Independent project report

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Killing-Yano tensors

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Abstract

In this project, we investigate the theory of Killing-Yano tensors and their application to the study of higher dimensional black holes. We will first develop the general theory of explicit and hidden symmetries for particles and fields, and relate them to conserved quantities and integrability of physical theories. The most general spacetime allowing the principal conformal Killing-Yano tensor is constructed, and we derive a suitable coordinate basis for this canonical metric. This we identify with the Kerr-NUT-(A)dS metric upon imposing the Einstein field equations, which describes a very general class of spacetimes with black holes with spherical topology of the event horizon. Focus is then turned onto those higher dimensional stationary black holes, which can all be considered special cases of the Kerr-NUT-(A)dS metric. This treatment includes especially the Kerr, Schwarzschild-Tangherlini and Myers-Perry black holes.

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1 Introduction

Symmetries have always been an important aspect of physics. Usually, they simplify calculations directly by giving constraints that the model under consideration must obey. There is a intimate connection between symmetries and conserved quantities. For the case of continuous symmetries, the famous Noether theorem states that for every continuous symmetry, there is a corresponding conservation law when the equations of motion of the system are obeyed [31]. This also generalizes outside of classical physics regime to the case of quantum physics by the Ward identities for quantum field theory.

Within the class of continuous spacetime symmetries, physicists have in the recent years been made aware of the fact that there are two kinds of such symmetries, which we could bold and name “explicit” and “hidden”. Explicit symmetries are isometries, diffeomorphisms of the metric, generated by a *Killing vector*, and their physical interpretation are in general rather intuitive - could for example be translation or rotational symmetries. On the other hand, hidden symmetries are symmetries that cannot be identified with isometries directly. Hidden symmetries are always described by tensors of rank two or greater, which we will call *Killing tensor*. They still give conserved quantities using Noethers theorem, and this is why we call them “hidden”, but their physical interpretation may in some cases be obscure. In other cases, as for the Kerr metric, which has a hidden symmetry they have a good physical meaning, here it can be interpreted as the total angular momentum of some particle or field in the asymptotically flat region. It turns out that in a Hamiltonian formulation on general manifolds we can naturally make sense of these hidden symmetries in phase space as we shall see. The first occurrence of such a hidden symmetry was indeed discovered for the Kerr metric. In 1968 Carter [8] discovered that the Kerr metric had some rather unexpected conserved quantity, and it was first in 1970 that Walker and Penrose [38] figured out that it originated from a Killing tensor of rank two. As a consequence of this, there were now a total of 4 constants of motion, 2 from Killing vectors and two from rank 2 Killing tensors (the other one being the metric itself), which renders the geodesic equation integrable - a remarkable result. However, Walker and Penrose were not able to develop a full theory that would explain the reason why some spacetimes admitted Killing tensors in this paper.

A rather complete solution to this problem has only emerged within the last 10 years or literature, culminating with the paper by Krtous et al. [27]. It was understood that for a large subclass of the solutions to the Einstein field equations admitting hidden symmetries, the integrability of the geodesic equation is very fundamentally related to that. It was realized that for this class of spacetimes, a further decomposition of the Killing tensors into contracted products of *Killing-Yano tensors* was indeed possible. The Killing-Yano tensors can be thought of as the “square root” of Killing tensors, and they have many interesting properties. The existence of a special Killing-Yano tensor, the so-called *principal conformal Killing-Yano tensor* turn out to exactly define the large class of integrable spacetimes. The principal conformal Killing-Yano tensor will generate a whole “tower” of Killing tensors and Killing vectors, that will again gives conserved quantities and secure integrability. These will also secure separability of field equations of interest, such as the Klein-Gordon and Dirac equations [29], which is a remarkable result as well.

In this project we shall study the theory of Killing-Yano tensors in detail and their relation to integrable spacetimes and symmetries. Our end goal will be to construct the most general metric solving Einsteins field equations allowing such a principal conformal Killing-Yano tensor, which turns out to be the very general *Kerr-NUT-(A)dS* metric in $D > 2$ spacetimes dimensions, first constructed by Chen et al. [9] in 2006. There will be given full proofs of most statements, but to contain the project to a reasonable size, we limit ourselves to give references to a some results and just describe the main ideas of the proofs. We will start with development of the mathematical background of Killing vectors and tensors, and their conformal generalizations. The relation of these to symmetries of theories of particles and fields, are constructed using both a Lagrangian and Hamiltonian formalism of mechanics, and two general versions of Noether's theorem are proven. Integrability and separability theory for particles and fields are then discussed, and we give a simple criterion for integrability of Hamiltonians, including the geodesic equation, along with a statement of a theorem on the existence of separability structures. We then move on to the theory of Killing-Yano tensors and their conformal generalizations, proving various number of theorems and lemmas. These are used for constructing the most general metric allowing a principal conformal Killing-Yano tensor, *the canonical metric*, using the theory. The detailed proof that we will be giving on this is somewhat simpler in some aspects than what can be found in the literature, and this is the main result of this project. We then show that this is in fact the Kerr-NUT-(A)dS metric when imposing the Einstein field equations. We then leave the most general case, and take a quick review of stationary black holes spacetimes and their relation to the Kerr-NUT-(A)dS metric, and take a look at important special cases of it. At last we discuss briefly how the theory that we have developed can be put into a greater perspective, applications and generalizations.

Notation and conventions Unless otherwise stated, we always use the Christoffel connection for the covariant derivative. The signature of the metric is in general arbitrary unless otherwise stated. We work in natural units with $G = c = \hbar = 1$. Bold symbols like $\boldsymbol{g}$, $\boldsymbol{h}$ refers to tensors that are written in component-free notation, i.e. including the basis.

2 Killing tensors

2.1 Isometries and the Lie derivative

In the following, let us work with a manifold M of dimension $D = 2n + \varepsilon$, $\varepsilon = 0, 1$ so we have explicitly odd ($\varepsilon = 1$) or even ($\varepsilon = 0$) dimensions, to which we can associate a metric $\boldsymbol{g}$, that is singular at a finite number of points at most.

The simplest kind of symmetry of such a manifold are isometries, active coordinate transformations that leaves the metric $\boldsymbol{g}$ invariant [7]. To make these statements more precise, let us first assume that we have some diffeomorphism $\varphi : M \to M$, i.e. a differentiable bijective mapping with an differentiable inverse. We then define the pullback[1] of the metric $\varphi^*\boldsymbol{g}$ as:

[1] For a diffeomorphism, we can pullback a general tensor, but contravariant indices transform oppositely, they are "pushed forward".

Definition 1 (Pullback)**.** For a diffeomorphism $\varphi : M \to M$ and a chart with coordinate point x of the manifold and with $y \equiv \varphi(x)$, we define the pullback of the metric $\varphi^* \boldsymbol{g}$ in local coordinates as

$$(\varphi^* g)_{\mu'\nu'}(x) \equiv \frac{\partial x^\mu}{\partial y(x)^{\mu'}} \frac{\partial x^\nu}{\partial y(x)^{\nu'}} g_{\mu\nu}(y(x)) \ . \tag{2.1}$$

In words, this is to say that the value of $\boldsymbol{g}$ at $y = \varphi(x)$ is mapped back ("pulled back") to be the value of $\boldsymbol{g}' \equiv \varphi^* \boldsymbol{g}$ at x, which both describes the same spacetime point of the manifold.

Now it is also easy to make a precise statement (the first of two equivalent ones) of what we mean by an isometry of the metric:

Definition 2 (Isometry 1)**.** We say that a diffeomorphism φ is an isometry if $\varphi^* \boldsymbol{g} = \boldsymbol{g}$.

This is what we should understand by an active coordinate transformation, that leaves the metric invariant. Say now that we have a one-parameter family of diffeomorphisms (not necessarily isometries) $\varphi_\lambda : M \times \mathbb{R} \to M$, where λ is the parameter[2] taken such that $\varphi_0 = I$ (the identity map), and the family is differentiable with respect to λ. The pullback of such a family is generated by the flow of some vector field because it defines a (differentiable) vector field as the tangent vector at each spacetime point for each value of the parameter. Conversely, given a vector field K^μ, the integral curves of its flow is the solution of the ODE given by setting the directional derivative of a curve $x^\mu(\lambda)$ equal to the vector field:

$$\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} = K^\mu(x(\lambda)) \ , \tag{2.2}$$

with boundary conditions $x^\mu(\lambda = 0) = x^\mu$ and $\left.\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0} = K^\mu(x)$ at each point of the manifold in some chosen chart x^μ. The existence and uniqueness of the solution is guaranteed by the Picard-Lindelöf theorem [35], given that K^μ is continuous - we will assume that it is as differentiable as needed in the following. We will call K^μ the generator of the one-parameter family of diffeomorphisms φ_λ.

We may then conclude that a vector field defines (or generates) a one-parameter family of diffeomorphisms, and the existence of one implies the other. Given such a generator $\boldsymbol{K}$ of a family φ_λ, we can define the Lie derivative $\mathcal{L}_{\boldsymbol{K}}$ along $\boldsymbol{K}$ of the metric (similarly for general tensors) as

$$\mathcal{L}_{\boldsymbol{K}} \boldsymbol{g}(x) \equiv \lim_{\lambda \to 0} \left[\frac{(\varphi^*_\lambda \boldsymbol{g})(x) - \boldsymbol{g}(x)}{\lambda} \right] , \tag{2.3}$$

which is a tensor itself, since it is just the difference of two tensors. The Lie derivative may be described in words as rate of change of the tensor as we move along the flow at each spacetime point. It has a lot of good properties and is one of the simplest constructions of a differential operator on a manifold, see [35].

[2]It is easy to see that under functional composition, they form a one-parameter group with this as the group product, because for two parameters λ, λ' $(\varphi_{\lambda'} \circ \varphi_\lambda)(x) = \varphi_{\lambda'}(\varphi_\lambda(x)) = \varphi_{\lambda'+\lambda}(x)$, and the remaining axioms may also easily be proven to hold.

Given now a one-parameter family of isometries φ_λ, we call the generator $\boldsymbol{K}$ a Killing vector (KV), which will turn out to be very fundamental quantity. We have $\varphi_\lambda^* \boldsymbol{g} = \boldsymbol{g}$ for any $\lambda \in \mathbb{R}$, and using the definition of the Lie derivative, this is equivalent to the vanishing of $\mathcal{L}_{\boldsymbol{K}} \boldsymbol{g}$, which gives us a theorem that connects isometries, Killing vectors and the Lie derivative, which we can formulate as below:

Theorem 3 (Isometry 2)**.** *$\boldsymbol{K}$ is a Killing vector if and only if $\mathcal{L}_{\boldsymbol{K}} \boldsymbol{g} = 0$.*

2.2 Killing vectors and tensors

As we have now established the role of Killing vectors in connection to isometries, the natural question is how to find them. The following famous equation singles them out as a special class of tensors.

Theorem 4 (The Killing equation)**.** *$\boldsymbol{K}$ is a Killing vector if and only if it satisfies*

$$\nabla_{(\mu} K_{\nu)} = 0\,. \tag{2.4}$$

Proof. This is most easily done expressing the Lie derivative $\mathcal{L}_{\boldsymbol{K}} \boldsymbol{g}$ through the covariant derivative. Assuming that $\boldsymbol{K}$ generates an isometry, we have by theorem 4 that

$$\begin{aligned} 0 &= \mathcal{L}_{\boldsymbol{K}} g_{\mu\nu} \\ &= K^\lambda \nabla_\lambda g_{\mu\nu} + \left(\nabla_\mu K^\lambda\right) g_{\lambda\nu} + \left(\nabla_\nu K^\lambda\right) g_{\lambda\mu} \\ &= 2\nabla_{(\mu} K_{\nu)}\,. \end{aligned} \tag{2.5}$$

This proves the claim. See appendix A for an alternative version of the proof. □

The Killing vector equation makes it easy to test if one has a generator of symmetry in a given spacetime, and in principle we could also determine Killing vectors by solving the corresponding PDE. Doing this in flat Minkowski space, we will find 10 Killing vectors corresponding the 4 translational, 3 rotational and 3 boost isometries [36]. Unfortunately it is not so simple in general geometries, and we must argue differently to find the Killing vectors.

The concept of a Killing vector can be generalized further. We say that a totally symmetric rank p tensor $\boldsymbol{K}$ is a Killing tensor (KT), if it satisfies the Killing tensor equation

$$\nabla_{(\mu} K_{\mu_1 \cdots \mu_p)} = 0\,. \tag{2.6}$$

The interpretation of the general Killing tensor is not as straight-forward as for the Killing vector. $\boldsymbol{K}$ is not associated with an isometry for $p > 1$, and we use the terminology that it generates a hidden symmetry in the meaning introduced earlier, that will imply that some quantities are conserved as we shall see soon in section 2.4. Here we will also find a different way to derive (2.6).

2.2.1 Conformal Killing vectors and tensors

Again, we may generalize the concept of a Killing tensor. The first generalization of this is the conformal Killing tensor (CKT), which is constructed by doing a local change of scale of the geometry so that the metric changes as

$$g'_{\mu\nu}(x) = \omega(x)^2 g_{\mu\nu}(x) , \tag{2.7}$$

where $\omega(x)$ is some arbitrary real and non-vanishing smooth function [7]. This is clearly a diffeomorphism, as the inverse conformal transformation is given by multiplying the metric by ω^{-2}. We say that $\boldsymbol{g}'$ is the conformal metric of the conformal frame, and $\boldsymbol{g}$ the original metric. In the language of diffeomorphisms, we say that the diffeomorphism φ_ω that (2.7) defines, is called the conformal transformation. Then we may write

$$\boldsymbol{g}' \equiv \varphi^*_\omega \boldsymbol{g} = \omega^2 \boldsymbol{g} . \tag{2.8}$$

We can then in general consider conformal transformed tensors, and relate them back to the original tensors and the original metric. The conformal Killing tensor equation is obtained in this way from (2.6), as we can formulate more precisely as a definition:

Definition 5 (Conformal Killing tensor)**.** A conformal Killing tensor of rank p is a totally symmetric tensor $K_{\mu_1\cdots\mu_p}$, that in the conformal frame obeys conformal Killing tensor equation

$$\nabla_{(\mu} K_{\mu_1\cdots\mu_p)} = p g_{(\mu\mu_1} \overline{K}_{\mu_2\cdots\mu_p)} , \tag{2.9}$$

where the tensor $\overline{K}_{\mu_2\cdots\mu_p}$ is some totally symmetric tensor of rank $p-1$, found by tracing both sides.

To see how this related to an ordinary Killing tensor, let us perform the conformal transformation explicitly. The covariant derivative in the conformal frame ∇ is related to the ordinary covariant derivative $\tilde{\nabla}$ by a change of the connection,

$$\Gamma^\nu_{\mu\lambda} = \tilde{\Gamma}^\nu_{\mu\lambda} + C^\nu{}_{\mu\lambda} , \tag{2.10}$$

where $C^\nu{}_{\mu\lambda}$ is a tensor, symmetric in lower indices, whose explicit form may be calculated in terms of ω. Assume that $K_{\mu_1\cdots\mu_p}$ is a Killing tensor. We then have

$$\begin{aligned} \nabla_\mu K_{\mu_1\cdots\mu_p} &= \tilde{\nabla}_\mu K_{\mu_1\cdots\mu_p} + C^\nu{}_{\mu\mu_1} K_{\nu\cdots\mu_p} + C^\nu{}_{\mu\mu_2} K_{\mu_1\nu\cdots\mu_p} + \ldots \\ &\equiv \tilde{\nabla}_\mu K_{\mu_1\cdots\mu_p} + Q_{\mu\mu_1\cdots\mu_p} . \end{aligned} \tag{2.11}$$

Note that $Q_{\mu\mu_1\cdots\mu_p} \equiv C^\nu{}_{\mu\mu_1} K_{\nu\cdots\mu_p} + C^\nu{}_{\mu\mu_2} K_{\mu_1\nu\cdots\mu_p} + \ldots$ is totally symmetric as well, because $C^\nu{}_{\mu\lambda}$ is symmetric in the lower indices. If we define $p g_{\mu(\mu_1} \overline{K}_{\mu_2)\cdots\mu_p} \equiv Q_{\mu\mu_1\cdots\mu_p}$, this is consistent with the definition of $Q_{\mu\mu_1\cdots\mu_p}$, when $\overline{K}_{\mu_2\cdots\mu_p}$ is a totally symmetric tensor of rank $p-1$. Using this and symmetrizing (2.11) using that $K_{\mu_1\cdots\mu_p}$ is a Killing tensor, we find

$$\begin{aligned}\nabla_{(\mu}K_{\mu_1\cdots\mu_p)} &= \tilde{\nabla}_{(\mu}K_{\mu_1\cdots\mu_p)} + pg_{(\mu(\mu_1}\overline{K}_{\mu_2)\cdots\mu_p)} \\ &= pg_{(\mu\mu_1}\overline{K}_{\mu_2\cdots\mu_p)}\,.\end{aligned}$$

Thus we have found the conformal Killing tensor equation as promised. If $\overline{K}_{\mu_2\cdots\mu_p}$ vanishes, then $K_{\mu_1\cdots\mu_p}$ is a normal Killing tensor. For any conformal Killing tensor, we can also conclude that there always exists an inverse conformal transformation, that takes it back to a normal Killing tensor.

2.3 The Lie and synmmetric Schouten–Nijenhuis brackets

The Lie bracket $[\cdot,\cdot]$ is defined as the action of the Lie derivative along the vector field $\boldsymbol{X}$ acting on a vector field $\boldsymbol{Y}$

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} \equiv [\boldsymbol{X},\boldsymbol{Y}]\,, \tag{2.12}$$

and it has a number of good properties listed below such as bilinearity and the Jacobi identity that can all be verified [35]:

$$\begin{aligned}[\boldsymbol{X},\boldsymbol{Y}] &= -[\boldsymbol{Y},\boldsymbol{X}] \\ [\boldsymbol{X},\alpha\boldsymbol{Y}+\beta\boldsymbol{Z}] &= \alpha[\boldsymbol{X},\boldsymbol{Y}]+\beta[\boldsymbol{X},\boldsymbol{Z}] \\ 0 &= [[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}]+[[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}]+[[\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}]\end{aligned} \tag{2.13}$$

where α,β are scalars, and $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ are vector fields. $[\boldsymbol{X},\boldsymbol{Y}]$ should be thought of as the rate of change of $\boldsymbol{Y}$ along the flow of $\boldsymbol{X}$, or equivalently minus the rate of change of $-\boldsymbol{X}$ along $\boldsymbol{Y}$. We may also find using the partial or covariant[3] derivative that it can be expressed as

$$[\boldsymbol{X},\boldsymbol{Y}] = (\boldsymbol{X}\cdot\nabla)\,\boldsymbol{Y} - (\boldsymbol{Y}\cdot\nabla)\,\boldsymbol{X}\,, \tag{2.14}$$

and in this form the interpretation given above is more clear. If we have some Lie group of diffeomorphisms[4] $\mathcal{G}$ of dimension n, then the generating vector fields $\{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n\}$ will form a Lie algebra $\mathfrak{g}$, which are algebras with a product that exactly fulfills (2.13). The Lie algebra fully describes the Lie group through the structure constants C_{ijk} that doesn't depend on spacetime. It is basically the infinitesimal flows generated around the identity transformation. In general we have

$$[\boldsymbol{X}_i,\boldsymbol{X}_j] = \sum_{k=1}^{n} C_{ijk}\boldsymbol{X}_k\,. \tag{2.15}$$

Notice that this expression is independent of any coordinates we might introduce, the

[3]Because of the antisymmetry $[\boldsymbol{X},\boldsymbol{Y}] = -[\boldsymbol{Y},\boldsymbol{X}]$, the connection terms drops out when the connection is torsion free - if the metric is not torsion free, then we must use partial derivatives in (2.14).

[4]For such a group the Lie algebra $\mathfrak{g}$ will automatically close, because applying two different diffeomorphisms of $\mathcal{G}$ must also be a diffeomorphism.

only indices are group indices related to $\mathcal{G}$. In general we could compare the calculated structure constants for one manifold to those calculated for other manifolds, and if they are the same, then the two diffeomorphism groups are isomorphic. This can be put to good use, as we then have a coordinate independent way of determining what kind of symmetries we have. This way we could for example determine that we would have spherical symmetry if we found three vectors that had commutation relations isomorphic to that of SO (3), i.e. $C_{ijk} = \epsilon_{ijk}$.

For a given spacetime of dimension D and a given basis $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_D\}$ of the tangent space $T(M)$, that is not necessarily a coordinate basis, the Lie bracket can actually tell us something about the possibility of introducing a coordinate basis:

Theorem 6 (Coordinate basis)**.** *Let $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_D\}$ be a basis for $T(M)$. We have $[\boldsymbol{X}_i, \boldsymbol{X}_j] = 0$ if and only if $\boldsymbol{X}_i = \boldsymbol{\partial}_{x_i}$, i.e. there exists a local coordinate basis for $T(M)$ defined by the flow of the basis vectors.*

The proof can be found in any reference book on differential geometry, see for example [35]. The intuition behind this is that if any two vector fields commute, then there is no change in the generators going along either flow, and the flow equation (A.1) that would define the coordinates is much easier to solve, as there is only change in one direction.

Particularly relevant for us is the restriction to isometries. The set of all the Killing vectors that would generate the isometries of a spacetime is called the isometry group. The maximal number of generators that we can have for a given spacetime is

$$n_{\max} = D + \binom{D}{2} = \frac{D}{2}(D+1) , \tag{2.16}$$

because this would correspond to the D translational Killing vectors, and $\binom{D}{2}$ rotational Killing vectors we have in flat space. The maximal number of rotational Killing vectors can be calculated as follows: we can choose the first coordinate axis x^i of the hyperplane that we rotate in, in D ways and the other x^j in $D-1$ ways, but as the hyperplane of (x^i, x^j) is the same as (x^j, x^i), we only have half of the product, which is $\binom{D}{2}$. Spacetimes with maximal number of Killing vectors are called maximal symmetric spacetimes, for example (A)dS spacetime. For such maximally symmetric spacetimes, the rotational part of the isometry group is SO (p, q), where $p + q = D$, and (p, q) is the signature, and the translational part is $\mathbb{R}^{p,q}$, and the full isometry group is SO $(p, q) \otimes \mathbb{R}^{p,q}$. For non-maximal symmetric spaces, the isometry group is a subgroup of this, as some of the Killing vectors may not be present. For example, the Robertson-Walker spacetimes doesn't have time-translational Killing vector when the scale factor depends on the time coordinate. Likewise, for axial symmetric spacetimes as are of interest to us in the following, there isn't full rotational symmetry, only along one rotational coordinate axis, so only one of the three SO (3) generators exists. This generalizes to higher dimensional hyperspherical and axisymmetrical symmetries as well as we discuss in section 5.2.

We can naturally generalize the above discussion to Killing tensors by introducing a more general bracket. Such a construction is given by the symmetric Schouten–Nijenhuis bracket [32], which is defined (in component form) as:

$$[X,Y]^{\mu_1\cdots\mu_{p+q-1}} \equiv pX^{\nu(\mu_1\cdots\mu_{p-1}}\nabla_\nu Y^{\mu_p\cdots\mu_{p+q-1})} - qY^{\nu(\mu_1\cdots\mu_{q-1}}\nabla_\nu X^{\mu_q\cdots\mu_{p+q-1})} \tag{2.17}$$

where the inputs are totally symmetric tensors $X^{\mu_1\cdots\mu_p} = X^{(\mu_1\cdots\mu_p)}$ and $Y^{\mu_1\cdots\mu_q} = Y^{(\mu_1\cdots\mu_q)}$. This definition holds when the connection is torsion free, but if this is not the case, we must replace them with partial derivatives[5]. $[X,Y]^{\mu_1\cdots\mu_{p+q-1}}$ is itself a symmetric tensor of rank $p+q-1$. We see that we have the Schouten–Nijenhuis bracket reduces to the Lie bracket (2.14) when both $\boldsymbol{X},\boldsymbol{Y}$ are vectors, $p=q=1$.

The symmetric Schouten–Nijenhuis (SSN) bracket defines a Lie algebra on the vector space of symmetric tensors, because one can easily check that it satisfies the Lie algebra axioms (2.13) of with $\boldsymbol{Z}$ beeing a rank r symmetric tensor. The vector space of symmetric tensors is in general infinite dimensional, because we can continue to generate symmetric tensors of higher and higher rank by symmetrizing tensor products of lower rank tensors.

We have that if $\boldsymbol{X}$ is a vector and $\boldsymbol{Y}$ is a general tensor, then

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X},\boldsymbol{Y}] \ , \tag{2.18}$$

which one can check using the coordinate expression of the Lie derivative acting on a tensor. The Leibniz property of the Lie derivative acting on multivectors is a special case of this equation and 2.17, as we can rewrite the expression as

$$\mathcal{L}_{\boldsymbol{X}}\left(\boldsymbol{Y}\otimes_+\boldsymbol{Z}\right) \ = \ \left(\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y}\right)\otimes_+\boldsymbol{Z} + \boldsymbol{Y}\otimes_+\left(\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Z}\right) \ , \tag{2.19}$$

where $\otimes_+$ is the symmetric part of the tensor product. Now, restricting the discussion to to Killing tensors, we may show the following result

Theorem 7 (SSN bracket and Killing tensor)**.** *$\boldsymbol{K}$ is a Killing tensor of rank p if and only if $[\boldsymbol{g},\boldsymbol{K}]=0$, when the connection is metric compatible.*

Proof. We do a direct calculation to show that the two notions are equivalent by using (2.17) with covariant derivatives:

$$\begin{aligned}
[\boldsymbol{g},\boldsymbol{K}]^{\mu_1\cdots\mu_{p+1}} \ &= \ 2g^{\nu(\mu_1}\nabla_\nu K^{\mu_2\cdots\mu_{p+1})} - pK^{\nu(\mu_1\cdots\mu_{p-1}}\nabla_\nu g^{\mu_p\mu_{p+1})} \\
(*) \ &= \ 2g^{\nu(\mu_1}\nabla_\nu K^{\mu_2\cdots\mu_{p+1})} \\
&= \ 2\nabla^{(\mu_1}K^{\mu_2\cdots\mu_{p+1})} \ ,
\end{aligned}$$

where we in (*) used that the connection was assumed to be metric compatible and then raised the index of the covariant index. Upon lowering all of the indices, we obtain exactly the Killing tensor equation (2.6) and proves the theorem. □

Of course, a special case of this theorem, is the case of a Killing vector stated in theorem 3. Also, we might begin to think about whether the set of Killing tensors is a

[5]When torsion free, the connection terms will cancel anyway, but it is just more convenient using covariant derivatives.

Lie algebra on its own with the SSN bracket $[\cdot,\cdot]$. We have the following nice result:

Theorem 8 (SSN bracket of KTs)**.** *Let $\boldsymbol{K}$ and $\boldsymbol{Q}$ be a Killing tensors of rank p and q. Then $[\boldsymbol{K},\boldsymbol{Q}]$ is a Killing tensor of rank $p+q-1$.*

Proof. We just have to verify that the Killing tensor equation (2.6) holds for $[\boldsymbol{K},\boldsymbol{Q}]$. This is most easily done using the Jacobi identity on $[[\boldsymbol{K},\boldsymbol{Q}],\boldsymbol{g}]$ and using theorem 7. We see

$$\begin{aligned} 0 &= [[\boldsymbol{K},\boldsymbol{Q}],\boldsymbol{g}] + [[\boldsymbol{Q},\boldsymbol{g}],\boldsymbol{K}] + [[\boldsymbol{g},\boldsymbol{K}],\boldsymbol{Q}] \\ &= [[\boldsymbol{K},\boldsymbol{Q}],\boldsymbol{g}] \,, \end{aligned}$$

where we used that $[\boldsymbol{g},\boldsymbol{Q}] = [\boldsymbol{g},\boldsymbol{K}] = 0$ by theorem 7. This shows that $[\boldsymbol{K},\boldsymbol{Q}]$ is a Killing tensor. □

This means that the SSN bracket of Killing tensors closes, and thus the set of Killing tensors of a given spacetime with the SSN bracket is a Lie algebra. In general it is infinite dimensional, and it is a subalgebra of the Lie algebra of all symmetric tensors. This concludes the treatment of the algebraic aspects of Killing tensors for now.

2.4 Conservation laws for particles

2.4.1 Conservation of Killing tensors

The great interest in finding such Killing tensors for a physicist is because they give us conserved quantities, that puts constraints on the equations of motion that we want to solve. Our first example of this is conservation of quantities along geodesics.

We may prove the following result:

Theorem 9 (Conservation and geodesics)**.** *Let $K_{\mu_1\cdots\mu_p}$ be a Killing tensor and $x^\mu(\tau)$ a geodesic. Then the scalar $J = K_{\mu_1\cdots\mu_p}p^{\mu_1}\cdots p^{\mu_p}$ is constant along a geodesic, where $p^\mu \equiv \dot{x}^\mu(\tau)$ is the tangent vector of the geodesic.*

Proof. We would like to show that $\frac{\mathrm{D}J}{\mathrm{d}\tau} = \frac{\mathrm{D}}{\mathrm{d}\tau}K_{\mu_1\cdots\mu_p}p^{\mu_1}\cdots p^{\mu_p} = 0$, where $\frac{\mathrm{D}}{\mathrm{d}\tau} \equiv p^\mu\nabla_\mu$ is the covariant directional derivative along the geodesic, which fulfills the geodesic equation $\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = 0$. We do a direct calculation using this and the total symmetry of the Killing tensor

$$\begin{aligned} \frac{\mathrm{D}J}{\mathrm{d}\tau} &= p^\mu\left(\nabla_\mu K_{\mu_1\cdots\mu_p}\right)p^{\mu_1}\cdots p^{\mu_p} \\ (*) &= \left(\nabla_{(\mu}K_{\mu_1\cdots\mu_p)}\right)p^\mu p^{\mu_1}\cdots p^{\mu_p} \\ (**) &= 0\,. \end{aligned}$$

In (*) we used that the product $p^\mu p^{\mu_1}\cdots p^{\mu_p}$ is totally symmetric in the indices, so only the totally symmetric part of $\nabla_\mu K_{\mu_1\cdots\mu_p}$ contributes. This gives us the Killing tensor equation that we can invoke in (**) to conclude what we wanted to show. □

One could take this as the motivation for the Killing tensor equation (2.6).

2.4.2 Noether's theorem

What we have considered so far have solely been spacetime symmetries. We can however relate these to theories defined on the spacetime, given by some Lagrangian [density], which gives another view on the role of the conservation law of theorem 9. The former can be seen as a special case of a more general theorem, the famous Noether theorem, which we will formulate shortly [35]. For a short review of Lagrangian and Hamiltonian mechanics on general manifolds, see appendix B.

Imagine now that the Lagrangian L has some continuous symmetry described by some representation of a Lie group $\mathcal{G}$, in the meaning that a transformation of L by some $\mathscr{T} \in \mathcal{G}$ leaves it invariant, formally $\mathscr{T}L = L$. The symmetry transformation will change the coordinates and directional derivatives in exactly such a way that the Lagrangian remains unchanged after the transformation[6]. Working in phase space[7] Γ, the most general infinitesimal transformation $\hat{\delta}x^{\mu}$ in local coordinates is given by the vertical (momentum) derivative of (B.5) [36]:

$$\hat{\delta}x^{\mu} \equiv \epsilon R^{\mu}(x,p) = \epsilon \left[K^{\mu}(x) + K^{\mu\nu}(x)\, p_{\nu} + \frac{1}{2!} K^{\mu\nu\sigma}(x)\, p_{\nu} p_{\sigma} + \dots \right], \tag{2.20}$$

where ϵ is infinitesimal and the family of tensors $K_{\mu\nu\sigma\dots}$ are totally symmetric with all indices lowered, and are independent of momenta. It depends on the structure of $\mathcal{G}$ how many terms there are. For ordinary diffeomorphisms, which is restricted to the configuration space part of Γ, the series truncates after the first term. For more general transformations there are more terms as there could be some "momentum dependent" symmetry transformation. We could go back to configuration space by imposing (B.16) if we like. The notation for the tensors in the expansion is not arbitrary - we will see soon that they are exactly Killing tensors for special Lagrangians.

The Noether theorem then says that when imposing the classical equations of motions, there are conserved quantities, given a continuous symmetry.

Theorem 10 (Noether's theorem for particles)**.** *Let a Lagrangian $L : T(M) \to \mathbb{R}$ have a continuous symmetry under a transformation given by* (2.20)*. Then there exists a corresponding conserved charge given by*

$$J = R^{\mu} \frac{\partial L}{\partial x^{\mu}}, \tag{2.21}$$

when x^{μ} obeys the equations of motion.

Proof. A symmetry transformation must give us $\delta L \equiv \mathscr{T}L - L = 0$ even when we are

[6] Up to boundary terms, that we assume are zero, so they doesn't contribute to the EOMs.
[7] For simplicity we study the case of a single particle here, but it is easily generalized.

off-shell. Using this, we can do an expansion in $\hat{\delta}x^\mu$ and $\hat{\delta}\dot{x}^\mu$, and do some rewriting:

$$\begin{aligned} 0 &= \delta L \\ &= L\left(x+\hat{\delta}x^\mu, \dot{x}+\hat{\delta}\dot{x}^\mu\right) - L\left(x,\dot{x}\right) \\ &= L\left(x+\epsilon R^\mu, \dot{x}+\epsilon \dot{R}^\mu\right) - L\left(x,\dot{x}\right) \\ &= L\left(x,\dot{x}\right) + \frac{\partial L\left(x,\dot{x}\right)}{\partial x^\mu}\epsilon R^\mu + \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu}\epsilon \dot{R}^\mu - L\left(x,\dot{x}\right) \\ &= \frac{\partial L\left(x,\dot{x}\right)}{\partial x^\mu}\epsilon R^\mu + \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu}\epsilon \dot{R}^\mu \,. \end{aligned}$$

Applying the equations of motion $\frac{\partial L(x,\dot{x})}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu}$, this can be written as a total derivative:

$$\begin{aligned} 0 &= \frac{\partial L\left(x,\dot{x}\right)}{\partial x^\mu} R^\mu + \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} \dot{R}^\mu \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} R^\mu + \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} \dot{R}^\mu \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} R^\mu\right) \,. \end{aligned}$$

Thus we have that

$$J \equiv \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} R^\mu \,, \tag{2.22}$$

is conserved along the equations of motion, as we wanted to show. □

One should especially notice that the Lagrangian which has the geodesic equation as EOM is the single-particle Lagrangian

$$L\left(x,\dot{x}\right) = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \,, \tag{2.23}$$

which can be seen as a "free particle" Lagrangian, with corresponding Hamiltonian

$$H\left(x,p\right) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \,. \tag{2.24}$$

In this case equation (2.22) tells us that the conserved quantity is

$$\begin{aligned} J &= \frac{\partial L\left(x,\dot{x}\right)}{\partial \dot{x}^\mu} R^\mu = p_\mu R^\mu \\ &= K^\mu p_\mu + K^{\mu\nu} p_\mu p_\nu + \frac{1}{2!} K^{\mu\nu\sigma} p_\mu p_\nu p_\sigma + \ldots \end{aligned} \tag{2.25}$$

To see that they are actually our Killing tensors, we notice that

$$
\begin{aligned}
\frac{\mathrm{D}J}{\mathrm{d}\tau} &= p_\lambda \nabla^\lambda \left(K^\mu p_\mu + K^{\mu\nu} p_\mu p_\nu + \frac{1}{2!} K^{\mu\nu\sigma} p_\mu p_\nu p_\sigma + \dots \right) \\
&= \nabla^{(\lambda} K^{\mu)} p_\lambda p_\mu + \nabla^{(\lambda} K^{\mu\nu)} p_\lambda p_\mu p_\nu + \frac{1}{2!} \nabla^{(\lambda} K^{\mu\nu\sigma)} p_\lambda p_\mu p_\nu p_\sigma + \dots \\
&= 0 \,.
\end{aligned} \tag{2.26}
$$

Because each term is different from another[8], we have that each tensor must satisfy the Killing tensor equation (2.6). Thus we have successfully connected a theory of free particles to the spacetime symmetries.

Interactions between particles of a non-quantum theory is given by a potential function $V : T(M) \to \mathbb{R}$, so the Lagrangian takes the form $L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V$, and the Hamiltonian $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V$, if there is no velocity dependence. The symmetries of L could then be thought to be constrained in number. This is however not physical - good physical potentials should as a minimum respect the spacetime symmetries if we are looking at a theory should be applicable anywhere. Otherwise the consequence would be that the result of an experiment could depend on the location, rotation, etc. in the universe. Let us formulate this more concisely:

Principle 1 (Symmetries of a Lagrangian)**:** The spacetime symmetries is a subset of all the symmetries of a Lagrangian.

This should serve as a guiding principle for constructing theories.

2.5 Conservation laws for fields

For fields we have additional degrees of freedom, as they both depend on spacetime, and may carry internal indices. We can write a general field as Λ_ℓ, where ℓ is short for both spacetime and internal indices, see appendix C for a short review of classical field theory on general manifolds. The representation of continuous symmetry transformation by $\mathscr{T} \in \mathcal{G}$ must now be extended, because along with spacetime symmetries, we could also have internal symmetries, where the tensor components changes:

$$
\mathscr{T} \Lambda_\ell (x) = \Lambda_\ell (\mathscr{T} x) = U_\ell{}^\kappa \Lambda_\kappa (\mathscr{T} x) \,, \tag{2.27}
$$

where U is a representation of the internal symmetry on the components, and $\mathscr{T} x^\mu$

[8]

Proof. Assume first that there are two non-zero contracted terms that would cancel in (2.26). We can factor out the p's such that we have something like $\left(K^{\mu\nu\cdots} + c K^{\mu\nu\cdots\lambda\kappa\cdots} p_\lambda p_\kappa \cdots\right) p_\mu p_\nu \cdots$, $c \in \mathbb{R}$, which would have to be zero, so $K^{\mu\nu\cdots} = -c K^{\mu\nu\cdots\lambda\kappa\cdots} p_\lambda p_\kappa$. But according to our initial definition of (2.20), $K^{\mu\nu\cdots}$ is independent of p's, and so we have a contradiction. This argument can be repeated with any number of terms, factoring out terms higher order in p's, and thus we have proven that the Killing tensor equation holds for all members of the family. $\square$

represents the spacetime transformation. Likewise we could do an infinitesimal variation

$$\delta\Lambda_\ell(x) \equiv \check{\delta}\Lambda_\ell\left(x+\hat{\delta}x^\mu\right) - \Lambda_\ell(x) \ , \tag{2.28}$$

where $\hat{\delta}x^\mu$ is given by (2.20), and $\check{\delta}\Lambda_\ell$ is an infinitesimal change of the components, which we could write like

$$\check{\delta}\Lambda_\ell(x) \equiv \epsilon\omega(x)_\ell^{\ \kappa}\Lambda_\kappa(x) \ , \tag{2.29}$$

where ϵ is infinitesimal and $\omega(x)_\ell^{\ \kappa}$ is a transformation of the components.

For the metric, which can be thought of as a classical field on its own, we would have that $\mathscr{T}\boldsymbol{g}(x) = \boldsymbol{g}'(x)$ carries some representation of the symmetry. Notice, that if the representation of the transformation is generated by a Killing vector, then we simply have $\mathscr{T}\boldsymbol{g} = \varphi^*\boldsymbol{g} = \boldsymbol{g}$ by definition of the Killing vector, and there is no variation in $\boldsymbol{g}$. Again we say that $\mathscr{T}$ is a symmetry if[9] $\mathscr{T}\mathcal{L} = \mathcal{L}$ and Noether's theorem will then generalize with essentially the same content and similar proof.

Theorem 11 (Noether's theorem for fields)**.** *Let a Lagrangian density $\mathcal{L} : T^p_q(M) \to \mathbb{R}$ have a continuous symmetry under a transformation given by* (2.27)*. Then there exists a corresponding conserved current given by*

$$J^\mu \equiv \frac{\partial\mathcal{L}}{\partial\left[\nabla_\mu\Lambda_\ell\right]}\left(\omega_\ell^{\ \ell'}\hat{\Lambda}_{\ell'} - \nabla_\nu\Lambda_\ell R^\nu\right) + \mathcal{L}R^\mu \,, \tag{2.30}$$

when Λ_ℓ obeys the equations of motion, and $\hat{\Lambda}_{\ell'}$ is the value of the fields at the transformed spacetime.

Proof. We can first split $\delta\Lambda_\ell(x) \equiv \check{\delta}\Lambda_\ell\left(x+\hat{\delta}x^\mu\right) - \Lambda_\ell(x)$ into two terms using definitions (2.28) and (2.29), an infinitesimal variation of the spacetime argument and an infinitesimal variation of the components by a expansion in ϵ to first order

$$\begin{aligned}
\delta\Lambda_\ell(x) &= \check{\delta}\Lambda_\ell\left(x+\hat{\delta}x^\mu\right) - \Lambda_\ell(x) \\
&= \Lambda_\ell(x) + \check{\delta}\Lambda_\ell(x) + \nabla_\mu\Lambda_\ell(x)\,\hat{\delta}x^\mu - \Lambda_\ell(x) + \mathcal{O}\left(\epsilon^2\right) \\
&\equiv \check{\delta}\Lambda_\ell(x) + \hat{\delta}\Lambda_\ell(x) \ ,
\end{aligned} \tag{2.31}$$

where $\check{\delta}$ is the purely internal variation, and $\hat{\delta}$ is the pure spacetime variation. A symmetry transformation must give us $\delta S = 0$ even when we are off-shell. We can do an expansion of this to first order in ϵ in the spacetime variation to do a rewriting of the integrations to write them as a single integral over the non-transformed coordinates:

[9] The theorem can actually easily be extended by allowing $\delta\mathcal{L}$ to be a total divergence.

$$
\begin{aligned}
0 &= \delta S \\
&= \int_{\mathscr{T}M} \mathcal{L}\left(\mathscr{T}\Lambda_\ell, \mathscr{T}\nabla\Lambda_\ell\right) \mathrm{d}^D \mathscr{T}x - \int_{\varphi(M)} \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \mathrm{d}^D x \\
&= \int_{\mathscr{T}M} \mathcal{L}\left(\Lambda_\ell + \delta\Lambda_\ell, \nabla\Lambda_\ell + \nabla\delta\Lambda_\ell\right) \mathrm{d}^D \mathscr{T}x - \int_{\varphi(M)} \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \mathrm{d}^D x \\
(*) &= \int_M \mathcal{L}\left(\Lambda_\ell + \check{\delta}\Lambda_\ell, \nabla\Lambda_\ell + \nabla\check{\delta}\Lambda_\ell\right) \det\left(\frac{\partial \mathscr{T}x}{\partial x}\right) \mathrm{d}^D x - \int_M \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \mathrm{d}^D x \\
&= \int_M \mathcal{L}\left(\Lambda_\ell + \check{\delta}\Lambda_\ell, \nabla\Lambda_\ell + \nabla\check{\delta}\Lambda_\ell\right) \left(1 - \nabla_\mu \hat{\delta}x^\mu\right) \mathrm{d}^D x - \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \mathrm{d}^D x + \mathcal{O}\left(\epsilon^2\right) \\
(**) &= \int_M \mathcal{L}\left(\Lambda_\ell + \check{\delta}\Lambda_\ell, \nabla\Lambda_\ell + \nabla\check{\delta}\Lambda_\ell\right) - \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) + \nabla_\mu \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \hat{\delta}x^\mu \mathrm{d}^D x + \mathcal{O}\left(\epsilon^2\right) \\
&= \int_M \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\Lambda_\ell\right]} \check{\delta}\Lambda_\ell + \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \nabla_\mu \check{\delta}\Lambda_\ell + \nabla_\mu \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \hat{\delta}x^\mu \mathrm{d}^D x + \mathcal{O}\left(\epsilon^2\right) ,
\end{aligned}
$$

where we in (*) did a change of variables from $\mathscr{T}x$ to x, which gives $\mathrm{d}^D \mathscr{T}x = \det\left(\frac{\partial \mathscr{T}x}{\partial x}\right) \mathrm{d}^D x = \left(1 - \nabla_\mu \hat{\delta}x^\mu + \mathcal{O}\left(\epsilon^2\right)\right) \mathrm{d}^D x$ in the first integral, so only the internal variations remains. In (**) we did an expansion and kept only $\mathcal{O}\left(\epsilon^2\right)$ terms, and a partial integration of the only surviving term from the determinant factor gave us the term $\nabla_\mu \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \hat{\delta}x^\mu$. In the last line did an expansion of $\mathcal{L}\left(\Lambda_\ell + \check{\delta}\Lambda_\ell, \nabla\Lambda_\ell + \check{\delta}\nabla\Lambda_\ell\right)$ to first order in the internal field variations only. We have $\check{\delta}\nabla_\mu\Lambda_\ell = \nabla_\mu\check{\delta}\Lambda_\ell$, exactly because they are evaluated at the same point (they act on different spaces), and using this we can do a further rewriting of the variation of the action. Notice that going on-shell using $\nabla_\mu \frac{\partial \mathcal{L}(\Lambda_\ell, \nabla\Lambda_\ell)}{\partial[\nabla_\mu\Lambda_\ell]} = \frac{\partial \mathcal{L}(\Lambda_\ell, \nabla\Lambda_\ell)}{\partial[\Lambda_\ell]}$ gives us

$$
\begin{aligned}
\nabla_\mu \left[\frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \check{\delta}\Lambda_\ell\right] &= \left(\nabla_\mu \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]}\right) \check{\delta}\Lambda_\ell + \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \left(\nabla_\mu \check{\delta}\Lambda_\ell\right) \\
&= \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\Lambda_\ell\right]} \check{\delta}\Lambda_\ell + \frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \left(\nabla_\mu \check{\delta}\Lambda_\ell\right) ,
\end{aligned}
$$

and we can exactly recognize the first two terms in our variation. We may then write

$$
\begin{aligned}
0 &= \int_M \nabla_\mu \left[\frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \check{\delta}\Lambda_\ell\right] + \nabla_\mu \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \hat{\delta}x^\mu \mathrm{d}^D x \\
&= \int_M \nabla_\mu \left[\frac{\partial \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right)}{\partial\left[\nabla_\mu\Lambda_\ell\right]} \check{\delta}\Lambda_\ell + \mathcal{L}\left(\Lambda_\ell, \nabla\Lambda_\ell\right) \hat{\delta}x^\mu\right] \mathrm{d}^D x .
\end{aligned}
$$

Now we notice that from (2.31) we have

$$\begin{aligned}
\check{\delta}\Lambda_{\ell}(x) &= \delta\Lambda_{\ell}(x) - \hat{\delta}\Lambda_{\ell}(x) \\
&= \check{\delta}\Lambda_{\ell}\left(x + \hat{\delta}x^{\mu}\right) - \nabla_{\mu}\Lambda_{\ell}\hat{\delta}x^{\mu} \\
&= \epsilon\left(\omega(x)_{\ell}^{\ \ell'}\Lambda_{\ell'}\left(x + \hat{\delta}x^{\mu}\right) - \nabla_{\mu}\Lambda_{\ell}R^{\mu}\right) , \\
&= \epsilon\left(\omega(x)_{\ell}^{\ \ell'}\hat{\Lambda}_{\ell'}(x) - R^{\nu}\nabla_{\mu}\Lambda_{\ell}\right)
\end{aligned}$$

where we also used that $\hat{\delta}x^{\mu} = \epsilon R^{\mu}$, and defined $\hat{\Lambda}_{\ell'} \equiv \Lambda_{\ell'}\left(x + \hat{\delta}x^{\mu}\right)$, which is the value of $\Lambda_{\ell'}$ at the variation spacetime point. Using this, we can factor out ϵ, and write

$$0 = \epsilon\int_M \nabla_{\mu}\left[\frac{\partial\mathcal{L}(\Lambda_{\ell}, \nabla\Lambda_{\ell})}{\partial[\nabla_{\mu}\Lambda_{\ell}]}\left(\omega_{\ell}^{\ \ell'}\hat{\Lambda}_{\ell'} - R^{\nu}\nabla_{\nu}\Lambda_{\ell}\right) + \mathcal{L}(\Lambda_{\ell}, \nabla\Lambda_{\ell})R^{\mu}\right]\mathrm{d}^D x\,.$$

We can then define a conserved quantity by the current

$$J^{\mu} \equiv \frac{\partial\mathcal{L}(\Lambda_{\ell}, \nabla\Lambda_{\ell})}{\partial[\nabla_{\mu}\Lambda_{\ell}]}\left(\omega_{\ell}^{\ \ell'}\hat{\Lambda}_{\ell'} - R^{\nu}\nabla_{\nu}\Lambda_{\ell}\right) + \mathcal{L}(\Lambda_{\ell}, \nabla\Lambda_{\ell})R^{\mu}\,, \tag{2.32}$$

that fulfills the covariant conservation law $\nabla_{\mu}J^{\mu} = 0$. □

A special case of great importance for our ambitions is of course the Einstein-Hilbert lagrangian for just the metric and some matter fields Ψ_I, given by

$$S[\boldsymbol{g}, \Psi_I] = \int \underbrace{\frac{1}{16\pi}R\sqrt{|g|}}_{\equiv\mathcal{L}_{\text{vac}}} + \mathcal{L}_{\text{mat}}\mathrm{d}^D x = S_{\text{vac}}[\boldsymbol{g}] + S_{\text{mat}}[\boldsymbol{g}, \Psi_I]\ , \tag{2.33}$$

where R is the Ricci scalar, that depends only on the metric $\boldsymbol{g}$ through the Christoffel symbols, and $\mathcal{L}_{\text{mat}}$ is the lagrangian density for interaction with matter, that might depend on other fields Ψ_I, for example gauge fields. Variation wrt. to the metric yields Einsteins equations for general relativity [7] with (Hilbert) energy-momentum tensor given by

$$T_{\mu\nu} \equiv -2\frac{1}{\sqrt{|g|}}\frac{\delta S_{\text{mat}}[\boldsymbol{g}, \Psi_I]}{\delta g^{\mu\nu}}\,. \tag{2.34}$$

The energy-momentum tensor is a strange beast in classical field theory [16]. We could also define a the so-called canonical energy-momentum tensor $T^{\mu\nu}_{\text{can.}}$ associated with spacetime translation invariant lagrangians, i.e. in local coordinates the full lagrangian is invariant under $x^{\mu} \to x^{\mu} + \epsilon a^{\mu}$, where a^{μ} is some constant. In this case we have $\omega_I^{\ I'} = 0$, no change of the tensor components of the matter fields Ψ_I, and (2.30) will give us

$$\begin{aligned}
J^{\mu} &= \frac{\partial\mathcal{L}}{\partial[\nabla_{\mu}\Psi_I]}a^{\nu}\nabla_{\nu}\Psi_I - a^{\mu}\mathcal{L} \\
&= \left[\frac{\partial\mathcal{L}}{\partial[\nabla_{\mu}\Psi_I]}\nabla^{\nu}\Psi_I - \mathcal{L}g^{\mu\nu}\right]a_{\nu} \\
&\equiv T^{\mu\nu}_{\text{can.}}a_{\nu}\,.
\end{aligned}$$

In general $T^{\mu\nu}_{\text{can.}}$ is not symmetric as required by the Einstein equations, and neither is it gauge invariant, which is required if Ψ_I are gauge fields. This is a bit awkward and can sometimes be fixed by adding extra terms to the current. One reason why it doesn't always hold, is because we don't always has full translation invariance, as is the case for the black hole spacetimes as we will look at later. A further discussion of the problem can be found in [16]. It is not really of concern for our future purposes, where we are either in the vacuum or have a cosmological constant.

As corollary benefit of this version of Noether's theorem, we see that if we just consider the spacetime symmetries, we will have exactly the same number of conserved quantities for fields as we had for particles. The symmetries that we have, must at least have the symmetries of the "free theory" with no matter fields, equivalent to the discussion that led to principle 1. The extra internal symmetries can give some extra conserved quantities, as for example if we have a non-abelian gauge theory defining the matter fields.

2.6 Classical integrability and separability

We may put the conservation of certain quantities into good use. Sometimes we will be able to prove that if there are enough conserved quantities, we may always in principle solve the geodesic equation and other classical equations of motion for particles and fields, like the curved space versions of the Klein-Gordon or Dirac equations. In this case we say that the equations of motion are integrable.

Of course, say we have N degrees of freedom in the equation under consideration, then it is clear that we would need at least N conserved quantities to solve them. It might not be that N is enough; they would all have to be "independent" and it might also be that in certain cases some of the conserved quantities are not useful. Sometimes we may not prove full integrability, but it might be that we could prove something less powerful such as separability. To put all of these aspects on more solid ground, we will have to develop a mathematical theory. Given some observable A and a Hamiltonian H, we have that the derivative of A wrt. the curve parameter τ, $\frac{\mathrm{d}A}{\mathrm{d}\tau} \equiv \dot{A}$, can be related to Poisson bracket $\{A, H\}$, when imposing the Hamiltonian equations of motion (B.14). Expanding this in the 1-form basis $\mathbf{d}\mathsf{x}^i$ of $T^*(\Gamma)$, we find

$$\begin{aligned} \dot{A} &= \frac{\partial A}{\partial \mathsf{x}^i}\frac{\mathrm{d}\mathsf{x}^i}{\mathrm{d}\tau} \\ &= \frac{\partial A}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \frac{\partial A}{\partial p_\nu}\frac{\mathrm{d}p_\nu}{\mathrm{d}\tau} \\ (*) \quad &= \frac{\partial A}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial A}{\partial p_\nu}\frac{\partial H}{\partial x^\mu} \\ &= \{A, H\} \, . \end{aligned} \tag{2.35}$$

If we have that A is a conserved quantity, $\dot{A} = 0$, then it simply commutes with the Hamiltonian, and the two statements are equivalent. We will use a slightly different language and say that the observable A commutes with H, then it is a first integral [12]. Because $A(x, p) = \text{constant}$ when x, p obeys the Hamilton equations, it defines a hyperplane in phase space Γ of dimension $2D-2$ (in some non-empty region of Γ at least). The solution of the EOMs must belong to this hyperplane, and thus effectively we have reduced the degrees of freedom of them by 2 (1 for position and 1 for momentum), so the equivalent Hamiltonian system is of dimension $2D-2$.

By the vague statement that two observables A, B should be "independent", we mean that they should be functionally independent:

Definition 12 (Functional independence)**.** Two observables A, B are functionally independent, if they are not related to each other by a constant factor, or equivalently, that their vertical derivatives are linearly independent along the EOMs.

Given two functionally independent observables A, B, then we would like to see which conditions they must obey for there to be 2 different first integrals, so we have a further restriction of the system. Using the properties of the Poisson bracket of theorem 37, especially the Jacobi identity, we can look at $\{\{A, B\}, H\}$:

$$0 = \{\{A, B\}, H\} + \{\{H, A\}, B\} + \{\{B, H\}, A\} \Rightarrow$$

$$\begin{aligned} \{\{A, B\}, H\} &= -\{\{B, H\}, A\} + \{\{A, H\}, B\} \\ &= \left\{A, \dot{B}\right\} + \left\{\dot{A}, B\right\} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\{A, B\} \\ &= 0\,, \end{aligned}$$

and thus if they are really functionally independent we don't have $\frac{\mathrm{d}}{\mathrm{d}\tau}\{A, B\} = 0$ in general. We can then conclude that we must have $\{A, B\} = 0$. By the discussion above we can characterize a system where we can find a unique solution, a completely integrable system, by the theorem below, a result called Lioville's theorem.

Theorem 13 (Lioville)**.** *We say that a Hamiltonian system is integrable if we have D functionally independent observables A_i that all Poisson commute, $\{A_i, A_j\} = 0$ for all i, j. If this is the case, we then say that the observables are in involution.*

For such a system there are $2D - 2D = 0$ degrees of freedom left, and we should be able to solve the equations of motion completely, at least in some region where there are no degeneracies between the observables. We always have that H itself is a first integral, and this corresponds to (classical) conservation of energy[10]. For the geodesic equation this translates to that the Hamiltonian $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ should Poisson commute with D functionally independent observables A_i that are in involution, for it to be integrable.

There is a deep connection between the Poisson brackets of conserved observables and Killing tensors that generates them. The following theorem holds:

Theorem 14 (SSN and Poisson brackets)**.** *We have $\{A, B\} = 0$ if and only if $[\boldsymbol{A}, \boldsymbol{B}] = 0$ for the corresponding Killing tensors.*

[10]As our system is defined on Γ, which is again is a structure defined on a general manifold M, it is not certain that H can really be interpreted as an energy function.

Proof. We first show that this holds for the restricted case $A = A^{\mu_1\cdots\mu_p}(x)\, p_{\mu_1}\cdots p_{\mu_p}$ and $B = B^{\nu_1\cdots\nu_q}(x)\, p_{\nu_1}\cdots p_{\nu_q}$. We do a direct calculation:

$$
\begin{aligned}
\{A,B\} &= \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B \\
&= \left(\nabla_\mu A^{\mu_1\cdots\mu_p}\right) p_{\mu_1}\cdots p_{\mu_p} B^{\nu_1\cdots\nu_q}\partial^\mu \left(p_{\nu_1}\cdots p_{\nu_q}\right) - A^{\mu_1\cdots\mu_p}\partial^\mu\left(p_{\mu_1}\cdots p_{\mu_p}\right)\left(\nabla_\mu B^{\nu_1\cdots\nu_q}\right) p_{\nu_1}\cdots p_{\nu_p} \\
(*) \quad &= \left[\sum_{i=1}^{q} \delta^\mu_{\nu_i} B^{\nu_i\mu_1\cdots\mu_{q-1}}\left(\nabla_\mu A^{\mu_q\cdots\mu_{p+q-1}}\right) - \sum_{j=1}^{p}\delta^\mu_{\mu_j} A^{\mu_j\mu_1\cdots\mu_{p-1}}\left(\nabla_\mu B^{\mu_p\cdots\mu_{p+q-1}}\right)\right]\left(p_{\mu_1}\cdots p_{\mu_{p+q-1}}\right) \\
(**) \quad &= \left[q B^{\mu(\mu_1\cdots\mu_{q-1}}\left(\nabla_\mu A^{\mu_q\cdots\mu_{p+q-1})}\right) - p A^{\mu(\mu_1\cdots\mu_{p-1}}\left(\nabla_\mu B^{\mu_p\cdots\mu_{p+q-1})}\right)\right]\left(p_{\mu_1}\cdots p_{\mu_{p+q-1}}\right) \\
&= -\left[A,B\right]^{\mu_1\cdots\mu_{p+q-1}}\left(p_{\mu_1}\cdots p_{\mu_{p+q-1}}\right)
\end{aligned}
$$

In (*) we rewrote the q and p terms resulting from the momenta derivatives relabeled the summation, and in (**) we used that the product of momenta is totally symmetric and then identified the SSN bracket. This shows for the restricted case that $\{A,B\} = 0 \Leftrightarrow [\boldsymbol{A},\boldsymbol{B}] = 0$. The general case of $A = \sum_{n=0}^{\infty}\frac{1}{n!}A^{\mu_1\cdots\mu_n}(x)\, p_{\mu_1}\cdots p_{\mu_n}$ and $B = \sum_{n=0}^{\infty}\frac{1}{n!}B^{\mu_1\cdots\mu_n}(x)\, p_{\mu_1}\cdots p_{\mu_n}$ then follows from linearity of the Poisson bracket. □

For field theories described by a lagrangian density, we should still think about conserved quantities of Noethers theorem as putting constraints on the equations of motion for the fields. However, as these equations are partial differential equations in contrast to the ordinary differential equations of the particle mechanics, and there might be internal indices as well, there are many more degrees of freedom. If we have a number of $d \times I$ internal degrees of freedom in D spacetime dimensions where the fields are rank p tensors, then there are a priori $\tilde{N} = dI + pD$ degrees of freedom at each spacetime point, but this number is clearly reduced by boundary conditions along with additional properties of the internal indices. In general, we do not have enough conserved quantities to constrain the equations enough so we would have the field theoretic analog of integrability of definition 13. We may sometimes use the symmetries to prove that certain equations of motions are separable in special coordinates.

One can develop a theory of separability structures, which helps putting these notions on a more rigorous ground [3]. A chart is said to be a r separability structure of the manifold if the Hamilton-Jacobi equation allows a additive separation of variables, where r of the coordinates are ignorable, i.e. we have r independent Killing vectors in some chart. One can prove the following result:

Theorem 15 (Separability)**.** *A manifold M of dimension D with metric $\boldsymbol{g}$ admits a r separability structure if and only if there exists r functionally independent Killing vectors $\bar{\boldsymbol{\epsilon}}^{(\bar{j})}$, $\bar{j} = 0,\ldots,r-1$ and $D-r$ functionally independent rank 2 Killing tensors $\boldsymbol{K}^{(i)}$, $j = 1,\ldots,D-r$ such that*

$$\left[\boldsymbol{K}^{(i)},\boldsymbol{K}^{(j)}\right] = 0\,, \tag{2.36}$$

$$\left[\boldsymbol{K}^{(i)},\bar{\boldsymbol{\epsilon}}^{(\bar{j})}\right] = 0\,, \tag{2.37}$$

$$\left[\overline{\boldsymbol{\epsilon}}^{(\bar{i})}, \overline{\boldsymbol{\epsilon}}^{(\bar{j})}\right] = 0\,, \tag{2.38}$$

all with respect to the symmetric Schouten-Nijenhuis bracket, and the Killing tensors $\boldsymbol{K}^{(i)}$ *has* $D - r$ *common eigenvectors* $\boldsymbol{x}^{(i)}$ *such that*

$$\left[\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}\right] = \left[\boldsymbol{x}^{(i)}, \overline{\boldsymbol{\epsilon}}^{(\bar{j})}\right] = 0\,, \tag{2.39}$$

$$\boldsymbol{x}^{(i)} \cdot \overline{\boldsymbol{\epsilon}}^{(\bar{j})} = 0\,. \tag{2.40}$$

The relation to field equations is a bit subtle, but the existence of a separability structure and theorem 15 implies that the Klein-Gordon equation is separable [29]. One can also show that a similar result for the Dirac equation holds [6]. We will find that the principal conformal Killing-Yano tensor dictates the existence of a tower of Killing vectors and a tower of Killing tensors that fulfills the requirement of theorem 15.

3 Killing-Yano tensors

3.1 Basic results

The motivation to construct the Killing tensors of before was that they gave us conserved quantities. We may introduce the Killing-Yano tensor (KYT) class, which can be thought of as the square root of a Killing tensor. They are even more fundamental than the Killing tensors, but their relation to symmetries is a bit obscure.

Definition 16 (Killing-Yano tensor)**.** A rank p KYT $\boldsymbol{f}$ is defined as a totally antisymmetric tensor $f_{\mu_1\cdots\mu_p} = f_{[\mu_1\cdots\mu_p]}$ that fulfills

$$\nabla_\mu f_{\mu_1\cdots\mu_p} = \nabla_{[\mu} f_{\mu_1\cdots\mu_p]}\,, \tag{3.1}$$

i.e. the covariant derivative acting on it becomes totally antisymmetric.

The KYTs have a number of nice properties that are allows us to do further developments. The reason why one can say that KYTs are the square root of Killing tensors, is because of their relation given by the following theorem:

Theorem 17 (Properties of KYTs)**. (1):** *For a KYT* $f_{\mu_1\cdots\mu_p}$*, we have that* $K_{\mu\nu} = f_{\mu\mu_2\cdots\mu_p} f_\nu{}^{\mu_2\cdots\mu_p}$ *is a Killing tensor of rank 2.* **(2):** $f_{\mu\mu_2\cdots\mu_p} p^{\mu_p}$ *is parallel transported along a geodesic with tangent vector* p^{μ_p}*.*

Proof. **(1):** It is easy to see that $K_{\mu\nu}$ as defined is symmetric, as we can just change the order of the contraction and interchange the lowered/rised indices. We then do a direct calculation of $\nabla_\rho K_{\mu\nu}$

$$\begin{aligned}
\nabla_\rho K_{\mu\nu} &= \nabla_\rho \left(f_{\mu\mu_2\cdots\mu_p} f_\nu{}^{\mu_2\cdots\mu_p}\right) \\
&= \left(\nabla_\rho f_{\mu\mu_2\cdots\mu_p}\right) f_\nu{}^{\mu_2\cdots\mu_p} + f_\mu{}^{\mu_2\cdots\mu_p}\left(\nabla_\rho f_{\nu\mu_2\cdots\mu_p}\right) \\
&= f_\nu{}^{\mu_2\cdots\mu_p} \nabla_{[\rho} f_{\mu\mu_2\cdots\mu_p]} + f_\mu{}^{\mu_2\cdots\mu_p} \nabla_{[\rho} f_{\nu\mu_2\cdots\mu_p]}\,.
\end{aligned}$$

Symmetrizing this (antisymmetrizing was done first) we find

$$\begin{aligned}\nabla_{(\rho}K_{\mu\nu)} &= f_{(\nu|}{}^{\mu_2\cdots\mu_p}\nabla_{[\rho}f_{\mu)\mu_2\cdots\mu_p]} + f_{(\mu|}{}^{\mu_2\cdots\mu_p}\nabla_{[\rho}f_{\nu)\mu_2\cdots\mu_p]}\\ &= 2f_{(\nu|}{}^{\mu_2\cdots\mu_p}\nabla_{[\rho}f_{\mu)\mu_2\cdots\mu_p]}\\ (\mu\leftrightarrow\rho)\quad &= -2f_{(\nu|}{}^{\mu_2\cdots\mu_p}\nabla_{[\mu}f_{\rho)\mu_2\cdots\mu_p]}\\ &= 0\,,\end{aligned}$$

The last conclusion follows because $\nabla_{(\rho}K_{\mu\nu)}$ is symmetric in μ,ρ, so we should not get a minus, but we do get one from the antisymmetrization that was done first, and thus it must vanish.

(2): We want to show that $\frac{\mathrm{D}}{\mathrm{d}\tau}f_{\mu\mu_2\cdots\mu_p}p^{\mu_p}=0$. To prove this, we do a direct calculation using the antisymmetry of the Killing-Yano tensor equation (3.2):

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}\tau}f_{\mu\mu_2\cdots\mu_p}p^{\mu_p} &= p^{\mu}\nabla_{\mu}f_{\mu\mu_2\cdots\mu_p}p^{\mu_p}\\ &= \nabla_{[\mu}f_{\mu\mu_2\cdots\mu_p]}p^{\mu_p}p^{\mu}\\ &= 0\,,\end{aligned}$$

where we in the last line used that $\nabla_{[\mu}f_{\mu\mu_2\cdots\mu_p]}$ is totally antisymmetric, while $p^{\mu_p}p^{\mu}$ is totally symmetric, and they must then vanish. □

The first part of the theorem is unfortunately not an if and only if theorem; it might very well be that we cannot "take the square root" of a Killing tensor and decompose it into a KYT.

We can again do a conformal transformation of them, and doing this, their definition becomes:

Definition 18 (Conformal Killing-Yano tensor)**.** A conformal Killing-Yano tensor (CKYT) of rank p is totally antisymmetric tensor $k_{\mu_1\cdots\mu_p}$, p-form, that fulfills

$$\nabla_{\mu}k_{\mu_1\cdots\mu_p}=\nabla_{[\mu}k_{\mu_1\cdots\mu_p]}+pg_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]}\,, \tag{3.2}$$

where $\overline{k}_{\mu_2\cdots\mu_p}$ is a antisymmetric tensor of rank $p-1$.

We find explicitly by doing a contraction of μ and μ_1 that[11]

$$\overline{k}_{\mu_2\cdots\mu_p}=\frac{1}{D-p+1}\nabla_{\mu}k^{\mu}{}_{\mu_2\cdots\mu_p}\,, \tag{3.3}$$

If $\overline{k}_{\mu_2\cdots\mu_p}$ vanishes, then we say that $k_{\mu_1\cdots\mu_p}=f_{\mu_1\cdots\mu_p}$ is a regular Killing-Yano tensor. For general rank CKYTs, we do not get that the contraction above gives a rank 2 CKT, but it is actually the case for a rank 2 CKYT [29], as we can prove:

[11] First we find that $g^{\mu\mu_1}\nabla_{\mu}k_{\mu_1\cdots\mu_p} = g^{\mu\mu_1}\nabla_{[\mu}k_{\mu_1\cdots\mu_p]} + pg^{\mu\mu_1}g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]} = pg^{\mu\mu_1}g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]}$. Now, using that $\overline{k}$ is antisymmetric, we can write $g^{\mu\mu_1}g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]} = \frac{1}{p!}g^{\mu\mu_1}\left(g_{\mu\mu_1}(p-1)!\overline{k}_{\mu_2\mu_3\cdots\mu_p} - g_{\mu\mu_2}(p-1)!\overline{k}_{\mu_1\mu_3\cdots\mu_p}\right) = \frac{1}{p}D\overline{k}_{\mu_2\cdots\mu_p} - \frac{1}{p}g^{\mu\mu_1}g_{\mu\mu_2}\overline{k}_{\mu_1\mu_3\cdots\mu_p} = \frac{1}{p}D\overline{k}_{\mu_2\cdots\mu_p} - \frac{1}{p}(p-1)\overline{k}_{\mu_2\cdots\mu_p} = \frac{1}{p}(D-p+1)\overline{k}_{\mu_2\cdots\mu_p}$, which then when inverted gives us (3.3).

Lemma 19 (CKTs and CKYTs). *Let $k_{\mu\nu}$ be a rank 2 CKYT. Then $K_{\mu\nu} \equiv k_{\mu\lambda}k_\nu^{\ \lambda}$ is a CKT.*

Proof. We verify directly that (2.9) is fulfilled by taking the covariant derivative and use (3.2)

$$\begin{aligned}\nabla_\rho K_{\mu\nu} &= \left(\nabla_\rho k_{\mu\lambda}\right) k_\nu^{\ \lambda} + k_\mu^{\ \lambda}\left(\nabla_\rho k_{\nu\lambda}\right)\\ &= \left(\nabla_{[\rho}k_{\mu\lambda]} + 2g_{\rho[\mu}\overline{k}_{\lambda]}\right) k_\nu^{\ \lambda} + k_\mu^{\ \lambda}\left(\nabla_{[\rho}k_{\nu\lambda]} + 2g_{\rho[\nu}\overline{k}_{\lambda]}\right) .\end{aligned}$$

Symmetrizing this, makes the first and third term vanish as they are antisymmetric in μ, ν.

$$\begin{aligned}\nabla_{(\rho}K_{\mu\nu)} &= 2\left(g_{(\rho[\mu}\overline{k}_{\lambda]}k_{\nu)}^{\ \lambda} + k_{(\mu}^{\ \lambda}g_{\rho[\nu}\overline{k}_{\lambda])}\right)\\ &= \left(g_{(\rho\mu}\overline{k}_{|\lambda|}k_{\nu)}^{\ \lambda} + g_{(\rho\nu}\overline{k}_{|\lambda|}k_{\mu)}^{\ \lambda}\right)\\ &= 2g_{(\rho\mu}\overline{k}_{|\lambda|}k_{\nu)}^{\ \lambda}\\ &\equiv 2g_{(\rho\mu}\overline{K}_{\nu)} ,\end{aligned}$$

where we in the last line identified $\overline{k}_\lambda k_\nu^{\ \lambda} = \overline{K}_\nu$ as the RHS of (2.9), which is seen to hold, because it has the correct form and a contraction of both sides would then give us $\overline{K}_\nu$. This proves the lemma. □

3.2 Antisymmetric Schouten-Nijenhuis bracket

We can define a bracket that works on totally antisymmetric tensor fields as well. This is given by the antisymmetric Schouten–Nijenhuis (ASN) bracket [11], which is defined (in component form) as

$$[X, Y\}^{\mu_1\cdots\mu_{p+q-1}} \equiv pX^{\nu[\mu_1\cdots\mu_{p-1}}\nabla_\nu Y^{\mu_p\cdots\mu_{p+q-1}]} + q\,(-1)^{pq}\,Y^{\nu[\mu_1\cdots\mu_{q-1}}\nabla_\nu X^{\mu_q\cdots\mu_{p+q-1}]} \qquad (3.4)$$

where both of the inputs are forms with all indices raised (multivectors) $X^{\mu_1\cdots\mu_p} = X^{[\mu_1\cdots\mu_p]}$ and $Y^{\mu_1\cdots\mu_q} = Y^{[\mu_1\cdots\mu_q]}$, and we have assumed that the connection is torsion free. If this is not the case, then we must use partial derivatives instead, the connection terms will cancel if it is torsion free.

The antisymmetric Schouten–Nijenhuis bracket defines a $\mathbb{Z}_2$-graded Lie algebra on the vector space of (anti)symmetric multivectors, because one can check that it satisfies the following graded Lie algebra axioms with $\boldsymbol{Z}$ beeing a rank r (anti)symmetric multivector

$$\begin{aligned}[\boldsymbol{X}, \boldsymbol{Y}\} &= (-1)^{pq}\,[\boldsymbol{Y}, \boldsymbol{X}]\\ [\boldsymbol{X}, \alpha\boldsymbol{Y} + \beta\boldsymbol{Z}\} &= \alpha\,[\boldsymbol{X}, \boldsymbol{Y}] + \beta\,[\boldsymbol{X}, \boldsymbol{Z}]\\ 0 &= (-1)^{p(r+1)}\,[[\boldsymbol{X}, \boldsymbol{Y}\}, \boldsymbol{Z}\} + (-1)^{q(p+1)}\,[[\boldsymbol{Y}, \boldsymbol{Z}\}, \boldsymbol{X}\} + (-1)^{r(q+1)}\,[[\boldsymbol{Z}, \boldsymbol{X}\}, \boldsymbol{Y}\} .\end{aligned} \qquad (3.5)$$

The vector space of antisymmetric multivectors is finite dimensional, because there can be no forms of rank larger than D, so the graded Lie algebra must be finite dimensional as well. We may show that the following product rule holds

$$[\boldsymbol{X}, \boldsymbol{Y} \otimes_- \boldsymbol{Z}\} = [\boldsymbol{X}, \boldsymbol{Y}\} \otimes_- \boldsymbol{Z} + (-1)^{q(p+1)} [\boldsymbol{X}, \boldsymbol{Z}\} \otimes_- \boldsymbol{Y} \tag{3.6}$$

where $\boldsymbol{Y} \otimes_- \boldsymbol{Z}$ is the antisymmetric part of the tensor product. We have that if $\boldsymbol{X}$ is a vector and $\boldsymbol{Y}$ is a multivector, then

$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}\} \ , \tag{3.7}$$

which one can check using the coordinate expression of the Lie derivative acting on a tensor. The Leibniz property of the Lie derivative acting on multivectors is a special case of this equation and (3.6), as we can rewrite the expression as

$$\mathcal{L}_{\boldsymbol{X}} (\boldsymbol{Y} \otimes_- \boldsymbol{Z}) \ = \ (\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y}) \otimes_- \boldsymbol{Z} + \boldsymbol{Y} \otimes_- (\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Z}) \ . \tag{3.8}$$

Let us now discuss what the ASN bracket implies on the set of all Killing-Yano tensors (with all indices raised). A good question would then be if they would form a graded Lie algebra, i.e. is the ASN bracket of two KYTs again a KYT. In general the answer is negative, as has been investigated in Kastor et al. [26]. However, we may prove a less general result:

Theorem 20 (ASN bracket of rank 1 and 2 KYTs)**.** *Let* $\boldsymbol{\xi}$ *be a Killing-Yano tensor of rank 1 (a Killing vector), and* $\boldsymbol{f}$ *a KYT of rank 2. Then* $\boldsymbol{q} \equiv [\boldsymbol{\xi}, \boldsymbol{f}\}$ *is a KYT of rank 2.*

Proof. In component form, we have $\boldsymbol{q} \equiv [\boldsymbol{\xi}, \boldsymbol{f}\}$ given by (3.4) with indices lowered

$$\begin{aligned} q_{\mu\nu} \ &= \ \xi^\lambda \nabla_\lambda f_{[\mu\nu]} + 2 f^\lambda{}_{[\mu} \nabla_{|\lambda|} \xi_{\nu]} \\ &= \ \xi^\lambda \nabla_\lambda f_{\mu\nu} + f^\lambda{}_\mu \nabla_\lambda \xi_\nu - f^\lambda{}_\nu \nabla_\lambda \xi_\mu \, , \end{aligned} \tag{3.9}$$

for which we want to show that $\nabla_\rho q_{\mu\nu} = \nabla_{[\rho} q_{\mu\nu]}$. Doing a direct calculation of $\nabla_\rho q_{\mu\nu}$ we may find using (3.1) and the relations of for the second order derivatives from appendix E, that we have

$$\nabla_\rho q_{\mu\nu} = -3 \nabla_\sigma \xi_{[\rho} \nabla^\sigma f_{\mu\nu]} + \frac{3}{2} \xi^\sigma R_{\sigma\lambda[\rho\mu} f_{\nu]}{}^\lambda \, . \tag{3.10}$$

This is manifestly antisymmetric in all free indices, so $\nabla_\rho q_{\mu\nu} = \nabla_{[\rho} q_{\mu\nu]}$, and thus it is a Killing-Yano tensor. This proves the theorem. □

In general, the Schouten–Nijenhuis bracket doesn't define graded Lie algebra of KYTs, because the bracket fails to close on something that is a KYT. Counter-examples of the closure can be found in [26], and includes important classes of spacetimes such as the general Kerr-NUT-(A)dS one. However, one can prove that for maximally symmetric spacetimes, they do form a graded Lie algebra [26].

3.3 Closed conformal Killing-Yano tensors

Next logical step is to go on and classify different CKYTs. We have three different classes that we can divide (3.2) into, that have different properties:

KYTs: Here we have $g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]} = 0$, which makes it a Killing-Yano tensor by definition.

Closed: Here we have $\nabla_{[\mu}k_{\mu_1\cdots\mu_p]} = 0$. This implies that $\boldsymbol{k} = \mathbf{d}\boldsymbol{b}$, where $\boldsymbol{b}$ is some $p-1$ form - this holds globally when the spacetime is simply connected and locally if the singularities are mild. Such closed CKYTs (CCKYTs) are very important for the theory we are about to build.

Both: In the case both terms vanishes, and we simply have $\nabla_\mu k_{\mu_1\cdots\mu_p} = 0$, which means that k is covariantly constant and is also both a CCKYT and a KYT.

It turns out that the Hodge duality transformation takes a CKYT that is not a KYT, into a Hodge dual that is a KYT [5]. Let us prove this:

Theorem 21 (KYTs and CCKYTs)**.** *The Hodge dual $\star\boldsymbol{k}$ of a rank p CCKYT $\boldsymbol{k}$ is a KYT $\boldsymbol{f} \equiv \star\boldsymbol{k}$ of rank $D-p$ and vice versa.*

Proof. Assume that $k_{\mu_1\cdots\mu_p}$ is a CCKYT. We do a direct calculation by taking the covariant derivative of $f_{\mu_{p+1}\cdots\mu_D} \equiv (\star k)_{\mu_{p+1}\cdots\mu_D} = \frac{1}{p!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}k_{\mu_1\cdots\mu_p}$ and simplify the expression by relating it to the CCKYT and its properties:

$$
\begin{aligned}
\nabla_\mu f_{\mu_{p+1}\cdots\mu_D} &= \nabla_\mu (\star\boldsymbol{k})_{\mu_{p+1}\cdots\mu_D} \\
&= \frac{1}{p!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\nabla_\mu k_{\mu_1\cdots\mu_p} \\
(*) \quad &= \frac{1}{p!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_p]} \\
&= \frac{1}{p!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\frac{1}{p}\left(g_{\mu\mu_1}\overline{k}_{\mu_2\cdots\mu_p} - (p-1)\, g_{\mu\mu_2}\overline{k}_{\mu_1\mu_3\cdots\mu_p}\right) \\
(**) \quad &= \frac{1}{p!p}\epsilon_\mu{}^{\mu_2\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\overline{k}_{\mu_2\cdots\mu_p} + \frac{(p-1)}{p!p}\epsilon_\mu{}^{\mu_1\mu_3\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\overline{k}_{\mu_1\mu_3\cdots\mu_p} \\
&= \frac{1}{p!}\epsilon_\mu{}^{\mu_2\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}\overline{k}_{\mu_2\cdots\mu_p}\,,
\end{aligned}
$$

where we in (*) used the definition of the CCKY, and then expanded the antisymmetrization. In (**) we interchanged $\mu_1 \leftrightarrow \mu_2$ to get a minus. In this form, we can see explicitly that $\nabla_\mu f_{\mu_{p+1}\cdots\mu_D} = \nabla_{[\mu}f_{\mu_{p+1}\cdots\mu_D]}$ because the Levi-Civita tensor is totally antisymmetric in its lower indices. Thus $\boldsymbol{f} \equiv \star\boldsymbol{k}$ is a KYT, and the converse follows from the bijective properties of the Hodge dual transformation and we have proven the theorem. □

One of the important properties of CCKYTs is that their wedge product is again a CCKY tensor of higher rank [28].

Theorem 22 (CCKYTs and wedge product)**.** *Let* $\boldsymbol{w}$ *be a CCKYT of rank* p *and* $\boldsymbol{v}$ *be a CCKYT of rank* q*. Then their wedge product* $\boldsymbol{k} \equiv \boldsymbol{w} \wedge \boldsymbol{v}$ *is also a CCKYT of rank* $p+q$*.*

Proof. We then need to show that $\boldsymbol{k}$ is a CKYT, i.e. that it obeys (3.2). Notice first that $\boldsymbol{k} \equiv \boldsymbol{w} \wedge \boldsymbol{v}$ is closed, because of the properties of the exterior derivative:

$$\begin{aligned} \mathbf{d}\boldsymbol{k} &= \mathbf{d}\boldsymbol{w} \wedge \boldsymbol{v} + (-1)^p\, \boldsymbol{w} \wedge \mathbf{d}\boldsymbol{v} \\ &= 0 + 0\,. \end{aligned} \tag{3.11}$$

We then only need to show that $\nabla_\mu k_{\mu_1\cdots\mu_{p+q}}$ has the right form. To do this, we use the product rule of the covariant derivative.

$$\begin{aligned} \nabla_\mu k_{\mu_1\cdots\mu_{p+q}} &= \nabla_\mu \left(\boldsymbol{w}\wedge\boldsymbol{v}\right)_{\mu_1\cdots\mu_{p+q}} \\ &= \nabla_\mu \left(\frac{(p+q)!}{p!q!} w_{[\mu_1\cdots\mu_p} v_{\mu_{p+1}\cdots\mu_{p+q}]}\right) \\ &= \frac{(p+q)!}{p!q!}\left[\left(\nabla_\mu w_{[\mu_1\cdots\mu_p}\right) v_{\mu_{p+1}\cdots\mu_{p+q}]} + w_{[\mu_1\cdots\mu_p}\left(\nabla_{|\mu|} v_{\mu_{p+1}\cdots\mu_{p+q}]}\right)\right] \\ (*) \quad &= \frac{(p+q)!}{p!q!}\left[p g_{\mu[\mu_1}\overline{w}_{\mu_2\cdots\mu_p} v_{\mu_{p+1}\cdots\mu_{p+q}]} + q w_{[\mu_1\cdots\mu_p} g_{\mu[\mu_{p+1}}\overline{v}_{\mu_{p+2}\cdots\mu_{p+q}}\right] \\ &\equiv (p+q)\, g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_{p+q}]}\,. \end{aligned}$$

In (*) we used the action of a covariant derivative on a CCKYT. (*) also shows that $\nabla_\mu k_{\mu_1\cdots\mu_{p+q}}$ has the correct form required by a CCKYT with a metric factor that is antisymmetrized. The expression we have found must therefore be equal to what we would find by a contraction, $(p+q)\, g_{\mu[\mu_1}\overline{k}_{\mu_2\cdots\mu_{p+q}]}$. This concludes the proof. □

3.4 The principal conformal Killing-Yano tensor

When it exists, even more fundamental is the principal conformal Killing-Yano tensor (PCKYT) $h_{\mu\nu}$, which is a special CCKYT of rank 2. We better give a proper definition:

Definition 23 (PCKYT)**.** $\boldsymbol{h}$ is called a PCKYT, if it is a CCKYT of rank 2, i.e. from (3.2) we have that it is antisymmetric and satisfies

$$\nabla_\rho h_{\mu\nu} = 2 g_{\rho[\mu}\xi_{\nu]} \quad , \quad \xi_\nu \equiv \frac{1}{D-1}\nabla_\rho h^\rho{}_\nu\,. \tag{3.12}$$

Further, $\boldsymbol{h}$ must be non-degenerate, in the meaning that it as a matrix have rank $2n$. $\boldsymbol{\xi}$ is called the primary vector.

The primary vector $\boldsymbol{\xi}$ is actually a Killing vector for the metric if we are in an Einstein space. To show this, we can do a small calculation on

$$\begin{aligned}\nabla_\mu \xi_\nu &= \frac{1}{D-1}\nabla_\mu \nabla_\rho h^\rho{}_\nu \\ &= \frac{1}{D-1} g^{\lambda\rho}\nabla_\mu \nabla_\lambda h_{\rho\nu}\,,\end{aligned}$$

where we used the definition of ξ_ν and factored out a metric. Using the results for the second order covariant derivative of the PCKYT given in appendix E, we find by inserting this and symmetrizing that

$$\begin{aligned}\nabla_{(\mu}\xi_{\nu)} &= \frac{3}{2}\frac{1}{D-1} g^{\lambda\rho}\left[R^\sigma{}_{(\mu|[\rho\lambda}h_{|\sigma||\nu)]}\right] \\ &= -\frac{3}{2}\frac{1}{D-1} R_{\sigma(\mu|}h^\sigma{}_{|\nu)},\end{aligned}$$

where we used that $R_{\sigma\nu\mu\lambda}$ is totally antisymmetric in the last three indices, and we could identify the Ricci tensor $R_{\sigma\mu} \equiv g^{\lambda\rho}R_{\rho\sigma\lambda\mu}$. Now using the Einstein space condition $R_{\sigma\mu} = \Lambda g_{\sigma\mu}$ we find

$$\begin{aligned}\nabla_{(\mu}\xi_{\nu)} &= -\frac{3\Lambda}{2}\frac{1}{D-1} g_{\sigma(\mu|}h^\sigma{}_{|\nu)} \\ &= -\frac{3\Lambda}{4}\frac{1}{D-1}\left(g_{\sigma\mu}h^\sigma{}_\nu + g_{\sigma\nu}h^\sigma{}_\mu\right) \\ &= -\frac{3\Lambda}{4}\frac{1}{D-1}\left(h_{\mu\nu} + h_{\nu\mu}\right) \\ &= 0\,, \end{aligned} \tag{3.13}$$

using the antisymmetry of $h_{\nu\mu}$ to make the final conclusion. Actually, this also holds "off-shell" without imposing the Einstein condition for the canonical metric, which we will show later. This is however more complicated to show and intimately related to the structure of the PCKYT and the kind of metrics that allows such one.

Again, since that $\boldsymbol{h}$ is closed, we have that there exists a "KY potential" 1-form $\boldsymbol{b}$ such that

$$\boldsymbol{h} = \mathbf{d}\boldsymbol{b}\,, \tag{3.14}$$

given the same assumptions as stated previously.

3.5 Killing-Yano towers

3.5.1 The tensor towers

By theorem 22, we can construct a many CCKY tensors from the PCKYT in particular by taking the wedge product of it with itself. This is known as the KY tensor tower - there will be a vector tower as well [28]. Let us define it properly:

Definition 24 (KY tensor tower)**.** We define the j'th CCKYT of the KY tensor tower as $\boldsymbol{h}^{(j)} \equiv \boldsymbol{h}^{\wedge j} = \bigwedge_{n=1}^{j} \boldsymbol{h}$.

There are a number of properties connected to the KY tensor tower, which we summarize and proof using mostly earlier obtained results in the following theorem:

Theorem 25 (Properties of KY tower)**. (1):** *$\boldsymbol{h}^{(j)}$ is a CCKY of rank $2j$.* **(2):** *For $\boldsymbol{b}^{(j)} \equiv \boldsymbol{b} \wedge \boldsymbol{h}^{(j-1)}$, we have $\boldsymbol{h}^{(j)} = \mathbf{d}\boldsymbol{b}^{(j)}$.* **(3)***:* $\boldsymbol{f}^{(j)} \equiv \star \boldsymbol{h}^{(j)}$ is a $D-2j$ form that is a KY tensor. **(4)***: To the j'th step of the tower, there is associated a step of the Killing tensor tower given by*

$$K^{(j)}_{\mu\nu} \equiv f^{(j)}_{\mu\mu_2\cdots\mu_{D-2j}} f^{(j)\ \mu_2\cdots\mu_{D-2j}}_{\nu} \,. \tag{3.15}$$

Proof. **(1):** This is simply a consequence of theorem 22.

(2): By the definition of the KY potential b, we see by direct calculation and use of the product law of the exterior derivative d that

$$\begin{aligned}
\mathbf{d}\boldsymbol{b}^{(j)} &= \boldsymbol{d}\left(\boldsymbol{b} \wedge \boldsymbol{h}^{(j-1)}\right) \\
&= (\mathbf{d}\boldsymbol{b}) \wedge h^{(j-1)} + (-1)^{2(j-1)}\, \boldsymbol{b} \wedge \left(\mathbf{d}h^{(j-1)}\right) \\
&= (\mathbf{d}\boldsymbol{b}) \wedge \boldsymbol{h}^{(j-1)} \\
&= \boldsymbol{h}^{(j)} \,,
\end{aligned}$$

where we used that $\mathbf{d}\boldsymbol{h}^{(j-1)} = 0$ since it is closed.

(3) and (4): This follows from theorem 17. □

Since we know that the PCKY is non-degenerate, we cannot have that any of the $\boldsymbol{h}^{(j)}$ steps in the tower vanishes, because the potential zero eigenvalue can at most give n zero columns/rows. The last step, the $\boldsymbol{h}^{(n)}$ is in even dimensions $D = 2n$ exactly proportional to the Levi-Civita tensor, the only possibility, while in odd dimensions $D = 2n+1$, we have that $\boldsymbol{h}^{(n)}$ is a $2n$ form that is dual to a Killing vector by theorem 25. $\boldsymbol{h}^{(n)}$ is not very interesting, and we will exclude it from our tower, because the Killing tensors that we will generate are either just constant or just a product of Killing vectors. We therefore take the allowed steps of the Killing tensor tower in any case to be $1 \leq j \leq n-1$. Thus we generate a total of $n-1$ Killing tensors that gives us conserved quantities. If we include the metric tensor (which is trivially a Killing tensor) as the $j = 0$ step, $\boldsymbol{K}^{(0)} \equiv \boldsymbol{g}$, then we have n Killing tensors of the (extended) Killing tensor tower $\boldsymbol{K}^{(i)}$, $0 \leq i \leq n-1$.

The explicit form of these Killing tensors can be obtained using the identities for contracted products of the Levi-Civita tensor in appendix E. We then find that (3.15) can be written as

$$
\begin{aligned}
K^{(j)\mu}{}_{\nu} &\equiv f^{(j)\mu\mu_2\cdots\mu_{D-2j}} f^{(j)}_{\nu\mu_2\cdots\mu_{D-2j}} \\
&= \frac{1}{(2j!)^2} \epsilon^{\nu_1\cdots\nu_{2j}\mu\mu_2\cdots\mu_{D-2j}} h^{(j)}_{\nu_1\cdots\nu_{2j}} \epsilon_{\nu'_1\cdots\nu'_{2j}\nu\mu_2\cdots\mu_{D-2j}} h^{(j)\nu'_1\cdots\nu'_{2j}} \\
(*) \quad &= \frac{(2j!)\,(D-2j)!}{(2j!)^2} \delta^{[\mu}_{\nu} \delta^{\nu_1}_{\nu'_1} \cdots \delta^{\nu_{2j}]}_{\nu'_{2j}} h^{(j)}_{\nu_1\cdots\nu_{2j}} h^{(j)\nu'_1\cdots\nu'_{2j}} \\
(**) \quad &= \frac{(2j!)\,(D-2j)!}{(2j!)^2} \delta^{[\mu}_{[\nu} \delta^{\nu_1}_{\nu'_1} \cdots \delta^{\nu_{2j}]}_{\nu'_{2j}]} h^{(j)}_{\nu_1\cdots\nu_{2j}} h^{(j)\nu'_1\cdots\nu'_{2j}} \\
&= \frac{(2j+1)!}{(2^j\,(j!))^2} \delta^{[\mu}_{[\nu} h^{\nu_1\nu'_1} \cdots h^{\nu_{2j}\nu'_{2j}]} h_{\nu_1\nu'_1} \cdots h_{\nu_{2j}\nu'_{2j}]} \\
(***) \quad &= \frac{(2j)!}{(2^j j!)^2} \left(\delta^{\mu}_{\nu} h^{[\nu_1\nu'_1} \cdots h^{\nu_{2j}\nu'_{2j}]} h_{[\nu_1\nu'_1} \cdots h_{\nu_{2j}\nu'_{2j}]} - 2j h^{\mu[\nu'_1} \cdots h^{\nu_{2j}\nu'_{2j}]} h_{\nu[\nu'_1} \cdots h_{\nu_{2j}\nu'_{2j}]} \right) \\
&\equiv A^{(j)} \delta^{\mu}_{\nu} - \tilde{K}^{(j)\mu}{}_{\nu} \,, \qquad (3.16)
\end{aligned}
$$

where we in (*) used (E.1), in (*) that the $\boldsymbol{h}^{(j)}$s are totally antisymmetric and products of hs for use in the following line. In (***) we use (E.2) to split up the expression. In the last line we have defined the convenient quantities

$$
A^{(j)} \equiv \frac{(2j)!}{(2^j j!)^2} h^{[\nu_1\nu'_1} \cdots h^{\nu_{2j}\nu'_{2j}]} h_{[\nu_1\nu'_1} \cdots h_{\nu_{2j}\nu'_{2j}]} \,, \tag{3.17}
$$

$$
\tilde{K}^{(j)\mu}{}_{\nu} = \frac{2j\,(2j)!}{(2^j j!)^2} h^{\mu[\nu'_1} \cdots h^{\nu_{2j}\nu'_{2j}]} h_{\nu[\nu'_1} \cdots h_{\nu_{2j}\nu'_{2j}]} \,. \tag{3.18}
$$

At the present stage, it is not clear whether all of the corresponding conserved quantities are functionally independent. We will eventually show that they in fact are in section 4.8, but have have a lot of work to do before we can make the conclusion.

3.5.2 The vector tower

We had a whole tower of tensors that gave rise to n conserved charges. We also find that there are $n+\varepsilon$ (Killing) vectors, that gives a “Killing vector tower” with steps $\bar{\epsilon}^{(\bar{j})}_{\mu}$, $0 \leq \bar{j} \leq n-1+\varepsilon$. The first step of the tower is the primary (Killing) vector (3.12) of the PCKY[12],

$$
\bar{\epsilon}^{(0)}_{\mu} \equiv \xi_{\mu} = K^{(j)}_{\mu\nu} \xi^{\nu} \,. \tag{3.19}
$$

The other steps of the tower is generated by contraction of ξ^{μ} with the remaining steps of the Killing tensor tower, which gives us $n-1$ steps from contraction of the primary (Killing) vector with tensors from the non-extended Killing tower (3.15):

[12] We must stress the fact that at present we have only shown that it is a Killing vector when the Einstein equations are imposed. We are going to show that it is actually a Killing vector for the canonical metric without the Einstein equations imposed.

$$\bar{\epsilon}_\mu^{(\bar{j})} \equiv K_{\mu\nu}^{(\bar{j})} \xi^\nu \,. \tag{3.20}$$

In even dimensions, there are no more steps, and there are now in total $D = 2n$ steps from the two towers. For odd dimensions, there is an extra step, exactly the one Killing vector from the Hodge dual of $h^{(n)}$:

$$\bar{\epsilon}_\mu^{(\bar{n})} \equiv f_\mu^{(n)} = (\star \boldsymbol{h})_\mu^{(n)} \,. \tag{3.21}$$

So we have for odd dimensions that there are also $D = 2n + \varepsilon$ steps from the two towers combined. Again, at the present stage it is not clear that they are independent, and thus yield different conserved quantities. We show this in section 4.8.

4 The canonical metric

We now restrict ourselves to spacetimes of euclidean signatures. The cases with indefinite signature, especially $(- + \ldots +)$ are obtained from the results below by a Wick rotation, which we will discuss later. This is mainly because of the computational advantages in this form, but it has no influence on the existence and uniqueness of the PCKYT, as the form of it doesn't change by this procedure.

4.1 Darboux basis

To study the PCKYT more closely, it is useful to see if we can bring a given PCKYT $\boldsymbol{h}$ to a simpler form by a change of basis (to a non-coordinate one). Taking

$$\overleftrightarrow{\mathbf{h}} \equiv h^\mu{}_\nu = g^{\mu\lambda} h_{\lambda\nu} \tag{4.1}$$

as a normal matrix of dimensions $D \times D$, an operator on the tangent space $T(M)$, which has real vector space structure. We can also endow it with an inner product "$\cdot$" by using the metric $\boldsymbol{g}$, so we define

$$\boldsymbol{X} \cdot \boldsymbol{Y} \equiv X_\mu Y^\mu = g_{\mu\nu} X^\mu Y^\nu \quad , \quad \boldsymbol{X}, \boldsymbol{Y} \in T(M) \,. \tag{4.2}$$

On the tangent space $\overleftrightarrow{\mathbf{h}}$ is then an antisymmetric operator, as we have

$$\begin{aligned} \boldsymbol{X} \cdot \left(\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{Y} \right) &= g_{\mu\nu} X^\mu \left(g^{\nu\lambda} h_{\lambda\rho} Y^\rho \right) \\ &= \delta_\mu^\lambda X^\mu h_{\lambda\rho} Y^\rho \\ &= -Y^\rho h_{\rho\lambda} X^\lambda \\ &= -\boldsymbol{Y} \cdot \left(\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{X} \right) \,. \end{aligned} \tag{4.3}$$

The associated CKT $\boldsymbol{H}$ from lemma 19 with one index raised defines another operator on $T(M)$ which we can write as

$$\overleftrightarrow{\mathbf{H}} \equiv H^{\mu}{}_{\nu} \equiv -h^{\mu\lambda}h_{\lambda\nu} = -\overleftrightarrow{\mathbf{h}} \cdot \overleftrightarrow{\mathbf{h}} \,. \tag{4.4}$$

Now thinking of $\overleftrightarrow{\mathbf{H}}$ as an operator on $T(M)$, this is clearly a symmetric operator, and it is also positive definite as

$$\boldsymbol{X} \cdot \left(\overleftrightarrow{\mathbf{H}} \cdot \boldsymbol{Y}\right) = -\boldsymbol{X} \cdot \left(\overleftrightarrow{\mathbf{h}} \cdot \overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{Y}\right) = \left(\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{X}\right) \cdot \left(\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{Y}\right) \geq 0 \,.$$

Since it is symmetric, it may be diagonalized by the spectral theorem of linear algebra, and we then know that there exists an orthonormal basis where it is diagonal. Say that we are in this orthonormal basis, which will be a non-coordinate (vielbein) basis in general as considered in appendix D, and consider the generic eigenvalue problem

$$\overleftrightarrow{\mathbf{H}} \cdot \boldsymbol{n} = A\boldsymbol{n} \,,$$

where we have normalized the vector $\boldsymbol{n}$ such that

$$\boldsymbol{n} \cdot \boldsymbol{n} = g_{\mu\nu} n^{\mu} n^{\nu} = +1 \,.$$

We can then prove the following lemma:

Lemma 26 (Conjugate eigenvectors properties)**.** *If $\boldsymbol{n}$ is normalized and an eigenvector of $\overleftrightarrow{\mathbf{H}}$, with eigenvalue $A \neq 0$, then we define the conjugate vector as*

$$\overline{\boldsymbol{n}} \equiv \frac{1}{\sqrt{|A|}} \overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{n} \,, \tag{4.5}$$

and this is also an eigenvector of $\overleftrightarrow{\mathbf{H}}$ with same eigenvalue. Further we have that that they are orthogonal, i.e. $\boldsymbol{n} \cdot \overline{\boldsymbol{n}} = 0$ and that $\overline{\boldsymbol{n}} \cdot \overline{\boldsymbol{n}} = 1$. If $A = 0$, then we set $\boldsymbol{n} = \overline{\boldsymbol{n}}$ and say that $\boldsymbol{n}$ is self-conjugate.

Proof. We do a direct calculation of $H^{\mu}{}_{\nu}\overline{n}^{\nu}$ using $H^{\mu}{}_{\nu} \equiv -h^{\mu\lambda}h_{\lambda\nu} = -h^{\mu}{}_{\lambda}h^{\lambda}{}_{\nu}$:

$$\begin{aligned}
H^{\mu}{}_{\nu}\overline{n}^{\nu} &= \left(-h^{\mu}{}_{\rho}h^{\rho}{}_{\nu}\right)\left(\frac{1}{\sqrt{|A|}}h^{\nu}{}_{\lambda}n^{\lambda}\right) \\
&= \frac{1}{\sqrt{|A|}}h^{\mu}{}_{\rho}\left(-h^{\rho}{}_{\nu}h^{\nu}{}_{\lambda}\right)n^{\lambda} \\
&= \frac{1}{\sqrt{|A|}}h^{\mu}{}_{\rho}\left(H^{\rho}{}_{\lambda}n^{\lambda}\right) \\
&= A\left(\frac{1}{\sqrt{|A|}}h^{\mu}{}_{\rho}n^{\rho}\right) \\
&= A\overline{n}^{\mu} \,,
\end{aligned}$$

as claimed. To show that the conjugate and regular eigenvectors are orthogonal, we do a small calculation using that $h_{\mu\lambda}$ is antisymmetric

$$
\begin{aligned}
g_{\mu\nu} n^{\mu} \overline{n}^{\nu} &= \frac{1}{\sqrt{|A|}} g_{\mu\nu} h^{\nu}{}_{\lambda} n^{\mu} n^{\lambda} \\
&= \frac{1}{\sqrt{|A|}} h_{\mu\lambda} n^{\mu} n^{\lambda} \\
&= 0\,,
\end{aligned}
$$

because the product $n^{\mu} n^{\lambda}$ is symmetric. To show the normalization of the conjugate eigenvectors, we use (4.5)

$$
\begin{aligned}
g_{\mu\nu} \overline{n}^{\mu} \overline{n}^{\nu} &= \frac{1}{|A|} g_{\mu\nu} h^{\mu}{}_{\rho} n^{\rho} h^{\nu}{}_{\lambda} n^{\lambda} \\
&= \frac{1}{|A|} h_{\nu\rho} h^{\nu}{}_{\lambda} n^{\rho} n^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|A|} H_{\rho\lambda} n^{\rho} n^{\lambda} \\
&= \frac{A}{|A|} n_{\lambda} n^{\lambda} \\
&= \operatorname{sgn}(A)\, n_{\lambda} n^{\lambda} \\
&= 1\,,
\end{aligned}
$$

where $\operatorname{sgn}(A) = 1$ because $\overleftrightarrow{\mathbf{H}}$ is positive definite[13]. This finishes the proof. □

The conclusion to draw from lemma 26 is that each eigenvalue has a multiplicity of at least two if non-zero, and thus the dimension of the corresponding eigenspace is at least 2. As the eigenvalues A_i are positive, we can write them as $A_i = x_i^2$. Now, as $\overleftrightarrow{\mathbf{h}}$ is an antisymmetric operator, to get $\overleftrightarrow{\mathbf{H}} = -\overleftrightarrow{\mathbf{h}} \cdot \overleftrightarrow{\mathbf{h}}$ diagonal in this basis, we must have that it is of the form

$$
\overleftrightarrow{\mathbf{h}} = \begin{pmatrix}
\ddots & & & & & \iddots \\
& 0 & x_i & & & \\
& -x_i & 0 & & & \\
& & & \ddots & & \\
& & & & 0 & \\
\iddots & & & & & \ddots
\end{pmatrix}. \tag{4.6}
$$

[13] This argument obviously doesn't hold for Lorentzian signatures. For such metrics we have that one eigenvalue will be negative, and give us a time-like vector instead of all spacelike.

Because of the rank of $\boldsymbol{h}$ assumed to be $2n$, there is at most one eigenvalue $A_i = 0$, which can then only be present in odd dimensions, and the remaining n x_is must be functionally independent, and $\boldsymbol{h}$ cannot be covariantly constant. Thus the number of 2×2 submatrices is n corresponding to $A_i = x_i^2$ with eigenvectors $\boldsymbol{n}_i$, $\overline{\boldsymbol{n}}_i$, and in odd dimensions we have additionally a 1×1 submatrix corresponding to the $A_i = 0$ eigenvalue with eigenvector $\boldsymbol{n}_0 = \overline{\boldsymbol{n}}_0$. The eigenvalues of $\overleftrightarrow{\mathbf{h}}$ is then be seen to be $\pm i x_i$.

From the spectral theorem we also know that eigenvectors of $\overleftrightarrow{\mathbf{H}}$ corresponding to different eigenvalues are orthogonal, and thus that the set of $D = 2n + \varepsilon$ vectors $\{\boldsymbol{n}_i, \overline{\boldsymbol{n}}_i\}$. In this notation, we may summarize the results of lemma 26 as

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \delta_{ij} \quad , \quad \overline{\boldsymbol{n}}_i \cdot \overline{\boldsymbol{n}}_j = \delta_{ij} \quad , \quad \boldsymbol{n}_i \cdot \overline{\boldsymbol{n}}_j = 0 \quad , \quad \overleftrightarrow{\mathbf{H}} \cdot \boldsymbol{n}_i = A_i \boldsymbol{n}_i \quad , \quad \overleftrightarrow{\mathbf{H}} \cdot \overline{\boldsymbol{n}}_i = A_i \overline{\boldsymbol{n}}_i \tag{4.7}$$

This is called the Darboux basis, and we define a ordering of the vector $\boldsymbol{n}_i$ and its conjugate $\overline{\boldsymbol{n}}_i$ in the orthonormal basis, such that we have

$$\boldsymbol{n}_i = \begin{pmatrix} \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad , \quad \overline{\boldsymbol{n}}_i = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ \vdots \end{pmatrix} . \tag{4.8}$$

We can also collect the eigenvectors into a single set of vectors $\hat{\boldsymbol{n}}_i = \hat{n}^\mu{}_i \partial_\mu$, defined as $\hat{\boldsymbol{n}}_0 = \boldsymbol{n}_0$ (in odd dimensions only), $\hat{\boldsymbol{n}}_{2i-1} = \boldsymbol{n}_i$, $\hat{\boldsymbol{n}}_{2i} = \overline{\boldsymbol{n}}_i$. With this notation we have $\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j = \delta_{ij}$, i.e. the components of the metric is diagonal. We can define a dual basis of covectors $\hat{\boldsymbol{n}}^i = \hat{n}_\mu{}^i$ as usual by requiring $\hat{\boldsymbol{n}}^i(\hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{n}}^i \cdot \hat{\boldsymbol{n}}_j = \delta^i_j$ [35]. The advantage of this is that the metric has been diagonalized. Writing the coordinate basis indices explicitly we have from (4.7) that $\delta_{ij} = \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j = g_{\mu\nu} \hat{n}^\mu{}_i \hat{n}^\nu{}_j$. A contraction with two covectors on the RHS on this gives $g_{\lambda\rho} \hat{n}_\mu{}^i \left(\hat{n}^\lambda{}_i\right) \hat{n}_\nu{}^j \left(\hat{n}^\rho{}_j\right) = g_{\lambda\rho} \delta^\lambda_\mu \delta^\rho_\nu = g_{\mu\nu}$, and on the LHS we simply have $\delta_{ij} \hat{n}_\mu{}^i \hat{n}_\nu{}^j$. Equating the two sides yields

$$g_{\mu\nu} = \delta_{ij} \hat{n}_\mu{}^i \hat{n}_\nu{}^j , \tag{4.9}$$

and without the coordinate indices by contracting with the coordinate one-form basis:

$$\boldsymbol{g} = \delta_{ij} \hat{\boldsymbol{n}}^i \hat{\boldsymbol{n}}^j \tag{4.10}$$

In the same way we could derive that the inverse metric is

$$\boldsymbol{g}^{-1} = \delta^{ij} \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j . \tag{4.11}$$

Now we turn to the PCKYT $\boldsymbol{h}$ and see how this looks in the Darboux basis. If we now act with (4.9) on $\overleftrightarrow{\mathbf{h}}$ we obtain the PCKYT 2-form $\boldsymbol{h}$. We find using how the covectors $\hat{\boldsymbol{n}}^i$ (which can be thought of as row-vectors) works on $\overleftrightarrow{\mathbf{h}}$ the last desired result of this section:

$$\begin{aligned}
h_{\mu\nu} &= g_{\mu\lambda} h^{\lambda}{}_{\nu} \\
&= \delta_{ij} \hat{n}_{\mu}{}^{i} \hat{n}_{\lambda}{}^{j} h^{\lambda}{}_{\nu} \\
&= \sum_{i=1}^{n} n_{\mu}{}^{i} \left(n_{\lambda}{}^{i} h^{\lambda}{}_{\nu} \right) + \overline{n}_{\mu}{}^{i} \left(\overline{n}_{\lambda}{}^{i} h^{\lambda}{}_{\nu} \right) + \varepsilon n_{\mu}{}^{0} \left(n_{\lambda}{}^{0} h^{\lambda}{}_{\nu} \right) \\
&= \sum_{i=1}^{n} n_{\mu}{}^{i} \left(x_i \overline{n}_{\nu}{}^{i} \right) + \overline{n}_{\mu}{}^{i} \left(-x_i n_{\nu}{}^{i} \right) + \varepsilon n_{\mu}{}^{0} \left(0 \right) \\
&= \sum_{i=1}^{n} x_i n_{\mu}{}^{i} \wedge \overline{n}_{\nu}{}^{i} \,. \qquad (4.12)
\end{aligned}$$

In coordinate free notation we have

$$\boldsymbol{h} = \sum_{i=1}^{n} x_i \boldsymbol{n}^i \wedge \overline{\boldsymbol{n}}^i \,. \qquad (4.13)$$

4.2 The Killing-Yano towers in Darboux basis

In the Darboux basis, the steps of the Killing tensor and vector towers have very simple expressions because of the simple structure [29]. We find by some simple combinatorics that we can express $\boldsymbol{h}^{(j)}$ of definition 24 as

$$\boldsymbol{h}^{(j)} = j! \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j} \boldsymbol{n}^{i_1} \wedge \overline{\boldsymbol{n}}^{i_1} \wedge \cdots \wedge \boldsymbol{n}^{i_k} \wedge \overline{\boldsymbol{n}}^{i_j} \,, \qquad (4.14)$$

because of the antisymmetric properties of the wedge product. One can also see by using the above that the scalar function $A^{(j)}$ given by (3.17) can be expressed as

$$A^{(j)} = \sum_{i_1 < \cdots < i_j} x_{i_1}^2 \cdots x_{i_j}^2 \,, \qquad (4.15)$$

when one does the combinatorics of the antisymmetric product. If we define

$$A_i^{(j)} \equiv \sum_{\substack{i_1 < \cdots < i_j \\ i \neq i_i}} x_{i_1}^2 \cdots x_{i_j}^2 \,, \qquad (4.16)$$

we can lower indices and write $\tilde{\boldsymbol{K}}^{(k)}$ defined in (3.18) as

$$\tilde{\boldsymbol{K}}^{(j)} = \sum_{i=1}^{n} x_i^2 A_i^{(j-1)} \left(\boldsymbol{n}^i \otimes \boldsymbol{n}^i + \overline{\boldsymbol{n}}^i \otimes \overline{\boldsymbol{n}}^i \right) \,. \qquad (4.17)$$

Notice that from the equations (4.15) and (4.16), we have the relation

$$A^{(j)} = A_i^{(j)} + x_i^2 A_i^{(j-1)} \,, \qquad (4.18)$$

because $x_i^2 A_i^{(j-1)}$ is the exactly the terms that are excluded in the summation of $A_i^{(j)}$. This allows us to explicitly write down the corresponding n steps of the Killing tensor tower (3.16) for $j = 0, \ldots, n-1$ in a simple form as

$$\boldsymbol{K}^{(j)} = \sum_{i=1}^{n} A_i^{(j)} \left(\boldsymbol{n}^i \otimes \boldsymbol{n}^i + \overline{\boldsymbol{n}}^i \otimes \overline{\boldsymbol{n}}^i \right) + \varepsilon A^{(j)} \boldsymbol{n}^0 \otimes \boldsymbol{n}^0 \,, \tag{4.19}$$

where the last term is only present in odd dimensions. In this form, the Killing tensors of the tower are all simultaneously diagonalized and thus have common eigenvectors. We may obtain a more explicit formula for the $n+\varepsilon$ steps of the Killing vector tower $\overline{\boldsymbol{\epsilon}}^{(\overline{j})}$for $\overline{j} = 0, \ldots, n-1+\varepsilon$, which in this notation are covectors, given by equations (3.19)-(3.21) using the above results if we like. For the n first steps $\overline{j} = 0, \ldots, n-1$ we have

$$\overline{\boldsymbol{\epsilon}}^{(\overline{j})} = \boldsymbol{\xi} \cdot \boldsymbol{K}^{(\overline{j})} \,, \tag{4.20}$$

while the last Killing vector[14] present in odd dimensions $\overline{\boldsymbol{\epsilon}}^{(\overline{n})}$ is given by (3.21).

4.3 Eigenvectors of $\overleftrightarrow{\mathbf{h}}$

We want to build the most general metric from $\overleftrightarrow{\mathbf{H}}$ and $\overleftrightarrow{\mathbf{h}}$ using their eigenvectors, eigenvalues, and other objects. Notice first that they did specify the metric $\boldsymbol{g}$ as well as the PCKYT $\boldsymbol{h}$ in a particular simple form. We can therefore reverse the construction and start out with a PCKYT $\boldsymbol{h}$ and the Darboux basis determined by it, along with the metric $\boldsymbol{g}$, given by equations (4.9) and (4.12). The result is then naturally going to be the most general metric with a PCKYT, where we will soon determine a proper coordinate basis so the structure of $\boldsymbol{g}$, $\boldsymbol{h}$ are more clear.

The eigenvalues of $\overleftrightarrow{\mathbf{h}}$ are $\pm i x_i$, and we can easily define the correspondingly (complex) eigenvectors as a linear combination of the Darboux basis vectors

$$\boldsymbol{m}_i \equiv \frac{1}{\sqrt{2}} \left(\overline{\boldsymbol{n}}_i + i \boldsymbol{n}_i \right) \quad , \quad \overline{\boldsymbol{m}}_i \equiv \left(\boldsymbol{m}_i \right)^{\dagger} = \frac{1}{\sqrt{2}} \left(\overline{\boldsymbol{n}}_i - i \boldsymbol{n}_i \right) \,. \tag{4.21}$$

This we will prove shortly. They are still eigenvectors of $\overleftrightarrow{\mathbf{H}}$, as they are just a linear combination of eigenvectors of the same eigenspace with eigenvalue $A_i = x_i^2$. The complex eigenvectors of $\overleftrightarrow{\mathbf{h}}$ complexifies the geometry if we want to use them as a basis instead (for odd dimensions $\boldsymbol{m}_0 \equiv \overline{\boldsymbol{m}}_0 \equiv \hat{\boldsymbol{n}}_0$, which is not null but spacelike) - which we will because then $\overleftrightarrow{\mathbf{H}}$, $\overleftrightarrow{\mathbf{h}}$ are simultaneously diagonalized, while $\boldsymbol{g}$ is no longer diagonal. This is just a convenient trick, because we use them as independent variables, but related by complex conjugation, so the manifold is still real, as there are no more degrees of freedom in this basis.

Lemma 27 (Properties of $\overline{\boldsymbol{m}}_i$, $\boldsymbol{m}_i$)**.** **(1)**: *We have* $\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{m}_i = -i x_i \boldsymbol{m}_i$ *and* $\overleftrightarrow{\mathbf{h}} \cdot \overline{\boldsymbol{m}}_i = +i x_i \overline{\boldsymbol{m}}_i$. **(2)**: *They are complex null vectors, i.e.* $\boldsymbol{m}_i \cdot \boldsymbol{m}_j = \overline{\boldsymbol{m}}_i \cdot \overline{\boldsymbol{m}}_j = 0$ *and* $\boldsymbol{m}_i \cdot \overline{\boldsymbol{m}}_j = \delta_{ij}$ *for even dimensions, and for odd dimensions we have additionally* $\overline{\boldsymbol{m}}_0 \cdot \overline{\boldsymbol{m}}_0 = \boldsymbol{m}_0 \cdot \boldsymbol{m}_0 = \overline{\boldsymbol{m}}_0 \cdot \boldsymbol{m}_0 = +1$. **(3)**: *For the dual vectors we have* $\delta^\mu_\nu = \overline{m}^\mu{}_i m_\nu{}^i + m^\mu{}_i \overline{m}_\nu{}^i$.

Proof. **(1):** We do a direct calculation using the properties of the $2n$ Darboux basis vectors and $\boldsymbol{m}_i \equiv \frac{1}{\sqrt{2}} \left(\overline{\boldsymbol{n}}_i + i \boldsymbol{n}_i \right)$ in the eigenspace corresponding to the x_i^2 eigenvalue of $\overleftrightarrow{\mathbf{H}}$:

[14]Yet again we must stress that we have not yet proven that they are Killing vectors.

$$\begin{aligned}
\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{m}_i &= \frac{1}{\sqrt{2}} \overleftrightarrow{\mathbf{h}} \cdot (\overline{\boldsymbol{n}}_i + i\boldsymbol{n}_i) \\
&= \frac{1}{\sqrt{2}} \left(\overleftrightarrow{\mathbf{h}} \cdot \overline{\boldsymbol{n}}_i + i \overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{n}_i \right) \\
&= \frac{1}{\sqrt{2}} (+x_i \boldsymbol{n}_i - i x_i \overline{\boldsymbol{n}}_i) \\
&= -i x_i \frac{1}{\sqrt{2}} (i\boldsymbol{n}_i + \overline{\boldsymbol{n}}_i) \\
&= -i x_i \boldsymbol{m}_i \, .
\end{aligned} \tag{4.22}$$

We have that $\overleftrightarrow{\mathbf{h}} \cdot \overline{\boldsymbol{m}}_i = +i x_i \overline{\boldsymbol{m}}_i$ follows from complex conjugation of this result.
(2): This we prove using the properties of $\boldsymbol{n}_i$, $\overline{\boldsymbol{n}}_j$ as given by (4.7).

$$\begin{aligned}
2\boldsymbol{m}_i \cdot \boldsymbol{m}_j &= (\overline{\boldsymbol{n}}_i + i\boldsymbol{n}_i) \cdot (\overline{\boldsymbol{n}}_j + i\boldsymbol{n}_j) \\
&= \overline{\boldsymbol{n}}_i \cdot \overline{\boldsymbol{n}}_j - \boldsymbol{n}_i \cdot \boldsymbol{n}_j \\
&= \delta_{ij} - \delta_{ij} \\
&= 0 \, .
\end{aligned} \tag{4.23}$$

We have $\overline{\boldsymbol{m}}_i \cdot \overline{\boldsymbol{m}}_j = 0$ by complex conjugation of the above. To show $\boldsymbol{m}_i \cdot \overline{\boldsymbol{m}}_j = 0$ we do a similar calculation:

$$\begin{aligned}
2\boldsymbol{m}_i \cdot \overline{\boldsymbol{m}}_j &= (\overline{\boldsymbol{n}}_i + i\boldsymbol{n}_i) \cdot (\overline{\boldsymbol{n}}_j - i\boldsymbol{n}_j) \\
&= \overline{\boldsymbol{n}}_i \cdot \overline{\boldsymbol{n}}_j - \boldsymbol{n}_i \cdot \boldsymbol{n}_j \\
&= \delta_{ij} + \delta_{ij} \\
&= 2\delta_{ij} \, ,
\end{aligned}$$

which was what we wanted for even dimensions. For the odd dimensions, the additional basis vector $\boldsymbol{m}_0 \equiv \overline{\boldsymbol{m}}_0$ is not null but spacelike, and the relations follows simply from the orthonormality of $\hat{\boldsymbol{n}}_0$.
(3): This can be shown directly using the dual Darboux basis and its properties:

$$\begin{aligned}
\overline{m}^\mu{}_i m_\nu{}^i + m^\mu{}_i \overline{m}_\nu{}^i &= \frac{1}{2} \left((\overline{n}^\mu{}_i - i n^\mu{}_i) \left(\overline{n}_\nu{}^i + i n_\nu{}^i \right) + (\overline{n}^\mu{}_i + i n^\mu{}_i) \left(\overline{n}_\nu{}^i - i n_\nu{}^i \right) \right) \\
&= \frac{1}{2} \left(\overline{n}^\mu{}_i \overline{n}_\nu{}^i + n^\mu{}_i n_\nu{}^i + \overline{n}^\mu{}_i \overline{n}_\nu{}^i + n^\mu{}_i n_\nu{}^i \right) \\
(*) \quad &= \hat{n}^\mu{}_i \hat{n}_\nu{}^i \\
&= \delta^\mu_\nu \, ,
\end{aligned}$$

where we in (*) used (4.10) with one index of the metric raised. This concludes the proof. □

In what is to come, we are going to need a covariant derivative in the non-coordinate Darboux basis. Defining the complexified covariant derivative along $\boldsymbol{m}_i = m^\mu{}_i\partial_\mu$ and $\overline{\boldsymbol{m}}_i = \overline{m}^\mu{}_i\partial_\mu$ as

$$D_i \equiv \nabla_{\boldsymbol{m}_i} \equiv m^\rho{}_i\nabla_\rho \quad , \quad \overline{D}_i \equiv \nabla_{\overline{\boldsymbol{m}}_i} \equiv \overline{m}^\rho{}_i\nabla_\rho \quad , \quad D_0 \equiv \overline{D}_0 \equiv \nabla_{\boldsymbol{m}_0} \equiv m^\rho{}_0\nabla_\rho \,, \tag{4.24}$$

where the spin connection $\chi_\mu{}^a{}_b$ and further details can be found in appendix D. We are going to use some lemmas, that gives us useful relations for the $2n$ eigenvectors $\overline{\boldsymbol{m}}_i$, $\boldsymbol{m}_i$ and $\boldsymbol{m}_0 \equiv \overline{\boldsymbol{m}}_0$.

Lemma 28. *We have for the null eigenvectors that*

$$\left(D_j \overleftrightarrow{\mathbf{h}}\right) \cdot \boldsymbol{m}_i = (\boldsymbol{m}_i \cdot \boldsymbol{\xi})\, \boldsymbol{m}_j + \boldsymbol{\xi}\delta_{j0}\delta_{i0}\,. \tag{4.25}$$

where $\boldsymbol{\xi}$ *is the primary vector of* (3.12) *in the null basis, and the last term can only be present in odd dimensions.*

Proof. We prove this using the definition of the PCKYT (3.12), $\nabla_\rho h_{\lambda\nu} = 2g_{\rho[\lambda}\xi_{\nu]}$. We can contract this with the vielbein $m^\rho{}_j g^{\mu\lambda}$, for which we find

$$m^\rho{}_j g^{\mu\lambda}\nabla_\rho h_{\lambda\nu} = 2m^\rho{}_j g^{\mu\lambda} g_{\rho[\lambda}\xi_{\nu]} \quad \Leftrightarrow$$

$$\begin{aligned}
D_j h^\mu_\nu &= 2m^\rho{}_j g^{\mu\lambda} g_{\rho[\lambda}\xi_{\nu]} \\
&= m^\rho{}_j g^{\mu\lambda}\left(g_{\rho\lambda}\xi_\nu - g_{\rho\nu}\xi_\lambda\right) \\
&= m^\rho{}_j\left(\delta^\mu_\rho\xi_\nu - g_{\rho\nu}\xi^\mu\right) \\
&= m^\mu{}_j\xi_\nu - m_{\nu j}\xi^\mu
\end{aligned}$$

Contracting with $m^\nu{}_i$, $i \neq 0$, gives

$$\begin{aligned}
D_j h^\mu_\nu m^\nu{}_i &= \left(m^\mu{}_j\xi_\nu - m\;{}_{j\nu}\xi^\mu\right) m^\nu{}_i \\
&= \left(m^\mu{}_j\xi_\nu m^\nu{}_i - m_{\nu j}m^\nu{}_i\xi^\mu\right)\,,
\end{aligned}$$

and translating the notation we have using lemma 27 that

$$\begin{aligned}
\left(D_j \overleftrightarrow{\mathbf{h}}\right) \cdot \boldsymbol{m}_i &= (\boldsymbol{m}_i \cdot \boldsymbol{\xi})\, \boldsymbol{m}_j + \underbrace{(\boldsymbol{m}_i \cdot \boldsymbol{m}_j)}_{=\delta_{j0}\delta_{i0}}\boldsymbol{\xi} \\
&= (\boldsymbol{m}_i \cdot \boldsymbol{\xi})\, \boldsymbol{m}_j + \boldsymbol{\xi}\delta_{j0}\,,
\end{aligned} \tag{4.26}$$

where the last term is only present for odd dimensions, where we have the extra basis vector $\boldsymbol{m}_0$ (which is spacelike). This proves the lemma. □

Lemma 29. *We have for the n relations for the null eigenvectors $\boldsymbol{m}_i$ that fulfills*

$$\left(\overleftrightarrow{\mathbf{h}} + ix_i I\right) \cdot D_j \boldsymbol{m}_i + i\left(D_j x_i\right) \boldsymbol{m}_i + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \boldsymbol{m}_j + \boldsymbol{\xi} \delta_{j0} \delta_{i0} = 0\,, \tag{4.27}$$

where I is the identity matrix.

Proof. Our starting point is the eigenvector equation of lemma 27 for general dimensional spacetimes, $\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{m}_i + ix_i \boldsymbol{m}_i = 0$. Acting with D_j on this and using the Leibniz properties of the covariant derivative gives us

$$\begin{aligned} 0 &= D_j\left(\overleftrightarrow{\mathbf{h}} \cdot \boldsymbol{m}_i + ix_i \boldsymbol{m}_i\right) \\ &= \left(D_j \overleftrightarrow{\mathbf{h}}\right) \cdot \boldsymbol{m}_i + i\left(D_j x_i\right) \boldsymbol{m}_i + \overleftrightarrow{\mathbf{h}} \cdot \left(D_j \boldsymbol{m}_i\right) + ix_i\left(D_j \boldsymbol{m}_i\right) \\ &= \left(D_j \overleftrightarrow{\mathbf{h}}\right) \cdot \boldsymbol{m}_i + i\left(D_j x_i\right) \boldsymbol{m}_i + \left(\overleftrightarrow{\mathbf{h}} + ix_i I\right) \cdot D_j \boldsymbol{m}_i\,, \end{aligned}$$

where I is the identity matrix. Now, using (4.25) we can rewrite the first term, and we then find

$$0 = \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \boldsymbol{m}_j + \boldsymbol{\xi} \delta_{j0} \delta_{i0} + i\left(D_j x_i\right) \boldsymbol{m}_i + \left(\overleftrightarrow{\mathbf{h}} + ix_i I\right) \cdot D_j \boldsymbol{m}_i\,. \tag{4.28}$$

This proves the lemma. □

A similar relation with $\overline{\boldsymbol{m}}_i$ is obtained by complex conjugation. A consequence of this last lemma is that we can say something about the eigenvalues x_i. This we formulate in yet another lemma.

Lemma 30. *We have for $i \neq j$ that*

$$D_j x_i = 0 \quad , \quad D_i x_i = i \boldsymbol{m}_i \cdot \boldsymbol{\xi} \quad , \quad D_0 x_i = 0 \qquad \textit{(no sum)}\,, \tag{4.29}$$

where the last equation is only present in odd dimensions.

Proof. To prove this, we first start with the case $i \neq 0$ contract (4.27) with $\overline{\boldsymbol{m}}_i$ (no sum over repeated indices) and use the results of lemma 27, especially $\boldsymbol{m}_i \cdot \overline{\boldsymbol{m}}_j = \delta_{ij}$:

$$\begin{aligned} 0 &= \left(\left(\overleftrightarrow{\mathbf{h}} + ix_i I\right) \cdot D_j \boldsymbol{m}_i + i\left(D_j x_i\right) \boldsymbol{m}_i + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \boldsymbol{m}_j + \boldsymbol{\xi} \delta_{j0} \delta_{i0}\right) \cdot \overline{\boldsymbol{m}}_i \\ &= \left(\overline{\boldsymbol{m}}_i \cdot \overleftrightarrow{\mathbf{h}} + ix_i \overline{\boldsymbol{m}}_i\right) \cdot D_j \boldsymbol{m}_i + i\left(D_j x_i\right) + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \delta_{ij} \\ (*) \quad &= \left(-ix_i + ix_i\right) \overline{\boldsymbol{m}}_i \cdot D_j \boldsymbol{m}_i + i\left(D_j x_i\right) + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \delta_{ij} \\ &= (0)\, \overline{\boldsymbol{m}}_i \cdot D_j \boldsymbol{m}_i + i\left(D_j x_i\right) + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \delta_{ij} \\ &= i\left(D_j x_i\right) + \left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \delta_{ij}\,, \end{aligned}$$

where we in (*) used the transpose of the eigenvalue equation of lemma 27 for $\overline{\boldsymbol{m}}_i$ and the fact that $\overleftrightarrow{\mathbf{h}}$ is antisymmetric, in the first term. This yields after a small rewriting

$$D_j x_i = i\left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right) \delta_{ij}\,, \tag{4.30}$$

and especially

$$D_0 x_i = i\left(\boldsymbol{m}_i \cdot \boldsymbol{\xi}\right)\delta_{i0} = 0\,,$$

which concludes the $i \neq 0$ case. For odd dimensions and $i = 0$ notice that $x_0 = 0$, and this automatically implies $D_0 x_i = 0$ and this concludes the proof of the lemma. □

4.4 The canonical and Kerr-NUT-(A)dS metric

Before finding the most general (euclidean) metric that allows an PCKYT, let us first state its form [27]:

Theorem 31 (Canonical metric)**.** *The most general metric in dimensions $D = 2n+\varepsilon > 2$ that allows a PCKYT is uniquely given by*

$$\boldsymbol{g} = \sum_{i=1}^{n}\left[\frac{\left(\mathbf{d}x_i\right)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] - \frac{\varepsilon c}{A^{(n)}}\left(\sum_{\overline{k}=0}^{n} A^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\,, \tag{4.31}$$

in coordinates $\left\{x_1,\ldots,x_n,\psi_{\overline{0}},\ldots,\psi_{\overline{D-n-1}}\right\}$, with quantities

$$A_i^{(\overline{k})} \equiv \sum_{\substack{i_1<\cdots<i_{\overline{k}}\\ i\neq i_i}} x_{i_1}^2\cdots x_{i_{\overline{k}}}^2\,, \tag{4.32}$$

$$A^{(\overline{k})} \equiv \sum_{i_1<\cdots<i_{\overline{k}}} x_{i_1}^2\cdots x_{i_{\overline{k}}}^2 \tag{4.33}$$

$$U_i \equiv \prod_{\substack{j=1\\ j\neq i}}^{n}\left(x_j^2 - x_i^2\right) \tag{4.34}$$

$$X_i \equiv X_i\left(x_i\right) \tag{4.35}$$

$$Q_i \equiv \frac{X_i}{U_i} \tag{4.36}$$

The equation (4.35) *for X_i holds off-shell, where it is understood that there is only a single coordinate dependence. When vacuum Einsteins field equations with a cosmological constant are imposed, we find that the metric is the Kerr-NUT-(A)dS with*

$$X_i = -2b_i x_i^{1-\varepsilon} + \frac{\varepsilon c}{x_i^2} + \sum_{k=\varepsilon}^{n} c_k x_i^{2k}\,, \tag{4.37}$$

where we have n parameters b_i, $n+1-\varepsilon$ parameters $c_k > 0$, and ε parameters c, where the total number of parameters is $2n+1$, and $D-\varepsilon$ are independent.

The structure of the metric is quite regular and simple. It is a bit simpler in even dimensions, where the last term is not there. The D coordinates $\left\{x_1,\ldots,x_n,\psi_{\overline{0}},\ldots,\psi_{\overline{D-n-1}}\right\}$ are the n non-zero x_is related to the eigenvalues of $\overleftrightarrow{\mathbf{h}}$, along with $D-n = n+\varepsilon$ extra coordinates, which we are going to show are exactly Killing coordinates generated by $\boldsymbol{h}$

also. The x_i coordinates can be thought of as directional cosine coordinates, while $\psi_{\overline{k}}$s can be thought of as azimuthal-like coordinates. It is exactly the assumption of functional independence of the x_is that allows us to use them as coordinates. The signature here is euclidean, but a Wick rotation will give us the the Lorentzian signature that we are interested in. A prescription for this will be discussed later on. When imposing the Einstein equation, one see by comparison to the Kerr-NUT-(A)dS metric found in for example Chen et al. [9], that the canonical metric is identical to it. As a corollary of the theorem, we have then proven that this very general metric has a PCKYT along with all of its good properties. All of the parameters, which are conserved Noether charges are related to the cosmological constant, angular momentum, mass and NUT charges. The relations are a bit subtle, as they are not all independent, and depends on whether the dimension is odd or even. The results can be found in Chen et al. [9], but we will consider the question of the interpretation of them later, and deduce special cases of relevance to us.

4.5 Proof of existence and uniqueness

The proof will take up the remainder of this section. First we do the even dimensions case, $D = 2n$, and then we generalize in the end to odd dimensions. The strategy is to define $n + n = D$ natural or canonical coordinates $\left\{x_1, \ldots, x_n, \psi_{\overline{0}}, \psi_{\overline{1}} \ldots, \psi_{\overline{n-1}}\right\}$ directly from what we have proven so far. It is essentially enough to prove that

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} = 0 \quad \text{and} \quad \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0\,, \tag{4.38}$$

where $\boldsymbol{\xi}$ is the primary vector given by the PCKYT equation (3.12). These equations tells us that $\boldsymbol{\xi}$ generates a family of diffeomorphisms that doesn't change the PCKYT $\boldsymbol{h}$ and the metric $\boldsymbol{g}$, and especially that it is a Killing vector. The proof is then subdivided into the corresponding parts, followed by additional parts where construct the coordinate basis and impose the Einstein equations.

$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} = 0$:

The result of lemma 30 for the x_is can be put to good use in the even dimensions case. If we now simply define the absolute square of the $i = j$ case $D_i x_i = i\boldsymbol{m}_i \cdot \boldsymbol{\xi}$ and its complex conjugate as

$$Q_i \equiv 2\left|D_i x_i\right|^2\,, \tag{4.39}$$

and we can also invert this equation to write

$$D_i x_i = \frac{i}{\sqrt{2}}\sqrt{Q_i}\,, \tag{4.40}$$

up to a complex phase, which we such that $D_i x_i$ is always imaginary. With the choice of phase in (4.40), we have that $\boldsymbol{m}_i \cdot \boldsymbol{\xi} = \frac{1}{\sqrt{2}}\sqrt{Q_i}$ and $\overline{\boldsymbol{m}}_i \cdot \boldsymbol{\xi} = \frac{1}{\sqrt{2}}\sqrt{Q_i}$. We can write $\boldsymbol{\xi}$ in terms of the null basis:

$$
\begin{aligned}
\boldsymbol{\xi} &= \sum_{i=1}^{n} (\boldsymbol{m}_i \cdot \boldsymbol{\xi})\, \boldsymbol{m}_i + (\overline{\boldsymbol{m}}_i \cdot \boldsymbol{\xi})\, \overline{\boldsymbol{m}}_i \\
&= \sum_{i=1}^{n} \frac{1}{i} \left(D_i x_i \boldsymbol{m}_i + D_i x_i \overline{\boldsymbol{m}}_i \right) \\
&= \sum_{i=1}^{n} \frac{1}{i} \frac{i}{\sqrt{2}} \sqrt{Q_i}\, (\boldsymbol{m}_i + \overline{\boldsymbol{m}}_i) \\
&= \sum_{i=1}^{n} \sqrt{Q_i} \frac{1}{2} \left((\overline{\boldsymbol{n}}_i + i\boldsymbol{n}_i) + (\overline{\boldsymbol{n}}_i - i\boldsymbol{n}_i) \right) \\
&= \sum_{i=1}^{n} \sqrt{Q_i} \overline{\boldsymbol{n}}_i \,, \qquad (4.41)
\end{aligned}
$$

which shows that is a projection onto the subspace spanned by the $\overline{\boldsymbol{n}}_i$s. This shows already now that $\boldsymbol{\xi}$ is not arbitrary. As our strategy is to use the x_is as coordinates, we take the exterior derivative $\mathbf{d}x_i$ of them to obtain a set of 1-forms. We do this for the components first:

$$
\begin{aligned}
(\mathbf{d}x_i)_\mu &= \nabla_\mu x_i \\
(*) \quad &= \left(m_\mu{}^j \overline{m}^\nu{}_j + \overline{m}_\mu{}^j m^\nu{}_j \right) \nabla_\nu x_i \\
&= m_\mu{}^j \overline{D}_j x_i + \overline{m}_\mu{}^j D_j x_i \\
(**) \quad &= \delta_{ij} \frac{i}{\sqrt{2}} \sqrt{Q_i} \left(-m_\mu{}^j + \overline{m}_\mu{}^j \right) \\
(***) \quad &= \frac{i}{2} \sqrt{Q_i} \left(-\left(\overline{n}_\mu{}^j + i n_\mu{}^j \right) + \left(\overline{n}_\mu{}^j - i n_\mu{}^j \right) \right) \\
&= \frac{i}{2} \sqrt{Q_i} \left(-2i n_\mu{}^i \right) \\
&= \sqrt{Q_i} n_\mu{}^i \qquad \text{(no sum)} \qquad (4.42)
\end{aligned}
$$

where we in (*) used property 3 of lemma 27. In (**) we used lemma 30 and (4.40), and in (***) we used the definition of the complex null covectors. In coordinate free notation we have

$$
\mathbf{d}x_i = \sqrt{Q_i} \boldsymbol{n}^i \qquad \text{(no sum)}\,. \tag{4.43}
$$

Notice that this equation gives us a relation for n of basis covectors $\boldsymbol{n}^i$ in relation to the canonical coordinates. A useful expression for $\boldsymbol{\xi} \cdot \boldsymbol{h}$ is obtained using the above results and (4.13)

$$
\begin{aligned}
\boldsymbol{\xi} \cdot \boldsymbol{h} &= \left(\sum_{j=1}^{n} \sqrt{Q_j} \overline{\boldsymbol{n}}_j \right) \cdot \left(\sum_{i=1}^{n} x_i \boldsymbol{n}^i \wedge \overline{\boldsymbol{n}}^i \right) \\
(*) &= \sum_{i,j=1}^{n} x_j \sqrt{Q_j} \left(\overline{\boldsymbol{n}}_j \cdot \overline{\boldsymbol{n}}^i \right) \boldsymbol{n}^i \\
&= \sum_{i=1}^{n} x_i \sqrt{Q_i} \boldsymbol{n}^i \\
(**) &= \mathbf{d} \left(\frac{1}{2} \sum_{i=1}^{n} x_i^2 \right)
\end{aligned} \tag{4.44}
$$

where we in (*) used the orthonormal relations of the conjugate vectors, and in (**) we used 4.43. This shows that the vector $\boldsymbol{\xi} \cdot \boldsymbol{h}$ is closed. We can then finally take the Lie derivative along $\boldsymbol{\xi}$ using this and the closedness of $\boldsymbol{h}$ (3.12) in the Cartan formula (E.3), which gives us

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} &= \boldsymbol{\xi} \cdot \mathbf{d} \boldsymbol{h} + \mathbf{d} \left(\boldsymbol{\xi} \cdot \boldsymbol{h} \right) \\
&= 0 + 0 \, ,
\end{aligned} \tag{4.45}
$$

which was what we wanted.

$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0$:

The claim is easy to verify on-shell as we have already done. It does also hold off-shell, which we will prove now. The process of proving this will also allow us to introduce the remaining $D - n = n$ coordinates in even dimensions.

First notice the very useful relation

$$
\begin{aligned}
\boldsymbol{\xi} \cdot \mathbf{d} x_i &= \left(\sum_{j=1}^{n} \sqrt{Q_j} \overline{\boldsymbol{n}}_j \right) \cdot \left(\sqrt{Q_i} \boldsymbol{n}^i \right) \\
&= 0 \, ,
\end{aligned} \tag{4.46}
$$

by the orthonormality relations of the Darboux basis (4.7). If we define

$$
q_i \equiv \boldsymbol{\xi} \cdot \mathbf{d} \ln \sqrt{Q_i} \, , \tag{4.47}
$$

then we have that using (4.46) we can express the Lie derivatives of $\boldsymbol{n}_i$ and $\overline{\boldsymbol{n}}_i$ along $\boldsymbol{\xi}$, which are going to be useful, in terms of this. We first do $\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{n}_i$ using (E.3):

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{n}_i &= \boldsymbol{\xi}\cdot\mathbf{d}\boldsymbol{n}_i + \mathbf{d}\left(\boldsymbol{n}_i\cdot\boldsymbol{\xi}\right)\\
\text{(i)} &= \boldsymbol{\xi}\cdot\mathbf{d}\boldsymbol{n}_i\\
\text{(ii)} &= \boldsymbol{\xi}\cdot\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\mathbf{d}x_i\right)\\
\text{(iii)} &= \boldsymbol{\xi}\cdot\left[\left(\mathbf{d}\frac{1}{\sqrt{Q_i}}\right)\wedge\mathbf{d}x_i\right]\\
&= \boldsymbol{\xi}\cdot\left[\left(\frac{1}{Q_i}\mathbf{d}\sqrt{Q_i}\right)\wedge\mathbf{d}x_i\right]\\
\text{(iv)} &= \left(\boldsymbol{\xi}\cdot\frac{1}{Q_i}\mathbf{d}\sqrt{Q_i}\right)\wedge\mathbf{d}x_i + (-1)^1\left(\frac{1}{Q_i}\mathbf{d}\sqrt{Q_i}\right)\boldsymbol{\xi}\cdot\mathbf{d}x_i\\
\text{(v)} &= \left(\boldsymbol{\xi}\cdot\frac{1}{\sqrt{Q_i}}\mathbf{d}\sqrt{Q_i}\right)\frac{1}{\sqrt{Q_i}}\mathbf{d}x_i\\
&= \left(\boldsymbol{\xi}\cdot\mathbf{d}\ln\sqrt{Q_i}\right)\frac{1}{\sqrt{Q_i}}\mathbf{d}x_i\\
&= +q_i\boldsymbol{n}_i\,.
\end{aligned}
\tag{4.48}
$$

In (i) we used that $\boldsymbol{\xi}$ is a linear combination of $\overline{\boldsymbol{n}}_j$s by (4.41) and the orthonormality of the Darboux basis. In (ii) we used (4.43), and in (iii) the closedness and Leibniz property of $\mathbf{d}$, in (iv) we use the interior product formula, which can be found in appendix E. In (v) we used (4.46) so that we could rewrite the wedge product, and then we could finally identify $q_i\boldsymbol{n}_i$. Likewise, but a little more involved we can express $\mathcal{L}_{\boldsymbol{\xi}}\overline{\boldsymbol{n}}_i$ in terms of q_i. We find

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\overline{\boldsymbol{n}}_i &= \boldsymbol{\xi}\cdot\mathbf{d}\overline{\boldsymbol{n}}_i + \mathbf{d}\left(\overline{\boldsymbol{n}}_i\cdot\boldsymbol{\xi}\right)\\
\text{(i)} &= \boldsymbol{\xi}\cdot\mathbf{d}\left(\frac{1}{x_i}\boldsymbol{h}\cdot\boldsymbol{n}_i\right) + \mathbf{d}\sqrt{Q_i}\\
\text{(ii)} &= \frac{(-1)^{1+1}}{x_i}\boldsymbol{\xi}\cdot\left(\boldsymbol{h}\cdot\mathbf{d}\boldsymbol{n}_i\right) + \mathbf{d}\sqrt{Q_i}\\
\text{(iii)} &= \frac{1}{x_i}\boldsymbol{\xi}\cdot\left(\boldsymbol{h}\cdot\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\mathbf{d}x_i\right)\right) + \mathbf{d}\sqrt{Q_i}\\
\text{(iv)} &= \boldsymbol{\xi}\cdot\left(\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\right)\wedge\left(\frac{1}{x_i}\boldsymbol{h}\cdot\mathbf{d}x_i\right)\right) + \mathbf{d}\sqrt{Q_i}\\
&= \left(\boldsymbol{\xi}\cdot\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\right)\right)\left(\frac{1}{x_i}\boldsymbol{h}\cdot\mathbf{d}x_i\right) + \mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\right)\left(\boldsymbol{\xi}\cdot\left(\frac{1}{x_i}\boldsymbol{h}\cdot\mathbf{d}x_i\right)\right) + \mathbf{d}\sqrt{Q_i}
\end{aligned}
$$

$$
\begin{aligned}
\text{(v)} &= \left(\boldsymbol{\xi}\cdot\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\right)\right)\left(\frac{1}{x_i}\boldsymbol{h}\cdot\mathbf{d}x_i\right) \\
&= \left(-\boldsymbol{\xi}\cdot\frac{1}{\sqrt{Q_i}}\left(\mathbf{d}\sqrt{Q_i}\right)\right)\left(\frac{1}{x_i}\boldsymbol{h}\cdot\boldsymbol{n}_i\right) \\
\text{(vi)} &= \left(-\boldsymbol{\xi}\cdot\frac{1}{\sqrt{Q_i}}\left(\mathbf{d}\sqrt{Q_i}\right)\right)\overline{\boldsymbol{n}}_i \\
&= -q_i\overline{\boldsymbol{n}}_i
\end{aligned}
\tag{4.49}
$$

where we in (i) used that $\overline{\boldsymbol{n}}_i = \frac{1}{x_i}\boldsymbol{h}\cdot\boldsymbol{n}_i$ by the definition given in lemma 26 and that $\overline{\boldsymbol{n}}_i\cdot\boldsymbol{\xi} = \sqrt{Q_i}$ by use of (4.41) and orthonormality of Darboux basis. In (ii) we used the chain-rule to conclude that $\boldsymbol{\xi}\cdot\left(\mathbf{d}\frac{1}{x_i}\right) \propto \boldsymbol{\xi}\cdot\mathbf{d}x_i = 0$ by (4.46), $\mathbf{d}\boldsymbol{h} = 0$ because it is closed, so we only get terms with the exterior derivative acting on $\boldsymbol{n}_i$ (using that it satisfies a modified Leibniz rule and commutes with contraction, there are 2 wedge products and two contractions here, which will give a plus sign), and $\boldsymbol{h}$ is understood to be contracted with the $\boldsymbol{n}_i$ part. In (iii) we used that $\mathbf{d}x_i = \sqrt{Q_i}\boldsymbol{n}^i$ can be inverted to give $\boldsymbol{n}_i = \delta_{ij}\boldsymbol{n}^j = \delta_{ij}\frac{1}{\sqrt{Q_j}}\mathbf{d}x_j = \frac{1}{\sqrt{Q_i}}\mathbf{d}x_i$. In (iv) we used that $\mathbf{d}x_i$ is closed to rewrite the exterior derivative and then do the contraction using the interior product formula of appendix E. To get the result in (v) we first identified $\mathbf{d}x_i = \sqrt{Q_i}\boldsymbol{n}_i$, then $\overline{\boldsymbol{n}}_i = \frac{1}{x_i}\boldsymbol{h}\cdot\boldsymbol{n}_i$, and then noticed that

$$
\begin{aligned}
\mathbf{d}\left(\frac{1}{\sqrt{Q_i}}\right)\left(\boldsymbol{\xi}\cdot\left(\frac{1}{x_i}\boldsymbol{h}\cdot\mathbf{d}x_i\right)\right) &= \frac{-1}{Q_i}\left(\mathbf{d}\sqrt{Q_i}\right)\left(\boldsymbol{\xi}\cdot\left(\frac{1}{x_i}\boldsymbol{h}\cdot\sqrt{Q_i}\boldsymbol{n}_i\right)\right) \\
&= \frac{-1}{\sqrt{Q_i}}\left(\mathbf{d}\sqrt{Q_i}\right)\left(\boldsymbol{\xi}\cdot\overline{\boldsymbol{n}}_i\right) \\
&= \frac{-1}{\sqrt{Q_i}}\left(\mathbf{d}\sqrt{Q_i}\right)\left(\sqrt{Q_i}\right) \\
&= -\mathbf{d}\sqrt{Q_i}\,,
\end{aligned}
$$

so that the last term in the expression would cancel this. In (vi) we then identified $\overline{\boldsymbol{n}}_i = \frac{1}{x_i}\boldsymbol{h}\cdot\boldsymbol{n}_i$ and q_i to finish the proof of the claim.

By some additional thinking, we can actually show that we have $Q_i = Q_i\left(x_1,\ldots,x_n\right)$. One could think that Q_i as defined by (4.39) in general depends on the spacetime point in some coordinate chart that has $\{x_1,\ldots,x_n\}$ as half of the coordinates because of the covariant derivative that enters in the definition, but this is not the case. As $\overline{\boldsymbol{n}}_i$s are the only other object besides the eigenvalues $\pm ix_i$ and $\boldsymbol{n}_i$s determined by $\boldsymbol{h}$, the remaining n yet undetermined coordinates that it must define, must be constructed from just the $\overline{\boldsymbol{n}}_i$s in some way, where the specific construction is not important for the argument. As we know that $\overline{\boldsymbol{n}}_i\cdot\boldsymbol{n}_i = 0$ by (4.7), this shows that the tangent space of the spacetime defined by $\boldsymbol{h}$ is the direct sum of two orthogonal subspaces, $T\left(M\right) = \{\overline{\boldsymbol{n}}_i\}\oplus\{\boldsymbol{n}_i\}$, $\{\overline{\boldsymbol{n}}_i\}\perp\{\boldsymbol{n}_i\}$. We have that Q_i is defined entirely from quantities that belongs to the subspace spanned by $\boldsymbol{n}_i$s, as we using (4.43) can write $Q_i = \left(\boldsymbol{n}_i\cdot\mathbf{d}x_i\right)^2$, and thus it cannot depend on anything but $\{x_1, \ldots, x_n\}$ coordinates.

This is a very useful result. As we have $\mathbf{d} \ln \sqrt{Q_i} \propto \sum_i c_i \mathbf{d} x_i$ by the chain-rule, where c_i are some functions determined by the chain-rule, we can then use (4.46) to simply conclude that we have

$$q_i = \boldsymbol{\xi} \cdot \mathbf{d} \ln \sqrt{Q_i} = 0 \,, \tag{4.50}$$

which allows us to to conclude that

$$\mathcal{L}_{\boldsymbol{\xi}} \overline{\boldsymbol{n}}_i = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{n}_i = 0 \,. \tag{4.51}$$

This proves the claim in even dimensions, because by raising indices we find for the dual basis that $\mathcal{L}_{\boldsymbol{\xi}} \overline{\boldsymbol{n}}^i = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{n}^i = 0$, and by the Leibniz property of the Lie derivative we simply have

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0 \,. \tag{4.52}$$

Killing vectors and tensors of the metric

If we take a look at the Killing towers generated by the PCKYT for the canonical metric and stated in section 4.2 in a convenient form:

$$\boldsymbol{K}^{(j)} = \sum_{i=1}^{n} A_i^{(j)} \left(\boldsymbol{n}^i \otimes \boldsymbol{n}^i + \overline{\boldsymbol{n}}^i \otimes \overline{\boldsymbol{n}}^i \right) \,, \tag{4.53}$$

$$\overline{\boldsymbol{\epsilon}}^{(\bar{j})} = \boldsymbol{\xi} \cdot \boldsymbol{K}^{(\bar{j})} \,, \tag{4.54}$$

we can with the above results prove that $\overline{\boldsymbol{\epsilon}}^{(\bar{j})}$ are Killing vectors. To do this we remember that $\boldsymbol{K}^{(\bar{j})}$ is constructed from Hodge duals of the wedge products of the PCKYT. As the Lie derivative commutes with the Hodge dual[15] and we can use the Leibniz property of the Lie derivative on the wedge products, we conclude first by using (4.45) that we have

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{K}^{(j)} = 0 \,. \tag{4.55}$$

Using that the SSN bracket commutes with the contraction and fulfills the Leibniz rule, we have

$$\begin{aligned}
\mathcal{L}_{\overline{\boldsymbol{\epsilon}}^{(\bar{j})}} \boldsymbol{g} &= \left[\overline{\boldsymbol{\epsilon}}^{(\bar{j})}, \boldsymbol{g} \right] \\
&= [\boldsymbol{\xi}, \boldsymbol{g}] \cdot \boldsymbol{K}^{(j)} + \boldsymbol{\xi} \cdot \left[\boldsymbol{K}^{(j)}, \boldsymbol{g} \right] \\
&= 0 + 0 \,,
\end{aligned} \tag{4.56}$$

which shows that we have n Killing vectors given by $\overline{\boldsymbol{\epsilon}}^{(\bar{j})}$. Using only that $\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} = 0$, one can prove by the method of introducing a specially chosen generating

[15] Because the Levi-Civita tensor is covariantly constant.

function[16] for the steps $\boldsymbol{K}^{(j)}$ [23, 28], that we have

Theorem 32. *For the n Killing vectors of the Killing vector tower $\bar{\boldsymbol{\epsilon}}^{(\bar{j})}$ and the n Killing tensors of the Killing tensors tower $\boldsymbol{K}^{(j)}$, we have*

$$\begin{aligned} \left[\boldsymbol{K}^{(i)}, \boldsymbol{K}^{(j)}\right] &= 0\,, && (4.57)\\ \left[\boldsymbol{K}^{(i)}, \boldsymbol{\xi}\right] &= 0\,, && (4.58)\\ \left[\boldsymbol{K}^{(i)}, \bar{\boldsymbol{\epsilon}}^{(\bar{j})}\right] &= 0\,. && (4.59) \end{aligned}$$

These results are some of the most crucial to what we want to prove now, but also integrability and separability as discussed previously. The n Killing vectors and n Killing tensors are all clearly independent, because they have different powers of x_is due to $A_i^{(j)}$, but this can also be shown directly using other arguments [29].

The Riemann tensor and the spin connection

The PCKYT equation (3.12) $\nabla_\rho h_{\mu\nu} = g_{\rho\mu}\xi_\nu - g_{\rho\nu}\xi_\mu$ can give us valuable information about the the Christoffel symbols and the covariant derivative, as it is over-constrained [39]. To relate it to the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$, we can use that the commutator of covariant derivatives is contracted sums of $\boldsymbol{h}$ with the Riemann tensor in a torsion-free connection. We have

$$\nabla_\sigma \nabla_\rho h_{\mu\nu} = g_{\rho\mu}\nabla_\sigma\xi_\nu - g_{\rho\nu}\nabla_\sigma\xi_\mu\,, \tag{4.60}$$

and thus

$$\left[\nabla_\sigma, \nabla_\rho\right] h_{\mu\nu} = g_{\rho\mu}\nabla_\sigma\xi_\nu - g_{\rho\nu}\nabla_\sigma\xi_\mu - g_{\sigma\mu}\nabla_\rho\xi_\nu + g_{\sigma\nu}\nabla_\rho\xi_\mu\,, \tag{4.61}$$

but we also have by the properties of the commutator [7]

$$\begin{aligned} \left[\nabla_\sigma, \nabla_\rho\right] h_{\mu\nu} &= -R^\lambda{}_{\mu\rho\sigma}h_{\lambda\nu} - R^\lambda{}_{\nu\rho\sigma}h_{\mu\lambda}\\ &= R_{\mu\rho\sigma}{}^\lambda h_{\lambda\nu} - R_{\nu\rho\sigma}{}^\lambda h_{\lambda\mu}\,, \end{aligned} \tag{4.62}$$

where we in the last line used the symmetry properties of $R^\rho{}_{\sigma\mu\nu}$ and $h_{\mu\nu}$. We can then write the combined equation as

$$R_{\mu\rho\sigma}{}^\lambda h_{\lambda\nu} - R_{\nu\rho\sigma}{}^\lambda h_{\lambda\mu} = g_{\rho\mu}\nabla_\sigma\xi_\nu - g_{\rho\nu}\nabla_\sigma\xi_\mu - g_{\sigma\mu}\nabla_\rho\xi_\nu + g_{\sigma\nu}\nabla_\rho\xi_\mu\,. \tag{4.63}$$

Contracting the indices with the Darboux vielbeins $\hat{n}^\mu{}_i$ and the inverses $\hat{n}_\mu{}^i$, we can express this equation equivalently in the Darboux basis as

$$R_{abdf}h^f{}_c - R_{abcf}h^f{}_d = \delta_{bc}\nabla_a\xi_d - \delta_{ac}\nabla_b\xi_d - \delta_{bd}\nabla_a\xi_c + \delta_{ad}\nabla_b\xi_c\,. \tag{4.64}$$

[16] Theorem 32 holds for the Killing towers of odd dimensional spacetimes as well.

Using the properties of the vielbeins and the symmetry properties of the non-vielbein indices of the Riemann tensor that we can extract some information about the covariant derivatives of $\boldsymbol{\xi}$. Taking for example $c = 2i-1$ and $d = 2i$, so that they correspond to the vectors $\hat{\boldsymbol{n}}_{2i-1} = \boldsymbol{n}_i$, $\hat{\boldsymbol{n}}_{2i} = \overline{\boldsymbol{n}}_i$, we find that the LHS of (4.64) vanishes by the simple form of $h^f{}_a = \overleftrightarrow{\mathbf{h}}$ in this basis and the antisymmetry of the Riemann tensor in the last two indicies, and we have the identity

$$0 = \delta_{b(2i-1)}\nabla_a\xi_{(2i)} - \delta_{a(2i-1)}\nabla_b\xi_{(2i)} - \delta_{b(2i)}\nabla_a\xi_{(2i-1)} + \delta_{a(2i)}\nabla_b\xi_{(2i-1)}\,. \tag{4.65}$$

Taking different projections of this equality, we get the following results:

$$\nabla_{(2i)}\xi_{(2j)} = \nabla_{(2i)}\xi_{(2j-1)} = \nabla_{(2i-1)}\xi_{(2j)} = \nabla_{(2i-1)}\xi_{(2j-1)} = 0 \quad \text{for } i \neq j\,, \tag{4.66}$$

$$\nabla_{(2i)}\xi_{(2i)} + \nabla_{(2i-1)}\xi_{(2i-1)} = 0\,. \tag{4.67}$$

Additional non-trivial relations can be also be derived using the other symmetry relations of the Riemann tensor.

With what we know so far, we can also calculate the spin connection coefficients for the torsion-free condition. After a long calculation, one arrives at some rather simple coefficients, that can be found in [25, 39].

Constructing the coordinate frames

Using the above results, we can calculate the Lie bracket of the Darboux basis. Doing a calculation of $\mathcal{L}_{\boldsymbol{\xi}}\overline{\boldsymbol{n}}_i$, we will find that $[\overline{\boldsymbol{n}}_j, \overline{\boldsymbol{n}}_i] = 0$, but a similar calculation shows that we in general can conclude that $[\boldsymbol{n}_i, \overline{\boldsymbol{n}}_j] \neq 0$ and $[\boldsymbol{n}_i, \boldsymbol{n}_j] \neq 0$. This means that they do not generate useful coordinates by their flow, because then they would depend implicitly on other coordinates defined by the other flows as discussed in section 2.3.

To have a useful coordinate basis, we must then find one where all of the basis vector does commute as stated in theorem 6. The right choice of a new basis $\{\boldsymbol{e}_i, \overline{\boldsymbol{e}}_{\overline{k}}\}$ is defined as the n vectors $\boldsymbol{\partial}_{x_i}$, which by (4.43) is proportional to $\boldsymbol{n}_i$, along with n vectors which are the corresponding Killing vectors of the Killing vector tower given by (4.20) [24]:

$$\boldsymbol{e}_i \equiv \frac{1}{\sqrt{Q_i}}\boldsymbol{n}_i = \boldsymbol{\partial}_{x_i}\,, \tag{4.68}$$

$$\overline{\boldsymbol{e}}_{\overline{k}} \equiv \boldsymbol{\epsilon}^{(\overline{k})} = \boldsymbol{\xi}\cdot\boldsymbol{K}^{(\overline{k})} = \sum_{i=1}^{n} A_i^{(\overline{k})}\sqrt{Q_i}\,\overline{\boldsymbol{n}}_i\,, \tag{4.69}$$

where again $i = 1, \ldots, n$ and $\overline{k} = 0, \ldots, n-1$. It really is a basis, because the vectors are all linearly independent as the $A_i^{(\overline{k})}$ factor gives different powers of x_is [39]. One should notice that $\boldsymbol{e}_i \cdot \overline{\boldsymbol{e}}_{\overline{k}} = 0$ for any choice of $i, \overline{k}$ because of the orthonormality relations of the Darboux basis. The $\overline{\boldsymbol{e}}_{\overline{k}}$s are exactly Killing vectors as we proved previously, while $\boldsymbol{e}_i$ are actually eigenvectors of the Killing tensors $\boldsymbol{K}^{(j)}$ with eigenvalue $A_i^{(j)}$, as

$$\begin{aligned} \boldsymbol{K}^{(j)} \cdot \boldsymbol{e}_i &= \sum_{k=1}^{n} A_k^{(j)} \left(\boldsymbol{n}^k \otimes \boldsymbol{n}^k + \overline{\boldsymbol{n}}^k \otimes \overline{\boldsymbol{n}}^k \right) \cdot \boldsymbol{e}_i \\ &= \sum_{k=1}^{n} A_k^{(j)} \frac{1}{\sqrt{Q_i}} \left(\boldsymbol{n}_i \cdot \boldsymbol{n}^k \right) \otimes \boldsymbol{n}^k + \left(\boldsymbol{n}_i \cdot \overline{\boldsymbol{n}}^k \right) \otimes \overline{\boldsymbol{n}}^k \\ &= \sum_{k=1}^{n} A_k^{(j)} \frac{1}{\sqrt{Q_i}} \delta_i^k \boldsymbol{n}^k \\ &= A_i^{(j)} \boldsymbol{e}_i \,. \end{aligned} \tag{4.70}$$

Using theorem 32 and using that the Lie bracket satisfies the Leibniz rule and commutes with contractions along with what we know about the covariant derivatives of $\boldsymbol{\xi}$ and the Jacobi identity, we can show that these vector fields all commute [27]:

$$\left[\overline{\boldsymbol{e}}_{\overline{i}}, \overline{\boldsymbol{e}}_{\overline{j}} \right] = \left[\boldsymbol{e}_i, \boldsymbol{e}_j \right] = \left[\boldsymbol{e}_i, \overline{\boldsymbol{e}}_{\overline{j}} \right] = 0 \,. \tag{4.71}$$

This shows that $\{\boldsymbol{e}_i,\, \overline{\boldsymbol{e}}_{\overline{k}}\}$ is the right basis that allows us to define a coordinate basis by theorem 6, with coordinates $\{x_i,\, \psi_{\overline{i}}\}$ such that

$$\boldsymbol{e}_i = \boldsymbol{\partial}_{x_i} \quad , \quad \overline{\boldsymbol{e}}_{\overline{i}} = \boldsymbol{\partial}_{\psi_{\overline{i}}} \,. \tag{4.72}$$

We find the cobasis by inverting (4.69) using the orthonormality of the Darboux basis and the definition of the dual basis $\boldsymbol{e}^i \left(\boldsymbol{e}_j \right) = \delta_j^i$ and $\overline{\boldsymbol{e}}^{\overline{i}} \left(\overline{\boldsymbol{e}}_{\overline{j}} \right) = \delta_{\overline{j}}^{\overline{i}}$. The result after some manipulation is

$$\boldsymbol{e}^i = \sqrt{Q_i} \boldsymbol{n}^i = \mathbf{d} x_i \,, \tag{4.73}$$

$$\overline{\boldsymbol{e}}^{\overline{j}} = \sum_{i=1}^{n} \frac{\left(-x_i^2 \right)^{n-1-\overline{j}}}{U_i \sqrt{Q_i}} \overline{\boldsymbol{n}}^i \,, \tag{4.74}$$

where we have defined

$$U_i \equiv \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(x_j^2 - x_i^2 \right) \,, \tag{4.75}$$

and expressed Q_i as

$$Q_i \equiv \frac{X_i}{U_i} \,, \tag{4.76}$$

where $X_i = X_i \left(x_i \right)$ is a function of just x_i. The form of X_i is a consequence of a lemma from algebra, which can be found in the appendix of [29]. We can think of $A_i^{(\overline{k})}$ as a $n \times n$ matrix, and then $B_{(\overline{j})}^i \equiv \frac{\left(-x_i^2 \right)^{n-1-\overline{j}}}{U_i}$ can be thought of as the inverse matrix that we need to invert (4.69), as we may show that

$$\sum_{i=1}^{n} B^{i}_{(\overline{j})} A_i^{(\overline{k})} = \delta^{\overline{k}}_{\overline{j}} \quad , \quad \sum_{\overline{k}=0}^{n-1} B^{i}_{(k)} A_j^{(\overline{k})} = \delta^{i}_{j} \,. \tag{4.77}$$

The corresponding n Killing coordinates $\psi_{\overline{k}}$ generated by the flow of these covectors fields $\left\{\boldsymbol{e}^i,\, \overline{\boldsymbol{e}}^{\overline{j}}\right\}$. One can show that the flow of Killing vectors gives closed curves, and that we have $\psi_{\overline{k}} \in [0, 2\pi]$ [9]. Thus we have obtained a coordinate basis for the metric as stated in theorem 31. We can also invert the equations (4.73) and (4.74) and obtain expressions for $\{\boldsymbol{n}^i, \overline{\boldsymbol{n}}^i\}$ in terms of $\left\{\boldsymbol{e}^i,\, \overline{\boldsymbol{e}}^{\overline{k}}\right\}$. We find in terms of the coordinate functions:

$$\boldsymbol{n}^i = \frac{1}{\sqrt{Q_i}} \mathbf{d}x_i \quad , \quad \overline{\boldsymbol{n}}^i = \sqrt{Q_i} \sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})} \mathbf{d}\psi_{\overline{k}} \,. \tag{4.78}$$

This result is very useful, as it is now just a matter of substitution to find the metric.

Inserting into the canonical metric

If we insert the above results in the vielbein metric of (4.10) then we obtain (4.73)

$$\begin{aligned}
\boldsymbol{g} &= \sum_{i=1}^{n} \boldsymbol{n}^i \otimes \boldsymbol{n}^i + \overline{\boldsymbol{n}}^i \otimes \overline{\boldsymbol{n}}^i \\
&= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{Q_i}} \mathbf{d}x_i \right)^2 + \left(\sqrt{Q_i} \sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})} \mathbf{d}\psi_{\overline{k}} \right)^2 \\
&= \sum_{i=1}^{n} \frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i \left(\sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})} \mathbf{d}\psi_{\overline{k}} \right)^2
\end{aligned}$$

This proves the off-shell result as this metric is exactly the form of the canonical metric (4.31), and the content of theorem 31. This finishes the proof of the existence and uniqueness part for even dimensions.

Generalizing to odd dimensions

For odd dimensions, we can give a heuristical proof by dimensional reduction from $D' = 2\,(n+1)$ to $D = 2n+1$ by setting $x_{n+1} = 0$ and then view the D dimensional canonical metric $\boldsymbol{g}$ as a pullback of the metric to this hyperplane from the D' dimensional spacetime with metric $\boldsymbol{g}'$. The 2×2 blocks on the diagonal of $\overleftrightarrow{\boldsymbol{h}}'$ all span independent subspaces, and thus everything that we generate from it will split up nicely, so the x_{n+1} coordinate is extra in this sense, as $\overleftrightarrow{\boldsymbol{h}}$ is a submatrix of $\overleftrightarrow{\boldsymbol{h}}'$ (an extra row and column with zeroes).

Let us show that we can put the metric in the correct form. First we do a rewriting:

$$
\begin{aligned}
\boldsymbol{g}' &= \sum_{i=1}^{n+1}\left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] \\
&= \sum_{i=1}^{n}\left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + \frac{(\mathbf{d}x_{n+1})^2}{Q_{n+1}} + Q_{n+1}\left(\sum_{\overline{k}=0}^{n} A_{n+1}^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2 \qquad (4.79) \\
&= \sum_{i=1}^{n}\left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + \sum_{i=1}^{n}\left[Q_i\left(A_i^{(\overline{n})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + \frac{(\mathbf{d}x_{n+1})^2}{Q_{n+1}} + Q_{n+1}\left(\sum_{\overline{k}=0}^{n} A_{n+1}^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2
\end{aligned}
$$

Now setting $x_{n+1} = 0$ and thus $\mathbf{d}x_{n+1} = 0$, we have that the metric $\boldsymbol{g}$ on this hyperplane reduces to

$$
\begin{aligned}
\boldsymbol{g} &= \sum_{i=1}^{n}\left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + \sum_{i=1}^{n}\left[Q_i\left(A_i^{(\overline{n})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + Q_{n+1}\left(\sum_{\overline{k}=0}^{n} A_{n+1}^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2 \\
&= \sum_{i=1}^{n}\left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i\left(\sum_{\overline{k}=0}^{n-1} A_i^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] - \frac{c}{A^{(n)}}\left(\sum_{\overline{k}=0}^{n} A^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2, \qquad (4.80)
\end{aligned}
$$

where $c = X_{n+1}\,(x_{n+1} = 0)$ is some constant. The claim

$$
\sum_{i=1}^{n}\left[Q_i\left(A_i^{(\overline{n})}\mathbf{d}\psi_{\overline{k}}\right)^2\right] + Q_{n+1}\left(\sum_{\overline{k}=0}^{n} A_{n+1}^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2 = -\frac{c}{A^{(n)}}\left(\sum_{\overline{k}=0}^{n} A^{(\overline{k})}\mathbf{d}\psi_{\overline{k}}\right)^2, \qquad (4.81)
$$

follows when setting $x_{n+1} = 0$ of the LHS, as we then have $A_{n+1}^{(\overline{k})} = A^{(\overline{k})}$(because the exclusion of products with x_{n+1} in the sum is then just $A^{(\overline{k})}$), $A_i^{(\overline{n})} = 0$ (because there is a x_{n+1} factor in all terms), and $U_{n+1} \propto A^{(\overline{n})}$ (constant of proportionality is absorbed in c). We know that $\overline{\boldsymbol{\epsilon}}^{(\overline{n})}$ is a Killing vector from theorem 21, so the coordinate it generates is consistent to use. This shows that the theorem holds for the odd dimensional case as well.

The PCKYT and potential

In the canonical coordinates introduced in general D dimensions, the PCKYT and its KY potential given by (3.12) and in the Darboux basis as (4.13), can be rewritten to take the following form:

$$
\begin{aligned}
\boldsymbol{h} &= \sum_{i=1}^{n} x_i \boldsymbol{n}^i \wedge \overline{\boldsymbol{n}}^i \\
&= \sum_{i=1}^{n} x_i \left(\frac{1}{\sqrt{Q_i}} \mathbf{d}x_i \right) \wedge \left(\sqrt{Q_i} \sum_{\overline{k}=0}^{n-1} A_i^{\left(\overline{k}\right)} \mathbf{d}\psi_{\overline{k}} \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \mathbf{d}x_i^2 \wedge \left(\sum_{\overline{k}=0}^{n-1} A_i^{\left(\overline{k}\right)} \mathbf{d}\psi_{\overline{k}} \right) \\
(*) &= \frac{1}{2} \sum_{\overline{k}=0}^{n-1} \mathbf{d}A_i^{\left(\overline{k}+1\right)} \wedge \mathbf{d}\psi_{\overline{k}} \,. \qquad (4.82)
\end{aligned}
$$

In (*) we used that the structure of $A_i^{\left(\overline{k}\right)}$ will reduce it to sums of one order lower, when taking derivatives wrt. x_i^2s. In this form it is also easy to see that the potential should be taken as

$$
\boldsymbol{b} = \frac{1}{2} \sum_{\overline{k}=0}^{n-1} A_i^{\left(\overline{k}+1\right)} \mathbf{d}\psi_{\overline{k}} \,. \qquad (4.83)
$$

4.6 Imposing Einsteins equations

Having calculated the spin connection coefficients, one can impose the Einstein equation, most easily done in the Cartan formalism. This can then be used to calculate the components of the Riemann tensor as discussed earlier, and then the Einstein equations can be imposed, which was done in [20]. This is most easily done in the Darboux basis, and the result of this is that X_i is no longer arbitrary and takes the form:

$$
X_i = -2b_i x_i^{1-\varepsilon} + \frac{\varepsilon c}{x_i^2} + \sum_{k=\varepsilon}^{n} c_k x_i^{2k} \,, \qquad (4.84)
$$

This was first performed by Hamamoto et al. [20] in 2006, and more extensively worked through in Houri et al. [24]. The parameters b_i, c, c_k found in the process have different interpretation. One finds that c_n is proportional to the cosmological constant, as for the Ricci tensor one finds

$$
R_{\mu\nu} = (-1)^n \left(D-1\right) c_n g_{\mu\nu} \,,
$$

which satisfies the Einstein space condition with cosmological constant given by $\Lambda = (-1)^n \left(D-1\right) c_n$. The remaining n parameters c_k, c (last one only present in odd dimensions) plus n parameters b_i are rotational and NUT parameters of the spacetime. The NUT charges can be thought of as higher-dimensional generalization of the parameter introduced by Newman et al. [30] for a generalization of the $D=4$ Schwarschild metric, which could be interpreted as being proportional to magnetic charge in euclidean signature[4].

Both in odd and even dimensions, there is a scaling symmetry of the metric coordinates that can be used to make one parameter (not c_n) redundant. This effectively reduces the number of independent parameters to $D - \varepsilon$, with the correct number of rotational parameters[17] as discussed in section 5.2, while the rest are then interpreted as NUT parameters.

4.7 Wick rotated metric

If we introduce a radial coordinate $r \equiv -ix_n$ in the metric (4.31), then we will change the signature of the metric so it becomes Lorentzian [9]. A consequence of this is that we get sign changes in the quantities that enters in the metric. For those depending on x_n^2, i.e. $A_i^{(\bar{k})}$, $A^{(\bar{k})}$, U_i, we should substitute $x_n = -r^2$ into the expressions. For X_n we have

$$\begin{aligned} X_n &= -2b_n x_n^{1-\varepsilon} + \frac{\varepsilon c}{x_n^2} + \sum_{k=\varepsilon}^{n} c_k x_n^{2k} \\ &= -2b_n \left(-i\right)^{1-\varepsilon} \left(r\right)^{1-\varepsilon} - \frac{\varepsilon c}{r^2} + \sum_{k=\varepsilon}^{n} c_k \left(-r^2\right)^k \\ &\equiv -2Mr^{1-\varepsilon} - \frac{\varepsilon c}{r^2} + \sum_{k=\varepsilon}^{n} c_k \left(-r^2\right)^k , \end{aligned} \tag{4.85}$$

where we have defined the mass parameter $M \equiv \left(-i\right)^{1+\varepsilon} b_n$, as the corresponding Wick rotated NUT parameter. What becomes the time coordinate, is the coordinate generated by generated by the Killing vector $\boldsymbol{\xi}$, $\psi_{\bar{0}}$. This is because as Q_n changes sign, we don't get a change of sign for the terms with r, but the $\psi_{\bar{0}}$ with $A_i^{(\bar{0})} = 1$ we get a change of sign. As $\psi_{\bar{0}} \in [0, 2\pi]$ we can do a rescaling so it becomes more physical if we like. The remaining coordinates are still interpreted as directional cosines and azimuthal coordinates. The Wick rotation as presented above is a highly formal procedure, which introduces coordinate singularities in the new form of the metric, while it was well-defined in the old coordinates [39].

Regarding the parameters, the discussion of section 5.2 we have one less independent rotation than for the euclidean signature, i.e. $n - 1 + \varepsilon$, when the spatial number of dimensions is even (D odd), as there are the same number of possible rotations as the former case with an odd number of spatial dimensions. In even spacetime dimensions, we will then have an extra NUT parameter, where one of them is the mass. We then have $n - 1 + \varepsilon$ rotational parameters, $n - 1 - \varepsilon$ non-mass NUT parameters, the mass and the cosmological constant that parametrizes the solution to Einsteins equations.

4.8 Integrability and separability

As we have proven that the Killing tensors and vectors of the KY towers all are independent and commute wrt. the SSN bracket by (4.57)-(4.59), it is now just a simple matter of applying theorem 14 to this result. We then have the following result for the canonical metric

[17] This discussion applies to Wick rotated metric, but here we just have an extra spatial dimension.

Theorem 33 (Conserved charges)**.** *The n steps of the Killing tensor tower $\boldsymbol{K}^{(j)}$, $0 \le j \le n-1$, are all independent and give rise to n different conserved charges. The $n+\varepsilon$ steps of $\boldsymbol{\xi}^{(\bar{j})}$ of the Killing vector tower, $0 \le \bar{j} \le n-1+\varepsilon$, are all independent and give rise to $n+\varepsilon$ different conserved charges. The total number of conserved quantities is D.*

The total number of conserved quantities being D it is exactly enough to render the geodesic equation integrable by definition 13. We also see explicitly that theorem 15 holds and we have a $(D-n)$-separability structure.

5 Stationary black holes

5.1 Aspects of black hole spacetimes

Black holes are objects that on some chosen length-scale are very "energetic" (to be made more precise), and thus tend to curve spacetime. Far away from the black hole, spacetime should be flat if we restrict ourselves to the class of so-called vacuum black holes with cosmological constant identical zero, which can be thought of as approximating the spacetime of the whole universe by just this single object. For non-zero cosmological constant, the situation is more involved, but one could also consider black holes that are asymptotically (A)dS spacetimes, which roughly means that taking the limit of size of the black hole going to zero, one should have that the geometry approaches (A)dS, see Ashtekar and Das [2] for more precise definitions. In any case, the special thing about black holes is that they curve spacetime so much, that if you are close enough to one and go along the geodesic, then you will at some point be trapped inside a hypersurface, where no physics (following an arbitrary future-directed curve) can help you escape again [7]. We now give a more precise definition of what we mean by this statement:

Definition 34 (Black hole)**.** By a black hole spacetime we mean a spacetime where there exist a closed hypersurface E, the event horizon, at which passing time-like curves becomes confined inside.

This is common to spacetimes of arbitrary dimension, and for any value of the cosmological constant. The consequence of this is that the event horizon is a closed null hypersurface, and this fact in particular makes helps find equations using the metric that determines it. More precise definitions of the event horizon can be found in [21]. We will first consider a more restrictive class of black holes, the stationary ones. For black holes that are asymptotically (A)dS, have the same symmetries as the asymptotically flat spacetimes, because (A)dS are maximally symmetric spacetimes. By stationary we mean the following:

Definition 35 (Stationary spacetime)**.** We say that an asymptotically flat or (A)dS spacetime is stationary, if there exists a Killing vector $\boldsymbol{T}$ that is time-like in the asymptotically flat or (A)dS region.

This definitions means that we in a stationary spacetime can always can find coordinates where the metric is independent of the time coordinate. These are the black holes that are physically of interest, because after any physical processes that would carry energy-momentum away from a confined spatial region has diminished, spacetime should be independent of time. The end state of any such process should thus be a black hole (if we are still in a vacuum region). If we have the metric in such a form that it is independent of the time coordinate t generated by the flow of $\boldsymbol{T}$, then we have that $\boldsymbol{T} = \boldsymbol{\partial}_t$, since this is the tangent vector of curve that only moves in the time-direction. There are some subtleties and differencies between asymptotically flat and (A)dS black holes, for example the possibility of an cosmological horizon in dS spacetime, and in AdS spacetime spatial infinite is a time-like Killing vector [13, 7]. Stationarity implies staticity, the existence of a time-like Killing vector that further at any point of spacetime is orthogonal to a spatial hypersurface (constant time coordinate), but not the other way around. In fact, a stationary spacetime doesn't need to be orthogonal to any spatial hypersurface, as is the case for the rotating black hole spacetimes we will consider later on.

Questions about the topology of the event horizon E are subtle. It has been known for a long time that in four spacetime dimensions, it is very restricted for stationary spacetimes under some technical but reasonable assumptions [34]:

Theorem 36 (Hawking)**.** *For a stationary spacetime in $D = 4$ dimensions, the event horizon E is a 2-sphere S^2.*

Likewise in higher dimensions, when rigorous result were still lacking, it was thought for some time that the topology were to be very restricted. In 2002 Emparan and Reall [14] demonstrated that there existed a slight generalization of the $D = 5$ Myers-Perry metric allowed a $S^2 \times S^1$ topology of the event horizon, which was to be called a (rotating) black ring. This generalization is not a special case of the general Kerr-NUT-(A)dS metric. A theorem on the topologies of of event horizons [18] that generalizes the Hawking theorem 36 shows that in general event horizons can indeed be product topologies in higher dimensions, and even consist of several disconnected components. Since the Emparan and Reall solution, several non-trivial topologies has been determined for $D = 5$ since, for example a $S^3 \cup (S^2 \times S^1)$ "black Saturn" event horizon in agreement with the more general theorem, along with other $D > 4$ dimensional black holes have been found [13], but no complete classification exists at the present time.

5.2 Energy and angular momentum

Say now that we are in a $D = 2n + \varepsilon$ dimensional spacetime with one time direction and $D - 1$ spatial directions. A euclidean rotation is an active or passive process that happens in the hyperplanes of two spatial coordinates $\{x^i, x^j\}$, keeping the distance to rotation axis and spacial distance as well fixed, but otherwise moving the points of the hyperplane around. In D dimensions, the maximal isometry group of rotations in Minkowski space is then $\mathrm{SO}\,(D-1)$, and the number of (not necessarily independent) rotation hyperplanes, which is exactly the number of generators of the group is

$$N = \binom{D-1}{2} = \frac{1}{2}\,(D-1)\,(D-2)\;, \tag{5.1}$$

because we can choose the first coordinate axis x^i of the hyperplane in $D-1$ ways, and the other x^j in $D-2$ ways, but as the hyperplane of (x^i, x^j) is the same as (x^j, x^i), we only have half of the product. The rank of $\mathrm{SO}(D-1)$ is $H \equiv n-1+\varepsilon$, so the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(D-1)$ is of dimension H [35]. From the theory of the Cartan subalgebra, we know that these generators will all commute wrt. the Lie bracket with themselves, and are therefore exactly the number of independent rotations we can have. We can therefore always chose a clever coordinate basis where each of the H independent hyperplanes[18] are associated with a single rotational Killing vector $R^i_\mu \equiv (\partial_{\varphi_i})_\mu$ that generates a coordinate by its flow, plus the remaining Killing vectors of the isometry group. In this basis we have that the metric $\boldsymbol{g}$ is independent of the coordinates φ_i, $1 \leq i \leq H$. In the case that we don't have maximally rotational symmetry, it is a subgroup of $\mathrm{SO}(D-1)$. If the only rotational symmetry left is exactly the one generated by Cartan subgroup of $\mathrm{SO}(D-1)$, then we say that we have axial symmetry. The Cartan subgroup would is then simply isomorphic to $U(1) \times \cdots \times U(1) = U(1)^N$.

In any case, given a rotational Killing vector $\boldsymbol{R}$, we have by the field-theoretic version of Noether's theorem 11, that there correspond a conserved charge to the Lagrangian density for general relativity, the Einstein-Hilbert lagrangian [22]. For asymptotically flat stationary spacetimes, this conserved quantity is what we will call angular momentum J_φ and is given by

$$J_\varphi \equiv -\frac{1}{8\pi} \int_{\partial\Sigma} n_\mu \sigma_\nu \nabla^\mu R^\nu \sqrt{g_{|\partial\Sigma}} \mathrm{d}^{D-2}x \,, \tag{5.2}$$

where Σ is some $D-1$ dimensional spacelike hypersurface at spatial infinity where spacetime is asymptotically flat, and $\partial\Sigma$ is the boundary of it, a $D-2$ dimensional hypersurface. $g_{|\partial\Sigma}$ is the determinant of the induced metric - the metric on $\partial\Sigma$ is the pullback of g from M to $\partial\Sigma$. Also n_μ is the unit normal vector to Σ, and σ_ν is an outwards pointing unit normal vector of $\partial\Sigma$. As these are conserved for the given spacetime, they play the role of a parametrization a whole set of solutions to the Einstein equations, as discussed for the Kerr-NUT-(A)dS metric. Since we know what angular momentum is in flat spacetimes, where we could define it as we have done in the above, and the spacetime is asymptotically flat, the conclusion is that J_φ should really be thought of as angular momentum. For asymptotically (A)dS spacetimes, the correspondingly conserved quantity must be calculated differently, because the above integral (5.2) in general diverges, see [2] for a discussion of this.

Likewise we find that the existence of an asymptotically time-like Killing vector $\boldsymbol{T}$ for stationary spacetimes gives us some sort of energy-conservation. The corresponding conserved charge M is given by

$$M \equiv \frac{1}{4\pi} \int_{\partial\Sigma} n_\mu \sigma_\nu \nabla^\mu T^\nu \sqrt{g_{|\partial\Sigma}} \mathrm{d}^{D-2}x \,, \tag{5.3}$$

which normally can be thought of the total mass or energy of the black hole including gravitational binding energy. Again we have to use a different definition of the conserved

[18] Of course, there N conserved quantities for each of the generators that are exactly the Killing vectors, but the point is that they are not all independent as they in general doesn't commute wrt. the Lie bracket. The Casimirs are exactly the independent ones.

charge in (A)dS spacetimes.

5.3 The PCKYT and black holes

The existence of the PCKYT gave rise to the canonical and the Kerr-NUT-(A)dS metric by theorem 31. The Kerr-NUT-(A)dS black holes are the most general (higher dimensional) black hole spacetime solutions of the vacuum Einsteins field equations (with a cosmological constant) known at the present time. Actually we also have that the only possibility for the event horizon topology, because in the Wick rotated version of (4.31), the event horizon is located at a fixed value of the radial coordinate $r = r_H$ [10]. If there is a solution, then it defines a S^{D-2} hypersphere, which is the event horizon. One can then take the point of view that the existence of the PCKYT is exactly what characterizes the (higher dimensional) black hole spacetimes with spherical topology of the event horizon [17]. This is also the thing that solutions of different dimensions have in common, along with the good integrability and separability properties, and in this way the PCKYT ensures that these black holes are similar in any dimension.

6 Special cases of Kerr-NUT-(A)dS

Name \ Parameters	NUT	Rotational	Mass	Cosm. const.	Dimensions
Kerr-NUT-(A)dS	X	X	X	X	D
Myers-Perry-(A)dS		X	X	X	D
Myers-Perry		X	X		D
Schwarzschild-AdS			X	X	D
Schwarzschild-Tangherlini			X		D
Kerr		X	X		4
Taub–NUT	X		X		4
Schwarzschild			X		4

Table 1: Some special cases of the Kerr-NUT-(A)dS metric classified according to which parameters that are present, and the name of these metrics in the literature.

Basically all stationary black holes of interest can be derived from the Kerr-NUT-(A)dS metric as claimed before. It is a matter of choosing the parameters correctly, but the solutions one obtains from the general form of the metric (4.31), will in general not be in the standard coordinates that one often derives these special cases in. It is then of course necessary to perform a change of coordinates to make a positive identification. We can get an overview of the possibilities and their corresponding names in relation to the NUT and rotation parameters, along with the mass and cosmological constant as outlined in the table 1 in increasing order of specialization.

6.1 The $D = 4$ Kerr solution

As a warm-up, we should consider the $D = 4$ solution for a electromagnetically neutral rotating black hole, the Kerr solution [37]. This will be useful so we have a "baseline" of comparison to differences in higher dimensions on both a quantitatively and qualitatively basis. The Kerr metric is in component form in Boyer–Lindquist coordinates[19] (t, r, θ, φ) given by

$$\boldsymbol{g} = -\frac{\Delta}{\Sigma}\left(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(\left[r^2 + a^2\right]\mathbf{d}\varphi - a\mathbf{d}t\right)^2 + \frac{\Sigma}{\Delta}\mathbf{d}r^2 + \Sigma\mathbf{d}\theta^2\,, \tag{6.1}$$

where the tensor product in squares is understood, and where

$$\Delta \equiv r^2 - 2Mr + a^2\,, \tag{6.2}$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta\,, \tag{6.3}$$

$$a \equiv J/M\,. \tag{6.4}$$

We see that g is independent of φ and t, so there at least two Killing vectors, those that are associated with invariance under translation of these. The Killing vector $R_\mu \equiv (\partial_\varphi)_\mu$ that generates φ-translations is a rotational Killing vector, because $\varphi \to \varphi + 2\pi$ is the same point in spacetime. Likewise the Killing vector $T_\mu \equiv (\partial_t)_\mu$ that generates t-translations is a asymptotically time-like Killing vector, as we may verify. Here J is the conserved charge (5.2) associated with the rotational Killing vector, and we should thus think of it as angular momentum and M is likewise the conserved charge (5.3) associated with the time-like Killing vector, the total mass. Both can be verified by direct calculation. The quantity $a = J/M$ is thus angular momentum per unit mass.

If we take the limit $r \to \infty$, it is easy to see that $\boldsymbol{g} \to \boldsymbol{\eta}$ in spherical coordinates, so it is asymptotically flat. It is not static, because taking $t \to -t$ gives a change of sign of the two cross terms $\left(\mathbf{d}t - a\sin^2\theta\mathbf{d}\varphi\right)^2$ and $\left([r^2 + a^2]\,\mathbf{d}\varphi - a\mathbf{d}t\right)^2$. Neither is it fully spherically symmetric, because translating θ doesn't leave the metric invariant.

The conclusion is that it is a stationary spacetime, that describes a rotating black hole with mass M and angular momentum J rotating the $\theta = 0$ hyperplane. The Boyer–Lindquist coordinates are already in a "spherical coordinates form" where we can easily find the event horizon(s), which are then given by

$$g^{rr} = \frac{\Delta}{\Sigma} = 0 \;\;\Rightarrow\;\; \Delta = r_H^2 - 2Mr_H + a^2 = 0\,. \tag{6.5}$$

The number of solutions for r_H of (6.5) depends on the relationship between the mass and angular momentum. $M^2 > J$ corresponds to the most physical relevant situation where the total energy is larger than the angular momentum, and this gives two solutions

$$r_H^\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{6.6}$$

[19] Ranges $t \in \mathbb{R}$, $r \in \mathbb{R}_+$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$.

In the Boyer–Lindquist coordinates, this will define a hypersurface that is of S^2 topology. Another interesting feature of the Kerr metric, is the existence of an ergosphere (ergohypersurface). This is a hypersurface, that is defined by $\boldsymbol{T}$ becoming null, i.e.

$$0 = g^{\mu\nu} T_\mu T_\nu = g^{tt} \,.$$

For general stationary spacetimes the ergosphere is not the same as the event horizon, because in general there might be linear combinations of other tangent vectors that will be time-like at the ergosurface, it is just the stationary curves generated by $\boldsymbol{T}$ that will become unphysical. The consequence is that inside of the ergosphere, objects are forced to move. This will further lead to particularities such as frame-dragging and the Penrose process [7].

6.1.1 Getting Kerr from the Kerr-NUT-(A)dS

We can get the Kerr metric from the general (euclidean) Kerr-NUT-(A)dS metric (4.31) by taking $D = 4$. The functions that enter have the expressions

$$A_1^{(0)} = A_2^{(0)} = 1 \quad , \quad A_1^{(1)} = x_2^2 \quad , \quad A_2^{(1)} = x_1^2 \,, \tag{6.7}$$

$$U_1 = -U_2 = x_2^2 - x_1^2 \,. \tag{6.8}$$

The metric then becomes

$$\begin{aligned}
\boldsymbol{g} &= \sum_{i=1}^{2} \left[\frac{(\mathbf{d}x_i)^2}{Q_i} + Q_i \left(\sum_{\overline{k}=0}^{1} A_i^{(\overline{k})} \mathbf{d}\psi_{\overline{k}} \right)^2 \right] \\
&= \frac{(\mathbf{d}x_1)^2}{Q_1} + Q_1 \left(A_1^{(0)} \mathbf{d}\psi_0 + A_1^{(1)} \mathbf{d}\psi_1 \right)^2 + \frac{(\mathbf{d}x_2)^2}{Q_2} + Q_2 \left(A_2^{(0)} \mathbf{d}\psi_0 + A_2^{(1)} \mathbf{d}\psi_1 \right)^2 \\
&= \frac{x_2^2 - x_1^2}{X_1} (\mathbf{d}x_1)^2 + \frac{X_1}{x_2^2 - x_1^2} \left(\mathbf{d}\psi_0 + x_2^2 \mathbf{d}\psi_1 \right)^2 - \frac{x_2^2 - x_1^2}{X_2} (\mathbf{d}x_2)^2 - \frac{X_2}{x_2^2 - x_1^2} \left(\mathbf{d}\psi_0 + x_1^2 \mathbf{d}\psi_1 \right)^2 \,.
\end{aligned}$$

If we now do the Wick rotation $x_2 \to ir$, and also define $\psi_0 \equiv \tau$, $\psi_1 \equiv \psi$, $x_1 \equiv y$, $X_1 \equiv Y$, $X_2 \equiv R$ for simplicity we obtain

$$\begin{aligned}
\boldsymbol{g} &= -\frac{r^2 + y^2}{X_1} \mathbf{d}y^2 - \frac{X_1}{r^2 + y^2} \left(\mathbf{d}\tau - r^2 \mathbf{d}\psi \right)^2 - \frac{r^2 + y^2}{X_2} \mathbf{d}r^2 + \frac{X_2}{r^2 + y^2} \left(\mathbf{d}\tau + y^2 \mathbf{d}\psi \right)^2 \\
&= \frac{1}{r^2 + y^2} \left[R \left(\mathbf{d}\tau + y^2 \mathbf{d}\psi \right)^2 - Y \left(\mathbf{d}\tau - r^2 \mathbf{d}\psi \right)^2 \right] - \left(r^2 + y^2 \right) \left[\frac{\mathbf{d}y^2}{Y} + \frac{\mathbf{d}r^2}{R} \right] \,.
\end{aligned} \tag{6.9}$$

We have that imposing the Einstein equations specifies $X_1 \equiv Y$, $X_2 \equiv R$ by (4.37), which then (with $M \equiv -ib_2$) takes the form

$$Y = -2b_1 y + c_0 + c_1 y^2 + c_2 y^4 \,, \tag{6.10}$$

$$R = -2Mr + c_0 - c_1 r^2 + c_2 r^4 \,. \tag{6.11}$$

The number of constants here is 5, but they are not all independent as we know from the general theory. The scaling symmetry given by

$$r \to \alpha r \quad , \quad y \to \alpha y \quad , \quad \tau \to \alpha^{-1}\tau \quad , \quad \psi \to \alpha^{-3}\psi \quad , \quad R \to \alpha^4 R \quad , \quad Y \to \alpha^4 Y \,, \tag{6.12}$$

as is easy to verify. We then define new parameters that obeys this scaling law directly, and we can write

$$\begin{aligned} Y &= -2b_1 y + c_0 + c_1 y^2 + c_2 y^4 \\ &\to -2b_1 \alpha y + c_0 + c_1 \alpha^2 y^2 + c_2 \alpha^4 y^4 \\ &= \underbrace{c_0^2}_{\equiv \alpha^4 a^2} - 2 \underbrace{b_1}_{\equiv -\alpha^3 N} \alpha y + \underbrace{c_0}_{\equiv \alpha^2 \left(a^2 \frac{\Lambda}{3} - 1\right)} \alpha^2 y^2 + \underbrace{c_2}_{\equiv \frac{\Lambda}{3}} \alpha^4 y^4 \\ &= \alpha^4 \left(a^2 + 2Ny + \left(a^2 \frac{\Lambda}{3} - 1\right) y^2 - \frac{\Lambda}{3} y^4\right) \\ &= \alpha^4 \left[\left(a^2 - y^2\right)\left(1 + \frac{\Lambda y^2}{3}\right) + 2Ny\right] \,, \end{aligned}$$

where the 3 independent parameters $\{a,\, N,\, \Lambda\}$ we have defined are exactly angular momentum pr. mass, NUT charge, and the cosmological constant. Their physically identification can be related to these scaling properties, as $a \to \alpha^2 a$, $N \to \alpha^3 N$, $\Lambda \to \alpha^0 \Lambda$ scales as we would expect their physical quantities does by a rescaling of the coordinates. Likewise with these definitions and a similar one for the mass M we find that we can

$$\begin{aligned} R &= -2Mr + c_0 - c_1 r^2 + c_2 r^4 \\ &\to -2b_1 \alpha y + c_0 + c_1 \alpha^2 y^2 + c_2 \alpha^4 y^4 \\ &\equiv \alpha^4 \left[\left(a^2 + r^2\right)\left(1 - \frac{\Lambda r^2}{3}\right) - 2Mr\right] \,, \end{aligned} \tag{6.13}$$

which would also scale as $M \to \alpha^3 M$. The Kerr metric is then recovered as a special case by setting $N = \Lambda = 0$, albeit not in the Boyer–Lindquist coordinates, which is related by a change of variables which can be found in Kubiznak [29].

6.1.2 Tower of Killing tensors

We have already found two Killing vectors $\boldsymbol{T}$, $\boldsymbol{R}$, and concluded that there cannot be any further[20] by the general theory. If there exists a PCKY for the Kerr spacetime, which we eventually will show that there does, we know that there should be two rank 2 Killing tensors of the extended Killing tensor tower. One is the metric itself, and the last one

[20] In principle, there could be more, but they could not be generated by the PCKY.

gives rise to a conserved charge first found by Carter in 1968. It was not understood until Walker and Penrose in 1970 showed that it originated from a rank 2 Killing tensor $\boldsymbol{\sigma}$ [38]. It is given by

$$\sigma_{\mu\nu} \equiv 2\Sigma^2 l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}\,, \tag{6.14}$$

where

$$l^\mu \equiv \frac{1}{\Delta}\left(r^2 + a^2, \Delta, 0, a\right) \quad , \quad n^\mu \equiv \frac{1}{2\Sigma^2}\left(r^2 + a^2, -\Delta, 0, a\right) \tag{6.15}$$

We can interpret $\sigma_{\mu\nu}$ as the total angular momentum of a particle, as in the asymptotically flat region $\Delta \sim r^2$, $\Sigma \sim r^2$, and we have

$$l^\mu \sim (1,1,0,0) \quad , \quad n^\mu \sim \frac{1}{2}(1,-1,0,0)\ . \tag{6.16}$$

then

$$\begin{aligned}
\sigma_{\mu\nu} &\sim 2r^2 l_{(\mu} n_{\nu)} + r^2 \eta_{\mu\nu} \\
&= \begin{pmatrix} +r^2 - r^2 & & & \\ & -r^2 + r^2 & & \\ & & 0 + r^2 & \\ & & & 0 + r^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & r^2 & \\ & & & r^2 \end{pmatrix} .
\end{aligned}$$

Contracting this with geodesics of the flat region in spherical coordinates, we find that

$$\sigma_{\mu\nu} p^\mu p^\nu \sim r^2 \left[r^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 \right] ,$$

which is what we would call total angular momentum per unit mass squared of a particle.

6.2 Schwarzschild-Tangherlini

The first higher dimensional solution to the Einstein field equations was the generalization of the spherically symmetric Schwarzschild solution, the Tangherlini solution [13] given by

$$\boldsymbol{g} = -\left(1 - \frac{\mu}{r^{D-3}}\right)\mathbf{d}t^2 + \left(1 - \frac{\mu}{r^{D-3}}\right)^{-1}\mathbf{d}r^2 + r^2 \mathbf{d}\Omega^2_{D-2}\,, \tag{6.17}$$

where

$$\mu \equiv \frac{16\pi M}{(D-2)\,\Omega_{D-2}}\,, \tag{6.18}$$

where M is the total mass of the black hole, and Ω_{D-2} is the hyperarea of a $(D-2)$-sphere. This was a rather straight forward generalization, more or less done by using the higher-dimensional generalization of Newtons law of universal gravitation the weak-field limit, for which the propagator of the Poisson equation $G(r) \propto 1/r^{D-3}$ changes with the number of spatial dimensions, and then solving the Einstein equation, which greatly simplifies using the $\mathrm{SO}(D-1)$ symmetry of spacetime. We most easily get the Schwarzschild-Tangherlini solution from the Myers-Perry solution that we consider in the next section. The Schwarzschild-Tangherlini solution is also a special case of the Wick-rotated Kerr-NUT-(A)dS metric (4.31), with only one non-zero parameter, the mass M. This means that all of the X_i functions (4.37) are very simple. We use the scaling symmetry to normalize $X_i = 1$ for $i \neq n$, as there are no degree of freedom left here, and we conclude that

$$X_i = 1 \quad (i \neq n) \quad , \quad X_n = -2Mr^{1-\varepsilon}\,. \tag{6.19}$$

The topology of the event horizon is of course S^{D-2}, and it is located at the coordinate singularity $\frac{\mu}{r_H^{D-3}} = 1 \Rightarrow r_H = \sqrt[D-3]{\mu}$, and thus always exists for $M > 0$. The generalization of the Schwarzschild solution to higher dimensions thus doesn't give any real surprises, which is due to the high degree of $(\mathrm{SO}(D-1))$ symmetry of the spacetime, that restricts the physics very much because of the number of Killing vectors, that can be enhanced from the number generated from the Kerr-NUT-(A)dS metric.

6.3 Myers-Perry spacetimes

Once upon a time, people started wondering how one should generalize the Kerr solution to higher dimensions in a way that was natural. The fact that there is not full spherical symmetry even rendered the Kerr solution had to find in the first place, and it was first in 1986 by Myers and Perry that a generalization was achieved. Of course, in higher dimensions there are more degrees of freedom for a rotation to take place, as we found in section 5.2, there are $H \equiv n - 1 + \varepsilon$ independent planes of rotation and H conserved angular momenta. So for an axialsymmetric higher dimensional black hole, that rotates in the maximal number of independent hyperplanes, we have the minimal number of rotational Killing vectors, just spanned by the Cartan subalgebra of $\mathrm{SO}(D-1)$.

The solution found by Myers and Perry has the general solution given by suitable coordinates

$$\boldsymbol{g} = -\mathbf{d}t^2 + \frac{U}{V-2M}\mathbf{d}r^2 + \frac{2M}{U}\left(\mathbf{d}t - \sum_{i=1}^{H} a_i\mu_i^2\mathbf{d}\varphi_i\right)^2 + \sum_{i=1}^{H}\left(r^2+a_i^2\right)\left(\mathbf{d}\mu_i^2 + \mu_i^2\mathbf{d}\varphi_i^2\right) + (1-\varepsilon)\,r^2\mathbf{d}\alpha^2\,, \tag{6.20}$$

where we define

$$V \equiv r^{-(\varepsilon+1)} \prod_{i=1}^{H} \left(r^2 + a_i^2 \right) , \tag{6.21}$$

$$U \equiv V \left(1 - \sum_{i=1}^{H} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \right) , \tag{6.22}$$

$$a_i \equiv J_i / M . \tag{6.23}$$

where it is understood that in odd dimensions when $\varepsilon = 1$, the α coordinate is not present. The $\{t, r, \varphi_i, \mu_i\}$ coordinates[21] are a variant of hyperspherical coordinates, where we have defined μ_i and α as the directional cosines, which in the asymptotically flat region in cartesian coordinates can be written as $\mu_i \equiv x_i / r$. The coordinates are thus not all independent and satisfies the constraint

$$\sum_{i=1}^{H} \mu_i^2 + (1 - \varepsilon) \, \alpha^2 = 1 ,$$

and one should eliminate one of the coordinates using this, but this will spoil the somewhat simple structure of the metric. The metric is independent of t and φ_i, so we should expect some isometries. The rotational coordinates φ_i are the coordinates of rotation in the H independent hyperplanes, and there are thus H rotational Killing vectors $K_\mu^{(i)} \equiv \left(\partial_{\varphi_i}\right)_\mu$ corresponding to them, with conserved angular momentum J_i associated with each one. Also the time-like Killing vector $T_\mu = \left(\partial_t\right)_\mu$ defines the conserved quantity, the mass of the black hole M. Again, the a_i parameters that are interpreted as angular momentum pr. mass in each independent hyperplane of rotation.

As we take $r \to \infty$ we have

$$V \sim r^{-(\varepsilon+1)} r^{2H} = r^{2H-(\varepsilon+1)} \quad , \quad U \sim V = r^{2H-(\varepsilon+1)} ,$$

and the metric reduces to

$$\boldsymbol{g} \sim -\mathbf{d}t^2 + \mathbf{d}r^2 + r^2 \sum_{i=1}^{H} \mathbf{d}\mu_i^2 + \mu_i^2 \mathbf{d}\varphi_i^2 + (1 - \varepsilon) \, r^2 \mathbf{d}\alpha^2 , \tag{6.24}$$

these are indeed just flat space in the variant of hyperspherical coordinates. Thus we can really conclude that the Myers-Perry spacetime is stationary and asymptotically flat.

6.3.1 Event horizon

The coordinates we have used are suitable for finding the event horizons that may appear. By definition of the event horizon, we should solve

$$0 = g^{rr} \left(r_H \right) = \frac{V - 2M}{U} \quad \Rightarrow$$

[21] Ranges $t \in \mathbb{R}$, $r \in \mathbb{R}_+$, $\mu_i \in [-1, 1)$, $\varphi_i \in [0, 2\pi)$.

$$2Mr_H^{\varepsilon+1} = \prod_{i=1}^{H} \left(r_H^2 + a_i^2 \right) . \tag{6.25}$$

This is a complicated polynomial equation that may have many different roots depending on the parameters a_i. We can see from the equation that the cosine directions doesn't enter, and it is only an equation for the radial coordinate r_H, and when it has a solution, the topology of the event horizon is then S^{D-2} as expected.

6.3.2 Getting Myers-Perry from Kerr-NUT-(A)dS

Looking at the Kerr-NUT-(A)dS metric (4.31), we see that the forms of the metrics look similar, but the definition of the functions that enters the metrics are not the same. We obtain the Myers-Perry spacetimes from the Kerr-NUT-(A)dS by setting all NUT charges and the cosmological constant equal to zero (keeping mass), which reduces the degrees of freedom in the parameters by $n - \varepsilon$. This will correspond to the mass, and H rotational parameters. The procedure for obtaining the Myers-Perry metric from the Kerr-NUT-(A)dS can be obtained from [9], which also gives us the coordinate transformation to get the metric in the form (6.20). In this paper the authors starts with the Kerr-(A)dS metric, which was found as a generalization of the Myers-Perry metric by Gibbons et al. [19], and introduces NUT parameters. Working in the opposite direction, one then obtains the Myers-Perry metric by choosing the extra parameters in a specific way and doing the inverse coordinate transformation, as a special case of the Kerr-NUT-(A)dS. This is in principle straight-forward, but one must pay attention to the domain of the coordinate transformation, and one then reaches the desired conclusion.

7 Discussion and further developments

In summary, what we have shown in the former sections, is that we can contribute many of the good, desirable and remarkable properties of a large class of the higher dimensional black hole solutions to the existence of the PCKYT. To put the theory into a bigger picture, one can ask how it is connected to other aspects of physics. It is a well-known fact that general relativity cannot be the correct theory of gravity, as it is a classical field theory. The correct theory has to be described by some quantum theory, if gravity as expected follows the historical development of theories of the other fundamental forces, that has been very successful upon quantization of the corresponding classical field theory. Thus the kind of black holes described by the Kerr-NUT-(A)dS metric can be thought of as a classical solution, subject to quantum corrections that becomes more important at high energies or small length scales. String theories are different candidates for a theory of quantum gravity, and those of interest these days are higher-dimensional ones, so it is important to understand what classical black holes looks like in higher dimensions, to improve the understanding of the quantum versions.

An open problem in the theory of the Killing-Yano tensors, is to give separability results for other field equations than the Klein-Gordon and Dirac equations [6] defined on the canonical spacetime. For example, separation of variables of the minimally coupled

Maxwell equations have never been proven, but is conjectured to be possible [39]. A general result about the separability of field equations is lacking at present time. Likewise it is also an open problem what the relation to black holes with non-spherical event horizon topologies is. It has been shown that even-though the black ring solution doesn't admit a Killing-Yano tensor, a dimensionally Kaluza-Klein reduced version of this admits a CKYT, which is not in general closed [1]. This explains some of the good properties of the 5D black ring metric, but further exploration hasn't been done at present time.

One can also consider product manifolds $M = L \times K$, where K is the canonical spacetime with a PCKYT, and L is some other manifold. M will then inherit the good properties of the PCKYT on K, but of course full integrability and separability etc. depends on the structure of L. In particular if we have $L = \mathbb{R}^p$, i.e. we add to K a total of p flat directions so the metric on M becomes

$$\boldsymbol{g} = \boldsymbol{g}_K + \sum_{i=1}^{p} \mathbf{d}z_i^2 \,, \tag{7.1}$$

then we get a p-brane solution, where all of the good properties of K carries over in a simple way because L is maximally symmetric. One could then for example apply the theory of blackfolds developed by Emparan et al. [15] to study such spacetimes in the limit of two widely different length scales of the spacetime.

It is also possible to generalize theorem 31 for the canonical metric beginning from the less restrictive case of just a CKYT and not necessarily a PCKYT. The x_is that occurred squared in the corresponding diagonalized Killing tensor are then in general fewer in number, $i = 1, \ldots, \ell \leq n$, but it is still possible to construct an orthonormal basis by the spectral theorem of linear algebra. The metric that occurs after a similar treatment to what lead to the Kerr-NUT-(A)dS metric is called the generalized Kerr-NUT-(A)dS metric, with a set of parameters that will in general have a different interpretation because of the degeneracy [39]. Results about the integrability and separability properties of these spacetimes appears not to have been studied so far in the literature.

8 Summary

In this project we have investigated the theory of Killing and Killing-Yano tensors, and their relations to explicit and hidden symmetries. We have successfully related the explicit symmetries to isometries, and hidden symmetries to symmetries of the phase space of the Hamiltonian that gives the geodesic equation. The existence of the principal conformal Killing-Yano tensor has been shown to just the thing that secures integrability and separability of the most important equations of motion defined on a large class of spacetimes, defined by the canonical metric. Imposing the Einstein field equations on this gives us the Kerr-NUT-(A)dS metric, which is a very general black hole solution that has a number of well-known spacetimes as special cases. All of these special cases have a spherical event horizon topology, which is a general feature of spacetimes with a PCKYT. Various generalizations of the theory was briefly discussed, but it is clear that a somewhat complete theory at present time with few open questions to be answered.

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A Alternative proof of theorem 4

An alternative proof of theorem 4 can be given by infinitesimal arguments, where the intuitive aspects are a bit more transparent. Equation (2.2) is equivalent to the integral equation

$$x^{\mu}(\lambda) = x^{\mu}(\lambda = 0) + \int_0^{\lambda} K^{\mu}(x(\tau))\,\mathrm{d}\tau\,. \tag{A.1}$$

Integrating to $\lambda = \epsilon$ infinitesimal, we have that the infinitesimal flow generated at $x^{\mu}(\lambda) \equiv y^{\mu}$ by (A.1) to first order in ϵ is equivalent to the equation

$$y^{\mu} = y^{\mu}(x) = x^{\mu} + \epsilon K^{\mu}(x)\;, \tag{A.2}$$

We have to first order in ϵ that

$$\begin{aligned} \epsilon K^{\mu}(x) &= \epsilon K^{\mu}(y) - \frac{\partial}{\partial \epsilon} K^{\mu}(y - \epsilon K)\bigg|_{\epsilon=0} \epsilon^2 + \dots \\ &= \epsilon K^{\mu}(y) + \mathcal{O}\left(\epsilon^2\right)\,. \end{aligned}$$

We will also have to use that

$$\begin{aligned} \epsilon \frac{\partial K^{\sigma}(y)}{\partial y^{\mu}} &= \epsilon \frac{\partial K^{\sigma}(x)}{\partial y^{\mu}} + \mathcal{O}\left(\epsilon^2\right) \\ &= \epsilon \frac{\partial K^{\sigma}(x)}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial y^{\mu}} + \mathcal{O}\left(\epsilon^2\right) \\ &= \epsilon \frac{\partial K^{\sigma}(x)}{\partial x^{\mu}} + \mathcal{O}\left(\epsilon^2\right)\,. \end{aligned}$$

We can write the transformed metric $\varphi^* \boldsymbol{g} \equiv \boldsymbol{g}'$ at x given by (2.1) to first order in ϵ:

$$\begin{aligned} g'_{\mu\nu}(x) &= \frac{\partial x^{\sigma}}{\partial y^{\mu}} \frac{\partial x^{\rho}}{\partial y^{\nu}} g_{\sigma\rho}(y) \\ &= \frac{\partial\left(y^{\sigma} - \epsilon K^{\sigma}(y)\right)}{\partial y^{\mu}} \frac{\partial\left(y^{\rho} - \epsilon K^{\rho}(y)\right)}{\partial y^{\nu}} g_{\sigma\rho}(y) \\ &= g_{\mu\nu}(y) - \epsilon\left(\frac{\partial K^{\sigma}(y)}{\partial y^{\mu}} \delta^{\rho}_{\nu} + \delta^{\sigma}_{\mu} \frac{\partial K^{\rho}(y)}{\partial y^{\nu}}\right) g_{\sigma\rho}(y) \\ &= g_{\mu\nu}(y) - \epsilon\left(\frac{\partial K^{\sigma}(x)}{\partial x^{\mu}} \delta^{\rho}_{\nu} + \delta^{\sigma}_{\mu} \frac{\partial K^{\rho}(x)}{\partial x^{\nu}}\right) g_{\sigma\rho}(x) + \mathcal{O}\left(\epsilon^2\right)\,. \end{aligned} \tag{A.3}$$

If we now also do an expansion of the metric at y in ϵ to first order, we find

$$\begin{aligned} g_{\mu\nu}(y) &= g_{\mu\nu}(y)\big|_{\epsilon=0} + \left.\frac{\partial g_{\mu\nu}(y)}{\partial\epsilon}\right|_{\epsilon=0}\epsilon + \mathcal{O}\left(\epsilon^2\right) \\ &= g_{\mu\nu}(x) + \epsilon \left.\frac{\partial x^\lambda}{\partial\epsilon}\frac{\partial g_{\mu\nu}(y)}{\partial x^\lambda}\right|_{\epsilon=0} + \mathcal{O}\left(\epsilon^2\right) \\ &= g_{\mu\nu}(x) - \epsilon K^\lambda(x)\frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} + \mathcal{O}\left(\epsilon^2\right) . \end{aligned} \tag{A.4}$$

Inserting this into (A.3), we find

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \epsilon\left[\left(\frac{\partial K^\sigma(x)}{\partial x^\mu}\delta^\rho_\nu + \delta^\sigma_\mu\frac{\partial K^\rho(x)}{\partial x^\nu}\right)g_{\sigma\rho}(x) + K^\lambda(x)\frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\right] + \mathcal{O}\left(\epsilon^2\right) . \tag{A.5}$$

Now, the requirement that this generates an isometry, is to say that $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$ by definition 2, and we must then have that the order ϵ term must be zero, which gives us the equation

$$g_{\sigma\nu}\partial_\mu K^\sigma + g_{\nu\rho}\partial_\mu K^\rho + K^\lambda\partial_\lambda g_{\mu\nu} = 0\,, \tag{A.6}$$

where all partial derivatives are wrt. the coordinate x. We want (A.6) to be equivalent to the Killing equation $\nabla_{(\mu}K_{\nu)} = 0$, which we will show that it is. If we take a look at $\nabla_\mu K_\nu$ and expand it using the definition of the Christoffel symbol

$$\begin{aligned} \nabla_\mu K_\nu &= g_{\nu\lambda}\nabla_\mu K^\lambda \\ &= g_{\nu\lambda}\left(\partial_\mu K^\lambda + \Gamma^\lambda_{\mu\kappa}K^\kappa\right) \\ &= g_{\nu\lambda}\left(\partial_\mu K^\lambda + \frac{1}{2}g^{\lambda\varphi}\left(\partial_\mu g_{\kappa\varphi} + \partial_\kappa g_{\mu\varphi} - \partial_\varphi g_{\mu\kappa}\right)K^\kappa\right) \end{aligned}$$

$$\begin{aligned} &= g_{\nu\lambda}\partial_\mu K^\lambda + \frac{1}{2}\delta^\varphi_\nu\left(\partial_\mu g_{\kappa\varphi} + \partial_\kappa g_{\mu\varphi} - \partial_\varphi g_{\mu\kappa}\right)K^\kappa \\ &= g_{\nu\lambda}\partial_\mu K^\lambda + \frac{1}{2}\left(\partial_\mu g_{\kappa\nu} - \partial_\nu g_{\mu\kappa} + \partial_\kappa g_{\mu\nu}\right)K^\kappa \\ &= g_{\nu\lambda}\partial_\mu K^\lambda + \left(\partial_{[\mu}g_{|\kappa|\nu]} + \frac{1}{2}\partial_\kappa g_{\mu\nu}\right)K^\kappa \end{aligned}$$

Symmetrizing this, we find

$$
\begin{aligned}
\nabla_{(\mu}K_{\nu)} &= \frac{1}{2}\left(\nabla_\mu K_\nu + \nabla_\nu K_\mu\right) \\
&= \frac{1}{2}\left(g_{\nu\lambda}\partial_\mu K^\lambda + \left(\partial_{[\mu}g_{|\kappa|\nu]} + \frac{1}{2}\partial_\kappa g_{\mu\nu}\right)K^\kappa + g_{\mu\lambda}\partial_\nu K^\lambda + \left(\partial_{[\nu}g_{|\kappa|\mu]} + \frac{1}{2}\partial_\kappa g_{\nu\mu}\right)K^\kappa\right) \\
&= \frac{1}{2}\left(g_{\nu\lambda}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda + \frac{1}{2}\left(\partial_\kappa g_{\mu\nu} + \partial_\kappa g_{\nu\mu}\right)K^\kappa\right) \\
&= \frac{1}{2}\left(g_{\nu\lambda}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda + K^\kappa\partial_\kappa g_{\mu\nu}\right) \\
&= \frac{1}{2}(0) \\
&= 0\,,
\end{aligned}
$$

where we used (A.6) in the third last line to connect the Killing equation to isometries, as we wanted to.

B Mechanics on general manifolds

B.1 Lagrangian formalism

Say that we have some theory defined on our spacetime by some action S, a functional of curves. The curves we consider all maps into n copies of the manifold $M^n \equiv \bigcup_{i=1}^n M_i$, each corresponding to the position of a given particle, n in total. We think of M^n as the configuration space of Lagrangian mechanics. Hence such a curve γ may be defined as $\gamma : I \subseteq \mathbb{R} \to M^n$, and we can take them to be as differentiable as we wish. To give a proper definition of the Lagrangian function L, we first define $T^n(M) \equiv \bigcup_{i=1}^n T(M_i)$, where $T(M_i)$ is the tangent bundle of the i'th copy of the manifold.

We then consider the action defined as an ordinary integral over some function $L : T^n(M) \to \mathbb{R}$, the Lagrangian, of differentiable curves $\gamma : I \subseteq \mathbb{R} \to M^n$ and the tangent vectors to the curve $\dot{\gamma} : I \subseteq \mathbb{R} \to T^n$, where $T^n \equiv \bigcup_{i=1}^n T_i$ is the corresponding n tangent spaces. Then $\tilde{\gamma} \equiv (\gamma, \dot{\gamma}) : I \subseteq \mathbb{R} \to T^n(M)$, the lift of the curve γ, traces out a curve in phase space, which may then be composed with the Lagrangian. We may then write

$$
\begin{aligned}
S[\gamma] &= \int_I L(\tilde{\gamma}(\tau))\,\mathrm{d}\tau = \int_I L(\gamma(\tau), \dot{\gamma}(\tau))\,\mathrm{d}\tau \\
&= \int_I L(x_1(\tau), \ldots, x_n(\tau), \dot{x}_1(\tau), \ldots \dot{x}_n(\tau))\,\mathrm{d}\tau\,, \qquad \text{(B.1)}
\end{aligned}
$$

where we in the last equality have inserted a chart mapping from the tangent bundle to parametrized local coordinates $x_i^\mu(\tau)$ and directional derivatives ("velocities") $\dot{x}_i^\mu(\tau) \equiv \frac{\mathrm{d}x_i^\mu(\tau)}{\mathrm{d}\tau}$ of the i'th particle. The classical equations of motion is given by stationary points of the action functional, and doing a variation of the integral, we obtain the n Euler-Lagrange equations

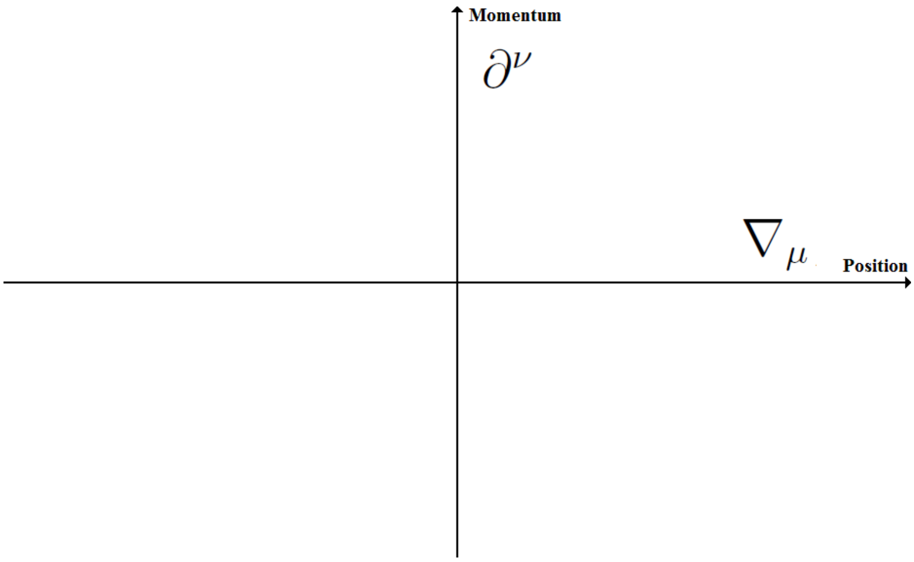

Figure 1: Phase space illustration, vertical and horizontal directions

$$\frac{\partial L\left(x_i,\dot{x}_i\right)}{\partial x_i^\mu}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L\left(x_i,\dot{x}_i\right)}{\partial \dot{x}_i^\mu}=0\,. \tag{B.2}$$

B.2 Hamiltonian formalism

The Euler-Lagrange equations are all second-order differential equations. It is more neat to have first order equations, and to do this we would have to rewrite the Euler-Lagrange equations as two (coupled) first-order equations. This is the Hamiltonian formalism, and here we will take take position x_i^μ and covariant momentum $p_{i\mu}$ as independent variables on the cotangent bundle $T^{*n}\left(M\right)\equiv\bigcup_{i=1}^{n}T^{*}\left(M_i\right)$, as on M_i we have that $\omega_{M_i}=p_{i\nu}\mathbf{d}x_i^\nu$ is an arbitrary one-form[22], $\left(x_i^\nu,p_{i\nu}\right)$ are coordinates on $T^{*n}\left(M\right)$. We should think of $\Gamma\equiv T^{*n}\left(M\right)$ as the diffeomorphic invariant phase space of n particles, consisting of n sets of points and momenta of the manifold. Γ is itself a manifold of dimension $\dim\left(\Gamma\right)=2nD$ that has some nice properties as we shall see. We now need to define equations om Γ that will give the same equations of motion as (B.2).

Functions $F:\Gamma\to\mathbb{R}$ are called observables, and we may write then as $F(x_i,p_i)$. We would like to have some kind of covariant derivative on Γ. As we have the covariant derivative ∇_i defined on M_i, we can extend it to Γ as a directional derivative along the x_i^μ coordinates. Given a vector $X_i^\mu\in T(M_i)$, we have that it defines a one-parameter family of flow by the equation $\frac{\mathrm{d}x_i^\mu(\lambda)}{\mathrm{d}\lambda}=X_i^\mu\left(x_i\left(\lambda\right)\right)$, where $x_i^\mu\left(\lambda=0\right)=x_i^\mu$. For this flow, $p_{i\mu}\in T^*(M_i)$ should be "kept constant", so it must be parallel transported by the equation $\frac{\mathrm{D}p_{i\mu}(\lambda)}{\mathrm{d}\lambda}=\frac{\mathrm{d}x_i^\nu(\lambda)}{\mathrm{d}\lambda}\nabla_{i\nu}p_{i\mu}\left(\lambda\right)=0$, with $p_{i\mu}\left(\lambda=0\right)=p_{i\mu}$. The directional derivative along X_i^μ of F (which we will call the horizontal direction) is defined as

$$\left[X^\mu\nabla_{i\mu}\right]F(x_i,p_i)\equiv\left.\frac{\mathrm{d}F(x_i\left(\lambda\right),p_i\left(\lambda\right))}{\mathrm{d}\lambda}\right|_{\lambda=0}\,. \tag{B.3}$$

[22]In phase space itself, this is called the canonical one-form.

For the momentum direction, we define for a 1-form at $x_i \in M_i$, $\omega_{i\nu} \in T^*_{x_i}(M_i)$, that the vertical derivative along $\omega_{i\nu}$, $\omega_{i\nu}\partial_i^\nu$ can be defined by the action on F as

$$[\omega_{i\nu}\partial_i^\nu] \, F(x_i, p_i) \equiv \left. \frac{\mathrm{d}F(x_i, p_i + \lambda\omega_i)}{\mathrm{d}\lambda} \right|_{\lambda=0} . \tag{B.4}$$

As $T^*_x(M_i)$ is a linear vector space of dimension D, we don't need to define any action on x_i. Let us now drop the particle index i for notational convenience; it can be thought of a part of μ if one likes, with nD coordinates. Without the directions X^μ, ω_ν, the horizontal and vertical derivatives may be written as $\nabla_\mu F$ and $\partial^\nu F$ in the meaning that they have the action of (B.3) and (B.4) in each coordinate and 1-form component.

For a general observable F, we may expand it as a power-series in p_μ as

$$F(x,p) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\mu_1 \cdots \mu_n}(x) \, p_{\mu_1} \cdots p_{\mu_n} , \tag{B.5}$$

where $f^{\mu_1 \cdots \mu_n}(x)$ are fully symmetric and only depends on x^μ. In this form, the action of the horizontal derivative are easy, as we simply have

$$\nabla_\mu F(x,p) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\nabla_\mu f^{\mu_1 \cdots \mu_n}(x)\right) p_{\mu_1} \cdots p_{\mu_n} , \tag{B.6}$$

because the $\nabla_\mu p_{\mu_i}$'s are covariantly constant when parallel transported along μ of M. Likewise, for the vertical derivative, we find using the product rule that

$$\begin{aligned} F(x,p) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{\mu_1 \cdots \mu_n}(x) \, \partial^\nu \left(p_{\mu_1} \cdots p_{\mu_n}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{\nu\mu_2 \cdots \mu_n}(x_i) \left(1 \cdot p_{\mu_2} \cdots p_{\mu_n}\right) + \sum_{n=0}^{\infty} \frac{1}{n!} f^{\mu_1 \nu \cdots \mu_n}(x) \left(p_{\mu_1} p_{\mu_3} \cdots p_{\mu_n}\right) + \end{aligned} \tag{B.7}$$

We can use the coordinates $\mathsf{x}^i \equiv \{x^\mu, p_\nu\}$ to form a 1-form basis for covectors of $T^*(\Gamma)$, by defining the basis elements as $\mathbf{d}\mathsf{x}^i \equiv \{\mathbf{d}x^\mu, \mathbf{d}p_\nu\}$, and we may likewise also define a basis for vector fields of $T(\Gamma)$ as $\eth_i \equiv \left\{\frac{\partial}{\partial x^\mu} \equiv \boldsymbol{\partial}_{x^\mu}, \frac{\partial}{\partial p_\nu} \equiv \boldsymbol{\partial}_{p_\nu}\right\}$. A general vector $X \in T(\Gamma)$ can then be written in component as

$$X = X^i \eth_i = (\boldsymbol{X}_M)^\mu \, \boldsymbol{\partial}_{x^\mu} + (\boldsymbol{X}_{T^*M})_\nu \, \boldsymbol{\partial}_{p_\nu} , \tag{B.8}$$

where $\boldsymbol{X}_M$ is a vector on M, the horizontal part, and $\boldsymbol{X}_{T^*M}$ is a covector on T^*M, the vertical part. A general covector field $\boldsymbol{\omega} \in T^*(\Gamma)$ with components ω_i can be written as

$$\boldsymbol{\omega} = \omega_i \mathbf{d}\mathsf{x}^i = (\boldsymbol{\omega}_M)_\mu \, \mathrm{d}x^\mu + (\boldsymbol{\omega}_{T^*M})^\nu \, \mathrm{d}p_\nu , \tag{B.9}$$

where $\boldsymbol{\omega}_M$ is a covector on M, and $\boldsymbol{X}_{T^*M}$ is a vector on T^*M (and a covector on M). As we see, this defines natural mappings from $T(\Gamma)$ to $T(M)$ and from $T^*(\Gamma)$ to $T^*(M)$.

The covariant derivative one-form operator on Γ can then be defined with components $\boldsymbol{\nabla}_i = \{\nabla_\mu, \partial^\nu\}$, and we can write

$$\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_i \mathbf{d}\mathsf{x}^i = \nabla_\mu \boldsymbol{d}x^\mu + \partial^\nu \mathbf{d}p_\nu \,. \tag{B.10}$$

We can define a 2-form on Γ, called the symplectic structure $\boldsymbol{\Omega}$, by

$$\boldsymbol{\Omega} \equiv \mathbf{d}x^\mu \wedge \mathbf{d}p_\mu \,, \tag{B.11}$$

which is a scalar on M. The symplectic structure is obviously closed, and it is invariant under a change of coordinates, because $\mathbf{d}x^\mu$, $\mathbf{d}p_\mu$ transforms oppositely on M. Because of this structure, we say that Γ is a symplectic manifold. Acting on vector fields $A^{,i}\eth_i = \frac{\partial A}{\partial x_\mu}\boldsymbol{\partial}_{x^\mu} + \frac{\partial A}{\partial p^\nu}\boldsymbol{\partial}_{p_\nu}$ and $B^{,i}\eth_i = \frac{\partial B}{\partial x_\mu}\boldsymbol{\partial}_{x^\mu} + \frac{\partial B}{\partial p^\nu}\boldsymbol{\partial}_{p_\nu}$, where A, B are observables, we find that

$$\begin{aligned}
\Omega\left(A_{,i}\mathbf{d}\mathsf{x}^i, B_{,i}\mathbf{d}\mathsf{x}^i\right) &= \mathbf{d}x^\mu\left(A^{,i}\eth_i\right) \wedge \mathbf{d}p_\mu\left(B^{,i}\eth_i\right) \\
&= \mathbf{d}x^\mu\left(\frac{\partial A}{\partial x_\rho}\boldsymbol{\partial}_{x^\rho}\right)\mathbf{d}p_\mu\left(\frac{\partial B}{\partial p^\nu}\boldsymbol{\partial}_{p_\nu}\right) - \mathbf{d}p_\mu\left(\frac{\partial A}{\partial p^\nu}\boldsymbol{\partial}_{p_\nu}\right)\mathbf{d}x^\mu\left(\frac{\partial B}{\partial x_\rho}\boldsymbol{\partial}_{x^\rho}\right) \\
&= \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x^\mu} \\
&= \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B \,.
\end{aligned} \tag{B.12}$$

We can define now the Poisson bracket $\{A, B\}$ of two observables A, B from the symplectic structure directly as

$$\{A, B\} \equiv \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B \,. \tag{B.13}$$

We can show that the Poisson bracket has a number of nice properties:

Theorem 37 (Properties of the Poisson bracket)**. (1):** $\{A + B, C\} = \{A, C\} + \{B, C\}$ *and* $\{\alpha A, B\} = \alpha\{A, B\}$, $\alpha \in \mathbb{R}$ *(Linearity).* **(2):** $\{A, B\} = -\{B, A\}$ *(Antisymmetry).* **(3):** $\{AB, C\} = A\{B, C\} + \{A, C\}B$ *(It is a derivation).* **(4):** $\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0$ *(Jacobi identity), all for arbitrary observables* A, B, C. **(5):** *The Poisson bracket is invariant under a change of coordinates.*

Proof. **(1):** This is easily proven using the linearity of ∇_μ, ∂^ν.

(2): Also easily proven:

$$\{A, B\} = \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B = -\left(\nabla_\mu B \partial^\mu A - \partial^\mu B \nabla_\mu A\right) = -\{B, A\}$$

(3): We find by using the product rule of covariant differentiation that

$$\begin{aligned}
\{AB, C\} &= \nabla_\mu (AB)\, \partial^\mu C - \partial^\mu (AB)\, \nabla_\mu C \\
&= \left(\nabla_\mu A \partial^\mu C\right) B - \left(\partial^\mu A \nabla_\mu C\right) B + A\left(\nabla_\mu B \partial^\mu C\right) - A\left(\partial^\mu B \nabla_\mu C\right) \\
&= A\{B, C\} + \{A, C\} B \,.
\end{aligned}$$

(4): This can be done explicitly by writing out all terms, but one can also just note that this follows because of associativity of compositions of observables.

(5): This follows directly from that Ω is a scalar on M. □

In other words, we have proven that the Poisson bracket on Γ is a Lie algebra, called the Poisson algebra. Let us further show that it doesn't matter which connection we use for the covariant derivative, as long as it is torsion-free:

Lemma 38 (Independence of connection)**.** *Let $\tilde{\nabla}_\mu$ and ∇_μ be two covariant derivatives with arbitrary torsion-free connections, and let the tensor $C^\rho{}_{\mu\nu}$ be the difference between the connections, i.e. $\tilde{\Gamma}^\rho_{\mu\nu} = C^\rho{}_{\mu\nu} + \Gamma^\rho_{\mu\nu}$. The Poisson bracket takes the same value with either one.*

Proof. Let $\widetilde{\{A,B\}}$ be the value using $\tilde{\nabla}_\mu$, and $\{A,B\}$ the value using ∇_μ. A direct calculation of $\tilde{\nabla}_\mu A$ using the expansion (B.5) gives

$$\begin{aligned}
\tilde{\nabla}_\mu A &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tilde{\nabla}_\mu a^{\mu_1\cdots\mu_n}\right) p_{\mu_1}\cdots p_{\mu_n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\nabla_\mu a^{\mu_1\cdots\mu_n} + C^{\mu_1}_{\mu\nu} a^{\nu\mu_2\cdots\mu_n} + C^{\mu_2}_{\mu\nu} a^{\mu_1\nu\cdots\mu_n} + \dots\right) p_{\mu_1}\cdots p_{\mu_n} \\
&= \nabla_\mu A + \sum_{n=0}^{\infty} \frac{1}{n!} \left(C^{\mu_1}_{\mu\nu} a^{\nu\mu_2\cdots\mu_n} + C^{\mu_2}_{\mu\nu} a^{\mu_1\nu\cdots\mu_n} + \dots\right) p_{\mu_1}\cdots p_{\mu_n} \\
&= \nabla_\mu A + p_\lambda C^\lambda{}_{\mu\nu} \partial^\nu \sum_{n=0}^{\infty} \frac{1}{n!} a^{\mu_1\cdots\mu_n} p_{\mu_1}\cdots p_{\mu_n} \\
&= \nabla_\mu A + p_\lambda C^\lambda{}_{\mu\nu} \partial^\nu A\,.
\end{aligned}$$

Using this we can do a calculation of $\widetilde{\{A,B\}}$:

$$\begin{aligned}
\widetilde{\{A,B\}} &= \tilde{\nabla}_\mu A \partial^\mu B - \partial^\mu A \tilde{\nabla}_\mu B \\
&= \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B + p_\lambda C^\lambda{}_{\mu\nu} \partial^\nu A \partial^\mu B - \partial^\mu A p_\lambda C^\lambda_{\mu\nu} B \\
(*) \quad &= \nabla_\mu A \partial^\mu B - \partial^\mu A \nabla_\mu B + p_\lambda C^\lambda{}_{\mu\nu} \left(\partial^\nu A \partial^\mu B - \partial^\mu A p_\lambda B\right) \\
&= \{A,B\}\ ,
\end{aligned}$$

where we in (*) used the assumption about torsion-free connections to factor out $C^\lambda{}_{\mu\nu}$, which is then symmetric in lower indices. ☐

Especially this also shows that the Poisson bracket has the same value in any conformal frame, as the extra term in the covariant derivative will vanish under the antisymmetric behavior of the Poisson bracket.

We will now proceed to formulate classical mechanics in the Hamiltonian formalism. Let us first define the Hamiltonian as an observable $H : \Gamma \to \mathbb{R}$, and claim that the equations of motion for a curve $\gamma : \mathbb{R} \to \Gamma$ parametrized by τ are given by Hamiltons equations

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \{x^\mu, H\} = \frac{\partial H}{\partial p_\mu} \quad , \quad \dot{p}_\mu \equiv \frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \{p_\mu, H\} = -\frac{\partial H}{\partial x^\mu}\,. \tag{B.14}$$

Theorem 39 (Hamilton)**.** *Let the Lagrangian L be given. Then one obtains the Hamiltonian H, which obeys Hamiltons equations and yields the same equations of motion, by a Legendre transformation.*

Proof. To see that these are equivalent to the Lagrangian formulation we will do a Legendre transformation and define

$$H(x,p) \equiv \sup_{\dot{x}} \left(p_\mu \dot{x}^\mu - L(x,\dot{x}) \right) , \tag{B.15}$$

where the supremum is found by solving $\frac{\partial}{\partial \dot{x}^\nu}\left(p_\mu \dot{x}^\mu - L(x,\dot{x})\right) = 0$, yielding

$$p_\mu = \frac{\partial L(x,\dot{x})}{\partial \dot{x}^\nu} . \tag{B.16}$$

The Euler-Lagrange equations can then be written as

$$\dot{p}_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L(x,\dot{x})}{\partial \dot{x}^\mu} = \frac{\partial L(x,\dot{x})}{\partial x^\mu} . \tag{B.17}$$

Taking the exterior derivative of H in coordinate expansion, we have that

$$\begin{aligned}
\mathrm{d}H &= \frac{\partial H}{\partial \mathsf{x}^i}\mathrm{d}\mathsf{x}^i \\
&= \frac{\partial H}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial H}{\partial p_\nu}\mathrm{d}p_\nu \\
(*) \quad &= -\dot{p}_\mu \mathrm{d}x^\mu + \dot{x}^\nu \mathrm{d}p_\nu \\
(**) \quad &= -\frac{\partial L(x,\dot{x})}{\partial x^\mu}\mathrm{d}x^\mu + \mathrm{d}\left(\dot{x}^\nu p_\nu\right) - p_\nu \mathrm{d}\dot{x}^\nu \\
(***) \quad &= \mathrm{d}\left(\dot{x}^\nu p_\nu\right) - \left[\frac{\partial L(x,\dot{x})}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial L(x,\dot{x})}{\partial \dot{x}^\nu}\mathrm{d}\dot{x}^\nu\right] \\
&= \mathrm{d}\left(\dot{x}^\nu p_\nu - L\right) ,
\end{aligned}$$

which shows that the two formalisms are equivalent, up to some constant of no importance. In (*) we invoked both of Hamiltons equations (B.14), in (**) we invoked the Euler-Lagrange equation (B.17) and used that $\mathrm{d}\left(\dot{x}^\nu p_\nu\right) - p_\nu \mathrm{d}\dot{x}^\nu = \dot{x}^\nu \mathrm{d}p_\nu$ using the product rule of the exterior derivative. Finally in (***) we invoked the supremum condition (B.16). □

As a consequence of this, we can always Legendre transform back to a Lagrangian, given a Hamiltonian.

C Classical field theory on general manifolds

Our results in the above may be generalized to general tensor fields $\psi_I : M \to T^p_q(M)$ defined on the manifold, where $T^p_q(M)$ is the (p,q) tensor bundle, and $I = \{\mu, \nu, \ldots\}$ is shorthand for all the tensor components and other indices. In this case we would consider theories defined by the Lagrangian density $\mathcal{L} : T^p_q(M) \to \mathbb{R}$, and the field action is given by composition of the lift of ψ_I and $\mathcal{L}$:

$$\begin{aligned} S[\psi_I] &\equiv \int_M \mathcal{L}(\psi_I, \nabla\psi_I, g) \\ &= \int_{\varphi(M)} \hat{\mathcal{L}}(\psi_I(x), \nabla\psi_I(x), g)\sqrt{|g|}\mathrm{d}^D x\,, \qquad \text{(C.1)} \\ &= \int_{\varphi(M)} \mathcal{L}(\psi_I(x), \nabla\psi_I(x), g)\,\mathrm{d}^D x \qquad \text{(C.2)} \end{aligned}$$

where we in the second line have inserted local coordinates. We have defined two lagrangians $\hat{\mathcal{L}}, \mathcal{L}$ here; the difference between them is that $\hat{\mathcal{L}}$ is a scalar, and $\mathcal{L}$ is a pseudo-scalar, that doesn't transform correctly, but with correct measure. If $\hat{\mathcal{L}} = \hat{\mathcal{L}}(\psi_I(x), \nabla\psi_I(x))$ is independent of the metric in the meaning that it is already defined on the manifold (ψ_I is decoupled from it), and we do variations that doesn't change the metric, the equations of motion is as always given by the Euler-Lagrange equations [21]:

$$\frac{\partial\hat{\mathcal{L}}}{\partial[\psi_I]} - \nabla_\mu \frac{\partial\hat{\mathcal{L}}}{\partial[\nabla_\mu\psi_I]} = 0\,. \qquad \text{(C.3)}$$

This derived is under the assumption that the variation at the boundary is set to zero. In general we may have that the field ψ_I would couple to the metric, as is for example the case for the Einstein-Hilbert action for general relativity, and in this case we should use $\mathcal{L} = \mathcal{L}(\psi_I(x), \nabla\psi_I(x), g)$ as our starting point. In this case the equations of motion are

$$\frac{\partial\mathcal{L}}{\partial[\psi_I]} - \nabla_\mu \frac{\partial\mathcal{L}}{\partial[\nabla_\mu\psi_I]} = 0\,, \qquad \text{(C.4)}$$

$$\frac{\partial\mathcal{L}}{\partial[g_{\rho\sigma}]} - \nabla_\mu \frac{\partial\mathcal{L}}{\partial[\nabla_\mu g_{\rho\sigma}]} = 0\,, \qquad \text{(C.5)}$$

as we would have to vary the metric itself as well, and everything that depends on the metric, including the Christoffel symbols.

We may also be able to define a Hamiltonian formalism and a Hamiltonian density. There might be problems with the interpretation and diffeomorphic invariance, as "time" and "space" would be treated differently in the most naive treatment, where we would simply do a Legendre transformation. There are cures for this, see for example [33], but this is well outside the main line of this project.

D Non-coordinate (vielbein) bases

Consider a differentiable manifold M of dimension D with signature $p+q$. Given vector field $X = X^\mu\partial_\mu$, we can do a change of basis from the coordinate basis to a special choice of vector fields $X = \tilde{X}^a\hat{\boldsymbol{n}}_a \equiv X^a\hat{\boldsymbol{n}}_a$, that satisfies that they span an pseudo-orthonormal basis, i.e. $g(\hat{\boldsymbol{n}}_a, \hat{\boldsymbol{n}}_b) = \eta_{ab}$, when we are working with a Lorentzian manifold M. The vectors $\{\hat{\boldsymbol{n}}_a\}$, called the vielbeins, are everywhere pseudo-orthonormal as defined, and the existence of such a basis for the vector bundle $T\left(M\right)$ comes from regular Linear algebra, because at each point $p \in M$ the tangent space $T_p\left(M\right)$ is a vector space of dimension $d = \dim M < \infty$, and so we can always choose a pseudo-orthonormal basis (we need a non-degenerate metric, to ensure this). We can define a pseudo-orthonormal basis $\left\{\hat{\boldsymbol{n}}^b\right\}$ for the covector bundle $T^*\left(M\right)$ by the requirement that $\hat{\boldsymbol{n}}^b\left(\hat{\boldsymbol{n}}_a\right) = \delta^b_a$, which also proves their existence as they are dual basis vectors.

The transformation between coordinate basis and the vielbein basis is given by linear transformation of the coordinate basis (because there is a general fiber bundle at point of the manifold, which is a vector space), at each point of the manifold, so we have

$$\hat{\boldsymbol{n}}_a = n^\mu{}_a \partial_\mu \,, \tag{D.1}$$

where $n^\mu{}_a = n^\mu{}_a\left(p\right)$, $p \in M$, of course is a invertible matrix at each point, because we are just doing a change of basis. The components then transforms as

$$X = X^a \hat{\boldsymbol{n}}_a = X^a n^\mu{}_a \partial_\mu \quad \Rightarrow$$

$$X^\mu = X^a n^\mu{}_a \quad \Leftrightarrow \quad X^a = n_\mu{}^a X^\mu \,, \tag{D.2}$$

where $n_\mu{}^a$ is the inverse of the matrix $n^\mu{}_a$, which then satisfies that

$$n_\mu{}^b n^\mu{}_a = \delta^b_a \quad , \quad n^\mu{}_a n_\nu{}^a = \delta^\mu_\nu \,. \tag{D.3}$$

The covectors $\hat{\boldsymbol{n}}^b$ will have to transform oppositely because they are defined by the dual basis requirement $\hat{\boldsymbol{n}}^b\left(\hat{\boldsymbol{n}}_a\right) = \delta^b_a$, so changing basis for the vielbeins

$$\delta^b_a = \hat{\boldsymbol{n}}^b\left(\hat{\boldsymbol{n}}_a\right) = \hat{\boldsymbol{n}}^b\left(n^\mu{}_a \partial_\mu\right) = n^\mu{}_a \hat{\boldsymbol{n}}^b\left(\partial_\mu\right) \equiv n^\mu{}_a M^b_\mu$$

$$\hat{\boldsymbol{n}}^b = \left(n^\mu{}_b\right)^{-1} \mathrm{d}x^\mu \equiv \left(n_\mu{}^b\right) \mathrm{d}x^\mu \,, \tag{D.4}$$

where we have defined $M^b_\mu \equiv \hat{\boldsymbol{n}}^b\left(\partial_\mu\right)$, because it is a linear transformation, but then we see that it is exactly the inverse matrix, and hence we get the wanted transformation law.

In general we cannot expect that a vielbein is a basis for all of the manifold, but we can make a smooth change of variables at each overlap, as we can always define them from a coordinate basis. Such a transformation between vielbeins is then called a local Lorentz transformation $\Lambda^a{}_{a'}\left(p\right)$, LLT, which depends on manifold, and we have

$$\hat{\boldsymbol{n}}_{a'} = \Lambda^a{}_{a'}\left(p\right) \hat{\boldsymbol{n}}_a \quad , \quad \hat{\boldsymbol{n}}^{b'} = \Lambda_b{}^{b'}\left(p\right) \hat{\boldsymbol{n}}^b \,, \tag{D.5}$$

where $\Lambda^{a'}_a\left(p\right)$ is the inverse of $\Lambda^a{}_{a'}\left(p\right)$. By definition they must leave the metric unchanged (they are symmetry transformations):

$$g = \eta_{ab} \hat{\boldsymbol{n}}^a \otimes \hat{\boldsymbol{n}}^b = \eta_{a'b'} \hat{\boldsymbol{n}}^{a'} \otimes \hat{\boldsymbol{n}}^{b'} = \eta_{a'b'} \Lambda^{a'}_a\left(p\right) \hat{\boldsymbol{n}}^a \otimes \Lambda^{b'}_b\left(p\right) \hat{\boldsymbol{n}}^b = \Lambda^{a'}_a\left(p\right) \Lambda^{b'}_b\left(p\right) \eta_{a'b'} \hat{\boldsymbol{n}}^a \otimes \hat{\boldsymbol{n}}^b \Rightarrow$$

$$\eta_{a'b'} = \Lambda^{a}{}_{a'}\left(p\right)\Lambda^{b}{}_{b'}\left(p\right)\eta_{ab}\,. \tag{D.6}$$

The number of isometries at each point is then 6 in $1+3$ dimensions (3 boosts and 3 rotations), as these will leave the metric unchanged. In general we have that the isometry group in D dimensions is locally SO (p,q). We can rise and lower indices with η_{ab} and $g_{\mu\nu}$ and the inverse metrics, because this is the same in any basis. We can write a general vector $\boldsymbol{\xi} = \xi^{\mu}\partial_{\mu}$ in the vielbein basis as

$$\boldsymbol{\xi} = \sum_{i=1}^{D}\left(g_{\nu\mu}\hat{n}^{\nu}{}_{i}\cdot\xi^{\mu}\right)\hat{\boldsymbol{n}}_{i} = \sum_{i=1}^{D}\left(\hat{\boldsymbol{n}}_{i}\cdot\boldsymbol{\xi}\right)\hat{\boldsymbol{n}}_{i} \tag{D.7}$$

In general we can take a tensor in a mixed basis, and we can still do contractions, rising and lowering of indices, and transform vielbeins by LLT and coordinate basis transformations by a general coordinate transformation GCT. The components of the tensor $A = A^{a\mu}{}_{b\nu}\hat{\boldsymbol{n}}_{a}\otimes\partial_{\mu}\otimes\hat{\boldsymbol{n}}^{b}\otimes \mathrm{d}x^{\nu} = A^{a'\mu'}{}_{b'\nu'}\hat{\boldsymbol{n}}_{a'}\otimes\partial_{\mu'}\otimes\hat{\boldsymbol{n}}^{b'}\otimes\mathrm{d}x^{\nu'}$ is then easily seen to transform as

$$A^{b'\mu'}{}_{a'\nu'} = \Lambda^{a'}{}_{a}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\Lambda^{b}{}_{b'}\frac{\partial x^{\nu}}{\partial x^{\nu'}}A^{a\mu}{}_{b\nu}\,. \tag{D.8}$$

A special tensor is the $(1,1)$ tensor

$$e = n_{\mu}{}^{a}\mathrm{d}x^{\mu}\otimes\hat{\boldsymbol{n}}_{a}\,. \tag{D.9}$$

This is actually the the identity map, because it just takes a vector in the coordinate basis and changes it to the vielbein basis, or takes a covector in vielbein basis and transforms it to the coordinate basis - the components change, but the tensors doesn't. Here we see the power of the formalism, namely that the metric is always simple and constant. What changes then is the covariant derivative and other kinds of differentiation. We know how to take covariant derivatives of tensors in coordinate basis, and this we can use to derive the transformation properties for vielbeins. First we notice, that if we take the covariant derivative ∇_{μ} of a vector X^{a}_{b}, it needs to be something of a partial derivative along with linear transformation $\chi_{\mu}{}^{a}{}_{b}$, the spin connection, terms to agree with the defining axioms of the covariant derivative . We can find that for a $(1,1)$ tensor we have that the covariant derivative has components given by [7]:

$$\nabla_{\mu}X^{a}_{b} = \partial_{\mu}X^{a}_{b} + \chi_{\mu}{}^{a}{}_{c}X^{c}_{b} - \chi_{\mu}{}^{c}{}_{b}X^{a}_{c}\,, \tag{D.10}$$

where the covariant index must transform with a same transformation except for a minus to ensure that the covariant derivative of a scalar contracted from vielbeins is just the partial derivative. If we now take a covariant derivative of the vector X^{a} and transform back to coordinate basis, where we know what is going on, we have that we can find an expression for the spin connection $\chi_{\mu}{}^{a}{}_{b}$ in terms of the connection $\Gamma^{\nu}_{\mu\lambda}$ (which we don't assume is metric compatible or torsion free):

$$\begin{aligned}\nabla_\mu X^a &= \partial_\mu X^a + \chi_\mu{}^a{}_b X^b \\ &= \partial_\mu \left(n_\nu{}^a X^\nu\right) + \chi_\mu{}^a{}_b n_\nu{}^b X^\nu \\ &= \left(\partial_\mu X^\nu\right) n_\nu{}^a + X^\nu \left(\partial_\mu n_\nu{}^a\right) + \chi_\mu{}^a{}_b n_\nu{}^b X^\nu \end{aligned} \tag{D.11}$$

Comparing with $n_\nu{}^a \nabla_\mu X^\nu = n_\nu{}^a \partial_\mu X^\nu + n_\nu{}^a \Gamma^\nu_{\mu\lambda} X^\lambda$, which is the components of the same tensor ∇X, we find that

$$\Gamma^\nu_{\mu\lambda} = n^\nu{}_a \partial_\mu n_\lambda{}^a + n^\nu{}_a n_\lambda{}^b \chi_\mu{}^a{}_b \,, \tag{D.12}$$

which can be inverted to yield the spin connection

$$\chi_\mu{}^a{}_b = n_\nu{}^a n^\lambda{}_b \Gamma^\nu_{\mu\lambda} - n^\lambda{}_b \partial_\mu n_\lambda{}^a \,. \tag{D.13}$$

The spin connection $\chi_\mu{}^a{}_b$ is not a tensor, but it does transform as tensor in the coordinate basis index μ. Under the vielbein indices it transforms non-tensorially as

$$\chi_\mu{}^{a'}{}_{b'} = \Lambda^{a'}_{\ a} \Lambda^b_{\ b'} \chi_\mu{}^a{}_b - \Lambda^c_{\ b'} \partial_\mu \Lambda^{a'}_{\ c} \,. \tag{D.14}$$

The covariant derivative of $e = n_\mu{}^a \mathrm{d}x^\mu \otimes \hat{\boldsymbol{n}}_a$ is of course zero, as this is just the identity map, which we can verify directly using the above, which gives us

$$\nabla e = \nabla \left(n_\mu{}^a \mathrm{d}x^\mu \otimes \hat{\boldsymbol{n}}_a\right) = \left(\nabla_\nu n_\mu{}^a\right) \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \otimes \hat{\boldsymbol{n}}_a = 0 \ \Rightarrow$$

$$\nabla_\nu n_\mu{}^a = 0 \,. \tag{D.15}$$

This is also called the tetrad postulate. The covariant derivative along a vector $\hat{\boldsymbol{n}}_b$ (the b-direction in the vielbein basis) can be written as

$$\nabla_b \equiv \nabla_{\hat{\boldsymbol{n}}_b} \equiv n_b{}^\mu \nabla_\mu \,, \tag{D.16}$$

and using (D.11), we may find that the covariant derivative of a vector X^a can be written as

$$\begin{aligned}\nabla_b X^a &= n_b{}^\mu \nabla_\mu X^a \\ &= n_b{}^\mu \left(\partial_\mu X^\nu\right) n_\nu{}^a + n_b{}^\mu X^\nu \left(\partial_\mu n_\nu{}^a\right) + n_b{}^\mu \chi_\mu{}^a{}_c n_\nu{}^c X^\nu \,.\end{aligned}$$

We can also find that as a special case, the covariant derivative of one of the basis vectors is

$$\nabla_b \hat{\boldsymbol{n}}_a = \chi^c{}_{ba} \hat{\boldsymbol{n}}_c \,. \tag{D.17}$$

We can view tensors with vielbein indices as each of these indices takes a vector or covector as an input. This view is very convenient. This Cartan formalism allows us to reformulate general relativity in the vielbein formalism, which is computationally simpler.

We will however not pursue this further.

E Useful theorems and identities

We use a number of identities to simplify the calculations, which can be found in [35].

First a identity for contracted products of Levi-Civita tensors in D dimensions:

$$\epsilon^{i_1\cdots i_r n_{r+1}\cdots n_D}\epsilon_{j_1\cdots j_r n_{r+1}\cdots n_D} = r!\,(D-r)!\delta^{[i_1}_{j_1}\cdots\delta^{i_r]}_{j_r} \tag{E.1}$$

Secondly, from [29] we have

$$(r+1)\,\delta^{[i}_{[j}\delta^{i_1}_{j_1}\cdots\delta^{i_r]}_{j_r]} = \delta^{i}_{j}\delta^{[i_1}_{[j_1}\cdots\delta^{i_r]}_{j_r]} - r\delta^{i}_{[j_1}\delta^{[i_1}_{|j|}\cdots\delta^{i_r]}_{j_r]} \tag{E.2}$$

We have also used Cartan's "magic formula" several times, which gives us a simple expression for the Lie derivative of forms:

Theorem 40 (Cartan's identity)**.** *For a vector $\boldsymbol{X}$ and a n-form $\boldsymbol{k}$, we have that*

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{k} = \boldsymbol{X}\cdot\mathbf{d}\boldsymbol{k} + \mathbf{d}\left(\boldsymbol{X}\cdot\boldsymbol{k}\right)\,, \tag{E.3}$$

where by "$\boldsymbol{a}\cdot\boldsymbol{b}$" we mean the interior product, i.e. the contraction of $\boldsymbol{b}$ with $\boldsymbol{a}$ in first variable.

Another useful relation is the interior product formula of wedge products of a p-form $\boldsymbol{w}$ and a q-form $\boldsymbol{v}$:

$$\boldsymbol{\xi}\cdot(\boldsymbol{w}\wedge\boldsymbol{v}) = (\boldsymbol{\xi}\cdot\boldsymbol{w})\wedge\boldsymbol{v} + (-1)^p\,\boldsymbol{w}\wedge(\boldsymbol{\xi}\cdot\boldsymbol{v})\ . \tag{E.4}$$

E.1 Second order covariant derivatives for KVs and KYTs

For a Killing vector ξ_ρ we have [26]:

$$\begin{aligned} 2\nabla_\mu\nabla_\lambda\xi_\rho &= -R^\sigma{}_{\rho\mu\lambda}\xi_\sigma + R^\sigma{}_{\rho\lambda\mu}\xi_\sigma \\ &= -2R^\sigma{}_{\rho\mu\lambda}\xi_\sigma\,. \end{aligned} \tag{E.5}$$

For a rank-2 CCKYT $h_{\mu\nu}$ we have [29]:

$$2\nabla_\mu\nabla_\lambda h_{\rho\nu} = 3R^\sigma{}_{\mu[\rho\lambda}h_{|\sigma|\nu]}\,. \tag{E.6}$$