauge theory is expressed in terms of $SU(2)_R \times U(1)_R$ invariants. The quiver gauge theory thus indeed has $\mathcal{N}=2$ supersymmetry.

The orbifold group action (1.2.1) breaks the gauge group $U(NM)$ of the $\mathcal{N}=4$ theory into

$$U(N)^{(1)} \times U(N)^{(2)} \times \cdots \times U(N)^{(M)} \tag{1.2.2}$$

which is thus the gauge group of $\mathcal{N}=2$ quiver gauge theory. We can see this as a manifestation of the fact that the quiver gauge theory is a low-energy effective field theory of open strings. Indeed, each of the M copies of $\mathbb{C}^3/\mathbb{Z}_M$ embedded in $\mathbb{C}^3$ will contain N coincident D3-branes, and an open string can attach its endpoints to any of the stacks. Finally, to conclude the enumeration of the symmetries of $\mathcal{N}=2$ quiver gauge theory, we note that it is known to be a conformally invariant theory like the parent $\mathcal{N}=4$ SYM theory [26].

In order to define the action of the orbifold group $\mathbb{Z}_M$ on the $\mathcal{N}=4$ SYM fields we first set up some notation. First $\mathbb{Z}_M$ is embedded into $U(NM)$ by defining the twist matrix

[6] To be precise, one has the breaking $SU(4) \cong SO(6) \longrightarrow SO(2) \times SO(4) \cong U(1)_R \times SU(2)_L \times SU(2)_R$ since the z_1 direction of $\mathbb{C}^3$ is left inert under the $\mathbb{Z}_M$ action (1.2.1). We will not be concerned with the $SU(2)_L$ symmetry (which is strictly speaking broken to a $U(1)_L$ symmetry) and just remark that the charges of all fields of $\mathcal{N}=2$ quiver gauge theory under a Cartan generator J_L of $SU(2)_L$ can be found in Table 1 of [31].

$\gamma \equiv \mathrm{diag}(1, \omega, \ldots, \omega^{M-1})$ and mapping $\mathbb{Z}_M \ni k \longmapsto \gamma^k \in U(NM)$. (Note that the entries ω^j of γ are really $N \times N$ matrices.)[7] The action of $\mathbb{Z}_M$ on the $\mathcal{N}=4$ SYM fields is then

$$\phi \longrightarrow (\gamma^k)^\dagger \, (\rho \cdot \phi) \, \gamma^k \tag{1.2.3}$$

where $\rho \cdot \phi$ equals a phase times the field ϕ. For the scalar fields the phase is determined by their identifications with the z_1, z_2 and z_3 directions in $\mathbb{C}^3$ and comparing with (1.2.1). For the gauge field the phase is 1. For the spinor fields the phase equals that of their bosonic superpartner. Thus the condition for the $\mathcal{N}=4$ SYM fields ϕ to be invariant under the action of $\mathbb{Z}_M$ is

$$\phi = \gamma^\dagger \, (\rho \cdot \phi) \, \gamma \, . \tag{1.2.4}$$

In the following we will obtain the Lagrangian density of $\mathcal{N}=2$ $U(N)^M$ quiver gauge theory by rewriting the $\mathcal{N}=4$ $U(NM)$ SYM Lagrangian density and require that all the fields satisfy the $\mathbb{Z}_M$-invariance condition (1.2.4).

We now consider $\mathcal{N}=4$ $U(NM)$ SYM theory on $\mathbb{R} \times S^3$ where the radius of S^3 is denoted by R. The scalar fields will couple conformally to the curvature of the S^3 through a quadratic term in the action. In the decompactification limit $R \to \infty$ this term will vanish. The action of $\mathcal{N}=4$ $U(NM)$ SYM theory on $\mathbb{R} \times S^3$ equipped with a metric of Euclidean signature reads

$$\begin{aligned} S^{\mathcal{N}=4} \;=\; & \int d^4x \; \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} (D_\mu \phi^i)(D_\mu \phi^i) + \frac{1}{2} R^{-2} \phi^i \phi^i - \frac{1}{4} g^2 \, [\phi^i, \phi^j] \, [\phi^i, \phi^j] \right. \\ & \left. + \frac{i}{2} \overline{\psi_p} \gamma_\mu D_\mu \psi_p - \frac{g}{2} \overline{\psi_p} \big[(\alpha^k_{pq} \phi^{2k-1} + i \beta^k_{pq} \gamma_5 \phi^{2k}), \psi_q \big] \right) \end{aligned} \tag{1.2.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + ig[A_\mu, \,\cdot\,]$. The traces are taken over the gauge indices. The indices have the ranges $\mu, \nu = 0, \ldots, 3$; $i, j = 1, \ldots, 6$; $p, q = 1, \ldots, 4$ and $k = 1, \ldots, 3$. Here ϕ^i are six real scalar fields and ψ_p are four 4-component Majorana spinors. Moreover, γ_μ are the 4-dimensional 4×4 gamma matrices and α^k and β^k are 4×4 matrices satisfying the relations

$$\{\alpha^k, \alpha^l\} = -2\delta^{kl} \mathbf{1}_4 \, , \qquad \{\beta^k, \beta^l\} = -2\delta^{kl} \mathbf{1}_4 \, , \qquad [\alpha^k, \beta^l] = 0 \, . \tag{1.2.6}$$

Explicit representations can be given as

$$\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} , \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} , \qquad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} , \tag{1.2.7}$$

$$\beta^1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} , \qquad \beta^2 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} , \qquad \beta^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} . \tag{1.2.8}$$

[7]Note that this representation of $\mathbb{Z}_M$ satisfies $\mathrm{Tr}\, \gamma^k = 0$ for all $k \in \mathbb{Z}_M \setminus \{0\}$. As pointed out in Ref. [29], this is needed for consistency (the cancellation of one-loop open string tadpole diagrams).

1.2.1 The bosonic part of the quiver action

To put the action of $\mathcal{N}=4$ SYM theory in a form suitable for performing the orbifold projection we now define three complex scalar fields

$$A=\frac{1}{\sqrt{2}}(\phi^1+i\phi^2)\,,\qquad B=\frac{1}{\sqrt{2}}(\phi^3+i\phi^4)\,,\qquad \Phi=\frac{1}{\sqrt{2}}(\phi^5+i\phi^6)\,. \tag{1.2.9}$$

The fields ϕ^i are Hermitian (since they transform in the adjoint representation of the gauge group $U(NM)$), so by Hermitian conjugation of (1.2.9) we find

$$\overline{A}=\frac{1}{\sqrt{2}}(\phi^1-i\phi^2)\,,\qquad \overline{B}=\frac{1}{\sqrt{2}}(\phi^3-i\phi^4)\,,\qquad \overline{\Phi}=\frac{1}{\sqrt{2}}(\phi^5-i\phi^6)\,. \tag{1.2.10}$$

The scalar field part of the $\mathcal{N}=4$ SYM Lagrangian density written in terms of these fields takes the form

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=4}_{\text{scalar}} &= \operatorname{Tr}\left(\frac{1}{2}(D_\mu\phi^i)(D_\mu\phi^i)+\frac{1}{2}R^{-2}\phi^i\phi^i-\frac{1}{4}g^2\,[\phi^i,\phi^j]\,[\phi^i,\phi^j]\right)\\ &= \operatorname{Tr}\left(D_\mu A\,D_\mu\overline{A}+D_\mu\overline{B}\,D_\mu B+D_\mu\Phi\,D_\mu\overline{\Phi}\right)\\ &\quad +R^{-2}\operatorname{Tr}\left(A\overline{A}+\overline{B}B+\Phi\overline{\Phi}\right)+\mathcal{L}^{\mathcal{N}=4}_D+\mathcal{L}^{\mathcal{N}=4}_F \end{aligned}\tag{1.2.11}$$

where the D and F terms are, respectively,

$$\mathcal{L}^{\mathcal{N}=4}_D = \frac{1}{2}g^2\operatorname{Tr}\left([A,\overline{A}]+[B,\overline{B}]+[\Phi,\overline{\Phi}]\right)^2 \tag{1.2.12}$$

$$\mathcal{L}^{\mathcal{N}=4}_F = -2g^2\operatorname{Tr}\left([A,B]\,[\overline{A},\overline{B}]+[A,\Phi]\,[\overline{A},\overline{\Phi}]+[B,\Phi]\,[\overline{B},\overline{\Phi}]\right). \tag{1.2.13}$$

The scalar fields Φ, A and B can be identified with the z_1, z_2 and z_3 directions of the $\mathbb{C}^3$ (because they are the Goldstone bosons associated with breaking the translational invariance in the directions transverse to the D3-branes), so we have the orbifold group action $\rho:(\Phi,A,B)\mapsto(\Phi,\omega^{-1}A,\omega B)$, and the condition for these fields to be invariant under the $\mathbb{Z}_M$-transformation is then

$$\gamma^\dagger\Phi\gamma=\Phi\,,\qquad \gamma^\dagger A\gamma=\omega A\,,\qquad \gamma^\dagger B\gamma=\omega^{-1}B\,. \tag{1.2.14}$$

One easily checks that these conditions are satisfied by splitting the $NM\times NM$ matrix fields of the $\mathcal{N}=4$ $U(NM)$ SYM theory up into $M\times M$ block matrices whose entries are $N\times N$ matrices:

$$A_\mu=\begin{pmatrix} A_{\mu1} & & & \\ & A_{\mu2} & & \\ & & \ddots & \\ & & & A_{\mu M}\end{pmatrix},\qquad A=\begin{pmatrix} 0 & A_{1,2} & & & \\ & 0 & A_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & A_{(M-1),M} \\ A_{M,1} & & & & 0\end{pmatrix}, \tag{1.2.15}$$

$$B=\begin{pmatrix} 0 & & & & B_{1,M} \\ B_{2,1} & 0 & & & \\ & B_{3,2} & \ddots & & \\ & & \ddots & 0 & \\ & & & B_{M,(M-1)} & 0 \end{pmatrix}, \qquad \Phi=\begin{pmatrix} \Phi_1 & & & \\ & \Phi_2 & & \\ & & \ddots & \\ & & & \Phi_M \end{pmatrix}. \tag{1.2.16}$$

Here $A_{\mu i}, A_{i,(i+1)}, B_{(i+1),i}$ and Φ_i are $N\times N$ matrices (where $i=1,\dots,M$ and we identify $i\simeq i+M$). Inserting the $\mathbb{Z}_M$-invariant forms of A_μ, A, B and Φ given in Eqs. (1.2.15)-(1.2.16) into (1.2.11)-(1.2.13) the scalar field part of the $\mathcal{N}=2$ quiver gauge theory Lagrangian density reads

$$\begin{aligned}
\mathcal{L}_{\rm scalar} \;=\; \sum_{i=1}^{M}\Bigg\{ & \operatorname{Tr}\Big[\Big(\partial_\mu A_{i,(i+1)} + igA_{\mu i}A_{i,(i+1)} - igA_{i,(i+1)}A_{\mu(i+1)}\Big) \\
& \qquad \times\Big(\partial_\mu\overline{A_{i,(i+1)}} + igA_{\mu(i+1)}\overline{A_{i,(i+1)}} - ig\overline{A_{i,(i+1)}}A_{\mu i}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu B_{(i+1),i} + igA_{\mu(i+1)}B_{(i+1),i} - igB_{(i+1),i}A_{\mu i}\Big) \\
& \qquad \times\Big(\partial_\mu\overline{B_{(i+1),i}} + igA_{\mu i}\overline{B_{(i+1),i}} - ig\overline{B_{(i+1),i}}A_{\mu(i+1)}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu\Phi_i + ig[A_{\mu i},\Phi_i]\Big)\Big(\partial_\mu\overline{\Phi_i} + ig[A_{\mu i},\overline{\Phi_i}]\Big)\Big] \\
& + R^{-2}\operatorname{Tr}\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} + \overline{B_{(i+1),i}}B_{(i+1),i} + \Phi_i\overline{\Phi_i}\Big) \\
& + \frac{1}{2}g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}}A_{(i-1),i} \\
& \qquad + B_{i,(i-1)}\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}B_{(i+1),i} + [\Phi_i,\overline{\Phi_i}]\Big)^2\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}B_{(i+1),i} - B_{i,(i-1)}A_{(i-1),i}\Big) \\
& \qquad \times\Big(\overline{A_{(i-1),i}}\;\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\;\overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\Phi_{i+1} - \Phi_i A_{i,(i+1)}\Big)\Big(\overline{A_{i,(i+1)}}\;\overline{\Phi_i} - \overline{\Phi_{i+1}}\;\overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(B_{(i+1),i}\Phi_i - \Phi_{i+1}B_{(i+1),i}\Big)\Big(\overline{B_{(i+1),i}}\;\overline{\Phi_{i+1}} - \overline{\Phi_i}\;\overline{B_{(i+1),i}}\Big)\Big]\Bigg\}
\end{aligned} \tag{1.2.17}$$

Inserting the form of A_μ given in (1.2.15) into (1.2.5), the gauge field part of the $\mathcal{N}=2$ quiver gauge theory Lagrangian density reads

$$\mathcal{L}_{\rm gauge} = \frac{1}{4}\sum_{i=1}^{M}\operatorname{Tr}F^i_{\mu\nu}F^i_{\mu\nu} \tag{1.2.18}$$

where of course $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig[A^i_\mu, A^i_\nu]$ and the trace is taken over the gauge indices of the $N\times N$ matrices.

1.2.2 The fermionic part of the quiver action

The fermionic part of the $\mathcal{N}=4$ SYM Lagrangian density reads

$$\mathcal{L}_{\text{ferm}}^{\mathcal{N}=4} = \text{Tr}\left(\frac{i}{2}\overline{\psi_p}\gamma_\mu D_\mu \psi_p - \frac{g}{2}\overline{\psi_p}\big[(\alpha^k_{pq}\phi^{2k-1} + i\beta^k_{pq}\gamma_5\phi^{2k}), \psi_q\big]\right) \tag{1.2.19}$$

where the gamma matrices are given by

$$\gamma_\mu \equiv \begin{pmatrix} 0 & \tau_\mu \\ \overline{\tau}_\mu & 0 \end{pmatrix}, \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{1.2.20}$$

$$\tau_\mu \equiv (1, i\boldsymbol{\sigma}), \qquad \overline{\tau}_\mu \equiv (1, -i\boldsymbol{\sigma}) \tag{1.2.21}$$

and representations of α^k and β^k are given in Eqs. (1.2.7) and (1.2.8), respectively. The fields ψ_p, $p = 1, \ldots, 4$ are 4-component Majorana spinors which can be decomposed in terms of 2-component Weyl spinors as follows

$$(\psi_p)^a \equiv \begin{pmatrix} (\lambda_p)_\alpha \\ (\overline{\lambda_p})^{\dot\alpha} \end{pmatrix}, \qquad (\overline{\psi_p})_a \equiv \begin{pmatrix} (\lambda_p)^\alpha \\ (\overline{\lambda_p})_{\dot\alpha} \end{pmatrix} \tag{1.2.22}$$

where $a = 1, \ldots, 4$ is the spinor index on ψ_p. The Majorana spinors are related to their conjugates through the Majorana condition

$$\psi_p = C\overline{\psi_p} \tag{1.2.23}$$

where the Majorana conjugation matrix is $C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$ with $\epsilon_{12} = -\epsilon_{21} = -1$.

Combining Eqs. (1.2.22) and (1.2.20)-(1.2.21) one finds

$$\frac{1}{2}\overline{\psi_p}\gamma_\mu D_\mu \psi_p = (\lambda_p)^\alpha (\tau_\mu)_{\alpha\dot\beta} \overleftrightarrow{D}_\mu (\overline{\lambda_p})^{\dot\beta}. \tag{1.2.24}$$

Here the operator $\overleftrightarrow{D}_\mu$ is defined by $\chi_p \overleftrightarrow{D}_\mu \chi_q \equiv \frac{1}{2}\big(\chi_p D_\mu \chi_q - (D_\mu\chi_p)\,\chi_q\big)$.

It will be useful for exhibiting the R-symmetry of the quiver gauge theory to express the fermionic Lagrangian density in terms of the following Weyl spinors

$$\chi_A \equiv \overline{\lambda_1}, \qquad \chi_B \equiv \overline{\lambda_2}, \qquad \psi \equiv \overline{\lambda_3}, \qquad \psi_\Phi \equiv \lambda_4. \tag{1.2.25}$$

Here $\chi_A, \chi_B, \psi, \psi_\Phi$ are the respective superpartners of A, B, A_μ, Φ. Note here that the bar used over the spinors in (1.2.25) is understood to mean the *Hermitian conjugate* whereas the bar over the λ_p in (1.2.24) denotes the usual conjugate of Weyl spinors. Explicitly, letting $\alpha = 1, 2$ be the spinor index and letting m, n be the gauge indices,

$$(\lambda_1)_{\alpha,mn} \equiv (\chi_A)^*_{\alpha,nm} = (\overline{\chi_A})_{\alpha,mn} \tag{1.2.26}$$

and

$$(\overline{\lambda_1})_{\dot\alpha,mn} = \big((\lambda_1)^*_{\alpha,mn}\big)^T = (\lambda_1)^*_{\alpha,nm} = (\chi_A)_{\alpha,mn} \tag{1.2.27}$$

and analogously for χ_B, ψ_Φ and ψ. In particular, note that all the Weyl spinors $\chi_A, \chi_B, \psi_\Phi$ and ψ have undotted indices.

Inserting the definitions (1.2.25) into the decomposition (1.2.24) we can write the kinetic part of the fermionic $\mathcal{N}=4$ SYM Lagrangian density (1.2.19) in the form

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4,\text{kin}}_{\text{ferm}} &= \frac{i}{2}\,\text{Tr}\left(\overline{\psi_p}\gamma_\mu D_\mu \psi_p\right) \\ &= i\,\text{Tr}\left(\overline{\chi_A}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overleftrightarrow{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overleftrightarrow{D}_\mu \overline{\psi_\Phi}\right). \end{aligned} \tag{1.2.28}$$

In order to find the potential part of the fermionic Lagrangian density we use the explicit forms of the α^k, β^k matrices given in Eqs. (1.2.7)-(1.2.8)

$$\begin{aligned} &\overline{\psi_p}\left[(\alpha^k_{pq}\phi^{2k-1} + i\beta^k_{pq}\gamma_5\phi^{2k}), \psi_q\right] \\ &= \frac{1}{\sqrt{2}}\Big(\alpha^1_{pq}\overline{\psi_p}\left[A+\overline{A},\psi_q\right] + \alpha^2_{pq}\overline{\psi_p}\left[B+\overline{B},\psi_q\right] + \alpha^3_{pq}\overline{\psi_p}\left[\Phi+\overline{\Phi},\psi_q\right] \\ &\qquad + \beta^1_{pq}\overline{\psi_p}\left[\gamma_5(A-\overline{A}),\psi_q\right] + \beta^2_{pq}\overline{\psi_p}\left[\gamma_5(B-\overline{B}),\psi_q\right] + \beta^3_{pq}\overline{\psi_p}\left[\gamma_5(\Phi-\overline{\Phi}),\psi_q\right]\Big) \\ &= \frac{1}{\sqrt{2}}\Bigg(\overline{\psi_1}\left[A+\overline{A},\psi_4\right] + \overline{\psi_2}\left[A+\overline{A},\psi_3\right] - \overline{\psi_3}\left[A+\overline{A},\psi_2\right] - \overline{\psi_4}\left[A+\overline{A},\psi_1\right] \\ &\quad - \overline{\psi_1}\left[B+\overline{B},\psi_3\right] + \overline{\psi_2}\left[B+\overline{B},\psi_4\right] + \overline{\psi_3}\left[B+\overline{B},\psi_1\right] - \overline{\psi_4}\left[B+\overline{B},\psi_2\right] \\ &\quad + \overline{\psi_1}\left[\Phi+\overline{\Phi},\psi_2\right] - \overline{\psi_2}\left[\Phi+\overline{\Phi},\psi_1\right] + \overline{\psi_3}\left[\Phi+\overline{\Phi},\psi_4\right] - \overline{\psi_4}\left[\Phi+\overline{\Phi},\psi_3\right] \\ &\quad + \overline{\psi_1}\left[\gamma_5(A-\overline{A}),\psi_4\right] - \overline{\psi_2}\left[\gamma_5(A-\overline{A}),\psi_3\right] + \overline{\psi_3}\left[\gamma_5(A-\overline{A}),\psi_2\right] - \overline{\psi_4}\left[\gamma_5(A-\overline{A}),\psi_1\right] \\ &\quad + \overline{\psi_1}\left[\gamma_5(B-\overline{B}),\psi_3\right] + \overline{\psi_2}\left[\gamma_5(B-\overline{B}),\psi_4\right] - \overline{\psi_3}\left[\gamma_5(B-\overline{B}),\psi_1\right] - \overline{\psi_4}\left[\gamma_5(B-\overline{B}),\psi_2\right] \\ &\quad - \overline{\psi_1}\left[\gamma_5(\Phi-\overline{\Phi}),\psi_2\right] + \overline{\psi_2}\left[\gamma_5(\Phi-\overline{\Phi}),\psi_1\right] + \overline{\psi_3}\left[\gamma_5(\Phi-\overline{\Phi}),\psi_4\right] - \overline{\psi_4}\left[\gamma_5(\Phi-\overline{\Phi}),\psi_3\right]\Bigg). \end{aligned} \tag{1.2.29}$$

We can simplify this expression by noting that for two arbitrary 4-component Majorana spinors

$$\overline{\chi}^a \equiv \begin{pmatrix} \chi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix}, \qquad \psi_a \equiv \begin{pmatrix} \psi^\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix} \tag{1.2.30}$$

we have the identity

$$\overline{\chi}^a\psi_a = \left(\chi_\alpha, \overline{\chi}^{\dot\alpha}\right)\begin{pmatrix} \psi^\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix} = \chi_\alpha\psi^\alpha + \overline{\chi}^{\dot\alpha}\overline{\psi}_{\dot\alpha} = -\left(\chi^\alpha\psi_\alpha + \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha}\right) \tag{1.2.31}$$

and analogously

$$\overline{\chi}^a\gamma_5\psi_a = -\left(\chi^\alpha\psi_\alpha - \overline{\chi}_{\dot\alpha}\overline{\psi}^{\dot\alpha}\right). \tag{1.2.32}$$

Applying these identities and leaving spinor indices implicit we find

$$\begin{aligned}
&\overline{\psi_p}\,[(\alpha^k_{pq}\phi^{2k-1} + i\beta^k_{pq}\gamma_5\phi^{2k}),\psi_q] \\
&= \sqrt{2}\Big(-\lambda_1\big([A,\lambda_4]-[\overline{B},\lambda_3]\big) - \lambda_2\big([\overline{A},\lambda_3]+[B,\lambda_4]\big) + \lambda_3\big([\overline{A},\lambda_2]-[\overline{B},\lambda_1]\big) \\
&\quad + \lambda_4\big([A,\lambda_1]+[B,\lambda_2]\big) - \overline{\lambda_1}\big([\overline{A},\overline{\lambda_4}]-[B,\overline{\lambda_3}]\big) - \overline{\lambda_2}\big([A,\overline{\lambda_3}]+[\overline{B},\overline{\lambda_4}]\big) \\
&\quad + \overline{\lambda_3}\big([A,\overline{\lambda_2}]-[B,\overline{\lambda_1}]\big) + \overline{\lambda_4}\big([\overline{A},\overline{\lambda_1}]+[\overline{B},\overline{\lambda_2}]\big) \\
&\quad - \lambda_1[\overline{\Phi},\lambda_2] + \lambda_2[\overline{\Phi},\lambda_1] - \lambda_3[\Phi,\lambda_4] + \lambda_4[\Phi,\lambda_3] \\
&\quad - \overline{\lambda_1}[\Phi,\overline{\lambda_2}] + \overline{\lambda_2}[\Phi,\overline{\lambda_1}] - \overline{\lambda_3}[\overline{\Phi},\overline{\lambda_4}] + \overline{\lambda_4}[\overline{\Phi},\overline{\lambda_3}] \Big) \qquad (1.2.33) \\
&= \sqrt{2}\Big(-\overline{\chi_A}\big([A,\psi_\Phi]-[\overline{B},\overline{\psi}]\big) - \overline{\chi_B}\big([\overline{A},\overline{\psi}]+[B,\psi_\Phi]\big) + \overline{\psi}\big([\overline{A},\overline{\chi_B}]-[\overline{B},\overline{\chi_A}]\big) \\
&\quad + \psi_\Phi\big([A,\overline{\chi_A}]+[B,\overline{\chi_B}]\big) - \chi_A\big([\overline{A},\overline{\psi_\Phi}]-[B,\psi]\big) - \chi_B\big([A,\psi]+[\overline{B},\overline{\psi_\Phi}]\big) \\
&\quad + \psi_\Phi\big([A,\chi_B]-[B,\chi_A]\big) + \overline{\psi_\Phi}\big([\overline{A},\chi_A]+[\overline{B},\chi_B]\big) \\
&\quad - \overline{\chi_A}[\overline{\Phi},\overline{\chi_B}] + \overline{\chi_B}[\overline{\Phi},\overline{\chi_A}] - \overline{\psi}[\Phi,\psi_\Phi] + \psi_\Phi[\Phi,\overline{\psi}] \\
&\quad - \chi_A[\Phi,\chi_B] + \chi_B[\Phi,\chi_A] - \psi[\overline{\Phi},\overline{\psi_\Phi}] + \overline{\psi_\Phi}[\overline{\Phi},\psi] \Big)\,. \qquad (1.2.34)
\end{aligned}$$

In the last equality we made the substitutions (1.2.25). We conclude that by expressing the 4-component Majorana spinors ψ_p in terms of the 2-component Weyl spinors λ_p (cf. (1.2.22)) and then making the substitutions (1.2.25) the fermionic part of the $\mathcal{N}=4$ SYM Lagrangian density takes the form

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=4}_{\text{ferm}} &= i\,\mathrm{Tr}\Big(\overline{\chi_A}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overleftrightarrow{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overleftrightarrow{D}_\mu \overline{\psi_\Phi}\Big) \\
&\quad + \frac{g}{\sqrt{2}}\,\mathrm{Tr}\Big(\overline{\chi_A}\big([A,\psi_\Phi]-[\overline{B},\overline{\psi}]\big) + \overline{\chi_B}\big([\overline{A},\overline{\psi}]+[B,\psi_\Phi]\big) \\
&\qquad - \overline{\psi}\big([\overline{A},\overline{\chi_B}]-[\overline{B},\overline{\chi_A}]\big) - \psi_\Phi\big([A,\overline{\chi_A}]+[B,\overline{\chi_B}]\big) \\
&\qquad + \chi_A\big([\overline{A},\overline{\psi_\Phi}]-[B,\psi]\big) + \chi_B\big([A,\psi]+[\overline{B},\overline{\psi_\Phi}]\big) \\
&\qquad - \psi\big([A,\chi_B]-[B,\chi_A]\big) - \overline{\psi_\Phi}\big([\overline{A},\chi_A]+[\overline{B},\chi_B]\big) \\
&\qquad + \overline{\chi_A}[\overline{\Phi},\overline{\chi_B}] - \overline{\chi_B}[\overline{\Phi},\overline{\chi_A}] + \overline{\psi}[\Phi,\psi_\Phi] - \psi_\Phi[\Phi,\overline{\psi}] \\
&\qquad + \chi_A[\Phi,\chi_B] - \chi_B[\Phi,\chi_A] + \psi[\overline{\Phi},\overline{\psi_\Phi}] - \overline{\psi_\Phi}[\overline{\Phi},\psi] \Big)\,. \qquad (1.2.35)
\end{aligned}$$

The Weyl spinor fields $\chi_A, \chi_B, \psi_\Phi, \psi$ are the respective superpartners of A, B, Φ, A_μ. Therefore they must satisfy the $\mathbb{Z}_M$-invariance conditions

$$\gamma^\dagger\chi_A\gamma = \omega\chi_A\,, \qquad \gamma^\dagger\chi_B\gamma = \omega^{-1}\chi_B\,, \qquad \gamma^\dagger\psi_\Phi\gamma = \psi_\Phi\,, \qquad \gamma^\dagger\psi\gamma = \psi\,. \qquad (1.2.36)$$

One easily checks that these conditions are satisfied by splitting the $NM\times NM$ matrix fields of the $\mathcal{N}=4$ $U(NM)$ SYM theory up into $M\times M$ block matrices whose entries are

$N \times N$ matrices:

$$\psi = \begin{pmatrix} \psi_1 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_M \end{pmatrix}, \qquad \chi_A = \begin{pmatrix} 0 & \chi_{A,1} & & & \\ & 0 & \chi_{A,2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \chi_{A,M-1} \\ \chi_{A,M} & & & & 0 \end{pmatrix}, \tag{1.2.37}$$

$$\chi_B = \begin{pmatrix} 0 & & & & \chi_{B,M} \\ \chi_{B,1} & 0 & & & \\ & \chi_{B,2} & \ddots & & \\ & & \ddots & 0 & \\ & & & \chi_{B,M-1} & 0 \end{pmatrix}, \qquad \psi_\Phi = \begin{pmatrix} \psi_{\Phi,1} & & & \\ & \psi_{\Phi,2} & & \\ & & \ddots & \\ & & & \psi_{\Phi,M} \end{pmatrix}. \tag{1.2.38}$$

Here $\psi_i, \chi_{A,i}, \chi_{B,i}$ and $\psi_{\Phi,i}$ are $N \times N$ matrices (where $i = 1, \dots, M$ and we identify $i \simeq i+M$). Inserting the $\mathbb{Z}_M$-invariant forms of ψ, χ_A, χ_B and ψ_Φ given in Eqs. (1.2.37)-(1.2.38) into (1.2.35), the spinor field part of the $\mathcal{N}=2$ quiver gauge theory Lagrangian density reads (summation over $i = 1, \dots, M$ implied)

$$\begin{aligned}
\mathcal{L}_{\text{ferm}} &= i \operatorname{Tr}\Big(\overline{\chi_{A,i}}\, \tau_\mu \overleftrightarrow{D}_\mu \chi_{A,i} + \overline{\chi_{B,i}}\, \tau_\mu \overleftrightarrow{D}_\mu \chi_{B,i} + \overline{\psi_i}\, \tau_\mu \overleftrightarrow{D}_\mu \psi_i + \psi_{\Phi,i}\, \tau_\mu \overleftrightarrow{D}_\mu \overline{\psi_{\Phi,i}}\Big) \\
&+ \frac{g}{\sqrt{2}} \operatorname{Tr}\Big(\overline{\chi_{A,i}} A_{i,(i+1)} \psi_{\Phi,(i+1)} - \overline{\chi_{A,i}} \psi_{\Phi,i} A_{i,(i+1)} - \overline{\chi_{A,i}}\, \overline{B_{(i+1),i}}\, \overline{\psi_{i+1}} + \overline{\chi_{A,i}}\, \overline{\psi_i}\, \overline{B_{(i+1),i}} \\
&\quad + \overline{\chi_{B,i}}\, \overline{A_{i,(i+1)}}\, \overline{\psi_i} - \overline{\chi_{B,i}}\, \overline{\psi_{i+1}}\, \overline{A_{i,(i+1)}} + \overline{\chi_{B,i}} B_{(i+1),i} \psi_{\Phi,i} - \overline{\chi_{B,i}} \psi_{\Phi,(i+1)} B_{(i+1),i} \\
&\quad - \overline{\psi_{i+1}}\, \overline{A_{i,(i+1)}}\, \overline{\chi_{B,i}} + \overline{\psi_i}\, \overline{\chi_{B,i}}\, \overline{A_{i,(i+1)}} + \overline{\psi_i}\, \overline{B_{(i+1),i}}\, \overline{\chi_{A,i}} - \overline{\psi_{i+1}}\, \overline{\chi_{A,i}}\, \overline{B_{(i+1),i}} \\
&\quad - \psi_{\Phi,i} A_{i,(i+1)} \overline{\chi_{A,i}} + \psi_{\Phi,(i+1)} \overline{\chi_{A,i}} A_{i,(i+1)} - \psi_{\Phi,(i+1)} B_{(i+1),i} \overline{\chi_{B,i}} + \psi_{\Phi,i} \overline{\chi_{B,i}} B_{(i+1),i} \\
&\quad + \chi_{A,i} \overline{A_{i,(i+1)}}\, \overline{\psi_{\Phi,i}} - \chi_{A,i} \overline{\psi_{\Phi,(i+1)}}\, \overline{A_{i,(i+1)}} - \chi_{A,i} B_{(i+1),i} \psi_i + \chi_{A,i} \psi_{i+1} B_{(i+1),i} \\
&\quad + \chi_{B,i} A_{i,(i+1)} \psi_{i+1} - \chi_{B,i} \psi_i A_{i,(i+1)} + \chi_{B,i} \overline{B_{(i+1),i}}\, \overline{\psi_{\Phi,(i+1)}} - \chi_{B,i} \overline{\psi_{\Phi,i}}\, \overline{B_{(i+1),i}} \\
&\quad - \psi_i A_{i,(i+1)} \chi_{B,i} + \psi_{i+1} \chi_{B,i} A_{i,(i+1)} + \psi_{i+1} B_{(i+1),i} \chi_{A,i} - \psi_i \chi_{A,i} B_{(i+1),i} \\
&\quad - \overline{\psi_{\Phi,(i+1)}}\, \overline{A_{i,(i+1)}} \chi_{A,i} + \overline{\psi_{\Phi,i}} \chi_{A,i} \overline{A_{i,(i+1)}} - \overline{\psi_{\Phi,i}}\, \overline{B_{(i+1),i}} \chi_{B,i} + \overline{\psi_{\Phi,(i+1)}} \chi_{B,i} \overline{B_{(i+1),i}} \\
&\quad + \overline{\chi_{A,i}}\, \overline{\Phi_i}\, \overline{\chi_{B,i}} - \overline{\chi_{A,i}}\, \overline{\chi_{B,i}}\, \overline{\Phi_{i+1}} - \overline{\chi_{B,i}}\, \overline{\Phi_{i+1}}\, \overline{\chi_{A,i}} + \overline{\chi_{B,i}}\, \overline{\chi_{A,i}}\, \overline{\Phi_i} \\
&\quad + \chi_{A,i} \Phi_{i+1} \chi_{B,i} - \chi_{A,i} \chi_{B,i} \Phi_i - \chi_{B,i} \Phi_i \chi_{A,i} + \chi_{B,i} \chi_{A,i} \Phi_{i+1} \\
&\quad + \overline{\psi_i} \left[\Phi_i, \psi_{\Phi,i}\right] - \psi_{\Phi,i} \left[\Phi_i, \overline{\psi_i}\right] + \psi_i \left[\overline{\Phi_i}, \overline{\psi_{\Phi,i}}\right] - \overline{\psi_{\Phi,i}} \left[\overline{\Phi_i}, \psi_i\right]\Big).
\end{aligned} \tag{1.2.39}$$

We conclude that the Lagrangian density of $\mathcal{N}=2$ $U(N)^M$ quiver gauge theory is

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{ferm}} \tag{1.2.40}$$

where $\mathcal{L}_{\text{scalar}}$, $\mathcal{L}_{\text{gauge}}$ and $\mathcal{L}_{\text{ferm}}$ are given in Eqs. (1.2.17), (1.2.18) and (1.2.39), respectively.

1.3 R-symmetry

The Lagrangian density of $\mathcal{N} = 2$ quiver gauge theory (given in Eqs. (1.2.40), (1.2.17), (1.2.18) and (1.2.39)) is invariant under global $SU(2)_R \times U(1)_R$ transformations. The $U(1)_R$ factor of the R-symmetry group acts on the fields as

$$A_{i,(i+1)} \longrightarrow A_{i,(i+1)} \ , \qquad B_{(i+1),i} \longrightarrow B_{(i+1),i} \ , \qquad \Phi_i \longrightarrow e^{i\zeta}\Phi_i \tag{1.3.1}$$

$$\chi_{A,i} \longrightarrow e^{-i\zeta/2}\chi_{A,i} \ , \qquad \chi_{B,i} \longrightarrow e^{-i\zeta/2}\chi_{B,i} \tag{1.3.2}$$

$$\psi_i \longrightarrow e^{i\zeta/2}\psi_i \ , \qquad \psi_{\Phi,i} \longrightarrow e^{-i\zeta/2}\psi_{\Phi,i} \ . \tag{1.3.3}$$

The $U(1)_R$ transformations of the Hermitian conjugate fields are obtained by flipping $\zeta \to -\zeta$. The Lagrangian density is manifestly invariant under the $U(1)_R$ transformation.

We now move to consider the $SU(2)_R$ transformations. Define the 2-component spinors

$$(\lambda_i)_a \equiv \begin{pmatrix} A_{i,(i+1)} \\ B_{(i+1),i} \end{pmatrix} , \qquad (\overline{\lambda_i})^a \equiv \begin{pmatrix} \overline{A_{i,(i+1)}} \\ \overline{B_{(i+1),i}} \end{pmatrix} . \tag{1.3.4}$$

Under $\sigma \in SU(2)_R$ these spinors have the transformations

$$(\lambda_i)_a \longrightarrow \sigma_a{}^b\,(\lambda_i)_b \tag{1.3.5}$$

$$(\overline{\lambda_i})^a \longrightarrow (\overline{\lambda_i})^b\,\overline{\sigma}_b{}^a \ . \tag{1.3.6}$$

Note that $(\overline{\lambda_i})_a = \epsilon_{ab}(\overline{\lambda_i})^b$ has the transformation

$$(\overline{\lambda_i})_a \longrightarrow \epsilon_{ab}\,\overline{\sigma}_c{}^b\,\epsilon^{dc}\,(\overline{\lambda_i})_d = \sigma_a{}^d(\overline{\lambda_i})_d \tag{1.3.7}$$

where the equality follows by using $\sigma \in SU(2)_R$. Thus, $(\lambda_i)_a$ and $(\overline{\lambda_i})_a$ are $SU(2)_R$ doublets. To exhibit the $SU(2)_R$ invariance of the Lagrangian density we define $SU(2)_R$ invariants such as

$$(\lambda_i)_a\,(\overline{\lambda_i})^a = -\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b = -A_{i,(i+1)}\overline{A_{i,(i+1)}} - \overline{B_{(i+1),i}}B_{(i+1),i} \tag{1.3.8}$$

and write the Lagrangian density in terms of these. For $\mathcal{N} = 2$ quiver gauge theory the bifundamental scalars and the adjoint fermions are organized into $SU(2)_R$ doublets as follows

$$(\lambda_i)_a \equiv \begin{pmatrix} A_{i,(i+1)} \\ B_{(i+1),i} \end{pmatrix} , \qquad (\overline{\lambda_i})_a \equiv \begin{pmatrix} -\overline{B_{(i+1),i}} \\ \overline{A_{i,(i+1)}} \end{pmatrix} \tag{1.3.9}$$

$$(\chi_i)_a \equiv \begin{pmatrix} \overline{\psi_i} \\ \psi_{\Phi,i} \end{pmatrix} , \qquad (\overline{\chi_i})_a \equiv \begin{pmatrix} -\overline{\psi_{\Phi,i}} \\ \psi_i \end{pmatrix} . \tag{1.3.10}$$

The scalar field Lagrangian density written in terms of the $SU(2)_R$ doublets takes the following form[8]

[8] Note that the term $R^{-2}\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b + \Phi_i\overline{\Phi_i}\right)$ describing the conformal coupling of the scalar fields to the curvature has been omitted here.

$$
\begin{aligned}
\mathcal{L}_{\rm scalar} \;=\; \sum_{i=1}^{M} \Big[& \mathrm{Tr}\left(\epsilon^{ab}(D_\mu\lambda_i)_a(D_\mu\overline{\lambda_i})_b + D_\mu\Phi_i D_\mu\overline{\Phi_i}\right) \\
& + \frac{1}{2}g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b - \epsilon^{ab}(\overline{\lambda_{i-1}})_a(\lambda_{i-1})_b + [\Phi_i, \overline{\Phi_i}]\right)^2 \\
& - 2g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b - \epsilon^{ab}(\overline{\lambda_{i-1}})_a(\lambda_{i-1})_b\right)^2 \\
& - 2g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\epsilon^{cd}(\overline{\lambda_{i-1}})_c(\lambda_{i-1})_d + \epsilon^{ab}(\lambda_{i-1})_a(\lambda_i)_b\epsilon^{cd}(\overline{\lambda_i})_c(\overline{\lambda_{i-1}})_d\right) \\
& + 2g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\overline{\lambda_i})_a(\lambda_i)_b\epsilon^{cd}(\overline{\lambda_i})_c(\lambda_i)_d + \epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\epsilon^{cd}(\lambda_i)_c(\overline{\lambda_i})_d\right) \\
& - 2g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a\Phi_{i+1}(\overline{\lambda_i})_b\overline{\Phi_i} + \epsilon^{ab}(\lambda_i)_a\overline{\Phi_{i+1}}(\overline{\lambda_i})_b\Phi_i\right) \\
& + 2g^2\,\mathrm{Tr}\left(\epsilon^{ab}(\lambda_i)_a\overline{\Phi_{i+1}}\Phi_{i+1}(\overline{\lambda_i})_b + \epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\overline{\Phi_i}\Phi_i\right)\Big]\,. \qquad (1.3.11)
\end{aligned}
$$

The spinor field Lagrangian density written in terms of the $SU(2)_R$ doublets takes the following form

$$
\begin{aligned}
\mathcal{L}_{\rm ferm} \;=\; \sum_{i=1}^{M} \Big[& i\,\mathrm{Tr}\left(\overline{\chi_{A,i}}\,\tau_\mu\overleftrightarrow{D}_\mu\chi_{A,i} + \overline{\chi_{B,i}}\,\tau_\mu\overleftrightarrow{D}_\mu\chi_{B,i} + \epsilon^{cd}(\chi_i)_c(\tau_\mu\overleftrightarrow{D}_\mu\overline{\chi_i})_d\right) \\
& + \frac{g}{\sqrt{2}}\,\mathrm{Tr}\Big(\epsilon^{cd}\big\{\chi_{A,i}(\overline{\lambda_i})_c,\,(\overline{\chi_i})_d\big\} + \epsilon^{cd}\big\{\chi_{A,i},\,(\overline{\chi_{i+1}})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\overline{\chi_{A,i}}\,(\lambda_i)_c,\,(\chi_{i+1})_d\big\} + \epsilon^{cd}\big\{\overline{\chi_{A,i}},\,(\chi_i)_c(\lambda_i)_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\chi_{B,i}(\lambda_i)_c,\,(\overline{\chi_{i+1}})_d\big\} + \epsilon^{cd}\big\{\chi_{B,i},\,(\overline{\chi_i})_c(\lambda_i)_d\big\} \\
& \qquad - \epsilon^{cd}\big\{\overline{\chi_{B,i}}\,(\overline{\lambda_i})_c,\,(\chi_i)_d\big\} - \epsilon^{cd}\big\{\overline{\chi_{B,i}},\,(\chi_{i+1})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{(\chi_i)_c\Phi_i,\,(\chi_i)_d\big\} + \epsilon^{cd}\big\{(\overline{\chi_i})_c\overline{\Phi_i},\,(\overline{\chi_i})_d\big\} \\
& \qquad + \big\{\chi_{A,i}\Phi_{i+1},\,\chi_{B,i}\big\} + \big\{\overline{\chi_{A,i}}\,\overline{\Phi_i},\,\overline{\chi_{B,i}}\big\} \\
& \qquad - \big\{\chi_{B,i}\Phi_i,\,\chi_{A,i}\big\} - \big\{\overline{\chi_{B,i}}\,\overline{\Phi_{i+1}},\,\overline{\chi_{A,i}}\big\}\Big)\Big]\,. \qquad (1.3.12)
\end{aligned}
$$

These results are conveniently summarized in Table 1 which lists the R-charges of all the fields in $\mathcal{N}=2$ quiver gauge theory. Here the generators of $\mathfrak{su}(2)_R$ in the fundamental representation are chosen as $\frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$. The R-charges of the corresponding Hermitian conjugate fields are obtained by simply changing the signs of the $U(1)_R$ and $SU(2)_R$ charges.

	$A_{i,(i+1)}$	$B_{(i+1),i}$	Φ_i	$A_{\mu i}$	$\chi_{A,i}$	$\chi_{B,i}$	$\psi_{\Phi,i}$	ψ_i
$U(1)_R$	0	0	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$SU(2)_R$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 1. *R-charges for the bosonic and fermionic fields*

We note that the R-charges of the scalar fields are consistent with the geometric intuition that Φ is associated with the z_1 direction of $\mathbb{C}^3$, which is inert under the action of the orbifold group $\mathbb{Z}_M$, and Φ should thus not be mixed under R-symmetry transformations with the fields A and B associated with the z_2 and z_3 directions in which $\mathbb{Z}_M$ acts. Therefore Φ should have zero $SU(2)_R$ charge, and A, B should have zero $U(1)_R$ charge.

1.4 R-symmetry chemical potentials

In this section we argue why, given a non-Abelian symmetry group, one should introduce chemical potentials conjugate to a maximal Abelian subgroup rather than the entire group itself. Consider a quantum mechanical system described by a Hamiltonian H with a symmetry group G that is assumed to be a compact and semisimple Lie group. Let $U(g)$ denote a unitary representation of a group element g. We define a generalized partition function by the expression

$$Z(\beta, g) \equiv \mathrm{Tr}\left(U(g)\, e^{-\beta H}\right) . \tag{1.4.1}$$

By assumption $U(g)$ commutes with H, and consequently $Z(\beta, g)$ is a class function,

$$Z(\beta, \eta^{-1} g \eta) = \mathrm{Tr}\left(U(g)\, U(\eta)\, e^{-\beta H}\, U(\eta^{-1})\right) = \mathrm{Tr}\left(U(g)\, e^{-\beta H}\right) = Z(\beta, g) . \tag{1.4.2}$$

Since G is compact and semisimple, there exists a maximal torus T of G. In particular, T is a compact and connected maximal Abelian subgroup with the property that any element g of G can be written in the form $g = \eta^{-1} t\, \eta$ with t an element of T. As a further property, using that T is compact and connected, any element t of T can be written as the exponential of some element of a corresponding Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Letting $\{Q_a \mid a = 1, \ldots, \mathrm{rank}\, G\}$ denote a complete set of generators[9] for $\mathfrak{t}$ we can thus write $t = e^{i\gamma_a Q_a}$ where γ_a are real numbers. Therefore $Z(\beta, g)$ can be expressed as a function of $\mathrm{rank}\, G$ real numbers γ_a and rewritten as

$$Z(\beta, \gamma_a) = \mathrm{Tr}\, e^{-\beta H + i\gamma_a Q_a} . \tag{1.4.3}$$

After analytically continuing $\gamma_a \to -i\beta\mu_a$ this takes the form

$$Z(\beta, \mu_a) = \mathrm{Tr}\, e^{-\beta(H - \mu_a Q_a)} . \tag{1.4.4}$$

We recognize this as the grand canonical partition function. The chemical potentials μ_a are conjugate to a maximal set of commuting conserved charges Q_a. This demonstrates why, given a non-Abelian symmetry group, one should introduce chemical potentials conjugate to a maximal torus of the group rather than the entire group itself.

[9] We adopt the convention that the generators Q_a are Hermitian.

1.5 Lagrangian density with R-symmetry chemical potentials

The maximal torus[10] of the R-symmetry group $SU(2)_R \times U(1)_R$ of $\mathcal{N}=2$ quiver gauge theory is $U(1)\times U(1)$. For the $U(1)$ corresponding to $U(1)_R$ the eigenvalues of the Cartan generators can directly be read off from Table 1. For the $U(1)\subset SU(2)_R$ we choose as a basis for the Cartan subalgebra the diagonal generator σ_z so that the $SU(2)_R$ doublets will have well-defined charges under $U(1)$. (We choose σ_z rather than $\frac{1}{2}\sigma_z$ as the generator Q_2 because we require $e^{iQ_2\theta}$ to be invariant under $\theta \to \theta + 2\pi$. Setting $Q_2 \equiv \sigma_z$ we have $e^{iQ_2\theta} = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ which is clearly invariant.) Therefore the charges under the maximal torus $U(1)$ of $SU(2)_R$ will be 2 times the $SU(2)_R$ charges.

Thus for the bosonic fields,

$$(\mu_a Q_a)\, A_{i,(i+1)} = \mu_2\, A_{i,(i+1)} \tag{1.5.1}$$
$$(\mu_a Q_a)\, B_{(i+1),i} = \mu_2\, B_{(i+1),i} \tag{1.5.2}$$
$$(\mu_a Q_a)\, \Phi_i = \mu_1 \Phi_i \tag{1.5.3}$$
$$(\mu_a Q_a)\, A_{\mu i} = 0\,, \tag{1.5.4}$$

and for the fermionic fields,

$$(\mu_a Q_a)\, \chi_{A,i} = -\tfrac{1}{2}\mu_1\, \chi_{A,i} \tag{1.5.5}$$
$$(\mu_a Q_a)\, \chi_{B,i} = -\tfrac{1}{2}\mu_1\, \chi_{B,i} \tag{1.5.6}$$
$$(\mu_a Q_a)\, \psi_i = \left(\tfrac{1}{2}\mu_1 - \mu_2\right) \psi_i \tag{1.5.7}$$
$$(\mu_a Q_a)\, \psi_{\Phi,i} = \left(-\tfrac{1}{2}\mu_1 - \mu_2\right) \psi_{\Phi,i}\,. \tag{1.5.8}$$

The corresponding expressions for the Hermitian conjugate fields are obtained by simply changing the signs of the chemical potentials.

To obtain the Lagrangian density of $\mathcal{N}=2$ quiver gauge theory with chemical potentials for the $SU(2)_R \times U(1)_R$ Cartan generators, we imagine gauging the global $SU(2)_R \times U(1)_R$ symmetry. This will introduce a fictitious gauge field. The time component A^R_{a0} of the fictitious gauge field will couple to the conserved $U(1)\times U(1)$ charge densities j^R_{a0} through the standard coupling term $A^R_{a\mu} j^R_{a\mu}$ giving rise to a term $ig\left(Q_a \frac{i\mu_a \delta_{\mu 0}}{g}\right)$ in the covariant derivative.[11] Thus, introducing R-symmetry chemical potentials is equivalent to coupling the R-charged fields to a fictitious constant gauge field $\frac{i\mu_a\delta_{\mu 0}}{g}$. In conclusion, the R-symmetry chemical potentials μ_a are introduced into the Lagrangian density by making

[10] Maximal tori are unique only up to conjugation, but the generators of Cartan algebras will appear inside traces, and by cyclicity of the trace such expressions are invariant under conjugation. We will therefore treat the maximal tori as identical.

[11] Since we are only interested in coupling the gauge field to the R-charge density, we set the spatial components to zero, i.e. $A^R_{a\mu} = \frac{i\mu_a\delta_{\mu 0}}{g}$. The factor i is natural since the appropriate Minkowskian gauge field is $\frac{\mu_a\delta_{\mu 0}}{g}$, and the corresponding Euclidean gauge field is obtained by multiplying a factor of i. We include the coupling constant in the denominator since we want to keep the term when taking the $g \to 0$ limit.

the substitution

$$D_\mu \longrightarrow D_\mu - \mu_a Q_a \delta_{\mu 0} \,. \tag{1.5.9}$$

Below we have written the Lagrangian densities for the fundamental scalar and spinor fields of $\mathcal{N}=2$ quiver gauge theory. This will be important for the analysis in the following sections in order to distinguish the adjoint from the bifundamental structures.

The Lagrangian density for the scalar fields with R-symmetry chemical potentials is

$$\begin{aligned}
\mathcal{L}_{\text{scalar}} = \sum_{i=1}^{M} \Bigg\{ & \operatorname{Tr}\Big[\Big(\partial_\mu A_{i,(i+1)} + igA_{\mu i}A_{i,(i+1)} - igA_{i,(i+1)}A_{\mu(i+1)} - \mu_2\delta_{\mu 0}A_{i,(i+1)}\Big) \\
& \qquad \times \Big(\partial_\mu \overline{A_{i,(i+1)}} + igA_{\mu(i+1)}\overline{A_{i,(i+1)}} - ig\overline{A_{i,(i+1)}}A_{\mu i} + \mu_2\delta_{\mu 0}\overline{A_{i,(i+1)}}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu B_{(i+1),i} + igA_{\mu(i+1)}B_{(i+1),i} - igB_{(i+1),i}A_{\mu i} - \mu_2\delta_{\mu 0}B_{(i+1),i}\Big) \\
& \qquad \times \Big(\partial_\mu \overline{B_{(i+1),i}} + igA_{\mu i}\overline{B_{(i+1),i}} - ig\overline{B_{(i+1),i}}A_{\mu(i+1)} + \mu_2\delta_{\mu 0}\overline{B_{(i+1),i}}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu\Phi_i + ig[A_{\mu i},\Phi_i] - \mu_1\delta_{\mu 0}\Phi_i\Big)\Big(\partial_\mu\overline{\Phi_i} + ig[A_{\mu i},\overline{\Phi_i}] + \mu_1\delta_{\mu 0}\overline{\Phi_i}\Big)\Big] \\
& + R^{-2}\operatorname{Tr}\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} + \overline{B_{(i+1),i}}B_{(i+1),i} + \Phi_i\overline{\Phi_i}\Big) \\
& + \frac{1}{2}g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}}A_{(i-1),i} \\
& \qquad + B_{i,(i-1)}\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}B_{(i+1),i} + [\Phi_i,\overline{\Phi_i}]\Big)^2\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}B_{(i+1),i} - B_{i,(i-1)}A_{(i-1),i}\Big) \\
& \qquad \times \Big(\overline{A_{(i-1),i}}\;\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\;\overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\Phi_{i+1} - \Phi_iA_{i,(i+1)}\Big)\Big(\overline{A_{i,(i+1)}}\;\overline{\Phi_i} - \overline{\Phi_{i+1}}\;\overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(B_{(i+1),i}\Phi_i - \Phi_{i+1}B_{(i+1),i}\Big)\Big(\overline{B_{(i+1),i}}\;\overline{\Phi_{i+1}} - \overline{\Phi_i}\;\overline{B_{(i+1),i}}\Big)\Big]\Bigg\}
\end{aligned} \tag{1.5.10}$$

Here the traces are always taken over the gauge indices of the $N\times N$ matrices. Observe that the chemical potentials μ_1 and μ_2 act like negative mass squares for Φ_i and $A_{i,(i+1)}, B_{(i+1),i}$. On a compact spatial manifold such as S^3, these terms are balanced by the positive mass square terms induced by the conformal coupling to curvature. We immediately observe from (1.5.10) that $\mathcal{N}=2$ quiver gauge theory on $S^1\times S^3$ is well-defined as long as $\mu_1,\mu_2 \le R^{-1}$. If the chemical potentials exceed this bound, the theory develops tachyonic modes and there exists no stable ground state.

The Lagrangian density for the spinor fields with R-symmetry chemical potentials is

$$
\begin{aligned}
\mathcal{L}_{\text{ferm}} = \sum_{i=1}^{M} \Bigg\{ & \frac{i}{2}\,\mathrm{Tr}\left(\overline{\chi_{A,i}}\,\tau_\mu\big(\partial_\mu\chi_{A,i} + igA_{\mu i}\chi_{A,i} - ig\chi_{A,i}A_{\mu(i+1)} + \tfrac{1}{2}\mu_1\delta_{\mu 0}\chi_{A,i}\big)\right) \\
& -\frac{i}{2}\,\mathrm{Tr}\left(\big(\partial_\mu\overline{\chi_{A,i}} + igA_{\mu(i+1)}\overline{\chi_{A,i}} - ig\overline{\chi_{A,i}}A_{\mu i} - \tfrac{1}{2}\mu_1\delta_{\mu 0}\overline{\chi_{A,i}}\big)\,\tau_\mu\,\chi_{A,i}\right) \\
& +\frac{i}{2}\,\mathrm{Tr}\left(\overline{\chi_{B,i}}\,\tau_\mu\big(\partial_\mu\chi_{B,i} + igA_{\mu(i+1)}\chi_{B,i} - ig\chi_{B,i}A_{\mu i} + \tfrac{1}{2}\mu_1\delta_{\mu 0}\chi_{B,i}\big)\right) \\
& -\frac{i}{2}\,\mathrm{Tr}\left(\big(\partial_\mu\overline{\chi_{B,i}} + igA_{\mu i}\overline{\chi_{B,i}} - ig\overline{\chi_{B,i}}A_{\mu(i+1)} - \tfrac{1}{2}\mu_1\delta_{\mu 0}\overline{\chi_{B,i}}\big)\,\tau_\mu\,\chi_{B,i}\right) \\
& +\frac{i}{2}\,\mathrm{Tr}\left(\overline{\psi_i}\,\tau_\mu\big(\partial_\mu\psi_i + ig[A_{\mu i},\psi_i] - \big(\tfrac{1}{2}\mu_1 - \mu_2\big)\,\delta_{\mu 0}\psi_i\big)\right) \\
& -\frac{i}{2}\,\mathrm{Tr}\left(\big(\partial_\mu\overline{\psi_i} + ig[A_{\mu i},\overline{\psi_i}] + \big(\tfrac{1}{2}\mu_1 - \mu_2\big)\,\delta_{\mu 0}\overline{\psi_i}\big)\,\tau_\mu\,\psi_i\right) \\
& +\frac{i}{2}\,\mathrm{Tr}\left(\psi_{\Phi,i}\,\tau_\mu\big(\partial_\mu\overline{\psi_{\Phi,i}} + ig[A_{\mu i},\overline{\psi_{\Phi,i}}] - \big(\tfrac{1}{2}\mu_1 + \mu_2\big)\,\delta_{\mu 0}\overline{\psi_{\Phi,i}}\big)\right) \\
& -\frac{i}{2}\,\mathrm{Tr}\left(\big(\partial_\mu\psi_{\Phi,i} + ig[A_{\mu i},\psi_{\Phi,i}] + \big(\tfrac{1}{2}\mu_1 + \mu_2\big)\,\delta_{\mu 0}\psi_{\Phi,i}\big)\,\tau_\mu\,\overline{\psi_{\Phi,i}}\right) \\
& +\frac{g}{\sqrt{2}}\,\mathrm{Tr}\Big(\epsilon^{cd}\big\{\chi_{A,i}(\overline{\lambda_i})_c,\,(\overline{\chi_i})_d\big\} + \epsilon^{cd}\big\{\chi_{A,i},\,(\overline{\chi_{i+1}})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\overline{\chi_{A,i}}(\lambda_i)_c,\,(\chi_{i+1})_d\big\} + \epsilon^{cd}\big\{\overline{\chi_{A,i}},\,(\chi_i)_c(\lambda_i)_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\chi_{B,i}(\lambda_i)_c,\,(\overline{\chi_{i+1}})_d\big\} + \epsilon^{cd}\big\{\chi_{B,i},\,(\overline{\chi_i})_c(\lambda_i)_d\big\} \\
& \qquad - \epsilon^{cd}\big\{\overline{\chi_{B,i}}(\overline{\lambda_i})_c,\,(\chi_i)_d\big\} - \epsilon^{cd}\big\{\overline{\chi_{B,i}},\,(\chi_{i+1})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{(\chi_i)_c\Phi_i,\,(\chi_i)_d\big\} + \epsilon^{cd}\big\{(\overline{\chi_i})_c\overline{\Phi_i},\,(\overline{\chi_i})_d\big\} \\
& \qquad + \big\{\chi_{A,i}\Phi_{i+1},\,\chi_{B,i}\big\} + \big\{\overline{\chi_{A,i}}\,\overline{\Phi_i},\,\overline{\chi_{B,i}}\big\} \\
& \qquad - \big\{\chi_{B,i}\Phi_i,\,\chi_{A,i}\big\} - \big\{\overline{\chi_{B,i}}\,\overline{\Phi_{i+1}},\,\overline{\chi_{A,i}}\big\}\Big)\Bigg\}\,. \qquad (1.5.11)
\end{aligned}
$$

Here the traces are always taken over the gauge indices of the $N \times N$ matrices. Note that the potential part of the Lagrangian density has been written in terms of the $SU(2)_R$ doublets given in Eqs. (1.3.9)-(1.3.10) for notational simplicity.

To recapitulate, (1.5.10) and (1.5.11) are obtained by making the substitution (1.5.9) in the Lagrangian density given in (1.1.3), (1.1.4) and (1.1.5), and finally making the substitutions given in (1.2.15)-(1.2.16) and (1.2.37)-(1.2.38).

Finally, as the gauge fields have zero charge under $SU(2)_R \times U(1)_R$, the gauge field part of the Lagrangian density is unaffected by introducing the R-symmetry chemical potentials. Nonetheless, we give the result here for convenience:

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4}\sum_{i=1}^{M} \mathrm{Tr}\, F^i_{\mu\nu}F^i_{\mu\nu} \qquad (1.5.12)$$

where of course $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig[A^i_\mu, A^i_\nu]$.

Chapter 2

Matrix model and phase structure

The matrix model we will consider is defined by integrating out the fluctuations of the quantum fields. Therefore we will compute in this section the one-loop quantum effective action.

2.1 One-loop quantum effective action

Mention (for thermal field theory): [32], [33]

The partition function for the grand canonical ensemble has the path integral representation

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\phi \, \mathcal{D}\psi \, e^{-\int_{S^1\times S^3} d^4x \, \sqrt{|g|} \, (\mathcal{L}_{\text{gauge}}+\mathcal{L}_{\text{scalar}}+\mathcal{L}_{\text{ferm}})} \tag{2.1.1}$$

with $\mathcal{L}_{\text{gauge}}, \mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{ferm}}$ being the Lagrangian densities with R-symmetry chemical potentials given by Eqs. (1.5.12), (1.5.10) and (1.5.11), respectively, and where the measures $\mathcal{D}A_\mu$, $\mathcal{D}\phi$ and $\mathcal{D}\psi$ are the products of the measures over all the gauge fields, scalar fields and spinor fields, respectively. We will obtain an effective action from this expression by taking the free limit $g \to 0$ of the tree-level action. However, since the theory is defined on a compact spatial S^3 one must impose the Gauss law constraint that all states be gauge invariant. We perform the projection onto gauge invariant states by using A_{0i} as a Lagrange multiplier,

$$A_{\mu i}(x) \longrightarrow \widetilde{A}_{\mu i}(x) + \delta_{\mu 0} a_i / g \tag{2.1.2}$$

where $\widetilde{A}_{0i}$ integrates to zero over $S^1 \times S^3$ and a_i are constant Hermitian matrices which by gauge invariance can be assumed diagonal, $a_i = \text{diag}(q_i^1, \ldots, q_i^N)$. To obtain the correct zero coupling limit one inserts the decomposition (2.1.2) into the action given through (1.5.10)- (1.5.12) and then takes the $g \to 0$ limit.

We will write up the resulting action in a bilinear form that is amenable to Gaussian path integrations. For this purpose, define (m, n and i are fixed)

$$\Box^{mn}_{A_{i,(i+1)}} = \big(\partial_\mu + \delta_{\mu 0}(\mu_2 + iq^n_{i+1} - iq^m_i)\big)^2 - R^{-2} \tag{2.1.3}$$

$$\Box^{mn}_{B_{(i+1),i}} = \big(\partial_\mu + \delta_{\mu 0}(\mu_2 + iq^n_i - iq^m_{i+1})\big)^2 - R^{-2} \tag{2.1.4}$$

$$\Box^{mn}_{\Phi_i} = \big(\partial_\mu + \delta_{\mu 0}(\mu_1 - iq^{mn}_i)\big)^2 - R^{-2} \tag{2.1.5}$$

where $q^{mn}_i \equiv q^m_i - q^n_i$. Then, by partial integration and discarding total derivatives, the zero coupling limit of the scalar field action (given through (1.5.10)) can be written

$$\begin{aligned}\lim_{g\to 0} S_{\text{scalar}} = \sum_{i=1}^{M}\sum_{m,n=1}^{N}\int_{S^1\times S^3} d^4x\,\sqrt{|g|}\,\Big(&(A_{i,(i+1)})_{mn}(-\Box^{mn}_{A_{i,(i+1)}})(\overline{A_{i,(i+1)}})_{nm}\\ &+(B_{(i+1),i})_{mn}(-\Box^{mn}_{B_{(i+1),i}})(\overline{B_{(i+1),i}})_{nm} + (\Phi_i)_{mn}(-\Box^{mn}_{\Phi_i})(\overline{\Phi_i})_{nm}\Big)\,. \end{aligned}\tag{2.1.6}$$

The path integrals over the scalar fields are now Gaussian integrals and can readily be done. For the scalar fields $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$ we have the respective contributions

$$\ln Z_{A_{i,(i+1)}} = -\,\text{Tr}\,\ln\big(-\Box_{A_{i,(i+1)}}\big) \tag{2.1.7}$$

$$\ln Z_{B_{(i+1),i}} = -\,\text{Tr}\,\ln\big(-\Box_{B_{(i+1),i}}\big) \tag{2.1.8}$$

$$\ln Z_{\Phi_i} = -\,\text{Tr}\,\ln\big(-\Box_{\Phi_i}\big)\,. \tag{2.1.9}$$

Here the traces are taken both over the gauge indices and over the Hilbert spaces of the scalar fields $A_{i,(i+1)}, B_{(i+1),i}$ and Φ_i, respectively.

In order to put the spinor field part of the Lagrangian density in a bilinear form we exploit the fact that these fields are Grassmann-values. Define

$$\mathbf{D}^{mn}_{\chi_{A,i}} = \begin{pmatrix}\big(i\partial_\mu - \delta_{\mu0}(q^n_{i+1}-q^m_i) - \frac{i}{2}\delta_{\mu0}\mu_1\big)\tau^{\text{T}}_\mu & 0\\ 0 & \tau_\mu\big(i\partial_\mu - \delta_{\mu0}(q^n_i - q^m_{i+1}) + \frac{i}{2}\delta_{\mu0}\mu_1\big)\end{pmatrix} \tag{2.1.10}$$

$$\mathbf{D}^{mn}_{\chi_{B,i}} = \begin{pmatrix}\big(i\partial_\mu - \delta_{\mu0}(q^n_i-q^m_{i+1}) - \frac{i}{2}\delta_{\mu0}\mu_1\big)\tau^{\text{T}}_\mu & 0\\ 0 & \tau_\mu\big(i\partial_\mu - \delta_{\mu0}(q^n_{i+1} - q^m_i) + \frac{i}{2}\delta_{\mu0}\mu_1\big)\end{pmatrix} \tag{2.1.11}$$

$$\mathbf{D}^{mn}_{\psi_i} = \begin{pmatrix}\big(i\partial_\mu - \delta_{\mu0}q^{nm}_i + i\delta_{\mu0}(\frac{1}{2}\mu_1-\mu_2)\big)\tau^{\text{T}}_\mu & 0\\ 0 & \tau_\mu\big(i\partial_\mu - \delta_{\mu0}q^{nm}_i - i\delta_{\mu0}(\frac{1}{2}\mu_1-\mu_2)\big)\end{pmatrix} \tag{2.1.12}$$

$$\mathbf{D}^{mn}_{\psi_{\Phi,i}} = \begin{pmatrix}\big(i\partial_\mu - \delta_{\mu0}q^{nm}_i + i\delta_{\mu0}(\frac{1}{2}\mu_1+\mu_2)\big)\tau^{\text{T}}_\mu & 0\\ 0 & \tau_\mu\big(i\partial_\mu - \delta_{\mu0}q^{nm}_i - i\delta_{\mu0}(\frac{1}{2}\mu_1+\mu_2)\big)\end{pmatrix} \tag{2.1.13}$$

where the superscript $^{\text{T}}$ denotes the transpose. Then the zero coupling limit of the spinor field Lagrangian density (1.5.11) can be written

$$\begin{aligned}
\lim_{g\to 0}\mathcal{L}_{\mathrm{ferm}} &= \sum_{i=1}^{M}\Bigg\{\sum_{m,n=1}^{N}\frac{1}{2}\big((\chi_{A,i})_{mn},(\overline{\chi_{A,i}})_{mn}\big)\,\mathbf{D}^{mn}_{\chi_{A,i}}\begin{pmatrix}(\overline{\chi_{A,i}})_{nm}\\(\chi_{A,i})_{nm}\end{pmatrix}\\
&\quad+\frac{1}{2}\big((\chi_{B,i})_{mn},(\overline{\chi_{B,i}})_{mn}\big)\,\mathbf{D}^{mn}_{\chi_{B,i}}\begin{pmatrix}(\overline{\chi_{B,i}})_{nm}\\(\chi_{B,i})_{nm}\end{pmatrix}\\
&\quad+\frac{1}{2}\big((\psi_i)_{mn},(\overline{\psi_i})_{mn}\big)\,\mathbf{D}^{mn}_{\psi_i}\begin{pmatrix}(\overline{\psi_i})_{nm}\\(\psi_i)_{nm}\end{pmatrix}\\
&\quad+\frac{1}{2}\big((\overline{\psi_{\Phi,i}})_{mn},(\psi_{\Phi,i})_{mn}\big)\,\mathbf{D}^{mn}_{\psi_{\Phi,i}}\begin{pmatrix}(\psi_{\Phi,i})_{nm}\\(\overline{\psi_{\Phi,i}})_{nm}\end{pmatrix}\Bigg\}\,. \qquad (2.1.14)
\end{aligned}$$

From the path integrations over the Grassmann-valued spinor fields we obtain

$$\ln Z_{\chi_{A,i}} = \tfrac{1}{2}\ln\det\left(\mathbf{D}^2_{\chi_{A,i}}\right) \qquad (2.1.15)$$
$$\ln Z_{\chi_{B,i}} = \tfrac{1}{2}\ln\det\left(\mathbf{D}^2_{\chi_{B,i}}\right) \qquad (2.1.16)$$
$$\ln Z_{\psi_i} = \tfrac{1}{2}\ln\det\left(\mathbf{D}^2_{\psi_i}\right) \qquad (2.1.17)$$
$$\ln Z_{\psi_{\Phi,i}} = \tfrac{1}{2}\ln\det\left(\mathbf{D}^2_{\psi_{\Phi,i}}\right)\,. \qquad (2.1.18)$$

Here the determinants are taken both over the gauge indices and over the Hilbert spaces of the Weyl spinor fields $\chi_{A,i}, \chi_{B,i}$, ψ_i and $\psi_{\Phi,i}$, respectively.

For the gauge field, define (i, m and n fixed)

$$\Box_i^{mn} = \left(\partial_\mu + i q_i^{mn}\delta_{\mu 0}\right)^2\,. \qquad (2.1.19)$$

Due to the decomposition in Eq. (2.1.2) we have (for a fixed i), up to total derivatives,

$$\frac{1}{4}\,\mathrm{Tr}\left(F^i_{\mu\nu}F^i_{\mu\nu}\right) = \frac{1}{2}\sum_{m,n=1}^{N}(\widetilde{A}^\mu_i)^*_{mn}(-\Box_i^{mn})(\widetilde{A}^\mu_i)_{mn} - \frac{1}{2}\,\mathrm{Tr}\left[(\partial^\mu\widetilde{A}^\mu_i + i[a_i,\widetilde{A}^0_i])^2\right] \qquad (2.1.20)$$

where the * denotes complex conjugation. We choose the gauge defined by adding the gauge-fixing term

$$\mathcal{L}_{\mathrm{g.f.}} = \frac{1}{2}\,\mathrm{Tr}\left[(\partial^\mu\widetilde{A}^\mu_i + i[a_i,\widetilde{A}^0_i])^2\right]\,. \qquad (2.1.21)$$

One then decomposes the spatial components of the gauge field into spherical harmonics on S^3 by writing them as a sum of a transverse (i.e. divergenceless) vector field $\mathbf{A}_i^\perp$ and a longitudinal vector field ∇F_i where F_i is a scalar function. That is, for $k = 1, 2, 3$ we decompose

$$\widetilde{A}^k_i = (A_i^\perp)^k + (\nabla F_i)^k\,. \qquad (2.1.22)$$

Using that $\mathbf{A}_i^{\perp}$ is divergenceless and ignoring total derivatives, the gauge fixed Lagrangian density written in terms of S^3 spherical harmonics takes the form

$$\begin{aligned}\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{g.f.}} &= \frac{1}{2} \sum_{m,n=1}^{N} \Big((\widetilde{A}_i^0)^*_{mn} \,(-\Box_i^{mn})\, (\widetilde{A}_i^0)_{mn} + (A_i^{\perp})^{k*}_{mn} \,(-\Box_i^{mn})\, (A_i^{\perp})^{k}_{mn} \\ &\qquad + (\nabla F_i)^{k*}_{mn} \,(-\Box_i^{mn})\, (\nabla F_i)^{k}_{mn} \Big) \,. \end{aligned} \tag{2.1.23}$$

Thus, the contribution to the path integral from the gauge field $\widetilde{A}_i^{\mu}$ and the Fadeev-Popov ghosts $\overline{c_i}, c_i$ is

$$\begin{aligned} Z_{\text{gauge}} &= \int \mathcal{D}A_i^{\perp k} \exp\left(-\frac{1}{2} \int d^4x \; (A_i^{\perp})^{k*}_{mn} \,(-\Box_i^{mn})\, (A_i^{\perp})^{k}_{mn} \right) \\ &\quad \times \left\{ \int \mathcal{D}(\nabla F_i)^k \;\exp\left(-\frac{1}{2} \int d^4x \; (\nabla F_i)^{k*}_{mn} \,(-\Box_i^{mn})\, (\nabla F_i)^{k}_{mn} \right) \right. \\ &\quad \cdot \int \mathcal{D}\widetilde{A}_i^0 \;\exp\left(-\frac{1}{2} \int d^4x \; (\widetilde{A}_i^0)^*_{mn} \,(-\Box_i^{mn})\, (\widetilde{A}_i^0)_{mn} \right) \\ &\quad \left. \cdot \int \mathcal{D}\overline{c_i}\, \mathcal{D}c_i \;\exp\left(- \int d^4x \; \overline{c_i} \,(-\Box_i^{mn})\, c_i \right) \right\} . \end{aligned} \tag{2.1.24}$$

The path integrals inside the brackets $\{\cdots\}$ immediately evaluate to

$$\exp\left(-\frac{1}{2} \,\text{Tr}\, \ln(-\Box_i^{mn}) \right) \exp\left(-\frac{1}{2} \,\text{Tr}\, \ln(-\Box_i^{mn}) \right) \exp\big(\text{Tr}\, \ln(-\Box_i^{mn}) \big) = 1 \,. \tag{2.1.25}$$

The last equality follows from the fact that the fields $(\nabla F_i)^k, \widetilde{A}_i^0, \overline{c_i}, c_i$ all have the same eigenvalues wrt. $\Box_i^{mn}$ and the same degeneracy corresponding to each eigenvalue. (To illustrate this, the eigenvalues are given explicitly in Eqs. (2.1.29)-(2.1.33) below. The corresponding degeneracies can be read off from Table 2.) Therefore,

$$\begin{aligned} Z_{\text{gauge}} &= \int \mathcal{D}A_i^{\perp k} \exp\left(-\frac{1}{2} \int d^4x \; (A_i^{\perp})^{k*}_{mn} \,(-\Box_i^{mn})\, (A_i^{\perp})^{k}_{mn} \right) \\ &= \exp\left(-2 \cdot \frac{1}{2}\, \text{Tr} \ln \big(-\Box_i \big) \right) \,. \end{aligned} \tag{2.1.26}$$

The factor 2 in the exponential comes from the fact that the transverse gauge field has 2 real degrees of freedom, hence there are 2 path integrations. Thus, the contribution to the quantum effective action from the gauge field $\widetilde{A}_i^{\mu}$ is

$$\ln Z_{\text{gauge}} = -\,\text{Tr} \ln \big(-\Box_i \big) \,. \tag{2.1.27}$$

Here the trace is taken both over the gauge indices and over the Hilbert space of transversal gauge fields $(A_i^{\perp})^k$.

2.1.1 Evaluating the trace over the Matsubara frequencies

Let τ denote the 0-direction. We will use the convention that any field ϕ defined on $S^1 \times S^3$ has the Fourier mode decomposition

$$\phi(\tau, \boldsymbol{x}) = \sum_{k=-\infty}^{\infty} e^{i\omega_k \tau} \phi^{[k]}(\boldsymbol{x}) \tag{2.1.28}$$

where the quantized Matsubara frequencies are $\omega_k = \frac{2\pi k}{\beta}$ for bosons and $\omega_k = \frac{(2k+1)\pi}{\beta}$ for fermions giving, respectively, periodic and antiperiodic boundary conditions around the thermal circle.[1] We now take the traces over both the gauge indices and the Matsubara frequencies in Eqs. (2.1.7)-(2.1.9), (2.1.15)-(2.1.18) and (2.1.27).

quantum field		eigenvalue	notation in text	degeneracy (D_h)
transverse vector	$\mathbf{A}^\perp$	$-(h+1)^2 R^{-2}$	$-\Delta_g^2$	$2h(h+2)$
longitudinal vector	∇F	$-h(h+2) R^{-2}$	$-\Delta_s^2$	$(h+1)^2$
real scalar	A^0, ϕ	$-h(h+2) R^{-2}$	$-\Delta_s^2$	$(h+1)^2$
Weyl spinor	ψ	$-\left(h+\frac{1}{2}\right)^2 R^{-2}$	$-\Delta_f^2$	$h(h+1)$

Table 2. *Eigenvalues and corresponding degeneracies of the S^3 spatial Laplacian $\nabla^2 \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ for various quantum fields defined on S^3. Here R denotes the radius of the S^3. The irreducible representations of the $SO(4)$ isometry group are labelled by the angular momentum h which has the range $h = 0, 1, 2, \ldots$ for all the fields except for the longitudinal vector field ∇F where h starts from 1.*[2]

Then (for fixed i) we have

$$\Box_i^{mn} \mathbf{A}_i^\perp = -\sum_{k=-\infty}^{\infty} \left(\Delta_g^2 + (\omega_k + q_i^{mn})^2\right) (\mathbf{A}_i^\perp)^{[k]} \tag{2.1.29}$$

$$\Box_i^{mn} (\nabla F_i) = -\sum_{k=-\infty}^{\infty} \left(\Delta_s^2 + (\omega_k + q_i^{mn})^2\right) (\nabla F_i)^{[k]} \tag{2.1.30}$$

$$\Box_i^{mn} \widetilde{A}_i^0 = -\sum_{k=-\infty}^{\infty} \left(\Delta_s^2 + (\omega_k + q_i^{mn})^2\right) (\widetilde{A}_i^0)^{[k]} \tag{2.1.31}$$

$$\Box_i^{mn} \overline{c_i} = -\sum_{k=-\infty}^{\infty} \left(\Delta_s^2 + (\omega_k + q_i^{mn})^2\right) (\overline{c_i})^{[k]} \tag{2.1.32}$$

$$\Box_i^{mn} c_i = -\sum_{k=-\infty}^{\infty} \left(\Delta_s^2 + (\omega_k + q_i^{mn})^2\right) (c_i)^{[k]} \,. \tag{2.1.33}$$

[1] However, for the Fadeev-Popov ghosts the boundary conditions are taken periodic.

[2] For further information on S^3 spherical harmonics, see also [34, 35].

Performing the trace over the gauge indices and over the Matsubara frequencies in Eq. (2.1.27) yields the expression

$$\ln Z_{\text{gauge}} = -\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\left(\Delta_g^2 + (\omega_k + q_i^{mn})^2\right) \tag{2.1.34}$$

where the eigenvalue in Eq. (2.1.29) has been inserted. Proceeding along the same lines one obtains the following expressions for the scalar fields

$$\ln Z_{A_{i,(i+1)}} = -\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\left[\Delta_s^2 + R^{-2} + (\omega_k - i(\mu_2 + iq_{i+1}^n - iq_i^m))^2\right] \tag{2.1.35}$$

$$\ln Z_{B_{(i+1),i}} = -\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\left[\Delta_s^2 + R^{-2} + (\omega_k - i(\mu_2 + iq_i^n - iq_{i+1}^m))^2\right] \tag{2.1.36}$$

$$\ln Z_{\Phi_i} = -\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\left[\Delta_s^2 + R^{-2} + (\omega_k - i(\mu_1 - iq_i^{mn}))^2\right] . \tag{2.1.37}$$

As indicated in Eqs. (2.1.15)-(2.1.18), for the spinor fields it is most convenient to square the differential operators in Eqs. (2.1.10)-(2.1.13) using that (for fixed μ) $\tau_\mu^2 = \left(\tau_\mu^{\text{T}}\right)^2 = \pm 1$ and thus obtain the eigenvalues of the Fourier modes. Applying the identity $\ln\det K = \text{Tr}\ \ln K$, the results take the form

$$\begin{aligned}\ln Z_{\chi_{A,i}} = \frac{1}{2}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\Big[&\left(\Delta_f^2 + \left(\omega_k + (q_{i+1}^n - q_i^m) + \tfrac{i\mu_1}{2}\right)^2\right)\\ &\times\left(\Delta_f^2 + \left(\omega_k + (q_i^n - q_{i+1}^m) - \tfrac{i\mu_1}{2}\right)^2\right)\Big]\end{aligned} \tag{2.1.38}$$

$$\begin{aligned}\ln Z_{\chi_{B,i}} = \frac{1}{2}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\Big[&\left(\Delta_f^2 + \left(\omega_k + (q_i^n - q_{i+1}^m) + \tfrac{i\mu_1}{2}\right)^2\right)\\ &\times\left(\Delta_f^2 + \left(\omega_k + (q_{i+1}^n - q_i^m) - \tfrac{i\mu_1}{2}\right)^2\right)\Big]\end{aligned} \tag{2.1.39}$$

$$\begin{aligned}\ln Z_{\psi_i} = \frac{1}{2}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\Big[&\left(\Delta_f^2 + \left(\omega_k + q_i^{nm} - i\left(\tfrac{1}{2}\mu_1 - \mu_2\right)\right)^2\right)\\ &\times\left(\Delta_f^2 + \left(\omega_k + q_i^{nm} + i\left(\tfrac{1}{2}\mu_1 - \mu_2\right)\right)^2\right)\Big]\end{aligned} \tag{2.1.40}$$

$$\begin{aligned}\ln Z_{\psi_{\Phi,i}} = \frac{1}{2}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}\ \ln\Big[&\left(\Delta_f^2 + \left(\omega_k + q_i^{nm} - i\left(\tfrac{1}{2}\mu_1 + \mu_2\right)\right)^2\right)\\ &\times\left(\Delta_f^2 + \left(\omega_k + q_i^{nm} + i\left(\tfrac{1}{2}\mu_1 + \mu_2\right)\right)^2\right)\Big] .\end{aligned} \tag{2.1.41}$$

The traces in Eqs. (2.1.34)-(2.1.41) are taken over S^3 transverse vector, scalar and spinor spherical harmonics. The summations over the Matsubara frequencies can be done explicitly:

Theorem 2.1.1 (Bosonic Matsubara frequency sum). *Let $\omega_k = \frac{2\pi k}{\beta}$. Then the following identity holds up to an additive constant that does not depend on Δ*

$$\sum_{k=-\infty}^{\infty} \ln\left[(\omega_k + q)^2 + \Delta^2\right] = \ln\left[(1 - e^{-\beta(\Delta+iq)})\,(1 - e^{-\beta(\Delta-iq)})\right] + \beta\Delta\,. \tag{2.1.42}$$

Proof. Differentiating the above identity wrt. Δ leads to the identity

$$\sum_{k=-\infty}^{\infty} \frac{2\Delta}{(w_k + q)^2 + \Delta^2} = \frac{\beta}{e^{\beta(\Delta+iq)} - 1} + \frac{\beta}{e^{\beta(\Delta-iq)} - 1} + \beta\,. \tag{2.1.43}$$

Therefore it is enough to show this identity. For this purpose we need the following two identities

$$\sum_{k=-\infty}^{\infty} \frac{1}{i(\omega_k + q) - \Delta} = -\frac{\beta}{e^{\beta(\Delta-iq)} - 1} - \frac{\beta}{2} \tag{2.1.44}$$

and

$$\sum_{k=-\infty}^{\infty} \frac{\omega_k + q}{(\omega_k + q)^2 + \Delta^2} = \frac{i\beta}{2}\left(\frac{1}{e^{\beta(\Delta+iq)} - 1} - \frac{1}{e^{\beta(\Delta-iq)} - 1}\right)\,. \tag{2.1.45}$$

To prove identity (2.1.44), define the meromorphic function

$$f(z)\,n_b(z) = \frac{1}{z - \Delta}\left(\frac{1}{e^{\beta(z-iq)} - 1}\right) \tag{2.1.46}$$

and note that it has a simple pole at $z = \Delta$ with residue

$$\mathrm{Res}(f(z)\,n_b(z), \Delta) = \frac{1}{e^{\beta(\Delta-iq)} - 1} \tag{2.1.47}$$

and simple poles at $z_k \equiv i(\omega_k + q)$ with residue

$$\mathrm{Res}(f(z)\,n_b(z), z_k) = \frac{1}{\beta}\left(\frac{1}{i(\omega_k + q) - \Delta}\right) \tag{2.1.48}$$

(use that $e^{\beta(z-iq)} - 1 = \beta(z - z_k) + \mathcal{O}((z - z_k)^2)$). From Cauchy's residue theorem we thus obtain[3]

$$-\frac{1}{2} = \frac{1}{2\pi i}\lim_{R\to\infty}\oint_{C_R(0)} dz\, f(z)\,n_b(z) = \sum_{k=-\infty}^{\infty}\frac{1}{\beta}\left(\frac{1}{i(\omega_k + q) - \Delta}\right) + \frac{1}{e^{\beta(\Delta-iq)} - 1}\,. \tag{2.1.49}$$

This completes the proof of Eq. (2.1.44).

To prove identity (2.1.45), define the meromorphic function

$$g(z)\,n_b(z) = \frac{iz}{z^2 - \Delta^2}\left(\frac{1}{e^{\beta(z-iq)} - 1}\right) \tag{2.1.50}$$

[3] The $R \to \infty$ limit of the integral can easily be found in Maple by setting $\beta = 1$ and $\Delta = q = 0$.

and note the residues

$$\begin{aligned} \mathrm{Res}(g(z)\, n_b(z), \pm\Delta) &= \frac{i}{2}\left(\frac{1}{e^{\beta(\pm\Delta - iq)} - 1}\right) \\ \mathrm{Res}(g(z)\, n_b(z), z_k) &= \frac{1}{\beta}\left(\frac{\omega_k + q}{(\omega_k + q)^2 + \Delta^2}\right) \end{aligned} \tag{2.1.51}$$

and the integral

$$\lim_{R\to\infty} \oint_{C_R(0)} \frac{iz}{z^2 - \Delta^2}\left(\frac{1}{e^{\beta(z-iq)} - 1}\right) = \pi\,. \tag{2.1.52}$$

After some algebra this yields Eq. (2.1.45).

With the identities (2.1.44) and (2.1.45) in hand one finds

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{\Delta}{(\omega_k + q)^2 + \Delta^2} &= \sum_{k=-\infty}^{\infty} \frac{\Delta + i(\omega_k + q)}{(\omega_k + q)^2 + \Delta^2} + \frac{\beta}{2}\left(\frac{1}{e^{\beta(\Delta + iq)} - 1} - \frac{1}{e^{\beta(\Delta - iq)} - 1}\right) \\ &= -\sum_{k=-\infty}^{\infty} \frac{1}{i(\omega_k + q) - \Delta} + \frac{\beta}{2}\left(\frac{1}{e^{\beta(\Delta + iq)} - 1} - \frac{1}{e^{\beta(\Delta - iq)} - 1}\right) \\ &= \frac{\beta}{2}\left(\frac{1}{e^{\beta(\Delta + iq)} - 1} + \frac{1}{e^{\beta(\Delta - iq)} - 1} + 1\right)\,. \end{aligned} \tag{2.1.53}$$

Multiplying both sides by 2 we get Eq. (2.1.43) as desired. □

Theorem 2.1.2 (Fermionic Matsubara frequency sum). *Let* $\omega_k = \frac{(2k+1)\pi}{\beta}$*. The following identity holds up to an additive constant that does not depend on* Δ

$$\sum_{k=-\infty}^{\infty} \ln\left[(\omega_k + q)^2 + \Delta^2\right] = \ln\left[(1 + e^{-\beta(\Delta + iq)})\,(1 + e^{-\beta(\Delta - iq)})\right] + \beta\Delta\,. \tag{2.1.54}$$

Proof. Simply substitute $q \to q + \frac{\pi}{\beta}$ in (2.1.42). □

2.1.2 Evaluating the trace over the S^3 spherical harmonics

To obtain the single-particle partition functions one proceeds as follows. First one applies the relevant Matsubara frequency sum.[4] Then one Taylor expands the ln appearing on the right hand side. Finally one performs the traces over the spherical harmonics on S^3 by replacing $\mathrm{Tr}\left[\cdots\right]$ with $\sum_{h=0}^{\infty} D_h\left[\cdots\right]$ (where D_h denotes the degeneracy of the eigenvalue labelled by h, see Table 2). Before proceeding, we make the observation that due to the decomposition (2.1.2) the Polyakov loop $U_i = \mathcal{P}\big(e^{ig\int_0^\beta dx^0 A_i^0}\big)$ for the gauge field A_i^μ has the $g \to 0$ limit

$$U_i = \begin{pmatrix} e^{i\beta q_i^1} & & \\ & \ddots & \\ & & e^{i\beta q_i^N} \end{pmatrix}\,. \tag{2.1.55}$$

[4] In applying either of the frequency sums (2.1.42) or (2.1.54) we ignore the term $\beta\Delta$ appearing on the right hand side as it will give rise to a Casimir energy times βN^2, yielding an infinite additive constant in the $N \to \infty$ limit. We will evaluate the Casimir energies via ζ-function regularization in subsection 2.1.3.

For the scalar field $A_{i,(i+1)}$ one obtains[5]

$$\begin{aligned}
\ln Z_{A_{i,(i+1)}} &= -\sum_{m,n=1}^{N} \operatorname{Tr}\ln\left(1-e^{-\beta\left(\sqrt{\Delta_s^2+1}+(\mu_2+iq_{i+1}^n-iq_i^m)\right)}\right) \\
&\quad -\sum_{m,n=1}^{N} \operatorname{Tr}\ln\left(1-e^{-\beta\left(\sqrt{\Delta_s^2+1}-(\mu_2+iq_{i+1}^n-iq_i^m)\right)}\right) \\
&= \sum_{m,n=1}^{N}\sum_{h=0}^{\infty}(h+1)^2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)+(\mu_2+iq_{i+1}^n-iq_i^m))} \\
&\quad +\sum_{m,n=1}^{N}\sum_{h=0}^{\infty}(h+1)^2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)-(\mu_2+iq_{i+1}^n-iq_i^m))} \\
&= \sum_{l=1}^{\infty}\frac{1}{l}\left(\sum_{h=0}^{\infty}(h+1)^2e^{-\beta l(h+1)}\right)e^{-\beta l\mu_2}\left(\sum_{m,n=1}^{N}e^{-i\beta l(q_{i+1}^n-q_i^m)}\right) \\
&\quad +\sum_{l=1}^{\infty}\frac{1}{l}\left(\sum_{h=0}^{\infty}(h+1)^2e^{-\beta l(h+1)}\right)e^{\beta l\mu_2}\left(\sum_{m,n=1}^{N}e^{i\beta l(q_{i+1}^n-q_i^m)}\right) \\
&= \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{e^{-\beta l}+e^{-2\beta l}}{(1-e^{-\beta l})^3}\right)e^{-\beta l\mu_2}\operatorname{Tr}U_i^l\operatorname{Tr}U_{i+1}^{-l} \\
&\quad +\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{e^{-\beta l}+e^{-2\beta l}}{(1-e^{-\beta l})^3}\right)e^{\beta l\mu_2}\operatorname{Tr}U_i^{-l}\operatorname{Tr}U_{i+1}^l\,.
\end{aligned} \tag{2.1.56}$$

In the last equality we made use of the identities

$$\ln(1-x) = -\sum_{l=1}^{\infty}\frac{x^l}{l} \tag{2.1.57}$$

$$\sum_{h=0}^{\infty}(h+1)^2e^{-\beta l(h+1)} = \frac{d^2}{d(\beta l)^2}\sum_{h=0}^{\infty}(e^{-\beta l})^{h+1} = \frac{e^{-\beta l}+e^{-2\beta l}}{(1-e^{-\beta l})^3}\,. \tag{2.1.58}$$

It is precisely the identity (2.1.58) that is responsible for the characteristic form of the single-particle partition function for scalar fields.

For the gauge field A_i^μ one obtains

$$\begin{aligned}
\ln Z_{\text{gauge}} &= -\sum_{m,n=1}^{N}\operatorname{Tr}\ln\left(1-e^{-\beta(\Delta_g+iq_i^{mn})}\right)-\sum_{m,n=1}^{N}\operatorname{Tr}\ln\left(1-e^{-\beta(\Delta_g-iq_i^{mn})}\right) \\
&= \sum_{m,n=1}^{N}\sum_{h=0}^{\infty}2h(h+2)\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)+iq_i^{mn})} \\
&\quad +\sum_{m,n=1}^{N}\sum_{h=0}^{\infty}2h(h+2)\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)-iq_i^{mn})}
\end{aligned}$$

[5] Here and in the following we will put $R=1$ to simplify the notation.

$$
\begin{aligned}
&= 2\sum_{l=1}^{\infty}\frac{1}{l}\left(\sum_{h=0}^{\infty} h(h+2)\,e^{-\beta l(h+1)}\right)\sum_{m,n=1}^{N} e^{-i\beta l q_i^{mn}} \\
&\qquad + 2\sum_{l=1}^{\infty}\frac{1}{l}\left(\sum_{h=0}^{\infty} h(h+2)\,e^{-\beta l(h+1)}\right)\sum_{m,n=1}^{N} e^{i\beta l q_i^{mn}} \\
&= \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{6e^{-2\beta l}-2e^{-3\beta l}}{(1-e^{-\beta l})^3}\right)\mathrm{Tr}\,U_i^l\;\mathrm{Tr}\,U_i^{-l}\,. \qquad (2.1.59)
\end{aligned}
$$

In the last equality we made use of the identity

$$
\sum_{h=0}^{\infty} h(h+2)\,e^{-\beta l(h+1)} = \sum_{h=0}^{\infty}(h+1)^2\,e^{-\beta l(h+1)} - \sum_{h=0}^{\infty} e^{-\beta l(h+1)} = \frac{3e^{-2\beta l}-e^{-3\beta l}}{(1-e^{-\beta l})^3} \qquad (2.1.60)
$$

where in the second equality we used Eq. (2.1.58). It is precisely the identity (2.1.60) that is responsible for the characteristic form of the single-particle partition function for gauge fields.

For the Weyl spinor $\chi_{A,i}$ the Taylor expansion of the ln on the right hand side of Eq. (2.1.54) will yield an alternating series; i.e., since

$$
\ln(1+x) = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\,x^l \qquad (2.1.61)
$$

one finds

$$
\begin{aligned}
\ln Z_{\chi_{A,i}} &= \frac{1}{2}\sum_{m,n=1}^{N}\mathrm{Tr}\,\ln\Big[\Big(1+e^{-\beta(\Delta_f+i(q_{i+1}^n-q_i^m)-\frac{\mu_1}{2})}\Big)\Big(1+e^{-\beta(\Delta_f-i(q_{i+1}^n-q_i^m)+\frac{\mu_1}{2})}\Big) \\
&\qquad\times\Big(1+e^{-\beta(\Delta_f+i(q_i^n-q_{i+1}^m)+\frac{\mu_1}{2})}\Big)\Big(1+e^{-\beta(\Delta_f-i(q_i^n-q_{i+1}^m)-\frac{\mu_1}{2})}\Big)\Big] \\
&= \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\sum_{h=0}^{\infty} h(h+1)\,e^{-\beta l(h+\frac{1}{2})}\right)e^{\beta l\mu_1/2}\left(\sum_{m,n=1}^{N} e^{-i\beta l\,(q_{i+1}^n-q_i^m)}\right) \\
&\qquad+\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\sum_{h=0}^{\infty} h(h+1)\,e^{-\beta l(h+\frac{1}{2})}\right)e^{-\beta l\mu_1/2}\left(\sum_{m,n=1}^{N} e^{i\beta l\,(q_{i+1}^n-q_i^m)}\right) \\
&= \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2e^{-\frac{3\beta l}{2}}}{(1-e^{-\beta l})^3}\right)e^{\beta l\mu_1/2}\;\mathrm{Tr}\,U_i^l\;\mathrm{Tr}\,U_{i+1}^{-l} \\
&\qquad+\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2e^{-\frac{3\beta l}{2}}}{(1-e^{-\beta l})^3}\right)e^{-\beta l\mu_1/2}\;\mathrm{Tr}\,U_{i+1}^l\,\mathrm{Tr}\,U_i^{-l}\,. \qquad (2.1.62)
\end{aligned}
$$

In the last equality we made use of the identity

$$
\begin{aligned}
\sum_{h=0}^{\infty} h(h+1)\,e^{-\beta l(h+\frac{1}{2})} &= e^{-\frac{\beta l}{2}}\left(\sum_{h=0}^{\infty} h^2 e^{-\beta l h} + \sum_{h=0}^{\infty} h e^{-\beta l h}\right) \\
&= e^{-\frac{\beta l}{2}}\left(\frac{e^{-\beta l}+e^{-2\beta l}}{(1-e^{-\beta l})^3} + \frac{e^{-\beta l}-e^{-2\beta l}}{(1-e^{-\beta l})^3}\right)
\end{aligned}
$$

$$= \frac{2e^{-\frac{3\beta l}{2}}}{(1-e^{-\beta l})^3}\,. \tag{2.1.63}$$

It is precisely the identity (2.1.63) that is responsible for the characteristic form of the single-particle partition function for spinor fields.

The effective actions for the remaining scalar and spinor fields are obtained in exactly the same way. Below we write the results in terms of the variables $x \equiv e^{-\beta}$ and $y_j \equiv e^{\beta\mu_j}$

$$\begin{aligned}\ln Z_{A_{i,(i+1)}} &= \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)y_2^l\ \mathrm{Tr}\,U_i^{-l}\,\mathrm{Tr}\,U_{i+1}^l \\ &\quad+\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)y_2^{-l}\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_{i+1}^{-l}\end{aligned} \tag{2.1.64}$$

$$\begin{aligned}\ln Z_{B_{(i+1),i}} &= \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)y_2^l\ \mathrm{Tr}\,U_i^{l}\,\mathrm{Tr}\,U_{i+1}^{-l} \\ &\quad+\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)y_2^{-l}\ \mathrm{Tr}\,U_i^{-l}\ \mathrm{Tr}\,U_{i+1}^{l}\end{aligned} \tag{2.1.65}$$

$$\ln Z_{\Phi_i} = \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)(y_1^l+y_1^{-l})\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_i^{-l} \tag{2.1.66}$$

$$\ln Z_{\mathrm{gauge}} = \sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{6x^{2l}-2x^{3l}}{(1-x^l)^3}\right)\mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_i^{-l} \tag{2.1.67}$$

$$\begin{aligned}\ln Z_{\chi_{A,i}} &= \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)y_1^{l/2}\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_{i+1}^{-l} \\ &\quad+\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)y_1^{-l/2}\ \mathrm{Tr}\,U_{i+1}^l\,\mathrm{Tr}\,U_i^{-l}\end{aligned} \tag{2.1.68}$$

$$\begin{aligned}\ln Z_{\chi_{B,i}} &= \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)y_1^{l/2}\ \mathrm{Tr}\,U_{i+1}^l\,\mathrm{Tr}\,U_i^{-l} \\ &\quad+\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)y_1^{-l/2}\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_{i+1}^{-l}\end{aligned} \tag{2.1.69}$$

$$\ln Z_{\psi_i} = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)(y_1^{l/2}y_2^{-l}+y_1^{-l/2}y_2^l)\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_i^{-l} \tag{2.1.70}$$

$$\ln Z_{\psi_{\Phi,i}} = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)(y_1^{l/2}y_2^{l}+y_1^{-l/2}y_2^{-l})\ \mathrm{Tr}\,U_i^l\ \mathrm{Tr}\,U_i^{-l}\,. \tag{2.1.71}$$

The quantum effective action $\Gamma[U_i]$ is defined by

$$e^{-\Gamma[U_i]} = \int \mathcal{D}A_\mu \, \mathcal{D}\phi \, \mathcal{D}\psi \, e^{-\int_{S^1\times S^3} d^4x \, \sqrt{|g|} \, (\mathcal{L}_{\text{gauge}}+\mathcal{L}_{\text{scalar}}+\mathcal{L}_{\text{ferm}})} \, . \tag{2.1.72}$$

with $\mathcal{L}_{\text{gauge}}, \mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{ferm}}$ given by Eqs. (1.5.12), (1.5.10) and (1.5.11), respectively. Thus, by adding Eqs. (2.1.64)-(2.1.71), we find

$$\begin{aligned} \Gamma[U_i] \;=\; & -\sum_{i=1}^{M}\sum_{l=1}^{\infty}\Bigg[\frac{1}{l}\left(\frac{6x^{2l}-2x^{3l}}{(1-x^l)^3}\right)+\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_1^l+y_1^{-l}\right) \\ & \qquad + \frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\left(y_2^l+y_2^{-l}\right)\Bigg]\left(\operatorname{Tr}U_i^l \, \operatorname{Tr}U_i^{-l}\right) \\ & -\sum_{i=1}^{M}\sum_{l=1}^{\infty}\Bigg[\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_2^l+y_2^{-l}\right)+\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\Bigg] \\ & \qquad \times\left(\operatorname{Tr}U_i^l \, \operatorname{Tr}U_{i+1}^{-l}+\operatorname{Tr}U_i^{-l}\, \operatorname{Tr}U_{i+1}^{l}\right) . \end{aligned} \tag{2.1.73}$$

Note that the adjoint holonomy factors come from the vector multiplets $(A_{\mu i}, \Phi_i, \psi_i, \psi_{\Phi,i})$, and the bifundamental factors come from the hypermultiplets $(A_{i,(i+1)}, B_{(i+1),i}, \chi_{A,i}, \chi_{B,i})$. For later convenience we define here the total single-particle partition functions for the bosonic and fermionic sectors of the vector and hypermultiplets:

$$z^B_{\text{ad}}(x;y_1,y_2) \equiv \frac{6x^2-2x^3}{(1-x)^3}+\frac{x+x^2}{(1-x)^3}\left(y_1+y_1^{-1}\right) \tag{2.1.74}$$

$$z^F_{\text{ad}}(x;y_1,y_2) \equiv \frac{2x^{3/2}}{(1-x)^3}\left(y_1^{1/2}+y_1^{-1/2}\right)\left(y_2+y_2^{-1}\right) \tag{2.1.75}$$

$$z^B_{\text{bi}}(x;y_1,y_2) \equiv \frac{x+x^2}{(1-x)^3}\left(y_2+y_2^{-1}\right) \tag{2.1.76}$$

$$z^F_{\text{bi}}(x;y_1,y_2) \equiv \frac{2x^{3/2}}{(1-x)^3}\left(y_1^{1/2}+y_1^{-1/2}\right) . \tag{2.1.77}$$

These results are consistent with Ref. [6], Eqs. (3.17)-(3.18), where the summation over representations is taken to run over the adjoint and the bifundamental representations, and the charges Q are taken as β times the Cartan charges Q_1, Q_2 given implicitly through (1.5.1)-(1.5.8).

2.1.3 Casimir energies

Let us consider the contributions to the quantum effective action $\Gamma[U_i]$ originating from the $\beta\Delta$ terms in the Matsubara frequency summations (2.1.42) and (2.1.54) that we discarded in the previous section. In particular, for a scalar field one finds the contribution

$$-\ln Z_{\text{scalar}} = \sum_{m,n=1}^{N} \operatorname{Tr}\left(\beta\Delta_s\right) \tag{2.1.78}$$

where the trace is taken over the S^3 scalar spherical harmonics. This trace is evaluated by regularization via the Riemann ζ-function as explained below. The free energy F is related to the partition function Z through the relation $\beta F = -\ln Z$. Therefore, since the right hand side of (2.1.78) is linear in β, and contributions to F containing $x = e^{-\beta}$ go to zero in the $T \to 0$ limit, we conclude that the right hand side of (2.1.78) represents the contribution to the Casimir energy of $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ from a scalar field. Using the eigenvalues and corresponding degeneracies of the Laplacian on S^3 given in Table 2 one finds the following contribution to the Casimir energy from a scalar field on $S^1 \times S^3$:

$$\begin{aligned} \mathcal{E}_{\text{scalar}} &= \frac{1}{2}\sum_{h=0}^{\infty}(h+1)^2(h+1)R^{-1}\Big|_{s=-1} = \frac{1}{2R}\sum_{h=0}^{\infty}(h+1)^{2-s}\Big|_{s=-1} \\ &= \frac{1}{2R}\zeta(s-2)\Big|_{s=-1} \\ &= \frac{1}{240R} \end{aligned} \tag{2.1.79}$$

where in the last equality we set $s = -1$ in the analytic continuation of ζ.

Analogously, the contribution from a vector field defined on $S^1 \times S^3$ is:

$$\begin{aligned} \mathcal{E}_{\text{vector}} &= \frac{1}{2}\sum_{h=0}^{\infty}2h(h+2)(h+1)R^{-1} = \frac{1}{R}\sum_{h=0}^{\infty}\Big((h+1)^2(h+1)-(h+1)\Big) \\ &= \frac{1}{R}\big(\zeta(-3)-\zeta(-1)\big) \\ &= \frac{11}{120R}\,. \end{aligned} \tag{2.1.80}$$

For a spinor field the contribution to the Casimir energy is obtained by regularization via the Hurwitz ζ-function:

$$\begin{aligned} \mathcal{E}_{\text{spinor}} &= -\sum_{h=0}^{\infty}h(h+1)(h+1/2)R^{-1} = -\frac{1}{R}\sum_{h=0}^{\infty}\Big(\big(h+1/2\big)^3-\tfrac{1}{4}\big(h+1/2\big)\Big) \\ &= -\frac{1}{R}\Big(\zeta\big(-3,\tfrac{1}{2}\big)-\tfrac{1}{4}\zeta\big(-1,\tfrac{1}{2}\big)\Big) \\ &= \frac{17}{960R}\,. \end{aligned} \tag{2.1.81}$$

From these contributions, the total Casimir energy of $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ evaluates to

$$\mathcal{E}_{\text{Casimir}} = M\left(6\cdot\frac{1}{240R}+\frac{11}{120R}+4\cdot\frac{17}{960R}\right) = \frac{3}{16R}M\,. \tag{2.1.82}$$

Note that for any finite radius R of S^3 the Casimir energy is non-vanishing, implying that supersymmetry is spontaneously broken. Thus, supersymmetry is broken even in the $T \to 0$ limit as a result of defining the field theory on a compact spatial manifold.

2.2 The matrix model

The matrix model we will consider is defined by the partition function

$$Z_{\rm MM} = \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\big(-\Gamma[U_i]\big) \tag{2.2.1}$$

where $\Gamma[U_i]$ is given in (2.1.73). It is convenient for taking the continuum limit to rewrite $\Gamma[U_i]$ directly in terms of the zero modes a_i. To simplify the notation, define the rescaled zero mode $\alpha_i \equiv \beta a_i$ so that $U_i = e^{i\alpha_i}$. We note that (for fixed i)

$$\mathrm{Tr}\, U_i^l \; \mathrm{Tr}\, U_i^{-l} = \sum_{m,n=1}^{N} \cos\big(l\alpha_i^m - l\alpha_i^n\big) \tag{2.2.2}$$

$$\mathrm{Tr}\, U_i^l \; \mathrm{Tr}\, U_{i+1}^{-l} + \mathrm{Tr}\, U_i^{-l} \; \mathrm{Tr}\, U_{i+1}^{l} = 2\sum_{m,n=1}^{N} \cos\big(l\alpha_i^m - l\alpha_{i+1}^n\big)\,. \tag{2.2.3}$$

Furthermore, the path integral measures $\big[\mathcal{D}U_i\big]$ and $\big[\mathcal{D}\alpha_i\big]$ are related by

$$\begin{aligned}\big[\mathcal{D}U_i\big] &= \prod_{m<n} \sin^2\left(\frac{\alpha_i^m - \alpha_i^n}{2}\right)\big[\mathcal{D}\alpha_i\big] \\ &= \exp\left(\sum_{m\neq n} \ln\left|\sin\left(\frac{\alpha_i^m - \alpha_i^n}{2}\right)\right|\right)\big[\mathcal{D}\alpha_i\big]\,. \end{aligned} \tag{2.2.4}$$

Hence the path integral of the matrix model takes the form

$$Z_{\rm MM} = \int \prod_{i=1}^{M} [\mathcal{D}\alpha_i] \, \exp\left(-\sum_{m\neq n}\Big(V_{\rm ad}(\alpha_i^m - \alpha_i^n) + V_{\rm bi}(\alpha_i^m - \alpha_{i+1}^n)\Big)\right) \tag{2.2.5}$$

where the adjoint and bifundamental potentials are, respectively

$$\begin{aligned} V_{\rm ad}(\theta) &\equiv -\ln\left|\sin\left(\frac{\theta}{2}\right)\right| - \sum_{l=1}^{\infty} \frac{1}{l}\Big(z_{\rm ad}^B(x^l; y_1^l, y_2^l) + (-1)^{l+1} z_{\rm ad}^F(x^l; y_1^l, y_2^l)\Big)\cos(l\theta) \\ &= \ln 2 + \sum_{l=1}^{\infty} \frac{1}{l}\Big(1 - z_{\rm ad}^B(x^l; y_1^l, y_2^l) - (-1)^{l+1} z_{\rm ad}^F(x^l; y_1^l, y_2^l)\Big)\cos(l\theta) \end{aligned} \tag{2.2.6}$$

$$V_{\rm bi}(\theta) \equiv -\sum_{l=1}^{\infty} \frac{2}{l}\Big(z_{\rm bi}^B(x^l; y_1^l, y_2^l) + (-1)^{l+1} z_{\rm bi}^F(x^l; y_1^l, y_2^l)\Big)\cos(l\theta)\,. \tag{2.2.7}$$

In the second equality in Eq. (2.2.6) we used the identity

$$\ln\left|2\sin\left(\frac{\theta}{2}\right)\right| = -\sum_{l=1}^{\infty}\frac{1}{l}\cos(l\theta) \tag{2.2.8}$$

which can be found, e.g., in Ref. [9] (p. 16, footnote 8). We will ignore the term $\ln 2$ on the right hand side of Eq. (2.2.6) since it will only give rise to an overall constant that can be absorbed into the normalization of $Z_{\rm MM}$.

We will now take the continuum limit $N \to \infty$. It is convenient to introduce eigenvalue distributions $\rho_i(\theta_i)$ proportional to the density of the eigenvalues $e^{i\theta_i}$ of U_i at the angle $\theta_i \in [-\pi, \pi]$. Here ρ_i must be everywhere non-negative, and we choose its normalization so that for any fixed i

$$\int_{-\pi}^{\pi} d\theta_i \, \rho_i(\theta_i) = 1 \, . \tag{2.2.9}$$

Furthermore we define the Fourier modes of ρ_i and $V_{\rm ad}$ and $V_{\rm bi}$:

$$\rho_i^l \equiv \int_{-\pi}^{\pi} d\theta_i \, \rho_i(\theta_i) \cos(l\theta_i) \tag{2.2.10}$$

$$V_{\rm ad}^l \equiv \int_{-\pi}^{\pi} d\theta \, V_{\rm ad}(\theta) \cos(l\theta) \tag{2.2.11}$$

$$V_{\rm bi}^l \equiv \int_{-\pi}^{\pi} d\theta \, V_{\rm bi}(\theta) \cos(l\theta) \tag{2.2.12}$$

so that, assuming ρ_i, $V_{\rm ad}$, $V_{\rm bi}$ to be even functions, we have the Fourier expansions

$$\rho_i(\theta_i) = \frac{1}{\pi} \sum_{l=1}^{\infty} \rho_i^l \cos(l\theta_i) \tag{2.2.13}$$

$$V_{\rm ad}(\theta) = \frac{1}{\pi} \sum_{l=1}^{\infty} V_{\rm ad}^l \cos(l\theta) \tag{2.2.14}$$

$$V_{\rm bi}(\theta) = \frac{1}{\pi} \sum_{l=1}^{\infty} V_{\rm bi}^l \cos(l\theta) \, . \tag{2.2.15}$$

The continuum limit is obtained by making the substitution[6]

$$\frac{1}{N} \sum_{n=1}^{N} \left[\cdots \right] \longrightarrow \int_{-\pi}^{\pi} d\theta_i \, \rho_i(\theta_i) \left[\cdots \right] \tag{2.2.16}$$

so that

$$\frac{1}{N^2} \sum_{m,n=1}^{N} \cos(l\alpha_i^m - l\alpha_i^n) \longrightarrow (\rho_i^l)^2 \tag{2.2.17}$$

$$\frac{1}{N^2} \sum_{m,n=1}^{N} \cos(l\alpha_i^m - l\alpha_{i+1}^n) \longrightarrow \rho_i^l \rho_{i+1}^l \, . \tag{2.2.18}$$

Furthermore we replace the path integral measure $\left[\mathcal{D}\alpha_i\right] \longrightarrow \left[\mathcal{D}\lambda_i\right]$. Thus, in the continuum limit the path integral of the matrix model takes the form

$$Z_{\rm MM} = \int \prod_{i=1}^{M} \left[\mathcal{D}\lambda_i\right] \exp\left(-S_{\rm MM}[\boldsymbol{\rho}]\right) \tag{2.2.19}$$

[6]Here it is implied that the content of the brackets $\left[\cdots\right]$ carries an i label.

where the action for the eigenvalue distribution functions $\boldsymbol{\rho}$ is

$$S_{\rm MM}[\boldsymbol{\rho}] = \frac{N^2}{\pi}\sum_{i=1}^{M}\sum_{l=1}^{\infty}\Big((\rho_i^l)^2 V_{\rm ad}^l(T;\mu_1,\mu_2) \,+\, \rho_i^l\rho_{i+1}^l V_{\rm bi}^l(T;\mu_1,\mu_2)\Big)\,. \tag{2.2.20}$$

Here $V_{\rm ad}^l$ and $V_{\rm bi}^l$ denote the l'th moment of the adjoint and bifundamental potentials. Explicitly,

$$V_{\rm ad}^l(T;\mu_1,\mu_2) \;=\; \frac{\pi}{l}\Big(1 - z_{\rm ad}^B(x^l;y_1^l,y_2^l) - (-1)^{l+1}z_{\rm ad}^F(x^l;y_1^l,y_2^l)\Big) \tag{2.2.21}$$

$$V_{\rm bi}^l(T;\mu_1,\mu_2) \;=\; -\frac{2\pi}{l}\Big(z_{\rm bi}^B(x^l;y_1^l,y_2^l) + (-1)^{l+1}z_{\rm bi}^F(x^l;y_1^l,y_2^l)\Big)\,. \tag{2.2.22}$$

To summarize, the matrix model under study is defined by Eqs. (2.2.19)-(2.2.22) along with Eqs. (2.1.74)-(2.1.77).

2.3 Order parameters of deconfinement phase transition

Before proceeding with the analysis of the phase structure of the matrix model established in Sections 2.1-2.2 we will first review some well-known order parameters for deconfinement in 4-dimensional thermal Yang-Mills theories defined on flat space $S^1\times\mathbb{R}^3$. (For a more thorough review, see [36].)

First of all it is very important to note that even though $U(N)$ gauge theory for N finite can only develop genuine phase transition in the infinite volume limit, $N=\infty$ theories may have phase transitions even in finite volume because the $N\to\infty$ limit acts as a thermodynamic limit. One well-known example of this type is the Gross-Witten phase transition in 2-dimensional $U(N)$ gauge theories [37].

It is a defining property of a confining theory (with no fields in the fundamental representation of $SU(N)$) that a single external particle in the fundamental representation (a “quark”) can be inserted into this theory only at an infinite cost in free energy. Heuristically, such a quark forms one end of a QCD string which is infinitely long because it has nowhere else to end. In the low temperature confining phase this long string also has infinite positive free energy. Let $F_q(T)$ denote the change of the free energy of the system induced by the presence of an external quark. It follows from the form of the coupling of an external quark to the gauge fields that $e^{-F_q(T)/T}=\langle\mathcal{P}\rangle$, where $\mathcal{P}=\frac{1}{N}\,\mathrm{Tr}\,P\exp\big(\oint A\big)$ is the so-called Polyakov loop [38, 39], the trace of a Wilson loop around the compactified Euclidean thermal time circle. Therefore, since $F_q(T)$ is infinite in a confined phase, $\langle\mathcal{P}\rangle=0$ in the low temperature confining phase, whereas in a deconfined phase $F_q(T)$ is finite and therefore $\langle W(C)\rangle\neq 0$.

Thus, $\langle\mathcal{P}\rangle$ constitutes an order parameter that sharply distinguishes the low temperature confining phase from the high temperature deconfined phase. From a low temperature point of view, the transition that separates these two phases is associated with the condensation of flux tubes whose effective free energy is driven negative at high enough temperatures (when the energy of these strings is overcome by the entropy of their vibrations).

In the large N limit (with fixed 't Hooft coupling, or equivalently fixed Λ_{QCD}) the deconfinement phase transition has yet another order parameter [40]. The confined phase is dominated by gauge-invariant bound states and so its free energy $F(T)$ scales like N^0 at large N. On the other hand, the deconfined phase is described by free gluons, and consequently its free energy scales as N^2 at large N. Thus, in the large N limit, $\lim_{N\to\infty} F(T)/N^2$ constitutes a second order parameter for deconfinement; like the Polyakov loop this new order parameter vanishes in the confined phase but is finite in the deconfined phase.

2.4 Low-temperature solution and phase transition

The term $-\ln\left|\sin\left(\frac{\theta}{2}\right)\right|$ in the adjoint potential (2.2.6) originating from the change of measure is a temperature-independent repulsive potential. On the other hand, the remaining parts of the adjoint and bifundamental potentials (2.2.6)-(2.2.7) provide an attractive force[7] which grows from zero to infinite strength as the temperature is raised from zero to infinity. One would therefore expect that at low temperatures, the stable saddle points of the matrix model are characterized by the eigenvalues of the holonomy matrices U_i spreading out uniformly over the unit circle, whereas at high temperatures the attractive potential causes them to localize. In this section and the next we will see that this intuition indeed is true.

We now consider the saddle points of the matrix model action (2.2.20),

$$0 = \frac{\partial S_{\rm MM}}{\partial \rho_i^l} = \frac{N^2}{\pi}\Big(2\rho_i^l V_{\rm ad}^l \;+\; \big(\rho_{i-1}^l + \rho_{i+1}^l\big)V_{\rm bi}^l\Big)\,. \tag{2.4.1}$$

For $M \geq 2$, this condition translates into M linear equations in M unknowns:

$$2\rho_i^l V_{\rm ad}^l \;+\; \big(\rho_{i-1}^l + \rho_{i+1}^l\big)V_{\rm bi}^l = 0\,. \tag{2.4.2}$$

The determinant of this system of equations is generically non-zero, so we find the unique solution $\rho_i^l = 0$, corresponding to the flat distribution $\rho_i = \frac{1}{2\pi}$. Thus we conclude that the eigenvalues of the holonomy matrices U_i are distributed uniformly on each of the M unit circles. This defines the low-temperature solution of the matrix model.

The leading $\mathcal{O}(N^2)$ contribution to the free energy computed from the path integral (2.2.19) comes from the action $S_{\rm MM}[\boldsymbol{\rho}]$. However, as $\rho_i^l = 0$, the first non-zero contribution to the free energy in this phase comes from a Gaussian integral over the fluctuations about the solution $\rho_i = \frac{1}{2\pi}$. The free energy is therefore of $\mathcal{O}(1)$ with respect to N, suggesting that the theory in this phase describes a non-interacting gas of color singlet states. Furthermore, we note that the Polyakov loop $W(C) \equiv {\rm Tr}\,\mathcal{P}\exp\big(ig\int_0^\beta dx^0 A_i^0\big)$ has zero expectation value

[7] The fact that the remaining parts of (2.2.6)-(2.2.7) are attractive potentials can be shown following the argument in [6], footnote 32.

since the trace averages to zero in the uniform eigenvalue distribution. In particular, this implies that the $\mathbb{Z}_N$ center symmetry is left unbroken in this phase. Accordingly, we label this phase "confining".

For $M \geq 2$ the solution $\rho_i = \frac{1}{2\pi}$ will be a minimum of the action until we reach values of $(T; \mu_1, \mu_2)$ for which

$$0 = \det H_{ij} = \left| \frac{\partial^2 S_{\text{MM}}}{\partial \rho_i^l \, \partial \rho_j^l} \right| \tag{2.4.3}$$

for any fixed l. When the temperature or the chemical potentials are raised above these critical values, the flat distribution becomes an unstable saddle point of the matrix model, and the model thus enters a new phase which we will discuss in the next section. For now we note that (2.4.3) defines a phase transition condition of the matrix model.

It will be convenient to express the Hessian matrix in terms of the variables $\xi_l \equiv 2V_{\text{ad}}^l$ and $\eta_l \equiv V_{\text{bi}}^l$. Note first that in the special case $M = 2$, due to the identification $i \simeq i + M = i + 2$, the Hessian matrix obtained from (2.2.20) takes the form[8]

$$H = \begin{pmatrix} \xi_l & 2\eta_l \\ 2\eta_l & \xi_l \end{pmatrix} . \tag{2.4.4}$$

The determinant factorizes as $\det H = -4\big(\eta_l - \frac{1}{2}\xi_l\big)\big(\eta_l + \frac{1}{2}\xi_l\big)$. For $M \geq 3$ the Hessian matrix is a tridiagonal, periodically continued matrix:

$$H_{ij} = \begin{cases} \xi_l & \text{for } j = i \\ \eta_l & \text{for } j = i \pm 1 \end{cases} \tag{2.4.5}$$

where, as usual, we make the identifications $i \simeq i + M$ and $j \simeq j + M$. The determinant of H factorizes as follows[9]

$$\det \begin{pmatrix} \xi & \eta & & \eta \\ \eta & \xi & \ddots & \\ & \ddots & \ddots & \eta \\ \eta & & \eta & \xi \end{pmatrix} = \prod_{j=1}^{M} \left(\xi + 2\cos\left(\frac{2\pi j}{M}\right) \eta \right) . \tag{2.4.6}$$

Thus, the determinant of H vanishes on any of the lines $\xi_l + 2\cos\left(\frac{2\pi j}{M}\right)\eta_l = 0$ for $j = 1, \ldots, M$. To single out the physically relevant condition for the vanishing of $\det H$ we will first consider the case $M = 12$ to gain intuition. For $M = 12$ the determinant in particular factorizes as

$$\det H = -36\,\xi_l^2 \left(\eta_l^2 - \xi_l^2\right)^2 \left(\eta_l^2 - \frac{\xi_l^2}{4}\right) \left(\eta_l^2 - \frac{\xi_l^2}{3}\right)^2 \tag{2.4.7}$$

where l is fixed. In Figure 2.1 we have divided the (ξ_l, η_l) plane into regions where H is positive-definite (denoted by $+$) and where H is indefinite (denoted by $-$).

[8] We omit here, and in the following, the overall factor of $\frac{N^2}{\pi}$ in Eq. (2.2.20) for notational simplicity.

[9] This formula is a special case of (3.1.11).

Thus regions marked by + correspond to a local extremum (minimum) of $S_{\rm MM}$, and regions marked by $-$ correspond to unstable saddle points. In Figure 2.1 we have furthermore marked the region occupied by the $\mathcal{N}=2$ quiver gauge theory matrix model in the low temperature phase by plotting (ξ_1,η_1) for $(T;\mu_1,\mu_2)=(0.1;0.8,0.8)$. For fixed chemical potentials, $z^B_{\rm ad}, z^F_{\rm ad}, z^B_{\rm bi}, z^F_{\rm bi}$ all increase monotonically with the temperature. Therefore, as the temperature increases, the dot in Figure 2.1 will move as indicated and hit the instability line $\eta_l = -\frac{1}{2}\xi_l$ at the phase transition temperature.

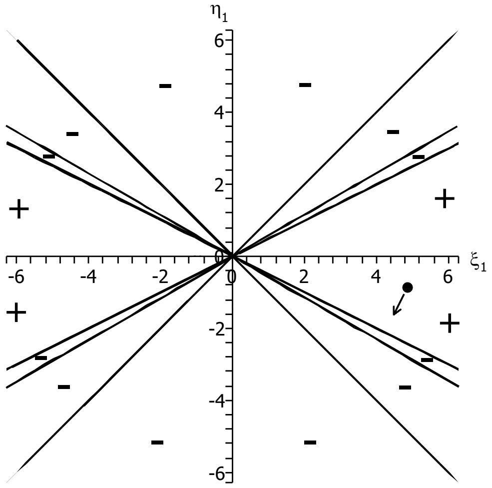


Figure 2.1: Regions of positive-definiteness and indefiniteness of H for the case $M=12$. Regions where H is positive-definite (corresponding to a local minimum of $S_{\rm MM}$) are marked by +; regions where H is indefinite (corresponding to an unstable saddle point of $S_{\rm MM}$) are marked by $-$. The lines represent the locus of $\det H = 0$. The physically accessible region of the (ξ_l,η_l) plane is bounded from above by the ξ_l axis and from below by the line of the numerically smallest negative slope. This is illustrated by the dot which corresponds to $(T;\mu_1,\mu_2)=(0.1;0.8,0.8)$ and $l=1$. The arrow shows how the dot will move as the temperature is increased, keeping μ_1 and μ_2 fixed.

By the same analysis, for any $M\geq 2$ the phase transition occurs at the instability line $\eta_l=\alpha(M)\xi_l$ where $\alpha(M)$ is the numerically smallest negative slope of the zero lines of the Hessian determinant. For all $M\geq 3$ we find from (2.4.6) that $\alpha(M)=-\frac{1}{2}$ (corresponding to $j=M$). For $M=2$ we also find $\alpha(M)=-\frac{1}{2}$. Indeed, note that for $M\geq 2$ the matrix obtained by substituting $\eta_l=-\frac{1}{2}\xi_l$ in Eqs. (2.4.4) and (2.4.5) will have a zero eigenvalue (with $(1,1,\ldots,1)$ as an eigenvector) and hence zero determinant.

2.4.1 The large M limit

As a consistency check, we can derive that $\lim_{M\to\infty}\alpha(M) = -\frac{1}{2}$ by a different route. We take the continuum limit $M\to\infty$ in the quiver direction. The quiver label i thus becomes a continuous angular parameter ϑ which we take to be 2π-periodic; i.e., we identify $\vartheta \simeq \vartheta + 2\pi$. Accordingly we make the substitutions

$$(\rho_i^l)^2 \longrightarrow (\rho^l(\vartheta))^2 \tag{2.4.8}$$

$$(\rho_i^l \rho_{i+1}^l) \longrightarrow -\frac{1}{2}\big(\dot{\rho}^l(\vartheta)\big)^2 + (\rho^l(\vartheta))^2 \tag{2.4.9}$$

where $\dot{}$ denotes $\frac{d}{d\vartheta}$. The matrix model action (2.2.20) thus becomes[10]

$$S_{\text{MM}}[\boldsymbol{\rho}] = \frac{N^2 M}{(2\pi)^2} \sum_{l=1}^{\infty} \int_0^{2\pi} d\vartheta \left[(\xi_l + 2\eta_l)(\rho^l(\vartheta))^2 - \eta_l \big(\dot{\rho}^l(\vartheta)\big)^2 \right] . \tag{2.4.10}$$

The Euler-Lagrange equations obtained from this action are those of a harmonic oscillator,

$$\eta_l \ddot{\rho}^l(\vartheta) + (\xi_l + 2\eta_l)\,\rho^l(\vartheta) = 0 \tag{2.4.11}$$

where $l = 1, 2, \ldots$. Note here that it is the bifundamental contribution in (2.2.20) that gives rise to the derivative term in (2.4.10) and in turn to the mass term for the harmonic oscillator. Thus, the harmonic oscillator EOM's in the large M limit is a pure 'quiver phenomenon'. Solutions to these equations will become unstable when the tension $\tau \equiv (\xi_l + 2\eta_l)$ goes from $\tau > 0$ to $\tau < 0$. Thus, for large M, the phase transition will occur when $\eta_l = -\frac{1}{2}\xi_l$, consistent with what we found above.

We now return to the phase transition condition $\eta_l = \alpha(M)\,\xi_l$. Since $z_{\text{ad}}^B, z_{\text{ad}}^F, z_{\text{bi}}^B, z_{\text{bi}}^F$ are all monotonically increasing as functions of x and $0 \leq x < 1$, the $l = 1$ condition is the strongest. Therefore, the phase transition condition for $M \geq 2$ is

$$\text{for } M \geq 2: \big(z_{\text{ad}}^B(x; y_1, y_2) + z_{\text{ad}}^F(x; y_1, y_2)\big) + 2\big(z_{\text{bi}}^B(x; y_1, y_2) + z_{\text{bi}}^F(x; y_1, y_2)\big) = 1\,. \tag{2.4.12}$$

Finally, in the special case $M = 1$ we immediately obtain $V_{\text{ad}}^l + V_{\text{bi}}^l = 0$ from (2.4.1) due to the identification $i \simeq i + M = i + 1$. Putting $l = 1$, this is precisely the phase transition condition (2.4.12). We thus conclude that for any M the phase transition condition is

$$\big(z_{\text{ad}}^B(x; y_1, y_2) + z_{\text{ad}}^F(x; y_1, y_2)\big) + 2\big(z_{\text{bi}}^B(x; y_1, y_2) + z_{\text{bi}}^F(x; y_1, y_2)\big) = 1\,. \tag{2.4.13}$$

In Figure 2.2 below we have plotted the curves in the (T, μ) plane obtained from this condition for the cases $(\mu_1, \mu_2) = (\mu, 0)$; $(\mu_1, \mu_2) = (0, \mu)$ and $(\mu_1, \mu_2) = (\mu, \mu)$. For each of these cases, the relevant curve defines the phase diagram of $\mathcal{N} = 2$ quiver gauge theory as a function of both temperature and chemical potential. Note that, as discussed in Section 1.5, if one or both of the chemical potentials are larger than the inverse radius of the spatial manifold S^3, the theory develops tachyonic modes and becomes ill-defined. Therefore the line $\mu = 1/R$ defines a boundary of the phase diagram.

[10] The extra prefactor $\frac{M}{2\pi}$ comes from changing the counting measure over i to the measure $d\vartheta$.

The phase transition condition (2.4.13) defines a phase transition temperature $T_H(\mu_1, \mu_2)$ as a function of the chemical potentials. We will refer to $T_H(\mu_1, \mu_2)$ as the Hagedorn temperature of $\mathcal{N} = 2$ quiver gauge theory. This terminology will be justified in Section 2.5. We remark that the Hagedorn temperature at zero chemical potential is

$$T_H = -\frac{1}{\ln(7 - 4\sqrt{3})} \approx 0.37966 \tag{2.4.14}$$

in units of R^{-1}, the inverse radius of the S^3. This is exactly the Hagedorn temperature for $\mathcal{N} = 4$ SYM theory (cf. [6, 7]). The origin of this fact can be traced to the observation in [30] that in the large N limit the correlation functions of $\mathcal{N} = 4$ $U(N)$ SYM theory equal the corresponding correlation functions of the $\mathcal{N} = 2, 1, 0$ quiver gauge theories obtained from orbifold projections. Since our computations rely on perturbation theory (namely, taking the $g \to 0$ limit of the action and then performing Gaussian path integrations), and we are furthermore taking the $N \to \infty$ limit, we should expect that the matrix model defined out of the quantum effective action will have the same behavior for the $\mathcal{N} = 2$ quiver gauge theory as for the $\mathcal{N} = 4$ SYM theory.

Furthermore, for small chemical potentials the Hagedorn temperature is given by

$$T_H(\mu_1, \mu_2) = \frac{1}{\beta_0} + c\big(\mu_1^2 + 2\mu_2^2\big) + c_{11}\mu_1^4 + c_{12}\mu_1^2\mu_2^2 + c_{22}\mu_2^4 + \mathcal{O}(\mu_i^6) \tag{2.4.15}$$

where the coefficients are

$$\beta_0 = -\ln(7 - 4\sqrt{3})\,, \quad c = -\frac{\sqrt{3}}{18}\,, \quad c_{11} = -\frac{\beta_0}{864}\left(\frac{362\beta_0 - 209\sqrt{3}\beta_0 + 2896\sqrt{3} - 5016}{-627 + 362\sqrt{3}}\right) \tag{2.4.16}$$

$$c_{12} = \frac{\beta_0}{216}\left(\frac{1810\beta_0 - 1045\sqrt{3}\beta_0 - 2896\sqrt{3} + 5016}{-627 + 362\sqrt{3}}\right), \quad c_{22} = \frac{\beta_0}{108}\left(\frac{362\beta_0 - 209\sqrt{3}\beta_0 - 1448\sqrt{3} + 2508}{-627 + 362\sqrt{3}}\right) \tag{2.4.17}$$

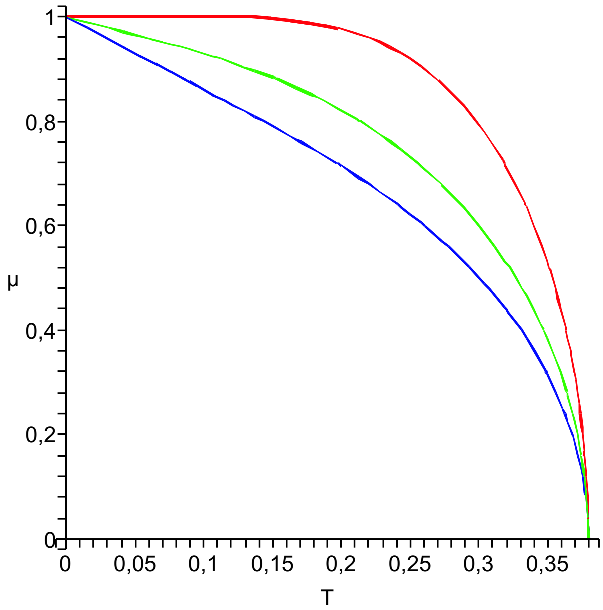


Figure 2.2: Phase diagram of $\mathcal{N}=2$ quiver gauge theory. The outermost curve is the transition line corresponding to $(\mu_1,\mu_2)=(\mu,0)$. It has slope 0 in the neighborhood of the point $(T,\mu)=(0,1)$. The inbetween curve corresponds to $(\mu_1,\mu_2)=(0,\mu)$, with slope $-\ln 2$ near $(0,1)$. The innermost curve corresponds to $(\mu_1,\mu_2)=(\mu,\mu)$, with slope $-\ln 4$ near $(0,1)$. The phase transition temperature at zero chemical potential is common for the three curves and equals $T=-\frac{1}{\ln(7-4\sqrt{3})}\approx 0.37966$ as in the $\mathcal{N}=4$ SYM case.

2.5 Solution above the Hagedorn temperature

As the temperature is increased beyond $T > T_H$, the attractive terms in the pairwise potential continue to increase in strength, and so the eigenvalues will become increasingly localized. The precise distribution can be determined, following [6], by the condition that a single additional eigenvalue α_i added on the i'th circle experiences no net force from the other eigenvalues on the circles $i-1$, i and $i+1$:

$$0 \;=\; \int_{-\pi}^{\pi} d\zeta\; 2V'_{\rm ad}(\alpha_i-\zeta)\,\rho_i(\zeta) \;+\; \int_{-\pi}^{\pi} d\zeta\; V'_{\rm bi}(\alpha_i-\zeta)\big(\rho_{i-1}(\zeta)+\rho_{i+1}(\zeta)\big) \tag{2.5.1}$$

where $V_{\rm ad}$ and $V_{\rm bi}$ are given in (2.2.6) and (2.2.7), respectively. Setting

$$a_l \;=\; z^B_{\rm ad}(x^l;y^l_1,y^l_2) \;+\; (-1)^{l+1} z^F_{\rm ad}(x^l;y^l_1,y^l_2) \tag{2.5.2}$$
$$b_l \;=\; z^B_{\rm bi}(x^l;y^l_1,y^l_2) \;+\; (-1)^{l+1} z^F_{\rm bi}(x^l;y^l_1,y^l_2) \tag{2.5.3}$$

we find

$$V'_{\rm ad}(\alpha_i - \zeta) \;=\; -\frac{1}{2}\cot\left(\frac{\alpha_i-\zeta}{2}\right) + \sum_{l=1}^{\infty} a_l \sin(l\alpha_i - l\zeta) \tag{2.5.4}$$

$$V'_{\rm bi}(\alpha_i - \zeta) \;=\; 2\sum_{l=1}^{\infty} b_l \sin(l\alpha_i - l\zeta)\,. \tag{2.5.5}$$

Moreover, using the Fourier expansion of ρ_i in (2.2.13) and furthermore using the orthonormality of $\left(1, \frac{1}{\sqrt{\pi}}\cos(n\zeta), \frac{1}{\sqrt{\pi}}\sin(n\zeta)\right)$ on $[-\pi,\pi]$, one finds

$$\int_{-\pi}^{\pi} d\zeta\; 2V'_{\rm ad}(\alpha_i-\zeta)\,\rho_i(\zeta) \;=\; -\int_{-\pi}^{\pi} d\zeta\;\cot\left(\frac{\alpha_i-\zeta}{2}\right)\rho_i(\zeta) \;+\; 2\sum_{l=1}^{\infty} a_l \rho_i^l \sin(l\alpha_i) \tag{2.5.6}$$

$$\int_{-\pi}^{\pi} d\zeta\; V'_{\rm bi}(\alpha_i-\zeta)\big(\rho_{i-1}(\zeta)+\rho_{i+1}(\zeta)\big) \;=\; 2\sum_{l=1}^{\infty} b_l \big(\rho_{i-1}^l + \rho_{i+1}^l\big)\sin(l\alpha_i)\,. \tag{2.5.7}$$

Thus the no-force condition (2.5.1) can be written

$$\int_{-\pi}^{\pi} d\zeta\;\cot\left(\frac{\alpha_i-\zeta}{2}\right)\rho_i(\zeta) \;=\; 2\sum_{l=1}^{\infty} a_l\rho_i^l\sin(l\alpha_i) \;+\; 2\sum_{l=1}^{\infty} b_l\sin(l\alpha_i)\big(\rho_{i-1}^l+\rho_{i+1}^l\big) \tag{2.5.8}$$

which provides M equilibrium conditions

$$\left.\frac{\partial S(\boldsymbol{\zeta})}{\partial\zeta_i}\right|_{\zeta_i=\alpha_i} + \int_{-\pi}^{\pi} d\zeta_i\;\rho_i(\zeta_i)\cot\left(\frac{\alpha_i-\zeta_i}{2}\right) \;=\; 0 \qquad \forall\, i=1,\ldots,M \tag{2.5.9}$$

for the lattice action

$$S_{\rm latt} \;=\; N\sum_{i=1}^{M}\sum_{l=1}^{\infty}\frac{a_l\rho_i^l + b_l\rho_{i-1}^l + b_l\rho_{i+1}^l}{l}\left(\mathrm{Tr}\,U_i^l + \mathrm{Tr}\,U_i^{-l}\right). \tag{2.5.10}$$

We will assume unbroken quiver translational invariance for the solution; i.e., that $\rho_i^l = \rho_{i+1}^l$ for any i. Our motivation for this is the non-perturbative equivalence between parent/daughter gauge theories related by orbifold projections [41, 42, 43, 44]. More specifically, it was shown in [42, 43] that a necessary and sufficient condition for the equivalence is unbroken invariance under the orbifold group. Since it was shown in [30] that there is a perturbative equivalence between $\mathcal{N}=4$ SYM theory and $\mathcal{N}=2,1,0$ quiver gauge theories obtained from orbifold projections in the planar limit, we expect the case of unbroken quiver translational invariance to be the one of physical interest. However, we note that non-perturbative effects might destroy the equivalence established in [30], potentially leading to i-dependent saddle points of (2.2.20).

To find the solutions ρ_i we rewrite[11] the equilibrium conditions (2.5.9) in terms of the complex variables $z_i \equiv e^{i\alpha_i}$ and $\tau \equiv e^{i\zeta}$. Defining $K(z_j) \equiv \frac{\partial S(\boldsymbol{\zeta})}{\partial\zeta_j}$, the conditions (2.5.9) can be written

$$K(z_j) + i \;=\; 2\oint_{S^1}\frac{\rho_j(\tau)\,d\tau}{\tau - z_j}\,. \tag{2.5.11}$$

[11] Use that $\cot\left(\frac{\alpha_i-\zeta}{2}\right) = i\frac{z_i+\tau}{z_i-\tau}$ and that $\oint_{S_1}\frac{d\tau}{\tau}\rho_i(\tau) = i$.

Assuming $\rho_i^l = \rho_{i+1}^l$, the exact inversion formula of this equation was found in [45]. As we will not need the exact solution for our purposes, we will not go further into its derivation, and we simply cite here the following approximate solution:

$$\rho_i(\theta) \;=\; \frac{1}{\pi s^2}\sqrt{s^2 - \sin^2\left(\frac{\theta}{2}\right)}\,\cos\left(\frac{\theta}{2}\right) \tag{2.5.12}$$

where

$$s^2 \;\equiv\; \sin^2\left(\frac{\theta_0}{2}\right) \;=\; 1 - \sqrt{1 - \frac{1}{a_1 + 2b_1}}\,. \tag{2.5.13}$$

The support of the solution (2.5.12) is $[-\theta_0, \theta_0]$. It is immediately clear from (2.5.13) that for temperatures above the Hagedorn temperature one has $\theta_0 < \pi$; i.e., the eigenvalue distribution becomes gapped.

2.5.1 Twisted partition function

In analogy with [46] we note the possibility that the quiver translational invariance can be broken when the boundary conditions for the spinor fields on the S^1 are taken to be periodic rather than antiperiodic.[12] In this case the Matsubara frequencies for the spinor fields will be the same as for the bosonic fields, and the twisted partition function[13] $\widetilde{Z} = \mathrm{Tr}(-1)^F e^{-\beta H} = e^{-\widetilde{\Gamma}[U_i]}$ may be obtained directly from (2.1.73) by replacing $(-1)^{l+1} \longrightarrow (-1)$. In order to exhibit the $\mathbb{Z}_M$ symmetry of the (twisted) partition function more clearly we rewrite the adjoint and bifundamental holonomy factors in terms of eigenvectors under quiver node displacements $i \to i+1$. Indeed, define for $\omega \equiv e^{2\pi i/M}$,

$$\Omega_k^l \;\equiv\; \sum_{j=1}^{M} \omega^{-kj}\, U_j^l\,. \tag{2.5.14}$$

Under the quiver node displacement $U_i^l \;\longrightarrow\; U_{i+1}^l$ we find

$$\Omega_k^l \;\longrightarrow\; \sum_{j=1}^{M} \omega^{-kj}\, U_{j+1}^l \;=\; \omega^k \sum_{j=1}^{M} \omega^{-k(j+1)}\, U_{j+1}^l \;=\; \omega^k \Omega_k^l \tag{2.5.15}$$

so that Ω_k^l is an eigenvector under the displacement with eigenvalue ω^k. The adjoint holonomy factor can be written in terms of Ω_k^l in the form

$$\sum_{i=1}^{M} \mathrm{Tr}\, U_i^l\, \mathrm{Tr}\, U_i^{-l} \;=\; \frac{1}{M}\sum_{k=1}^{M} \mathrm{Tr}\,\Omega_k^l\, \mathrm{Tr}\,\Omega_{-k}^{-l}\,. \tag{2.5.16}$$

[12]Note that the choices of periodic and antiperiodic boundary conditions exhaust the possible choices on S^1. This follows since the spin structures on S^1 are in 1-1 correspondence with the elements of $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$. Thus, there are 2 spin structures, corresponding to 2 distinct choices of boundary conditions for spinor fields.

[13]The twisted partition function is also known as the Witten index. Letting the trace run over the entire Hilbert space of gauge invariant states, the Witten index defines a topological invariant of a supersymmetric field theory. In particular, it may be shown to be independent of the inverse temperature β.

Likewise, the bifundamental holonomy factor can be written

$$\sum_{i=1}^{M} \mathrm{Tr}\, U_i^l \,\mathrm{Tr}\, U_{i+1}^{-l} \;=\; \frac{1}{M}\sum_{k=1}^{M} \omega^{-k}\, \mathrm{Tr}\,\Omega_k^l \,\mathrm{Tr}\,\Omega_{-k}^{-l} \,. \tag{2.5.17}$$

Thus, the twisted quantum effective action takes the form

$$\begin{aligned}
\widetilde{\Gamma}[U_i] \;=\; & -\frac{1}{M}\sum_{k=1}^{M}\sum_{l=1}^{\infty}\Bigg[\frac{1}{l}\left(\frac{6x^{2l}-2x^{3l}}{(1-x^l)^3}\right)+\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_1^l+y_1^{-l}\right) \\
& \qquad -\frac{1}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\left(y_2^l+y_2^{-l}\right)\Bigg]\Big(\mathrm{Tr}\,\Omega_k^l\,\mathrm{Tr}\,\Omega_{-k}^{-l}\Big) \\
& -\frac{1}{M}\sum_{k=1}^{M}\sum_{l=1}^{\infty}\Bigg[\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_2^l+y_2^{-l}\right)-\frac{1}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\Bigg] \\
& \qquad \times\,\omega^{-k}\Big(\mathrm{Tr}\,\Omega_k^l\,\mathrm{Tr}\,\Omega_{-k}^{-l}+\mathrm{Tr}\,\Omega_k^{-l}\,\mathrm{Tr}\,\Omega_{-k}^{l}\Big)\,.
\end{aligned} \tag{2.5.18}$$

As in Ref. [46], we expect that for the effective action $\widetilde{\Gamma}[U_i]$ there exist phases that break the translational invariance along the quiver direction. It would be interesting to study this further.

2.5.2 Free energy slightly above the Hagedorn temperature

Using the Hagedorn temperature for small chemical potentials given in (2.4.15)-(2.4.17) we can compute the free energy slightly above the Hagedorn temperature in analogy with [10]. Defining $\Delta T \equiv T - T_H(\mu_1,\mu_2)$, we find for $0<\Delta T \ll 1$ the perturbative expansion

$$\begin{aligned}
\frac{F}{N^2 M} \;=\; & -\beta_0\frac{3}{8}\left(1-\beta_0\frac{2\sqrt{3}+\beta_0}{36}\left(\mu_1^2+2\mu_2^2\right)+\mathcal{O}(\mu_i^4)\right)\Delta T \\
& -\beta_0^2\sqrt{\frac{3}{8}}\left(1-\beta_0\frac{4+\sqrt{3}\beta_0}{24\sqrt{3}}\left(\mu_1^2+2\mu_2^2\right)+\mathcal{O}(\mu_i^4)\right)\Delta T^{3/2}+\mathcal{O}(\Delta T^2)\,.
\end{aligned} \tag{2.5.19}$$

2.5.3 High-temperature behavior of free energy

Let us examine the free energy in the limit $T\to\infty$ where the chemical potentials are kept fixed. We will first treat the $\mu_1,\mu_2=0$ case. First note that from (2.2.21) and (2.1.74)-(2.1.75) one obtains, for large T,

$$V_{\mathrm{ad}}^l(T;0,0) = -\frac{2\pi T^3}{l^4}\left(4+4(-1)^{l+1}\right)\,, \tag{2.5.20}$$

whereas from (2.2.22) and (2.1.76)-(2.1.77) one obtains, for large T,

$$V_{\mathrm{bi}}^l(T;0,0) = -\frac{2\pi T^3}{l^4}\left(4+4(-1)^{l+1}\right)\,. \tag{2.5.21}$$

In the $T \to \infty$ limit the pairwise attractive potentials grow to infinite strength, so the eigenvalues of the holonomy matrices U_i localize to extremely small intervals; i.e. the eigenvalue distribution functions will become delta functions, $\rho_i(\theta_i) \to \delta(\theta_i)$. (This is also clear from (2.5.13) since for $T \to \infty$ one has $a_1, b_1 \to \infty$ and thus $\theta_0 \to 0$. The normalization condition (2.2.9) then implies $\rho_i(\theta_i) \to \delta(\theta_i)$.) Therefore $\rho_i^l \to 1$, and so from (2.2.20) we find that the free energy in the $T \to \infty$ limit is

$$\begin{aligned} F &= TS_{\mathrm{MM}} = -2N^2MT^4\sum_{l=1}^{\infty}\frac{1}{l^4}\big(8+8(-1)^{l+1}\big) \\ &= -2N^2MT^4\zeta(4)\left(8+\left(1-\tfrac{1}{8}\right)8\right) \qquad (2.5.22) \\ &= -\frac{\pi^2}{6}N^2MT^4\mathrm{Vol}(S^3) \qquad (2.5.23) \end{aligned}$$

where in (2.5.22) we used that $\zeta(4) = \frac{\pi^4}{90}$ and $\mathrm{Vol}(S^3) = 2\pi^2$ (putting $R = 1$) and finally that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^d} = \eta(d) = \left(1-\frac{1}{2^{d-1}}\right)\zeta(d) \qquad (2.5.24)$$

where η is the Dirichlet eta function. We note that the free energy scales as N^2M as $N \to \infty$. This is to be expected from the orbifold projection invariant form of the fields (1.2.15)-(1.2.16) and (1.2.37)-(1.2.38), given that the free energy scales as N^2 for $\mathcal{N} = 4$ $U(N)$ SYM theory for high temperatures in the $N \to \infty$ limit (cf. Eq. (5.62) of [6]).

Note that the weight factor for the number of bosonic degrees of freedom (namely, $\mathcal{N}_B^{\mathrm{DOF}} = 8$) in (2.5.22) is 1, whereas the weight factor for the number of fermionic degrees of freedom (namely, $\mathcal{N}_F^{\mathrm{DOF}} = 8$) is $\frac{7}{8}$. The origin of this difference in weight factors is the antiperiodic boundary conditions for the fermionic fields. Thus we explicitly see that supersymmetry is broken as a result of the antiperiodic boundary conditions for fermions.

For finite values of the chemical potentials μ_1, μ_2 one finds from (2.2.21) and (2.1.74)-(2.1.75) for large T

$$V_{\mathrm{ad}}^l(T;\mu_1,\mu_2) = -\frac{2\pi T^3}{l^4}\Big(2+\big(y_1^l+y_1^{-l}\big)+(-1)^{l+1}\big(y_1^{l/2}+y_1^{-l/2}\big)\big(y_2^l+y_2^{-l}\big)\Big)\,, \qquad (2.5.25)$$

whereas from (2.2.22) and (2.1.76)-(2.1.77) one obtains, for large T,

$$V_{\mathrm{bi}}^l(T;\mu_1,\mu_2) = -\frac{2\pi T^3}{l^4}\Big(2\big(y_2^l+y_2^{-l}\big)+2(-1)^{l+1}\big(y_1^{l/2}+y_1^{-l/2}\big)\Big)\,. \qquad (2.5.26)$$

Once again, since $\rho_i^l \to 1$ in the $T \to \infty$ limit, one finds from (2.2.20) that the free energy in the $T \to \infty$ limit is given by

$$F \;=\; TS_{\rm MM} \;=\; -2N^2MT^4\sum_{l=1}^{\infty}\frac{1}{l^4}\Big(2+\big(y_1^l+y_1^{-l}\big)+(-1)^{l+1}\big(y_1^{l/2}+y_1^{-l/2}\big)\big(y_2^l+y_2^{-l}\big) + 2\big(y_2^l+y_2^{-l}\big)+2(-1)^{l+1}\big(y_1^{l/2}+y_1^{-l/2}\big)\Big) \tag{2.5.27}$$

$$= -2N^2MT^4\Big(2\zeta(4)+2B_4(\beta\mu_1)-2F_4\left(\beta\left(\tfrac{\mu_1}{2}+\mu_2\right)\right)-2F_4\left(\beta\left(\tfrac{\mu_1}{2}-\mu_2\right)\right) + 4B_4(\beta\mu_2)-4F_4\left(\tfrac{\beta\mu_1}{2}\right)\Big) \tag{2.5.28}$$

$$= -N^2M\left(\frac{\pi^2T^4}{6}+\frac{T^2}{4}\big(\mu_1^2+2\mu_2^2\big)-\frac{1}{32\pi^2}\big(\mu_1^4-4\mu_1^2\mu_2^2\big)\right)\mathrm{Vol}(S^3)\,. \tag{2.5.29}$$

Here $B_n(z)$ and $F_n(z)$ are given, respectively, in (E.6) and (E.7) of Ref. [47], and the polylogarithm regularization procedure described there has been used to obtain (2.5.29).[14]

The fact that the free energies (2.5.19) and (2.5.29) are both of $\mathcal{O}(N^2M)$ with respect to N suggests that the gauge theory in the phase above the Hagedorn temperature describes a non-interacting plasma of color non-singlet states. Furthermore, from the fact that the eigenvalue distribution (2.5.12)-(2.5.13) is gapped we can immediately conclude that the Polyakov loop $W(C)$ has non-zero expectation value as the trace does not average to zero in this case. In particular, this implies that the $\mathbb{Z}_N$ center symmetry is spontaneously broken in this phase. Accordingly, we label this phase "deconfined". Thus, we conclude that the phase transition defined by Eq. (2.4.13) is a confinement/deconfinement phase transition. Since furthermore the derivative of the free energy with respect to the temperature is discontinuous at the phase transition temperature $T_H(\mu_1,\mu_2)$, we conclude that the transition is of first order. Furthermore, cf. [7, 6], we identify it with a Hagedorn phase transition, and $T_H(\mu_1,\mu_2)$ is thus the Hagedorn temperature of $\mathcal{N}=2$ quiver gauge theory.

2.5.4 Order of the phase transition

Before closing this section we raise the issue of the order of the phase transition when the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ is non-zero.

[...]

$$S_{\rm MM} = \frac{N^2}{\pi}\big(m_1^2(\rho^1)^2 + b\lambda^2(\rho^1)^4\big) \tag{2.5.30}$$

where b is of order $\mathcal{O}(\lambda^0)$, and where $m_1^2 \equiv V^1_{\rm ad}(T;\mu_1,\mu_2)+V^1_{\rm bi}(T;\mu_1,\mu_2)$ is interpreted as the mass squared of the mode ρ^1. Since for fixed chemical potentials, $z^B_{\rm ad}, z^F_{\rm ad}, z^B_{\rm bi}, z^F_{\rm bi}$ all increase monotonically with the temperature, it is clear that m_1^2 changes sign across the phase transition (2.4.13): it is respectively positive, vanishing and negative when the temperature is below, equal to and above T_H. Depending on the sign of b there can either be two phase transitions as the temperature is increased (a second order deconfinement transition followed by a third order transition) or a single first order transition.

[14] Note that there is a minus sign missing on the right hand side of (E.4) for the $n \neq 1$ case.

First, consider the case $b > 0$. For temperatures below T_H we have $m_1^2 > 0$, and so $\rho^1 = 0$, corresponding to the uniform eigenvalue distribution, is a global minimum of the action (2.5.30). For $T > T_H$, however, $m_1^2 < 0$, and so $\rho^1 = 0$ becomes unstable, and $S_{\rm MM}$ has a minimum at $(\rho^1)^2 = \frac{|m_1|^2}{2b}$. At this minimum the action takes the value $S_{\rm MM} = -\frac{N^2|m_1|^4}{4b}$, and since $m_1^2 \sim K(T_H - T)$ for temperatures in a neighborhood of T_H, we have $S_{\rm MM} \sim (T - T_H)^2$, so the phase transition at $T = T_H$ is of second order. As the temperature is increased above T_H, the eigenvalue distribution becomes non-uniform, but still non-gapped. This continues to hold until $m_1^2 = -b/2$ where $|\rho^1| = 1/2$, and the eigenvalue distribution develops a gap. This happens at the temperature $T = T_H + \frac{b}{2K}$ which thus defines the threshold temperature of a second phase transition which is of third order [6].

Now consider the case where b is negative. For temperatures where $m_1^2 < |b|/2$ there are two minima, one at $\rho^1 = 0$ and one at $|\rho^1| = \frac{1}{2}$. When $m_1^2 > |b|/4$ the free energy at $|\rho^1| = \frac{1}{2}$ is positive, and so $\rho^1 = 0$ is thermodynamically preferred, having zero free energy. However, when the temperature is increased to $m_1^2 < |b|/4$, the free energy at $|\rho^1| = \frac{1}{2}$ becomes negative, and so this minimum dominates the thermal ensemble. We conclude that at $T = T_H - \frac{|b|}{2K}$, $|\rho^1|$ jumps discontinuously from 0 to $\frac{1}{2}$, and the theory undergoes a first order phase transition. Above this temperature the eigenvalue distribution becomes gapped. This behavior is qualitatively similar to that of the free theory which we have analysed in the previous sections, with the exception that the phase transition now occurs below T_H.

Settling the issue of the order and pattern of the phase transitions thus requires knowing whether b is positive or negative which in turn requires a 3-loop computation. This has so far only been carried out for pure $U(N)$ Yang-Mills theory [8] where it was found that the deconfinement phase transition is of first order.

Finally we remark that, in the context of $\mathcal{N} = 4$ $U(N)$ SYM theory on $S^1 \times S^3$, it is known that the Hagedorn singularity persists for non-zero 't Hooft coupling λ and chemical potentials, at least for $\lambda \ll 1$ [48, 10].

2.6 Comparison to the gravity dual phase transition

In the previous two sections we have studied the phase transition occuring for $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ at zero gauge coupling. From the point of view of the AdS/CFT correspondence one should expect that in the gravity dual theory there should be some analogous phase transition separating the gravity solutions that have the conformal boundary $S^1 \times S^3$ at a finite temperature. Indeed, this expectation holds true, as was first realized by Witten in [4]. In this article, dealing with thermal $\mathcal{N} = 4$ SYM theory and its dual gravity description on $AdS_5 \times S^5$, Witten argued that the *strongly* coupled gauge theory exhibits a deconfinement phase transition, and that this transition manifests itself in the dual bulk gravity theory as the Hawking-Page phase transition [5] originally studied in the context of quantum gravity on AdS spaces.

In this section we will therefore briefly review the Hawking-Page phase transition in thermal AdS_{n+1}. In [5] Hawking and Page identified two solutions to the Einstein equations whose boundary at infinity is $S^1 \times S^{n-1}$. The first one is the manifold X_1 which is obtained as a quotient of AdS_{n+1} by a specific subgroup of $SO(1, n+1)$ that is isomorphic to $\mathbb{Z}$. The metric (with Euclidean signature) can be written

$$ds^2 = \left(\frac{r^2}{b^2} + 1\right) dt^2 + \frac{b^2}{r^2 + b^2} dr^2 + r^2 d\Omega^2 \,, \tag{2.6.1}$$

with $d\Omega^2$ the metric of a round sphere S^{n-1} of unit radius. Here t is a periodic variable of arbitrary period. The topology of X_1 is $S^1 \times B^n$. Here (2.6.1) has been normalized so that the Einstein equations read

$$R_{ij} = -nb^{-2} g_{ij} \,. \tag{2.6.2}$$

The second solution, X_2, is the Schwarzschild black hole in AdS_{n+1}. The metric is

$$ds^2 = \left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}} + r^2 d\Omega^2 \,. \tag{2.6.3}$$

Here w_n is defined as the constant

$$w_n \equiv \frac{16\pi G_N}{(n-1)\mathrm{Vol}(S^{n-1})} \,. \tag{2.6.4}$$

Here G_N is the $(n+1)$-dimensional Newton's constant and $\mathrm{Vol}(S^{n-1})$ is the volume of a unit $(n-1)$-sphere; the factor w_n has been included here so that M is the mass of the black hole (as we will verify below). Note that the spacetime (2.6.3) is restricted to the region $r \geq r_+$ where r_+ denotes the largest solution to the equation

$$\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} = 0 \,. \tag{2.6.5}$$

The metric (2.6.3) will have a conical singularity at $r = r_+$ unless t is periodic with the period

$$\beta_0 = \frac{4\pi b^2 r_+}{n r_+^2 + (n-2) b^2} \,. \tag{2.6.6}$$

The inverse β_0^{-1} is referred to as the Hawking temperature of the black hole (2.6.3).

2.6.1 Entropy of AdS Schwarzschild black holes

With the normalization (2.6.2) of the cosmological constant, the bulk Einstein action is

$$I = -\frac{1}{16\pi G_N} \int d^{n+1}x \sqrt{|g|} \left(R + \frac{n(n-1)}{b^2}\right) \,. \tag{2.6.7}$$

For a solution to the equations of motion one has $R = -\frac{1}{2}n(n+1)/b^2$, and the action becomes

$$I = \frac{n}{8\pi G_N} \int d^{n+1}x \sqrt{|g|} \,, \tag{2.6.8}$$

that is, the volume of the spacetime times $\frac{n}{8\pi G_N}$. The action additionally has a surface term [49, 50], but the surface term vanishes for the AdS Schwarzschild black hole, as noted in [5], because the black hole correction to the AdS metric vanishes too rapidly at infinity.

Actually, both the AdS spacetime (2.6.1) and the black hole spacetime (2.6.3) have infinite volume. As in [5], one subtracts the two volumes to get a finite result. Putting an upper cutoff R on the radial integrations, the regularized volume of the AdS spacetime is

$$V_1(R) = \int_0^{\beta'} dt \int_0^R dr \int_{S^{n-1}} d\Omega \, r^{n-1} \,, \tag{2.6.9}$$

and the regularized volume of the AdS black hole spacetime is

$$V_2(R) = \int_0^{\beta_0} dt \int_{r_+}^R dr \int_{S^{n-1}} d\Omega \, r^{n-1} \,. \tag{2.6.10}$$

Note that the radial integration for the AdS black hole starts at $r \geq r_+$, while in the AdS spacetime $r \geq 0$. Another difference between the two integrals is that one must use different periodicities β' and β_0 for the t integrals in the two cases. The black hole spacetime is smooth only if β_0 has the value given in (2.6.6), but for the AdS spacetime, any value of β' is possible. One must adjust β' so that the geometry of the hypersurface $r = R$ is the same in the two cases. More precisely, the proper circumference of the S^1 at $r = R$ must be the same as the proper length on X_2 of an orbit of the Killing vector $\frac{\partial}{\partial t}$, also at $r = R$. This is done by setting $\beta'\sqrt{r^2/b^2+1} = \beta_0\sqrt{r^2/b^2+1-w_n M/r^{n-2}}$. After doing so, one finds that the action difference is

$$I = \frac{n}{8\pi G_N} \lim_{R\to\infty} \big(V_2(R) - V_1(R)\big) = \frac{\mathrm{Vol}(S^{n-1})(b^2 r_+^{n-1} - r_+^{n+1})}{4G_N\big(n r_+^2 + (n-2)b^2\big)} \,. \tag{2.6.11}$$

This is positive for small r_+ and negative for large r_+, showing that the thermodynamically preferred spacetime for low temperatures is the AdS spacetime (2.6.1), whereas for high temperatures the spacetime dominating the thermal ensemble is the AdS Schwarzschild black hole (2.6.3). These two geometries are separated by a phase transition occurring at a temperature β_0 for which the action difference in (2.6.11) is zero. This is the Hawking-Page phase transition first identified in [5].

One then computes the regularized energy

$$E = \frac{\partial I}{\partial \beta_0} = \frac{(n-1)\mathrm{Vol}(S^{n-1})(r_+^n b^{-2} + r_+^{n-2})}{16\pi G_N} = M \,, \tag{2.6.12}$$

which in particular shows that the M appearing in (2.6.3) is the mass of the black hole, and the regularized entropy

$$S = \beta_0 E - I = \frac{r_+^{n-1}\mathrm{Vol}(S^{n-1})}{4G_N} \tag{2.6.13}$$

of the black hole. The entropy can be written

$$S = \frac{A}{4G_N} \tag{2.6.14}$$

where A denotes the volume of the horizon; i.e., the surface at $r = r_+$.

Now, the black hole entropy (2.6.13) should be compared to the boundary conformal field theory on $S^1 \times S^{n-1}$ where the S^1 has circumference β_0; that is to say, the temperature of the boundary field theory is identified with the Hawking temperature of the black hole. Conformal invariance dictates that the coupling constant is dimensionless, and therefore the entropy density on S^{n-1} scales as $\beta_0^{-(n-1)}$ as β_0 is taken to zero. From (2.6.6), in the limit $\beta_0 \to 0$ one finds $r_+ \to \infty$ (the $r_+ \to 0$ branch is thermodynamically unfavored [5]) with $\beta_0 \sim 1/r_+$. Hence the boundary conformal field theory predicts that the entropy of this system is of order r_+^{n-1}, and is thus asymptotically a fixed multiple of the horizon volume which appears in (2.6.14). Note that the discussion here assumes $\beta_0 \ll 1$, meaning that $r_+ \gg b$; therefore, it applies only to AdS black holes whose Schwarzschild radius is much greater than the radius of curvature of AdS space. However, in this limit, one does get a simple explanation of why the black hole entropy is proportional to the area of its horizon that is entirely 'holographic'. Fixing the constant of proportionality between entropy and horizon volume presumably requires some more detailed knowledge of the boundary quantum field theory.

2.7 Quantum mechanical sectors

Since $\mathcal{N} = 2$ quiver gauge theory is a conformal field theory, we can exploit the state/operator correspondence and map the Hamiltonian H to the dilatation operator D.[15] As a consequence, the partition function of thermal $\mathcal{N} = 2$ quiver gauge theory in the grand canonical ensemble takes the form

$$Z(T; \mu_1, \mu_2) = \mathrm{Tr}_{\mathcal{H}} \left(e^{-\beta D + \beta \mu_i Q_i} \right) . \tag{2.7.1}$$

Here the trace is taken over the entire Hilbert space $\mathcal{H}$ of gauge invariant operators. For weak 't Hooft coupling $\lambda \ll 1$, the dilatation operator D can be expanded perturbatively[16]

$$D = D_0 + \sum_{n=2}^{\infty} \lambda^{n/2} D_n . \tag{2.7.2}$$

We let Q denote the total charge with respect to the Cartan generators of $SU(2)_R \times U(1)_R$, $Q = Q_1 + Q_2$, with μ as the associated chemical potential.[17] Taking $\lambda = 0$, the partition function (2.7.1) can be rewritten as

$$Z(T; \mu) = \mathrm{Tr}_{\mathcal{H}} \exp \Big(-\beta(D_0 - Q) - \beta(1-\mu)Q \Big) . \tag{2.7.3}$$

[15]To be more specific, the conformal dimension of some operator $\mathcal{O}$ on $\mathbb{R}^4$ is mapped to the energy of the associated state $\mathcal{O}|0\rangle$ on $\mathbb{R} \times S^3$ (where $|0\rangle$ denotes the vacuum state).

[16]This was shown for $\mathcal{N} = 4$ $U(N)$ SYM theory in [14, 51].

[17]Recall that in Section 1.5 we defined the generator of the Cartan subalgebra of $SU(2)_R$ to be σ_z rather than $\frac{1}{2}\sigma_z$ so that we have the associated charges Q_1, Q_2 implicitly given through Eqs. (1.5.1)-(1.5.8). It is these charges we are referring to here, rather than the R-charges given in Table 1.

Following [10], we now consider the region of small temperature and near-critical chemical potential

$$T \ll 1\,, \qquad 1-\mu \ll 1\,. \tag{2.7.4}$$

In this region, the Hilbert space of gauge invariant operators of $\mathcal{N}=2$ quiver gauge theory truncates to certain subsectors. To show this, first observe that in the region (2.7.4), operators with $D_0 > Q$ appear with an extremely small weight factor in the partition function (2.7.3) since $\beta \gg 1$. On the other hand, for operators with $D_0 = Q$, the weight factor is non-negligible precisely because $1-\mu \ll 1$. Therefore, the partition function (2.7.3) is dominated by contributions from operators belonging to the subsector

$$\mathcal{H}_0 \equiv \left\{ \mathcal{O} \in \mathcal{H} \,\middle|\, (D_0 - Q)\mathcal{O} = 0 \right\}. \tag{2.7.5}$$

We thus conclude that by taking the near-critical limit

$$x \longrightarrow 0\,, \qquad xy \;\text{ fixed}\,, \tag{2.7.6}$$

the full Hilbert space $\mathcal{H}$ of gauge-invariant operators effectively truncates to the subsector $\mathcal{H}_0$. We will consider three concrete examples of this truncation below, obtained by either turning off one of the R-symmetry chemical potentials, or by putting them equal. As we remark below, the resulting subsectors are in a certain sense quantum mechanical.

2.7.1 Case 1: The 1/2 BPS sector

We take $(\mu_1, \mu_2) = (\mu, 0)$, and thus the total Cartan charge is $Q = Q_1$. Taking the near-critical limit (2.7.6) of the partition function (2.2.1) then yields

$$Z(x;y) \;\longrightarrow\; \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\left(\sum_{i=1}^{M} \sum_{l=1}^{\infty} \frac{(xy)^l}{l} \operatorname{Tr} U_i^l \operatorname{Tr} U_i^{-l} \right). \tag{2.7.7}$$

Since the scalar field Φ_i has $D_0 = Q = 1$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to the 1/2 BPS sector spanned by multi-trace operators of the form

$$\operatorname{Tr}\left(\Phi_{i_1}^{J_1}\right) \operatorname{Tr}\left(\Phi_{i_2}^{J_2}\right) \cdots \operatorname{Tr}\left(\Phi_{i_k}^{J_k}\right). \tag{2.7.8}$$

It is clear that in the near-critical limit (2.7.6) all operators with covariant derivatives decouple. Thus all modes originating from defining a field theory on the spatial manifold S^3 are removed, and the locality of the field theory is lost. In this sense the resulting subsector of the field theory is quantum mechanical.

2.7.2 Case 2: The $SU(2)$ sector

We take $(\mu_1, \mu_2) = (0, \mu)$, and thus the total Cartan charge is $Q = Q_2$. Taking the near-critical limit (2.7.6) of the partition function (2.2.1) then yields

$$Z(x;y) \;\longrightarrow\; \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\left(\sum_{i=1}^{M} \sum_{l=1}^{\infty} \frac{2(xy)^l}{l} \operatorname{Tr} U_i^l \operatorname{Tr} U_{i+1}^{-l} \right). \tag{2.7.9}$$

Since the scalar fields $A_{i,(i+1)}$ and $B_{(i+1),i}$ both have $D_0 = Q = 1$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to the $SU(2)$ sector spanned by multi-trace operators of the form

$$\prod_{j=1}^{k} \operatorname{Tr}\left(Z^{(j)}_{1\rightarrow} Z^{(j)}_{2\rightarrow} \cdots Z^{(j)}_{J_j\rightarrow}\right) \tag{2.7.10}$$

where any letter $Z^{(j)}_{i_j\rightarrow}$ is one of the scalars $A_{i,(i+1)}$ or $B_{(i+1),i}$. The subscripts '$\rightarrow$' denote that the quiver labels on the fields in question must trace out a closed loop on the quiver diagram in Figure 1.1 so as to ensure gauge invariance. I.e., an example of a gauge invariant single-trace operator is $\operatorname{Tr}\left(A_{i,(i+1)} A_{(i+1),(i+2)} B_{(i+2),(i+1)} B_{(i+1),i}\right)$.

2.7.3 Case 3: The $SU(2|3)/\mathbb{Z}_M$ sector

We take $(\mu_1, \mu_2) = (\mu, \mu)$ and thus the total Cartan charge is $Q = Q_1 + Q_2$. Taking the near-critical limit (2.7.6) of the partition function (2.2.1) then yields

$$\begin{aligned} Z(x;y) \quad \longrightarrow \quad \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp \Bigg[\sum_{i=1}^{M} \sum_{l=1}^{\infty} \left(\frac{(xy)^l + 2(-1)^{l+1}(xy)^{3l/2}}{l} \right) \operatorname{Tr} U_i^l \operatorname{Tr} U_i^{-l} \\ + \sum_{i=1}^{M} \sum_{l=1}^{\infty} \frac{2(xy)^l}{l} \operatorname{Tr} U_i^l \operatorname{Tr} U_{i+1}^{-l} \Bigg] \,. \end{aligned} \tag{2.7.11}$$

Since the scalar fields $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$ all have $D_0 = Q = 1$, and the Weyl spinor field $\overline{\psi_{\Phi,i}}$ has $D_0 = Q = \frac{3}{2}$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to a subsector spanned by multi-trace operators of the form

$$\prod_{j=1}^{k} \operatorname{Tr}\left(W^{(j)}_{1\rightarrow} W^{(j)}_{2\rightarrow} \cdots W^{(j)}_{J_j\rightarrow}\right) \tag{2.7.12}$$

where any letter $W^{(j)}_{i_j\rightarrow}$ is either one of the scalars $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$, or the Weyl spinor $\overline{\psi_{\Phi,i}}$. Otherwise, the notation is as explained below (2.7.10).

It would be interesting to study this subsector further and determine its symmetry group. This group is presumably a subgroup of the $SU(2|3)$ symmetry observed in the $\mathcal{N} = 4$ SYM case [10], and determined by the way the $\mathbb{Z}_M$ orbifolding breaks the embedding of $SU(2|3)$ into the full $\mathcal{N} = 4$ superconformal group $PSU(2,2|4)$.

In [10] the authors considered weakly coupled $\mathcal{N} = 4$ $U(N)$ SYM theory on $S^1 \times S^3$ with R-symmetry chemical potentials in similar near-critical regions of the phase diagram as studied here. It was found that the Hilbert space of gauge invariant operators truncates to similar subsectors as identified here, namely the 1/2 BPS sector, the $SU(2)$ subsector or the $SU(2|3)$ subsector, depending on which chemical potentials are turned on. Furthermore, the analysis in [10] was generalized to small, but non-zero 't Hooft coupling λ by utilizing the 1-loop correction D_2 to the dilatation operator (cf. the perturbative expansion (2.7.2)).

In the large N limit, D_2 restricted to the $SU(2)$ subsector becomes the Hamiltonian of an $SU(2)$ spin chain; and restricted to the $SU(2|3)$ subsector it becomes the Hamiltonian of an $SU(2|3)$ spin chain. What is remarkable is that in both these cases, the spin chains are integrable [13, 14, 51], and that the truncated Hilbert spaces can be identified with subsectors of the complete dilatation operator of $\mathcal{N} = 4$ $U(N)$ SYM theory that are expected to be closed to any order in perturbation theory.

For $\mathcal{N} = 2$ quiver gauge theory, the full dilatation operator along with possible integrable subsectors is not yet completely settled, so we are not able to immediately generalize our results to small, but non-zero 't Hooft coupling λ. However, we note that much progress has been made in this area. In particular, anomalous dimensions of various operators, the anomalous dimension matrix restricted to various subsectors, Bethe ansätze and integrability have been investigated in [52, 53, 54, 55, 31, 56, 57, 58, 59, 60, 61, 62].

Chapter 3

One-loop quantum effective action with scalar VEV's

In this chapter we will extend the matrix model for $\mathcal{N}=2$ quiver gauge theory on $S^1 \times S^3$ in Sections 2.1-2.2 to include non-zero VEV's for the scalar fields. To this end we calculate the quantum effective action at weak 't Hooft coupling to 1 loop in a slice of the configuration space of the background fields. To simplify the calculation we restrict to the case of zero R-symmetry chemical potentials. The potential we compute will be valid within the temperature range $0 \leq TR \ll \lambda^{-1/2}$. The origin of the bound $TR \ll \lambda^{-1/2}$ is that the color-electric Debye screening length $(\lambda^{1/2}T)^{-1}$ of the Yang-Mills plasma[1] must be much less than the radius R of the S^3 in order for the perturbative scheme to be reliable.[2]

The method employed for computing the effective potential will be the standard background field formalism. That is, we expand the quantum fields about classical background fields and path integrate over the fluctuations, discarding terms of cubic or higher order in the fluctuations. The background fields will be taken to be static and spatially homogeneous; thus, the potential obtained from the computation will be a static effective potential. Furthermore, we carry out the computation only in a slice of the configuration space in which the background fields are mutually "commuting" in a sense that conforms to the quiver structure.

We now proceed with a more detailed description of the calculation. For convenience we first rescale all the fields in the $\mathcal{N}=2$ quiver gauge theory Lagrangian density (as given in Eqs. (1.2.40), (1.2.17), (1.2.18) and (1.2.39)) with a factor of g_{YM} as follows

$$\phi \quad\longrightarrow\quad \frac{1}{g_{\mathrm{YM}}}\,\phi\,. \tag{3.0.1}$$

We then expand the quantum fields about classical background fields by applying the

[1] For details about the computation of the Debye screening length of an $SU(N)$ Yang-Mills plasma, see (e.g.) [63] and the references therein.

[2] For a careful explanation of this point, see for example Section 3.3 of [19].

following transformations to the Lagrangian density

$$A_{i,(i+1)} \longrightarrow A_{i,(i+1)} + a_{i,(i+1)} \tag{3.0.2}$$
$$B_{(i+1),i} \longrightarrow B_{(i+1),i} + b_{(i+1),i} \tag{3.0.3}$$
$$\Phi_i \longrightarrow \Phi_i + \phi_i \tag{3.0.4}$$
$$A_{\mu i} \longrightarrow A_{\mu i} + \delta_{\mu 0}\,\alpha_i\,. \tag{3.0.5}$$

The background fields $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are assumed to solve the Euler-Lagrange EOM's so that they are the VEV's of the corresponding fluctuating fields. We take the background fields to be static and spatially homogeneous, i.e. constant on $S^1 \times S^3$. This is to preserve the $SO(4)$ isometry of S^3 as we will not examine the more exotic phases in which the vacuum spontaneously breaks rotational invariance.

The terms of the Lagrangian density arising after the transformations (3.0.2)-(3.0.5) are grouped by their order in the fluctuating fields. The terms of zeroth order are grouped into a tree-level Lagrangian density. The terms linear in the fluctuating fields combine to vanish as the background fields are solutions to the Euler-Lagrange EOM's. We discard terms containing fluctuating fields to cubic or higher order.[3] The quantum corrections to the tree-level Lagrangian density thus arise from path integrations over the terms quadratic in the fluctuations. The result will thus be valid to 1-loop order in the loop expansion.

Since the zero modes $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are constant over $S^1 \times S^3$, the tree-level action is obtained from the tree-level Lagrangian density by simply multiplying the volume[4] of $S^1 \times S^3$, yielding the result

$$\begin{aligned} S^{(0)} = \frac{2\pi^2\beta R^3}{g_{\rm YM}^2}\sum_{i=1}^{M}\mathrm{Tr}\Bigg[& -\Big(\alpha_i\, a_{i,(i+1)} - a_{i,(i+1)}\,\alpha_{i+1}\Big)\Big(\alpha_{i+1}\,\overline{a_{i,(i+1)}} - \overline{a_{i,(i+1)}}\alpha_i\Big) \\ & -\Big(\alpha_{i+1}\, b_{(i+1),i} - b_{(i+1),i}\,\alpha_i\Big)\Big(\alpha_i\,\overline{b_{(i+1),i}} - \overline{b_{(i+1),i}}\,\alpha_{i+1}\Big) \\ & -\big[\alpha_i,\phi_i\big]\big[\alpha_i,\overline{\phi_i}\big] + R^{-2}\Big(a_{i,(i+1)}\,\overline{a_{i,(i+1)}} + \overline{b_{(i+1),i}}\,b_{(i+1),i} + \phi_i\,\overline{\phi_i}\Big) \\ & +\frac{1}{2}\Big(a_{i,(i+1)}\,\overline{a_{i,(i+1)}} - \overline{a_{(i-1),i}}\,a_{(i-1),i} + b_{i,(i-1)}\,\overline{b_{i,(i-1)}} \\ & \qquad -\overline{b_{(i+1),i}}\,b_{(i+1),i} + \big[\phi_i,\overline{\phi_i}\big]\Big)^2 \\ & -2\Big(\big(a_{i,(i+1)}\,b_{(i+1),i} - b_{i,(i-1)}\,a_{(i-1),i}\big)\big(\overline{a_{(i-1),i}}\,\overline{b_{i,(i-1)}} - \overline{b_{(i+1),i}}\,\overline{a_{i,(i+1)}}\big) \\ & \quad + \big(a_{i,(i+1)}\,\phi_{i+1} - \phi_i\,a_{i,(i+1)}\big)\big(\overline{a_{i,(i+1)}}\,\overline{\phi_i} - \overline{\phi_{i+1}}\,\overline{a_{i,(i+1)}}\big) \\ & \quad + \big(b_{(i+1),i}\,\phi_i - \phi_{i+1}\,b_{(i+1),i}\big)\big(\overline{b_{(i+1),i}}\,\overline{\phi_{i+1}} - \overline{\phi_i}\,\overline{b_{(i+1),i}}\big)\Big)\Bigg]\,. \end{aligned} \tag{3.0.6}$$

[3]Note that with the redefinition of fields in (3.0.1), discarding terms of cubic or higher order in the fluctuations is the analog of taking the $g_{\rm YM} \longrightarrow 0$ limit in Section 2.1.

[4]To be specific, one obtains $S^{(0)} = \int_{S^1\times S^3} d^4x\,\sqrt{|g|}\,\mathcal{L}^{(0)} = \mathrm{Vol}(S^1)\mathrm{Vol}(S^3)\mathcal{L}^{(0)} = 2\pi^2\beta R^3\mathcal{L}^{(0)}$ where $\mathrm{Vol}(S^k)$ is to be interpreted as the k-dimensional volume of a k-sphere of some radius. Explicitly, letting r denote the radius, the k-dimensional volume is given by $\mathrm{Vol}(S^k) = \frac{2\pi^{(k+1)/2}}{\Gamma(\frac{k+1}{2})}\,r^k$.

It is technically difficult to compute the quantum corrections to the effective potential for arbitrary background fields. We will therefore impose the constraints given below which are analogous to requiring that the background fields commute, while at the same time they respect the quiver structure of the theory.

First, the Polyakov loops must "commute" with the scalar VEV's:

$$\begin{aligned}
\alpha_i\, a_{i,(i+1)} - a_{i,(i+1)}\, \alpha_{i+1} &= 0\,, & \alpha_{i+1}\, \overline{a_{i,(i+1)}} - \overline{a_{i,(i+1)}}\, \alpha_i &= 0 \\
\alpha_{i+1}\, b_{(i+1),i} - b_{(i+1),i}\, \alpha_i &= 0\,, & \alpha_i\, \overline{b_{(i+1),i}} - \overline{b_{(i+1),i}}\, \alpha_{i+1} &= 0 \\
\left[\alpha_i, \phi_i\right] &= 0\,, & \left[\alpha_i, \overline{\phi_i}\right] &= 0\,.
\end{aligned} \tag{3.0.7}$$

Second, the scalar VEV's must "commute" among themselves:

$$\begin{aligned}
a_{i,(i+1)}\, \overline{a_{i,(i+1)}} - \overline{a_{(i-1),i}}\, a_{(i-1),i} &= 0\,, & b_{i,(i-1)}\, \overline{b_{i,(i-1)}} - \overline{b_{(i+1),i}}\, b_{(i+1),i} &= 0 \\
\left[\phi_i, \overline{\phi_i}\right] &= 0\,, & a_{(i-1),i}\, \overline{b_{(i+1),i}} - \overline{b_{i,(i-1)}}\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, b_{i,(i-1)} - b_{(i+1),i}\, \overline{a_{(i-1),i}} &= 0\,, & a_{i,(i+1)}\, \overline{\phi_{i+1}} - \overline{\phi_i}\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, \phi_i - \phi_{i+1}\, \overline{a_{i,(i+1)}} &= 0\,, & b_{(i+1),i}\, \overline{\phi_i} - \overline{\phi_{i+1}}\, b_{(i+1),i} &= 0 \\
\overline{b_{(i+1),i}}\, \phi_{i+1} - \phi_i\, \overline{b_{(i+1),i}} &= 0\,, & a_{i,(i+1)}\, b_{(i+1),i} - b_{i,(i-1)}\, a_{(i-1),i} &= 0 \\
\overline{a_{(i-1),i}}\, \overline{b_{i,(i-1)}} - \overline{b_{(i+1),i}}\, \overline{a_{i,(i+1)}} &= 0\,, & a_{i,(i+1)}\, \phi_{i+1} - \phi_i\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, \overline{\phi_i} - \overline{\phi_{i+1}}\, \overline{a_{i,(i+1)}} &= 0\,, & b_{(i+1),i}\, \phi_i - \phi_{i+1}\, b_{(i+1),i} &= 0 \\
\overline{b_{(i+1),i}}\, \overline{\phi_{i+1}} - \overline{\phi_i}\, \overline{b_{(i+1),i}} &= 0\,.
\end{aligned} \tag{3.0.8}$$

After imposing the constraints (3.0.7)-(3.0.8) the tree-level action reduces to

$$S^{(0)} = \frac{2\pi^2 \beta R}{g_{\rm YM}^2} \sum_{i=1}^{M} {\rm Tr}\left(a_{i,(i+1)}\, \overline{a_{i,(i+1)}} + b_{(i+1),i}\, \overline{b_{(i+1),i}} + \phi_i\, \overline{\phi_i} \right). \tag{3.0.9}$$

We choose an R_ξ gauge defined by adding the gauge fixing action

$$\begin{aligned}
S_{\rm g.f.} = \frac{1}{g_{\rm YM}^2} \frac{1}{2\xi} \sum_{i=1}^{M} \int d^4x \sqrt{|g|}\, {\rm Tr} \Big[\partial_\mu A_{\mu i} + i[\alpha_i, A_{0i}] + i\xi \Big(\Big(\overline{a_{(i-1),i}} A_{(i-1),i} - A_{i,(i+1)} \overline{a_{i,(i+1)}} \Big) \\
+ \Big(a_{i,(i+1)}\, \overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}}\, a_{(i-1),i} \Big) + \Big(\overline{b_{(i+1),i}}\, B_{(i+1),i} - B_{i,(i-1)}\, \overline{b_{i,(i-1)}} \Big) \\
+ \Big(b_{i,(i-1)}\, \overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\, b_{(i+1),i} \Big) + [\overline{\phi_i}, \Phi_i] + [\phi_i, \overline{\Phi_i}] \Big) \Big]^2 .
\end{aligned} \tag{3.0.10}$$

We will furthermore choose the Feynman gauge $\xi = 1$ for convenience. The virtue of this gauge fixing action is that, using (3.0.7)-(3.0.8), it cancels terms appearing in the Lagrangian density after the transformations (3.0.2)-(3.0.5) that contain both gauge field and scalar field fluctuations. Thus, one can do the path integrations over the gauge field fluctuations and over the scalar field fluctuations separately.

Specification of the vacuum

We will restrict to the case where all the zero modes $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are taken to be diagonal $N \times N$ matrices.[5] The most general ansatz satisfying all the constraints (3.0.7)-(3.0.8) is given by

$$\begin{aligned} a_{i,(i+1)} &= \mathrm{diag}\big(e^{i\theta_1^i}, \ldots, e^{i\theta_N^i}\big)\, a_{(i-1),i} && (3.0.16) \\ b_{(i+1),i} &= \mathrm{diag}\big(e^{-i\theta_1^i}, \ldots, e^{-i\theta_N^i}\big)\, b_{i,(i-1)} && (3.0.17) \\ \phi_i &= \phi_{i+1} && (3.0.18) \\ \alpha_i &= \alpha_{i+1}\,. && (3.0.19) \end{aligned}$$

However, the gauge invariance and quiver translation invariance of the action places strong constraints on the moduli θ_j^i as we will now consider. Under a gauge transformation

$$a_{i,(i+1)} \longrightarrow U a_{i,(i+1)} U^\dagger \qquad \forall\, i = 1, \ldots, M\,, \tag{3.0.20}$$

Eq. (3.0.16) becomes

$$U a_{i,(i+1)} U^\dagger = \mathrm{diag}\big(e^{i\theta_1^i}, \ldots, e^{i\theta_N^i}\big)\, U a_{(i-1),i} U^\dagger \tag{3.0.21}$$

and analogously for (3.0.17). Thus, in order for (3.0.16)-(3.0.17) to be invariant under gauge transformations, we must have

$$U \mathrm{diag}\big(e^{i\theta_1^i}, \ldots, e^{i\theta_N^i}\big) U^\dagger = \mathrm{diag}\big(e^{i\theta_1^i}, \ldots, e^{i\theta_N^i}\big)\,. \tag{3.0.22}$$

This is equivalent to requiring $\theta_1^i = \cdots = \theta_N^i$. Similarly, since the action is invariant under quiver translations $a_{i,(i+1)} \longrightarrow a_{(i+1),(i+2)}$, the moduli θ_j^i must also be independent of

[5] When the VEV's are allowed to be off-diagonal, satisfying the constraints (3.0.7)-(3.0.8) along with the quiver M-periodicity (i.e., $a_{(i+M),(i+M+1)} = a_{i,(i+1)}$ etc.) ultimately leads to relations between N and M, as the following example illustrates.

Define the "raising" and "lowering" operators

$$(J^+)_{kl} = \begin{cases} 1 & \text{if } l = k+1 \\ 0 & \text{otherwise} \end{cases}\,, \qquad (J^-)_{kl} = \begin{cases} 1 & \text{if } l = k-1 \\ 0 & \text{otherwise} \end{cases}\,. \tag{3.0.11}$$

Assume that α_i and ϕ_i are diagonal and that $a_{i,(i+1)}$ and $b_{(i+1),i}$ are above and below diagonal, respectively (i.e., $(a_{i,(i+1)})_{kl} = 0$ unless $l = k+1$ and $(b_{(i+1),i})_{kl} = 0$ unless $l = k-1$). Then the constraints (3.0.7)-(3.0.8) are satisfied by imposing the following relations,

$$\begin{aligned} a_{i,(i+1)} &= J^- a_{(i-1),i} J^+\,, & \overline{a_{i,(i+1)}} &= J^- \overline{a_{(i-1),i}} J^+ && (3.0.12) \\ b_{(i+1),i} &= J^- b_{i,(i-1)} J^+\,, & \overline{b_{(i+1),i}} &= J^- \overline{b_{i,(i-1)}} J^+ && (3.0.13) \\ \phi_{i+1} &= J^- \phi_i J^+\,, & \overline{\phi_{i+1}} &= J^- \overline{\phi_i} J^+\,. && (3.0.14) \end{aligned}$$

By the quiver M-periodicity we must have

$$(J^+)^M = (J^-)^M = \mathbf{1}_N \tag{3.0.15}$$

which is true if, and only if, $N \,|\, M$ (since $J^\pm$ thought of as group elements have order N).

i. Finally, because of the identification $a_{i,(i+1)} = a_{(i+M),(i+M+1)}$, the moduli θ^i_j must be integer multiples of $\frac{2\pi}{M}$.

We conclude that upon restricting to diagonal VEV's $a_{i,(i+1)}, b_{(i+1),i}, \phi_i, \alpha_i$, the most general vacuum satisfying (3.0.7)-(3.0.8) and respecting the gauge invariance and quiver translational invariance of the action along with the quiver M-periodicity is

$$
\begin{aligned}
a_{i,(i+1)} &= \omega^k a_{(i-1),i} && (3.0.23)\\
b_{(i+1),i} &= \omega^{-k} b_{i,(i-1)} && (3.0.24)\\
\phi_i &= \phi_{i+1} && (3.0.25)\\
\alpha_i &= \alpha_{i+1} && (3.0.26)
\end{aligned}
$$

where $\omega = e^{2\pi i/M}$ and $k \in \mathbb{Z}$. This is the vacuum we will adhere to in the computations throughout this section. We will find that the expression for the quantum effective action is independent of the value of k in (3.0.23)-(3.0.24).

3.1 Quantum corrections from bosonic fluctuations

There are radiative corrections to the tree-level potential coming from path integrations over the part of the action that is quadratic in the bosonic fluctuations. Below we present in a bilinear form the part of the action that is quadratic in the bosonic fluctuations, as it appears after being added to the gauge fixing action (3.0.10) and the Fadeev-Popov ghost action, and the constraints (3.0.7)-(3.0.8) have been imposed. The path integrals will then be Gaussian and can be evaluated easily.

First we introduce some notation. Define

$$
\mathbf{A}_{\mu mn} \equiv \begin{pmatrix} (A_{\mu 1})_{mn} \\ \vdots \\ (A_{\mu M})_{mn} \end{pmatrix}, \qquad \mathbf{A}_{mn} \equiv \begin{pmatrix} (A_{1,2})_{mn} \\ \vdots \\ (A_{M,1})_{mn} \end{pmatrix}, \qquad (3.1.1)
$$

$$
\mathbf{B}_{mn} \equiv \begin{pmatrix} (B_{1,M})_{mn} \\ \vdots \\ (B_{M,(M-1)})_{mn} \end{pmatrix}, \qquad \mathbf{\Phi}_{mn} \equiv \begin{pmatrix} (\Phi_1)_{mn} \\ \vdots \\ (\Phi_M)_{mn} \end{pmatrix} \qquad (3.1.2)
$$

so that, e.g.,

$$
(\mathbf{A}^T)_{mn} = \Big((A_{1,2})_{mn}, \ldots, (A_{M,1})_{mn}\Big) \qquad \text{and} \qquad \mathbf{A}^*_{mn} = \begin{pmatrix} (\overline{A_{1,2}})_{nm} \\ \vdots \\ (\overline{A_{M,1}})_{nm} \end{pmatrix}. \qquad (3.1.3)
$$

Furthermore, define for fixed m, n the fluctuation operators $\Box^{mn}_g, \Box^{mn}_{\mathbf{A}}, \Box^{mn}_{\mathbf{B}}$ and $\Box^{mn}_{\mathbf{\Phi}}$ as the following $M \times M$ matrices (labelled by $i, j = 1, \ldots, M$)

$$(\Box_g^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j = i-1 \\ -\partial^2 - 2i(\alpha_i^n - \alpha_i^m)\,\partial_0 + (\alpha_i^n - \alpha_i^m)^2 & \\ \quad +2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{(i-1),i})_{mm}\,(\overline{a_{(i-1),i}})_{mm} & \\ \qquad + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j = i \\ -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j = i+1 \end{cases} \tag{3.1.4}$$

and

$$(\Box_{\mathbf{A}}^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{i,(i+1)}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{(i+1),i})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j = i-1 \\ -\partial^2 - 2i(\alpha_{i+1}^n - \alpha_i^m)\,\partial_0 + (\alpha_{i+1}^n - \alpha_i^m)^2 + R^{-2} & \\ \quad +\ 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{i,(i+1)})_{mm}\,(\overline{a_{i,(i+1)}})_{mm} & \\ \qquad + (b_{(i+1),i})_{nn}\,(\overline{b_{(i+1),i}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_{i+1})_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_{i+1}})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j = i \\ -2\Big((a_{(i+1),(i+2)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+2),(i+1)}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j = i+1 \end{cases} \tag{3.1.5}$$

and

$$(\Box_{\mathbf{B}}^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{i,(i+1)})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{(i+1),i}})_{mm}\Big) & \text{for } j = i-1 \\ -\partial^2 - 2i(\alpha_i^n - \alpha_{i+1}^m)\,\partial_0 + (\alpha_i^n - \alpha_{i+1}^m)^2 + R^{-2} & \\ \quad +\ 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{i,(i+1)})_{mm}\,(\overline{a_{i,(i+1)}})_{mm} & \\ \qquad + (b_{(i+1),i})_{nn}\,(\overline{b_{(i+1),i}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_i)_{nn} - (\phi_{i+1})_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_{i+1}})_{mm}\big)\Big) & \text{for } j = i \\ -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{(i+1),(i+2)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+2),(i+1)})_{mm}\Big) & \text{for } j = i+1 \end{cases} \tag{3.1.6}$$

and

$$(\Box^{mn}_{\mathbf{\Phi}})_{ij} \;=\; \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j=i-1 \\ -\partial^2 - 2i(\alpha_i^n - \alpha_i^m)\,\partial_0 + (\alpha_i^n - \alpha_i^m)^2 + R^{-2} & \\ \quad +\ 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{(i-1),i})_{mm}\,(\overline{a_{(i-1),i}})_{mm} & \\ \qquad + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j=i \\ -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j=i+1 \end{cases} \tag{3.1.7}$$

Then the part of the action that is quadratic in the bosonic fluctuations (including the Fadeev-Popov ghosts $\overline{c_i}, c_i$) can be written in the form ($k = 1, 2, 3$)

$$\begin{aligned} S_b \;=\; \frac{1}{g^2_{\rm YM}} \sum_{m,n=1}^{N} \int d^4x \sqrt{|g|}\, \Big(& \tfrac{1}{2}(\mathbf{A}_k^{\perp\,T})_{mn}\,\Box^{mn}_g\,(\mathbf{A}_k^{\perp})_{nm} + \tfrac{1}{2}(\partial_k \mathbf{F}^T)_{mn}\,\Box^{mn}_g\,(\partial_k \mathbf{F})_{nm} \\ & + \tfrac{1}{2}\,(\mathbf{A}_0^T)_{mn}\,\Box^{mn}_g\,\mathbf{A}_{0\,nm} + (\overline{\mathbf{c}}^T)_{mn}\,\Box^{mn}_g\,\mathbf{c}^*_{mn} \\ & + (\mathbf{A}^T)_{mn}\,\Box^{mn}_{\mathbf{A}}\,\mathbf{A}^*_{mn} + (\mathbf{B}^T)_{mn}\,\Box^{mn}_{\mathbf{B}}\,\mathbf{B}^*_{mn} + (\mathbf{\Phi}^T)_{mn}\,\Box^{mn}_{\mathbf{\Phi}}\,\mathbf{\Phi}^*_{mn}\Big) \end{aligned} \tag{3.1.8}$$

where, as in Section 2.1, the spatial components of the gauge field have been decomposed into a transversal (i.e., divergenceless) part $(A_i^{\perp})^k$ and a longitudinal part $(\nabla F_i)^k$. Thereby all the fields have been written in terms of S^3 spherical harmonics. The path integrations over the bosonic fluctuations $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$ and $A_{\mu\,i}$ can now readily be done and yield the formal expression[6]

$$\begin{aligned} \Gamma_{\rm bos}\big[\alpha_i, a_{i,(i+1)}, b_{(i+1),i}, \phi_i\big] \;=\; & \frac{1}{2}\sum_{m,n=1}^{N} {\rm Tr}\ln\det \Box^{mn}_g + \sum_{m,n=1}^{N} {\rm Tr}\ln\det \Box^{mn}_{\mathbf{A}} \\ & + \sum_{m,n=1}^{N} {\rm Tr}\ln\det \Box^{mn}_{\mathbf{B}} + \sum_{m,n=1}^{N} {\rm Tr}\ln\det \Box^{mn}_{\mathbf{\Phi}}\,. \end{aligned} \tag{3.1.9}$$

Here the traces are taken over the Matsubara frequencies and over the S^3 spherical harmonics, and the determinants are taken over the i, j indices of the operators $\Box^{mn}_g, \Box^{mn}_{\mathbf{A}}, \Box^{mn}_{\mathbf{B}}$ and $\Box^{mn}_{\mathbf{\Phi}}$. In the general vacuum (3.0.16)-(3.0.19) these operators are tridiagonal, periodically continued matrices (assuming $M \geq 3$). The determinant of this class of matrices was

[6]We are using a rather sloppy notation here as the term involving $\Box^{mn}_g$ is to be interpreted as the total contribution from the path integrations over the transversal and longitudinal parts of the spatial components of the gauge field, the time component of the gauge field and the Fadeev-Popov ghosts. The individual contributions are explicitly written out in (3.1.13) below.

considered in Ref. [64] (Appendix B) who found the following result, valid for $M \geq 3$:

$$\begin{aligned} \det \Box^{mn} &= \operatorname{tr} \prod_{i=M}^{1} \begin{pmatrix} (\Box^{mn})_{ii} & -(\Box^{mn})_{i,(i-1)}(\Box^{mn})_{(i-1),i} \\ 1 & 0 \end{pmatrix} \\ &\quad + (-1)^{M+1} \operatorname{tr} \prod_{i=M}^{1} \begin{pmatrix} (\Box^{mn})_{i,(i-1)} & 0 \\ 0 & (\Box^{mn})_{(i-1),i} \end{pmatrix} . \end{aligned} \tag{3.1.10}$$

The inverse order of the initial and final indices on the product symbol indicates that the matrix with the highest index i is on the left of the product.

Fortunately, in the vacuum (3.0.23)-(3.0.26) the fluctuation determinants take a much simpler form. Namely, using (3.0.23)-(3.0.26), the operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}, \Box_{\mathbf{\Phi}}^{mn}$ (for fixed m, n) can be written in the particular form below, and there is a simple closed expression for the determinant.[7] That is, defining $\omega \equiv e^{2\pi i/M}$, we have the determinant formula

$$\det \begin{pmatrix} \xi & -\eta & & & -\omega^{-k(M-1)}\overline{\eta} \\ -\overline{\eta} & \xi & -\omega^{k}\eta & & \\ & -\omega^{-k}\overline{\eta} & \xi & \ddots & \\ & & \ddots & \ddots & -\omega^{k(M-2)}\eta \\ -\omega^{k(M-1)}\eta & & & -\omega^{-k(M-2)}\overline{\eta} & \xi \end{pmatrix} = \prod_{i=1}^{M} \left(\xi - \omega^{i}\eta - \omega^{-i}\overline{\eta} \right) . \tag{3.1.11}$$

Note in particular that the phases ω^k on the left hand side cancel out. Therefore, for any value of $k \in \mathbb{Z}$ in (3.0.23)-(3.0.24), one obtains the same result for the fluctuation determinants.

Let us define here for convenience

$$\begin{aligned} v_{i,j;\,n,m} \equiv\ & 2\Big(\big((a_{i,(i+1)})_{nn} - \omega^{-j}\,(a_{i,(i+1)})_{mm}\big)\big((\overline{a_{i,(i+1)}})_{nn} - \omega^{j}\,(\overline{a_{i,(i+1)}})_{mm}\big) \\ & + \big((b_{(i+1),i})_{nn} - \omega^{j}\,(b_{(i+1),i})_{mm}\big)\big((\overline{b_{(i+1),i}})_{nn} - \omega^{-j}\,(\overline{b_{(i+1),i}})_{mm}\big) \\ & + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) . \end{aligned} \tag{3.1.12}$$

Now we apply the determinant formula (3.1.11) to the formal expression (3.1.9) for Γ_{bos}. Then we take the traces over the Matsubara frequencies and over the S^3 spherical harmon-

[7] To prove the formula, note first that the powers of ω^k appearing in the super- and subdiagonal mutually cancel according to (3.1.10), so the determinant is independent of the value of k. Putting $k = 0$, the formula (3.1.10) is a special case of Eq. (A.1) in Ref. [65].

ics, labelled by the angular momentum h (see Table 2). This yields the following result

$$\begin{aligned}\Gamma_{\rm bos} \;=\; & \frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_g^2+v_{i,j;\,n,m}\right]\\ & +\frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h>0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+v_{i,j;\,n,m}\right]\\ & +\left(\frac{1}{2}-1\right)\frac{1}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+v_{i,j;\,n,m}\right]\\ & +\frac{3}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+R^{-2}+v_{i,j;\,n,m}\right].\end{aligned} \tag{3.1.13}$$

Here the first line comes from the path integrations over the transverse part of the spatial gauge field, and the second line from the integrations over the longitudinal part. The third line comes from integrating over the temporal component of the gauge field and the Fadeev-Popov ghosts, contributing with the weights $\frac{1}{2}$ and -1, respectively. Finally, the fourth line comes from path integrating over the conformally coupled scalar fluctuations. Note that there is an exact cancellation between the contributions of all $h > 0$ spherical harmonics in the second and third line. As we will see in Section 4.2, the surviving contribution from the $h = 0$ scalar spherical harmonic will be the dominating radiative correction in the low-temperature regime.

The summations over the (bosonic) Matsubara frequencies $\omega_k \equiv \frac{2\pi k}{\beta}$ can be performed explicitly by applying the following identity which holds up to an additive constant that does not depend on Δ

$$\sum_{k=-\infty}^{\infty}\ln\left[(\omega_k+\alpha)^2+\Delta^2\right] = \ln\left[\left(1-e^{-\beta(\Delta+i\alpha)}\right)\left(1-e^{-\beta(\Delta-i\alpha)}\right)\right]+\beta\Delta \tag{3.1.14}$$

$$= 2\ln\left|2\sinh\frac{\beta\Delta+i\beta\alpha}{2}\right| \tag{3.1.15}$$

$$= \beta\Delta-2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l\Delta}\cos(\beta l\alpha)\,. \tag{3.1.16}$$

Here (3.1.14) is the Matsubara frequency sum (2.1.42) and (3.1.16) is proven by Taylor expanding ln in (3.1.15) and using the definition of sinh along with the identity $\ln|z| = \mathrm{Re}\ln z$.

After performing the summations over the Matsubara frequencies by use of (3.1.16) and writing out the traces over the S^3 spherical harmonics with the appropriate eigenvalues of

∇^2 and their degeneracies (cf. Table 2) we find

$$
\begin{aligned}
\Gamma_{\rm bos} = \frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\Bigg[& -\beta\big(v_{i,j;\,n,m}\big)^{1/2} + 2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l(v_{i,j;\,n,m})^{1/2}}\cos\big(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\big) \\
& + \sum_{h=0}^{\infty} 2h(h+2)\Bigg(\beta\big((h+1)^2R^{-2}+v_{i,j;\,n,m}\big)^{1/2} \\
& \qquad - 2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)^2R^{-2}+v_{i,j;\,n,m})^{1/2}}\cos\big(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\big)\Bigg) \\
& + 6\sum_{h=0}^{\infty}(h+1)^2\Bigg(\beta\big((h+1)^2R^{-2}+v_{i,j;\,n,m}\big)^{1/2} \\
& \qquad - 2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)^2R^{-2}+v_{i,j;\,n,m})^{1/2}}\cos\big(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\big)\Bigg)\Bigg]
\end{aligned}
\tag{3.1.17}
$$

where $v_{i,j;\,n,m}$ is defined in (3.1.12). This is the complete result for the contribution to the quantum effective action coming from bosonic fluctuations.

3.2 Quantum corrections from fermionic fluctuations

The fluctuating fermionic fields will also give rise to radiative corrections that can be computed much along the lines of the bosonic corrections. It is convenient to carry out the calculation using $\mathcal{N}=4$ SYM notation for the Weyl spinor fields. The quiver structure of the action is taken into account by the Ω_c factors as explained below.

Once again, we will need to introduce some notation in order to write up the part of the action that is quadratic in the fermionic fluctuations. Define

$$
J_k^+(\omega) \equiv \begin{pmatrix} 0 & 1 & & & \\ & 0 & \omega^k & & \\ & & 0 & \ddots & \\ & & & \ddots & \omega^{k(M-2)} \\ \omega^{k(M-1)} & & & & 0 \end{pmatrix}, \quad J_k^-(\omega) \equiv \begin{pmatrix} 0 & & & & \omega^{k(M-1)} \\ 1 & 0 & & & \\ & \omega^k & 0 & & \\ & & \ddots & \ddots & \\ & & & \omega^{k(M-2)} & 0 \end{pmatrix}
\tag{3.2.1}
$$

and

$$
\begin{aligned}
w_1 &\equiv a = \langle A\rangle\,, & \Omega_1 &\equiv J_k^+(\omega) & (3.2.2)\\
w_2 &\equiv b = \langle B\rangle\,, & \Omega_2 &\equiv J_k^-(\omega^{-1}) & (3.2.3)\\
w_3 &\equiv \phi = \langle \Phi\rangle\,, & \Omega_3 &\equiv \mathbf{1}_M & (3.2.4)
\end{aligned}
$$

where it is implied that A, B, Φ take the $\mathbb{Z}_M$ projection invariant forms given in (1.2.15)-(1.2.16). The fermionic part of the Lagrangian density can be written in $\mathcal{N}=4$ SYM notation (cf. (1.2.25)) in the following bilinear form

$$\mathcal{L}_{\rm ferm} = \frac{1}{g_{\rm YM}^2}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\big((\overline{\lambda_p})_{i;mn},(\lambda_p)_{i;mn}\big)\,\mathbf{D}_{ij}^{mn}\begin{pmatrix}(\lambda_q)_{j;nm}\\(\overline{\lambda_q})_{j;nm}\end{pmatrix} \tag{3.2.5}$$

where $(c,d=1,2,3)$

$$\mathbf{D}_{ij}^{mn} \equiv \begin{pmatrix} \frac{i}{2}\delta_{pq}\overline{\tau_\mu}\big(\partial_\mu+\delta_{\mu 0}(\alpha_i^n-\alpha_i^m)\big)\delta_{ij} & \begin{matrix}-\frac{1}{\sqrt{2}}\alpha_{pq}^c\left[\big((w_c)_{nn}+(\overline{w_c})_{nn}\big)-\big((w_c)_{mm}\Omega_c+(\overline{w_c})_{mm}\Omega_c^{-1}\big)\right]_{ij}\\ +\frac{1}{\sqrt{2}}\beta_{pq}^c\left[\big((w_c)_{nn}-(\overline{w_c})_{nn}\big)-\big((w_c)_{mm}\Omega_c-(\overline{w_c})_{mm}\Omega_c^{-1}\big)\right]_{ij}\end{matrix} \\ \begin{matrix}-\frac{1}{\sqrt{2}}\alpha_{pq}^d\left[\big((w_d)_{nn}+(\overline{w_d})_{nn}\big)-\big((w_d)_{mm}\Omega_d+(\overline{w_d})_{mm}\Omega_d^{-1}\big)\right]_{ij}\\ -\frac{1}{\sqrt{2}}\beta_{pq}^d\left[\big((w_d)_{nn}-(\overline{w_d})_{nn}\big)-\big((w_d)_{mm}\Omega_d-(\overline{w_d})_{mm}\Omega_d^{-1}\big)\right]_{ij}\end{matrix} & \frac{i}{2}\delta_{pq}\tau_\nu\big(\partial_\nu+\delta_{\nu 0}(\alpha_i^n-\alpha_i^m)\big)\delta_{ij} \end{pmatrix} \tag{3.2.6}$$

The reason why the w_c entries labelled by the gauge index m have additional factors of Ω_c compared to the entries labelled by n comes from the commutator structure of the Yukawa coupling (see (1.2.35)). Namely, when taking the trace over the gauge indices, the w_c entries labelled by n correspond to the terms where a scalar field appears between two spinor fields, whereas those labelled with m correspond to the terms where the scalar field appears to the right of both spinor fields. After substituting the orbifold projection invariant forms given in Eqs. (1.2.15)-(1.2.16) and (1.2.37)-(1.2.38), the bifundamental scalar VEV's will couple different pairs of spinor fields depending on whether the VEV appears between the spinor fields or to the right of them in the Yukawa coupling. Since the scalar VEV's are mutually related through the vacuum (3.0.23)-(3.0.26), this can be compensated for by appropriately multiplying factors of Ω_c.

To compute the result of the path integrations it is convenient to define (for a fixed c)

$$F_c \equiv \big((w_c)_{nn}+(\overline{w_c})_{nn}\big)-\big((w_c)_{mm}\Omega_c+(\overline{w_c})_{mm}\Omega_c^{-1}\big) \tag{3.2.7}$$
$$G_c \equiv \big((w_c)_{nn}-(\overline{w_c})_{nn}\big)-\big((w_c)_{mm}\Omega_c-(\overline{w_c})_{mm}\Omega_c^{-1}\big)\,. \tag{3.2.8}$$

Noting that

$$\big[F_c,F_d\big]=0\,,\qquad \big[F_c,G_d\big]=0\,,\qquad \big[G_c,G_d\big]=0 \tag{3.2.9}$$

one finds, by using the (anti)commutation relations (1.2.6) for α^c and β^d, that the result of the path integrations over the fermionic fluctuations $(\lambda_p)_i, (\overline{\lambda_p})_i$ is

$$\det(\mathbf{D}_{ij}^{mn}) = \det\Big(-\big(\partial_\mu+i\delta_{\mu 0}(\alpha_i^n-\alpha_i^m)\big)^2 - \tfrac{1}{2}\big(\tfrac{1}{2}\{\alpha^c,\alpha^d\}_{pr}F_cF_d+[\alpha^c,\beta^d]_{pr}F_cG_d-\tfrac{1}{2}\{\beta^c,\beta^d\}_{pr}G_cG_d\big)\Big) \tag{3.2.10}$$
$$= \det\Big(\big(i\partial_\mu-\delta_{\mu 0}(\alpha_i^n-\alpha_i^m)\big)^2+\tfrac{1}{2}(F_cF_d-G_cG_d)\,\delta_{cd}\delta_{pr}\Big) \tag{3.2.11}$$
$$= \det\Delta_{ij}\,. \tag{3.2.12}$$

Here we have defined the $M \times M$ matrix (labelled by $i, j = 1, \dots, M$)

$$\Delta_{ij} = \begin{cases} -2\Big((a_{1,2})_{nn}\,(\overline{a_{1,2}})_{mm} + (\overline{b_{2,1}})_{nn}\,(b_{2,1})_{mm}\Big)\,\omega^{-(i-2)k} & \text{for } j = i-1 \\ -\partial^2 - 2i(\alpha_1^n - \alpha_1^m)\partial_0 + (\alpha_1^n - \alpha_1^m)^2 & \\ \quad + 2\Big((a_{1,2})_{nn}\,(\overline{a_{1,2}})_{nn} + (a_{1,2})_{mm}\,(\overline{a_{1,2}})_{mm} & \\ \qquad + (b_{2,1})_{nn}\,(\overline{b_{2,1}})_{nn} + (b_{2,1})_{mm}\,(\overline{b_{2,1}})_{mm} & \\ \qquad + \big((\phi_1)_{nn} - (\phi_1)_{mm}\big)\big((\overline{\phi_1})_{nn} - (\overline{\phi_1})_{mm}\big)\Big) & \text{for } j = i \\ -2\Big((\overline{a_{1,2}})_{nn}\,(a_{1,2})_{mm} + (b_{2,1})_{nn}\,(\overline{b_{2,1}})_{mm}\Big)\,\omega^{(i-1)k} & \text{for } j = i+1 \end{cases} \tag{3.2.13}$$

where we have used (3.0.23)-(3.0.26) to arrive at the equality (3.2.12). Applying the determinant formula (3.1.11) and using (3.0.23)-(3.0.26) again one finds, after taking the traces over the fermionic Matsubara frequencies $\omega_k \equiv \frac{(2k+1)\pi}{\beta}$ and over the S^3 spherical harmonics,

$$\Gamma_{\text{ferm}} = -\frac{4}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty} \text{Tr}_{h\geq 0} \ln\Big[\big(\omega_k + (\alpha_i^{nn} - \alpha_i^{mm})\big)^2 + \Delta_f^2 + v_{i,j;\,n,m}\Big]. \tag{3.2.14}$$

The factor 4 comes from the 4 path integrations. After carrying out the summation over k by use of the identity[8]

$$\sum_{k=-\infty}^{\infty} \ln\left[(\omega_k + \alpha)^2 + \Delta^2\right] = \beta\Delta + 2\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}e^{-\beta l \Delta}\cos(\beta l \alpha) \tag{3.2.15}$$

and writing out the trace over the S^3 spherical harmonics with the appropriate eigenvalues of ∇^2 and their degeneracies one finds the result

$$\begin{aligned}\Gamma_{\text{ferm}} = -\frac{4}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{h=0}^{\infty} h(h+1)\Bigg(&\beta\left(\left(h+\tfrac{1}{2}\right)^2 R^{-2} + v_{i,j;\,n,m}\right)^{1/2} \\ &+ 2\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}e^{-\beta l\left(\left(h+\frac{1}{2}\right)^2 R^{-2} + v_{i,j;\,n,m}\right)^{1/2}}\cos\left(\beta l(\alpha_i^{nn} - \alpha_i^{mm})\right)\Bigg)\end{aligned} \tag{3.2.16}$$

where $v_{i,j;\,n,m}$ is defined in (3.1.12). This is the complete result for the contribution to the quantum effective action coming from fermionic fluctuations.

We conclude that the quantum effective action of $\mathcal{N} = 2$ quiver gauge theory with constant scalar field VEV's satisfying (3.0.7)-(3.0.8) is given by

$$\Gamma_{\text{eff}} = S^{(0)} + \Gamma_{\text{bos}} + \Gamma_{\text{ferm}} \tag{3.2.17}$$

[8] Once again, the identity (3.2.15) is only valid up to an additive constant that does not depend on Δ. It is immediately obtained from (3.1.16) by substituting $\alpha \to \alpha + \frac{\pi}{\beta}$.

where $S^{(0)}$ is the tree-level action

$$S^{(0)} = \frac{2\pi^2\beta R}{g_{\rm YM}^2}\sum_{i=1}^{M}\sum_{n=1}^{N}\Big((a_{i,(i+1)})_{nn}(\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn}(\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn}(\overline{\phi_i})_{nn}\Big)\,. \tag{3.2.18}$$

and $\Gamma_{\rm bos}$ and $\Gamma_{\rm ferm}$ are given in (3.1.17) and (3.2.16), respectively, with $v_{i,j;\,n,m}$ given in (3.1.12).

Note that the tree-level potential (3.2.18) is attractive, whereas the 1-loop quantum corrections in (3.1.17) and (3.2.16) are repulsive. As we will see in the Section 4.2, the competition between an attractive and a repulsive part of the potential will cause the equilibrium configurations of the eigenvalues of the scalar VEV's to be hypersurfaces.

3.3 Generalization to other $\mathbb{Z}_M$ orbifold field theories

The computations in the preceding sections of this chapter can immediately be generalized to field theories obtained as $\mathbb{Z}_M$ projections of $\mathcal{N}=4$ $U(NM)$ SYM theory where the action of $\mathbb{Z}_M$ is that in (1.2.1) with ω replaced by ω^p for $p\in\mathbb{Z}$. For these theories[9], the quantum fields must satisfy the $\mathbb{Z}_M$ invariance conditions obtained from (1.2.14) and (1.2.36) by replacing $\omega\to\omega^p$. In turn, the fields will take $\mathbb{Z}_M$ projection invariant forms analogous to (1.2.15)-(1.2.16) and (1.2.37)-(1.2.38), except that the bifundamental fields will have non-zero entries on the p'th super- or sub-diagonal. That is, A and B will have the non-zero entries $A_{i,(i+p)}$ and $B_{(i+p),i}$, respectively, and analogously for the respective superpartners χ_A and χ_B. As a result, the fluctuation operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}$ and $\Box_{\mathbf{\Phi}}^{mn}$ in (3.1.4)-(3.1.7) and Δ_{ij} in (3.2.13) will have non-zero entries on the p'th super- and sub-diagonals. Therefore, using the generalized determinant formula[10] (where $\omega\equiv e^{2\pi i/M}$)

$$\det\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_M \\ z_M & z_1 & z_2 & \cdots & z_{M-1} \\ z_{M-1} & z_M & z_1 & \cdots & z_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_2 & z_3 & z_4 & \cdots & z_1 \end{pmatrix} = \prod_{j=1}^{M}\big(z_1+\omega^j z_2+\omega^{2j}z_3+\cdots+\omega^{(M-1)j}z_M\big) \tag{3.3.1}$$

we see that the fluctuation determinants factorize as in (3.1.11), with ω^{pj} replacing ω^j. We conclude that the quantum effective action of these more general $\mathbb{Z}_M$ orbifold field theories is given by the expression (3.2.17) where $S^{(0)}$ is given in (3.2.18) and $\Gamma_{\rm bos}$ and $\Gamma_{\rm ferm}$ are given in (3.1.17) and (3.2.16), respectively. The only change is that $v_{i,j;\,n,m}$ now takes the

[9]These theories have also been considered in, e.g., Refs. [26, 29, 30, 58].

[10]We emphasize that the entries of the $M\times M$ matrix in (3.3.1) are allowed to be complex numbers.

form

$$\begin{aligned} v_{i,j;\,n,m} \;\equiv\;& 2\Big(\big((a_{i,(i+p)})_{nn} - \omega^{-pj}\,(a_{i,(i+p)})_{mm}\big)\big((\overline{a_{i,(i+p)}})_{nn} - \omega^{pj}\,(\overline{a_{i,(i+p)}})_{mm}\big) \\ &+ \big((b_{(i+p),i})_{nn} - \omega^{pj}\,(b_{(i+p),i})_{mm}\big)\big((\overline{b_{(i+p),i}})_{nn} - \omega^{-pj}\,(\overline{b_{(i+p),i}})_{mm}\big) \\ &+ \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big)\,. \end{aligned} \tag{3.3.2}$$

Chapter 4

Topology transition and emergent spacetime

The discovery that the eigenvalues of scalar VEV's reconstruct the dual spacetime geometry was originally made by Berenstein *et al.* in [66], [67], [68], providing a very concrete realization of emergent geometry within AdS/CFT. The general approach was to set up matrix models for the different sectors of BPS operators in the chiral ring.

In Section 4.1 we review ...

In Section 4.2 ......

4.1 The notion of emergent spacetime

In this section we review some parts of Berenstein's article "Large N BPS states and emergent quantum gravity" [68], in particular the matrix models of 1/4 BPS and 1/8 BPS states in $\mathcal{N} = 4$ SYM theory, the solutions of these matrix models, and the dual spacetime interpretation of these solutions. The purpose of this review is to justify how the distribution of eigenvalues of the scalar field VEV's can be interpreted as the emergence of the compact part of the dual AdS spacetime.

First, in the conventions of [68], the 1/2 BPS operators are those which saturate the BPS bound $\Delta \geq J$ with $J = J_1$ and $J_2 = J_3 = 0$ where J_i denote the eigenvalues under the Cartan generators of the R-symmetry group $SU(4)_R$. The 1/4 BPS operators saturate the bound with $J = J_1 + J_2$ and $J_3 = 0$. The 1/8 BPS operators saturate the bound with $J = J_1 + J_2 + J_3$.

The matrix models of 1/4 and 1/8 BPS operators in [68] are based on previous work in [66]. Namely, it was observed in [66] that all 1/8 BPS states are built out of the s-wave of three complex scalar fields on S^3, plus a single partial wave for fermions (these can have the two different polarizations of spin up or down), plus the s-wave of the time component of the gauge field (which imposes the Gauss constraint that all the operators in the model are gauge invariant). Thus the model of 1/8 BPS states consists of the three Hermitian matrices Z, X, Y, plus two fermionic matrices W_α. This forms a closed subsector of the Hilbert space of $\mathcal{N} = 4$ SYM operators with $SU(2|3)$ symmetry in the sense that the action of the dilation operator $\mathcal{D}$ closes on this subspace. Similarly, the model of 1/4 BPS states consists of the two Hermitian matrices Z, X and no fermionic matrices, forming the closed $SU(2)$ subsector.

In complex notation the Lagrangian density of the scalar fields reads

$$\mathcal{L} = \sum_i \mathrm{Tr}(D_t\phi_i D_t\overline{\phi_i} - \phi_i\overline{\phi_i}) - \frac{1}{2}\sum_i \mathrm{Tr}[\phi_i, \phi_i][\overline{\phi_i}, \overline{\phi_i}] - \frac{1}{2}\,\mathrm{Tr}\left(\sum_i [\phi_i, \overline{\phi_i}]\right)^2 \tag{4.1.1}$$

where $(\phi_1, \phi_2, \phi_3) = (Z, X, Y)$ and the summation runs over $i = 1, 2$ for the 1/4 BPS model and over $i = 1, 2, 3$ for the 1/8 BPS model. The decomposition in (4.1.1) mirrors the F-terms and D-terms of the $\mathcal{N} = 4$ SYM theory. By a Legendre transformation, the corresponding Hamiltonian density is

$$\mathcal{H} = \sum_i \mathrm{Tr}(p_{\phi_i} p_{\overline{\phi_i}} + \phi_i\overline{\phi_i}) + \frac{1}{2}\sum_i \mathrm{Tr}[\phi_i, \phi_i][\overline{\phi_i}, \overline{\phi_i}] + \frac{1}{2}\,\mathrm{Tr}\left(\sum_i [\phi_i, \overline{\phi_i}]\right)^2 . \tag{4.1.2}$$

The $U(1)_R$ charge is given by the generator

$$R = (-i)\sum_i \mathrm{Tr}(\phi_i\dot{\overline{\phi_i}} - \overline{\phi_i}\dot{\phi_i}) \tag{4.1.3}$$

so that for example Z has charge 1 and $\overline{Z}$ has charge (-1). We now want to understand how the classical BPS bound $\Delta \geq J$ can be saturated. Here Δ is the energy of the state, while J is the R-charge. Treating the commutators in (4.1.2) as a small perturbation and ignoring them for the moment, we get a sum of harmonic oscillators with $\omega = 1$, and we can decompose the modes as follows

$$Z(t) = Z_+ e^{it} + Z_- e^{-it} \tag{4.1.4}$$

and similarly for X, Y while $\overline{Z}, \overline{X}, \overline{Y}$ are obtained by Hermitian conjugation. Inserting (4.1.4) into (4.1.2), one finds the energy

$$H = \sum_i \mathrm{Tr}\left(|(\phi_i)_-|^2 + |(\phi_i)_+|^2\right) \tag{4.1.5}$$

while the R-charge is found to be

$$J = \sum_i \mathrm{Tr}\left(|(\phi_i)_-|^2 - |(\phi_i)_+|^2\right) \tag{4.1.6}$$

so that we need $Z_+ = X_+ = Y_+ = 0$ if we want $H = J$. If we now include the commutator terms as a perturbation of the dynamics, the energy increases if the commutators between Z_-, X_-, Y_- don't vanish, whereas the value of J stays invariant. Thus to saturate the classical BPS bound we need to require that Z_-, X_- and Y_- commute. *This is a crucial feature of the model for 1/4 and 1/8 BPS states.*

Adding the fermions to obtain all the 1/8 BPS states is straightforward, leading to a (2|3) dimensional harmonic oscillator per eigenvalue.

We now turn to consider the solutions of the 1/4 BPS and 1/8 BPS matrix models. Since Z, X, Y are commuting Hermitian matrices, they can be simultaneously diagonalized. This will reduce the dynamics to that of N eigenvalues for each of the matrices Z, X and Y. Performing the transformation from commuting Hermitian to diagonal matrices gives rise to a measure factor

$$\mu^2 = \prod_{i<j} |\vec{\phi}_i - \vec{\phi}_j|^2 \tag{4.1.7}$$

in the path integral measure. Including this quantum correction (in analogy with the steps (2.2.1)-(2.2.5) in Section 2.2), the matrix models for the 1/4 BPS and 1/8 BPS states are given by the Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{1}{2\mu^2} \nabla_i \mu^2 \nabla_i + \frac{1}{2}\left(\vec{\phi}_i\right)^2 . \tag{4.1.8}$$

The arrow denotes the direction in $\mathbb{C}^2$ or $\mathbb{C}^3$ for the 1/4 BPS and 1/8 BPS matrix models, respectively. Using that Z, X, Y are diagonal, to each diagonal element of the triplet of matrices we can assign a point in $\mathbb{C}^3$. Therefore (4.1.8) can be interpreted as a Hamitonian for N non-relativistic particles. The measure factor μ^2 generates an effective repulsive potential between the eigenvalues so that the N bosons are strongly interacting.

One now does a coarse graining of the degrees of freedom of the matrix models by going to densities $\rho(\vec{z})$ of particles in $\mathbb{C}^2$ or $\mathbb{C}^3$. The function ρ must be positive-definite and satisfy $\int d^D z\, \rho(z) = N$ where $D = 2, 3$ for the 1/4 and 1/8 BPS models, respectively. After coarse graining, the EOM's obtained from the Hamiltonian (4.1.8) take the form

$$x^2 + C - 2\int d^{2D}y\, \rho(\vec{y}) \ln(|\vec{x} - \vec{y}|) = 0 \tag{4.1.9}$$

where we have put $z^1 = x^1 + iy^1, z^2 = x^2 + iy^2$ and $z^3 = x^3 + iy^3$ and the arrows denote the direction in $\mathbb{R}^4$ or $\mathbb{R}^6$. Here C is a Lagrange multiplier enforcing the constraint that ρ integrates to N. Now, if ρ is smooth at x, then we can take derivatives of (4.1.9). Indeed, for the 1/4 BPS matrix model

$$(\nabla_x^2)^2 \ln(|\vec{x} - \vec{y}|) \sim \delta^{(4)}(\vec{x} - \vec{y}) \tag{4.1.10}$$

so that $\rho(\vec{x}) = 0$ at such points. For the 1/8 BPS model we use the fact that $(\nabla_x^2)^3 \ln(|\vec{x} - \vec{y}|) \sim \delta^{(6)}(\vec{x} - \vec{y})$ to reach the same conclusion. From this calculation we find that the distribution of particles on $\mathbb{R}^4$ or $\mathbb{R}^6$ is singular. The simplest singular behavior we can imagine is that the distributions $\rho(\vec{z})$ are supported on hypersurfaces in $\mathbb{R}^4$ or $\mathbb{R}^6$. Because of the rotational invariance of (4.1.8) one must assume that these hypersurfaces are spheres and that ρ is constant over its support; i.e., the eigenvalues are uniformly distributed. The radius r_0 of the sphere can be estimated by balancing the mutual repulsive force between the particles, which is of order Nr_0^{-1}, and the confining harmonic oscillator force, which is of order r_0. Thus, at $r_0 \sim \sqrt{N}$, the configuration will be in equilibrium.

To summarize, for the 1/4 BPS matrix model the eigenvalues of Z, X are uniformly distributed over a round S^3, whereas for the 1/8 BPS model the eigenvalues of Z, X, Y are uniformly distributed over a round S^5. As we will discuss in detail immediately, one can think of these eigenvalue distributions as fluids made up of charged particles flowing at the speed of sound along certain fibrations on the spheres in an external magnetic field.

For the 1/8 BPS case it is tantalizing to identify the S^5 distribution of particles with the S^5 factor of the dual $AdS_5 \times S^5$ geometry. Indeed, there are 4 directions of the AdS_5 that are already part of the field theory description. These are the angles of the S^3 boundary at infinity, and the time direction. Our claim now is that the S^5 is reproduced from the dynamics of the 1/8 BPS states (and similarly, the dynamics of the 1/4 BPS states reproduces an S^3 embedded in the S^5). We will now go back to the complex coordinates $z^i = x^i + iy^i$ so that S^3 and S^5 are embedded in the phase spaces $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively.

In the 1/4 BPS model there is a symplectic form ω on the phase space $\mathbb{C}^2$. We have the embedding $i : S^3 \to \mathbb{C}^2$. The pullback $i^*\omega$ of the symplectic form can be interpreted as a strong magnetic field. These magnetic field lines foliate the S^3 along the Hopf fibration. The particles can be seen as being charged in this magnetic field, and transport of particles can only happen along the magnetic field lines. The motion transverse to these lines is confined due to magnetic effects. The particle trajectories are BPS, and motion along them happens at the speed of sound. In [68] it is noted that this phenomenon is very similar to the quantum Hall effect of particles in the lowest Landau level[1] in higher dimensions [69], [70], [71], [72].

In many senses this dynamics corresponding to the 1/4 BPS states is natural. On the AdS side, the 1/4 BPS gravitons for some associated R-charge will all flow along a geodesic on a particular S^3 equator of S^5, and they will do so along the direction specified by the vector field on the sphere which rotates the configuration by R-transformations. This is done along a Hopf fibration of the S^3 which is adapted to the R-charge associated to the 1/4 BPS states. There is a particular J associated to an $SO(2)$ rotation in $SO(6)$ which is a symmetry of all these configurations. This $SO(2)$ has fixed points on the S^3 manifold that we associated above for 1/4 BPS states.

We can do the same analysis for the 1/8 BPS states. The equivalent of the Hopf fibration is the statement that S^5 is a circle bundle over $\mathbb{CP}^2$. In this case, the circle never shrinks to zero size on the S^5, and the circles are all diameters of the S^5. This fibration is associated to a vector field on the S^5 along which the excitations flow.

So far we have only described the ground state of the 1/4 BPS and 1/8 BPS matrix

[1]Defining the shifted Hamiltonian $H' \equiv \Delta - J$ the BPS bound translates into $H' \geq 0$, and BPS states can be thought of as occupying the lowest Landau level.

models. To understand more completely the dynamics of these models and to be able to compare it with supergravity, we must look at the spectrum of excitations of the models and the corresponding holographic dual descriptions on $AdS_5 \times S^5$.

This is easy to do for BPS excitations. Traces of polynomials in z^1, z^2, z^3 provide the variables that correspond to gravitational quanta [2, 3]. We multiply the ground state wavefunction by a truncated coherent state with small parameter. That is, we consider the wave function

$$\exp(\alpha_{nm} N_{nm} \operatorname{Tr}((z^1)^n (z^2)^m))\psi_0 \,. \tag{4.1.11}$$

To find the shape of the eigenvalue distribution (E-brane) corresponding to the state, the procedure roughly consists of writing out $|\psi_0|^2$ and taking the exponential term in the coherent state wave function and treating it like a small correction to the background potential. This will change the balance of forces, and the shape of the E-brane will change to compensate for it. The potential one gets in this way is not the most general potential, because the wavefunction is holomorphic. This also shows that these deformations do not diffuse the E-brane: the E-brane should still be infinitesimally thin. The argument is along the same lines as when we showed that the eigenvalue distribution is singular for the ground state configuration. This is because the Laplacian (squared or cubed) acting on a holomorphic function is zero, so the particle distribution density cannot have smooth support. This is a very important point when one wants to embed the E-brane geometry in the AdS spacetime.[2]

The potential for the individual eigenvalues is then given by the real part of the exponential term in the coherent state. If we let time evolve, then different terms in the coherent state expression will have different energies which are shifted by integers. This makes the shape of the perturbation evolve uniformly in time, as these changes can be absorbed into a phase rotation of all the z_i. In essence, the initial shape of the geometry is kept fixed up to some trivial rotations of the configuration.

Again, we have a constant speed of sound associated to these shape perturbations, which we want to identify with the speed of light on the supergravity geometry. Moreover, we have as many deformations as there are BPS gravitons with the given quantum numbers. This follows from the AdS/CFT correspondence established in [2] and [3].

We see that the excited states of the 1/4 BPS and 1/8 BPS matrix models are certain geometric objects with some peculiar hydrodynamics on them. The hydrodynamics is such that the collective motion of particles occur at the speed of sound, and this motion coincides with the motion of gravitons at the origin of AdS_5 which lie on an S^3 equator and which are BPS with respect to a particular R-charge, or which follow a particular fibration of the S^5 geometry with respect to some R-charge. It is natural then to conjecture that the S^3 we found in phase space in the 1/4 BPS model embeds itself in the S^5 factor of $AdS_5 \times S^5$ in some particular way.

[2]This is similar to stating that in the 1/2 BPS geometry the classical configurations are incompressible, and the edge of the droplet is sharp.

In conclusion, we found that the eigenvalue distributions which arose as the ground state solutions of the 1/4 BPS and 1/8 BPS matrix models could be interpreted as fluids made up of particles flowing at the speed of sound along certain fibrations on the spheres in such a way that the motion occurs along BPS trajectories preserving some specific supercharges. On the AdS side, these configurations correspond to gravitons propagating in $AdS_5 \times S^5$ along specific BPS trajectories at the origin of AdS_5 at the speed of light. Moreover, we found that also for the excited states of the 1/4 BPS and 1/8 BPS models there is a precise mapping between the motion of the particles on the E-branes and the motion of gravitons at the origin of AdS_5. Therefore, we conclude that the S^5 eigenvalue distribution found in the ground state of the 1/8 BPS matrix model can be identified with the S^5 factor of the holographically dual spacetime $AdS_5 \times S^5$. In other words, the dynamics of the 1/8 BPS states reproduces the S^5 factor of the dual geometry. This phenomenon is referred to in [68] as 'emergent spacetime'.

4.2 Low-temperature eigenvalue distribution

In this section we will find the solutions minimizing the effective potential computed in Chapter 3 (given in (3.2.17), (3.2.18), (3.1.17), (3.2.16) and (3.1.12)) within the temperature range $0 \leq TR \ll \lambda^{-1/2}$. We stress that, since the effective action of Section 5 is only valid within a sector of constant background fields satisfying (3.0.7)-(3.0.8), the minima we find in this section are not the absolute minima of the gauge theory, and the phase transitions within this sector of background fields do not necessarily extend to phase transitions in the full gauge theory (cf. [73]). Nonetheless, we will see that the matrix model of Section 5 exhibits some interesting dynamics.

The resulting distributions of eigenvalues will preserve the $SU(2)\times U(1)$ R-symmetry of $\mathcal{N} = 2$ quiver gauge theory. As in Ref. [21] we believe that due to the preserved R-symmetry, the minima found here are indeed the global minima of the effective action (within the sector of constant "commuting" VEV's). The key observation needed for obtaining the solutions is that both in the low-temperature regime and above the Hagedorn temperature T_H, the eigenvalue distributions for the scalar VEV's and the Polyakov loop can be solved for separately. As we will see, the Hagedorn transition causes a change in the topology of the joint eigenvalue distribution when the temperature is raised above T_H.

For temperatures low compared to the inverse radius of the S^3 (i.e., $TR \ll 1$), one can consistently discard terms in the quantum effective potential that are suppressed by Boltzmann factors[3], and so one obtains the following low-temperature limit of the effective

[3]We will verify a posteriori that this procedure is valid for all temperatures below the Hagedorn temperature.

potential

$$\begin{aligned}
\Gamma_{TR\ll 1} &= \frac{2\pi^2\beta R}{g_{\mathrm{YM}}^2}\sum_{i=1}^{M}\sum_{n=1}^{N}\Big((a_{i,(i+1)})_{nn}(\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn}(\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn}(\overline{\phi_i})_{nn}\Big) \\
&\quad -\frac{\beta}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\Big[2\Big(\big((a_{i,(i+1)})_{nn} - \omega^{-j}(a_{i,(i+1)})_{mm}\big)\big((\overline{a_{i,(i+1)}})_{nn} - \omega^{j}(\overline{a_{i,(i+1)}})_{mm}\big) \\
&\qquad + \big((b_{(i+1),i})_{nn} - \omega^{j}(b_{(i+1),i})_{mm}\big)\big((\overline{b_{(i+1),i}})_{nn} - \omega^{-j}(\overline{b_{(i+1),i}})_{mm}\big) \\
&\qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big)\Big]^{1/2}. \qquad (4.2.1)
\end{aligned}$$

We observe that the eigenvalues of the Polyakov loop are not coupled to the eigenvalues of the scalar VEV's. Therefore, for low temperatures, the distribution of the Polyakov loop eigenvalues will be the same as in the case with zero scalar VEV's treated in Sections 2.4-2.5. Thus, we immediately conclude from Section 2.4 that the eigenvalues $e^{i\alpha_i^{nn}}$ of the Polyakov loop (for i fixed) are uniformly distributed over S^1 for any temperature below the Hagedorn temperature. Note that for a uniform distribution of the angles α_i^{nn}, the terms multiplied by Boltzmann factors in (3.1.17) and (3.2.16) vanish exactly. Therefore we can consistently discard these terms as long as the temperature is below T_H.

In order to find the saddle points of (4.2.1) we make the observation that by making the identifications

$$a_{i,(i+1)} \cong \omega^{-1}a_{i,(i+1)} \qquad (4.2.2)$$

$$b_{(i+1),i} \cong \omega\, b_{(i+1),i} \qquad (4.2.3)$$

$$\phi_i \cong \phi_i \qquad (4.2.4)$$

and applying them recursively to (4.2.1), the low-temperature effective potential reduces to

$$\begin{aligned}
\Gamma_{TR\ll 1} &= \frac{2\pi^2\beta R}{g_{\mathrm{YM}}^2}\sum_{i=1}^{M}\sum_{n=1}^{N}\Big((a_{i,(i+1)})_{nn}(\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn}(\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn}(\overline{\phi_i})_{nn}\Big) \\
&\quad -\frac{\beta}{2}\sum_{i=1}^{M}\sum_{m,n=1}^{N}\Big[2\Big(\big((a_{i,(i+1)})_{nn} - (a_{i,(i+1)})_{mm}\big)\big((\overline{a_{i,(i+1)}})_{nn} - (\overline{a_{i,(i+1)}})_{mm}\big) \\
&\qquad + \big((b_{(i+1),i})_{nn} - (b_{(i+1),i})_{mm}\big)\big((\overline{b_{(i+1),i}})_{nn} - (\overline{b_{(i+1),i}})_{mm}\big) \\
&\qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big)\Big]^{1/2}. \qquad (4.2.5)
\end{aligned}$$

It is important to note that the identifications (4.2.2)-(4.2.4) correspond uniquely to the effective potential. That is, if one replaces ω by ω^q in (4.2.2)-(4.2.3), the potential (4.2.1) will not reduce to (4.2.5) for general M. To see this, note that, since all M powers of ω appear in (4.2.1), the order of ω^q must be M. Thus we must have $\gcd(q, M) = 1$ for all M which implies $q = 1$.

We now proceed with finding the saddle points of (4.2.5). These will be saddle points of (4.2.1) where the identifications (4.2.2)-(4.2.4) have been made. It is convenient to introduce the dimensionless variables

$$(\theta_i)_n \equiv \beta(\alpha_i)_{nn} , \tag{4.2.6}$$

$$(z_i)_{n,1} \equiv \beta(\phi_i)_{nn} , \qquad (z_i)_{n,2} \equiv \beta(a_{i,(i+1)})_{nn} , \qquad (z_i)_{n,3} \equiv \beta(b_{(i+1),i})_{nn} \tag{4.2.7}$$

and

$$(\boldsymbol{z}_i)_n \equiv \big((z_i)_{n,1}, (z_i)_{n,2}, (z_i)_{n,3}\big) \tag{4.2.8}$$

so that $(\boldsymbol{z}_i)_n \in \mathbb{C}^3$ for fixed i and n. Furthermore we introduce a norm on $\mathbb{C}^3$ defined by

$$\|\boldsymbol{w} - \boldsymbol{z}\| \equiv \left(\sum_{c=1}^{3} \big|(w_c) - (z_c)\big|^2 \right)^{1/2} \tag{4.2.9}$$

where $|\cdot|$ denotes the modulus. Written in this notation, (4.2.5) takes the form

$$\Gamma_{TR\ll 1} = \frac{2\pi^2 R}{g_{\mathrm{YM}}^2 \beta} \sum_{i=1}^{M} \sum_{n=1}^{N} \big\|(\boldsymbol{z}_i)_n\big\|^2 - \frac{1}{\sqrt{2}} \sum_{i=1}^{M} \sum_{m,n=1}^{N} \big\|(\boldsymbol{z}_i)_n - (\boldsymbol{z}_i)_m\big\| . \tag{4.2.10}$$

We will now take the continuum limit $N \to \infty$ and describe the eigenvalues of the Polyakov loop and the scalar VEV's by a joint eigenvalue distribution $\rho_i(\theta_i, \boldsymbol{z}_i)$ proportional to the density of eigenvalues at the point $(\theta_i, \boldsymbol{z}_i)$ (for some fixed i) and normalized as $\int d\theta_i\, d^3\boldsymbol{z}_i\, \rho_i(\theta_i, \boldsymbol{z}_i) = 1$. The continuum limit is obtained by applying the substitution

$$\frac{1}{N} \sum_{n=1}^{N} \big[\cdots\big] \longrightarrow \int d\theta_i\, d^3\boldsymbol{z}_i\, \rho_i(\theta_i, \boldsymbol{z}_i) \big[\cdots\big] \tag{4.2.11}$$

in analogy with (2.2.16). Here it is implied that the content of the brackets $\big[\cdots\big]$ carries an i label. In the continuum limit, the equation of motion for $\boldsymbol{z}_i$ obtained from (4.2.10) reads

$$\frac{\sqrt{2}\pi R}{\lambda\beta} \boldsymbol{z}_i = \int_{D_i} d^3\boldsymbol{z}_i'\, \rho_i(\boldsymbol{z}_i') \frac{\boldsymbol{z}_i - \boldsymbol{z}_i'}{\|\boldsymbol{z}_i - \boldsymbol{z}_i'\|} . \tag{4.2.12}$$

Here $\rho_i(\cdot)$ is defined as the average $\rho_i(\boldsymbol{z}_i) \equiv \int_{-\pi}^{\pi} d\theta_i\, \rho_i(\theta_i, \boldsymbol{z}_i)$, and $D_i \subseteq \mathbb{C}^3$ denotes the support for ρ_i. The solution to (4.2.12) is given by the eigenvalue distribution

$$\rho_i(\boldsymbol{z}_i) = \frac{\delta(\|\boldsymbol{z}_i\| - r_i)}{2\pi^4 r_i^5} \tag{4.2.13}$$

where the radius r_i is given by

$$r_i = \frac{\lambda\beta}{\sqrt{2}\pi^3 R} \frac{1024}{945} \tag{4.2.14}$$

as can be checked straightforwardly. That is, (4.2.12) is satisfied for any $\boldsymbol{z}_i$ when the eigenvalues are distributed uniformly over an S^5 with the radius (4.2.14). Since (4.2.10) was obtained from the low-temperature effective potential (4.2.1) by making the orbifold identifications (4.2.2)-(4.2.4), we thus conclude that the saddle point of (4.2.1) is a uniform distribution of the eigenvalues of the scalar VEV's over $S^5/\mathbb{Z}_M$ where the action of $\mathbb{Z}_M$ is precisely as in (1.2.1). This is consistent with [74], as one should expect in the low temperature limit where thermal effects are small. Since we found that the eigenvalues of the Polyakov loop are distributed uniformly over an S^1 for temperatures below the Hagedorn temperature, we conclude furthermore that the joint eigenvalue distribution of the scalar VEV's and the Polyakov loop is $S^5/\mathbb{Z}_M \times S^1$ in this temperature range.

It is remarkable that the eigenvalues of the scalar VEV's localize to a hypersurface in $\mathbb{C}^3$ rather than spreading out over the configuration space. The physical origin of the localization is essentially common for the matrix model developed here and the matrix model of [74], namely the competition between an attractive part of the quantum effective potential, and a repulsive part where the latter is generated by the path integrations. We interpret the eigenvalue distribution of the scalar VEV's as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ string theory geometry. Finally we note that the hypersurface $S^5/\mathbb{Z}_M$ has the isometry group $SU(2)\times U(1)$, resulting from breaking the $SU(4)$ isometry via the orbifold identifications (4.2.2)-(4.2.4). Since this is the full R-symmetry group $SU(2)_R \times U(1)_R$ of $\mathcal{N}=2$ quiver gauge theory we believe (cf. [21]) that the minimum found here is indeed the global minimum of the effective action of Chapter 3.

4.3 Eigenvalue distribution above the Hagedorn temperature

In the matrix model treated in Chapter 2 where the VEV's of the scalar fields were zero we observed that as the temperature is increased above $T_H \approx 0.38\,R^{-1}$, the Polyakov loop eigenvalue distributions open a gap. In this section we will examine how this phase transition manifests itself in the general case with non-zero scalar VEV's.

From the radius (4.2.14) one in particular finds that for low temperatures $\|\boldsymbol{z}_i\| \gg \lambda$, so that the tree-level term dominates over the quantum correction by a factor $\sim \frac{\|\boldsymbol{z}_i\|}{\lambda} \gg 1$. On the other hand, around the Hagedorn temperature T_H one finds $\|\boldsymbol{z}_i\| \sim \lambda$, and the tree-level term and the quantum corrections come within the same order of magnitude. It is therefore natural to re-express the effective potential in terms of the new variables

$$(\zeta_i)_{n,1} \equiv \lambda^{-1}(z_i)_{n,1}\,, \qquad (\zeta_i)_{n,2} \equiv \lambda^{-1}(z_i)_{n,2}\,, \qquad (\zeta_i)_{n,3} \equiv \lambda^{-1}(z_i)_{n,3}\,. \qquad (4.3.1)$$

The computations in this section will be valid for temperatures in the range $0 \le TR \ll \lambda^{-1/2}$. Since we can no longer neglect the terms multiplied by Boltzmann factors, we have to consider the full quantum effective action as computed in Chapter 3 (given in (3.2.17), (3.2.18), (3.1.17), (3.2.16) and (3.1.12)). Once again, we apply the orbifold identifications (4.2.2)-(4.2.4), and express the result in terms of the variables $\theta_i, \boldsymbol{\zeta}_i$. However, the rescaling with the 't Hooft coupling λ in (4.3.1) will reorganize the perturbative expansion of the effective potential into

$$\Gamma_{\text{eff}} \;=\; \Gamma^{(0)}[\theta_i] + \lambda\Gamma^{(1)}[\theta_i, \boldsymbol{\zeta}_i] + \mathcal{O}(\lambda^2)\,. \tag{4.3.2}$$

Here the 0-loop term is

$$\begin{aligned}\Gamma^{(0)}[\theta_i] \;=\; & \sum_{i=1}^{M}\sum_{m,n=1}^{N}\sum_{l=1}^{\infty}\frac{1}{l}\Big[1-\Big(z_{\text{ad}}^{B}(e^{-\beta l R^{-1}};1,1)\,+\,2z_{\text{bi}}^{B}(e^{-\beta l R^{-1}};1,1)\Big)\\ & -(-1)^{l+1}\Big(z_{\text{ad}}^{F}(e^{-\beta l R^{-1}};1,1)\,+\,2z_{\text{bi}}^{F}(e^{-\beta l R^{-1}};1,1)\Big)\Big]\cos\big(l(\theta_i)_n - l(\theta_i)_m\big)\end{aligned} \tag{4.3.3}$$

where $z_{\text{ad}}^{B}, z_{\text{ad}}^{F}, z_{\text{bi}}^{B}, z_{\text{bi}}^{F}$ are given in Eqs. (2.1.74), (2.1.75), (2.1.76), (2.1.77), respectively, and $y_1 = y_2 = 1$ in this case since we are taking $\mu_1 = \mu_2 = 0$ here.

The 1-loop term in (4.3.2) is given by

$$\begin{aligned}\Gamma^{(1)}[\theta_i, \boldsymbol{\zeta}_i] \;=\; & \frac{2\pi^2 RN}{\beta}\sum_{i=1}^{M}\sum_{n=1}^{N}\big\|(\boldsymbol{\zeta}_i)_n\big\|^2\\ & -\frac{1}{\sqrt{2}}\sum_{i=1}^{M}\sum_{m,n=1}^{N}\big\|(\boldsymbol{\zeta}_i)_n - (\boldsymbol{\zeta}_i)_m\big\|\left(1+2\sum_{l=1}^{\infty}\cos\big(l(\theta_i)_n - l(\theta_i)_m\big)\right).\end{aligned} \tag{4.3.4}$$

From the expansion (4.3.2) it is immediately obvious that to leading order in λ the θ_i are unaffected by the $\boldsymbol{\zeta}_i$. Therefore, to leading order, the eigenvalue distributions of the θ_i are the same as they were in the case with zero scalar VEV's treated in Sections 2.4-2.5. The eigenvalue distributions of the scalar VEV's can therefore be found by minimizing $\Gamma^{(1)}[\theta_i, \boldsymbol{\zeta}_i]$. Taking the large N limit of (4.3.4) according to (4.2.11) one finds

$$\begin{aligned}\frac{1}{N^2}\Gamma^{(1)} \;=\; & \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\, d^3\boldsymbol{\zeta}_i\, \rho_i(\theta_i, \boldsymbol{\zeta}_i)\,\|\boldsymbol{\zeta}_i\|^2\\ & -\sqrt{2}\,\pi\sum_{i=1}^{M}\int d\theta_i\, d^3\boldsymbol{\zeta}_i\, d^3\boldsymbol{\zeta}_i'\, \rho_i(\theta_i, \boldsymbol{\zeta}_i)\rho_i(\theta_i, \boldsymbol{\zeta}_i')\,\|\boldsymbol{\zeta}_i - \boldsymbol{\zeta}_i'\|\,.\end{aligned} \tag{4.3.5}$$

Here we have used the identity $1 + 2\sum_{l=1}^{\infty}\cos\big(l(\theta_i)_n - l(\theta_i)_m\big) = 2\pi\delta\big((\theta_i)_n - (\theta_i)_m\big)$ which is simply the Fourier expansion of the delta function.

Now we proceed to minimize the action (4.3.5). Since the eigenvalue distributions for the Polyakov loop and the scalar VEV's can be solved for separately, the joint eigenvalue distribution factorizes:

$$\rho_i(\theta_i, \boldsymbol{\zeta}_i) = \frac{\rho_i(\theta_i)\,\delta\big(\|\boldsymbol{\zeta}_i\| - r_i(\theta_i)\big)}{\|\boldsymbol{\zeta}_i\|^5\,(1+(dr_i/d\theta_i)^2)^{1/2}\,\text{Vol}(S^5)}\,. \tag{4.3.6}$$

Inserting (4.3.6) into the 1-loop term (4.3.5) one finds

$$\begin{aligned}\frac{1}{N^2}\Gamma^{(1)} \;=\; & \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\,\rho_i(\theta_i)\,r_i(\theta_i)^2 - 2\pi C\sum_{i=1}^{M}\int d\theta_i\,\rho_i(\theta_i)^2 r_i(\theta_i)\\ =\; & \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\left[\rho_i(\theta_i)\left(r_i(\theta_i) - \frac{C\beta}{2\pi R}\rho_i(\theta_i)\right)^2 - \frac{C^2\beta^2}{4\pi^2 R^2}\rho_i(\theta_i)^3\right]\end{aligned} \tag{4.3.7}$$

where $C = \frac{2048\sqrt{2}}{945\pi}$. The final term only contributes to the 2-loop order distribution of the Polyakov loop eigenvalues and can therefore be ignored cf. [21]. However, it might potentially play a role in the higher loop computations needed for determining the order of the phase transition at finite coupling (cf. the discussion in Section 2.5.4). Hence for a minimum we have

$$r_i(\theta_i) = \frac{C\beta}{2\pi R}\,\rho_i(\theta_i)\,. \tag{4.3.8}$$

As we know from Section 2.5, when the temperature is raised above the Hagedorn temperature T_H, the Polyakov loop eigenvalue distribution becomes gapped and is thus an interval $[-\theta_0, \theta_0]$. The scalar VEV eigenvalues are now distributed uniformly over an $S^5/\mathbb{Z}_M$ fibered over this interval, with the radius of the $S^5/\mathbb{Z}_M$ at any point θ_i in the interval being proportional to the density of Polyakov loop eigenvalues at θ_i (for fixed TR). The $S^5/\mathbb{Z}_M$ thus shrinks to zero radius at the endpoints $\pm\theta_0$ of the interval: the topology of the joint eigenvalue distribution is an $S^6/\mathbb{Z}_M$ where the $\mathbb{Z}_M$ is understood to act on the S^5 transverse to an S^1 diameter. Thus, the Hagedorn phase transition manifests itself in the general case of non-zero scalar VEV's as a change in the topology of the joint eigenvalue distribution $S^5/\mathbb{Z}_M \times S^1 \longrightarrow S^6/\mathbb{Z}_M$.

In order to understand how the $S^6/\mathbb{Z}_M$ eigenvalue distribution may be realized in the dual AdS spacetime we first need to consider the S^1 part of the low-temperature distribution $S^1 \times S^5/\mathbb{Z}_M$. The eigenvalues of the Wilson line wound around the thermal circle give the positions of D2-branes[4] on the T-dual of the thermal circle in thermal AdS_5. As the temperature is raised higher and higher beyond T_H, the Polyakov loop eigenvalues become localized to smaller and smaller intervals. On the AdS side one therefore finds a localized D2-brane configuration. It was noted in [21] that a similar localization of D2-branes on a *spatial* circle, at finite temperature, was investigated in [76] where it was observed to produce a near-horizon geometry containing a non-contractible S^6. Moreover, it was predicted in [76] from supergravity that a $S^1 \times S^5 \to S^6$ topological transition of a Gregory-Laflamme type should take place. In the present case, where the dual spacetime is $AdS_5 \times S^5/\mathbb{Z}_M$, we expect the appearance of an $S^6/\mathbb{Z}_M$ in the near-horizon geometry of the localized configuration of D2-branes on the T-dual of the thermal circle.

4.3.1 Generalization to other $\mathbb{Z}_M$ orbifold field theories

The computations in this section immediately carry over to the more general $\mathbb{Z}_M$ orbifold field theories considered in Section 3.3. In this paragraph we remark on the theory defined by letting the action of $\mathbb{Z}_M$ be that of (1.2.1) with ω replaced by ω^p for some fixed $p \in \mathbb{Z}$. The quantum effective action of the corresponding field theory is obtained from that of $\mathcal{N} = 2$ quiver gauge theory by defining $v_{i,j;\,n,m}$ to be as given in (3.3.2). The saddle points of this effective action are found by making the orbifold identifications

$$a_{i,(i+1)} \cong \omega^{-p} a_{i,(i+1)}\,, \qquad b_{(i+1),i} \cong \omega^{p}\, b_{(i+1),i}\,, \qquad \phi_i \cong \phi_i\,. \tag{4.3.9}$$

[4]The D2-branes here are T-dual to the original D3-branes. See [75], pp. 263-270, for a careful explanation of this point.

The resulting expression for the effective action is then precisely the same as in the case of $\mathcal{N} = 2$ quiver gauge theory treated in this section, and the conclusions carry directly over. In particular, having made the orbifold identifications (4.3.9), one finds the low-temperature joint eigenvalue distribution $S^5 \times S^1$ and the high-temperature distribution S^6. Alternatively, the joint eigenvalue distributions are $S^5/\mathbb{Z}_M \times S^1$ and $S^6/\mathbb{Z}_M$, respectively, where the action of $\mathbb{Z}_M$ is precisely the orbifold action defining the $\mathbb{Z}_M$ orbifold theory. It is important to note that the orbifold identifications (4.3.9) correspond uniquely to the quantum effective action of the field theory. Indeed, assume that we make the identifications (4.3.9) with some ω^q replacing ω^p. In order for the quantum effective action to reduce to an expression involving norms on $\mathbb{C}^3$ we must require ω^q to have the same order as ω^p. That is, we must have $\forall M \in \mathbb{N} : \gcd(q, M) = \gcd(p, M)$ which implies $q = p$. Identifying the above $S^5/\mathbb{Z}_M$ distribution with the $S^5/\mathbb{Z}_M$ part of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ spacetime, this shows in particular that, within this class of $\mathbb{Z}_M$ orbifold field theories, the geometry of the dual AdS spacetime is mirrored in the structure of the quantum effective action in a precise way.

Chapter 5

The $SO(6)$ sector of $\mathcal{N} = 4$ $SU(N)$ SYM theory

In this chapter we present a detailed derivation of the anomalous dimension matrix for $\mathcal{N} = 4$ $SU(N)$ SYM theory single-trace scalar operators to 1-loop order in the large N limit. In the planar limit we only need to consider single-trace operators because they completely decouple from multi-trace operators due to large N factorization (cf. [77]). Our motivation for carrying out the computation in the large N limit is a different one, however: as first noted by Minahan and Zarembo in [13], the anomalous dimension matrix for this sector of operators can, up to a constant, be identified with the Hamiltonian of an *integrable* $SO(6)$ spin chain. However, for finite N the integrability is lost. We will return to the importance of integrability later in this section.

We will restrict attention to the single-trace scalar operators of $\mathcal{N} = 4$ $SU(N)$ SYM theory defined as

$$\mathcal{O}(x) \equiv \mathrm{Tr}(\Phi_{i_1}(x) \cdots \Phi_{i_J}(x)) \,. \tag{5.0.1}$$

Under renormalization these operators will *mix*. That is, the relation between the bare and renormalized operators is given by

$$(\Phi_{i_1}(x) \cdots \Phi_{i_J}(x))_B = Z^{\cdots j_l j_{l+1} \cdots}_{\cdots i_l i_{l+1} \cdots} (\Phi_{j_1}(x) \cdots \Phi_{j_J}(x)) \,. \tag{5.0.2}$$

This implies the relation between bare and renormalized 1PI correlation functions

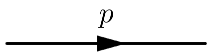
$$\Gamma^{(2)}_{i_1 \cdots i_J} = Z^{\cdots j_l j_{l+1} \cdots}_{\cdots i_l i_{l+1} \cdots} (\Gamma^{(2)}_{j_1 \cdots j_J})_B \tag{5.0.3}$$

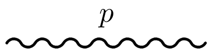
which can be used to determine the anomalous dimension matrix.

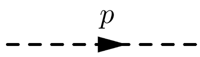
5.1 Feynman rules and Feynman integrals

The Feynman rules for $\mathcal{N} = 4$ $SU(N)$ SYM theory in 4-dimensional Euclidean spacetime (in covariant gauge) are listed below.

1) Propagators

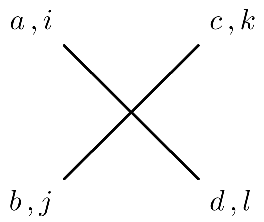
scalar propagator: $\Delta^{ab}_{ij}(p) = a\,,i$  $b\,,j = \delta^{ab}\frac{\delta_{ij}}{p^2}$

gauge boson propagator: $D^{ab}_{\mu\nu}(p) = a\,,\mu$  $b\,,\nu = \delta^{ab}\frac{\delta_{\mu\nu}}{p^2}$

fermion propagator: $(S^{ab}(p))_{\beta\alpha} = a\,,\alpha$  $b\,,\beta = -\delta^{ab}\frac{p_\mu(\Gamma^\mu)_{\beta\alpha}}{p^2}$

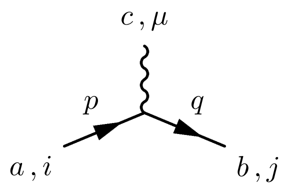
2) Vertices

4-scalar vertex:



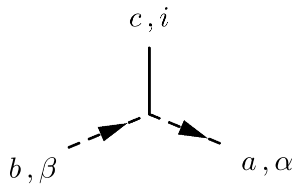
$$= -g^2\Big[f^{eab}f^{ecd}\big(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\big) + f^{eac}f^{edb}\big(\delta_{il}\delta_{kj}-\delta_{ij}\delta_{kl}\big) + f^{ead}f^{ebc}\big(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}\big)\Big]$$

scalar-scalar-gauge boson vertex:



$$= -igf^{abc}\delta_{ij}(p+q)_\mu$$

fermion-fermion-scalar vertex:



$$= -igf^{abc}(\Gamma^{i+4})_{\alpha\beta}$$

We note also the following Feynman integrals which hold in $D = 2\omega$ Euclidean space-time for $m \geq 1$:

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{1}{q^{2n}}\frac{1}{(q-p)^{2m}} = \frac{(-1)^{m+n}}{(4\pi)^\omega}\frac{\Gamma(m+n-\omega)\,B(\omega-n,\omega-m)}{\Gamma(m)\,\Gamma(n)}(p^2)^{\omega-m-n} \quad (5.1.1)$$

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}q_\mu\frac{1}{q^{2n}}\frac{1}{(q-p)^{2m}} = \frac{(-1)^{m+n}}{(4\pi)^\omega}\frac{\Gamma(m+n-\omega)\,B(\omega-n+1,\omega-m)}{\Gamma(m)\,\Gamma(n)}(p^2)^{\omega-m-n}p_\mu \quad (5.1.2)$$

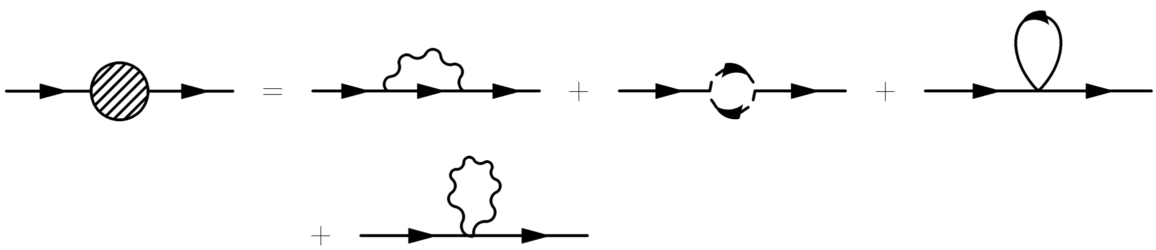
$$\begin{aligned}\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\, q_\mu q_\nu \,\frac{1}{q^{2n}}\,\frac{1}{(q-p)^{2m}} \;=\; & \frac{(-1)^{m+n}}{(4\pi)^\omega}\frac{(p^2)^{\omega-m-n}}{\Gamma(m)\,\Gamma(n)} \\ & \times\Big(p_\mu p_\nu \Gamma(m+n-\omega)\,B(\omega-m,\omega-n+2) \\ & \quad + \frac{1}{2}\delta_{\mu\nu}p^2\,\Gamma(m+n-\omega-1)\,B(\omega-m+1,\omega-n+1)\Big) \end{aligned} \tag{5.1.3}$$

where Euler's beta function B is given by

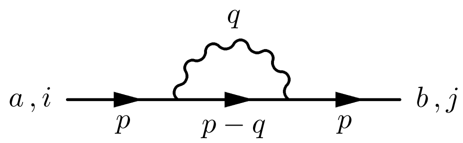
$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}\,. \tag{5.1.4}$$

5.2 The scalar self-energy diagrams

To 1-loop order we have a priori the following corrections to the scalar propagator:



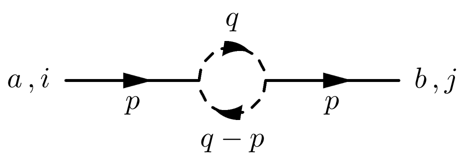
1) The gauge boson loop correction:



$$
\begin{aligned}
&= (-ig)^2 \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \Delta^{aa'}_{ii'}(p)\, f^{a'a''c}\,(2p-q)_\mu\, D^{\mu\nu}_{cc'}(q)\, \Delta^{a''b''}_{i''j''}(p-q)\, f^{b''b'c'}\,(2p-q)_\nu\, \Delta^{b'b}_{j'j}(p) \\
&= (-ig)^2 \delta_{ij}\,(-N\delta^{ab}) \left(\frac{1}{p^2}\right)^2 \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{(2p-q)^2}{q^2\,(p-q)^2} \\
&= 2g^2 N\delta^{ab}\delta_{ij} \frac{\Gamma(2-\omega)\,\Gamma(\omega-1)^2}{(4\pi)^\omega\,\Gamma(2\omega-2)}\,(p^2)^{\omega-3} \qquad (5.2.1)
\end{aligned}
$$

where in the second equality we also made use of the identity $f^{acd}f^{bcd}=N\delta^{ab}$.

2) The one-fermion loop correction:



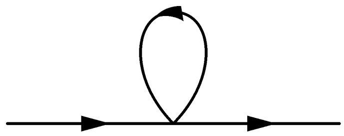
$$
\begin{aligned}
&= (-ig)^2(-1)\left(\frac{1}{2}\right) \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \Big(\Delta^{aa'}_{ii'}(p) f^{cda'}\big(\Gamma^{i'+4}\big)_{\alpha\beta}\big(S^{cc'}(q)\big)_{\alpha'\alpha}\big(S^{d'd}(q-p)\big)_{\beta\beta'} \\
&\qquad \times f^{d'c'b'}\big(\Gamma^{j'+4}\big)_{\beta'\alpha'}\Delta^{b'b}_{j'j}(p)\Big) \\
&= -\frac{1}{2}g^2N\delta^{ab}\left(\frac{1}{p^2}\right)^2 \Big(\big(\Gamma^\mu\big)_{\alpha'\alpha}\big(\Gamma^{i+4}\big)_{\alpha\beta}\big(\Gamma^\nu\big)_{\beta\beta'}\big(\Gamma^{j+4}\big)_{\beta'\alpha'}\Big) \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{q_\mu(q_\nu-p_\nu)}{q^2(q-p)^2} \\
&= -4g^2N\delta^{ab}\delta_{ij} \frac{\Gamma(2-\omega)\,\Gamma(\omega-1)^2}{(4\pi)^\omega\,\Gamma(2\omega-2)}\,\big(p^2\big)^{\omega-3}\,. \qquad (5.2.2)
\end{aligned}
$$

In the first equality we inserted an extra factor (-1) because of the fermion loop and the factor $\frac{1}{2}$ which is the weight factor of the diagram. In the third equality we used the following identity for gamma matrices of size $\Delta\times\Delta$

$$
\mathrm{Tr}\,\Gamma^\mu\Gamma^\nu\Gamma^\rho\Gamma^\sigma = \Delta\cdot\big(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\big)\,. \qquad (5.2.3)
$$

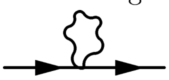
In our case $\Delta = 16$, and so in particular

$$\Big((\Gamma^\mu)_{\alpha'\alpha}(\Gamma^{i+4})_{\alpha\beta}(\Gamma^\nu)_{\beta\beta'}(\Gamma^{j+4})_{\beta'\alpha'}\Big) = \mathrm{Tr}\left(\Gamma^\mu\Gamma^{i+4}\Gamma^\nu\Gamma^{j+4}\right) = -16\,\delta^{\mu\nu}\delta_{ij}\,. \qquad (5.2.4)$$

3) The scalar tadpole:  $\propto \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{1}{q^2}$

Letting $D = 2\omega = 4 - 2\epsilon$, from Eq. (B.16) in [83] one finds

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{1}{q^2+m^2} = \frac{\Gamma(-1+\epsilon)}{(4\pi)^\omega}\,(m^2)^{1-\epsilon}\,. \qquad (5.2.5)$$

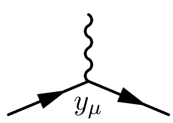
Putting $m = 0$ for $\epsilon > 0$, we find that the above diagram vanishes for any $\epsilon > 0$. Hence the tadpole does not give any contribution in dimensional regularization. Analogously, the diagram  does not give any contribution.

5.3 The four-point correlation functions

In the following we will need some of the Feynman rules of $\mathcal{N} = 4$ $SU(N)$ SYM theory in 2ω-dimensional (Euclidean) coordinate space. We list the Feynman rules we need for our calculations below.

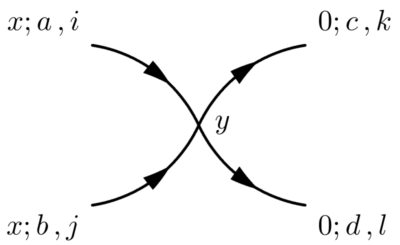
scalar propagator: $\Delta^{ab}_{ij}(x) \;=\; x; a\,, i$  $0; b\,, j \;=\; \delta^{ab}\delta_{ij}\frac{\Gamma(\omega-1)}{4\pi^\omega(x^2)^{\omega-1}}$

gauge boson propagator: $D^{ab}_{\mu\nu}(x) \;=\; x; a\,, \mu$ $0; b\,, \nu \;=\; \delta^{ab}\delta_{\mu\nu}\frac{\Gamma(\omega-1)}{4\pi^\omega(x^2)^{\omega-1}}$

scalar-scalar-gauge boson vertex:  $= -igf^{abc}\delta_{ij}\frac{\partial}{\partial y^\mu}$

Here the derivative $\frac{\partial}{\partial y^\mu}$ in the vertex is understood to act on the entire expression obtained from applying the Feynman rules to a diagram; i.e., on the product of all propagators containing the point y.

5.3.1 Four-scalar interaction



After contracting the indices on the four-scalar vertex with the indices of the scalar propagators one finds the following index structure of the diagram

$$-g^2\left[\, f^{eab}f^{ecd}\big(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\big)+f^{eac}f^{edb}\big(\delta_{il}\delta_{kj}-\delta_{ij}\delta_{kl}\big)+f^{ead}f^{ebc}\big(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}\big)\right]\,. \tag{5.3.1}$$

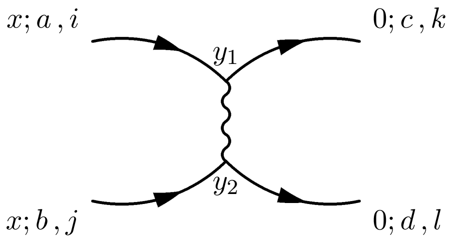
Ignoring the index structure, the diagram evaluates to

$$\begin{aligned}\int d^{2\omega}y\,\Delta^2(x-y)\,\Delta^2(y) &= \left(\frac{\Gamma(\omega-1)}{4\pi^\omega}\right)^4\int d^{2\omega}y\,\frac{1}{(y^2)^{2\omega-2}}\frac{1}{(x-y)^{2(2\omega-2)}}\\ &= \left(\frac{\Gamma(\omega-1)}{4\pi^\omega}\right)^4\frac{(2\pi)^{2\omega}}{(4\pi)^\omega}\frac{\Gamma(3\omega-4)\,\Gamma(2-\omega)^2}{\Gamma(2\omega-2)^2\,\Gamma(4-2\omega)}\,(x^2)^{4-3\omega}\\ &= \frac{\Gamma(2-\omega)^2\,\Gamma(\omega-1)^2\,\Gamma(3\omega-4)}{16\pi^\omega\,\Gamma(2\omega-2)^2\,\Gamma(4-2\omega)}\,(x^2)^{2-\omega}\Delta^2(x)\,.\end{aligned} \tag{5.3.2}$$

Thus, the final result for the diagram is

$$\begin{aligned}&-g^2\left[\, f^{eab}f^{ecd}\big(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\big)+f^{eac}f^{edb}\big(\delta_{il}\delta_{kj}-\delta_{ij}\delta_{kl}\big)+f^{ead}f^{ebc}\big(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}\big)\right]\\ &\qquad\times\frac{\Gamma(2-\omega)^2\,\Gamma(\omega-1)^2\,\Gamma(3\omega-4)}{16\pi^\omega\,\Gamma(2\omega-2)^2\,\Gamma(4-2\omega)}\,(x^2)^{2-\omega}\Delta^2(x)\,.\end{aligned} \tag{5.3.3}$$

5.3.2 Gauge boson exchange



$$
\begin{aligned}
&= \int d^{2\omega}y_1 \int d^{2\omega}y_2\, \frac{\partial}{\partial y_2^{\nu}} \frac{\partial}{\partial y_1^{\mu}} \Big(\Delta_{ii'}^{aa'}(x-y_1)\,\big(-igf^{a'c'e'}\delta_{i'k'}\big)\,\Delta_{k'k}^{c'c}(y_1)\,D_{e'f'}^{\mu\nu}(y_1-y_2) \\
&\qquad \times \Delta_{jj'}^{bb'}(x-y_2)\,\big(-igf^{b'd'f'}\delta_{j'l'}\big)\,\Delta_{l'l}^{d'd}(y_2)\Big) \\
&= (-ig)^2\delta_{ik}\delta_{jl}f^{ace}f^{bde}\left(\frac{\Gamma(\omega-1)}{4\pi^{\omega}}\right)^5 \int d^{2\omega}y_1 \int d^{2\omega}y_2\, \frac{\partial}{\partial y_2^{\mu}} \frac{\partial}{\partial y_1^{\mu}} \frac{1}{(y_1^2)^{\omega-1}(x-y_1)^{2(\omega-1)}} \\
&\quad \times \frac{1}{(y_2^2)^{\omega-1}(x-y_2)^{2(\omega-1)}} \frac{1}{(y_1-y_2)^{2(\omega-1)}} \\
&= (-ig)^2\delta_{ik}\delta_{jl}f^{ace}f^{bde}\left(\frac{\Gamma(\omega-1)}{4\pi^{\omega}}\right)^5 \\
&\quad \times \Bigg[\int d^{2\omega}y_1 \int d^{2\omega}y_2 \Bigg(\frac{8}{\big[(x-y_1)^{4(\omega-1)}y_1^{2(\omega-1)}\big]\big[y_2^{4(\omega-1)}(y_1-y_2)^{2(\omega-1)}\big]} \\
&\qquad + \frac{8}{\big[(x-y_1)^{4(\omega-1)}y_1^{2(\omega-1)}\big]\big[y_2^{2(\omega-1)}(y_1-y_2)^{4(\omega-1)}\big]} \\
&\qquad - \frac{4x^{2(\omega-1)}}{y_1^{4(\omega-1)}(x-y_1)^{2(\omega-1)}y_2^{2(\omega-1)}(x-y_2)^{4(\omega-1)}(y_1-y_2)^{2(\omega-1)}} \\
&\qquad - \frac{8}{y_1^{2(\omega-1)}(x-y_1)^{2(\omega-1)}y_2^{2(\omega-1)}(x-y_2)^{2(\omega-1)}(y_1-y_2)^{4(\omega-1)}} \Bigg) \Bigg] \\
&= g^2\delta_{ik}\delta_{jl}f^{ace}f^{bde} \frac{\Gamma(\omega-1)}{4\pi^{\omega}(2-\omega)(2\omega-3)}\,(x^2)^{2-\omega}\left(\frac{\Gamma(\omega-1)}{4\pi^{\omega}x^{2(\omega-1)}}\right)^2 . \qquad (5.3.4)
\end{aligned}
$$

In performing the y_1, y_2 differentiations under the integral signs we have simplified the result using that the integrals are invariant under the interchange of y_1 with y_2 and furthermore under the combined transformation $y_1 \to y_1' \equiv x - y_1$ and $y_2 \to y_2' \equiv x - y_2$.

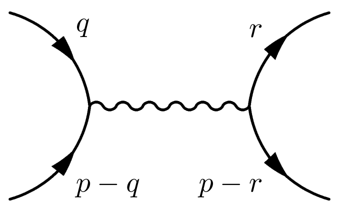
The first two integrals inside $\big[\cdots\big]$ can be done by appying (5.1.1) two times, first performing the integration over y_2 and then over y_1. The remaining two integrals can be found in Ref. [82].

5.3.3 Annihilation

It is easiest to evaluate this diagram in momentum space. To do this, we must assign 4-momenta correctly to the diagram. We consider the 4-scalar interaction and evaluate the Fourier transform of the result obtained in coordinate space:

$$\begin{aligned}\int d^{2\omega}x\, e^{-ip\cdot x}\left(\int d^{2\omega}y\, \Delta^2(x-y)\,\Delta^2(y)\right) \\ &= \left(\frac{\Gamma(2-\omega)^2\,\Gamma(\omega-1)^4}{(4\pi)^{2\omega}\,\Gamma(2\omega-2)^2}\right)\frac{1}{(p^2)^{4-2\omega}} \\ &= \int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\int\frac{d^{2\omega}r}{(2\pi)^{2\omega}}\,\Delta(q)\,\Delta(p-q)\,\Delta(r)\,\Delta(p-r)\,. \qquad (5.3.5)\end{aligned}$$

The external legs of any connected 4-point diagram must exactly match those of the 4-scalar interaction diagram since we are adding the contributions of these two diagrams. Therefore we conclude that the annihilation diagram in momentum space is



$$\begin{aligned} &= \int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\int\frac{d^{2\omega}r}{(2\pi)^{2\omega}}\Big(\Delta^{aa'}_{ii'}(q)\,\Delta^{bb'}_{jj'}(p-q)\,(-igf^{a'b'e})\,\delta_{i'j'}\,(2q-p)_\mu\, D^{\mu\nu}_{ee'}(p) \\ &\qquad (-igf^{d'c'e'})\,\delta_{l'k'}\,(2r-p)_\nu\,\Delta^{c'c}_{k'k}(r)\,\Delta^{d'd}_{l'l}(p-r)\Big) \\ &= (-ig)^2 f^{eab} f^{edc}\delta^{\mu\nu}\delta_{ij}\delta_{kl}\left(\frac{1}{p^2}\right)\left(\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{(2q-p)_\mu}{q^2(q-p)^2}\right)\left(\int\frac{d^{2\omega}r}{(2\pi)^{2\omega}}\frac{(2r-p)_\nu}{r^2(r-p)^2}\right) \\ &= 0 \qquad (5.3.6)\end{aligned}$$

where the last equality follows from

$$\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{(2q-p)_\mu}{q^2\,(q-p)^2} = 0\,. \qquad (5.3.7)$$

5.4 Renormalization

We will carry out the renormalization in the $\overline{\text{MS}}$ scheme, using dimensional regularization. That is, we will set the spacetime dimension to $D=2\omega=4-2\epsilon$. In order to proceed with finding the wavefunction renormalization factors we shall need:

- **Laurent expansion of Γ.** The meromorphic extension of the Γ function to the complex plane has simple poles in $-n$ where $n = 0, 1, 2, \ldots$. The Laurent expansion around these poles is

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} + \psi(n+1) + \frac{\epsilon}{2}\left(\frac{\pi^2}{3} + \psi(n+1)^2 - \psi'(n+1)\right) + \mathcal{O}(\epsilon^2)\right) . \tag{5.4.1}$$

Here the *digamma function* ψ and its derivative are given by

$$\psi(z) \equiv \frac{1}{\Gamma(z)}\frac{d\Gamma(z)}{dz} \tag{5.4.2}$$

$$\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma \tag{5.4.3}$$

$$\psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2} \tag{5.4.4}$$

where $\gamma = 0.57721\ldots$ denotes the Euler-Mascheroni constant.

- **Trace identities.** Let A, B, C, D be arbitrary $N \times N$ matrices and let T^a, $a = 1, \ldots, N^2 - 1$ be the generators of $su(N)$ in the fundamental representation. The conventions for the generators and structure constants are[1]

$$[T^a, T^b] = i f^{abc} T^c , \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} , \qquad (T^a)^{\alpha}{}_{\beta}(T^a)^{\gamma}{}_{\delta} = \frac{1}{2}\delta^{\alpha}{}_{\delta}\delta^{\gamma}{}_{\beta} . \tag{5.4.5}$$

Then the following identities hold

$$\mathrm{Tr}(T^a A T^b B)\,\mathrm{Tr}(T^c C T^d D) f^{ead} f^{ebc} = \frac{1}{8}\Big(\mathrm{Tr}\,A\,\mathrm{Tr}\,C\,\mathrm{Tr}\,BD + \mathrm{Tr}\,B\,\mathrm{Tr}\,D\,\mathrm{Tr}\,AC - \mathrm{Tr}(ACBD) - \mathrm{Tr}(ADBC)\Big) \tag{5.4.6}$$

$$\mathrm{Tr}(T^a A T^b B)\,\mathrm{Tr}(T^c C T^d D) f^{eac} f^{ebd} = \frac{1}{8}\Big(\mathrm{Tr}\,A\,\mathrm{Tr}\,D\,\mathrm{Tr}\,BC + \mathrm{Tr}\,B\,\mathrm{Tr}\,C\,\mathrm{Tr}\,AD\Big) - \mathrm{Tr}(ADBC) - \mathrm{Tr}(ACBD)\Big) \tag{5.4.7}$$

$$\mathrm{Tr}(T^a A T^b B)\,\mathrm{Tr}(T^c C T^d D) f^{eab} f^{ecd} = \frac{1}{8}\Big(\mathrm{Tr}\,A\,\mathrm{Tr}\,D\,\mathrm{Tr}\,BC + \mathrm{Tr}\,B\,\mathrm{Tr}\,C\,\mathrm{Tr}\,AD - \mathrm{Tr}\,A\,\mathrm{Tr}\,C\,\mathrm{Tr}\,BD - \mathrm{Tr}\,B\,\mathrm{Tr}\,D\,\mathrm{Tr}\,AC\Big) . \tag{5.4.8}$$

The proof of the first identity is given below. The second identity follows from substituting $C \longleftrightarrow D$ and $c \longleftrightarrow d$ and using the cyclicity of the trace. The third identity follows from the first two via the Jacobi identity.

As a consistency check, note that inserting $A = B = C = D = I$ in Eq. (5.4.6) one finds

$$\frac{1}{4} f^{eac} f^{eac} = \frac{1}{4} N(N^2 - 1) . \tag{5.4.9}$$

[1] Note that these are the same conventions as in [78] so that (5.4.6) above coincides with (5.3) there. For further information on trace identities for $su(N)$ generators, also see [79] and [80].

Indeed, this is a true identity, as immediately follows from the identity $f^{acd}f^{bcd} = N\delta^{ab}$ since $e = 1, \dots, N^2-1$.

(Also note that the RHS of Eq. (5.4.8) can be written as a product of two factors, one containing only T^e, A, B, and the other factor containing only T^e, C, D. This is easily seen by applying Eq. (5.4.27).)

Proof of Eq. (5.4.6). Using that $f^{abc} = -2i\,\mathrm{Tr}([T^a,T^b]T^c)$ and the normalization $(T^a)^\alpha{}_\beta (T^a)^\gamma{}_\delta = \frac{1}{2}\delta^\alpha{}_\delta\delta^\gamma{}_\beta$ we find

$$
\begin{aligned}
&\mathrm{Tr}(T^aAT^bB)\,\mathrm{Tr}(T^cCT^dD)f^{ead}f^{ebc}\\
&= (-2i)^2\,\mathrm{Tr}(T^aAT^bB)\,\mathrm{Tr}(T^cCT^dD)\,\mathrm{Tr}([T^a,T^d]T^e)\,\mathrm{Tr}([T^b,T^c]T^e)\\
&= (-4)\Big((T^a)^\alpha{}_\beta A^\beta{}_\gamma (T^b)^\gamma{}_\delta B^\delta{}_\alpha\Big)\Big((T^c)^\kappa{}_\lambda C^\lambda{}_\mu (T^d)^\mu{}_\nu D^\nu{}_\kappa\Big)\\
&\qquad\times\Big((T^a)^\epsilon{}_\eta (T^d)^\eta{}_\pi (T^e)^\pi{}_\epsilon - (T^d)^\epsilon{}_\eta (T^a)^\eta{}_\pi (T^e)^\pi{}_\epsilon\Big)\\
&\qquad\times\Big((T^b)^\rho{}_\sigma (T^c)^\sigma{}_\tau (T^e)^\tau{}_\rho - (T^c)^\rho{}_\sigma (T^b)^\sigma{}_\tau (T^e)^\tau{}_\rho\Big)\\
&= (-4)\Big((T^a)^\alpha{}_\beta (T^a)^\epsilon{}_\eta (T^b)^\gamma{}_\delta (T^b)^\rho{}_\sigma (T^c)^\kappa{}_\lambda (T^c)^\sigma{}_\tau\\
&\qquad\qquad\times (T^d)^\mu{}_\nu (T^d)^\eta{}_\pi (T^e)^\pi{}_\epsilon (T^e)^\tau{}_\rho A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad - (T^a)^\alpha{}_\beta (T^a)^\epsilon{}_\eta (T^b)^\gamma{}_\delta (T^b)^\sigma{}_\tau (T^c)^\kappa{}_\lambda (T^c)^\rho{}_\sigma\\
&\qquad\qquad\times (T^d)^\mu{}_\nu (T^d)^\eta{}_\pi (T^e)^\pi{}_\epsilon (T^e)^\tau{}_\rho A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad - (T^a)^\alpha{}_\beta (T^a)^\eta{}_\pi (T^b)^\gamma{}_\delta (T^b)^\rho{}_\sigma (T^c)^\kappa{}_\lambda (T^c)^\sigma{}_\tau\\
&\qquad\qquad\times (T^d)^\mu{}_\nu (T^d)^\epsilon{}_\eta (T^e)^\pi{}_\epsilon (T^e)^\tau{}_\rho A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad + (T^a)^\alpha{}_\beta (T^a)^\eta{}_\pi (T^b)^\gamma{}_\delta (T^b)^\sigma{}_\tau (T^c)^\kappa{}_\lambda (T^c)^\rho{}_\sigma\\
&\qquad\qquad\times (T^d)^\mu{}_\nu (T^d)^\epsilon{}_\eta (T^e)^\pi{}_\epsilon (T^e)^\tau{}_\rho A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\Big)\\
&= (-4)\Big(\frac{1}{2^5}(\delta^\alpha{}_\eta\delta^\epsilon{}_\beta)\,(\delta^\gamma{}_\sigma\delta^\rho{}_\delta)\,(\delta^\kappa{}_\tau\delta^\sigma{}_\lambda)\,(\delta^\mu{}_\pi\delta^\eta{}_\nu)\,(\delta^\pi{}_\rho\delta^\tau{}_\epsilon)\,A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad - \frac{1}{2^5}(\delta^\alpha{}_\eta\delta^\epsilon{}_\beta)\,(\delta^\gamma{}_\tau\delta^\sigma{}_\delta)\,(\delta^\kappa{}_\sigma\delta^\rho{}_\lambda)\,(\delta^\mu{}_\pi\delta^\eta{}_\nu)\,(\delta^\pi{}_\rho\delta^\tau{}_\epsilon)\,A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad - \frac{1}{2^5}(\delta^\alpha{}_\pi\delta^\eta{}_\beta)\,(\delta^\gamma{}_\sigma\delta^\rho{}_\delta)\,(\delta^\kappa{}_\tau\delta^\sigma{}_\lambda)\,(\delta^\mu{}_\eta\delta^\epsilon{}_\nu)\,(\delta^\pi{}_\rho\delta^\tau{}_\epsilon)\,A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\\
&\qquad + \frac{1}{2^5}(\delta^\alpha{}_\pi\delta^\eta{}_\beta)(\delta^\gamma{}_\tau\delta^\sigma{}_\delta)\,(\delta^\kappa{}_\sigma\delta^\rho{}_\lambda)\,(\delta^\mu{}_\eta\delta^\epsilon{}_\nu)\,(\delta^\pi{}_\rho\delta^\tau{}_\epsilon)A^\beta{}_\gamma B^\delta{}_\alpha C^\lambda{}_\mu D^\nu{}_\kappa\Big)\\
&= \frac{1}{8}\Big(-\mathrm{Tr}(ACBD) + \mathrm{Tr}\,A\,\mathrm{Tr}\,C\,\mathrm{Tr}(BD) + \mathrm{Tr}(AC)\,\mathrm{Tr}\,B\,\mathrm{Tr}\,D - \mathrm{Tr}(ADBC)\Big)\,.
\end{aligned}
$$

□

- **Fourier transforms**

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{e^{ip\cdot x}}{(p^2)^\alpha} = \frac{\Gamma(\omega-\alpha)}{4^\alpha \pi^\omega \Gamma(\alpha)} \frac{1}{(x^2)^{\omega-\alpha}} \tag{5.4.10}$$

$$\int d^{2\omega}x \, e^{-ip\cdot x} \frac{1}{(x^2)^\alpha} = \frac{4^{\omega-\alpha}\pi^\omega \Gamma(\omega-\alpha)}{\Gamma(\alpha)} \frac{1}{(p^2)^{\omega-\alpha}} \tag{5.4.11}$$

We now consider the renormalization of the scalar operators

$$\mathcal{O}(x) \equiv \mathrm{Tr}\left(\Phi_{i_1}(x)\cdots\Phi_{i_J}(x)\right). \tag{5.4.12}$$

The two-point correlation function is given by

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \langle \phi^{a_1}_{i_1}(x)\cdots\phi^{a_J}_{i_J}(x)\,\phi^{b_1}_{j_1}(0)\cdots\phi^{b_J}_{j_J}(0)\rangle\ \mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\mathrm{Tr}\left(T^{b_1}\cdots T^{b_J}\right). \tag{5.4.13}$$

The tree-level contribution to $\langle \phi^{a_1}_{i_1}(x)\cdots\phi^{a_J}_{i_J}(x)\phi^{b_1}_{j_1}(0)\cdots\phi^{b_J}_{j_J}(0)\rangle$ comes from J scalar propagators connecting x and 0:

$$\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\left(\frac{1}{4\pi^\omega x^2}\right)^J. \tag{5.4.14}$$

5.4.1 Insertion of self-energy diagram

Now consider the diagram obtained from the above tree level diagram by replacing the i_l to j_l propagator (where $l=1,\ldots,J$) by the self-energy correction described in Section 5.2. To compute this diagram we now translate the self-energy diagram results Eqs. (5.2.1) and (5.2.2) obtained in momentum space to coordinate space. Adding Eqs. (5.2.1) and (5.2.2) and using (5.4.10) we find

$$\begin{aligned} &\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{ip\cdot x}\left(-2g^2N\delta^{a_lb_l}\delta_{i_lj_l}\frac{\Gamma(2-\omega)\,\Gamma(\omega-1)^2}{(4\pi)^\omega\,\Gamma(2\omega-2)}\,(p^2)^{\omega-3}\right) \\ &\quad = -g^2N\delta^{a_lb_l}\delta_{i_lj_l}\,\frac{\Gamma(\omega-1)^2}{32\pi^{2\omega}\,(2-\omega)\,(2\omega-3)}\,\frac{1}{(x^2)^{2\omega-3}}. \end{aligned} \tag{5.4.15}$$

Therefore the full diagram with $J-1$ propagators connecting x and 0 with the self-energy diagram inserted on the l'th spot evaluates to

$$-g^2N\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\frac{\Gamma(\omega-1)^2}{8\pi^\omega(2-\omega)(2\omega-3)}\left(\frac{1}{4\pi^\omega x^2}\right)^J. \tag{5.4.16}$$

The divergent part of this is

$$-g^2N\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\frac{1}{8\pi^2\epsilon}\left(\frac{1}{4\pi^\omega x^2}\right)^J. \tag{5.4.17}$$

Therefore, the renormalized 1PI Green's function $\Gamma^{(2)}_{i_1\cdots i_J}$ is given by

$$\left(1+\frac{g^2N}{8\pi^2\epsilon}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\left(\frac{1}{4\pi^\omega x^2}\right)^J \tag{5.4.18}$$

i.e. the sum of the tree-level result and the counterterm associated with the one-loop result obtained from inserting the self-energy diagram. Comparing this to Eq. (5.0.3) we conclude that

$$Z^{\cdots j_l j_{l+1} \cdots}_{\cdots i_l i_{l+1} \cdots} = \left(1 + \frac{g^2 N}{8\pi^2 \epsilon}\right) \delta_{i_l j_l} \delta_{i_{l+1} j_{l+1}} \tag{5.4.19}$$

where we have explicitly shown the $i_l, i_{l+1}, j_l, j_{l+1}$ components. This notation will prove useful for the 4-point one-loop renormalizations.

5.4.2 Insertion of the 4-scalar interaction diagram

The diagram obtained from the tree-level diagram by removing the i_l to j_l propagator and the i_{l+1} to j_{l+1} propagator and inserting the four-scalar interaction diagram[2] evaluates to:

$$\begin{aligned}
&-g^2 \frac{\Gamma(2-\omega)^2\,\Gamma(\omega-1)^2\,\Gamma(3\omega-4)}{16\pi^\omega\,\Gamma(2\omega-2)^2\,\Gamma(4-2\omega)} \Big(f^{ea_l a_{l+1}} f^{eb_l b_{l+1}} \big(\delta_{i_l j_l}\delta_{i_{l+1} j_{l+1}} - \delta_{i_l j_{l+1}}\delta_{i_{l+1} j_l}\big) \\
&\qquad + f^{ea_l b_l} f^{eb_{l+1} a_{l+1}} \big(\delta_{i_l j_{l+1}}\delta_{j_l i_{l+1}} - \delta_{i_l i_{l+1}}\delta_{j_l j_{l+1}}\big) \\
&\qquad + f^{ea_l b_{l+1}} f^{ea_{l+1} b_l} \big(\delta_{i_l i_{l+1}}\delta_{j_l j_{l+1}} - \delta_{i_l j_l}\delta_{i_{l+1} j_{l+1}}\big)\Big) \\
&\quad \times \big(\delta^{a_1 b_1}\cdots\delta^{a_{l-1} b_{l-1}}\delta^{a_{l+2} b_{l+2}}\cdots\delta^{a_J b_J}\big)\big(\delta_{i_1 j_1}\cdots\delta_{i_{l-1} j_{l-1}}\delta_{i_{l+2} j_{l+2}}\cdots\delta_{i_J j_J}\big) \\
&\quad \times \mathrm{Tr}(T^{a_1}\cdots T^{a_J})\,\mathrm{Tr}(T^{b_1}\cdots T^{b_J}) \left(\frac{1}{4\pi^\omega x^2}\right)^J
\end{aligned} \tag{5.4.20}$$

$$\begin{aligned}
= &-g^2 \frac{\Gamma(2-\omega)^2\,\Gamma(\omega-1)^2\,\Gamma(3\omega-4)}{16\pi^\omega\,\Gamma(2\omega-2)^2\,\Gamma(4-2\omega)} \Big(\big(f^{ea_l b_l} f^{eb_{l+1} a_{l+1}} - f^{ea_l a_{l+1}} f^{eb_l b_{l+1}}\big) \delta_{i_l j_{l+1}}\delta_{i_{l+1} j_l} \\
&\qquad + \big(f^{ea_l a_{l+1}} f^{eb_l b_{l+1}} - f^{ea_l b_{l+1}} f^{ea_{l+1} b_l}\big) \delta_{i_l j_l}\delta_{i_{l+1} j_{l+1}} \\
&\qquad + \big(f^{ea_l b_{l+1}} f^{ea_{l+1} b_l} - f^{ea_l b_l} f^{eb_{l+1} a_{l+1}}\big) \delta_{i_l i_{l+1}}\delta_{j_l j_{l+1}}\Big) \\
&\quad \times \big(\delta^{a_1 b_1}\cdots\delta^{a_{l-1} b_{l-1}}\delta^{a_{l+2} b_{l+2}}\cdots\delta^{a_J b_J}\big)\big(\delta_{i_1 j_1}\cdots\delta_{i_{l-1} j_{l-1}}\delta_{i_{l+2} j_{l+2}}\cdots\delta_{i_J j_J}\big) \\
&\quad \times \mathrm{Tr}(T^{a_1}\cdots T^{a_J})\,\mathrm{Tr}(T^{b_1}\cdots T^{b_J}) \left(\frac{1}{4\pi^\omega x^2}\right)^J
\end{aligned} \tag{5.4.21}$$

Using the trace identities (5.4.6)-(5.4.8) we find the following relations:

$$\begin{aligned}
&\big(f^{ea_l b_l} f^{eb_{l+1} a_{l+1}} - f^{ea_l a_{l+1}} f^{eb_l b_{l+1}}\big) \\
&\qquad \times \mathrm{Tr}\left(T^{a_l} I T^{a_{l+1}} (T^{a_{l+2}}\cdots T^{a_{l-1}})\right) \mathrm{Tr}\left(T^{b_l} I T^{b_{l+1}} (T^{b_{l+2}}\cdots T^{b_{l-1}})\right) \\
= &\ \frac{1}{8}\Big((N^2+2)\,\mathrm{Tr}\left(T^{a_{l+2}}\cdots T^{a_{l-1}} T^{b_{l+2}}\cdots T^{b_{l-1}}\right) \\
&\qquad - 3N\,\mathrm{Tr}\left(T^{a_{l+2}}\cdots T^{a_{l-1}}\right)\mathrm{Tr}\left(T^{b_{l+2}}\cdots T^{b_{l-1}}\right)\Big)
\end{aligned} \tag{5.4.22}$$

[2]The reason why we restrict ourselves to interactions taking place only between neighboring sites l and $l+1$ is that we consider only *planar* diagrams; general diagrams with interactions between sites l and $l+p$ will be non-planar and hence subleading in the $1/N$ expansion.

$$\begin{aligned}
&\left(f^{ea_l a_{l+1}} f^{eb_l b_{l+1}} - f^{ea_l b_{l+1}} f^{ea_{l+1} b_l}\right) \\
&\qquad \times \operatorname{Tr}\left(T^{a_l} I T^{a_{l+1}} (T^{a_{l+2}} \cdots T^{a_{l-1}})\right) \operatorname{Tr}\left(T^{b_l} I T^{b_{l+1}} (T^{b_{l+2}} \cdots T^{b_{l-1}})\right) \\
&= -\frac{1}{4}\Big(\left(N^2-1\right) \operatorname{Tr}\left(T^{a_{l+2}} \cdots T^{a_{l-1}} T^{b_{l+2}} \cdots T^{b_{l-1}}\right)\Big)
\end{aligned} \tag{5.4.23}$$

$$\begin{aligned}
&\left(f^{ea_l b_{l+1}} f^{ea_{l+1} b_l} - f^{ea_l b_l} f^{eb_{l+1} a_{l+1}}\right) \\
&\qquad \times \operatorname{Tr}\left(T^{a_l} I T^{a_{l+1}} (T^{a_{l+2}} \cdots T^{a_{l-1}})\right) \operatorname{Tr}\left(T^{b_l} I T^{b_{l+1}} (T^{b_{l+2}} \cdots T^{b_{l-1}})\right) \\
&= \frac{1}{8}\Big(\left(N^2-4\right) \operatorname{Tr}\left(T^{a_{l+2}} \cdots T^{a_{l-1}} T^{b_{l+2}} \cdots T^{b_{l-1}}\right) \\
&\qquad\qquad + 3N \operatorname{Tr}\left(T^{a_{l+2}} \cdots T^{a_{l-1}}\right) \operatorname{Tr}\left(T^{b_{l+2}} \cdots T^{b_{l-1}}\right)\Big) .
\end{aligned} \tag{5.4.24}$$

To proceed further, we note the following theorem.

Theorem 5.4.1. *Let T^a, T^b, T^c and T^d be generators of $su(N)$ and let C, D be arbitrary $N \times N$ matrices. Then the following identities hold.*

$$\delta^{ab}\delta^{cd} \operatorname{Tr}(T^a T^c C) \operatorname{Tr}(T^b T^d D) = \frac{1}{4} \operatorname{Tr} C \operatorname{Tr} D \tag{5.4.25}$$

$$\delta^{ab}\delta^{cd} \operatorname{Tr}(T^a T^c C T^b T^d D) = \frac{1}{4} \operatorname{Tr}(CD) \tag{5.4.26}$$

Proof. First note the fusion and fission rules

$$\operatorname{Tr} T^a A \operatorname{Tr} T^a B = \frac{1}{2} \operatorname{Tr} AB \tag{5.4.27}$$

$$\operatorname{Tr} T^a A T^a B = \frac{1}{2} \operatorname{Tr} A \operatorname{Tr} B . \tag{5.4.28}$$

Both of these equations are easily obtained from the normalization condition on the $su(N)$ generators:

$$(T^a)^\alpha{}_\beta (T^a)^\gamma{}_\delta = \frac{1}{2}\delta^\alpha{}_\delta \delta^\gamma{}_\beta . \tag{5.4.29}$$

Using Eqs. (5.4.27) and (5.4.28) we find

$$\begin{aligned}
\delta^{ab}\delta^{cd} \operatorname{Tr}\left(T^a (T^c C)\right) \operatorname{Tr}\left(T^b (T^d D)\right) &= \delta^{cd} \operatorname{Tr}\left(T^a (T^c C)\right) \operatorname{Tr}\left(T^a (T^d D)\right) \\
&= \frac{1}{2}\delta^{cd} \operatorname{Tr}(T^c C T^d D) \\
&= \frac{1}{2} \operatorname{Tr}(T^c C T^c D) \\
&= \frac{1}{4} \operatorname{Tr} C \operatorname{Tr} D
\end{aligned} \tag{5.4.30}$$

which proves Eq. (5.4.25). Similarly for Eq. (5.4.26). □

In particular we find that

$$\operatorname{Tr}\left(T^{a_{l+2}} \cdots T^{a_{l-1}}\right) \operatorname{Tr}\left(T^{b_{l+2}} \cdots T^{b_{l-1}}\right) = 4\delta^{a_l b_l}\delta^{a_{l+1} b_{l+1}} \operatorname{Tr}\left(T^{a_1} \cdots T^{a_J}\right) \operatorname{Tr}\left(T^{b_1} \cdots T^{b_J}\right) . \tag{5.4.31}$$

Moreover, when J is even **(but oddly enough, not when J is odd!!)** we have

$$\begin{aligned} & N^2\,\mathrm{Tr}\left(T^{a_{l+2}}\cdots T^{a_{l-1}}T^{b_{l+2}}\cdots T^{b_{l-1}}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_{l-1}b_{l-1}}\delta^{a_{l+2}b_{l+2}}\cdots\delta^{a_Jb_J}\right) \\ = \; & N\,\mathrm{Tr}\left(T^{a_{l+2}}\cdots T^{a_{l-1}}\right)\mathrm{Tr}\left(T^{b_{l+2}}\cdots T^{b_{l-1}}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_{l-1}b_{l-1}}\delta^{a_{l+2}b_{l+2}}\cdots\delta^{a_Jb_J}\right) \quad (5.4.32)\end{aligned}$$

since both sides evaluate to $\left(\frac{1}{4}\right)^{\frac{J-2}{2}}N^3$.

With these trace identities, we can now write the result for the diagram obtained from the tree-level diagram by removing the i_l to j_l propagator and the i_{l+1} to j_{l+1} propagator and inserting the four-scalar interaction diagram:

$$\begin{aligned} & -g^2N\frac{\Gamma(2-\omega)^2\Gamma(\omega-1)^2\Gamma(3\omega-4)}{16\pi^\omega\Gamma(2\omega-2)^2\Gamma(4-2\omega)}\Big(-\delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}-\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}+2\delta_{i_li_{l+1}}\delta_{j_lj_{l+1}}\Big) \\ & \quad\times\left(\delta_{i_1j_1}\cdots\delta_{i_{l-1}j_{l-1}}\delta_{i_{l+2}j_{l+2}}\cdots\delta_{i_Jj_J}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right) \\ & \quad\times\mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\mathrm{Tr}\left(T^{b_1}\cdots T^{b_J}\right)\left(\frac{1}{4\pi^\omega x^2}\right)^J . \qquad (5.4.33)\end{aligned}$$

The divergent part of this is

$$\begin{aligned} & -\frac{g^2N}{8\pi^2\epsilon}\Big(-\delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}-\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}+2\delta_{i_li_{l+1}}\delta_{j_lj_{l+1}}\Big)\left(\delta_{i_1j_1}\cdots\delta_{i_{l-1}j_{l-1}}\delta_{i_{l+2}j_{l+2}}\cdots\delta_{i_Jj_J}\right) \\ & \quad\times\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\mathrm{Tr}\left(T^{b_1}\cdots T^{b_J}\right)\left(\frac{1}{4\pi^2x^2}\right)^J . \qquad (5.4.34)\end{aligned}$$

Therefore, the renormalized 1PI Green's function $\Gamma^{(2)}_{i_1\cdots i_J}$ is given by

$$\begin{aligned} & \left[\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}+\frac{g^2N}{8\pi^2\epsilon}\Big(-\delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}-\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}+2\delta_{i_li_{l+1}}\delta_{j_lj_{l+1}}\Big)\right] \\ & \quad\times\left(\delta_{i_1j_1}\cdots\delta_{i_{l-1}j_{l-1}}\delta_{i_{l+2}j_{l+2}}\cdots\delta_{i_Jj_J}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right) \\ & \quad\times\mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\mathrm{Tr}\left(T^{b_1}\cdots T^{b_J}\right)\left(\frac{1}{4\pi^2x^2}\right)^J . \qquad (5.4.35)\end{aligned}$$

i.e. the sum of the tree-level result and the counterterm associated with the one-loop result obtained from inserting the 4-scalar interaction. Comparing this to Eq. (5.0.3) we conclude that

$$Z^{\cdots j_lj_{l+1}\cdots}_{\cdots i_li_{l+1}\cdots}=\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}-\frac{g^2N}{8\pi^2\epsilon}\left(\delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}+\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}-2\delta_{i_li_{l+1}}\delta_{j_lj_{l+1}}\right). \quad (5.4.36)$$

This is, however, in disagreement with the expression $Z^{(b)\cdots j_lj_{l+1}\cdots}_{\cdots i_li_{l+1}\cdots}$ in Ref. [13] (p. 7)!!

5.4.3 Insertion of the gauge boson exchange diagram

Now we consider the diagram obtained from the tree-level diagram by removing the i_l to j_l propagator and the i_{l+1} to j_{l+1} propagator and inserting the gauge boson interaction diagram. More specifically, we consider only the divergent part of this diagram which evaluates to:

$$\begin{aligned}
&\frac{g^2}{4\pi^2\epsilon}\left(f^{ea_lb_l}f^{ea_{l+1}b_{l+1}}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_{l-1}b_{l-1}}\delta^{a_{l+2}b_{l+2}}\cdots\delta^{a_Jb_J}\right)\\
&\qquad\times\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\mathrm{Tr}\left(T^{b_1}\cdots T^{b_J}\right)\left(\frac{1}{4\pi^2x^2}\right)^J\\
=\;&\frac{g^2N}{4\pi^2\epsilon}\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\mathrm{Tr}\left(T^{a_1}\cdots T^{a_J}\right)\left(T^{b_1}\cdots T^{b_J}\right)\left(\frac{1}{4\pi^2x^2}\right)^J \qquad (5.4.37)
\end{aligned}$$

where we have used Eqs. (5.4.7) and (5.4.31).

Therefore, the renormalized 1PI Green's function $\Gamma^{(2)}_{i_1\cdots i_J}$ is given by

$$\left(1-\frac{g^2N}{4\pi^2\epsilon}\right)\left(\delta^{a_1b_1}\cdots\delta^{a_Jb_J}\right)\left(\delta_{i_1j_1}\cdots\delta_{i_Jj_J}\right)\left(\frac{1}{4\pi^2x^2}\right)^J \qquad (5.4.38)$$

i.e. the sum of the tree-level result and the counterterm associated with the one-loop result obtained from inserting the self-energy diagram. Comparing this to Eq. (5.0.3) we conclude that

$$Z^{\cdots j_lj_{l+1}\cdots}_{\cdots i_li_{l+1}\cdots} = \left(1-\frac{g^2N}{4\pi^2\epsilon}\right)\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}}\,. \qquad (5.4.39)$$

This is, however, in disagreement with the expression $Z^{(a)\cdots j_lj_{l+1}\cdots}_{\cdots i_li_{l+1}\cdots}$ in Ref. [13] (p. 7)!!

5.5 The anomalous dimension matrix

Multiplying Eqs. (5.4.19), (5.4.36) and (5.4.39) we find

$$Z^{\cdots j_lj_{l+1}\cdots}_{\cdots i_li_{l+1}\cdots} = \delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}} + \frac{\lambda}{16\pi^2\epsilon}\Big(\delta_{i_li_{l+1}}\delta_{j_lj_{l+1}} + 2\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}} - 2\delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}\Big)\,. \qquad (5.5.1)$$

The total Z factor is the product over all links of the expression in (5.5.1). For convenience we can rewrite the result in terms of two elementary operators which act on each link: the trace operator

$$K^{j_lj_{l+1}}_{i_li_{l+1}} = \delta_{i_li_{l+1}}\delta_{j_lj_{l+1}}\,, \qquad (5.5.2)$$

and the permutation operator

$$P^{j_lj_{l+1}}_{i_li_{l+1}} = \delta_{i_lj_{l+1}}\delta_{i_{l+1}j_l}\,. \qquad (5.5.3)$$

With this notation, the total Z factor can be written in the form

$$Z = \prod_{l=1}^{J}\left[\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}} + \frac{\lambda}{16\pi^2\epsilon}\Big(K^{j_lj_{l+1}}_{i_li_{l+1}} + 2\delta_{i_lj_l}\delta_{i_{l+1}j_{l+1}} - 2P^{j_lj_{l+1}}_{i_li_{l+1}}\Big)\right]\,. \qquad (5.5.4)$$

The renormalization factor Z determines the matrix Γ of anomalous dimensions through the renormalization group equation

$$\Gamma = \frac{1}{Z}\frac{dZ}{d\mu}\,. \tag{5.5.5}$$

Discarding terms of order λ^2 and higher, the anomalous dimension matrix can be written

$$\Gamma \;=\; \frac{\lambda}{16\pi^2}\sum_{l=1}^{J}\left(K^{j_l j_{l+1}}_{i_l i_{l+1}} + 2\delta_{i_l j_l}\delta_{i_{l+1} j_{l+1}} - 2P^{j_l j_{l+1}}_{i_l i_{l+1}}\right). \tag{5.5.6}$$

As observed in [13], this is the Hamiltonian of an integrable $SO(6)$ spin chain.

Integrability means that there exists a set of "higher" charges $Q_2, Q_3, \ldots$ that commute among themselves, $[Q_r, Q_s] = 0$, and that are conserved in the sense that they commute with the generators of the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$; i.e., $[\mathfrak{J}_0, Q_r] = 0$. Crucially, the generator of dilatations $\mathfrak{D}_2$ must be included in the set of conserved charges; i.e. $\mathfrak{D}_2 = Q_2$ for instance. A well-known sufficient condition for integrability is that the R-matrix satisfies a Yang-Baxter equation.

The $SO(6)$ sector is a closed subsector. By a 'closed' subsector we mean a subspace $\mathcal{H}_0$ of the total Hilbert space $\mathcal{H}$ of gauge invariant operators of $\mathcal{N} = 4$ $SU(N)$ SYM theory, such that the action of the dilatation operator $\mathcal{D}$ closes on $\mathcal{H}_0$; that is, that the image of $\mathcal{H}_0$ under $\mathcal{D}$ is contained in $\mathcal{H}_0$, i.e. $\mathcal{D}(\mathcal{H}_0) \subseteq \mathcal{H}_0$.

Conclusion

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some useful references:

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