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Master's thesis

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Explicit amplitudes for scattering of open strings on the worldvolume of a single D-brane

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Abstract

Motivated by recent progress in amplitude calculations in Yang-Mills and gravitational theories, the goal of this thesis is to calculate explicit open string scattering amplitudes in four and six dimensions from the Dirac-Born-Infeld action. This is an effective action describing electromagnetic fields on the worldvolume of a single D-brane. Scattering amplitudes are important in particle physics and string theory since they provide a direct connection between theory and experiment and because the scattering amplitude is really the measured physical quantity in detectors at particle accelerators as for instance the Large Hadron Collider (LHC) at CERN. This thesis introduces the basic concepts from string theory and the relevant methods from field theory used in scattering amplitude calculations are reviewed. Especially the spinor-helicity formalism is central and it is introduced in detail in order to streamline considerably the calculations in four dimensions. The simple structure in four dimensions invites an extension of studies into higher dimensions and the spinor-helicity formalism is also employed in a modified form in six dimensions in a search for an appropriate way of expressing six-dimensional amplitudes. The amplitudes are calculated for specific polarizations of external states with four-dimensional calculations of both four-point and six-point amplitudes and six-dimensional calculations of four-point amplitudes. A method of evaluating contractions of the electromagnetic field strength tensor in terms of traces of block matrices is developed in connection with the calculations in six dimensions and also a generic four-point amplitude is calculated in generality without specification of external polarization states and independent of the number of dimensions. In six dimensions, four-point and six-point pure scalar amplitudes are calculated and the cross section for scattering of four scalars is estimated for the current maximal energy at the LHC.

Table of contents

Introduction

Unification of theories has always been a fundamental concept in theoretical physics. When Einstein formulated the theory of general relativity in 1916 it was a successful generalization of his theory of special relativity of 1905 and he kept on pursuing an even more general theory in which he could formulate equations describing electromagnetism as well as gravity. Einstein did not succeed but his strive for unification is characteristic for a physicist's way of thinking. This is exemplified in modern high energy physics where unification plays a central role for the understanding of the fundamental particles and interactions observed in nature. Four fundamental interactions are observed with a huge difference in strength spanning over 39 orders of magnitude. One intriguing problem is to understand why the strengths of the fundamental interactions are so different. The difference in strengths can be illustrated by the following example where the electromagnetic interaction is compared to gravity. One can rub a comb on a piece of clothing to add static electricity to the comb surface. The comb will then be able to lift a piece of paper due to the net difference in charge between the comb and the paper and this shows how the electromagnetic attraction due to the small net charge difference easily overcomes the gravitational interaction between the entire earth and the paper.

The relative weakness of gravity is reflected in the standard model of particle physics which unifies all fundamental interactions except gravity. It is believed that both gravitational and quantum effects are important at scales near the Planck energy $E_{\mathrm{P}} \sim 10^{19}$ Gev, determined uniquely by combining the fundamental physical constants $\hbar$, G, and c, associated respectively with quantum phenomena and gravity. Although the standard model provides a correct description of nature at more moderate energies it is not a correct theory near the Planck energy since it does not incorporate gravity. The standard model is therefore an example of an effective theory which is defined in general as a theory with a validity only within some certain energy scale. This is manifest in many branches of physics where the physical understanding of different energy regimes is based on effective theories. Although a full description valid for any energy scale might not exist, the different energy regimes can be very well understood individually in the frame of effective theories. This thesis is no exception from the comprehensive use of effective theories since it is written in the framework of an effective theory.

Effective theories can be studied from scattering amplitudes which are directly connected to experiments performed at particle accelerators. The connection between theory and experiment goes through the scattering cross section and this makes scattering amplitudes naturally interesting. This thesis is motivated by the recent progress in the field of scattering amplitude calculations in Yang-Mills and gravitational theories where developments have resulted in new technology. The results presented in [1] are part of these interesting developments and this reference contains among other things calculations of string theory scattering amplitudes in four-dimensions. These amplitudes are calculated from the Dirac-Born-Infeld action which is a famous effective action describing gauge fields on D-branes in the low-energy limit of superstring theory. The calculations are performed with use of the spinor-helicity formalism which in this context appears to be the right language for four-dimensional calculations since the results for the amplitudes and the calculations themselves simplify considerably due to the formalism. Just as the spinor-helicity formalism seems to be the right language for four-dimensional calculations, another motivation for this thesis is to study a possible similar language in six dimensions in which calculations and results are more streamlined. In order to look for traces of the simplifications in four dimensions it is therefore interesting to study scattering amplitudes in higher dimensions using a modified form of the spinor-helicity formalism. This is addressed among other things in this thesis where the goal is to extend amplitude calculations from four dimensions and to perform amplitude calculations in six dimensions which have have not previously been done.

The thesis is structured as follows. Chapter 1 contains an introduction to the basic concepts of string theory with focus on quantization of the classical string. Also the concept of supersymmetry and the important objects in the open string sector known as D-branes are discussed. The use of effective theories is discussed along with an introduction to the Dirac-Born-Infeld effective action. In chapter 2 the useful spinor helicity formalism is introduced as a continuation of a more general discussion of spinors and representations of tensors. Also a review of the basic field theory methods used throughout the thesis is presented with focus on the use of functional methods. These methods and the observations from the first two chapters are put to use in chapter 3 where scattering amplitudes with specific configuration of external polarizations are calculated in four dimensions. These calculations are performed along the lines of [1]. Chapter 4 contains calculations of scattering amplitudes in six dimensions including calculations which have previously not been performed. These calculations are based on manipulations of the four-point scattering term in the Dirac-Born-Infeld Lagrangian and the developments in this context are also discussed in this chapter. Finally, chapter 5 contains a discussion of the results obtained in chapters 3 and 4 along with some general concluding remarks. Additional details of computations are found in the first appendix. The second appendix contains as a service to the reader the new results for the amplitudes which have not previously been calculated.

Chapter 1

String theory background

String theory is a huge subject in theoretical physics and a deep discussion of the field is beyond the scope of this thesis. This section presents therefore the concepts that are most important in order to place scattering amplitude calculations in a bigger picture of an effective theory emerging from string theory.

String theory is a candidate for a unified theory describing the fundamental interactions and elementary particles in nature. If string theory is a more fundamental theory than the standard model of particle physics, the results of the standard model must emerge somehow from it. An interesting aspect in this context arises in connection with the objects in string theory known as D-branes. The standard model is a quantum field theory of interacting gauge fields and it is found in string theory that D-branes have gauge fields living on them. First order corrections to these gauge fields are studied in this thesis where calculations of scattering amplitudes for photon scattering in the frame of an effective theory is the main tool.

1.1 Effective theories

The importance of effective theories was addressed in the introduction. An example of an effective theory is the Fermi theory for β decay which was developed in 1933 by Enrico Fermi as a theory of weak interactions. One problem at that time was to describe the observed continuous spectrum for the electron energy in a β-decay process and in 1930 Wolfgang Pauli suggested the existence of the neutrino in order to solve the problem. A typical β-decay process could then be

$$n \to p + e^- + \bar{\nu}_e \ , \tag{1.1.1}$$

according to the suggestion by Pauli. Fermi assumed in his formulation of the theory that the interaction responsible for this decay was pointlike. This provided a good qualitative description of the decay process and the theory had great phenomenological success. Early experiments for the Fermi theory carried out at energies below 10 Mev were consistent with the assumption of a pointlike interaction but a deeper structure has been revealed as experiments at larger energies have been carried out with newer generations of particle accelerators. Consequently it is known today that the interaction is not pointlike and that it takes place with an intermediate W^- boson decaying into the electron and the electron antineutrino. The W is a heavy particle with mass $m_W \sim 80$ GeV and it was not discovered until 1983. This example with the Fermi theory illustrates the typical use of an effective theory: It provides a good approximation at lower energies but becomes insufficient in order to describe the deeper structure at higher energies.

In the discussion of low-energy effective theories in particle physics it is natural to introduce the propagator which will be discussed in detail section 2.2. The propagator in a quantum field theory describes the amplitude for a particle to propagate in spacetime and it is of the form

$$\frac{1}{p^2 + m^2} \ , \tag{1.1.2}$$

where p is the momentum and m the particle rest mass. In the low-energy limit the momentum is negligible compared to the rest mass so that

$$\frac{p}{m} \ll 1 \ , \tag{1.1.3}$$

and hence (1.1.2) becomes

$$\frac{1}{m^2}\left[1 - \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right)\right] \ . \tag{1.1.4}$$

This can be understood heuristically from a comparison with the law of inertia in classical physics where a larger force is required in order to accelerate a heavier body. As for a massive particle in a quantum field theory, a sufficiently high energy is required in order to "move" the particle and make it take place in interactions. For sufficiently low energies the massive particles can therefore be ignored so that only massless particles are considered in interactions. In this sense one can say that high energy structure is hidden in the effective theory describing the low-energy limit. The effective action is defined in principle by the functional integral which will be discussed in detail in section 2.2.4. The ordinary action is formally replaced by the effective action

$$S_{\text{full}} = \int \mathrm{d}^n x \, \mathcal{L}_{\text{full}} \rightarrow S_{\text{eff}} = \int \mathrm{d}^n x \, \mathcal{L}_{\text{eff}} \ , \tag{1.1.5}$$

in which only variables important for the particular scale under study are taken into account. It should be noted that this procedure is only used in principle. For practical purposes, the appropriate effective action is constructed with desired dependence on relevant variables.

1.1.1 Physics at different energy scales

As discussed in the introduction, one problem in physics is to understand why the strengths of the four fundamental interactions are so different. This is known as the problem of separation of scales. A rough comparison for the coupling constants for the fundamental interactions is

$$\alpha_{\text{strong}} = 1 \ , \quad \alpha_{\text{electromagnetism}} \sim 10^{-2}, \quad \alpha_{\text{weak}} \sim 10^{-6} \ , \quad \alpha_{\text{gravity}} \sim 10^{-39} \ , \tag{1.1.6}$$

and it is apparent that gravity is by far the weakest of the interactions. The strong, the weak and the electromagnetic interaction tend to unify at energies $E_{\text{un}} \sim 10^{14}$ Gev. This discussion involves an interesting aspect of supersymmetry which is addressed in section 1.2.4. It should be noted that the comparison of the coupling constants for the fundamental interactions is more complicated than what is apparent from (1.1.6). This is because the interactions have different ranges so that a comparison of the coupling constants is only meaningful on a given energy. For (1.1.6) the scale is the mass $m_Z \sim 91$ Gev of the Z boson. This will not be discussed in further detail.

Gravity is so weak that it can be neglected in calculations in the standard model. However it is believed that gravity becomes important in physics at the Planck scale which is given by the Planck length defined uniquely in terms of the fundamental constants $\hbar$, G and c as

$$\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \sim 2 \times 10^{-33} \text{ cm} . \tag{1.1.7}$$

Because the Planck length is constructed from the fundamental units associated with quantum mechanics and gravity it is believed that both interactions are important at this scale. The Planck energy is defined uniquely as

$$E_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G}} = m_{\mathrm{P}} c^2 \sim 10^{19} \text{ Gev} , \tag{1.1.8}$$

and it is the natural energy scale associated with the Planck length. As both quantum phenomena and gravity are important at the Planck scale, a theory which unifies these two interactions is necessary in order to describe physics at this scale. One problem in the unification of gravity with the other three fundamental interactions is that general relativity is not power counting renormalizable because it has a dimensionfull coupling constant. This means that a theory providing a unified description of quantum mechanics and gravity cannot be constructed in the most straightforward way.

1.1.2 String theory and the standard model

The standard model of particle physics is the theory which accounts for the description of elementary particles and their interactions. The standard model is able to explain all experimentally observed particles and their interactions as an internally consistent theory of quantum fields. The theory is a non-Abelian gauge theory with gauge group $U(1) \times SU(2) \times SU(3)$ and it explains three of the four fundamental interactions; the electromagnetic interaction, the weak interaction and the strong interaction. The standard model has proven very successful and at the time of writing, no high energy particle experiment has yielded results in disagreement with the standard model. However, one piece is still missing in the unified picture as the standard model cannot include gravity which is described by the entirely different theory of general relativity. This is one reason why it is widely believed that the standard model is an effective theory describing the low-energy limit of a more fundamental theory which also includes gravity. An interesting aspect in this context is that string theory contains naturally a massless spin-2 particle which is the quantum of gravity, known as the graviton. It is the currently hypothetical carrier of the gravitational interaction just as the photon is the carrier of the electromagnetic interaction. String theory is therefore a candidate for such a more fundamental and unifying theory. It is formulated by the assumption that all particles and their physical properties are vibrations of very tiny physical strings and in this way, both the fermionic particles that make up matter and bosonic particles that transmit the fundamental interactions are unified. This property that everything is build into the theory from the beginning is the power of string theory. The simplicity is manifest by a comparison with the standard model which has almost 20 adjustable parameters that have to be determined from experiments in order to get a consistent theory.

1.2 Foundation of string theory

An interesting historical aspect is associated with string theory since it was not conceived as a theory of strings but emerged as a consequence of a postulate by Gabrielle Veneziano in 1968.

1.2.1 The Veneziano amplitude

In the late 1960's interactions of π-mesons were studied intensively along with scattering amplitudes. Today it is known that the π-meson is a bound state of a quark and an antiquark but at that time π-mesons were considered to be elementary particles just as the proton and the neutron were. For the study of scattering of four particles it is convenient to introduce

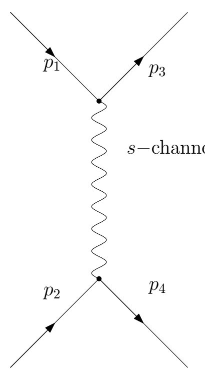

Figure 1.1: A scattering process through the s-channel.

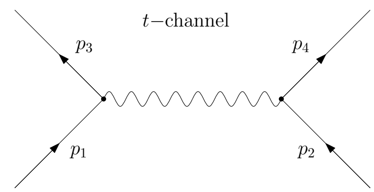

Figure 1.2: A scattering process through the t-channel.

the Mandelstam variables

$$s = (p_1 + p_2)^2 \ , \quad t = (p_1 + p_4)^2 \ , \quad u = (p_1 + p_3)^2 \ , \tag{1.2.1}$$

which for massless particles become

$$s = 2\, p_1 \cdot p_2 \ , \quad t = 2\, p_1 \cdot p_4 \ , \quad u = 2\, p_1 \cdot p_3 \ , \tag{1.2.2}$$

defined in terms of the four-momenta of the four interacting particles. The scattering of four particles can be described as taking place through the s, t or the u-channel as shown in figures 1.1 and 1.2 for the first two channels. The physical interpretation of particle scattering through the s-channel is different from that of interactions through the t-channel. However, the two interaction channels are just two ways of describing the same physical interaction where particles 1 and 2 interact and produce particle 3 and 4. The poles in the transition amplitude must therefore be the same regardless of channel. Veneziano searched for a mathematical formula which described this kind of interaction for π-mesons and he simply wrote down the amplitude

$$\mathcal{A}(p_1, p_2, p_3, p_4) = g_0^2 \int_0^1 \mathrm{d}x\, x^{2\alpha' p_1 \cdot p_2} \left(1 - x\right)^{2\alpha' p_2 \cdot p_3} \ , \tag{1.2.3}$$

as a postulate. The p_i's are the particle momenta. The Veneziano amplitude can as well be rewritten in terms of the β-function and the Γ-function as

$$\mathcal{A}(s,t) = g_0^2 \, \frac{\Gamma\left(-\alpha\left(s\right)\right)\Gamma\left(-\alpha\left(t\right)\right)}{\Gamma\left(-\alpha\left(s\right) - \alpha\left(t\right)\right)} = g_0^2 \ \beta\left(-\alpha(s)\, , -\alpha\left(t\right)\right) \ , \tag{1.2.4}$$

with

$$\alpha(s) \equiv \alpha' s + 1 \ , \tag{1.2.5}$$

and g_0 denoting the strength of the interaction. It follows from properties of the β-function that the amplitude can be expressed as

$$\begin{aligned} \mathcal{A}(s,t) &= -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)\,(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!}\frac{1}{\alpha(t)-n} \\ &= -\sum_{n=0}^{\infty} \frac{(\alpha(t)+1)\,(\alpha(t)+2)\cdots(\alpha(t)+n)}{n!}\frac{1}{\alpha(s)-n} \ , \end{aligned} \tag{1.2.6}$$

which can be found in [2]. The amplitude postulated by Veneziano resulted in much activity in the research field. This research culminated in the realization that elementary particles modeled as vibrational modes of one-dimensional strings instead of zero-dimensional particles are subject to interactions described exactly by the amplitude written down by Veneziano. The physical interpretation of the theory was due to Yoichiro Nambu, Holger Bech Nielsen and this realization was essentially the birth of string theory.

A pole in a scattering amplitude corresponds to the exchange of a physical particle. The poles in the Veneziano amplitude can be read of directly from (1.2.6) and it is apparent that the amplitude has the same poles in the s-channel as in the t-channel. Because of this symmetry where the two channels represent two ways of looking at the same scattering process, Veneziano's model became known as the dual resonance model.

1.2.2 Constructing the string action

The fundamental assumption in string theory is that the basic objects are tiny strings with a physical extension in one spatial dimension. This is very different from the notion in quantum field theory where particles are considered as zero-dimensional points. The fundamental scale in string theory is the length scale of the strings given by the string parameter

$$\alpha' \sim \ell_{\mathrm{P}}^2 \ , \tag{1.2.7}$$

with the Planck length given in (1.1.7). This parameter is also known as the slope parameter and it is the only parameter that enters the theory. A good pictorial way to think of string theory is to compare a relativistic tiny string with a classical guitar string which has certain resonant frequencies depending on its length and tension. The different resonances of a guitar string are perceived by the human ear as different musical notes whereas the different vibrational modes of the relativistic string constitute particles and their basic properties. This means that a particle in string theory is just a particular oscillation mode of a string and that different particles simply correspond to different oscillation modes. The different vibrational modes have different energies which lead to different masses for the particles through Einstein's famous mass-energy relation. Examples of different oscillation modes are given in figure 1.3. All other properties of a particles such as charge and spin are also governed by the vibrational patterns of strings. As described above, this is exactly the power of string theory. Since everything is just vibrational modes of the same string, matter and forces are unified and all particles and forces are intrinsically build into the theory from the beginning.

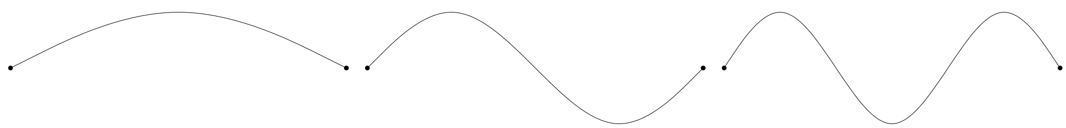

Figure 1.3: Different string oscillation modes. A string can undergo an infinite number of different resonances and the length and tension of the string determine which resonances are allowed. The connection between string vibrations and particle properties can be illustrated for the mass of a particle. Because of Einstein's equivalence principle between mass and energy, a low oscillation mode corresponds to a small mass whereas high energy oscillations correspond to a large particle mass. The leftmost string oscillation is the lowest possible oscillation mode and it corresponds therefore to the lowest possible particle mass in the string spectrum. The examples in the middle and to the right correspond to the next two masses in the spectrum.

String theory is formulated by writing down the appropriate action and quantize it by imposing quantum mechanical momentum and position commutation relations. In this context, the action for a relativistic point particle is a good starting point for the discussion of the string action. A relativistic point particle propagating in spacetime traces out a world line which is parametrized by the proper time of the particle. The proper time τ is a Lorentz invariant and is connected to the world line of the point particle by the integral

$$\tau = \int \frac{\mathrm{d}t}{\gamma} = \int \mathrm{d}s \ , \tag{1.2.8}$$

with the usual relativistic γ-factor

$$\gamma = \frac{1}{\sqrt{1-v^2}} \ , \tag{1.2.9}$$

where $c = 1$. From (1.2.8) it follows that the infinitesimal proper time is connected to the infinitesimal line element of the world line by

$$\mathrm{d}\tau = \mathrm{d}s \ . \tag{1.2.10}$$

All Lorentz observes must agree on the value of the action for any world line of the particle. Since the proper time is a Lorentz invariant and connected to the world line it is natural to construct the action for the point particle proportional to the proper time. Equation (1.2.8) holds for natural units where length has the inverse dimension of time and in order to ensure that the point particle action is dimensionless the proper time is multiplied by the rest mass which is also a Lorentz invariant. The relativistic point particle action is hence written as

$$S_{\mathrm{rel}} = -m \int \mathrm{d}s \ , \tag{1.2.11}$$

where the minus sign turns out to be correct in order to recover the right expression for the kinetic energy when the Lagrangian is expanded in the classical limit of low velocity. The action for a one-dimensional string can be constructed as a generalization of the point particle action. The string propagating in spacetime traces out a two-dimensional world sheet and just as all Lorentz observes will agree on the elapsed proper time of the point particle, all Lorentz observes will agree on the size of the area of the world sheet traced out by the string. Hence it is natural to construct the action for the string being proportional to the integral over the world sheet area. To ensure that the action is dimensionless it must be multiplied with a

Lorentz invariant quantity with dimension of inverse length squared. The fundamental string parameter is one such object and the resulting action is known as the Nambu-Goto action

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}A \ . \tag{1.2.12}$$

Equation (1.2.12) is fundamental in the sense that it shows how the fundamental string parameter α' is the only parameter that enters the theory.

1.2.3 String quantization

In order to obtain a quantum theory, the string action is quantized by imposing quantum commutation relations on the string momentum and position. It should be noted that in practice, often an action known as the Polyakov action will be used instead of the Nambu-Goto action when a string theory is quantized. These two actions are classically equivalent but the Polyakov action is more convenient for a quantum formulation. Quantization of the string action yields the different oscillation modes with corresponding masses in the string theory. This is known as the string spectrum. As described above, the only parameter of the theory is the string parameter α' and one must therefore expect that the mass scale is set by this parameter. In order to have the right dimension of the mass

$$m^2 \sim \frac{1}{\alpha'} \ , \tag{1.2.13}$$

must hold. A bosonic string theory is a theory that contains only bosons. It can be shown that a bosonic string theory requires 26 spacetime dimensions in order to be physically consistent. Such a theory can be quantized in four different ways depending on choice of string boundary conditions. Strings can be open with free ends or they can be closed with the ends joined together. Furthermore strings can be considered orientable or unorientable. An orientable string has two different directions to travel along whereas an unorientable string has only one direction. All bosonic theories include a particle known as the tachyon which has the lowest mass in the string theory spectrum. The mass square is

$$m^2_{\rm tachyon} = -\frac{1}{\alpha'} \ , \tag{1.2.14}$$

so that the tachyon mass is imaginary. The existence of the tachyon with imaginary mass signals an instability of the theory which can be seen from the potential for the tachyon field T. The potential is

$$V(T) = \frac{1}{2}m^2T^2 < 0 \ , \quad \text{for} \quad m^2 < 0 \ , \tag{1.2.15}$$

which is just a parabola with an unstable maximum. The existence of the tachyon combined with the existence of bosons only are two features in a bosonic string theory that make it unattractive as a candidate for a real theory. The Veneziano amplitude as discussed in section 1.2.1 is interpreted in string theory as the scattering of four open string tachyons.

To be considered as a theory of everything, string theory must contain fermions. Fermions obey Pauli statistics where two identical particles cannot be in the same quantum state. This causes fermionic theories to be more complicated than bosonic theories. It can be shown that a fermionic theory lives naturally in 10 spacetime dimensions and that a concept known as supersymmetry is necessary in order to make it physically consistent. Supersymmetry is a suggested fundamental symmetry in nature between fermions and bosons which will be discussed below.

1.2.4 Supersymmetry

Nature is considered to have a number of symmetries. One example is Einsteins equivalence principle according to which the physical laws are the same in all local inertial frames. Beside the observed symmetries in nature one can think of a possible symmetry which relates bosons and fermions. This symmetry is known as supersymmetry (SUSY) and it relates to any elementary particle a supersymmetric partner with the same mass and a spin quantum number which is decreased by half a unit of spin. Supersymmetry is an independent concept and can exist in nature independent of string theory. However, there is an interesting interplay between supersymmetry and string theory since supersymmetry is necessary in a string theory that includes fermions. A string theory with supersymmetry is known as a superstring theory.

An interesting aspect in connection with supersymmetry is the possible unification of the coupling constants of the electromagnetic interaction, the weak interaction and the strong interaction. These coupling constants depend on the energy as discussed in section 1.1.1. As seen in [3, 4, 5] the coupling constants almost unify in the standard model around $E_{\text{un}} \sim 10^{14}$ GeV. But only almost. If supersymmetry is included, it is found that the three coupling constants will unify at $E_{\text{un,SUSY}} \sim 10^{16}$ Gev which is known as gauge unification. This is naturally an indication for the presence of supersymmetry in nature.

As string theory is believed to be a unifying theory, the physics of the standard model should emerge from string theory somehow. There has been no experimental evidence of supersymmetric properties for the particles of the standard model. One possible explanation is that if supersymmetry is part of nature it must be spontaneously broken at low energies by some unknown mechanism. The undiscovered supersymmetric partners to the elementary particles must therefore be very heavy. Some of these supersymmetric partners have predicted mass ranges which should be visible at the LHC at CERN and the existence of supersymmetry in nature could therefore be suggested by future LHC experiments. This is exciting since supersymmetry is needed in order to ensure string theories to be physically consistent.

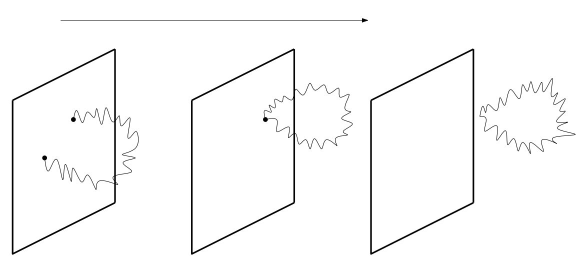

Figure 1.4: An interaction process between a D-brane and a string. Both ends of the open string are subject to Neumann boundary conditions on the D-brane. The ends can join to form a closed string which can leave the D-brane. The process can as well be reversed so that a closed string hits the D-brane. The closed string sector is not considered so interactions like this is ignored.

1.2.5 D-branes and gauge theories

Naturally there is a difference between open and closed strings. However, a closed string can break up into an open string and conversely the ends of an open string can join to form a closed string. Only the open string sector will be considered in this thesis. The ends of an open string are naturally subject to certain boundary conditions of which there exist two different types. A string with its endpoints free to move is subject to Neumann boundary conditions in which case momentum is conserved at the endpoints. A string having its endpoints fixed is subject to Dirichlet boundary conditions where momentum transfer takes place at the string endpoints. Objects on which open strings can end are known as D-branes and play an important role in string theory.

A D-brane is defined as a hypersurface onto which strings can end with Dirichlet boundary conditions. A D-brane is often written as a Dp-brane where p is an integer and denotes the number of spatial dimensions of the hypersurface. The integer p can take any value from 0 to $d-1$. A D$(d-1)$-brane is known as a space filling brane and since superstring theories live naturally in ten dimensions, a D9-brane is a space filling brane in a superstring theory. In the case of a space filling brane, the string endpoints are fixed on a hypersurface which fills the entire space. This corresponds therefore to a free open string subject to Neumann boundary conditions. For a general Dp-brane in d dimensions, open strings are subject to boundary conditions according to

$$p+1 \quad \text{directions with Neumann boundary conditions} \tag{1.2.16}$$

$$d-(p+1) \quad \text{directions with Dirichlet boundary conditions .} \tag{1.2.17}$$

It follows that an open string which ends on a D3-brane is subject to Neumann boundary conditions in 4 dimensions and Dirichlet boundary conditions in 6 dimensions.

An arrangement of several closely spaced D-branes enforces some constraints on which string states can be found in a system. For two D-branes close to each other, strings can stretch with an endpoint on each brane. A string stretching between the two branes has a certain minimum length which equals the brane separation. When a string is pulled, energy is added to the string since work is done on the string as it is pulled against its tension. Adding energy to the string is equivalent to adding mass. The separation of the D-branes thus controls the minimum mass of the resonance modes of open strings. In this sense the arrangement of D-branes controls which particles are present in the string theory.

The simplest case occurs when a string has both endpoints attached to the same D-brane. This is shown in figure 1.5. One can analyze this situation by quantizing the relevant string action and find that the photon is among the particles of the spectrum where it is recognized as the lowest oscillation mode. In this sense an electromagnetic field is living on the D-brane. It is found in general that a Dp-brane has an electromagnetic field obeying a p-dimensional generalization of Maxwell's equation living on it. From (1.2.16) and (1.2.17) it is apparent that the study of strings with both endpoints on a single D3-brane neglecting all string oscillation modes except for the lowest, leads to electromagnetic interactions in $3+1=4$ dimensions. Likewise, the study of open strings with both endpoints on a D5-brane leads to a generalization of electromagnetism in six dimensions.

The situation can be studied in the more general case with N closely spaced D-branes and open strings with endpoints on the branes. It can be shown that in the limit where all branes are put on top of each other, this corresponds exactly to a $U(N)$ gauge theory which is therefore in general non-Abelian. The discussion above with one single D-brane is hence a special case with $N=1$ which therefore corresponds to a generalization of electromagnetism with gauge group $U(1)$.

The effective action describing electromagnetism on the worldvolume of a single Dp-brane is

$$S_{\mathrm{Dp}} = \int \mathrm{d}^{p+1}x \left(\mathcal{L}_{\mathrm{DBI}} + \cdots\right) \ , \tag{1.2.18}$$

to leading order where $\mathcal{L}_{\text{DBI}}$ is the Dirac-Born-Infeld Lagrangian. Only the contributions from the DBI-action will be studied.

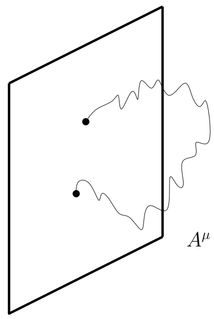

Figure 1.5: The simple case where both ends of an open string are subject to Neumann boundary conditions on one single D-brane. The lowest oscillation mode of the string corresponds to a massless gauge field and the D-brane has therefore an electromagnetic field living on its worldvolume. For a Dp-brane the electromagnetic field lives in $p+1$ dimensions.

1.3 String theory and experiment

The link between theory an physical observations in nature goes through experiments and measurements. In order to be able to test a theoretical model it has to contain parameters whose numerical values can be measured from an experiment. A good theory is even able to give a sharp prediction which can be tested. In turn, the interesting quantities to calculate are those that can actually be experimentally determined or at least be connected somehow to experiment. The following is a quick comparison of relevant energy scales.

1.3.1 Energy scales

In order to study phenomena at the Planck scale, at least naïvely, energies comparable with the Planck energy (1.1.8) are necessary. The string scale is of the order of the Planck scale and it can therefore in principle be studied directly by using energies comparable to the Planck energy. At the Large Hadron Collider (LHC) at CERN, proton beams will be collided with a maximal energy of $E_{\text{beam}} \sim 7$ Tev so that the total maximal energy is $E_{\text{max}} \sim 14$ Tev. A comparison yields

$$\frac{E_{\text{max}}}{E_{\text{P}}} \sim 1.4 \times 10^{-15} \ , \tag{1.3.1}$$

so the maximal energy at the LHC is roughly 15 orders of magnitude too small for the purpose of string theory experiments. This is naturally a very rough estimate which however gives a good indication of how far the string scale is from the available energies in the present generation of particle generators. An interesting estimate can be made for the size of the accelerator ring at the LHC. The radius of the ring is $r_{\text{LHC}} \sim 27$ km and the maximal possible energy is $E_{\text{max}} \sim 14$ Tev. By assuming that the maximal energy scales linearly with the radius of the ring, it can be estimated that an accelerator with a maximal energy $E_{\text{max}} = E_{\text{P}}$ equal to the Planck energy would require a ring with radius $r \sim 10^3$ parsec which is approximately thirty times smaller than the diameter of the Milky Way.

In principle, there could be very large unknown factors which have to be accounted for in an estimate like (1.3.1). This is considered in section 4.5 where an estimate for a cross section will be given.

1.3.2 Interaction cross sections

In the discussion of experiments and string theory it is natural to address the cross-section. Basically it is a measure of the likelihood of an interaction of particles which is independent of beam characteristics. More specifically, the cross section is defined by considering two cylindrical colliding beams of certain particles. These beams are referred to respectively as a and b and have the respective particle number densities ρ_a and ρ_b. For each beam, only a slice of length l is considered such that two bunches of particles with respective lengths l_a and l_b are collided. If A denotes the area where the two beams collide, the cross section is defined as the total number of scattering events N divided by the beam quantities

$$\sigma = \frac{N}{A\rho_a l_a \rho_b l_b} \ , \tag{1.3.2}$$

where the studied scattering events can be of whatever type desired. From (1.3.2) the cross section has dimension

$$\Big([\text{area}]\,[\text{volume}]^{-2}\,[\text{length}]^2\Big)^{-1} = [\text{area}] \ . \tag{1.3.3}$$

It is interpreted as the effective area of the target particle as seen from the incoming particle with the assumption that the particles will scatter with 100 % certainty if this area is hit. The definition (1.3.2) is symmetric in a and b as it should be since the scattering process is not affected by choice of reference frame.

In order to probe the behavior of elementary particles, beams with well-defined particle momenta are collided and the final-state particles and their momenta are detected. When doing so, the cross section becomes infinitesimal

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^3p_1\cdots\mathrm{d}^3p_n} \ , \tag{1.3.4}$$

and dependent on the momentum of the outgoing particles. An integration over any small momentum d^3p_i determines the cross section for scattering into that particular final-state momentum. The situation simplifies for the scattering of four particles due to four-momentum conservation and the two particles in the final state are constrained in such a way that only two components of the final-state momenta are independent. These two components can be specified with two spherical angles and the differential cross section is therefore expressed using the solid angle $\mathrm{d}\Omega$. For four identical particles with identical masses the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{A}|^2}{64\pi^2 E_{\text{cm}}^2} \ , \tag{1.3.5}$$

and is determined by the center of mass energy in the collision and the square of the amplitude for the particular scattering process. This is essentially the reason why calculations of scattering amplitudes are so interesting since it is apparent from (1.3.5) how the scattering amplitude is the direct connection between theory and experiment. It is also apparent that the cross section is independent of beam characteristics as it should be.

1.4 Born-Infeld theory

The Dirac-Born-Infeld action was discussed briefly in a string theory context in section 1.2.5 as the effective action describing a gauge field living on the worldvolume of a D-brane. In this thesis the DBI-action is used as an effective action from which scattering amplitudes are calculated. However, this is far from the original purpose for the action which was founded by Max Born in 1933 [6] and developed further in 1934 in collaboration with Leopold Infeld [7] many years before string theory. The Born-Infeld theory was formulated for the purpose of solving the problem in ordinary Maxwell theory that a charged point particle has an infinite self energy at the origin. In 1960 Dirac elaborated [8] on the original work by Born and Infeld and this is where the name Dirac-Born-Infeld (DBI) comes about.

1.4.1 Historical motivation

At the time of the paper [7] by Born and Infeld the relations between matter and electromagnetic fields were interpreted from two opposite viewpoints; the unitarian viewpoint versus the dualistic one. In the unitarian viewpoint, the electromagnetic fields are assumed to be the only physical entities and thus particles are considered as singularities of the fields. According to this viewpoint, the mass of a particle is a derived notion which is expressed in terms of the energy of the electromagnetic field. The dualistic viewpoint operates with particles and fields as two distinct entities where particles are the sources of the fields. The particles are as well acted upon by the fields. The dualistic viewpoint was widely accepted when the Born-Infeld theory was formulated. In particular it was supported by the theory of general relativity as well as quantum mechanics which is essentially based on a dualistic point of view. In ordinary Maxwell theory the dualistic viewpoint suffices as long as the wavelengths of the fields are large compared to the electron radius. On smaller length scales the theory breaks down and leads to an infinite electric field at the origin and thereby an infinite self energy of a charged point particle. The motivation for the formulation of the modified theory was to avoid these infinities which have later been removed with the principle of renormalization. Born and Infeld used the principle of finiteness according to which a satisfactory physical theory should not allow any physical quantity to become infinite. The infinite self energy for a charged point particle is discussed below.

1.4.2 Self energy for a charged point particle in Maxwell theory

The discussion begins with the operation of dualization which can be defined in four dimensions as

$$\tilde{F}_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ . \tag{1.4.1}$$

With the metric (2.4.1) and the convention that

$$\varepsilon_{0123} = +1 \ , \tag{1.4.2}$$

the electromagnetic field strength tensor and its dual are given explicitly for $c = 1$ in matrix form as

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} , \quad \tilde{F}_{\mu\nu} = i \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix} , \tag{1.4.3}$$

in terms of the spatial field components for the magnetic and electric fields. By use of the usual three-vector notation

$$\mathbf{E} = (E_x, E_y, E_z) \ , \quad E^2 = \mathbf{E} \cdot \mathbf{E} \ , \quad \mathbf{E} \cdot \mathbf{B} = E_x B_x + E_y B_y + E_z B_z \ , \tag{1.4.4}$$

the objects C_1 and C_2 can be constructed from $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ as

$$C_1 \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \left(E^2 - B^2 \right) \ , \quad C_2 \equiv -\frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} = -i \, \mathbf{E} \cdot \mathbf{B} \ , \tag{1.4.5}$$

such that both objects are fully contracted and hence Lorentz invariant. The field strength tensor is by itself gauge invariant and both C_1 and C_2 are therefore gauge invariant. It can be shown that C_1 and C_2 are the only independent naturally invariant objects that can be constructed from $F_{\mu\nu}$ without using its derivatives. This is discussed in A.1 with use of the spinor-helicity formalism from section 2.1.7. The ordinary Maxwell Lagrangian is

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = C_1 \ , \tag{1.4.6}$$

and in vacuum where $\mathbf{D} = \mathbf{E}$, the energy density is given by

$$\mathcal{H}_{\text{Maxwell}} = \mathbf{E} \cdot \mathbf{D} - \mathcal{L}_{\text{Maxwell}} = \frac{1}{2} \left(E^2 + B^2 \right) \ , \tag{1.4.7}$$

with $\mathbf{D}$ denoting the electric displacement field. In the electrostatic case $\mathbf{B} = \mathbf{0}$, the energy density is $\mathcal{H} \sim E^2$ and the self energy of a charged point particle is calculated by integrating the energy density. In a spherically symmetric field from a charged point particle, the electric field can only have radial components

$$\mathbf{E} = E_r \, \hat{\mathbf{r}} \ , \tag{1.4.8}$$

so that Maxwell's equation yields

$$0 = \nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 E_r \ , \tag{1.4.9}$$

whereby it follows that

$$|\mathbf{E}| \sim \frac{1}{r^2} \ , \tag{1.4.10}$$

for the field magnitude as function of the distance r from the origin. The self energy is given by

$$\mathcal{E} = \int \mathrm{d}^3 x \, \mathcal{H}_{\text{Maxwell}} = \frac{1}{2} \int \mathrm{d}^3 x \, E^2 \ , \tag{1.4.11}$$

and with the volume element $\mathrm{d}^3 \, x = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$ the integrand becomes

$$\mathrm{d}^3 x E^2 \sim \mathrm{d}r \, r^2 \frac{1}{r^4} = \mathrm{d}r \, \frac{1}{r^2} \ . \tag{1.4.12}$$

This result diverges for small r and it is apparent how the self energy for a charged point particle in ordinary Maxwell theory becomes infinite.

1.4.3 Modification of ordinary Maxwell theory

The modification of Maxwell theory according to Born and Infeld is formulated by replacing the ordinary Lagrangian (1.4.6) by a new non-linear Lagrangian. This can be done by incorporating a maximal value for the electric field and is another example of an effective theory. The theory of special relativity has an incorporated maximal value since nothing can move with a velocity greater than c. This property is reflected in the Lagrangian describing a relativistic point particle which is given in (1.2.11). Using (1.2.8) and restoring factors of c yields the relativistic point particle Lagrangian

$$\mathcal{L}_{\text{rel}} = -mc^2\sqrt{1-\frac{v^2}{c^2}} \ , \tag{1.4.13}$$

where the maximal possible velocity is $v_{\text{max}} = c$ since the argument of the square root is required positive. This property of maximal velocity in the theory of special relativity was used by Born and Infeld as an inspiration. The maximal value for the electric field is incorporated in the theory by writing the Lagrangian

$$\mathcal{L}_{\text{Maxwell}} \to \mathcal{L}' = -b^2\sqrt{1-\frac{2C_1}{b^2}} + b^2 = -b^2\sqrt{1-\frac{E^2-B^2}{b^2}} + b^2 \ , \tag{1.4.14}$$

which was originally proposed by Born [6] in 1934. For $\mathbf{B} = \mathbf{0}$ it follows that

$$|\mathbf{E}| \le b \ , \tag{1.4.15}$$

to ensure a positive argument under the square root. For a small electric field $C_1 \ll b^2$, equation (1.4.14) is expanded as

$$\mathcal{L}' = -b^2\left(1-\frac{C_1}{b^2}\right) + b^2 + \mathcal{O}\left(C_1^2\right) = C_1 + \mathcal{O}\left(C_1^2\right) \ , \tag{1.4.16}$$

and the Born-Infeld theory resembles the ordinary Maxwell theory in the limit of small fields,

$$\mathcal{L}'\Big|_{C_1 \ll b^2} \sim \mathcal{L}_{\text{Maxwell}} \ . \tag{1.4.17}$$

As was proposed by Born and Infeld [7] later in 1934, (1.4.14) can be modified further by the inclusion of one additional term under the square root

$$\mathcal{L}_{\text{DBI}} = -b^2\sqrt{1-\frac{2C_1}{b^2}+\frac{C_2^2}{b^4}} + b^2 = -b^2\sqrt{1-\frac{E^2-B^2}{b^2}-\frac{(\mathbf{E}\cdot\mathbf{B})^2}{b^4}} + b^2 \ . \tag{1.4.18}$$

This is the Lagrangian known as the Dirac-Born-Infeld Lagrangian which is both Lorentz and gauge invariant since it is constructed from C_1 and C_2. For small fields C_1 and C_2 are comparable and hence

$$\mathcal{L}_{\text{DBI}} \sim C_1 \ , \tag{1.4.19}$$

holds in the weak field limit just as (1.4.16). The modification from (1.4.14) to (1.4.18) is preferred because the latter expression can be generalized. The generalization reads

$$-\det\left(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right) = 1 - \frac{2C_1}{b^2} + \frac{C_2^2}{b^4} \ , \tag{1.4.20}$$

which can be checked explicitly in Mathematica by writing the metric and the field strength tensor explicitly as matrices. This particular computation can be found in detail in section A.1. From (1.4.20) it follows that

$$\mathcal{L}_{\text{DBI}} = -b^2 \sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{b} F_{\mu\nu}\right)} + b^2 \ , \tag{1.4.21}$$

which allows for a generalization to any number of dimensions. It follows straightforwardly from a Lorentz transformation

$$M \to M' = \Lambda M \Lambda^T \ , \tag{1.4.22}$$

with

$$\Lambda \Lambda^T = \mathbb{1} \ , \tag{1.4.23}$$

for the transformation matrix Λ that the determinant (1.4.21) is Lorentz invariant and hence also the DBI Lagrangian.

The classical Maxwell equations incorporate the electric displacement vector field **D** and the auxiliary magnetic field **H** in order to describe electromagnetism in materials. In a nonlinear theory, the vacuum itself behaves as some kind of material. Born-Infeld theory describes electromagnetism in vacuum and since it is a nonlinear theory there is a nontrivial relationship between **E** and **D**. From computations along the lines of section 1.4.2 for the displacement field **D** it can be shown [9] that the DBI Lagrangian leads to a finite self energy for a charged point particle as was the original purpose for the modification of the Lagrangian.

Chapter 2

Field theory background

The low-energy limit of string theory can be viewed effectively as a quantum field theory. On a low-energy scale, the smallest structure of string theory cannot be resolved and string theory appears therefore effectively as a quantum field theory. Consequently, scattering amplitudes in the next chapters will be calculated by use of perturbative methods from field theory.

This chapter contains a discussion of the relevant field theory methods used in amplitude calculations. It is discussed how correlation functions are computed by use of Wick contractions and the formalism of functional integrals are introduced. Furthermore, spinors and in particular the spinor-helicity formalism is discussed in detail. The spinor-helicity formalism is a method to express tensors in terms of spinors and it will be useful in explicit calculations of Wick contractions in connection with scattering amplitudes in the next chapters.

2.1 Spinor-helicity formalism in four dimensions

This section is based on references [10, 11, 12, 13, 14] and it deals with the so-called spinor-helicity formalism, a formalism that enables a translation between tensors and products of spin vectors. A modern review on the spinor helicity formalism can be found in [15]. It turns out that this correspondence between tensors and spinors is very useful especially in connection with the study of scattering amplitudes in four dimensions where the calculations simplify when tensors are expressed as spinors. The correspondence is introduced in the following sections and it is natural to begin with a discussion of the helicity quantum number.

2.1.1 Particle spin and the helicity quantum number

The spin of a massive particle is found by studying the transformation properties under rotations in the rest frame. The angular momentum operator, or the spin operator, in the rest frame is denoted by $\mathbf{J}$ and it does not have simple transformation properties under Lorentz transformations. It is therefore not the best operator to use in the description of particle states. A natural object to use is the polarization operator

$$W^{\mu} = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} P_{\nu} \Sigma_{\rho\sigma} \ , \tag{2.1.1}$$

with

$$\left(\Sigma_{\rho\sigma}\right)_{\mu\nu} = i\left(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\sigma\mu}\eta_{\rho\nu}\right) \ , \tag{2.1.2}$$

denoting the generators of Lorentz transformations i.e. boosts and rotations in the vector representation of the Poincaré algebra. The polarization operator is known as the Pauli-Lubanski vector and it transforms as a pseudo vector under Lorentz transformations. It commutes with the momentum operator

$$[W^{\mu}, P^{\nu}] = 0 \ , \tag{2.1.3}$$

and it is related to the spin operator by

$$W^0 = 0 \ , \quad W^1 = mJ_1 \ , \quad W^2 = mJ_2 \ , \quad W^3 = mJ_3 \ , \tag{2.1.4}$$

where m is the particle mass. The relations (2.1.4) can be obtained explicitly in the rest frame. In a specific representation the eigenvalues of the square of the Pauli Lubanski vector are

$$W_{\mu}W^{\mu} = -m^2 j\,(j+1) \ , \tag{2.1.5}$$

in terms of the mass and the particle spin j. It can be shown explicitly that the operator (2.1.5) commutes with all of the generators of the Lorentz algebra

$$\left[W_{\lambda}W^{\lambda}, P_{\mu}\right] = \left[W_{\lambda}W^{\lambda}, \Sigma_{\mu\nu}\right] = 0 \ , \tag{2.1.6}$$

which means that it is a Casimir operator. The other Casimir operator is the invariant mass operator,

$$P_{\mu}P^{\mu} = m^2 \ , \tag{2.1.7}$$

so that the eigenvalues of (2.1.5) and (2.1.7) can be used to characterize an arbitrary physical spin system of elementary particles. The helicity quantum number can be used instead of the spin in the description of elementary particles. This is particularly useful for a massless particle where the polarization operator,

$$W^{\mu}W_{\mu} \propto m^2 = 0 \ , \tag{2.1.8}$$

does not provide useful information. The helicity operator is defined as the projection of the spin on the direction of the momentum

$$\mathcal{H} = \mathbf{J} \cdot \hat{\mathbf{p}} \ , \tag{2.1.9}$$

with the unit vector

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|} \ , \tag{2.1.10}$$

pointing in the propagating direction of the momentum. An observer of a massive particle will always be able to perform a boost into a frame in which the particle momentum is reversed. It follows that the helicity quantum number for a massive particle is not Lorentz invariant but only invariant under spatial rotations. The situation is different for a massless particle since it has no rest frame. An observer cannot boost to reverse the direction of the particle momentum and a massless particle will therefore always appear to spin in the same direction along its direction of momentum. It follows that the helicity for a massless particle is independent of frame, meaning that it is a Lorentz invariant. A massless particle can be boosted to a frame in which its momentum vector has the form

$$p^\mu = (p, 0, 0, p) \ . \tag{2.1.11}$$

This vector is invariant under spatial rotations in the $(1,2)$ plane and it follows that the little group is $SO(2)$ which is generated by the angular momentum operator in the 3-direction. The angular momentum operator has eigenvalues given by the helicity λ and a massless particle state can therefore be characterized with the quantum state

$$|p, \lambda\rangle \ , \tag{2.1.12}$$

where λ is the eigenvalue of the helicity operator and p is the eigenvalue of the momentum operator.

2.1.2 Spatial rotations

In order to express tensors in terms of spinors it is natural to consider representations. A Lorentz transformation on an arbitrary four-vector will mix all vector components meaning that the vector representation of the Lorentz group is irreducible. A tensor with arbitrarily many indices transforms as

$$T^{\mu_1\mu_2\cdots\mu_n} \to T^{\mu_1'\mu_2'\cdots\mu_n'} = \Lambda^{\mu_1'}{}_{\mu_1}\Lambda^{\mu_2'}{}_{\mu_2}\cdots\Lambda^{\mu_n'}{}_{\mu_n}T^{\mu_1\mu_2\cdots\mu_n} \ , \tag{2.1.13}$$

which is a tensor product of transformations on each index. From (2.1.13) it is apparent that the tensor representation of the Lorentz group is just a tensor product of vector representations. Consequently the vector representation is said to be the fundamental representation of $SO(3,1)$. The group of spatial rotations $SO(3)$ is a subgroup of the group of Lorentz transformations. Under spatial rotations governed by $SO(3)$, a tensor representation will be labeled by its angular momentum j and contain a total number of $2j+1$ states. For $SO(3)$ the angular momentum will be a non-negative integer. A spatial rotation of the four-vector $v^\mu = \left(v^0, \mathbf{v}\right)$ does not mix the spatial components with the time component which means that the time component of the vector is invariant under spatial rotations and hence has angular momentum $j = 0$. From the point of view of $SO(3)$ rotations, the four-vector is then a reducible representation since it can be decomposed into

$$v^\mu \in \mathbf{0} \oplus \mathbf{1} \ , \tag{2.1.14}$$

which is a direct sum of a scalar representation with angular momentum $j = 0$ and a vector representation with angular momentum $j = 1$.

From non-relativistic quantum mechanics it is known that the tensor representation is not the only representation of spatial rotations. In particular the spinorial representation is of physical interest since fermionic particles are described by spinors which are the elements of a spinor representation. The spinor representations are as well labeled by the angular momentum j which in this case take non-zero half-integer values. The group for the spinor representation is $SU(2)$. Both $SU(2)$ and $SO(3)$ have the angular momentum algebra as Lie algebra but the groups are only locally isomorphic since a rotation of 2π in $SU(2)$ is minus the identity. In $SO(3)$ a rotation of 2π is identical to the identity. Physical systems with integer or half-integer spin can be constructed as composite systems of spin $j = 1/2$ particles and in particular the composite system

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1} \ , \tag{2.1.15}$$

is a direct sum of a scalar representation and a vector representation. The right hand side of (2.1.15) is equal to that of (2.1.14) and it is apparent how the vector representation is equivalent to the product of two spin representations. This equivalence is the starting point to establish the spinor helicity formalism as the correspondence between tensors and spinors.

2.1.3 Vector representation and spinor representation

The equivalence discussed above between the vector representation and the product of two spin representations is related to the group isomorphism

$$SO(4) \sim SU(2) \times SU(2) \ . \tag{2.1.16}$$

To make this explicit, a Lorentz index μ in four dimensions can just as well be written as the composite index $\alpha\dot{\alpha}$ where each index can take two values. This equivalence between two ways of writing indices can be discussed by considering an arbitrarily chosen four-vector K^μ which is written as

$$K^\mu = \begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \end{pmatrix} \ , \tag{2.1.17}$$

in a particular reference frame. The four-vector is fully determined by its four components. A hermitian 2×2 matrix

$$\mathcal{M} = \mathcal{M}^\dagger = \begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} \ , \tag{2.1.18}$$

with $\alpha = \alpha^*$, $\delta = \delta^*$ is also fully determined by four independent components and it is therefore suggested that a one-one correspondence between real world vectors and 2×2 hermitian matrices can be established. This is then the correspondence $\mu \leftrightarrow \alpha\dot{\alpha}$ discussed above. The correspondence is written as

$$K_{\alpha\dot{\alpha}} \equiv K_\mu \sigma^\mu_{\alpha\dot{\alpha}} \ , \tag{2.1.19}$$

where $\sigma^\mu_{\alpha\dot{\alpha}}$ are the Clebsch-Gordan coefficients for the transition between the vector representation and the spin representation. The Clebsch-Gordan coefficients are given by the Pauli matrices

$$(\sigma^0)_{\alpha\dot{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\sigma^1)_{\alpha\dot{\alpha}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad (\sigma^2)_{\alpha\dot{\alpha}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (\sigma^3)_{\alpha\dot{\alpha}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.1.20}$$

This can be seen by considering the Lorentz transformation

$$u'_\mu = \Lambda_\mu{}^\nu u_\nu \ , \tag{2.1.21}$$

expressed in spinor indices

$$u'_{\alpha\dot{\alpha}} = u'_\mu\,\sigma^\mu_{\alpha\dot{\alpha}} = \Lambda_\mu^{\ \nu} u_\nu\,\sigma^\mu_{\alpha\dot{\alpha}} \ . \tag{2.1.22}$$

Since the spinor representation is equivalent to the tensor representation, there exist transformation spinors denoted as $\zeta_\alpha^{\ \beta}$ so that

$$u'_{\alpha\dot{\alpha}} = \zeta_\alpha^{\ \beta}\,\zeta_{\dot{\alpha}}^{\ \dot{\beta}}\,u_{\beta\dot{\beta}} = \zeta_\alpha^{\ \beta}\,\zeta_{\dot{\alpha}}^{\ \beta}\,u_\nu\,\sigma^\nu_{\beta\dot{\beta}} \ . \tag{2.1.23}$$

It follows from a comparison of (2.1.22) and (2.1.23) that

$$\Lambda_\mu^{\ \nu}\,\sigma^\mu_{\alpha\dot{\alpha}} = \zeta_\alpha^{\ \beta}\,\zeta_{\dot{\alpha}}^{\ \beta}\,\sigma^\nu_{\beta\dot{\beta}} \ , \tag{2.1.24}$$

which relates the transformation matrix in tensor indices to the transformation matrices in spinor indices. The generalization of (2.1.19) for a tensor with arbitrarily many Lorentz indices is

$$T_{\mu_1\cdots\mu_n}\,\sigma^{\mu_1}_{\alpha_1\dot{\alpha}_1}\cdots\sigma^{\mu_n}_{\alpha_n\dot{\alpha}_n} = T_{\alpha_1\dot{\alpha}_1\cdots\alpha_n\dot{\alpha}_n} \ , \tag{2.1.25}$$

where $T_{\alpha_1\dot{\alpha}_1\cdots\alpha_n\dot{\alpha}_n}$ is an outer product of n (2×2) matrices. Equation (2.1.19) has the explicit matrix form

$$K_{\alpha\dot{\alpha}} = \begin{bmatrix} K^0 + K^3 & K^1 + iK^2 \\ K^1 - iK^2 & K^0 - K^3 \end{bmatrix}, \tag{2.1.26}$$

so that the determinant of the hermitian matrix

$$\det\, K_{\alpha\dot{\alpha}} = \left(K^0\right)^2 - \left(K^1\right)^2 - \left(K^2\right)^2 - \left(K^3\right)^2 = K_\mu K^\mu \ , \tag{2.1.27}$$

is the invariant length of the four-vector K^μ. As in (2.1.11), for K^μ massless, one can boost to a frame in which the vector has components

$$K^\mu = (K, 0, 0, K) \ , \tag{2.1.28}$$

and it is apparent from (2.1.26) that

$$K_{\alpha\dot{\alpha}} = \begin{bmatrix} 2K & 0 \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} \sqrt{2K} \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{2K}, & 0 \end{pmatrix} \ , \tag{2.1.29}$$

holds when K^μ is massless. It follows that a massless vector can always be written as the outer product

$$K_{\alpha\dot{\alpha}} = \lambda_\alpha \lambda_{\dot{\alpha}} \ . \tag{2.1.30}$$

Writing the general spin vector

$$\kappa_\alpha = \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \ , \tag{2.1.31}$$

the matrix $\mathcal{Q}$ can be constructed as the outer product

$$\kappa_\alpha \kappa_{\dot{\alpha}} = \begin{pmatrix} \zeta \\ \xi \end{pmatrix} \begin{pmatrix} \zeta^*, & \xi^* \end{pmatrix} = \begin{bmatrix} \zeta\zeta^* & \zeta\xi^* \\ \xi\zeta^* & \xi\xi^* \end{bmatrix} \equiv \mathcal{Q} \ , \tag{2.1.32}$$

such that

$$\det \mathcal{Q} = \zeta\zeta^*\xi\xi^* - \xi\zeta^*\zeta\xi^* = 0 \ . \tag{2.1.33}$$

This is equivalent to

$$\det K_{\alpha\dot{\alpha}} = 0 \ , \tag{2.1.34}$$

for K^μ massless.

2.1.4 Covariant and contravariant spinor indices

Equation (2.1.19) has the inversion

$$K_{\alpha\dot{\alpha}}\,\sigma^{\mu,\alpha\dot{\alpha}} = \kappa\,K^{\mu} = K_{\nu}\,\sigma^{\nu}_{\alpha\dot{\alpha}}\,\sigma^{\mu,\alpha\dot{\alpha}}\ , \tag{2.1.35}$$

such that

$$\sigma^{\nu}_{\alpha\dot{\alpha}}\,\sigma^{\mu,\alpha\dot{\alpha}} = \kappa\,\eta^{\mu\nu}\ . \tag{2.1.36}$$

This involves both covariant and contravariant spinor indices and it is therefore natural to discuss how spinor indices are raised and lowered. The vertical position of an index is changed by the metric spinor which is the spin space analogue of the metric tensor in spacetime. The metric tensor $\eta_{\mu\nu}$ is invariant under Lorentz transformations and it defines the invariant inner product

$$a^{\mu}b_{\mu} = a^{\mu}b^{\nu}\eta_{\mu\nu} = a'^{\mu}b'^{\nu}\eta_{\mu\nu} = a'^{\mu}b'_{\mu}\ , \tag{2.1.37}$$

between tensors. Furthermore, $\eta_{\mu\nu}$ acts as the link between covariant and contravariant tensors by raising and lowering Lorentz indices. The element in spin space which raises and lowers indices is the metric spinor denoted $\varepsilon_{\alpha\beta}$. It defines the bilinear form

$$\lambda_{\alpha}\kappa^{\alpha} = \lambda^{\alpha}\kappa^{\beta}\varepsilon_{\alpha\beta} = \lambda'_{\alpha}\kappa'^{\alpha} = \lambda'^{\alpha}\kappa'^{\beta}\varepsilon_{\alpha\beta}\ , \tag{2.1.38}$$

for the spinors λ and κ. Spinor products like (2.1.38) are invariant under $SU(2)$. In tensor language, if $t^{\mu}{}_{\nu}$ denotes the transformation matrix for some coordinate transformation, K^{μ} transforms as

$$K'^{\mu} = t^{\mu}{}_{\nu}K^{\nu}\ , \tag{2.1.39}$$

with an identical transformation in case of the covariant tensor K_{ν}. For tensors of more indices the transformation is just a tensor product of transformations (2.1.39) for each index. Similarly the transformation of an arbitrary spinor reads

$$\zeta'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\,\zeta^{\beta}\ , \tag{2.1.40}$$

and the metric spinor transforms according to

$$\varepsilon'_{\alpha\beta} = \Lambda_{\alpha}{}^{\gamma}\Lambda_{\beta}{}^{\delta}\varepsilon_{\gamma\delta}\ . \tag{2.1.41}$$

The metric spinor is required to be invariant under spin transformations

$$\begin{aligned}\varepsilon_{\alpha\beta} &= \varepsilon'_{\alpha\beta}\\ &= \Lambda_{\alpha}{}^{0}\Lambda_{\beta}{}^{1}\varepsilon_{01} + \Lambda_{\alpha}{}^{1}\Lambda_{\beta}{}^{0}\varepsilon_{10}\ ,\end{aligned} \tag{2.1.42}$$

where the transformation matrix is unimodular.

$$1 = \det\Lambda = \Lambda_{0}{}^{0}\Lambda_{1}{}^{1} - \Lambda_{1}{}^{0}\Lambda_{0}{}^{1}\ . \tag{2.1.43}$$

If the metric spinor is antisymmetric

$$\varepsilon_{00} = \varepsilon_{11} = 0\ ,\quad \varepsilon_{01} = -\varepsilon_{10}\ , \tag{2.1.44}$$

it follows from (2.1.42) that

$$\varepsilon'_{\alpha\beta} = \varepsilon_{01}\left(\Lambda_\alpha{}^0\Lambda_\beta{}^1 - \Lambda_\alpha{}^1\Lambda_\beta{}^0\right) \ , \tag{2.1.45}$$

so that an antisymmetric metric spinor is invariant under unimodular spin transformations

$$\varepsilon_{\alpha\beta} = \varepsilon'_{\alpha\beta} \ . \tag{2.1.46}$$

The metric spinor can be written explicitly as

$$\varepsilon_{\alpha\beta} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \ , \tag{2.1.47}$$

in matrix form where a choice of "1" as the entries has been made. Since $\varepsilon_{\alpha\beta}$ is antisymmetric, the bilinear form (2.1.38) is as well antisymmetric

$$\lambda_\alpha\kappa^\alpha = -\lambda^\alpha\kappa_\alpha \ , \tag{2.1.48}$$

and it follows that the contraction of a spin vector with itself necessarily vanishes

$$\lambda_\alpha\lambda^\alpha = \lambda^\alpha\lambda^\beta\varepsilon_{\alpha\beta} = -\lambda^\alpha\lambda^\beta\varepsilon_{\beta\alpha} = -\lambda^\alpha\lambda_\alpha = 0 \ . \tag{2.1.49}$$

As a consequence of the antisymmetry it is necessary to adopt a sign convention for the procedure of raising and lowering spinor indices. As indicated in (2.1.38) and (2.1.49) the convention is that spinor indices descent from left to right such that

$$\zeta_\alpha = \zeta^\beta\varepsilon_{\beta\alpha} \ , \quad \zeta^\alpha = \varepsilon^{\alpha\beta}\zeta_\beta \ . \tag{2.1.50}$$

For the purpose of determining the explicit matrix expression of the metric spinor with upper indices $\varepsilon^{\alpha\beta}$ it follows from (2.1.50) that

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \varepsilon^{\gamma\delta}\,\varepsilon_{\gamma\alpha}\,\varepsilon_{\delta\beta} \\ &= \varepsilon^{01}\,\varepsilon_{0\alpha}\,\varepsilon_{1\beta} + \varepsilon^{10}\,\varepsilon_{1\alpha}\,\varepsilon_{0\beta} \ , \end{aligned} \tag{2.1.51}$$

and thereby

$$\varepsilon_{01} = \varepsilon^{01} = -\varepsilon^{10} \ . \tag{2.1.52}$$

In terms of matrices, $\varepsilon_{\alpha\beta}$ is thereby identical to $\varepsilon^{\alpha\beta}$. Equivalent relations and matrix expressions are found for the spinors $\varepsilon_{\dot\alpha\dot\beta}$ and, $\varepsilon^{\dot\alpha\dot\beta}$ in the conjugate space. A useful result is

$$\varepsilon_{\alpha\beta}\,\varepsilon^{\alpha\beta} = \varepsilon_{01}\,\varepsilon^{01} + \varepsilon_{10}\,\varepsilon^{10} = 2 \ . \tag{2.1.53}$$

2.1.5 Relating the metric tensor to the metric spinor

Individual spinor indices are raised and lowered with the metric spinor just as individual Lorentz indices are raised and lowered with the metric tensor. A single Lorentz index μ corresponds to a pair of spinor indices $\alpha\dot\alpha$ and a change in the vertical position of an index μ corresponds therefore to a change in the vertical position of the index pair $\alpha\dot\alpha$. The object in spin space which raises or lowers a pair of indices must therefore be equivalent to the metric tensor in spacetime. The purpose of this section is to determine the relation between this particular object in spin space and the metric tensor in spacetime. The object $\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}$ is symmetric under $(\alpha\dot\alpha) \leftrightarrow \left(\beta\dot\beta\right)$ and from

$$K^{\alpha\dot\alpha} = \varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta} K_{\beta\dot\beta}\ , \tag{2.1.54}$$

it is apparent that

$$\eta^{\mu\nu} \leftrightarrow \varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta}\ . \tag{2.1.55}$$

For some constant C it holds that

$$\eta^{\mu\nu} K_\mu K_\nu = C\,\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} K_{\alpha\dot\alpha} K_{\beta\dot\beta}\ , \tag{2.1.56}$$

since both sides are invariants. The choice $C = 1/2$ yields

$$\begin{aligned}\eta^{\mu\nu} K_\mu K_\nu &= \frac{1}{2}\left[2\,\varepsilon^{01}\,\varepsilon^{\dot0\dot1} K_{0\dot0} K_{1\dot1} + 2\,\varepsilon^{01}\,\varepsilon^{\dot1\dot0} K_{0\dot1} K_{1\dot0}\right] \\ &= \left(K^0\right)^2 - \left(K^1\right)^2 - \left(K^2\right)^2 - \left(K^3\right)^2\ ,\end{aligned} \tag{2.1.57}$$

by use of (2.1.26). Equation (2.1.56) then becomes

$$\eta^{\mu\nu} K_\mu K_\nu = \frac{1}{2}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta} K_{\alpha\dot\alpha} K_{\beta\dot\beta} = \frac{1}{2}\,K_\mu K_\nu\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^{\beta\dot\beta}\ , \tag{2.1.58}$$

using (2.1.19) and it is subsequently found that

$$\eta^{\mu\nu} = \frac{1}{2}\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^{\nu,\alpha\dot\alpha}\ . \tag{2.1.59}$$

From the definition it holds that

$$\eta_{\alpha\beta\dot\alpha\dot\beta} = \eta_{\mu\nu}\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^\nu_{\beta\dot\beta} = \tilde{C}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\ , \tag{2.1.60}$$

and contracting both sides with $\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta}$ yields

$$4\tilde{C} = \eta_{\mu\nu}\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^{\nu,\alpha\dot\alpha} = 8\ , \tag{2.1.61}$$

using (2.1.59). The constant is therefore $\tilde{C} = 1/2$ and it follows that

$$\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} = \frac{1}{2}\,\eta_{\mu\nu}\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^\nu_{\beta\dot\beta}\ . \tag{2.1.62}$$

By considering

$$\begin{aligned}\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta} &= \left(\frac{1}{2}\,\eta_{\mu\nu}\,\sigma^\mu_{\alpha\dot\alpha}\,\sigma^\nu_{\beta\dot\beta}\right)\left(\tilde\kappa\eta_{\lambda\kappa}\,\sigma^{\lambda,\alpha\dot\alpha}\,\sigma^{\kappa,\beta\dot\beta}\right) \\ &= \frac{1}{2}\,\tilde\kappa\,\eta_{\mu\nu}\,\eta_{\lambda\kappa}\,2^2\eta^{\mu\lambda}\,\eta^{\nu\kappa} = 2\,\tilde\kappa\ ,\end{aligned} \tag{2.1.63}$$

it is apparent that $\tilde\kappa = 2$ and hence

$$\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta} = 2\,\eta_{\mu\nu}\,\sigma^{\mu,\alpha\dot\alpha}\,\sigma^{\nu,\beta\dot\beta}\ . \tag{2.1.64}$$

When Lorentz indices are expressed as spinor indices the correspondence is therefore

$$\eta_{\mu\nu} \leftrightarrow 2\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\ , \quad \eta^{\mu\nu} \leftrightarrow \frac{1}{2}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta}\ , \tag{2.1.65}$$

which will become important in order to obtain the right constant factors in calculations when Lorentz tensors are translated into spinors.

Momentum bilinears

For arbitrary spinors ϕ and ψ, the momentum bilinears

$$\langle\phi\psi\rangle \equiv \phi^{\dot\alpha}\psi_{\dot\alpha} = \phi^{\dot\alpha}\psi^{\dot\beta}\varepsilon_{\dot\beta\dot\alpha}\ , \quad [\phi\psi] \equiv \phi_\alpha\psi^\alpha = \phi^\alpha\psi^\beta\varepsilon_{\alpha\beta}\ , \tag{2.1.66}$$

defines respectively the holomorphic spinor product and the anti-holomorphic spinor product. If p^μ and k^μ are both massless $p_\mu p^\mu = k_\mu k^\mu = 0$, it follows that

$$(p^\mu + k^\mu)^2 = 2p_\mu k_\nu \eta^{\mu\nu} = 2\,\kappa_\alpha\kappa_{\dot\alpha}\lambda_\beta\lambda_{\dot\beta}\,\frac{1}{2}\,\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\ . \tag{2.1.67}$$

The dot product of two massless four-vectors can thereby be expressed in terms of spinor bilinears as

$$2\,p\cdot k = \langle\lambda\kappa\rangle\,[\kappa\lambda]\ , \tag{2.1.68}$$

with

$$k_{\alpha\dot\alpha} = \lambda_\alpha\lambda_{\dot\alpha}\ , \quad p_{\alpha\dot\alpha} = \kappa_\alpha\kappa_{\dot\alpha}\ . \tag{2.1.69}$$

Dot products of massless momenta as (2.1.68) will be written with the notation

$$2\,p_i\cdot p_j = \langle ij\rangle\,[ji]\ . \tag{2.1.70}$$

2.1.6 Symmetry properties of spinors

This section contains a discussion of spinors with certain symmetric properties. In particular spinors which are antisymmetric in some indices are considered and it is found that these spinors can be expressed in a simpler form. This is useful in order to express the electromagnetic field strength tensor in spinor indices.

If $\mathcal{M}$ is an arbitrary antisymmetric 2×2 matrix it is of the form

$$\mathcal{M} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.1.71}$$

and it is proportional to the matrix form (2.1.47) of $\varepsilon_{\alpha\beta}$. It is apparent from (2.1.71) that the matrix on the right hand side is necessarily proportional to any antisymmetric 2×2 matrix. In the space of 2×2 matrices, the matrix on the right hand side is therefore the only antisymmetric one up to a constant. In terms of indices, this property is manifest such that any object which is antisymmetric in two indices e.g. (α,β) must be proportional to the metric spinor $\varepsilon_{\alpha\beta}$ in the same two indices. For an arbitrary antisymmetric spinor $S_{\alpha\beta}$ it follows therefore that

$$S_{\alpha\beta} = S_{[\alpha\beta]} = \kappa\,\varepsilon_{\alpha\beta}\ , \tag{2.1.72}$$

where $[\cdots]$ denotes antisymmetrization as in (2.4.5). Contracting with $\varepsilon^{\alpha\beta}$ determines the constant

$$\kappa = \frac{1}{2}\,S_\alpha{}^\alpha\ , \tag{2.1.73}$$

and $S_{\alpha\beta}$ can subsequently be written as

$$S_{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta}\,S_{\gamma}{}^{\gamma} \ . \tag{2.1.74}$$

The spinor with only two indices is a special case of the general picture. A spinor with arbitrarily many indices with a certain antisymmetry in two of the indices may be expressed in a way analogous to (2.1.74). In order to find this expression and as a continuation of the discussion of antisymmetric spinors, the Schouten identity for products of metric spinors will now be derived.

Symmetry considerations leads to

$$\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta} \ - \varepsilon_{\delta\beta}\,\varepsilon_{\gamma\alpha} = \kappa\,\varepsilon_{\alpha\delta}\,\varepsilon_{\beta\gamma} \ . \tag{2.1.75}$$

By construction, the left hand side is antisymmetric under $(\alpha \leftrightarrow \delta)$ and as well under $(\beta \leftrightarrow \gamma)$. The right hand side has the same antisymmetric properties and (2.1.75) holds for some appropriate numerical constant κ. Contracting (2.1.75) with $\varepsilon^{\alpha\beta}\,\varepsilon^{\gamma\delta}$ yields $\kappa = -1$ and thereby

$$\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta} + \varepsilon_{\alpha\gamma}\,\varepsilon_{\delta\beta} + \varepsilon_{\alpha\delta}\,\varepsilon_{\beta\gamma} = 0 \ . \tag{2.1.76}$$

This is known as the Schouten identity and can as well be obtained from the antisymmetrization $\varepsilon_{\alpha[\beta}\,\varepsilon_{\gamma\delta]}$. In this case the antisymmetrization is performed over three indices taking only two values and the result necessarily vanishes. Equation (2.1.76) can be checked directly by some explicit choice of index values or by a contraction with any of the spinors $\varepsilon^{\alpha\beta}$, $\varepsilon^{\alpha\gamma}$, $\varepsilon^{\alpha\delta}$, $\varepsilon^{\beta\gamma}$ or $\varepsilon^{\beta\delta}$. Contracting (2.1.76) with upper indices with $\varepsilon_{\phi\alpha}\,\varepsilon_{\psi\beta}$ leads to

$$\varepsilon_{\alpha\beta}\,\varepsilon^{\gamma\delta} = \varepsilon_{\alpha}{}^{\gamma}\varepsilon_{\beta}{}^{\delta} - \varepsilon_{\alpha}{}^{\delta}\varepsilon_{\beta}{}^{\gamma} \ , \tag{2.1.77}$$

which will be useful in the derivation of an expression similar to (2.1.74) but for spinors with arbitrarily many indices. The spinor

$$S_{\dots\gamma\delta\dots} = S_{\dots[\gamma\delta]\dots} \ , \tag{2.1.78}$$

is defined with arbitrarily many indices represented by the dots and with the explicit property that it is antisymmetric in the indices (γ, δ). A contraction of (2.1.77) with $S_{\dots\gamma\delta\dots}$ yields

$$S_{\dots\alpha\beta\dots} = \frac{1}{2}\,\varepsilon_{\alpha\beta}\,S_{\dots\gamma}{}^{\gamma}{}_{\dots} \ , \tag{2.1.79}$$

which is the generalization of (2.1.74) for more than two indices.

In general, an arbitrary square matrix $\mathcal{N}$ can be expanded as a sum of its symmetric and antisymmetric components as

$$\mathcal{N} = \mathcal{N}^{(\mathrm{s})} + \mathcal{N}^{(\mathrm{a})} \ , \tag{2.1.80}$$

where the symmetric and antisymmetric components are given respectively

$$\mathcal{N}^{(\mathrm{s})} = \frac{1}{2}\left(\mathcal{N} + \mathcal{N}^{T}\right) \ , \quad \mathcal{N}^{(\mathrm{a})} = \frac{1}{2}\left(\mathcal{N} - \mathcal{N}^{T}\right) \ . \tag{2.1.81}$$

It follows from (2.1.25) that an arbitrary second-rank tensor $\xi_{\mu\nu}$ is written as

$$\xi_{\mu\nu} \leftrightarrow \xi_{\alpha\dot{\alpha}\beta\dot{\beta}} \ , \tag{2.1.82}$$

in spinor indices as an outer product of two 2×2 matrices. For an outer product of two matrices $\mathcal{N}_1$ and $\mathcal{N}_2$, an expansion of each of the matrices in symmetric and antisymmetric components as (2.1.80) leads to

$$\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}_1^{(\mathrm{s})}\mathcal{N}_2^{(\mathrm{s})} + \mathcal{N}_1^{(\mathrm{a})}\mathcal{N}_2^{(\mathrm{a})} + \mathcal{N}_1^{(\mathrm{s})}\mathcal{N}_2^{(\mathrm{a})} + \mathcal{N}_1^{(\mathrm{a})}\mathcal{N}_2^{(\mathrm{s})} \ , \tag{2.1.83}$$

with four possibilities for combining the symmetric and antisymmetric components. Viewing $\xi_{\alpha\dot{\alpha}\beta\dot{\beta}}$ as an outer product of two matrices allows an expression of the form (2.1.83) such that the spinor can be expanded as

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \xi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \xi_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} + \xi_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + \xi_{[\alpha\beta](\dot{\alpha}\dot{\beta})} \ , \tag{2.1.84}$$

in terms of the four possible ways of combining symmetrization and antisymmetrization over the indices. The notation $(\cdots)$ denotes symmetrization. Employing (2.1.79) for the antisymmetric elements in (2.1.84) leads to

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \xi_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \frac{1}{4}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,\xi_{\gamma}{}^{\gamma}{}_{\dot{\gamma}}{}^{\dot{\gamma}} + \frac{1}{2}\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,\xi_{(\alpha\beta)\dot{\gamma}}{}^{\dot{\gamma}} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\xi_{\gamma}{}^{\gamma}{}_{(\dot{\alpha}\dot{\beta})} \ . \tag{2.1.85}$$

If $\xi_{\mu\nu}$ is an antisymmetric tensor it holds that

$$\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} = -\xi_{\beta\alpha\dot{\beta}\dot{\alpha}} \ . \tag{2.1.86}$$

It is apparent from (2.1.85) that each of the first two terms are symmetric under the interchange $(\alpha\beta) \leftrightarrow \left(\dot{\alpha}\dot{\beta}\right)$ whereas the last two terms considered as one single object are antisymmetric under this interchange of indices. The first two terms constitute therefore the vanishing symmetric part of the tensor while the last two terms constitute the antisymmetric part of $\xi_{\alpha\beta\dot{\alpha}\dot{\beta}}$. The tensor $\xi_{\mu\nu}$ can hence be written in spinor indices as

$$\begin{aligned}\xi_{\alpha\beta\dot{\alpha}\dot{\beta}} &= \frac{1}{2}\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,\xi_{(\alpha\beta)\dot{\gamma}}{}^{\dot{\gamma}} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\xi_{\gamma}{}^{\gamma}{}_{(\dot{\alpha}\dot{\beta})} \\ &\equiv \varepsilon_{\dot{\alpha}\dot{\beta}}\,\phi_{\alpha\beta} + \varepsilon_{\alpha\beta}\,\psi_{\dot{\alpha}\dot{\beta}} \ ,\end{aligned} \tag{2.1.87}$$

with $\phi_{\alpha\beta}$ and $\psi_{\dot{\alpha}\dot{\beta}}$ symmetric.

2.1.7 Spinor expression for the field strength tensor

The Dirac-Born-Infeld action is build from the electromagnetic field strength tensor and its dual. It is of interest to study these two objects in spinor indices since calculations of scattering amplitudes from the Dirac-Born-Infeld action are simplified if the field strengths are expressed this way. The dual of the electromagnetic field strength tensor is defined in four dimensions using the totally antisymmetric four-dimensional tensor $\varepsilon_{\mu\nu\rho\sigma}$. In order to express the dual of the field strength in spinor indices it is therefore natural to study the totally antisymmetric four-dimensional tensor $\varepsilon_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}}$ in spinor indices. The correspondence is

$$\varepsilon_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \varepsilon_{\mu\nu\rho\sigma}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma^{\nu}_{\beta\dot{\beta}}\,\sigma^{\rho}_{\gamma\dot{\gamma}}\,\sigma^{\sigma}_{\delta\dot{\delta}} \ , \tag{2.1.88}$$

but it can be useful to instead consider another approach. Equation (2.1.76) can be used to check explicitly that

$$\begin{aligned}\left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\,\varepsilon_{\dot{\beta}\dot{\delta}}\right) &= \left(\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,\varepsilon_{\dot{\gamma}\dot{\delta}}\right) \\ &= \left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,\varepsilon_{\dot{\gamma}\dot{\delta}} - \varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\varepsilon_{\dot{\beta}\dot{\delta}}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\right) ,\end{aligned} \tag{2.1.89}$$

holds. The bracket on the left hand side is antisymmetric under the interchange of indices $(\alpha\dot{\alpha}) \leftrightarrow \left(\beta\dot{\beta}\right)$ as well as under the interchange $(\gamma\dot{\gamma}) \leftrightarrow \left(\delta\dot{\delta}\right)$. The upper bracket on the right hand side is antisymmetric under the interchange of indices $(\alpha\dot{\alpha}) \leftrightarrow (\gamma\dot{\gamma})$ and as well under $\left(\beta\dot{\beta}\right) \leftrightarrow \left(\delta\dot{\delta}\right)$ Finally, the lower bracket on the right hand side is antisymmetric under the interchange $(\alpha\dot{\alpha}) \leftrightarrow \left(\delta\dot{\delta}\right)$ and as well under $\left(\beta\dot{\beta}\right) \leftrightarrow (\gamma\dot{\gamma})$. It follows that

$$\varepsilon_{\mu\nu\rho\sigma} \leftrightarrow C\left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\,\varepsilon_{\dot{\beta}\dot{\delta}}\right) \ , \tag{2.1.90}$$

holds because the bracket has the correct antisymmetric properties in all indices. It is observed that

$$\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon_{\lambda\kappa\tau\upsilon}\,\eta^{\mu\lambda}\,\eta^{\nu\kappa}\,\eta^{\rho\tau}\,\eta^{\sigma\upsilon} = -24 \ , \tag{2.1.91}$$

with

$$\varepsilon_{0123} = -\,\varepsilon^{0123} = +1 \ , \tag{2.1.92}$$

which is used below. Evaluating explicitly the self contraction of the bracket without the constant C on the left hand side of (2.1.89) yields

$$\begin{aligned}\left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\,\varepsilon_{\dot{\beta}\dot{\delta}}\right)\frac{1}{2^4}\left(\varepsilon^{\alpha\gamma}\,\varepsilon^{\beta\delta}\,\varepsilon^{\dot{\alpha}\dot{\delta}}\,\varepsilon^{\dot{\gamma}\dot{\beta}} - \varepsilon^{\alpha\delta}\,\varepsilon^{\gamma\beta}\,\varepsilon^{\dot{\alpha}\dot{\gamma}}\,\varepsilon^{\dot{\beta}\dot{\delta}}\right)&\\ = \frac{1}{16}\left(2\times 16 - 2\times 4\right) = \frac{24}{16} \ ,&\end{aligned} \tag{2.1.93}$$

with the factor $1/2^4$ originating from the four metric tensors in (2.1.91). Hence

$$C^2 = -16 \ , \quad C = 4i \ , \tag{2.1.94}$$

where a choice of the positive solution has been made. The expression for $\varepsilon_{\mu\nu\rho\sigma}$ in spinor indices is therefore

$$\varepsilon_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = 4i\left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\,\varepsilon_{\dot{\beta}\dot{\delta}}\right) \ , \tag{2.1.95}$$

which will be used in the discussion of the dual of the electromagnetic field strength in spinor indices.

The electromagnetic field strength tensor is real and antisymmetric and can be written on the form (2.1.87). For an antisymmetric rank-two tensor in four dimensions, the dual is defined in (1.4.1) as the contraction of the tensor with the total antisymmetric symbol. It follows from (2.1.87) that the dual of $F_{\mu\nu}$ is given in spinor indices as

$$\begin{aligned}\tilde{F}_{\alpha\beta\dot{\alpha}\dot{\beta}} &= \frac{i}{2}\,4i\left(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot{\alpha}\dot{\delta}}\,\varepsilon_{\dot{\gamma}\dot{\beta}} - \varepsilon_{\alpha\delta}\,\varepsilon_{\gamma\beta}\,\varepsilon_{\dot{\alpha}\dot{\gamma}}\,\varepsilon_{\dot{\beta}\dot{\delta}}\right)\frac{1}{4}\left(\varepsilon^{\dot{\gamma}\dot{\delta}}\,\phi^{\gamma\delta} + \varepsilon^{\gamma\delta}\,\psi^{\dot{\gamma}\dot{\delta}}\right)\\ &= \varepsilon_{\alpha\beta}\,\psi_{\dot{\alpha}\dot{\beta}} - \varepsilon_{\dot{\alpha}\dot{\beta}}\,\phi_{\alpha\beta} \ .\end{aligned} \tag{2.1.96}$$

The field strength itself is given as

$$F_{\alpha\beta\dot{\alpha}\dot{\beta}} = \varepsilon_{\alpha\beta}\,\psi_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}}\,\phi_{\alpha\beta} \ , \tag{2.1.97}$$

and in terms of

$$\psi_{\dot\alpha\dot\beta} = {}^+F_{\dot\alpha\dot\beta} \ , \quad \phi_{\alpha\beta} = {}^-F_{\alpha\beta} \ , \tag{2.1.98}$$

the field strength and its dual can be rewritten as

$$F_{\alpha\beta\dot\alpha\dot\beta} = \varepsilon_{\alpha\beta}\, {}^+F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\, {}^-F_{\alpha\beta} \ , \tag{2.1.99}$$

$$\tilde F_{\alpha\beta\dot\alpha\dot\beta} = \varepsilon_{\alpha\beta}\, {}^+F_{\dot\alpha\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\, {}^-F_{\alpha\beta} \ , \tag{2.1.100}$$

which is nothing but a rescaling corresponding to the normalization

$${}^+F_{\mu\nu} \leftrightarrow \sqrt{2}\varepsilon_{\alpha\beta}{}^+F_{\dot\alpha\dot\beta} \ . \tag{2.1.101}$$

Since ${}^+F_{\alpha\beta}$ and ${}^-F_{\dot\alpha\dot\beta}$ are symmetric, the right hand sides of (2.1.99) and (2.1.100) are clearly antisymmetric under the interchange $\left(\alpha\dot\alpha \leftrightarrow \beta\dot\beta\right)$ as they should be. This particular antisymmetric form could have been guessed based on pure antisymmetry considerations. It follows from (2.1.99) for the selfdual and anti-selfdual that

$${}^+F_{\dot\alpha\dot\beta} = \frac{1}{2}\, F_{\mu\nu}\, \sigma^\mu_{\alpha\dot\alpha}\, \sigma^\nu_{\beta\dot\beta}\, \varepsilon^{\alpha\beta} = \frac{1}{2}\, F_{\alpha\beta\dot\alpha\dot\beta}\, \varepsilon^{\alpha\beta} \ , \quad {}^-F_{\alpha\beta} = \frac{1}{2}\, F_{\mu\nu}\, \sigma^\mu_{\alpha\dot\alpha}\, \sigma^\nu_{\beta\dot\beta}\, \varepsilon^{\dot\alpha\dot\beta} = \frac{1}{2}\, F_{\alpha\beta\dot\alpha\dot\beta}\, \varepsilon^{\dot\alpha\dot\beta} \ , \tag{2.1.102}$$

which will be used later in manipulations of the Dirac-Born-Infeld action and in specific calculations of scattering amplitudes.

2.1.8 Massive vectors

Equation (2.1.30) was obtained for a massless vector $K_{\alpha\dot\alpha}$ and it is natural also to consider the general case where $K_{\alpha\dot\alpha}$ is massive. The result will be used later in chapter 4. For any massive vector p^μ, one can construct the vector

$$b^\mu = p^\mu - \frac{p^2}{2\, p\cdot q}\, q^\mu \ , \tag{2.1.103}$$

in terms of the massless vector q^μ and with

$$p\cdot q \neq 0 \ . \tag{2.1.104}$$

It follows from

$$b^2 = p^2 - 2\, \frac{p^2}{2\, p\cdot q}\, q^\mu p_\mu = 0 \ , \tag{2.1.105}$$

that b^μ is massless and hence it can be considered as the massless part of p

$$b_\mu \to p^\flat_\mu \ . \tag{2.1.106}$$

Rearranging (2.1.103) yields in spinor indices

$$p_{\alpha\dot\alpha} = \lambda_\alpha\lambda_{\dot\alpha} - \frac{p^2}{2\, p\cdot q}\, \zeta_\alpha\zeta_{\dot\alpha} \ , \tag{2.1.107}$$

where

$$\lambda_\alpha \lambda_{\dot\alpha} = p^\flat_{\alpha\dot\alpha} \leftrightarrow p^\flat_\mu \ , \quad \zeta_\alpha \zeta_{\dot\alpha} = q_{\alpha\dot\alpha} \leftrightarrow q_\mu \ , \tag{2.1.108}$$

are massless spinors. It should be noted that in the case where p_μ is massless, (2.1.107) reduces properly

$$p_{\alpha\dot\alpha} = \lambda_\alpha \lambda_{\dot\alpha} \ , \tag{2.1.109}$$

in agreement with (2.1.30). It is observed that

$$p \cdot q = \frac{1}{2}\, p_{\alpha\dot\alpha} \zeta^\alpha \zeta^{\dot\alpha} = \frac{1}{2}\, \lambda_\alpha \lambda_{\dot\alpha} \zeta^\alpha \zeta^{\dot\alpha} = p^\flat \cdot q \ , \tag{2.1.110}$$

which can be substituted in the denominator in (2.1.107).

2.1.9 Polarization vectors

From (2.2.63) the polarization vector enters the amplitude calculations as

$$A = \frac{1}{2}\, \varepsilon^\pm_{\alpha\dot\alpha} A^{\alpha\dot\alpha} \ . \tag{2.1.111}$$

For on-shell photon fields in four dimensions the corresponding polarization vectors are

$$\varepsilon^+_{\alpha\dot\alpha} = \sqrt{2}\, \frac{\zeta_\alpha \lambda_{\dot\alpha}}{[\zeta\lambda]} \ , \quad \varepsilon^-_{\alpha\dot\alpha} = \sqrt{2}\, \frac{\zeta_{\dot\alpha} \lambda_\alpha}{\langle\zeta\lambda\rangle} \ , \tag{2.1.112}$$

where the spinor λ is the corresponding photon momentum as in (2.1.109) and ζ is a massless spinor which can be chosen freely due to the freedom of on-shell gauge transformations [15]. The spinor expression for the polarization vectors can be found explicitly as in [16] however here it will only be checked that (2.1.112) are both transverse to the momentum and that a contraction of two polarization vectors of the same kind vanishes. The transversality is apparent from

$$\frac{1}{2}\, \lambda^\alpha \lambda^{\dot\alpha} \left(\sqrt{2}\, \frac{\zeta_\alpha \lambda_{\dot\alpha}}{[\zeta\lambda]} \right) = \frac{1}{2}\, \lambda^\alpha \lambda^{\dot\alpha} \left(\sqrt{2}\, \frac{\zeta_{\dot\alpha} \lambda_\alpha}{\langle\zeta\lambda\rangle} \right) = 0 \ , \tag{2.1.113}$$

while

$$\frac{1}{2}\, \varepsilon^+_{i,\alpha\dot\alpha}\, \varepsilon^{+,\alpha\dot\alpha}_j = \frac{\zeta_\alpha \lambda_{i,\dot\alpha}}{[\zeta\lambda_i]} \frac{\zeta^\alpha \lambda^{\dot\alpha}_j}{[\zeta\lambda_j]} = \frac{1}{2}\, \varepsilon^-_{i,\alpha\dot\alpha}\, \varepsilon^{-,\alpha\dot\alpha}_j = \frac{\zeta_{\dot\alpha} \lambda_{i,\alpha}}{\langle\zeta\lambda_i\rangle} \frac{\zeta^{\dot\alpha} \lambda^{\dot\alpha}_j}{\langle\zeta\lambda_j\rangle} = 0 \ , \tag{2.1.114}$$

shows how the contraction of two polarization vectors of the same polarization state vanishes. This is the case even for different particles. The square root in (2.1.112) ensures the normalization

$$\frac{1}{2}\, \varepsilon^+_{\alpha\dot\alpha}\, \varepsilon^{-,\alpha\dot\alpha} = \frac{\zeta_\alpha \lambda_{\dot\alpha}}{[\zeta\lambda]} \frac{\zeta^{\dot\alpha} \lambda^\alpha}{\langle\zeta\lambda\rangle} = 1 \ . \tag{2.1.115}$$

In Lorentz indices the polarization vectors can be written in the light cone frame as

$$\varepsilon^{+,\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \ , \quad \varepsilon^{-,\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ +i \\ 0 \end{pmatrix} \ , \tag{2.1.116}$$

which becomes

$$\varepsilon^{+}_{\alpha\dot{\alpha}} = \sqrt{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \quad \varepsilon^{-}_{\alpha\dot{\alpha}} = \sqrt{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ , \tag{2.1.117}$$

in spinor notation.

2.2 Field theoretical methods

The fundamental objects, i.e. the strings in a string theory have finite size of the order of the Planck length $\ell_{\mathrm{P}} \simeq 10^{-33}$ cm. This is a crucial difference between string theory and quantum field theory where particles are point like. Small distances correspond to high energies and if a string theory is studied in the low energy limit, the finite extension of the strings is invisible such that the string theory therefore appears to be equivalent to a quantum field theory. In this case, the string theory can be studied by using quantum field theory as an effective theory in the low-energy limit so that computations are performed with quantum field theory methods. This section presents the important concepts which will be used in computations in chapters 3 and 4. The discussion in the subsequent sections 2.2.1 - 2.2.6 are based on [17, 18, 19, 20].

2.2.1 Propagators and integration contours

The propagator is a central object in computations in a field theory. The notion of the propagator will be discussed in this section where the Klein-Gordon field is used as an example. In particular the Feynman propagator will be discussed since it is useful in perturbative calculations for interacting fields.

In the Heisenberg picture the Klein Gordon field has the expansion

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^{\dagger}_{\mathbf{p}} e^{+ip\cdot x} \right) \ , \tag{2.2.1}$$

where $a_{\mathbf{p}}$ and $a^{\dagger}_{\mathbf{p}}$ are the usual time independent ladder operators acting on the quantum states of the Hilbert space. The vacuum state in this space is denoted by $|0\rangle$ and in the free Klein-Gordon theory the amplitude for a particle to propagate from the spacetime point y to the spacetime point x is determined by the vacuum expectation value

$$D(x-y) \equiv \langle 0|\, \phi(x)\, \phi(y)\, |0\rangle \ , \tag{2.2.2}$$

where $D(x-y)$ is referred to as the propagator. The object $\phi(x)\,\phi(y)$ contains four products of ladder operators but since it is placed inside two vacuum states the term

$$\langle 0|a_{\mathbf{p}} a^{\dagger}_{\mathbf{q}}|0\rangle = (2\pi)^3\, \delta^{(3)}(\mathbf{p}-\mathbf{q}) \ , \tag{2.2.3}$$

is the only one which is nonzero. This normalization and the three-dimensional delta function yields for the propagator

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}}\, e^{-ip\cdot(x-y)} \ . \tag{2.2.4}$$

In order to study different propagators it is observed that a four-dimensional momentum integral can be expressed as

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2}\, e^{-ip\cdot(x-y)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{i}{(p^0 + E_{\mathbf{p}})\,(p^0 - E_{\mathbf{p}})}\, e^{-ip^0\left(x^0 - y^0\right)} \ , \tag{2.2.5}$$

using the relativistic relation

$$m^2 = E_{\mathbf{p}}^2 - |\mathbf{p}|^2 \ , \tag{2.2.6}$$

such that

$$p^2 - m^2 = \left(p^0\right)^2 - E_{\mathbf{p}}^2 = \left(p^0 - E_{\mathbf{p}}\right)\left(p^0 + E_{\mathbf{p}}\right) \ . \tag{2.2.7}$$

The p^0 integral in (2.2.5) has poles at $p^0 = \pm E_{\mathbf{p}}$ and it can be evaluated as a contour integral in the complex plane for some appropriate choice of integration contour. In the specific case where $x^0 > y^0$ the contour shown in figure 2.1 is used and according to Jordan's lemma the convergence of the exponential on the integration contour is ensured by closing it in the lower half plane. This integration picks up both poles and the integral gives

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{i}{(p^0 + E_{\mathbf{p}})\,(p^0 - E_{\mathbf{p}})}\, e^{-ip^0\left(x^0 - y^0\right)} = \frac{1}{2E_{\mathbf{p}}}\, e^{-iE_{\mathbf{p}}\left(x^0 - y^0\right)} + \frac{1}{-\,2E_{\mathbf{p}}}\, e^{+E_{\mathbf{p}}\left(x^0 - y^0\right)} \ . \tag{2.2.8}$$

A substitution of (2.2.8) into (2.2.5) with a shift $\mathbf{p} \to -\mathbf{p}$ in integration variable for the second term yields

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2}\, e^{-p\cdot(x-y)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left[e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)}\right] \ , \tag{2.2.9}$$

since $E_{\mathbf{p}} = E_{-\mathbf{p}}$. In the case $x^0 < y^0$ the integral is zero because the integration contour

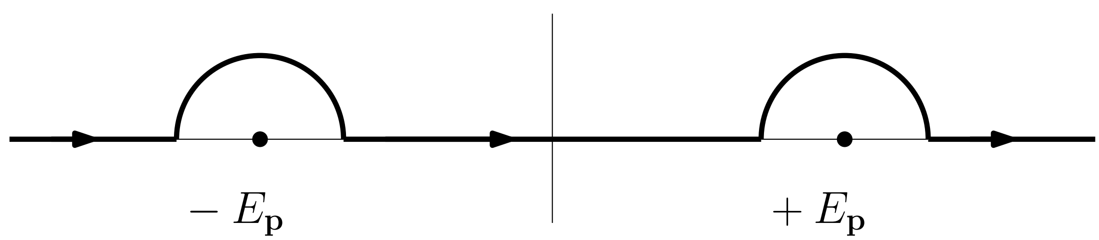

Figure 2.1: The integration contour for the p^0-integration corresponding to the retarded propagator. For $x^0 > y^0$ the integration contour can be closed in the lower half plane to pick up both poles. For $x^0 < y^0$ the contour has to be closed in the upper half plane and the integration is zero.

has to be closed in the upper half plane and hence encloses no poles. The right hand side of (2.2.9) is equal to the vacuum expectation value of the commutator

$$\langle 0|\,[\phi(x)\,,\phi(y)]\,|0\rangle = [\phi(x)\,,\phi(y)] \ , \tag{2.2.10}$$

which is a complex number and can be evaluated from (2.2.1). The normalization $\langle 0|0\rangle = 1$ has been used. In terms of the Heaviside step function

$$\theta(x) = \begin{cases} 1\ , & \text{for } x > 0 \\ 0\ , & \text{for } x < 0\ , \end{cases} \tag{2.2.11}$$

the expectation value (2.2.10) defines the retarded propagator as

$$D_R(x-y) \equiv \theta\big(x^0 - y^0\big)\, \langle 0|\, [\phi(x)\,,\phi(y)]\, |0\rangle\ , \tag{2.2.12}$$

which is the right hand side of (2.2.9) since the integral is zero for $x^0 < y^0$ as discussed above. The result of the Klein-Gordon operator acting on (2.2.12) can be obtained as

$$\big(\partial^2 + m^2\big)\, D_R(x-y) = \,-\, i\delta^{(4)}(x-y)\ , \tag{2.2.13}$$

which shows that the retarded propagator is a Green's function for the Klein-Gordon operator. If the retarded propagator is Fourier expanded and acted on by the Klein-Gordon operator it follows that

$$D_R(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2}\, e^{-ip\cdot(x-y)}\ , \tag{2.2.14}$$

with the same prescription for going around the two poles. Other prescriptions for the integration can be chosen and for the case $x^0 < y^0$ the contour is closed in the upper half plane picking up both poles. The propagator associated with this prescription is known as the advanced propagator.

When interacting fields in perturbation theory will be discussed in section 2.2.2 it turns out that a more physical prescription for the integration contour is the Feynman prescription in which the associated propagator is causal. This is not the case for neither the retarded or the advanced propagator as both have support outside the light cone. As is the case for the retarded propagator also the advanced propagator and the Feynman propagator are Green's functions with different boundary conditions for the Klein-Gordon operator. The Feynman prescription is written in terms of time ordering which is defined for two fields $\phi(x)$ and $\phi(y)$ as

$$\begin{aligned} T\{\phi(x)\,\phi(y)\} &= \begin{cases} \phi(x)\,\phi(y)\ , & \text{for } x^0 > y^0 \\ \phi(y)\,\phi(x)\ , & \text{for } x^0 < y^0 \end{cases} \\ &= \theta\big(x^0 - y^0\big)\, \phi(x)\,\phi(y) + \theta\big(y^0 - x^0\big)\, \phi(y)\,\phi(x)\ , \end{aligned} \tag{2.2.15}$$

placing the field with the latest time to the left. In (2.2.15) time ordering is written for just two fields but the operation has a straightforward generalization to an arbitrary number of fields. The Feynman prescription is associated with the Feynman propagator

$$D_F\,(x-y) = \langle 0|\, T\{\phi(x)\,\phi(y)\}\, |0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\varepsilon}\ , \tag{2.2.16}$$

where the inclusion of the term $i\varepsilon$ with infinitesimal ε is a convenient way to remember the contour prescription. The integral should be evaluated in the limit $\varepsilon \to 0$ whereby the denominator in (2.2.16) becomes

$$\big(p^0 + E_{\mathbf{p}} - i\tilde{\varepsilon}\big)\,\big(p^0 - E_{\mathbf{p}} + i\tilde{\varepsilon}\big) = \big(p^0 + E_{\mathbf{p}}\big)\,\big(p^0 - E_{\mathbf{p}}\big) + i\varepsilon\ , \tag{2.2.17}$$

with

$$\varepsilon = 2\tilde{\varepsilon} E_{\mathbf{p}} \ , \tag{2.2.18}$$

neglecting second order terms in ε. The poles are displaced infinitesimally from the real axis to positions at $p^0 = \pm(E - i\tilde{\varepsilon})$ such that the contour of the integration along the real axis runs below and above the poles respectively. An explicit evaluation of the p^0 integral in the Feynman propagator according to the prescription shown on figure 2.2.2 yields

$$D_F(x-y) = \begin{cases} D(x-y) \ , & \text{for } x^0 > y^0 \\ D(y-x) \ , & \text{for } x^0 < y^0 \ , \end{cases} \tag{2.2.19}$$

by a comparison with (2.2.4). For the integral $x^0 < y^0$ the integration variable has to be shifted $\mathbf{p} \to -\mathbf{p}$ as was the case above.

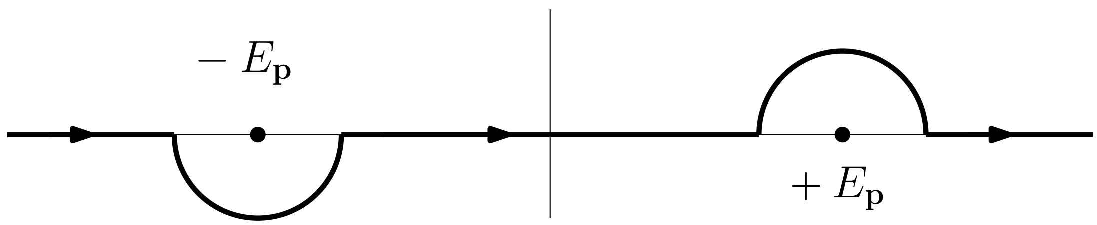

Figure 2.2: The Feynman prescription for the p^0-integration in the complex plane. The integration contour is closed in the lower half plane for $x^0 > y^0$ whereas the contour is closed in the upper half plane for $x^0 < y^0$.

2.2.2 Field interactions

In order to calculate real physical quantities such as cross sections of particle scattering one needs to compute amplitudes for certain interaction processes. An essential part of an amplitude is the correlation function which correlates fields on spacetime and has the physical interpretation of the amplitude for a particle to propagate between two spacetime points. The two-point correlation function of the free theory has already been discussed above and is just the Feynman propagator. In this section the general correlation function, i.e. the time ordered expectation value of fields between vacuum states of the interacting theory will be expressed as a time ordered expectation value of interacting fields between vacuum states of the free theory. Using ϕ^4-theory as an example, interactions are included in the theory as a correction to the non-interacting Hamiltonian

$$H = H_0 + H_{\text{interaction}} \ , \tag{2.2.20}$$

with

$$H_{\text{interaction}} = \int \mathrm{d}^3x \, \frac{\lambda}{4!} \, \phi^4 \ , \tag{2.2.21}$$

such that the interaction can be evaluated perturbatively as a power series in λ. The fields and the states in the free theory can be manipulated straightforwardly whereas the interacting fields and states are harder to manipulate. The fields and states of the interacting theory are therefore expressed perturbatively in terms of fields and states of the free theory.

The interaction Hamiltonian enters the interacting theory in two places namely in the field operator itself and in the vacuum state $|\Omega\rangle$ of the interacting field. The Heisenberg field in the interacting theory is defined as

$$\phi(x) = e^{iHt}\phi(\mathbf{x})\,e^{-iHt}\ , \tag{2.2.22}$$

with the Hamiltonian (2.2.20). What is known as the interacting field is defined subsequently as

$$\phi(x)\,|_{\lambda=0} \equiv \phi_I(x) = e^{iH_0(t-t_0)}\phi(t_0,\mathbf{x})\,e^{+iH_0(t-t_0)}\ , \tag{2.2.23}$$

which can be constructed explicitly as an expansion using creation and annihilation operators as in (2.2.1). When $\lambda = 0$, H becomes H_0 and there is no interaction. But as λ is assumed to be a small parameter, (2.2.23) is still an expression for the important part of the time dependence of the interacting field. Since $H_{\text{interaction}}$ is taken as a small perturbation it can be assumed that the vacuum states $|\Omega\rangle$ and $|0\rangle$ have some overlap which is a crucial assumption in order to relate the vacuum expectation value to the general expectation value. It is found that

$$\langle\Omega|\,T\{\phi(x_1)\,\phi(x_2)\}\,|\Omega\rangle = \frac{\langle 0|\,T\left\{\phi_I(x_1)\,\phi_I(x_2)\exp\left[-i\int \mathrm{d}t\,H_{\text{interaction}}\right]\right\}|0\rangle}{\langle 0|\,T\left\{\exp\left[-i\int \mathrm{d}t\,H_{\text{interaction}}\right]\right\}|0\rangle}\ , \tag{2.2.24}$$

and hence that

$$\langle\Omega|\,T\{\phi(x_1)\cdots\phi(x_n)\}\,|\Omega\rangle \to \langle 0|\,T\{\phi_I(x_1)\cdots\phi_I(x_n)\}\,|0\rangle\ , \tag{2.2.25}$$

holds. This should be understood in the sense that evaluating a n-point correlation function of Heisenberg fields in the interacting theory corresponds to evaluating a n-point correlation function of interacting fields as defined in (2.2.23) in the free theory. That the right hand side of (2.2.25) is a vacuum expectation value means that the Feynman propagator enters in computations of such correlation functions. This is an important observation which will be discussed in the next section.

2.2.3 Wick contractions

Interactions will naturally always involve the interaction field. For notational reasons the interaction subscript $\phi_I(x)$ will be dropped on fields in the following though the fields are still interacting fields as (2.2.23). When computing correlation functions of interacting fields, Wick's theorem is useful since it expresses a given correlation function as products of Feynman propagators.

In order to establish the relation between correlation functions and the Feynman propagator the operation of normal ordering of operators and the contraction of two fields will be defined. However the first step is to split up the operator of the interaction field into positive and negative frequency parts as

$$\phi(x) = \phi^+(x) + \phi^-(x)\ , \tag{2.2.26}$$

such that.

$$0 = \phi^+(x)\,|0\rangle = \langle 0|\,\phi^-(x)\ . \tag{2.2.27}$$

For $x^0 > y^0$ the time ordered product of two fields can be written with a commutator as

$$\begin{aligned} T\{\phi(x_1)\,\phi(x_2)\} = & \left[\phi^+(x_1)\,,\phi^-(x_2)\right] + \phi^-(x_2)\,\phi^+(x_1) \\ & + \phi^+(x_1)\,\phi^+(x_2) + \phi^-(x_1)\,\phi^+(x_2) + \phi^-(x_1)\,\phi^-(x_2)\ \ , \end{aligned} \tag{2.2.28}$$

while the time ordered product for $x^0 < y^0$ gives the same result but with x and y interchanged in the commutator. The operation of normal ordering is defined as

$$: a_{\mathbf{p}} a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{k}} a_{\mathbf{s}} : = a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{k}} a_{\mathbf{p}} a_{\mathbf{s}}\ , \tag{2.2.29}$$

where respectively the creation and annihilation operators commute mutually and their order is therefore irrelevant. An important observation is that the vacuum expectation value of a collection of normal ordered operators is zero. In (2.2.28) all terms except the commutator are normal ordered which means that the commutator is the only nonzero contribution of the vacuum expectation value of the time ordered product of two fields.

The Wick contraction of two fields is defined as

$$\overbrace{\phi(x)\,\phi}(y) = \begin{cases} [\phi^+(x)\,,\phi^-(y)]\ , & \text{for } x^0 > y^0 \\ [\phi^+(y)\,,\phi^-(x)]\ , & \text{for } x^0 < y^0\ , \end{cases} \tag{2.2.30}$$

and it is seen that

$$D_F(x-y) = \overbrace{\phi(x)\,\phi}(y)\ \ . \tag{2.2.31}$$

From (2.2.28) the relation between time ordering and normal ordering can be written in terms of a field contraction as

$$T\{\phi(x)\,\phi(y)\} = :\phi(x)\,\phi(y): + \overbrace{\phi(x)\,\phi}(y)\ \ , \tag{2.2.32}$$

which can be generalized to an arbitrary number of fields

$$T\{\phi(x_1)\cdots\phi(x_n)\} = :\phi(x_1)\cdots\phi(x_n): + \sum(\text{all possible contractions})\ \ , \tag{2.2.33}$$

where the sum contains a term for each way of contracting the fields. An example with four fields is convenient in order to show the structure

$$\begin{aligned} T\{\phi_a\phi_b\phi_c\phi_d\} = & :\phi_a\phi_b\phi_c\phi_d: + (D_{ab}:\phi_c\phi_d:) + (D_{ac}:\phi_b\phi_d:) + (D_{ad}:\phi_b\phi_c:) \\ & + (D_{bc}:\phi_a\phi_d:) + (D_{bd}:\phi_a\phi_c:) + (D_{cd}:\phi_a\phi_b:) \\ & + D_{ab}D_{cd} + D_{ac}D_{bd} + D_{ad}D_{bc}\ , \end{aligned} \tag{2.2.34}$$

which is a useful result. If the vacuum expectation value of (2.2.34) is evaluated, only the three fully contracted terms in the last line survive. The conclusion is that correlation functions are computed by evaluating all possible full Wick contractions of the involved fields. The terms that are not fully contracted simply vanish in the vacuum expectation value.

2.2.4 Path integral formulation

A quantum theory is the result of a quantization of a classical theory according to some quantization procedure. The same theory can be quantized in different ways and one speaks of different formulations of the same theory. Usually ordinary quantum mechanics is formulated using the procedure of canonical quantization where the classical variables such as position and momentum are promoted to quantum mechanical operators. An alternative formulation of a quantum theory is the path integral formulation in which a certain classical theory is quantized using path integrals. The path integral formulation can be used for quantum mechanics as well as for quantum field theories and it is due to Richard Feynman based on earlier work by Paul Dirac. In this formulation the fields in a quantum field theory remain ordinary functions instead of operators. The creation and annihilation operators in the canonical quantization provide a good understanding of the notion of particles which is not the case for the path integral. However, the path integral formulation has certain advantages. This comes about because the canonical quantization uses the Hamiltonian formalism where time has a special role and Lorentz invariance is therefore broken. In the path integral formulation of a quantum field theory, the Lagrangian is used instead of the Hamiltonian as the most fundamental way of specifying the theory. There is nothing special about time in the Lagrangian and it has therefore a build in manifest Lorentz invariance. Furthermore, the path integral method preserves all other symmetries which the Lagrangian may have. With the path integral method, computations can be done directly from the Lagrangian without invoking the Hamiltonian. The Hamiltonian dynamics are therefore taken to be defined by the path integral of the Lagrangian.

A natural way to introduce the path integral is to consider a double slit experiment where a quantum mechanical particle propagates from a source to a detector. Along the way of propagation the particle passes a screen with two closely spaced slits in it; a double slit. In a classical description of the propagation path the particle passes the double slit through either one or the other of the two slits whereas quantum mechanics has a fundamentally different interpretation in terms of wave functions. The particle is described by a wave function and as a wave it propagates through both slits to create an interference pattern with itself on the detector. This interference pattern is determined by the superposition of the two wave contributions from the slits. Since in this particular case only two possible propagation paths exist, only two contributions in the superposition sum are present. In general the number of possible paths can be infinite in which case the space of paths becomes continuous and the discrete total sum of superposition contributions becomes an integral over all possible paths. This integral is exactly the path integral.

The path integral provides the transition amplitude $\mathcal{A}(x_i, x_f; t)$ for some particle to propagate from a spacetime point x_i to the point x_f and it is the continuous limit of the sum of amplitudes for each of all possible paths on which the particle can propagate. The total sum of amplitudes is basically the sum of different phases for the different paths and hence the total propagation amplitude is written in terms of the path integral as

$$\mathcal{A}(x_i, x_f; t) = \sum e^{i\phi} \rightarrow \int \mathcal{D}x(t)\, e^{i\phi} \;, \tag{2.2.35}$$

where the arrow indicates the continuous limit. The integration measure $\mathcal{D}x(t)$ states that the integration is over the continuous space of all the coordinate functions that connect the points x_i and x_f. Each coordinate function is a function of time.

The path integral can be viewed as part of a generalization of calculus from spaces of numbers to spaces of functions. In this sense, a functional is defined as a function that maps functions into numbers. The path integral associates a complex number with each function $x(t)$ and the path integral is therefore a functional. In the classical limit the transition amplitude should have only one contribution from the path integral namely the classical path.

Considering the classical limit of (2.2.35) using a physicist hand waving arguments motivates

$$\int \mathcal{D}x(t)\, e^{i\phi} = \int \mathcal{D}x(t)\, e^{(i/\hbar)S[x(t)]} \ , \tag{2.2.36}$$

in the sense that the classical limit corresponds formally to

$$S[x\,(t)] \gg \hbar \ , \quad \text{or} \quad \hbar \to 0 \ , \tag{2.2.37}$$

such that the integrand on the right hand side of (2.2.36) oscillates wildly in the classical limit. These wild oscillations integrate to zero and hence the classical path can be identified as the contribution to the transition amplitude with a stationary phase. Mathematically speaking this argumentation is poor but from a physical point of view it makes sense. According to the action principle, the classical path is the path for which the action is a stationary minimum and this is exactly the reason why the phase is identified with the action as in (2.2.36). It can be checked explicitly that the right hand side of this expression provides the correct interference pattern for the double slit experiment.

The generalization of the path integral formulation to an infinite number of paths $x(t)$ is carried out by discretizing the time interval and approximating the path in each time interval by a straight line. In each time interval an integration over the coordinate is performed and the general form of the path integral is found as the limit where the number of time steps becomes infinite and the length of each step approaches zero. This procedure will not be discussed in further detail here.

When the functional integral formalism is applied to a quantum field theory of real scalar fields it turns out that correlation functions for the interacting theory can be computed in a way which has a certain similarity to (2.2.24). The two point correlation function is computed as the path integral

$$\langle\Omega|\, T\,\{\phi(x_1)\,\phi(x_2)\}\,|\Omega\rangle \;=\; \frac{\displaystyle\int \mathrm{D}\phi\,\phi(x_1)\,\phi(x_2)\exp\left[i\int \mathrm{d}^4x\,\mathcal{L}\right]}{\displaystyle\int \mathcal{D}\phi\,\exp\left[i\int \mathrm{d}^4x\,\mathcal{L}\right]} \ , \tag{2.2.38}$$

and it is noted that the path integral depends on the Lagrangian rather than on the Hamiltonian as discussed above. The Wick contraction as discussed in section 2.2.3 is defined in terms of the path integral

$$\phi(x_1)\,\phi(x_2) = \frac{\displaystyle\int \mathcal{D}\phi\,\phi(x_1)\,\phi(x_2)\exp\left[i\int \mathrm{d}^4x\,\mathcal{L}_0\right]}{\displaystyle\int \mathcal{D}\phi\exp\left[i\int \mathrm{d}^4x\,\mathcal{L}_0\right]} = D_F(x_1 - x_2) \ , \tag{2.2.39}$$

by considering the non-interacting Klein-Gordon field. In fact, different n-point functions for the free theory can be computed using path integrals with the rules of Gaussian integration and the result is the same as obtained with Wick's theorem.

2.2.5 Functional methods

Another method to compute correlation functions is the formal one of functional differentiation of the generating functional. This method is convenient when the non-interacting Lagrangian is replaced by an interacting one. The generating functional for a scalar field is defined as

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[i \int \mathrm{d}^4x \, [\mathcal{L} + J(x)\,\phi(x)] \right], \tag{2.2.40}$$

with the inclusion of the source term $J(x)\,\phi(x)$ in the exponential. Functional differentiation can be viewed as a continuous generalization of differentiation of discrete vectors and it is defined in four dimensions as

$$\frac{\delta}{\delta J(x)}\, J(y) = \delta^{(4)}(x-y) \ , \tag{2.2.41}$$

such that

$$\frac{\delta}{\delta J(y)} \int \mathrm{d}^4x \, J(x)\,\phi(x) = \phi(y) \ . \tag{2.2.42}$$

The two-point correlation function in the free theory is then computed by differentiating the generating functional with respect to the source as

$$\langle 0|\, T\{\phi(x_1)\,\phi(x_2)\}\,|0\rangle \ = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\,\Big|_{J=0} , \tag{2.2.43}$$

with

$$Z_0 = Z[J=0] = \int \mathcal{D}\phi \, \exp\left[i \int \mathrm{d}^4x \mathcal{L} \right]. \tag{2.2.44}$$

The source is put $J=0$ after the differentiations have been carried out. Equation (2.2.43) is just a special case of the generalization

$$\begin{aligned} \langle 0|\, T\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle \ &= \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\,\Big|_{J=0} \\ &= \frac{\displaystyle\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n) \exp\left[i \int \mathrm{d}^4x \, \mathcal{L}\right]}{\displaystyle\int \mathcal{D}\phi \, \exp\left[i \int \mathrm{d}^4x \, \mathcal{L}\right]} \ , \end{aligned} \tag{2.2.45}$$

which is a basic formula for computations. The fields on the left hand side are operators and hence this formula connects the operator formalism with the path integral formalism.

For the free Klein Gordon Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 \ , \tag{2.2.46}$$

the integral in the exponent of (2.2.40) can be rewritten by a partial integration as

$$\int \mathrm{d}^4x \, \frac{1}{2}\left[\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2 + J\phi\right] = \int \mathrm{d}^4x \frac{1}{2}\left[\phi\left(-\partial^2 + m^2\right)\phi + J\phi\right] \ . \tag{2.2.47}$$

By substituting the shifted field

$$\phi'(x) = \phi(x) - i \int \mathrm{d}^4y \, D_F(x-y)\, J(y) \ , \tag{2.2.48}$$

into (2.2.47), using that the Feynman propagator is a Green's function of the Klein-Gordon operator and changing integration variable back to $\phi(x)$ yields for the free field generating functional

$$Z[J] = Z_0 \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\, D_F(x-y)\, J(y)\right] . \tag{2.2.49}$$

It follows that

$$\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0} = Z_0\, D_F(x_1 - x_2) \ , \tag{2.2.50}$$

and hence

$$D_F(x_1 - x_2) = \frac{\int \mathcal{D}\phi\, \phi(x_1)\, \phi(x_2) \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}_0\right]}{\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}_0\right]} . \tag{2.2.51}$$

This is in agreement with (2.2.45).

The interaction part of the theory is introduced as a perturbation to the free theory. One can consider ϕ^4-theory as an example in which the interaction comes from the term

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \ , \tag{2.2.52}$$

in the Lagrangian where λ is a small parameter. That λ is small allows for the expansion

$$\exp\left[i\int \mathrm{d}^4x\, \mathcal{L}\right] = \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}_0\right]\left(1 - i\int \mathrm{d}^4x \frac{\lambda}{4!}\phi^4\right) \ , \tag{2.2.53}$$

of the exponential which should be performed in both the numerator and denominator in the four-point equivalent to (2.2.51) in the interacting theory. The expansion (2.2.53) of the denominator does only contribute with vacuum diagrams which are not relevant when only tree-level diagrams will be studied.

After the introduction of functional methods, this is the place for a comment in connection with section 1.1 on effective theories. In this context the path integral is used in the definition of the effective action. The massive fields which are neglected in the effective theory are said to be integrated out in the low-energy effective theory. This procedure is outlined in the following. Symbolically ϕ represents all the fields in a particular theory such that

$$\phi = \{\phi_0, \phi_m\} \ , \tag{2.2.54}$$

with ϕ_0 representing a massless field and ϕ_m representing all massive fields. The generating functional Z is used in calculations of expectation values as discussed above and in terms of the ϕ-fields

$$Z = \int \mathcal{D}\phi\, e^{iS[\phi]} = \int \mathcal{D}\phi_0 \mathcal{D}\phi_m\, \phi_0\, \phi_m e^{iS[\phi_0,\phi_m]} \ , \tag{2.2.55}$$

is a trivial expansion. In the low-energy limit, the interactions of the massive fields are neglected so that the integration is independent of the massive fields

$$Z = C\int \mathcal{D}\phi_0\, \phi_0\, e^{iS_{\mathrm{eff}}[\phi_0]} . \tag{2.2.56}$$

Hence the massive fields have been integrated out and the theory is described in the low-energy limit by the effective action as a function only of the massless field.

2.2.6 LSZ reduction formalism

In principle, the Lagrangian for some physical theory provides all information on the dynamics of the system. It has been discussed above how the path integral formalism with the Lagrangian is a neat way of computing relevant physical quantities. However the way from the Lagrangian of a particular theory to actual predictions of measurable physical quantities is still not straightforward. The LSZ reduction formalism is a useful step on this way. The name is due to the three German physicist Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann and the reduction formula is basically a way of relating the scattering amplitude for some interaction of particles to the vacuum expectation value of a time ordered product of fields.

The scattering matrix, or S-matrix, is defined as

$$S = 1 + iT \ , \tag{2.2.57}$$

and it relates the initial and final states in a particle interaction. It is a unitary matrix that connects the asymptotic particle states before and after the interaction. The "1" is just a trivial part representing no interaction while the "iT" part governs the interaction. The S-matrix element

$$_{\text{out}}\langle \mathbf{p}_1 \cdots \mathbf{p}_n | \, S \, | \mathbf{k}_1 \cdots \mathbf{k}_m \rangle_{\text{in}} \ , \tag{2.2.58}$$

appears in measurable physical quantities like the cross section and the LSZ formalism relates this matrix element to the vacuum expectation value

$$\langle 0 | \, T\{\phi(x_1) \cdots \phi(x_n) \, \phi(y_1) \cdots \phi(y_2)\} \, | 0 \rangle \ , \tag{2.2.59}$$

which can be calculated. In a general form the LSZ-formula is written in momentum space as

$$\begin{aligned} &\prod_{i=1}^{m} \int \mathrm{d}^n x_i \, e^{-ik_i \cdot x_i} \prod_{j=1}^{4} \int \mathrm{d}^4 y_j \, e^{+ip_j \cdot y_j} \langle 0 | \, T\{\phi(x_1) \cdots \phi(x_n) \, \phi(y_1) \cdots \phi(y_2)\} \, | 0 \rangle \\ &\quad = \left(\prod_{i=1}^{m} \frac{i}{k_i^2 - m^2} \right) \left(\prod_{j=1}^{n} \frac{i}{p_j^2 - m^2} \right) {}_{\text{out}}\langle \mathbf{p}_1 \cdots \mathbf{p}_n | \, S \, | \mathbf{k}_1 \cdots \mathbf{k}_m \rangle_{\text{in}} \ , \end{aligned} \tag{2.2.60}$$

and it serves to represent the unknown interactions in terms of well-known free asymptotic fields at time $t = \pm\infty$. The intermediate states between the asymptotic states are defined off mass shell but as time approaches infinity the interacting particles go on shell and they become free. In this limit where the interacting theory becomes a free theory, the fields are written as

$$\begin{aligned} \phi(x) &\to \phi_{\text{in}}(x) \quad \text{as} \quad t \to -\infty \ , \\ \phi(x) &\to \phi_{\text{out}}(x) \quad \text{as} \quad t \to +\infty \ . \end{aligned} \tag{2.2.61}$$

For interactions with four particles (2.2.60) reduces to

$$\begin{aligned} {}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 | \, S \, | \mathbf{k}_1 \mathbf{k}_2 \rangle_{\text{in}} &= \left(p_1^2 - m^2\right) \left(p_2^2 - m^2\right) \left(k_1^2 - m^2\right) \left(k_2^2 - m^2\right) \\ &\times \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \, \mathrm{d}^4 y_1 \, \mathrm{d}^4 y_2 \, e^{-ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} e^{ip_1 \cdot y_1} e^{ip_2 \cdot y_2} \langle 0 | \, T\{\phi(x_1) \, \phi(x_2) \, \phi(y_1) \, \phi(y_2)\} \, | 0 \rangle \ , \end{aligned} \tag{2.2.62}$$

where in particular the appearance of the squared momenta is of interest and will be discussed below.

2.2.7 Contractions of massless vector fields

In the subsequent sections the LSZ reduction formula has a central role and is used extensively in computations of photon amplitudes. This section presents the notation and basic methods which will be used for the computations.

Interactions will be studied at tree-level and the incoming fields will be photon fields

$$A_i^{\pm} = \varepsilon_i^{\pm,\mu} A_\mu(x_i) \ , \tag{2.2.63}$$

with polarization vector ε_i^μ projecting out a certain polarization state. For the photon field amplitudes the notation

$$\langle A_1 \cdots A_n \rangle = \langle 0 | T \{A_1 \cdots A_n\} |0\rangle \ , \tag{2.2.64}$$

will be used. The path integral formalism can be generalized to any field theory and therefore equation (2.2.45) will be the main expression with the scalar fields replaced by photon fields. When only interactions at tree-level are considered, the photon four-point function is

$$\langle A_i A_j A_k A_l \rangle = \frac{\delta^4}{\delta J(x_i)\, J(x_j)\, J(x_k)\, J(x_l)} \int \mathcal{D}A \exp\left[i \int \mathrm{d}^4 x \, (\mathcal{L} + JA) \right] \Big|_{J=0} \ , \tag{2.2.65}$$

where the Lagrangian is not specified. In the particular case of interest, the interactions are governed by the Dirac-Born-Infeld Lagrangian $\mathcal{L}_{\mathrm{DBI}}$ which was discussed in section 1.4 and will be again in section 3.1.2. From (2.2.65) using $\mathcal{L} \to \mathcal{L}_{\mathrm{DBI}}$ it follows that

$$\begin{aligned} &\Big\langle A(x_i) A(x_j)\, A(x_k)\, A(x_l) \Big\rangle \\ &\quad = \int \mathcal{D}A \, \left(\varepsilon_i^\mu A_\mu(x_i)\right) \left(\varepsilon_j^\nu A_\nu(x_j)\right) \left(\varepsilon_k^\rho A_\rho(x_k)\right) \left(\varepsilon_l^\sigma A_\sigma(x_l)\right) \exp\left[i \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{DBI}} \right] \\ &\quad = \varepsilon_i^\mu\, \varepsilon_j^\nu\, \varepsilon_k^\rho\, \varepsilon_l^\sigma\, i \int \mathrm{d}^4 x \int \mathcal{D}A A_{i,\mu} A_{j,\nu} A_{k,\rho} A_{l,\sigma} \mathcal{L}_{\mathrm{DBI}} \ , \end{aligned} \tag{2.2.66}$$

where in the last step the exponential has been expanded to first nontrivial order. The expansion parameter in the DBI-Lagrangian is the inverse string tension α'. Since the Lagrangian is a function of field strength tensors

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{2.2.67}$$

it is seen from (2.2.51) that (2.2.66) will be evaluated by performing Wick contractions of photon fields

$$\langle A_{i,\mu} A_{j,\nu} \rangle = \contraction{}{A_{i,\mu}}{}{A_{j,\nu}} A_{i,\mu} A_{j,\nu} = D_{\mu\nu}(x_i - x_j) \ , \tag{2.2.68}$$

where $D_{\mu\nu}(x_i - x_j)$ is the photon propagator. The photon propagator is found from the action integral for the free electromagnetic field which can be written as

$$S_{\mathrm{em,free}} = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \frac{1}{2} \int \mathrm{d}^4 x A_\mu \left[\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu \ , \tag{2.2.69}$$

by a partial integration. A Fourier transformation yields

$$S_{\mathrm{em,free}} = \frac{1}{2} \int \mathrm{d}^4 x \, \tilde{A}_\mu(k) \left[-k^2 \eta^{\mu\nu} + k^\mu k^\nu \right] \tilde{A}_\nu(-k) \ , \tag{2.2.70}$$

and the photon propagator is defined in position and momentum space by

$$\left(\partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu\right) D_{\nu\rho}(x-y) = i\delta^{(4)}(x-y)\,\delta^\mu_\rho \ , \tag{2.2.71}$$

$$\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right) \tilde{D}_{\nu\rho}(k) = i\delta^\mu_\nu \ , \tag{2.2.72}$$

as the inverse of the operator on the right hand side of (2.2.69) and (2.2.70) respectively. It is observed that

$$\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right) k_\mu = 0 \ , \tag{2.2.73}$$

so that

$$\tilde{A}_\mu(k) = k_\mu a(k) \ , \tag{2.2.74}$$

is a zero mode of $\tilde{D}_{\nu\rho}$ for any scalar function $a(k)$. The 4×4 matrix in (2.2.73) is therefore singular and (2.2.72) has no solution for the propagator. This problem of inverting the photon propagator arises because a gauge transformation is not physical and the functional integration of the free electromagnetic action is therefore performed over a continuous infinity of physically equivalent states. The solution is to change the integrand in order to perform the functional integration such that each physical state is counted only once. To break the gauge invariance the Lagrangian for the free electromagnetic field is modified into

$$\mathcal{L}_{\text{em,free}} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\,\partial_\mu A^\mu \partial_\nu A^\nu \ , \tag{2.2.75}$$

by adding the gauge fixing term. The photon propagator then becomes

$$D_{\mu\nu}(k) = \frac{-i}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right) \ , \tag{2.2.76}$$

with different choices of gauge corresponding to different values for the parameter ξ. The choice $\xi = 1$ corresponds to the Feynman-t' Hooft gauge in which the photon propagator takes the simple form

$$D_{\mu\nu}(k) = -\frac{i}{k^2}\,\eta_{\mu\nu} \ . \tag{2.2.77}$$

Using (2.2.77) for the photon propagator yields for a contraction of a photon field with a field strength tensor

$$\begin{aligned} \overbracket{A_i^{\pm} F}_{\mu\nu} &= (-i)\left(p_{i,\mu}\overbracket{A_\nu A}_{i,\rho} - p_{i,\nu}\overbracket{A_\mu A}_{i,\rho}\right)\varepsilon_i^{\pm,\rho} \\ &= -\frac{1}{p^2}\left(p_{i,\mu}\,\varepsilon^{\pm}_{i,\nu} - p_{i,\nu}\,\varepsilon^{\pm}_{i,\mu}\right) \ , \end{aligned} \tag{2.2.78}$$

where $\partial^\mu = -ip^\mu$ has been used. The four-point function for photons is computed by evaluating all possible full Wick contractions of four photon fields and from (2.2.78) it is seen that the expectation value has the form

$$\langle A_i A_j A_k A_l \rangle \to \frac{1}{p_i^2}\frac{1}{p_j^2}\frac{1}{p_k^2}\frac{1}{p_l^2} \ . \tag{2.2.79}$$

The amplitude is obtained when this expectation value is substituted into (2.2.62) whereby an exact cancellation of the momentum poles takes place. Hence the useful result

$$\overline{A_i^{\pm} F_{\mu\nu}} = \left\langle A_i^{\pm} \middle| F_{\mu\nu} \right\rangle \rightarrow - \left(p_{i,\mu}\, \varepsilon^{\pm}_{i,\nu} - p_{i,\nu}\, \varepsilon^{\pm}_{i,\mu} \right) \ , \tag{2.2.80}$$

is obtained for a contraction of a photon field into a field strength tensor. This result is also found in [21]. For the four-point function, computations will have the general form

$$\langle A_i A_j A_k A_l | F^4 \rangle = \lim_{p \to 0} p_i^2\, p_j^2\, p_k^2\, p_l^2\, \varepsilon_i^{\mu}\, \varepsilon_j^{\nu}\, \varepsilon_k^{\rho}\, \varepsilon_l^{\sigma} \int \mathcal{D}A\, A_{i,\mu} A_{j,\nu} A_{k,\rho} A_{l,\sigma}\, e^{iS} \ , \tag{2.2.81}$$

where all the poles coming from the photon propagators of the Wick contractions are cancelled by the factors of p^2. The F^4 in the expectation value indicates that the four-point function is controlled by terms in the action with a structure of four field strengths. This structure also includes selfdual and anti-selfdual field strengths as discussed in section 2.1.7. The limit means that the particles are on shell after the interaction. Equation (2.2.81) will be the basis for computations of amplitudes throughout the remaining chapters.

2.3 Compactification and dimensional reduction

This section discusses the concept of compactification and the related procedure of dimensional reduction. A dimensional reduction of a field theory under study is basically a redefinition of the theory in a lower number of dimensions. If the theory is formulated in d dimensions it can be dimensionally reduced to $\tilde{d} = d - n$ dimensions by taking all fields to be independent of the coordinates in the extra n dimensions. In terms of the action integral the dimensional reduction from 10 to 4 dimensions may be described as the procedure

$$S = \int \mathrm{d}^{10}x\, \mathcal{L}(x_1 \cdots , x_{10}) \rightarrow \int \mathrm{d}^{10}x\, \mathcal{L}\left(x_1, x_2, x_3, x_4\right) = \mathcal{C} \int \mathrm{d}^4 x\, \mathcal{L}\left(x_1, x_2, x_3, x_4\right) \ , \tag{2.3.1}$$

where the Lagrangian is taken to depend only on the four dimensions into which all dynamic variables such as momenta and polarization vectors are embedded. The Lagrangian as therefore independent of the six auxiliary dimensions and the integral can be factorized as above where the overall constant $\mathcal{C}$ does not matter.

A compactification of a given theory means that it is changed with respect to one dimension. Strictly speaking, a dimensional reduction is then the limit of a compactification where the size of a compactified dimension goes to zero. When a theory is compactified, one infinite dimension is taken to be finite and often also periodic. Figure 2.3 shows an example where a theory is formulated on the full space $M \times C$ and where the dimension C is compact. In the limit where the size of the compact dimension goes to zero, the theory can be described effectively as a theory in the space M independently of C. Compactification is an important concept in connection to string theory. String theory operates with ten spacetime dimensions and the universe appears to have four dimensions. In order to have a string theory which is consistent with observations, it is therefore necessary to explain why the extra dimensions are not observed. A possible explanation is that the extra dimensions are compact and so small that their existence can not be resolved from experiments. A field in a compact periodic dimension can always be written as a Fourier series

$$\phi(x) = \sum_n A_n e^{(2\pi i n / L)x} \ , \tag{2.3.2}$$

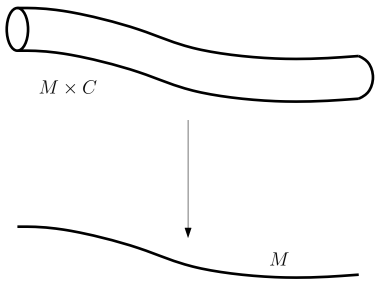

Figure 2.3: The principle of compactification. A theory is formulated on the full combined space $M \times C$ where C is compact. Upon a compactification of the full space, the theory is reformulated as an effective theory on the space M.

where L is the size of the dimension and n is an integer. The momentum is therefore

$$p \sim \frac{n\hbar}{L} \ , \tag{2.3.3}$$

whereby

$$E \sim p \to \infty \ , \quad \text{for} \quad L \to 0 \ . \tag{2.3.4}$$

In the special case $n = 0$, the field (2.3.2) is independent of the x-coordinate and thus independent of the compact dimension. For $n \neq 0$ it is apparent that the energy approaches infinity as the size of the dimension becomes very small. The conclusion is that if the compact dimension is very small it takes an infinite energy to resolve it. The compact dimension can therefore be neglected which is the idea of dimensional reduction.

The idea of compactification goes back to Theodor Kaluza in 1921 where he sought for a unified formulation of gravity and electromagnetism by extending gravity to five-dimensional spacetime. Oskar Klein continued the work and proposed in 1926 that the extra dimension was tiny and curled up. The result is known as Kaluza-Klein theory and is a five dimensional pure gravity which is compactified to four dimensions. The compactification can be outlined by the following where M, N are Lorentz indices in five dimension. The metric in five dimensions can be written

$$g_{MN} = g_{\mu\nu} + 2g_{5\mu} + g_{55} \ , \tag{2.3.5}$$

where $g_{\mu\nu}$ represents the four-dimensional gravitational field, $g_{5\mu}$ represents the electromagnetic field and g_{55} is a four-dimensional scalar. The five-dimensional gravity theory is thereby compactified to a four-dimensional gravity theory coupled to electromagnetism and a scalar.

2.4 Notation and conventions

This section is a brief presentation of conventions and notation used throughout the thesis.

Choice of metric

The metric for the flat spacetime is chosen to be mostly negative and reads on matrix form

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \tag{2.4.1}$$

Natural units and dimensions

Unless otherwise is stated, the natural units

$$\hbar = c = 1 \ , \tag{2.4.2}$$

are used. From the relations

$$E = \hbar\omega \ , \quad E^2 = \left(mc^2\right)^2 + p^2c^2 \ , \tag{2.4.3}$$

it follows that the dimension of length equals the dimension of time and that energy, mass and momentum have the same dimension. The dimensions of energy and time are inverse of each other and therefore

$$[\text{energy}] = [\text{mass}] = [\text{momentum}] = [\text{length}]^{-1} = [\text{time}]^{-1} \ . \tag{2.4.4}$$

Symmetrization and antisymmetrization

The operations of symmetrization and antisymmetrization are defined respectively as

$$A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right) \ , \quad A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\mu\nu}\right) \ , \tag{2.4.5}$$

for two indices. The operations can be generalized to arbitrarily many indices. According to (2.4.5), if $P_{\mu\nu}$ is some fully antisymmetric rank-two tensor and $Q_{\mu\nu}$ is a fully symmetric rank-two tensor the tensors can be written

$$P_{\mu\nu} = P_{[\mu\nu]} \ , \quad Q_{\mu\nu} = Q_{(\mu\nu)} \ . \tag{2.4.6}$$

Indices and dimensions

Lorentz tensors appear with different indices according to their dimensionality. Capital Latin letters, $(M, N, R, \ldots)$, denote six-dimensional Lorentz indices taking values $(0, 1, 2, 3, 4, 5)$ while Greek letters $(\mu, \nu, \rho, \ldots)$ denote the usual four-dimensional Lorentz indices taking values, $(0, 1, 2, 3)$. Latin letters $(m, n, r, \ldots)$, denote the auxiliary two dimensions in Lorentz space-time and thus take the values $(4, 5)$. An arbitrary tensor in six-dimensional spacetime can therefore be written as

$$Y_{MN} = Y_{\mu\nu} + Y_{mn} + Y_{\mu n} + Y_{m\nu} \ . \tag{2.4.7}$$

Spinors have dotted or undotted indices $(\alpha, \beta, \gamma, \ldots)$ and $\left(\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \ldots\right)$ denoted with Greek letters. These indices take values $(1, 2)$.

Inner products

Inner products are denoted

$$a \cdot b = a^\mu b_\nu \ , \quad \tilde{a} \cdot \tilde{b} = a^m b_m \ , \tag{2.4.8}$$

where especially the $\tilde{a} \cdot \tilde{b}$ is used in calculations in six dimensions.

Scattering amplitudes

For scattering amplitudes the notation

$$\mathcal{A}(\text{external fields}) \to \Big\langle \text{external fields} \Big| \text{internal fields} \Big\rangle \ , \tag{2.4.9}$$

will occur. The arrow simply indicates that the amplitude under study is constructed from all possible wick contractions on the right hand side but that front factors of π and α' are absent in that particular expression.

Chapter 3

Vector boson amplitudes in four dimensions

In this chapter the developments from chapters 1 and 2 will be put to use. The Dirac-Born-Infeld action will be the starting point as it is the effective action describing first order corrections to electromagnetic fields on a D-brane. Photon scattering amplitudes will be calculated in four dimensions and in this context, the spinor-helicity formalism from section 2.1 is central. Both four-point and six-point amplitudes will be calculated. The first part of the chapter involves calculations of amplitudes with specific configurations of external particles whereas the second part involves more general computations where a scattering amplitude is calculated as a generic result without considering any specific external polarizations. This is also a step towards a higher number of dimensions and in particular studies of amplitudes in six dimensions which are addressed in the next chapter.

3.1 Employing spinor-helicity

Amplitude calculations are basically just Wick contractions of external fields into field strength tensors. When the field strength tensor is split into its selfdual and anti-selfdual components and expressed in spinor indices, it turns out that the contraction of an external photon field into the field strength tensor simplifies. The subjects discussed in this chapter have previously been discussed in [1] where also the scattering amplitudes have been calculated.

3.1.1 Dimensional considerations

Results for amplitudes will be expressed in terms of momentum products as defined in section 2.1.8. Before going into amplitude calculations it is convenient with an analysis of dimensions of units. The relation for massless particles

$$2\,p_i \cdot p_j = [ij]\,\langle ji\rangle \ , \tag{3.1.1}$$

as in (2.1.70) leads to a relation for momentum bilinears so that the objects

$$\langle ij\rangle \ , \quad [ij] \ , \quad p \ , \tag{3.1.2}$$

have the same dimensionality

$$[p] = [\text{Energy}] \ . \tag{3.1.3}$$

The string parameter $\alpha' \sim l^2$ is associated with the square of the fundamental length of a string and it follows that the dimension of the string parameter is the inverse of momentum squared

$$\left[\alpha'\right] = [l]^2 = [p]^{-2} \ . \tag{3.1.4}$$

Amplitude calculations yield terms containing objects as (3.1.2) which are multiplied by factors of $(\pi\alpha')^n$. An amplitude has to be dimensionless and it is apparent for instance that the different terms

$$\frac{1}{[\zeta 1]}\langle jk\rangle\,(p_i{\cdot}p_j)\,\langle jk\rangle\langle\zeta i\rangle \ , \quad \langle jk\rangle^2\frac{[\zeta 1]\,\langle\zeta 1\rangle}{[\zeta i]^2}\,\tilde{p}_i^2 \ , \quad p^4 \ , \tag{3.1.5}$$

have the same dimensionality and must be multiplied by a factor $(\pi\alpha')^2$ in order to be dimensionless.

3.1.2 Dirac-Born-Infeld in four dimensions

The Dirac-Born-Infeld action in ten dimensions has the form [1, 22]

$$S_{\mathrm{DBI}} = -1 + \frac{1}{\pi^2 g_s \alpha'^5}\int \mathrm{d}^{10}x\sqrt{-\det\left(\eta_{MN} + \pi\alpha' F_{MN}\right)} \ . \tag{3.1.6}$$

As in [1] the string coupling constant is put $g_s = 1$ and the term "-1" is dropped from the action since this term is irrelevant for particle interactions. Along the lines of the discussion in section 2.3 the DBI-action can be dimensionally reduced such that the integration measure is simply taken $\mathrm{d}^{10}x \to \mathrm{d}^4x$ and the indices are taken as ordinary four-dimensional Lorentz indices $M \to \mu$. It follows from section 1.4.3 that the action takes the form

$$-\det\left(\eta_{\mu\nu} + \pi\alpha' F_{\mu\nu}\right) = 1 + \frac{\pi^2\alpha'^2}{2}F_{\mu\nu}F^{\mu\nu} + \frac{\pi^4\alpha'^4}{16}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 \ . \tag{3.1.7}$$

The right hand side can be expressed in terms of selfdual and anti-selfdual components of the field strength tensor and for this purpose (2.1.102) is useful. An explicit calculation for each of the relevant contractions yields

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= \frac{1}{4}\left(\varepsilon_{\alpha\beta}{}^{+}F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}{}^{-}F_{\alpha\beta}\right)\left(\varepsilon^{\alpha\beta+}F^{\dot\alpha\dot\beta} + \varepsilon^{\dot\alpha\dot\beta-}F^{\alpha\beta}\right) \\ &= \frac{1}{2}\left({}^{+}F^2 + {}^{-}F^2\right) \ , \end{aligned} \tag{3.1.8}$$

and

$$\begin{aligned} \left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 &= \left[\frac{1}{4}\left(\varepsilon_{\alpha\beta}{}^{+}F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}{}^{-}F_{\alpha\beta}\right)\left(\varepsilon^{\alpha\beta+}F^{\dot\alpha\dot\beta} - \varepsilon^{\dot\alpha\dot\beta-}F^{\alpha\beta}\right)\right]^2 \\ &= \frac{1}{4}\left[{}^{+}F^4 + {}^{-}F^4 - 2\left({}^{+}F^{2-}F^2\right)\right] \ , \end{aligned} \tag{3.1.9}$$

whereby the determinant takes the form

$$\begin{aligned} &-\det\left(\eta_{\mu\nu} + \pi\alpha' F_{\mu\nu}\right) = \\ &\qquad 1 + \frac{\pi^2\alpha'^2}{4}\left[{}^{+}F^2 + {}^{-}F^2\right] + \frac{\pi^4\alpha'^4}{64}\left[{}^{+}F^4 + {}^{-}F^4 - 2\left({}^{+}F^{2-}F^2\right)\right] \ . \end{aligned} \tag{3.1.10}$$

The determinant is a function of the string parameter which is small and the square root of the determinant can therefore be expanded as a Taylor series in α'. In practice the determinant is expanded in Mathematica by explicitly constructing (3.1.10). To tenth order the Taylor expansion in α' becomes

$$\begin{aligned}\sqrt{-\det\left(\eta_{\mu\nu}+\pi\alpha' F_{\mu\nu}\right)} = 1 + \frac{\pi^2\alpha'^2}{4}{}^{+}F^2 - \frac{\pi^4\alpha'^4}{32}{}^{+}F^{2\,-}F^2 + \frac{\pi^6\alpha'^6}{256}\left({}^{+}F^{4\,-}F^2 + {}^{-}F^{4\,+}F^2\right) \\ - \frac{\pi^8\alpha'^8}{2048}\left({}^{+}F^{6\,-}F^2 + {}^{-}F^{6\,+}F^2 + 3\,{}^{+}F^{4\,-}F^4\right) + \mathcal{O}\left(\alpha'^{10}\right) \ ,\end{aligned} \tag{3.1.11}$$

where the topological density $1/4 F_{\mu\nu}\tilde{F}^{\mu\nu} = 1/8\left({}^{+}F^2 - {}^{-}F^2\right)$ has been added after the expansion has been performed. Since the topological density is a total derivative it can be added in the action without affecting the equations of motion. Adding the topological density cancels a ${}^{-}F^2$-term in (3.1.11).

Equation (3.1.11) is the starting point for scattering amplitude calculations and it can be read of for instance that the four-point amplitude is controlled entirely by the term $\left(\pi^4\alpha'^4/32\right){}^{+}F^{2\,-}F^2$. Before turning into explicit amplitude computations it is appropriate to examine more generally the Wick contractions of external fields and $F^{\pm}$.

3.1.3 Wick contractions of field strengths

External contractions

From (2.1.102) the selfdual and the anti-selfdual components of the field strength are given respectively as

$$ {}^{+}F_{\dot\alpha\dot\beta} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\left(\partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha}\right) = \frac{1}{2}\left(\partial_{\alpha\dot\alpha}A^{\alpha}{}_{\dot\beta} + \partial_{\alpha\dot\beta}A^{\alpha}{}_{\dot\alpha}\right) \ , \tag{3.1.12}$$

$$ {}^{-}F_{\alpha\beta} = \frac{1}{2}\,\varepsilon^{\dot\alpha\dot\beta}\left(\partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha}\right) = \frac{1}{2}\left(\partial_{\alpha\dot\alpha}A_{\beta}{}^{\dot\alpha} + \partial_{\beta\dot\alpha}A_{\alpha}{}^{\dot\alpha}\right) \ , \tag{3.1.13}$$

which is also found in [23]. Equation (2.2.77) becomes simply

$$\langle A_{\alpha\dot\beta}A_{\beta\dot\beta}\rangle \ = \ -2\frac{i}{p^2}\,\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} \ , \tag{3.1.14}$$

in spinor indices and the contraction of a photon field as (2.1.111) with plus-helicity quantum number into a selfdual field strength is found as

$$\begin{aligned}\overbrace{A^{+}\,{}^{+}F}_{\dot\alpha\dot\beta} \equiv {}^{+}F_{\dot\alpha\dot\beta}\left[A^{+}\right] &= \frac{1}{2^2}\,i\left(\varepsilon^{+}\right)^{\gamma\dot\gamma}\left(\partial_{\alpha\dot\alpha}\overbrace{A^{\alpha}{}_{\dot\beta}A}_{\gamma\dot\gamma} + \partial_{\alpha\dot\beta}\overbrace{A^{\alpha}{}_{\dot\alpha}A}_{\gamma\dot\gamma}\right) \\ &= \frac{i}{4}\left(-i\right)\left(-2i\right)\left(\sqrt{2}\,\frac{\zeta^{\gamma}\lambda^{\dot\gamma}}{[\zeta\lambda]}\right)\left(p_{\alpha\dot\alpha}\,\varepsilon^{\alpha}{}_{\gamma}\,\varepsilon_{\dot\beta\dot\gamma} + p_{\alpha\dot\beta}\,\varepsilon^{\alpha}{}_{\gamma}\,\varepsilon_{\dot\alpha\dot\gamma}\right) \\ &= i\sqrt{2}\,\lambda_{\dot\alpha}\lambda_{\dot\beta} \ .\end{aligned} \tag{3.1.15}$$

It has been used that

$$\partial_{\mu} = -ip_{\mu} \leftrightarrow -ip_{\alpha\dot\alpha} \ , \tag{3.1.16}$$

and a factor of $1/p^2$ has been omitted in (3.1.15) because it is cancelled by the LSZ formula from (2.2.61). Since (3.1.15) is a tree-level calculation with particles on mass shell

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \lambda_{\dot{\alpha}} \ , \tag{3.1.17}$$

holds by (2.1.30) for massless particles. Calculations similar to the steps involved in (3.1.15) lead to an analogous result for the contraction of a photon field with minus-helicity quantum number into an anti-selfdual field strength

$$\overbrace{A^- \, {}^-F}_{\alpha\beta} \equiv {}^-F_{\alpha\beta}\left[A^-\right] = -i\sqrt{2}\,\lambda_\alpha \lambda_\beta \ , \tag{3.1.18}$$

where the sign is the opposite compared to [1]. A contraction of a photon field with minus-helicity quantum number into a selfdual field strength spinor yields the substitution

$$\frac{\zeta^\gamma \lambda^{\dot{\gamma}}}{[\zeta\lambda]} \rightarrow \frac{\zeta^{\dot{\gamma}} \lambda^\gamma}{\langle \zeta\gamma \rangle} \ , \tag{3.1.19}$$

in the second line of (3.1.15). Hence the vanishing contraction $\lambda^\alpha \lambda_\alpha$ is obtained and consequently

$$\overbrace{A^- \, {}^+F}_{\dot{\alpha}\dot{\beta}} \equiv {}^+F_{\dot{\alpha}\dot{\beta}}\left[A^-\right] = 0 \ , \tag{3.1.20}$$

holds. The opposite result for the contraction of a photon field with plus-helicity into an anti-selfdual field strength spinor is obtained using (3.1.19) with a reversed arrow in the calculation towards (3.1.18). Analogously the vanishing contraction $\lambda^{\dot{\alpha}} \lambda_{\dot{\alpha}}$ is obtained in this case and

$$\overbrace{A^+ \, {}^-F}_{\alpha\beta} \equiv {}^-F_{\alpha\beta}\left[A^+\right] = 0 \ , \tag{3.1.21}$$

follows. Equations (3.1.15), (3.1.18), (3.1.20) and (3.1.21) are on-shell relations useful for the calculations of four-point amplitudes directly from the action where only external contractions occur.

Internal contractions

In order to calculate six-point amplitudes it is necessary to consider internal contractions between field strength spinors. In this case the involved particles are off-shell and (3.1.20) and (3.1.21) are therefore not valid. Instead the internal contractions has to be worked out explicitly. The internal contraction between two field strengths of different types is

$$\begin{aligned} &\overbrace{{}^+F_{\dot{\alpha}\dot{\beta}} \, {}^-F}_{\alpha\beta} = \\ &(-i)^2\,(-2i)\frac{1}{4}\frac{1}{p^2}\left(p_{\gamma\dot{\alpha}}\,p_{\alpha\dot{\delta}}\,\varepsilon^\gamma{}_\beta\,\varepsilon_{\dot{\beta}}^{\ \dot{\delta}} + p_{\gamma\dot{\alpha}}\,p_{\beta\dot{\delta}}\,\varepsilon^\gamma{}_\alpha\,\varepsilon_{\dot{\beta}}^{\ \dot{\delta}} + p_{\gamma\dot{\beta}}\,p_{\alpha\dot{\delta}}\,\varepsilon^\gamma{}_\beta\,\varepsilon_{\dot{\alpha}}^{\ \dot{\delta}} + p_{\gamma\dot{\beta}}\,p_{\beta\dot{\delta}}\,\varepsilon^\gamma{}_\alpha\,\varepsilon_{\dot{\alpha}}^{\ \dot{\delta}}\right) \\ &\qquad = -\frac{i}{p^2}\left(p_{\alpha\dot{\beta}}p_{\beta\dot{\alpha}} + p_{\alpha\dot{\alpha}}p_{\beta\dot{\beta}}\right), \end{aligned} \tag{3.1.22}$$

as is calculated in detail in (A.1.22). It should be noticed that the internal contraction of field strengths of opposite type is nonlocal as this contraction contains a pole in the propagating momentum. The internal contraction of two selfdual field strength spinors is

$$
\begin{aligned}
&{}^+\overline{F_{\dot\alpha\dot\beta}\,{}^+F}_{\dot\gamma\dot\delta} = \\
&(-i)^2\,(-2i)\frac{1}{4}\frac{1}{p^2}\left(p_{\alpha\dot\alpha}\,p_{\beta\dot\gamma}\,\varepsilon^{\alpha\beta}\,\varepsilon_{\dot\beta\dot\delta} + p_{\alpha\dot\alpha}\,p_{\beta\dot\delta}\,\varepsilon^{\alpha\beta}\,\varepsilon_{\dot\beta\dot\gamma} + p_{\alpha\dot\beta}p_{\beta\dot\gamma}\,\varepsilon^{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\delta} + p_{\alpha\dot\beta}\,p_{\beta\dot\delta},\varepsilon^{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\gamma}\right)\;,
\end{aligned}
\tag{3.1.23}
$$

where the structure of each of the terms in the bracket is the same. Each term is antisymmetric in the two dotted indices so that for instance

$$
p_{\alpha\dot\alpha}\,p_{\beta\dot\gamma}\,\varepsilon^{\alpha\beta} = -p_{\alpha\dot\gamma}\,p_{\beta\dot\alpha}\,\varepsilon^{\alpha\beta} = \kappa\,\varepsilon_{\dot\alpha\dot\gamma}\;. \tag{3.1.24}
$$

This leads to

$$
2\,\kappa = p_{\alpha\dot\alpha}\,p_{\beta\dot\gamma}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\gamma} = 2\,p^2\;, \tag{3.1.25}
$$

when contracted with $\varepsilon^{\dot\alpha\dot\gamma}$. Substituting $\kappa = p^2$ in (3.1.24) and using this in (3.1.23) yields

$$
{}^+\overline{F_{\dot\alpha\dot\beta}\,{}^+F}_{\dot\gamma\dot\delta} = i\left(\varepsilon_{\dot\alpha\dot\gamma}\,\varepsilon_{\dot\beta\dot\delta} + \varepsilon_{\dot\alpha\dot\delta}\,\varepsilon_{\dot\beta\dot\gamma}\right)\;. \tag{3.1.26}
$$

The internal contraction between two anti-selfdual field strengths can be worked out using the same steps that lead to (3.1.26) and it follows that

$$
{}^-\overline{F_{\alpha\beta}\,{}^-F}_{\gamma\delta} = i\,(\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\,\varepsilon_{\beta\gamma})\,. \tag{3.1.27}
$$

From (3.1.26) and (3.1.27) it is apparent that the internal contraction of two field strengths of the same type is local since the pole factors are cancelled. The cancellation of pole factors occur off-shell and hence this local property does not depend on any on-shell conditions.

3.2 Scattering amplitudes with specified external polarizations

The developments in the previous section will now be employed in amplitude calculations where four-point and six-point amplitudes will be computed.

3.2.1 Four-point amplitudes

Equations (3.1.15), (3.1.18), (3.1.20) and (3.1.21) serve as the basis for calculations of four-point scattering amplitudes. Contractions between external states and vertices simplify considerably according to these equations and certain vanishing amplitudes can be read of directly from the expanded action. Since the four-point amplitude is controlled solely by the term $\left(-\pi^2\alpha'^2/32\right)\,{}^+F^2\,{}^-F^2$ in (3.1.11) it can be deduced right away that the following four-point amplitudes necessarily vanish

$$
\mathcal{A}\big(1^+2^+3^+4^+\big) = \mathcal{A}\big(1^-2^-3^-4^-\big) = \mathcal{A}\big(1^-2^+3^+4^+\big) = \mathcal{A}\big(1^+2^-3^-4^-\big) = 0\;. \tag{3.2.1}
$$

As an example the first amplitude in (3.2.1) is calculated

$$
\mathcal{A}\big(1^+2^+3^+4^+\big) \to \left\langle A^+A^+A^+A^+\middle|\,{}^+F_{\dot\alpha\dot\beta}\,{}^+F^{\dot\alpha\dot\beta}\,{}^-F_{\alpha\beta}\,{}^-F^{\alpha\beta}\right\rangle\;, \tag{3.2.2}
$$

and it is apparent that (3.1.21) will appear in all possible full contractions. Likewise it can be deduced that the second amplitude in (3.2.1) contains (3.1.20) in all contractions while each of the last two amplitudes in (3.2.1) respectively contain both (3.1.20) and (3.1.21). From the above discussion it can be concluded that the four-point amplitude

$$\mathcal{A}\big(1^+2^+3^-4^-\big) \to \Big\langle A^+ A^+ A^- A^- \Big| {}^+F_{\dot{\alpha}\dot{\beta}} {}^+F^{\dot{\alpha}\dot{\beta}} \, {}^-F_{\alpha\beta} {}^-F^{\alpha\beta} \Big\rangle \ , \tag{3.2.3}$$

is the only one which is nonzero. In order to evaluate this amplitude explicitly with the right numerical constants, (3.1.20) and (3.1.21) with upper indices

$${}^+F^{\dot{\alpha}\dot{\beta}}\left[A^-\right] = i\sqrt{2}\,\lambda^{\dot{\alpha}}\lambda^{\dot{\beta}} \ , \quad {}^-F^{\alpha\beta}\left[A^-\right] = -i\sqrt{2}\,\lambda^{\alpha}\lambda^{\beta} \ , \tag{3.2.4}$$

are used and the result is

$$\begin{aligned} \mathcal{A}\big(1^+2^+3^-4^-\big) &= \left(-\frac{\pi^2\alpha'^2}{32}\right)\left(4\,i^2\,(-i)^2\left(\sqrt{2}\right)^4 1_{\dot{\alpha}}\,1_{\dot{\beta}}\,2^{\dot{\alpha}}\,2^{\dot{\beta}}\,3_{\alpha}\,3_{\beta}\,4^{\alpha}\,4^{\beta}\right) \\ &= -\frac{\pi^2\alpha'^2}{2}\,\langle 12\rangle^2\,[34]^2 \ , \end{aligned} \tag{3.2.5}$$

where the proper numerical front factor from the action has been taken into account. The factor “4” comes about because all the possible four Wick contractions are identical.

3.2.2 Six-point amplitudes

In calculations of the six-point amplitudes two contributions have to be taken into account. One contribution is the direct one from the vertex $\left(\pi^6\alpha'^6/256\right)\left({}^+F^4\,{}^-F^2 + {}^-F^4\,{}^+F^2\right)$ in the action while the other contribution consists of two four-point vertices controlled by the square of the term $\left(-\pi^2\alpha'^2/32\right){}^+F^2\,{}^-F^2$ where two four-point vertices are contracted. This latter term originates from the second order contribution in the expansion series of the exponentiated action which involves internal contractions of field strengths. Therefore (3.1.22), (3.1.26) and (3.1.27) will be used. It can be read of directly from the action that the amplitudes

$$\begin{aligned} \mathcal{A}\big(1^+2^+3^+4^+5^+6^+\big) = \mathcal{A}\big(1^-2^-3^-4^-5^-6^-\big) = \mathcal{A}\big(1^+2^-3^-4^-5^-6^-\big) &= \mathcal{A}\big(1^-2^+3^+4^+5^+6^+\big) \\ &= 0 \ , \end{aligned} \tag{3.2.6}$$

vanish. This is simply due to the fact that the action does not contain any vertices with the structure

$${}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\mp F \ , \quad {}^\mp F\,{}^\mp F\,{}^\mp F\,{}^\mp F\,{}^\mp F\,{}^\pm F \ , \tag{3.2.7}$$

or

$${}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\pm F\,{}^\pm F \ . \tag{3.2.8}$$

Furthermore, no vertex with this structure is found when internal contractions of field strengths are carried out on the second-order terms in the expansion of the action.

In general, the n-point amplitude

$$\mathcal{A}\big(1^+2^+3^-\cdots n^-\big) \ , \tag{3.2.9}$$

is known as a maximally helicity violating amplitude (MHV) since in Yang-Mills theory it violates conservation of helicity [24] to the maximum possible extend at tree level. The particular amplitude from the DBI-action

$$\mathcal{A}\big(1^+2^+3^-4^-5^-6^-\big) \ , \tag{3.2.10}$$

is therefore an example of an MHV amplitude and it is evaluated by the Wick contractions

$$\Big\langle 1^+2^+3^-4^-5^-6^- \Big| {}^+F^{2\,+}F^{2\,-}F^{2\,-}F^2 \Big\rangle \ , \tag{3.2.11}$$

of the squared term from the action. In this case it is apparent that one internal contraction of two selfdual field strengths must be performed in order to obtain a nonzero full contraction. Equation (3.1.26) yields

$${}^+\overline{F_{\dot\alpha\dot\beta}\,{}^+F^{\dot\alpha\dot\beta}\,{}^+F}_{\dot\gamma\dot\delta}\,{}^+F^{\dot\gamma\dot\delta} = 2i\,{}^+F_{\dot\alpha\dot\beta}\,{}^+F^{\dot\alpha\dot\beta} \ , \tag{3.2.12}$$

for an internal contraction of selfdual field strengths. Since four identical internal contractions of the selfdual field strengths can be made, a factor of four is obtained and

$$\Big\langle 1^+2^+3^-4^-5^-6^- \Big| {}^+F^{2\,+}F^{2\,-}F^{2\,-}F^2 \Big\rangle = 8i\Big\langle 1^+2^+3^-4^-5^-6^- \Big| {}^+F^{2\,-}F^4 \Big\rangle \ , \tag{3.2.13}$$

holds for an internal contraction of selfdual field strengths. The internal structure of the contribution from the two four-point vertices is identical to that of the direct contribution ${}^+F^{2\,-}F^4$ and these two contributions only differ by a constant. The prefactor of the contribution from the two four-point vertices is

$$\frac{1}{2}\left(-\,i\,\frac{\pi^2\alpha'^2}{32}\right)^2 8i = -\,i\,\frac{\pi^4\alpha'^4}{256} \ , \tag{3.2.14}$$

where the factor $1/2$ is from the expansion of the action and the factor $8i$ is from (3.2.13). Equation (3.2.14) is exactly identical to the prefactor of the term in the action with six fields but with the opposite sign. The sum of these two contributions is exactly the amplitude (3.2.10) which therefore vanishes

$$\mathcal{A}\big(1^+2^+3^-4^-5^-6^-\big) = 0 \ . \tag{3.2.15}$$

It is interesting that the six-point contribution exactly cancels the contribution from the contraction of two four-point vertices.

The six-point helicity conserving NMHV amplitude

$$\mathcal{A}\big(1^+2^+3^+4^-5^-6^-\big) \ , \tag{3.2.16}$$

has only one contribution which is the one where two field strengths of the opposite type are contracted between two four-point vertices. There exist nine possible permutations of particles where two are shown in figures 3.1 and 3.2. The particular configuration of external particles in figure 3.1 corresponds to the propagating momentum

$$p_{\alpha\dot\alpha} = (1+2+4)_{\alpha\dot\alpha} \ , \tag{3.2.17}$$

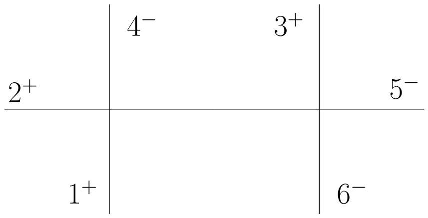

Figure 3.1: A contraction of two four-point vertices. The contraction for this configuration of external particles is calculated in (3.2.18).

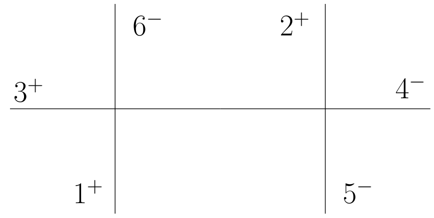

Figure 3.2: A contraction of two four-point vertices with another configuration of particles compared to figure 3.1.

and it follows from (3.1.22) that the particular full contraction becomes

$$\begin{aligned}
&\overline{{}^{+}F_{\dot\alpha\dot\beta}\,{}^{-}F_{\alpha\beta}}\;{}^{+}F^{\dot\alpha\dot\beta}\left[A_3^+\right]\,{}^{+}F_{\dot\gamma\dot\delta}\left[A_1^+\right]\,{}^{+}F^{\dot\gamma\dot\delta}\left[A_2^+\right]\,{}^{-}F^{\alpha\beta}\left[A_4^-\right]\,{}^{-}F_{\gamma\delta}\left[A_5^-\right]\,{}^{-}F^{\gamma\delta}\left[A_6^-\right] \\
&\qquad = -\frac{i}{p^2}\left(p_{\alpha\dot\alpha}\,p_{\beta\dot\beta}+p_{\alpha\dot\beta}\,p_{\beta\dot\alpha}\right)3^{\dot\alpha}\,3^{\dot\beta}\,1_{\dot\gamma}\,1_{\dot\delta}\,2^{\dot\gamma}\,2^{\dot\delta}\,4^{\alpha}\,4^{\beta}\,5_{\gamma}\,5_{\delta}\,6^{\gamma}\,6^{\delta} \\
&\qquad = -\frac{2i}{p^2}\langle 12\rangle^2\,[56]^2\,4^{\alpha}4^{\beta}p_{\alpha\dot\alpha}\,p_{\beta\dot\beta}3^{\dot\alpha}3^{\dot\beta} \\
&\qquad = -\frac{2i}{\left(p_1+p_2+p_4\right)^2}\langle 12\rangle^2\,[56]^2\left([4\,|\,(1+2+4)\,|3\rangle\right)^2\ ,
\end{aligned} \tag{3.2.18}$$

where the notational abbreviation

$$[i\,|k|l\rangle = i^{\alpha}j_{\alpha\dot\alpha}k^{\dot\alpha}\ , \tag{3.2.19}$$

is used. The full amplitude is a sum of the nine permutations of (3.2.18) and reads

$$\mathcal{A}\left(1^+2^+3^+4^-5^-6^-\right)=\frac{1}{4}\left(-\,i\,\frac{\pi^4\alpha'^4}{128}\right)\sum_{\sigma(l,m,n)}\,\sum_{\sigma(i,j,k)}\langle lm\rangle^2\,[ij]^2\,\frac{\left([k\,|\,(l+m+k)\,|n\rangle\right)^2}{\left(p_l+p_m+p_k\right)^2}\ , \tag{3.2.20}$$

where the sums are performed over the permutations of indices

$$\sigma\left(l,m,n\right)=\sigma\left(1,2,3\right)\ ,\quad \sigma\left(l,m,n\right)=\sigma\left(2,3,1\right)\ ,\quad \sigma\left(l,m,n\right)=\sigma\left(3,2,1\right)\ , \tag{3.2.21}$$

$$\sigma\left(i,j,k\right)=\sigma\left(4,5,6\right)\ ,\quad \sigma\left(i,j,k\right)=\sigma\left(5,6,4\right)\ ,\quad \sigma\left(i,j,k\right)=\sigma\left(4,5,6\right)\ . \tag{3.2.22}$$

The numerical factor in (3.2.20) is calculated as

$$\frac{1}{2}\left(-\,i\,\frac{\pi^2\alpha'^2}{32}\right)^2\left(\sqrt{2}\right)^6(2i)=\left(-\,i\,\frac{\pi^4\alpha'^4}{128}\right)\ , \tag{3.2.23}$$

which is in agreement with [1] apart from a factor i.

3.3 A step towards six dimensions

The previous section contains calculations of amplitudes in four dimensions. Amplitudes in six dimensions are not calculated in a similar straightforward way and it is necessary with some preliminary considerations.

3.3.1 Dirac-Born-Infeld in higher dimensions

The approach to calculate six-dimensional scattering amplitudes begins with the Dirac-Born-Infeld Lagrangian. In four dimensions it can be expanded [22] as

$$\mathcal{L}_{\mathrm{DBI}} = I_2 + I_4\left[1 + \mathcal{O}\left(F^2\right)\right] \ , \tag{3.3.1}$$

with abbreviations

$$I_2 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \tag{3.3.2}$$

$$I_4 = -\frac{1}{8}\left[F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right] = -\frac{1}{8}\left({}^{+}F\right)^2\left({}^{-}F\right)^2 \ , \tag{3.3.3}$$

where the string tension T has been put equal to one,

$$T = \frac{1}{2\pi\alpha'} \equiv 1 \ . \tag{3.3.4}$$

In an arbitrary number of dimensions (3.3.1) is instead

$$\mathcal{L}_{\mathrm{DBI}} = I_2 + I_4 + \mathcal{O}\left(F^6\right) \ , \tag{3.3.5}$$

with the same abbreviations used. That (3.3.5) holds has been checked explicitly by writing the field strengths as matrices in Mathematica. This is discussed in appendix A.1.

Whether four, six or any number of dimensions are considered, especially the I_4 term is of interest since it contains products of four field strengths and hence it controls the four-point amplitudes at tree-level. In the following the constant will be ignored and

$$I_4' \equiv F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \ , \tag{3.3.6}$$

will be studied. The operation of dualization was defined in four dimensions in (1.4.1) and the selfdual ${}^{+}F$ and anti-selfdual ${}^{-}F$ components of the field strength tensor was introduced in 3.1.2. The rightmost equality in (3.3.3) holds in four dimensions but it has no straightforward generalization to higher dimensions. In six dimensions one would define the dual of F_{MN} as

$$\tilde{F}_{MNRS} = \frac{i}{2}\varepsilon_{MNRSKL}F^{KL} \ , \tag{3.3.7}$$

which is obviously not a two-form. In order to obtain a two-form (3.3.7) must be contracted into some antisymmetric object with two indices and the dual will then depend on this particular object. The conclusion is that the dual in six dimensions is not uniquely defined. The simplicity of calculations in four dimensions as a consequence of the use of the dual can therefore not be transferred to six dimensions.

A general four-point amplitude with unspecified external polarizations

The purpose of this section is to calculate the photon four-point scattering amplitude as a general result in terms of generic polarization vectors. Only tree-level amplitudes will be studied and hence only external contractions are taken into account.

3.3.2 Constructing the amplitude

The result

$$\varepsilon_i^{\pm} \cdot \varepsilon_j^{\pm} = 0 \ , \tag{3.3.8}$$

is important in calculations of scattering amplitudes with specific external polarizations. However, when an amplitude is calculated in generality as a function of generic polarization vectors, (3.3.8) cannot be used since, in general, all dot products of polarization vectors are non-vanishing. The general amplitude is calculated from the four-point term

$$I_4' = F_{MN}F^{NR}F_{RS}F^{RM} - \frac{1}{4}\,F_{MN}F^{MN}F_{RS}F^{RS} \ , \tag{3.3.9}$$

in the Dirac-Born-Infeld Lagrangian as discussed in the previous section.

The general calculation can be outlined as a process of three steps. Step one is to compute one arbitrarily chosen full contraction. Step two is to construct in Mathematica the result of this computation as generic momenta and polarization tensors with indices and then perform the summation over all possible permutations of these indices. Step three is to employ momentum conservation to simplify the expression. These three steps are described below. The amplitude is computed by the Wick contractions of (3.3.9)

$$\begin{aligned}
\mathcal{A}(A_iA_jA_kA_l) &\to \varepsilon_i^{\alpha}\,\varepsilon_j^{\beta}\,\varepsilon_k^{\gamma}\,\varepsilon_l^{\delta}\left\langle A_{i,\alpha}A_{j,\beta}A_{k,\gamma}A_{l,\delta}\middle| F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right\rangle \\
&= \varepsilon_i^{\alpha}\,\varepsilon_j^{\beta}\,\varepsilon_k^{\gamma}\,\varepsilon_l^{\delta}\ \Big[\left\langle A_{i,\alpha}A_{j,\beta}A_{k,\gamma}A_{l,\delta}\middle| F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\right\rangle \\
&\qquad -\frac{1}{4}\left\langle A_{i,\alpha}A_{j,\beta}A_{k,\gamma}A_{l,\delta}\middle| F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right\rangle\Big] \ ,
\end{aligned} \tag{3.3.10}$$

suggesting that each of the two terms in (3.3.10) are treated separately. For each of the two terms one arbitrarily chosen full contraction

$$\chi_{(ijkl)} \equiv \varepsilon_i^{\alpha}\,\varepsilon_j^{\beta}\,\varepsilon_k^{\gamma}\,\varepsilon_l^{\delta}\,A_{i,\alpha}A_{j,\beta}A_{k,\gamma}A_{l,\delta}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \ , \tag{3.3.11}$$

and

$$\omega_{(ijkl)} \equiv \varepsilon_i^{\alpha}\,\varepsilon_j^{\beta}\,\varepsilon_k^{\gamma}\,\varepsilon_l^{\delta}\,A_{i,\alpha}A_{j,\beta}A_{k,\gamma}A_{l,\delta}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \ , \tag{3.3.12}$$

is labeled according to (3.3.11) and (3.3.12). Both objects $\chi_{(ijkl)}$ and $\omega_{(ijkl)}$ consist of sixteen terms and can be evaluated respectively as

$$\begin{aligned}
\chi_{(ijkl)} = \ & (p_i\!\cdot\!\varepsilon_l)\,(p_j\!\cdot\!\varepsilon_i)\,(p_k\!\cdot\!\varepsilon_j)\,(p_l\!\cdot\!\varepsilon_k) + (p_i\!\cdot\!p_l)\,(p_j\!\cdot\!\varepsilon_i)\,(p_k\!\cdot\!\varepsilon_l)\,(\varepsilon_j\!\cdot\!\varepsilon_k) \\
& - (p_k\!\cdot\!p_l)\,(p_i\!\cdot\!\varepsilon_l)\,(p_j\!\cdot\!\varepsilon_i)\,(\varepsilon_k\!\cdot\!\varepsilon_j) - (p_i\!\cdot\!p_l)\,(p_j\!\cdot\!\varepsilon_i)\,(p_k\!\cdot\!\varepsilon_j)\,(\varepsilon_k\!\cdot\!\varepsilon_l) \\
& + (p_i\!\cdot\!\varepsilon_j)\,(p_j\!\cdot\!p_k)\,(p_l\!\cdot\!\varepsilon_k)\,(\varepsilon_i\!\cdot\!\varepsilon_l) + (p_i\!\cdot\!\varepsilon_j)\,(p_j\!\cdot\!\varepsilon_k)\,(p_k\!\cdot\!\varepsilon_l)\,(p_l\!\cdot\!\varepsilon_i) \\
& - (p_k\!\cdot\!p_l)\,(p_i\!\cdot\!\varepsilon_j)\,(p_j\!\cdot\!\varepsilon_k)\,(\varepsilon_i\!\cdot\!\varepsilon_l) - (p_i\!\cdot\!\varepsilon_j)\,(p_j\!\cdot\!p_k)\,(p_l\!\cdot\!\varepsilon_i)\,(\varepsilon_k\!\cdot\!\varepsilon_l) \\
& - (p_i\!\cdot\!p_j)\,(p_k\!\cdot\!\varepsilon_j)\,(p_l\!\cdot\!\varepsilon_k)\,(\varepsilon_i\!\cdot\!\varepsilon_l) - (p_i\!\cdot\!p_j)\,(p_k\!\cdot\!\varepsilon_l)\,(p_l\!\cdot\!\varepsilon_i)\,(\varepsilon_k\!\cdot\!\varepsilon_j) \\
& + (p_i\!\cdot\!p_j)\,(p_k\!\cdot\!p_l)\,(\varepsilon^i\!\cdot\!\varepsilon_l)\,(\varepsilon_k\!\cdot\!\varepsilon_j) + (p_i\!\cdot\!p_j)\,(p_k\!\cdot\!\varepsilon^j)\,(p_l\!\cdot\!\varepsilon_i)\,(\varepsilon_k\!\cdot\!\varepsilon_l) \\
& - (p_i\!\cdot\!\varepsilon_l)\,(p_l\!\cdot\!\varepsilon_k)\,(p_j\!\cdot\!p_k)\,(\varepsilon_i\!\cdot\!\varepsilon_j) - (p_i\!\cdot\!p_l)\,(p_j\!\cdot\!\varepsilon_k)\,(p_k\!\cdot\!\varepsilon_l)\,(\varepsilon_i\!\cdot\!\varepsilon_j) \\
& + (p_k\!\cdot\!p_l)\,(p_i\!\cdot\!\varepsilon_l)\,(p_j\!\cdot\!\varepsilon_k)\,(\varepsilon_i\!\cdot\!\varepsilon_j) + (p_i\!\cdot\!p_l)\,(p_j\!\cdot\!p_k)\,(\varepsilon_i\!\cdot\!\varepsilon_j)\,(\varepsilon_k\!\cdot\!\varepsilon_l) \ ,
\end{aligned} \tag{3.3.13}$$

and

$$\omega_{(ijkl)} = 4\Big[(p_i\cdot p_j)\,(p_k\cdot p_l)\,(\varepsilon_i\cdot\varepsilon_j)\,(\varepsilon_k\cdot\varepsilon_l) + (p_i\cdot\varepsilon_j)\,(p_j\cdot\varepsilon_i)\,(p_k\cdot\varepsilon_l)\,(p_l\cdot\varepsilon_k)$$
$$- (p_k\cdot p_l)\,(p_i\cdot\varepsilon_j)\,(p_j\cdot\varepsilon_i)\,(\varepsilon_k\cdot\varepsilon_l) - (p_i\cdot p_j)\,(p_k\cdot\varepsilon_l)\,(p_l\cdot\varepsilon_k)\,(\varepsilon_i\cdot\varepsilon_j)\Big]\,. \tag{3.3.14}$$

Interchanging the contractions of A_i and A_j in (3.3.11) simply interchanges the indices i and j in (3.3.13). A similar structure is found for (3.3.12) and (3.3.14) and the full amplitude in (3.3.10) can therefore be written formally as the sum over all possible permutations of the indices i, j, k, l as

$$\mathcal{A}(A_iA_jA_kA_l) = \sum_{\sigma(i,j,k,l)} \left(\chi_{(ijkl)} - \frac{1}{4}\,\omega_{(ijkl)}\right) \equiv \mathcal{S}\ , \tag{3.3.15}$$

containing 4! contributions of the form (3.3.13) and just as many of the form (3.3.14). It follows that $24 \times (16+4) = 480$ is the total number of terms in the sum which will be referred to as $\mathcal{S}$. In order to evaluate and simplify $\mathcal{S}$, the full expressions for $\chi_{(ijkl)}$ and $\omega_{(ijkl)}$ are constructed individually in Mathematica such that for instance the second term in (3.3.13) is written as

$$(p_i\cdot p_l)\,(p_j\cdot\varepsilon_i)\,(p_k\cdot\varepsilon_l)\,(\varepsilon_j\cdot\varepsilon_k) \to (\mathtt{pp})_{i,l}\,(\mathtt{pe})_{j,i}\,(\mathtt{pe})_{k,l}\,(\mathtt{ee})_{j,k}\ \ , \tag{3.3.16}$$

with each dot product represented as one variable having two indices. The name of each variable carries the information of whether the dot product is between two momentum vectors, two polarization vectors or between one momentum vector and one polarization vector. The sum $\mathcal{S}$ is explicitly evaluated in Mathematica and the output is shown in figure A.3. As a consequence of the definitions of variables[1] (3.3.16) Mathematica distinguishes between terms such that

$$(\mathtt{pp})_{i,j} \neq (\mathtt{pp})_{j,i}\ \ ,\quad (\mathtt{ee})_{i,j} \neq (\mathtt{ee})_{j,i}\ \ , \tag{3.3.17}$$

even though these terms are identical. To obtain the proper cancellation of terms, the operation

$$(\mathtt{pp})_{j,i} \to (\mathtt{pp})_{i,j}\ \ ,\quad (\mathtt{ee})_{j,i} \to (\mathtt{ee})_{i,j}\ \ , \tag{3.3.18}$$

is performed for every combination of the indices i, j, k and l. The Mandelstam variables from (1.2.2) are

$$s = 2\,p_1\cdot p_2\ ,\quad t = 2\,p_1\cdot p_4\ ,\quad u = 2\,p_1\cdot p_3\ , \tag{3.3.19}$$

and due to conservation of momentum

$$s + t + u = 0\ , \tag{3.3.20}$$

holds. This simplifies $\mathcal{S}$ into a form of 60 term as shown in figure A.4.

[1] This definition could have been done more clever to avoid the problems described along (3.3.13).

3.3.3 Simplifying the overall sum of contributions

One finds from the Mathematica output that all the terms in $\mathcal{S}$ can be grouped into one of three distinct categories with certain characteristics. The first category consists of nine terms with the structure

$$s^2\,(\varepsilon_1\cdot\varepsilon_2)\,(\varepsilon_3\cdot\varepsilon_4)\ \ , \tag{3.3.21}$$

of two dot products between polarization vectors and the square of a Mandelstam variable. The second category consists of 42 terms with the structure

$$u\,(\varepsilon_2\cdot\varepsilon_3)\,(p_1\cdot\varepsilon_4)\,(p_3\cdot\varepsilon_1)\ \ , \tag{3.3.22}$$

having one Mandelstam variable, one dot product between polarization vectors and two dot products between a momentum vector and a polarization vector. The third category consists of 9 terms with the structure

$$(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_3) \qquad \text{or} \qquad (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)\ \ , \tag{3.3.23}$$

of four dot products between one momentum vector and one polarization vector. In the following each category of terms will be considered individually in order to simplify $\mathcal{S}$.

Terms from the first category

The nine terms in this category are manipulated using the rearrangement of (3.3.20)

$$s^2 = (t+u)^2 = t^2 + u^2 + 2tu\ , \tag{3.3.24}$$

whereby the three terms with the common coefficient $(\varepsilon_1\cdot\varepsilon_2)\,(\varepsilon_3\cdot\varepsilon_4)$ can be rewritten as

$$(\varepsilon_1\cdot\varepsilon_2)\,(\varepsilon_3\cdot\varepsilon_4)\left[-2s^2 + 2t^2 + 2u^2\right] = \,-\,4\ (\varepsilon_1\cdot\varepsilon_2)\,(\varepsilon_3\cdot\varepsilon_4)\,tu\ . \tag{3.3.25}$$

Equation (3.3.24) is symmetric in s, t and u and hence the method leading to (3.3.25) can be applied straightforwardly to the three terms proportional to $(\varepsilon_1\cdot\varepsilon_3)\,(\varepsilon_2\cdot\varepsilon_4)$ as well as for the three terms proportional to $(\varepsilon_1\cdot\varepsilon 4)\,(\varepsilon_2\cdot\varepsilon_3)$. This yields for the six remaining terms

$$(\varepsilon_1\cdot\varepsilon_3)\,(\varepsilon_2\cdot\varepsilon_4)\left[2s^2 + 2t^2 - 2u^2\right] = \,-\,4\ (\varepsilon_1\cdot\varepsilon_3)\,(\varepsilon_2\cdot\varepsilon_4)\,st\ , \tag{3.3.26}$$

$$(\varepsilon_1\cdot\varepsilon_4)\,(\varepsilon_2\cdot\varepsilon_3)\left[2s^2 - 2t^2 + 2u^2\right] = \,-\,4\ (\varepsilon_1\cdot\varepsilon_4)\,(\varepsilon_2\cdot\varepsilon_3)\,su\ , \tag{3.3.27}$$

such that the original nine terms have been rewritten as the three terms on the right hand sides of (3.3.25)—(3.3.27).

Terms from the second category

The four different polarization vectors form six different dot products each being a common factor in seven terms in the second category. In order to show how simplifications occur, the seven terms proportional to $(\varepsilon_1\cdot\varepsilon_2)$ are considered. Substituting s from (3.3.20) yields the expansion

$$
\begin{aligned}
4\Big[&-s(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)\;-s\,(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+u\,(p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)+t\,(p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)\\
&+t\,(p_1\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+u\,(p_2\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)-s\,(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\Big]\\
&=\;4t\Big[(p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)+(p_1\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)\\
&\quad+(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\Big]\\
&\quad+4u\Big[(p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)+(p_2\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)\\
&\quad+(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\Big]\,,
\end{aligned}
\tag{3.3.28}
$$

for the seven terms proportional to $(\varepsilon_1\cdot\varepsilon_2)$. Considering explicitly in (3.3.28) the sum of term number one, two, three and five in the square bracket proportional to $4t$ gives

$$
\begin{aligned}
(p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)&+(p_1\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(p_{2,\mu}p_{3,\nu}+p_{4,\mu}p_{1,\nu}+p_{2,\mu}p_{1,\nu}+p_{4,\mu}p_{3,\nu})\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(p_{2,\mu}+p_{4,\mu})\,(p_{1,\nu}+p_{3,\nu})\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(-p_{1,\mu}-p_{3,\mu})\,(-p_{2,\nu}-p_{4,\nu})\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,p_{1,\mu}\,p_{2,\nu}\\
&=(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)\;\;,
\end{aligned}
\tag{3.3.29}
$$

where momentum conservation has been employed along with transversality of the momentum. It is apparent that the right hand side of (3.3.29) is identical to term number four in the square bracket proportional to $4t$ in (3.3.28).

Identical manipulations are used in the square bracket proportional to $4u$ in (3.3.28) and hence the sum of term number one, two, four and five is

$$
\begin{aligned}
(p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)&+(p_2\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(p_{1,\mu}p_{3,\nu}+p_{4,\mu}p_{2,\nu}+p_{1,\mu}p_{2,\nu}+p_{4,\mu}p_{3,\nu})\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(p_{1,\mu}+p_{4,\mu})\,\big(p_{3,\nu}+p^2_{2,\nu}\big)\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,(-p_{2,\mu}+p_{3,\mu})\,(-p_{1,\nu}-p_{4,\nu})\\
&=\varepsilon_3^\mu\varepsilon_4^\nu\,p_{2,\mu}\,p_{1,\nu}\\
&=(p_2\cdot\varepsilon_3)\,(p_1\cdot\varepsilon_4)\;\;,
\end{aligned}
\tag{3.3.30}
$$

which is identical to term number four in the square bracket. From (3.3.29) and (3.3.30) it is possible to rewrite (3.3.28) in the much more compact form

$$
\begin{aligned}
4t\Big[&(p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)+(p_1\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)\\
&+(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\Big]\\
&+4u\Big[(p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_4)+(p_2\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_4)\,(p_2\cdot\varepsilon_3)\\
&+(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_3)\Big]\\
&=8t\,(p_1\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_4)+8u\,(p_2\cdot\varepsilon_3)\,(p_1\cdot\varepsilon_4)\;\;,
\end{aligned}
\tag{3.3.31}
$$

which is the final simplification.

Only seven terms have been considered in the manipulations (3.3.28) — (3.3.31) but the remaining 35 terms in this category can be manipulated in the same way. For each of the dot products $(\varepsilon_1\cdot\varepsilon_3)$,$(\varepsilon_1\cdot\varepsilon_4)$,$(\varepsilon_2\cdot\varepsilon_3)$,$(\varepsilon_2\cdot\varepsilon_4)$,$(\varepsilon_3\cdot\varepsilon_4)$ respectively, an expression equivalent to (3.3.31) can be obtained for the seven terms proportional to this particular dot product and hence the 42 terms in the second category are reduced to 12 terms.

Terms from the third category

Conservation of momentum yields

$$(p_4\cdot\varepsilon_2)\,(p_3\cdot\varepsilon_1) = (p_1\cdot\varepsilon_2)\,(p_2\cdot\varepsilon_1) + (p_1\cdot\varepsilon_2)\,(p_4\cdot\varepsilon_1) + (p_3\cdot\varepsilon_2)\,(p_2\cdot\varepsilon_1) + (p_3\cdot\varepsilon_2)\,(p_4\cdot\varepsilon_1)\ , \tag{3.3.32}$$

$$(p_4\cdot\varepsilon_3)\,(p_2\cdot\varepsilon_1) = (p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_1) + (p_1\cdot\varepsilon_3)\,(p_4\cdot\varepsilon_1) + (p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_1) + (p_2\cdot\varepsilon_3)\,(p_4\cdot\varepsilon_1)\ , \tag{3.3.33}$$

which will be used to expand the nine terms of the third category. For convenience and to introduce a certain labeling the nine terms are written explicitly

$$\begin{aligned}
&-\underbrace{(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)}_{\alpha_1}-\underbrace{(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)}_{\beta_1}-\underbrace{(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)}_{\gamma_1}\\
&+\underbrace{(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)}_{\beta_2}+\underbrace{(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)}_{\gamma_2}+\underbrace{(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)}_{\alpha_2}\\
&+\underbrace{(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)\,(p_4\cdot\varepsilon_2)\big(p_2\cdot\varepsilon^1\big)}_{\beta_3}+\underbrace{(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)}_{\gamma_3}+\underbrace{(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)}_{\alpha_3}.
\end{aligned} \tag{3.3.34}$$

Substituting (3.3.32) and (3.3.33) respectively in the expressions for α_2 and α_3, the sum of α_1, α_2 and α_3 becomes

$$\begin{aligned}
&-(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)+(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)\\
&\quad+(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)\\
&\qquad= (p_1\cdot\varepsilon_4)\Big\{(p_2\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_2)\,[(p_4\cdot\varepsilon_1)+(p_2\cdot\varepsilon_1)+(p_3\cdot\varepsilon_1)]\\
&\qquad\qquad+\ (p_2\cdot\varepsilon_3)\,(p_1\cdot\varepsilon_2)\,(p_4\cdot\varepsilon_1)+(p_2\cdot\varepsilon_3)\,(p_1\cdot\varepsilon_2)\,(p_2\cdot\varepsilon_1)\\
&\qquad\qquad+(p_3\cdot\varepsilon_2)\,(p_1\cdot\varepsilon_3)\,(p_4\cdot\varepsilon_1)+(p_3\cdot\varepsilon_2)\,(p_1\cdot\varepsilon_3)\,(p_3\cdot\varepsilon_1)\Big\}\\
&\qquad= (p_1\cdot\varepsilon_4)\,\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\,[p_{4,\mu}\,p_{1,\nu}\,p_{2,\rho}+p_{2,\mu}\,p_{1,\nu}\,p_{2,\rho}+p_{4,\mu}\,p_{3,\nu}\,p_{1,\rho}+p_{3,\mu}\,p_{3,\nu}\,p_{1,\rho}]\\
&\qquad= (p_1\cdot\varepsilon_4)\,\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\,[(p_{4,\mu}+p_{2,\mu})\,p_{1,\nu}\,p_{2,\nu}+(p_{4,\mu}+p_{3,\mu})\ p_{3,\nu}\,p_{1,\rho}]\\
&\qquad=\ -(p_1\cdot\varepsilon_4)\,\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\,[p_{3,\mu}\,p_{1,\nu}\,p_{2,\rho}+p_{2,\mu}\,p_{3,\nu}\,p_{1,\rho}]\ ,
\end{aligned} \tag{3.3.35}$$

where momentum conservation has been used in combination with transversality to obtain

$$(p_4\cdot\varepsilon_1)+(p_2\cdot\varepsilon_1)+(p_3\cdot\varepsilon_1) = -p_1\cdot\varepsilon_1 = 0\ . \tag{3.3.36}$$

An identical procedure can be used for the sum of $\beta_1, \beta_2, \beta_3$ as well as for the sum of $\gamma_1, \gamma_2, \gamma_3$. For the sum of $\beta_1, \beta_2, \beta_3$, the terms β_2 and β_3 are expanded and explicit calculations yield

$$\begin{aligned}
&-(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)+(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)\\
&\quad+\left(p^1\cdot\varepsilon^3\right)\left(p^3\cdot\varepsilon^4\right)\left(p^4\cdot\varepsilon^2\right)\left(p^2\cdot\varepsilon^1\right)\\
&\qquad\qquad=\,-(p_1\cdot\varepsilon_3)\,\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_4^\rho\left[p_{2,\mu}\,p_{4,\nu}\,p_{1,\rho}+p_{4,\mu}\,p_{1,\nu}p_{2,\rho}\right]\ .
\end{aligned}\tag{3.3.37}$$

For the sum of γ_1, γ_2 and γ_3 it is found that

$$\begin{aligned}
&-(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)+(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)\\
&\quad+(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)\\
&\qquad\qquad=\,-(p_1\cdot\varepsilon_2)\,\varepsilon_1^\mu\varepsilon_3^\nu\varepsilon_4^\rho\left[p_{3,\mu}\,p_{4,\nu}\,p_{1,\rho}+p_{4,\mu}\,p_{1,\nu}\,p_{3,\rho}\right]\ ,
\end{aligned}\tag{3.3.38}$$

in a similar way. Equations (3.3.35), (3.3.37) and (3.3.38) are summed up to give

$$\begin{aligned}
&-\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma\left[p_{3,\mu}p_{1,\nu}p_{2,\rho}p_{1,\sigma}+p_{2,\mu}p_{3,\nu}p_{1,\rho}p_{1,\sigma}\right]-\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma\left[p_{2,\mu}p_{4,\nu}p_{1,\rho}p_{1,\sigma}+p_{4,\mu}p_{1,\nu}p_{1,\rho}p_{2,\sigma}\right]\\
&\qquad-\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma\left[p_{3,\mu}p_{1,\nu}p_{4,\rho}p_{1,\sigma}+p_{4,\mu}p_{1,\nu}p_{1,\rho}p_{3,\sigma}\right]\\
&=-\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma\left[p_{3,\mu}p_{1,\nu}\left(p_{2,\rho}+p_{4,\rho}\right)p_{1,\sigma}+p_{2,\mu}\left(p_{3,\nu}+p_{4,\nu}\right)p_{1,\rho}p_{1,\sigma}+p_{4,\mu}p_{1,\nu}p_{1,\rho}\left(p_{2,\sigma}+p_{3,\sigma}\right)\right]\\
&=+\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma\left[p_{3,\mu}p_{1,\nu}\left(p_{1,\rho}+p_{3,\rho}\right)p_{1,\sigma}+p_{2,\mu}\left(p_{1,\nu}+p_{2,\nu}\right)p_{1,\rho}p_{1,\sigma}+p_{4,\mu}p_{1,\nu}p_{1,\rho}\left(p_{1,\sigma}+p_{4,\sigma}\right)\right]\\
&=\varepsilon^{1\,\mu}\varepsilon^{2\,\nu}\varepsilon^{3\,\rho}\varepsilon^{4\,\sigma}\left[p_{3,\mu}p_{1,\nu}p_{1,\rho}p_{1,\sigma}+p_{2,\mu}p_{1,\nu}p_{1,\rho}p_{1,\sigma}+p_{4,\mu}p_{1,\nu}p_{1,\rho}p_{1,\sigma}\right]\\
&=\,-\,\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\rho\varepsilon_4^\sigma p_1^\mu p_1^\nu p_1^\rho p_1^\sigma\\
&=0\ ,
\end{aligned}\tag{3.3.39}$$

such that the nine terms (3.3.34) add to zero.

Collecting the pieces

The entire amplitude can be written in terms of the contributions (3.3.25)—(3.3.27) together with six contributions of the form (3.3.31) where each contribution is multiplied be the appropriate dot product of polarization vectors. The final result for the amplitude is

$$\begin{aligned}
&\mathcal{A}\left(A_iA_jA_jA_k\right)\\
&\quad=\,-4\Big[(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)\,tu+(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)\,st+(\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\,su\Big]\\
&\qquad+8s\Big[(p_1\cdot\varepsilon_3)(p_4\cdot\varepsilon_2)(\varepsilon_1\cdot\varepsilon_4)+(p_1\cdot\varepsilon_4)(p_3\cdot\varepsilon_2)(\varepsilon_1\cdot\varepsilon_3)\\
&\qquad\qquad+(p_2\cdot\varepsilon_4)(p_3\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)+(p_2\cdot\varepsilon_3)(p_4\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_4)\Big]\\
&\qquad+8t\Big[(p_1\cdot\varepsilon_2)(p_3\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_3)+(p_1\cdot\varepsilon_3)(p_2\cdot\varepsilon_4)(\varepsilon_1\cdot\varepsilon_2)\\
&\qquad\qquad+(p_3\cdot\varepsilon_1)(p_4\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)+(p_2\cdot\varepsilon_1)(p_4\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)\Big]\\
&\qquad+8u\Big[(p_1\cdot\varepsilon_2)(p_4\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_4)+(p_1\cdot\varepsilon_4)(p_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2)\\
&\qquad\qquad+(p_4\cdot\varepsilon_1)(p_3\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)+(p_2\cdot\varepsilon_1)(p_3\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\Big]\,,
\end{aligned}\tag{3.3.40}$$

which is identical to the "kinematic factor" calculated in [25] apart from an overall multiplicative factor of 16. Furthermore, (3.3.40) is found in [26]. The different overall factor is due to different choices of conventions regarding normalization of the string parameter α'. It follows directly from (3.3.40) combined with (2.1.114) that the amplitudes $\mathcal{A}(\pm\pm\pm\pm)$ vanish identically regardless of dimension. This an interesting result which is in agreement with (3.2.1).

In higher dimensions one can chose specific polarization vectors and compute directly from (3.3.40). However, another approach starting from the Dirac-Born-Infeld action is used in order to gain more insight in the six-dimensional case. This approach will be the topic for the next chapter where one must expect that the $\mathcal{A}(\pm\pm\pm\pm)$ amplitude vanishes due to (3.3.40).

Chapter 4

Vector boson amplitudes in six dimensions

In the previous chapter, various scattering amplitudes were calculated in four dimensions with diligent use of the spinor-helicity formalism. The calculations were streamlined considerably due to the formalism. This chapter presents calculations of amplitudes in six dimensions where the situation is more complicated. A first observation is that the spinor-helicity formalism is not generalized straightforwardly to six dimensions and this suggests another approach. However, the four-dimensional spinor-helicity formalism will still be used to express the four dimensional part of six-dimensional amplitudes. This approach is based on some simplifying assumptions and the use of auxiliary dimensions. In fact it is part of the discussion of the appropriate language in six dimensions as addressed in the introduction.

4.1 Six dimensions from a four-dimensional perspective

The topic of the subsequent sections is to calculate different scattering amplitudes in six dimensions. The external states of these amplitudes are specified which in this sense is much along the lines of section 3.2 where specific amplitudes were calculated in four dimensions. However, in order to take the step from four to six dimensions some developments are required. These developments will be made below and are introduced in order to simplify the calculations. The general expression (3.3.40) will be left and instead the situation will be considered more specific using a bottom-up approach. One similarity however is that (3.3.6) is still the interesting term to consider.

4.1.1 Auxiliary dimensions

The overall motivation for the developments mentioned above is to simplify the calculations by removing some degrees of freedom. The problem under consideration is six-dimensional and one can think of the four and five-directions as being auxiliary dimensions with respect to the usual four dimensions. Introducing suitable constraints on vector components in the auxiliary dimensions will result in a splitting of the auxiliary dimensions from the usual four dimensions and in this way the overall kinematics can be treated in a simpler way. One can think of the auxiliary dimensions in terms of a scattering experiment in N dimensions. The scattered beams can be prepared in a suitable way so that momentum and polarization vectors are embedded in m dimensions where $N > m$. Of course momentum can be scattered into the $N - m$ dimensions as long as overall momentum is conserved but these dimensions are in some sense auxiliary to the m dimensions.

Two different approaches will be taken in the context described above and each of them uses its own constraint in the auxiliary dimensions. Scattering amplitudes in sections 4.3 and 4.4 are calculated using the different approaches respectively. These two approaches are discussed below.

4.1.2 Constraining gauge field components

In the first approach, scattering amplitudes in six dimensions will be calculated under the constraint

$$A_4 = A_5 = 0 \ , \tag{4.1.1}$$

on the gauge field components in the auxiliary dimensions. This constraint is used in (3.3.6) where it leads to a simplification of this term. The simplification does not occur straightforwardly but is obtained by writing the field strength tensors as explicit matrices. This procedure is discussed below in section 4.2.1

If one studies the four and five directions as two extra dimensions with respect to the usual four dimensions it is convenient to express the momentum square of the usual four directions in terms of momentum components in the extra directions. From a massless momentum vector

$$0 = p^2 = \left(p^0\right)^2 - \left(p^1\right)^2 - \left(p^2\right)^2 - \left(p^3\right)^2 - \left(p^4\right)^2 - \left(p^5\right)^2 \ , \tag{4.1.2}$$

one can define the momentum square

$$p^2_{(d=4)} \equiv \tilde{p}^2 = \left(p^0\right)^2 - \left(p^1\right)^2 - \left(p^2\right)^2 - \left(p^3\right)^2 = + \left(p^4\right)^2 + \left(p^5\right)^2 \ . \tag{4.1.3}$$

This definition suggests a four-dimensional interpretation. Let p^M be the momentum vector for a massless particle in six dimensions. By (4.1.3) it follows then that the same particle can be considered from a four-dimensional perspective as massive with

$$m^2_{(4\text{d})} = \left(p^4\right)^2 + \left(p^5\right)^2 \ . \tag{4.1.4}$$

The momentum vector in four dimensions should therefore be expressed as a massive spinor according to (2.1.107).

Another simplification used in this approach is regarding polarization of the scattered photons. A massless particle in d dimensions has $d - 2$ physical degrees of freedom because one degree of freedom is removed by the equations of motion and one is removed by the gauge condition. These degrees of freedom are reflected by a photon in four dimensions having two possible polarization states. In six dimensions, a photon has four possible polarization states of which two are the same as the four-dimensional states. The simplification used is to ignore the two extra polarization states in six dimensions. In this way the photon in six dimensions can be described using the same polarization vectors as in four dimensions. This turns out to be useful since the spinor-helicity formalism can then be used to describe the four-dimensional parts of results obtained in six-dimensional calculations.

4.1.3 Constraining momentum components

The second approach uses the assumption to neglect momentum components in the auxiliary dimensions

$$p_4 = p_5 = 0 \ . \tag{4.1.5}$$

This constraint leads to an interesting four-dimensional perspective on the fields in the theory. In order to see this, the Lagrangian for the free electromagnetic field

$$\mathcal{L}_{\text{em}}^{(6\text{d})}\left(A^M\right) = \frac{1}{4}\, F_{MN}F^{MN} \ , \tag{4.1.6}$$

is considered in six dimensions. The six-dimensional indices can be decomposed simply into lower-dimensional indices so that the field strength tensor becomes

$$F_{MN} = F_{\mu\nu} + F_{mn} + F_{\mu n} + F_{m\nu} \ , \tag{4.1.7}$$

and subsequently

$$F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} + 2F_{45}F^{45} + 2F_{\mu 4}F^{\mu 4} + 2F_{\mu 5}F^{\mu 5} \ . \tag{4.1.8}$$

For the case (4.1.5) the field strength components are

$$F_{45} = 0 \ , \quad F_{\mu i}F^{\mu i} = \, + \, \partial_\mu A_i \partial^\mu A^i = \, - \, \partial_\mu A_i \partial^\mu A_i \ , \tag{4.1.9}$$

for the metric (2.4.1) generalized to six dimensions. Equation (4.1.8) becomes

$$F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} - 4\, \partial_\mu \phi \partial^\mu \bar{\phi} \ , \tag{4.1.10}$$

for the definitions of scalars

$$\phi \equiv \frac{1}{\sqrt{2}}\left(A_4 + iA_5\right) \ , \quad \bar{\phi} \equiv \frac{1}{\sqrt{2}}\left(A_4 - iA_5\right) \ , \tag{4.1.11}$$

with the inversion

$$A_4 = \frac{1}{\sqrt{2}}\left(\phi + \bar{\phi}\right) \ , \quad A_5 = \, - \, \frac{i}{\sqrt{2}}\left(\phi - \bar{\phi}\right) \ . \tag{4.1.12}$$

It follows from (4.1.10) that the requirement (4.1.5) turns the six-dimensional Lagrangian with six-dimensional gauge fields into a Lagrangian with four-dimensional gauge fields and two complex scalars

$$\mathcal{L}_{\text{em}}^{(6\text{d})}\left(A^M\right) \rightarrow \mathcal{L}_{\text{em}}^{(4\text{d})}\left(A^\mu, \phi, \bar{\phi}\right) \ . \tag{4.1.13}$$

This is exactly the four-dimensional perspective on the theory as considered above which will be used in section 4.4 where scattering amplitudes involving the scalars (4.1.11) will be calculated.

4.2 Preparing amplitude calculations

The first amplitude calculations will be based on the gauge field constraint (4.1.1) as discussed above. The four-point amplitudes are controlled by the term

$$I_4' = F_{MN}F^{NR}F_{RS}F^{RM} - \frac{1}{4}F_{MN}F^{MN}F_{RS}F^{RS} \ , \tag{4.2.1}$$

which can be simplified by the gauge field constraint in the auxiliary dimensions. In order to simplify (4.2.1) it is useful to write the field strengths explicitly as block matrices. This is done in order to split the usual four-dimensional part of the tensor from the part in the auxiliary dimensions and it leads to an expression for (4.2.1) which is suitable for Wick contractions.

4.2.1 Decomposing the field strength tensor

When the field strengths are considered as matrices the two terms on the right hand side of (4.2.1) are written as the traces

$$F_{MN}F^{NR}F_{RS}F^{SM} = F^M{}_N F^N{}_R F^R{}_S F^S{}_M = \mathrm{Tr}\, F^4 \ , \tag{4.2.2}$$

$$F_{MN}F^{MN}F_{RS}F^{RS} = \big(-F^M{}_N F^N{}_M\big)\big(-F^R{}_S F^S{}_R\big) = +\,\mathrm{Tr}^2 F^2 \ , \tag{4.2.3}$$

of ordinary matrix products. In order to simplify (4.2.2) and (4.2.3) the matrix expression for the field strength tensor will be decomposed into block matrices. This is straightforward since an arbitrary matrix can be interpreted as a block matrix where the entries are grouped according to a certain block structure. In six dimensions the electromagnetic field strength tensor can be written as the 6×6 block matrix

$$F^M_N = \begin{bmatrix} \mathcal{A}_{(4\times 4)} & \mathcal{B}_{(4\times 2)} \\ -\mathcal{B}^T_{(2\times 4)} & \mathcal{D}_{(2\times 2)} \end{bmatrix}, \tag{4.2.4}$$

with the dimensionality of each of the matrices $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$ written explicitly. The notation of (4.2.4) is obviously insufficient since $\mathcal{A}^\mu{}_\nu$, $\mathcal{B}^\mu{}_n$ and $\mathcal{D}^m_{\ n}$ have Lorentz indices. In terms of these indices (4.2.4) becomes explicitly

$$F^M_N = \begin{bmatrix} \mathcal{A}^\mu{}_\nu & \mathcal{B}^\mu{}_n \\ -\mathcal{B}_\nu{}^m & \mathcal{D}^m_{\ n} \end{bmatrix}, \tag{4.2.5}$$

where the matrices are

$$\mathcal{B}_{\mu n} = \begin{bmatrix} \partial_0 A_4 - \partial_4 A_0 & \partial_0 A_5 - \partial_5 A_0 \\ \partial_1 A_4 - \partial_4 A_1 & \partial_1 A_5 - \partial_5 A_1 \\ \partial_2 A_4 - \partial_4 A_2 & \partial_2 A_5 - \partial_5 A_2 \\ \partial_3 A_4 - \partial_4 A_3 & \partial_3 A_5 - \partial_5 A_3 \end{bmatrix}, \quad \mathcal{D}_{mn} = \begin{bmatrix} 0 & \partial_4 A_5 - \partial_5 A_4 \\ \partial_5 A_4 - \partial_4 A_5 & 0 \end{bmatrix}, \tag{4.2.6}$$

and

$$\mathcal{A}^\mu{}_\nu = {}^{(4\mathrm{d})}F^\mu{}_\nu \equiv f^\mu{}_\nu \ , \tag{4.2.7}$$

is just the usual four-dimensional field strength.

4.2.2 Tensor contractions as traces of matrix products

In matrix notation

$$F^2 = F^M_{\ N} F^N_{\ R} = (FF)^M_{\ R} \ , \tag{4.2.8}$$

and from (4.2.5) the products of field strength matrices in (4.2.2) and (4.2.3) are

$$F^2 = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^T & \mathcal{D} \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^T & \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A}^2 - \mathcal{B}\mathcal{B}^T & \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{D} \\ -\mathcal{B}^T\mathcal{A} - \mathcal{D}\mathcal{B}^T & -\mathcal{B}^T\mathcal{B} + \mathcal{D}^2 \end{bmatrix}, \tag{4.2.9}$$

and

$$F^4 = \begin{bmatrix} \mathcal{A}^2 - \mathcal{B}\mathcal{B}^T & \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{D} \\ -\mathcal{B}^T\mathcal{A} - \mathcal{D}\mathcal{B}^T & -\mathcal{B}^T\mathcal{B} + \mathcal{D}^2 \end{bmatrix} \begin{bmatrix} \mathcal{A}^2 - \mathcal{B}\mathcal{B}^T & \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{D} \\ -\mathcal{B}^T\mathcal{A} - \mathcal{D}\mathcal{B}^T & -\mathcal{B}^T\mathcal{B} + \mathcal{D}^2 \end{bmatrix} = \begin{bmatrix} \phi & \chi \\ \psi & \omega \end{bmatrix}, \tag{4.2.10}$$

with the abbreviations

$$\phi = \mathcal{A}^4 + \mathcal{B}\mathcal{B}^T\mathcal{B}\mathcal{B}^T - \mathcal{A}^2\mathcal{B}\mathcal{B}^T - \mathcal{B}\mathcal{B}^T\mathcal{A}^2 - \mathcal{A}\mathcal{B}\mathcal{B}^T\mathcal{A} - \mathcal{B}\mathcal{D}^2\mathcal{B}^T - \mathcal{A}\mathcal{B}\mathcal{D}\mathcal{B}^T - \mathcal{B}\mathcal{D}\mathcal{B}^T\mathcal{A} \ , \tag{4.2.11}$$

$$\omega = -\mathcal{B}^T\mathcal{A}^2\mathcal{B} - \mathcal{D}\mathcal{B}^T\mathcal{B}\mathcal{D} - \mathcal{B}^T\mathcal{A}\mathcal{B}\mathcal{D} - \mathcal{D}\mathcal{B}^T\mathcal{A}\mathcal{B} + \mathcal{B}^T\mathcal{B}\mathcal{B}^T\mathcal{B} + \mathcal{D}^4 - \mathcal{B}^T\mathcal{B}\mathcal{D}^2 - \mathcal{D}^2\mathcal{B}^T\mathcal{B} \ , \tag{4.2.12}$$

$$\chi = \mathcal{A}^3\mathcal{B} - \mathcal{B}\mathcal{B}^T\mathcal{B}\mathcal{D} + \mathcal{A}^2\mathcal{B}\mathcal{D} - \mathcal{B}\mathcal{B}^T\mathcal{A}^2 + \mathcal{A}\mathcal{B}\mathcal{D}\mathcal{B} + \mathcal{B}\mathcal{D}^3 + \mathcal{A}\mathcal{B}\mathcal{D}^2 - \mathcal{B}\mathcal{D}\mathcal{B}^T\mathcal{B} \ , \tag{4.2.13}$$

$$\psi = -\mathcal{B}^T\mathcal{A}^3 + \mathcal{D}\mathcal{B}^T\mathcal{B}\mathcal{B}^T - \mathcal{D}\mathcal{B}^T\mathcal{A}^2 + \mathcal{B}^T\mathcal{B}\mathcal{B}^T\mathcal{A} - \mathcal{B}^T\mathcal{A}\mathcal{B}\mathcal{B}^T - \mathcal{D}^3\mathcal{B}^T + \mathcal{B}^T\mathcal{B}\mathcal{D}\mathcal{B}^T - \mathcal{D}^2\mathcal{B}^T\mathcal{A} \ . \tag{4.2.14}$$

This is nothing but a decomposition which by itself provides no simplification. The crucial step is to use (4.1.1) whereby

$$\mathcal{D}_{mn} = 0 \ , \tag{4.2.15}$$

and hence (4.2.9) and (4.2.10) simplify.

The trace of a product of arbitrary $n \times n$ matrices is cyclic in the order of matrices and by (4.2.15) the trace of (4.2.9) becomes

$$\mathrm{Tr}\, F^2 = \mathrm{Tr}\, \mathcal{A}^2 - 2\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T \ . \tag{4.2.16}$$

The trace of (4.2.10) is

$$\begin{aligned} \mathrm{Tr}\, F^4 &= \mathrm{Tr}\, \phi + \mathrm{Tr}\, \omega \\ &= \mathrm{Tr}\, \mathcal{A}^4 - 4\, \mathrm{Tr}\, \mathcal{A}^2\mathcal{B}\mathcal{B}^T + 2\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T\mathcal{B}\mathcal{B}^T \ , \end{aligned} \tag{4.2.17}$$

from (4.2.11) together with (4.2.12). Squaring (4.2.16) yields

$$\mathrm{Tr}^2 F^2 = \mathrm{Tr}^2 \mathcal{A}^2 + 4\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T - 4\, \mathrm{Tr}\, \mathcal{A}^2\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T \ , \tag{4.2.18}$$

and by (4.2.2), (4.2.3), (4.2.17) and (4.2.18), equation (4.2.1) becomes

$$\begin{aligned} I'_4 &= \mathrm{Tr}\, F^4 - \frac{1}{4}\, \mathrm{Tr}\, F^2 \\ &= \mathrm{Tr}\, \mathcal{A}^4 - \frac{1}{4}\, \mathrm{Tr}^2\mathcal{A}^2 + 2\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T\mathcal{B}\mathcal{B}^T - 4\, \mathrm{Tr}\, \mathcal{A}^2\mathcal{B}\mathcal{B}^T - \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T + \mathrm{Tr}\, \mathcal{A}^2\, \mathrm{Tr}\, \mathcal{B}\mathcal{B}^T \ . \end{aligned} \tag{4.2.19}$$

The matrix $\mathcal{A}$ is exactly the four-dimensional part of F_{MN} and hence

$$\operatorname{Tr}\mathcal{A}^4 - \frac{1}{4}\operatorname{Tr}^2\mathcal{A}^2 = F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \left({}^{+}F\right)^2\left({}^{-}F\right)^2 \ , \tag{4.2.20}$$

holds for the four-dimensional matrices. Subsequently (4.2.19) becomes

$$\begin{aligned} I_4' = \left({}^{+}F\right)^2\left({}^{-}F\right)^2 + 2\left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\mu} - 4\,F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\mu} \\ - \left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\mu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\nu} - F_{\mu\nu}\,F^{\mu\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda} \ , \end{aligned} \tag{4.2.21}$$

which appears almost as a four-dimensional expression because only four-dimensional Lorentz indices are present. The dependence on the auxiliary two dimensions is in the matrix product $\mathcal{B}\mathcal{B}^T$ where for instance

$$\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\sigma} = \mathcal{B}^{\rho n}\left(\mathcal{B}^T\right)_{n\sigma} = \mathcal{B}^{\rho n}\mathcal{B}_{\sigma n} \ . \tag{4.2.22}$$

By (4.1.1) equation (4.2.22) becomes

$$\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\sigma} = \partial^n A^{\rho}\partial_n A_{\sigma} \ , \tag{4.2.23}$$

which will be used in section 4.3 in order to compute Wick contractions of certain external fields into (4.2.21).

4.3 Amplitudes with gauge field constraints

This section contains the calculations based on the developments in section 4.2. Four-point amplitudes with specific configuration of external polarization will be calculated with the use of the constraints discussed in section 4.1.2. The calculated amplitudes are $\mathcal{A}(++++)$, $\mathcal{A}(-+++)$, $\mathcal{A}(--++)$. The explicit expression of the term I_4' in (4.2.21) is the starting point for all amplitudes calculations in this section. For convenience and to make references to certain terms easier throughout the calculations, the expression (4.2.20) is written again as

$$\begin{aligned} I_4' = \underbrace{\left({}^{+}F\right)^2\left({}^{-}F\right)^2}_{\chi_1} + 2\underbrace{\left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\mu}}_{\chi_2} - 4\underbrace{F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\mu}}_{\chi_3} \\ - \underbrace{\left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\mu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\nu}}_{\chi_4} + \underbrace{F_{\mu\nu}\,F^{\mu\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda}}_{\chi_5} \ , \end{aligned} \tag{4.3.1}$$

where a labeling of terms is included. For each of the χ_i-terms all non-vanishing Wick contractions have to be calculated. This is done below for each of the amplitudes. Before the amplitudes are calculated explicitly, it is natural to evaluate some particular Wick contractions which are relevant for the computations.

4.3.1 Relevant Wick contractions

The results obtained in (3.1.15) and (3.1.18) will still be used in the six-dimensional calculations. This is due to the discussion in section 4.1.2 of polarization states in six dimensions. The outcome is that since the two extra polarization states in six dimensions are neglected, the four-dimensional spinor-helicity formalism can be used to describe the non-auxiliary dimensions in the six-dimensional problem.

The contraction of a (+) photon field and a selfdual field strength is

$$\overbracket{A^{++}F}_{\dot\alpha\dot\beta} = \frac{1}{2^2}\,(-2i)\,(-i)\,\sqrt{2}\,\frac{\zeta^{\gamma}\lambda^{\dot\gamma}}{[\zeta\lambda]}\left(p_{\alpha\dot\alpha}\,\varepsilon^{\alpha}{}_{\gamma}\,\varepsilon_{\dot\beta\dot\gamma} + p_{\alpha\dot\beta}\,\varepsilon^{\alpha}{}_{\gamma}\,\varepsilon_{\dot\alpha\dot\gamma}\right)\ , \tag{4.3.2}$$

where the factor of $1/p^2$ from the photon propagator is cancelled due to (2.2.80) and has been omitted. Contracting the first term in the bracket of (4.3.2) with the numerator yields

$$\zeta^{\gamma}\lambda^{\dot\gamma}\,p_{\gamma\dot\alpha} = \zeta^{\gamma}\lambda^{\dot\gamma}\left(\lambda_{\gamma}\lambda_{\dot\alpha} + \frac{\tilde{p}^2}{2\,p^{\flat}\cdot q}\,\zeta_{\gamma}\zeta_{\dot\alpha}\right) = [\lambda\zeta]\,\lambda_{\dot\alpha}\lambda_{\dot\beta}\ , \tag{4.3.3}$$

which is also the result when the numerator is contracted with the rightmost term in the brackets. Similar calculations for the contraction $\overbracket{A^{--}F}_{\dot\alpha\dot\beta}$ yield in six dimensions

$$^{+}F_{\dot\alpha\dot\beta}\left[A^{+}\right]_{6d} = i\sqrt{2}\lambda_{\dot\alpha}\lambda_{\dot\beta}\ , \tag{4.3.4}$$

$$^{-}F_{\alpha\beta}\left[A^{-}\right]_{6d} = -i\sqrt{2}\lambda_{\alpha}\lambda_{\beta}\ . \tag{4.3.5}$$

It is important to note that

$$\overbracket{A^{-+}F}_{\dot\alpha\dot\beta} \neq 0\ , \tag{4.3.6}$$

$$\overbracket{A^{+-}F}_{\dot\alpha\dot\beta} \neq 0\ , \tag{4.3.7}$$

hold in the six-dimensional case. The contraction (4.3.6) is computed by substituting

$$\frac{\zeta^{\gamma}\lambda^{\dot\gamma}}{[\zeta\lambda]} \to \frac{\zeta^{\dot\gamma}\lambda^{\gamma}}{\langle\zeta\gamma\rangle}\ , \tag{4.3.8}$$

in (4.3.2) which yields

$$\frac{1}{\langle\zeta\lambda\rangle}\left[\left(\lambda_{\alpha}\lambda_{\dot\alpha} + \frac{\tilde{p}^2}{2\,p^{\flat}\cdot q}\,\zeta_{\alpha}\zeta_{\dot\alpha}\right)\zeta_{\dot\beta}\lambda^{\alpha} + \left(\lambda_{\alpha}\lambda_{\dot\beta} + \frac{\tilde{p}^2}{2\,p^{\flat}\cdot q}\,\zeta_{\alpha}\zeta_{\dot\beta}\right)\zeta_{\dot\alpha}\lambda^{\alpha}\right] = 2\,\frac{[\zeta\lambda]}{\langle\zeta\lambda\rangle}\frac{\tilde{p}^2}{2\,p^{\flat}\cdot q}\,\zeta_{\dot\alpha}\zeta_{\dot\beta}\ . \tag{4.3.9}$$

It follows from (2.1.68) that

$$\frac{[\zeta\lambda]}{2\,p^{\flat}\cdot q} = \frac{1}{\langle\lambda\zeta\rangle}\ , \tag{4.3.10}$$

whereby

$$^{+}F_{\dot\alpha\dot\beta}\left[A^{-}\right]_{6d} = i\sqrt{2}\frac{\tilde{p}^2}{\langle\zeta\lambda\rangle^2}\,\zeta_{\dot\alpha}\zeta_{\dot\beta}\ , \tag{4.3.11}$$

is obtained using (4.3.2) and (4.3.9). The opposite relation is derived in a similar fashion and reads

$$^{-}F_{\alpha\beta}\left[A^{+}\right]_{6d} = -i\sqrt{2}\frac{\tilde{p}^2}{[\zeta\lambda]^2}\,\zeta_{\alpha}\zeta_{\beta}\ . \tag{4.3.12}$$

Expressions (4.3.4), (4.3.5), (4.3.11) and (4.3.12) will be used in the following in calculations of the terms (4.3.1).

The amplitude $\mathcal{A}(++++)$

The all-plus amplitude

$$\mathcal{A}(++++) \rightarrow \left\langle A_1^+ A_2^+ A_3^+ A_4^+ \,\middle|\, I_4' \right\rangle , \tag{4.3.13}$$

is the most simple amplitude and according to the general amplitude (3.3.40) it is expected to vanish. It follows from (4.2.23) that

$$\left(\mathcal{B}\mathcal{B}^T\right)_{\mu\nu} \left(\mathcal{B}\mathcal{B}^T\right)^{\mu\nu} = \partial^m A_\mu \partial_m A_\nu \partial^n A^\mu \partial_n A^\nu , \tag{4.3.14}$$

so that

$$\left\langle I_4' \,\middle|\, \chi_2 \right\rangle = \left\langle I_4' \,\middle|\, \chi_4 \right\rangle = \left\langle I_4' \,\middle|\, \chi_5 \right\rangle = 0 , \tag{4.3.15}$$

because of metric contractions which lead to contractions of polarization vectors of the same type. The result

$$\left\langle I_4' \,\middle|\, \chi_1 \right\rangle = 0 , \tag{4.3.16}$$

follows straightforwardly since it is identical to (3.2.1). By (2.2.80) and (4.2.23) it follows that all contractions of the term χ_3 in the all-plus amplitude is of the form

$$\left(p_i^\mu \, \varepsilon_i^\nu - p_i^\nu \, \varepsilon_j^\mu\right) \left(p_{j,\nu} \, \varepsilon_{j,\rho} - p_{j,\rho} \, \varepsilon_{j,\nu}\right) p_k^n \, p_{l,n} \, \varepsilon_k^\rho \, \varepsilon_l^\mu , \tag{4.3.17}$$

where signs have been ignored. In the case of four identical polarizations

$$\varepsilon_i^\pm \ , \ \varepsilon_j^\pm \ , \ \varepsilon_k^\pm \ , \ \varepsilon_l^\pm \ , \tag{4.3.18}$$

equation (4.3.17) vanishes due to (2.1.114) and hence

$$\left\langle I_4' \,\middle|\, \chi_3 \right\rangle = 0 . \tag{4.3.19}$$

It has thus been found that

$$\mathcal{A}(++++) = 0 , \tag{4.3.20}$$

as expected.

The amplitude $\mathcal{A}(-+++)$

The amplitude with one polarization different vanishes in four dimensions. The situation is different in six dimensions as will be found below. The amplitude is calculated from all possible Wick contractions

$$\mathcal{A}(-+++) \rightarrow \left\langle A_1^- A_2^+ A_3^+ A_4^+ \,\middle|\, I_4' \right\rangle , \tag{4.3.21}$$

of (4.3.1).

4.3.2 Computing χ_i terms

The Wick contractions of external fields into (4.3.1) are considered term by term and will be collected after all terms have been calculated.

The terms χ_2 and χ_4

First, the χ_2 term given by (4.3.14) is considered. In order to support the following argument, the external photon fields are written as

$$\varepsilon_i^\rho\, \varepsilon_j^\sigma\, \varepsilon_k^\kappa\, \varepsilon_l^\lambda A_\rho(p_i)\, A_\sigma(p_j)\, A_\kappa(p_k)\, A_\lambda(p_l) \ , \tag{4.3.22}$$

without specifying the polarizations. Independent of the way the external photon fields are contracted into (4.3.14), a factor of metric tensors with the structure

$$\eta_{\lambda\mu}\eta_{\kappa\nu}\eta_\rho{}^\nu\eta_\sigma{}^\mu = \eta_{\sigma\lambda}\eta_{\rho\kappa} \ , \tag{4.3.23}$$

will be the outcome. From (4.3.23) it follows that any full Wick contraction will contain two dot products of polarization vectors

$$(\varepsilon_i \cdot \varepsilon_k)\,(\varepsilon_j \cdot \varepsilon_l) \ , \tag{4.3.24}$$

for some permutation of the indices. Since there are three external fields with polarization $(+)$ and one field with polarization $(-)$, every possible Wick contraction will produce a dot product $\varepsilon_i^+ \cdot \varepsilon_j^+ = 0$ of two polarization vectors with identical polarization and therefore all contractions of the χ_2 term must necessarily vanish. The term χ_4 is explicitly expressed as

$$\left(\mathcal{B}\mathcal{B}^T\right)^\mu{}_\mu \left(\mathcal{B}\mathcal{B}^T\right)^\nu{}_\nu = \partial^n A^\mu \partial_n A_\mu \partial^m A^\nu \partial_m A_\nu \ , \tag{4.3.25}$$

and identical arguments show that all contractions into χ_4 vanish as well so that

$$\left\langle I_4' \,\middle|\, \chi_2 \right\rangle = \left\langle I_4' \,\middle|\, \chi_4 \right\rangle = 0 \ , \tag{4.3.26}$$

holds.

The term χ_1

The Wick contractions of the term χ_1 are computed straightforwardly as follows. Contracting the external field A^- into one of the ${}^+F$ field strengths will force two contractions of a A^+ into a ${}^-F$. This particular contraction involves

$$\zeta_\alpha\zeta_\beta\zeta^\alpha\zeta^\beta = 0 \ , \tag{4.3.27}$$

and hence vanishes. By this argument it can be concluded that the only non-vanishing full contraction has the external field A^- contracted into one ${}^-F$ field strength. Employing the formulae derived above yields an explicit contraction

$$\begin{aligned} A_1^- A_2^+ A_3^+ A_4^+ \Big| {}^+F {}^+F {}^-F {}^-F &= \left(i\sqrt{2}\right)^4 [1_\alpha 1_\beta] \left[\frac{\tilde{p}_2^2}{[\zeta\, 2]^2}\, \zeta^\alpha\zeta^\beta\right] \left[3_{\dot\alpha} 3_{\dot\beta}\right] \left[4^{\dot\alpha} 4^{\dot\beta}\right] \\ &= 4\,\langle 34\rangle^2 \frac{[\zeta\, 1]^2}{[\zeta\, 2]^2}\, \tilde{p}_2^2 \ , \end{aligned} \tag{4.3.28}$$

where the square brackets respectively contain the results from each of the contractions of external photon fields into field strengths. It is apparent from (4.3.28) that neither of the particle permutations $\left(A_1^- \leftrightarrow A_2^+\right)$ or $\left(A_3^+ \leftrightarrow A_4^+\right)$ will alter the result of the contraction and hence a symmetry factor of four is obtained. The remaining two possible distinct contractions are obtained by permuting $\left(A_2^+ \leftrightarrow A_3^+\right)$ and $\left(A_2^+ \leftrightarrow A_4^+\right)$ respectively. In each of the calculations, a symmetry factor of four is obtained as is the case above. The full expression for the sum of all full contractions of the term χ_1 becomes

$$\begin{aligned}\left\langle A_1^- A_2^+ A_3^+ A_4^+ \middle| {}^+F^2\, {}^-F^2 \right\rangle &= 16\left(\langle 34\rangle^2 \frac{[\zeta 1]^2}{[\zeta 2]^2}\,\tilde{p}_2^2 + \langle 24\rangle^2 \frac{[\zeta 1]^2}{[\zeta 3]^2}\,\tilde{p}_3^2 + \langle 23\rangle^2 \frac{[\zeta 1]^2}{[\zeta 4]^2}\,\tilde{p}_4^2\right) \\ &= 16\,[\zeta 1]^2 \left(\frac{\langle 34\rangle^2}{[\zeta 2]^2}\,\tilde{p}_2^2 + \frac{\langle 24\rangle^2}{[\zeta 3]^2}\,\tilde{p}_3^2 + \frac{\langle 23\rangle^2}{[\zeta 4]^2}\,\tilde{p}_4^2\right) . \end{aligned} \tag{4.3.29}$$

As could be expected, the result is symmetric under permutations of particle momenta $1, 2$ and 3.

The term χ_5

The term χ_5 contains the trace

$$\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda} = \partial^n A^{\lambda} \partial_n A_{\lambda} \ . \tag{4.3.30}$$

Contracting two photon fields A_i and A_j into this term will produce the dot product $\varepsilon_i \cdot \varepsilon_j$ which vanish if the two photon fields are identically polarized. In order to obtain a nonzero contraction the two photon fields which are contracted into (4.3.30) must therefore have opposite polarizations. Hence the two remaining Wick contractions for the term χ_5 must be of the type

$$A_i^+ F_{\mu\nu} = -\left(p_{i,\mu}\,\varepsilon_{i,\nu}^+ - p_{i,\nu}\,\varepsilon_{i,\mu}^+\right) \ . \tag{4.3.31}$$

Furthermore it is obtained that

$$A_i^+ A_j^+ F_{\mu\nu} F^{\mu\nu} = -\ 2\, p_j^{\mu}\,\varepsilon_{i,\mu}^+\, p_i^{\nu}\,\varepsilon_{j,\nu}^+ \ , \tag{4.3.32}$$

due to the vanishing dot products $\varepsilon_i^+ \cdot \varepsilon_j^+$. In terms of spinor indices it can be obtained that

$$\begin{aligned} p_j^{\mu}\,\varepsilon_{i,\mu}^+ &= \frac{1}{2}\,\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\left(\lambda_{j,\beta}\lambda_{j,\dot{\beta}} + \frac{\tilde{p}_j^2}{2\,p^{\flat}\cdot\zeta}\,\zeta_{\beta}\zeta_{\dot{\beta}}\right)\sqrt{2}\,\frac{\zeta_{\alpha}\lambda_{i,\dot{\alpha}}}{[\zeta\lambda_i]} \\ &= \frac{1}{\sqrt{2}}\,\frac{[\zeta\lambda_j]}{[\zeta\lambda_i]}\langle\lambda_j\lambda_i\rangle \ , \end{aligned} \tag{4.3.33}$$

such that (4.3.32) becomes

$$A_i^+ A_j^+ F_{\mu\nu} F^{\mu\nu} = \langle\lambda_i\lambda_j\rangle^2 \ , \tag{4.3.34}$$

in terms of spinor products. A contraction of the photon fields A^+ and A^- into ${\mathcal{B}\mathcal{B}^T}^\lambda{}_\lambda$ is given by

$$A_i^- A_j^+ \partial^n A^\lambda \partial_n A_\lambda = p_i^n \, p_{j,n} \, \varepsilon_{i,\mu}^- \, \varepsilon_j^{+,\mu} \ , \tag{4.3.35}$$

which involves a contraction of polarization vectors that can be expressed in terms of spinor products as

$$\begin{aligned} \varepsilon_{i,\mu}^- \, \varepsilon_j^{+,\mu} &= \frac{1}{2} \, \varepsilon^{\alpha\beta} \varepsilon^{\dot\beta\dot\alpha} \sqrt{2} \, \frac{\zeta_{\dot\beta} \lambda_{i,\beta}}{\langle \zeta \lambda_i \rangle} \sqrt{2} \, \frac{\zeta_\alpha \lambda_{j,\dot\alpha}}{[\zeta \lambda_j]} \\ &= \frac{[\zeta\lambda_i] \, \langle \zeta \lambda_j \rangle}{\langle \zeta\lambda_i \rangle \, [\zeta\lambda_j]} \ . \end{aligned} \tag{4.3.36}$$

Both equations (4.3.32) and (4.3.35) are symmetric in the Wick contractions and therefore a symmetry factor of four is obtained. It follows from (4.3.34) combined with (4.3.35) and (4.3.36) that

$$A_i^- A_j^+ A_k^+ A_l^+ \, \Big| \, F_{\mu\nu} F^{\mu\nu} \partial^n A^\lambda \partial_n A_\lambda = \langle \lambda_k \lambda_l \rangle^2 \frac{\langle \zeta\lambda_j \rangle \, [\zeta\lambda_i]}{\langle \zeta\lambda_i \rangle \, [\zeta\lambda_j]} \, p_i^n \, p_{j,n} \ , \tag{4.3.37}$$

which is one of three possible different full contractions. The full sum of all full contractions is obtained by putting $i = 1$ and permuting the values $(2, 3, 4)$ between the indices (j, k, l) with the result

$$\begin{aligned} \Big\langle I_4 \, \Big| \, \chi_5 \Big\rangle &= 4 \left[\langle 34 \rangle^2 \frac{\langle \zeta 2 \rangle \, [\zeta 1]}{\langle \zeta 1 \rangle \, [\zeta 2]} \, (\tilde p_1 \cdot \tilde p_2) + \langle 24 \rangle^2 \frac{\langle \zeta 3 \rangle \, [\zeta 1]}{\langle \zeta 1 \rangle \, [\zeta 3]} \, (\tilde p_1 \cdot \tilde p_3) + \langle 23 \rangle^2 \frac{\langle \zeta 4 \rangle \, [\zeta 1]}{\langle \zeta 1 \rangle \, [\zeta 4]} \, (\tilde p_1 \cdot \tilde p_4) \right] \\ &= 4 \, \frac{[\zeta 1]}{\langle \zeta 1 \rangle} \left[\langle 34 \rangle^2 \frac{\langle \zeta 2 \rangle}{[\zeta 2]} \, (\tilde p_1 \cdot \tilde p_2) + \langle 24 \rangle^2 \frac{\langle \zeta 3 \rangle}{[\zeta 3]} \, (\tilde p_1 \cdot \tilde p_3) + \langle 23 \rangle^2 \frac{\langle \zeta 4 \rangle}{[\zeta 4]} \, (\tilde p_1 \cdot \tilde p_4) \right] , \end{aligned} \tag{4.3.38}$$

where the dot products of momenta have been written as

$$\tilde p_i \cdot \tilde p_j = p_i^n p_{j,n} = p_i^4 p_{j,4} + p_i^5 p_{j,5} \ . \tag{4.3.39}$$

The term χ_3

The calculations connected to the χ_3-term are a bit more complicated. As this term does not contain neither the trace of $\mathcal{B}\mathcal{B}^T$ or $\left(\mathcal{B}\mathcal{B}^T\right)^2$ it is not as straightforward as for the other χ_i-terms above to determine the vanishing contractions. Therefore a more systematic study of all the possible contractions is needed.

As a method to perform Wick contractions in a systematic way, the following calculations will distinguish between two types of contractions. One type of contractions has two A^+ photon fields contracted into $F^{\mu\nu} F_{\nu\rho}$ whereas the other type of contractions has one A^+ field and one A^- field contracted into $F^{\mu\nu} F_{\nu\rho}$. The first type of full contractions will be referred to as $\mathcal{T}_1$ while the second type will be referred to as $\mathcal{T}_2$. The $\mathcal{T}_1$-type of contractions is evaluated as

$$F^{\mu\nu} F_{\nu\rho} \left(\mathcal{B}\mathcal{B}^T \right)^\rho{}_\mu \to \left[p_k^\mu \, \varepsilon_k^{+,\nu} - p_k^\nu \, \varepsilon^{+,\mu} \right] \left[p_{l,\nu} \, \varepsilon_{l,\rho}^+ - p_{l,\rho} \, \varepsilon_{l,\nu}^+ \right] p_i^n \, \varepsilon_i^{\pm,\rho} p_{j,n} \, \varepsilon_{j,\mu}^\mp \ , \tag{4.3.40}$$

where the arrow represent one possible contraction and where the notation of the two polarization vectors on the right indicates the two possible configurations of polarization. The relation (4.2.23) has been used and factors of i as well as signs have been omitted. Equation (4.3.40) is equivalent to (4.3.17). Due to the mutual contractions of polarization vectors, one term vanishes when the two square brackets are expanded and one more term is cancelled from the contraction of the polarization vectors to the right. This takes place independently of which of the two possible polarizations is chosen. Hence (4.3.40) reduces to

$$\rightarrow \Big(p_k^\mu\, \varepsilon_k^{+,\nu}\, p_{l,\nu}\, \varepsilon_{l,\rho}^{+} + p_k^\nu\, \varepsilon_k^{+,\mu}\, p_{l,\rho}\, \varepsilon_{l,\nu}^{+}\Big)\, p_i^n\, \varepsilon_i^{\pm,\rho}\, p_{j,n}\, \varepsilon_{j,\mu}^{\mp}\ . \tag{4.3.41}$$

One of the terms in the parenthesis necessarily vanishes when contracted with the two polarization vectors to the right; which one depends again on the configuration of polarizations. It follows from (4.3.41) that

$$A_i^- A_j^+ A_k^+ A_l^+ \Big| F^{\mu\nu} F_{\nu\rho}\, \partial^n A^\rho \partial_n A_\mu = A_i^- A_j^+ A_k^+ A_l^+ \Big| F^{\mu\nu} F_{\nu\rho}\, \partial^n A^\rho \partial_n A_\mu\ , \tag{4.3.42}$$

is valid. If one particular full Wick contraction is considered, this contraction is identical to the full contraction where the four individual contractions are interchanged two by two in such a way that the new full contraction is still of the type $\mathcal{T}_1$.

The sum of all possible full Wick contractions of the type $\mathcal{T}_1$ can now be obtained from (4.3.40) by evaluating this expression for all possible permutations of this type of contractions. This sum is

$$\begin{aligned}\Sigma_1 = 2\Big\{&(\tilde p_1\!\cdot\!\tilde p_2)\Big[(\varepsilon_1\!\cdot\!\varepsilon_4)(p_4\!\cdot\!\varepsilon_3)(p_3\!\cdot\!\varepsilon_2) + (\varepsilon_1\!\cdot\!\varepsilon_3)(p_3\!\cdot\!\varepsilon_4)(p_4\!\cdot\!\varepsilon_2)\Big]\\ &+ (\tilde p_1\!\cdot\!\tilde p_3)\Big[(\varepsilon_1\!\cdot\!\varepsilon_2)(p_2\!\cdot\!\varepsilon_4)(p_4\!\cdot\!\varepsilon_3) + (\varepsilon_1\!\cdot\!\varepsilon_4)(p_4\!\cdot\!\varepsilon_2)(p_2\!\cdot\!\varepsilon_3)\Big]\\ &+ (\tilde p_1\!\cdot\!\tilde p_4)\Big[(\varepsilon_1\!\cdot\!\varepsilon_2)(p_2\!\cdot\!\varepsilon_3)(p_3\!\cdot\!\varepsilon_4) + (\varepsilon_1\!\cdot\!\varepsilon_3)(p_3\!\cdot\!\varepsilon_2)(p_2\!\cdot\!\varepsilon_4)\Big]\Big\}\,,\end{aligned} \tag{4.3.43}$$

where the factor of two stems from (4.3.42) and the terms have been ordered with mutual dot products of momentum vectors as the common factors. Again the short hand notation (4.3.39) has been used. Employing (4.3.33) and (4.3.36) leads to an expression for (4.3.43) in terms of spinor products

$$\begin{aligned}\Sigma_1 = \frac{[\zeta 1]}{\langle\zeta 1\rangle}\Big\{&(\tilde p_1\!\cdot\!\tilde p_2)\,\frac{\langle 34\rangle}{[\zeta 2]}\,[\langle 23\rangle\langle\zeta 4\rangle + \langle 42\rangle\langle\zeta 3\rangle] + (\tilde p_1\!\cdot\!\tilde p_3)\,\frac{\langle 24\rangle}{[\zeta 3]}\,[\langle 43\rangle\langle\zeta 2\rangle + \langle 32\rangle\langle\zeta 4\rangle]\\ &+ (\tilde p_1\!\cdot\!\tilde p_4)\,\frac{\langle 23\rangle}{[\zeta 4]}\,[\langle 42\rangle\langle\zeta 3\rangle + \langle 34\rangle\langle\zeta 2\rangle]\Big\}\,,\end{aligned} \tag{4.3.44}$$

with a little factorization occurring.

The evaluation of contractions of the type $\mathcal{T}_2$ goes on as follows. Analogously to (4.3.40), the arrow indicates that one possible contraction is evaluated and it follows that

$$F^{\mu\nu} F_{\nu\rho}\,(\mathcal{B}\mathcal{O})^\rho{}_\mu \rightarrow \Big(p_i^\mu\, \varepsilon_i^{\pm,\nu}\, p_{j,\nu}\, \varepsilon_{j,\rho}^{\mp} + p_i^\nu\, \varepsilon_i^{\pm,\mu}\, p_{j,\rho}\, \varepsilon_{j,\nu}^{\mp} - p_i^\mu\, \varepsilon_i^{\pm,\nu}\, p_{j,\rho}\, \varepsilon_{j,\nu}^{\mp}\Big)\, p_k^n\, \varepsilon_k^{+,\rho}\, p_{l,n}\, \varepsilon_{l,\mu}^{+}\ . \tag{4.3.45}$$

In (4.3.45) the term

$$-p_i^\nu\left(\varepsilon^\pm\right)_i^\mu p_{j\nu}\left(\varepsilon^\mp\right)_{j,\rho}\ , \tag{4.3.46}$$

has been omitted since it necessarily vanishes in the contraction with the polarization vectors to the right independent of the configuration of polarizations. Either the first or the second term in the bracket will vanish depending on the configuration of polarizations. Because two terms in the bracket survive the mutual contractions of polarization vectors, the total sum Σ_2 of full contractions of the type $\mathcal{T}_2$ contains twice as many terms as the sum Σ_1. The twelve different full contractions are calculated by considering respectively all terms with the same common factor of a certain momentum vector dot product. If the terms are as well collected with these dot products of momentum vectors as common factors, the sum is

$$\begin{aligned}\Sigma_2 = 2\Big\{(\tilde p_2\cdot\tilde p_3)\Big[&-(\varepsilon_1\cdot\varepsilon_4)(p_1\cdot\varepsilon_3)(p_4\cdot\varepsilon_2)-(\varepsilon_1\cdot\varepsilon_4)(p_1\cdot\varepsilon_2)(p_4\cdot\varepsilon_3)\\&+(\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)+(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)\Big]\\(\tilde p_2\cdot\tilde p_4)\Big[&-(\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_4)(p_3\cdot\varepsilon_2)-(\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_2)(p_3\cdot\varepsilon_4)\\&+(\varepsilon_1\cdot\varepsilon_4)(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)+(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)\Big]\\(\tilde p_3\cdot\tilde p_4)\Big[&-(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_4)(p_2\cdot\varepsilon_3)-(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_3)(p_2\cdot\varepsilon_4)\\&+(\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)+(\varepsilon_1\cdot\varepsilon_4)(p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)\Big]\Big\}\,.\end{aligned} \tag{4.3.47}$$

Rewriting this result in terms of spinor products yields

$$\begin{aligned}\Sigma_2 = \frac{[\zeta 1]^2}{\langle\zeta 1\rangle}\Big\{&(\tilde p_2\cdot\tilde p_3)\,\frac{1}{[\zeta 2]\,[\zeta 3]}\Big[-\langle\zeta 4\rangle\left(\langle 24\rangle\langle 31\rangle+\langle 21\rangle\langle 34\rangle\right)+\langle\zeta 3\rangle\langle 24\rangle\langle 41\rangle+\langle\zeta 2\rangle\langle 34\rangle\langle 41\rangle\Big]\\&+(\tilde p_2\cdot\tilde p_4)\,\frac{1}{[\zeta 2]\,[\zeta 4]}\Big[-\langle\zeta 3\rangle\left(\langle 23\rangle\langle 41\rangle+\langle 21\rangle\langle 43\rangle\right)+\langle\zeta 4\rangle\langle 23\rangle\langle 31\rangle+\langle\zeta 2\rangle\langle 43\rangle\langle 31\rangle\Big]\\&+(\tilde p_3\cdot\tilde p_4)\,\frac{1}{[\zeta 3]\,[\zeta 4]}\Big[-\langle\zeta 2\rangle\left(\langle 32\rangle\langle 41\rangle+\langle 42\rangle\langle 31\rangle\right)+\langle\zeta 3\rangle\langle 12\rangle\langle 24\rangle+\langle\zeta 4\rangle\langle 32\rangle\langle 21\rangle\Big]\Big\}\,.\end{aligned} \tag{4.3.48}$$

The results obtained in (4.3.29) (4.3.38), (4.3.44) and (4.3.48) are collected and determine the final result for the amplitude as

$$\begin{aligned}\mathcal{A}\big(1^-i^+j^+k^+\big) = \frac{\pi^4\alpha'^4}{8}\frac{[\zeta 1]}{\langle\zeta 1\rangle}\sum_{\sigma(i,j,k)}\frac{1}{[\zeta i]}\Big\{&+4\,\langle jk\rangle^2\frac{[\zeta 1]\,\langle\zeta 1\rangle}{[\zeta i]}\,\tilde p_i^2\\+\frac{[\zeta i]}{[\zeta j]}\Big[-\langle\zeta k\rangle\left(\,\langle ik\rangle\langle j1\rangle+\langle jk\rangle\langle i1\rangle\,\right)&+\langle\zeta j\rangle\langle ik\rangle\langle k1\rangle+\langle\zeta i\rangle\langle jk\rangle\langle k1\rangle\Big]\,(\tilde p_i\cdot\tilde p_j)\Big\}\,,\end{aligned} \tag{4.3.49}$$

where the sum contains three cyclic permutations of indices given by

$$\sigma\,(i,j,k)=\sigma\,(2,3,4)\ ,\quad \sigma\,(i,j,k)=\sigma\,(4,2,3)\ ,\quad \sigma\,(i,j,k)=\sigma\,(3,4,2)\ , \tag{4.3.50}$$

so that the amplitude contains three contributions of the form on the right hand side of (4.3.49). One term has cancelled out due to the Schouten identity (2.1.76). The details in this calculation can be found in appendix A.1.

The term which has been cancelled is the factorized sum of the respective contributions from the contractions of the term χ_5 and the term Σ_1. These two sums have the common factor "4". For the term Σ_1 this factor originates from the original expression (4.3.1) whereas the factor comes about as a symmetry factor from the contractions of the term χ_5. The remaining terms in (4.3.49) are respectively the contributions from the contractions of the term χ_1 and Σ_2. Not much factorization occur for these two terms.

It is interesting to consider the amplitude (4.3.49) in the four-dimensional limit where momentum components in the auxiliary dimensions are put to zero

$$p^4 = p^5 = 0 \ . \tag{4.3.51}$$

This corresponds to

$$\tilde{p}_i^2 = \tilde{p}_i \cdot \tilde{p}_j = 0 \ , \tag{4.3.52}$$

for momenta in the auxiliary dimensions and it follows that

$$\mathcal{A}(-+++)\Big|_{d=4} = 0 \ . \tag{4.3.53}$$

This is in agreement with (3.2.1) as it should be in the four-dimensional limit. The result for the amplitude (4.3.49) is also written in apendix B.1 where the new amplitude results are collected.

The amplitude $\mathcal{A}(--++)$

In this section the symmetric four-point amplitude

$$\mathcal{A}\,(--++) \rightarrow \left\langle A_1^- A_2^- A_3^+ A_4^+ \middle| \, I_4 \right\rangle \ , \tag{4.3.54}$$

having two $(-)$ polarization photons and two $(+)$ polarization photons will be computed in six dimensions. The same constraints on the gauge field components in the auxiliary dimensions as discussed in section 4.1.2 will be used. As in the previous section, the amplitude is computed from (4.3.1). In order to distinguish present calculations from calculations in the previous section the five terms in the expression will be labeled as $\tilde{\chi}_i$.

4.3.3 Computing $\tilde{\chi}_i$ terms

As in section 4.3.2 the Wick contractions of external fields into (4.3.1) will be considered term by term.

The term $\tilde{\chi}_1$

The first observation for the term $\tilde{\chi_1}$ is that a contraction of both A^+ fields respectively into an anti-selfdual field strength produces $\zeta_\alpha\zeta_\beta\zeta^\alpha\zeta^\beta$ and thus vanishes. The contraction of both A^- fields respectively into a selfdual field strength vanishes as well since this contraction produces $\zeta_{\dot\alpha}\zeta_{\dot\beta}\zeta^{\dot\alpha}\zeta^{\dot\beta}$. The simplest nonzero contraction is therefore when each A^+ is contracted into a ${}^+F$ and each A^- is contracted into a ${}^-F$ which gives

$$\left(-i\sqrt{2}\right)^4 (1_\alpha 1_\beta) \left(2^\alpha 2^\beta\right) \left(3_{\dot\alpha} 3_{\dot\beta}\right) \left(4^{\dot\alpha} 4^{\dot\beta}\right) = 4\ [12]^2\, \langle 34 \rangle^2 \ . \tag{4.3.55}$$

This contraction is symmetric in the interchange of the two A^- fields as well as the two A^+ fields and hence a symmetry factor of four is obtained. The remaining nonzero contractions of the χ_1-term are of the form

$$\begin{aligned} & A_1^- A_2^- A_3^+ A_4^+ \Big| \, {}^+F_{\dot\alpha\dot\beta} \, {}^+F^{\dot\alpha\dot\beta} \, {}^-F_{\alpha\beta} \, {}^-F^{\alpha\beta} \\ & \qquad = \left(i\sqrt{2}\, \frac{\tilde p_1^2}{\langle\zeta 1\rangle^2} \zeta_{\dot\alpha}\zeta_{\dot\beta} \right) \left(-i\sqrt{2}\, 2_\alpha 2_\beta \right) \left(i\sqrt{2}\, 3^{\dot\alpha} 3^{\dot\beta} \right) \left(-i\sqrt{2}\, \frac{\tilde p_4^2}{[\zeta 4]^2} \zeta^\alpha \zeta^\beta \right) \\ & \qquad = +\,4\, \tilde p_1^2\, \tilde p_4^2\, \frac{[\zeta 2]^2 \langle\zeta 3\rangle^2}{\langle\zeta 1\rangle^2 [\zeta 4]^2} \,, \end{aligned} \tag{4.3.56}$$

which is symmetric under interchange of the contractions of $\left(A_1^- \leftrightarrow A_3^+\right)$ as well as under $\left(A_2^- \leftrightarrow A_4^+\right)$. Hence there exist four contributions of the form (4.3.56). The remaining contractions are obtained from permutations of the fields in (4.3.56) such that the contribution from the term $\tilde\chi_1$ is

$$\begin{aligned} \left\langle \tilde\chi_1 \right\rangle = {} & 16\, [12]^2 \langle 34\rangle^2 \\ & + 16 \left[\tilde p_1^2 \tilde p_4^2 \frac{[\zeta 2]^2 \langle\zeta 3\rangle^2}{\langle\zeta 1\rangle^2 [\zeta 4]^2} + \tilde p_1^2 \tilde p_3^2 \frac{[\zeta 2]^2 \langle\zeta 4\rangle^2}{\langle\zeta 1\rangle^2 [\zeta 3]^2} + \tilde p_2^2 \tilde p_4^2 \frac{[\zeta 1]^2 \langle\zeta 3\rangle^2}{\langle\zeta 2\rangle^2 [\zeta 4]^2} + \tilde p_2^2 \tilde p_3^2 \frac{[\zeta 1]^2 \langle\zeta 4\rangle^2}{\langle\zeta 2\rangle^2 [\zeta 3]^2} \right] . \end{aligned} \tag{4.3.57}$$

The terms $\tilde\chi_2$ and $\tilde\chi_4$

For the term $\tilde\chi_2$, combinatorics for the contraction of photon fields into

$$\left(\mathcal{B}\mathcal{B}^T\right)^\mu{}_\nu \left(\mathcal{B}\mathcal{B}^T\right)^\nu{}_\mu = +\,\partial^m A^\mu \partial_m A_\nu \partial^n A^\nu \partial_n A_\mu \,, \tag{4.3.58}$$

have to be considered. Because of the given polarization of the external fields the only nonzero contractions of polarization vectors are $(\varepsilon_1 \cdot \varepsilon_3)$, $(\varepsilon_1 \cdot \varepsilon_4)$,$(\varepsilon_2 \cdot \varepsilon_3)$ and $(\varepsilon_2 \cdot \varepsilon_4)$ such that only four different nonzero full contractions exist. These are given by

$$\begin{aligned} \left\langle \left(\mathcal{B}\mathcal{B}^T\right)^\mu{}_\nu \left(\mathcal{B}\mathcal{B}^T\right)^\nu{}_\mu \right\rangle = {} & 4\, (\varepsilon_1 \cdot \varepsilon_3)\, (\varepsilon_2 \cdot \varepsilon_4) \left[(\tilde p_1 \cdot \tilde p_2)\, (\tilde p_3 \cdot \tilde p_4) + (\tilde p_1 \cdot \tilde p_4)\, (\tilde p_2 \cdot \tilde p_3) \right] \\ & + 4\, (\varepsilon_1 \cdot \varepsilon_4)\, (\varepsilon_2 \cdot \varepsilon_3) \left[(\tilde p_1 \cdot \tilde p_2)\, (\tilde p_3 \cdot \tilde p_4) + (\tilde p_1 \cdot \tilde p_3)\, (\tilde p_2 \cdot \tilde p_4) \right] , \end{aligned} \tag{4.3.59}$$

with the symmetry factor of four appearing. For the term $\tilde\chi_4$ given by

$$\left(\mathcal{B}\mathcal{B}^T\right)^\mu{}_\mu \left(\mathcal{B}\mathcal{B}^T\right)^\nu{}_\nu = +\,\partial^m A^\mu \partial_m A_\mu \partial^n A^\nu \partial_n A_\nu \,, \tag{4.3.60}$$

there exist only two distinct nonzero full contractions. With the appropriate symmetry factor these contractions read

$$\left\langle \left(\mathcal{B}\mathcal{B}^T\right)^\mu{}_\mu \left(\mathcal{B}\mathcal{B}^T\right)^\nu{}_\nu \right\rangle = 8 \left[(\tilde p_1 \cdot \tilde p_3)(\tilde p_2 \cdot \tilde p_4)(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) + (\tilde p_1 \cdot \tilde p_4)(\tilde p_2 \cdot \tilde p_3)(\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \right] . \tag{4.3.61}$$

Using (4.3.36) the contractions of polarization vectors can written in terms of spinor products as

$$\left(\varepsilon_1^- \cdot \varepsilon_3^+\right) \left(\varepsilon_2^- \cdot \varepsilon_4^+\right) = \frac{[\zeta 1] \langle\zeta 3\rangle}{\langle\zeta 1\rangle [\zeta 3]} \frac{[\zeta 2] \langle\zeta 4\rangle}{\langle\zeta 2\rangle [\zeta 4]} \,, \quad \left(\varepsilon_1^- \cdot \varepsilon_4^+\right) \left(\varepsilon_2^- \cdot \varepsilon_3^+\right) = \frac{[\zeta 1] \langle\zeta 4\rangle}{\langle\zeta 1\rangle [\zeta 4]} \frac{[\zeta 2] \langle\zeta 3\rangle}{\langle\zeta 2\rangle [\zeta 3]} \,, \tag{4.3.62}$$

and therefore

$$\left(\varepsilon_1^- \cdot \varepsilon_3^+\right)\left(\varepsilon_2^- \cdot \varepsilon_4^-\right) - \left(\varepsilon_1^- \cdot \varepsilon_4^+\right)\left(\varepsilon_2^- \cdot \varepsilon_3^+\right) = 0 \; . \tag{4.3.63}$$

The final result for the contractions is obtained as

$$\left\langle 2\left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\mu} - \left(\mathcal{B}\mathcal{B}^T\right)^{\mu}{}_{\mu}\left(\mathcal{B}\mathcal{B}^T\right)^{\nu}{}_{\nu}\right\rangle = 16\,(\tilde{p}_1\cdot\tilde{p}_2)(\tilde{p}_3\cdot\tilde{p}_4)\,\frac{[\zeta 1]\,[\zeta 2]\,\langle\zeta 3\rangle\langle\zeta 4\rangle}{\langle\zeta 1\rangle\langle\zeta 2\rangle\,[\zeta 3]\,[\zeta 4]} \; , \tag{4.3.64}$$

where the rightmost term in each square bracket of (4.3.59) is cancelled against (4.3.61) due to the factor “2” in (4.3.1).

The term $\tilde{\chi}_5$

From

$$\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda} = \partial^n A^\lambda \partial_n A_\lambda \; , \tag{4.3.65}$$

it follows that the contraction of two identically polarized photon fields into $\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda}$ necessarily vanish. Therefore all nonzero full contractions have one A^+ field and one A^- field contracted respectively into a field strength $F_{\mu\nu}$. Writing the field strengths in terms of selfdual and anti-selfdual components as

$$F_{\mu\nu}F^{\mu\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda} = \left({}^+F^2 + {}^-F^2\right)\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda} \; , \tag{4.3.66}$$

leads to the conclusion that all nonzero contractions are of the form

$${}^+F_{\dot\alpha\dot\beta}\left[A_k^+\right]{}^+F^{\dot\alpha\dot\beta}\left[A_i^-\right](\tilde{p}_j\cdot\tilde{p}_l)\,(\varepsilon_j\cdot\varepsilon_l) + {}^-F_{\alpha\beta}\left[A_k^+\right]{}^-F^{\alpha\beta}\left[A_i^-\right](\tilde{p}_j\cdot\tilde{p}_l)\,(\varepsilon_j\cdot\varepsilon_l) \; . \tag{4.3.67}$$

One particular full contraction for the ${}^+F^2\,(\mathcal{B}\mathcal{O})^{\lambda}{}_{\lambda}$ part is evaluated as

$$A_1^- A_2^- A_3^+ A_4^+ \Big| {}^+F_{\dot\alpha\dot\beta}\,{}^+F^{\dot\alpha\dot\beta} p^n A^\lambda p_n A_\lambda = 2\,\tilde{p_1}^2\,\frac{\langle\zeta 3\rangle^2}{\langle\zeta 1\rangle^2}\frac{[\zeta 2]\,\langle\zeta 4\rangle}{\langle\zeta 2\rangle\,[\zeta 4]}\,(\tilde{p}_2\cdot\tilde{p}_4) \; , \tag{4.3.68}$$

with the same result if the respective contractions of fields A_1^- and A_3^+ are interchanged. This result is again obtained if the contractions of the fields A_2^- and A_4^- are interchanged and hence a symmetry factor of 4 exists.

Performing all possible permutations of contractions of the form (4.3.68) for both the ${}^+F^2$ part and the ${}^-F^2$ in (4.3.66) yields the result

$$\begin{aligned}
&\left\langle A_1^- A_2^- A_3^+ A_4^+ \Big| F_{\mu\nu}F^{\mu\nu}\left(\mathcal{B}\mathcal{B}^T\right)^{\lambda}{}_{\lambda}\right\rangle \\
&= 8\left\{\frac{[\zeta 1]\,\langle\zeta 3\rangle}{\langle\zeta 1\rangle\,[\zeta 3]}\,(\tilde{p}_1\cdot\tilde{p}_3)\left[\tilde{p}_2^2\frac{\langle\zeta 4\rangle^2}{\langle\zeta 2\rangle^2} + \tilde{p}_4^2\frac{[\zeta 2]^2}{[\zeta 4]^2}\right] + \frac{[\zeta 1]\,\langle\zeta 4\rangle}{\langle\zeta 1\rangle\,[\zeta 4]}\,(\tilde{p}_1\cdot\tilde{p}_4)\left[\tilde{p}_2^2\frac{\langle\zeta 3\rangle^2}{\langle\zeta 2\rangle^2} + \tilde{p}_3^2\frac{[\zeta 2]^2}{[\zeta 3]^2}\right]\right. \\
&\quad\left. + \frac{[\zeta 2]\,\langle\zeta 3\rangle}{\langle\zeta 2\rangle\,[\zeta 3]}\,(\tilde{p}_2\cdot\tilde{p}_3)\left[\tilde{p}_1^2\frac{\langle\zeta 4\rangle^2}{\langle\zeta 1\rangle^2} + \tilde{p}_4^2\frac{[\zeta 1]^2}{[\zeta 4]^2}\right] + \frac{[\zeta 2]\,\langle\zeta 4\rangle}{\langle\zeta 2\rangle\,[\zeta 4]}\,(\tilde{p}_2\cdot\tilde{p}_4)\left[\tilde{p}_1^2\frac{\langle\zeta 3\rangle^2}{\langle\zeta 1\rangle^2} + \tilde{p}_3^2\frac{[\zeta 1]^2}{[\zeta 3]^2}\right]\right\} ,
\end{aligned} \tag{4.3.69}$$

where the structure of (4.3.67) is apparent.

The term $\tilde{\chi}_3$

As was the case for the amplitude $\mathcal{A}(-+++)$ the term $\tilde{\chi}_3$ is the most complicated. Because of its structure of one long trace, many full contractions are nonzero and must therefore be computed. The expansion of the $\tilde{\chi}_3$ term has the structure

$$F^{\mu\nu}F_{\nu\rho}\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\mu} \to \left(p^{\mu}\varepsilon^{\nu}p_{\nu}\varepsilon_{\rho} + p^{\nu}\varepsilon^{\mu}p_{\rho}\varepsilon_{\nu} - p^{\mu}\varepsilon^{\nu}p_{\rho}\varepsilon_{\nu} - p^{\nu}\varepsilon^{\mu}p_{\nu}\varepsilon_{\rho}\right)p^{m}\varepsilon^{\rho}p_{m}\varepsilon_{\mu} \ , \tag{4.3.70}$$

for some full Wick contraction. The way to compute all nonzero contributions is simply to consider every single term of (4.3.70). It is apparent that the two terms to the left gives identical contributions while the rightmost term has the simplest structure. The entire expression for the sum of all nonzero contractions is presented in (A.1.27) whereas the expression below is written on a more compact form as a sum of four permutations of indices. The result is

$$\begin{aligned}
&\left\langle A_1^- A_2^- A_3^+ A_4^+ \middle| F^{\mu\nu}F_{\nu\rho}\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\mu}\right\rangle \\
&\quad = \sum_{\sigma(i,j,k,l)} \Big\{ 2\,(\varepsilon_i\cdot\varepsilon_k)\Big[\Big((p_i\cdot\varepsilon_j)(p_k\cdot\varepsilon_l) + (p_k\cdot\varepsilon_j)(p_i\cdot\varepsilon_l)\Big)(\tilde{p}_j\cdot\tilde{p}_l) - (p_i\cdot\varepsilon_j)(p_j\cdot\varepsilon_l)(\tilde{p}_k\cdot\tilde{p}_l) \\
&\qquad\qquad - (p_l\cdot\varepsilon_j)(p_i\cdot\varepsilon_l)(\tilde{p}_j\cdot\tilde{p}_k) - (p_k\cdot\varepsilon_j)(p_j\cdot\varepsilon_l)(\tilde{p}_i\cdot\tilde{p}_l) - (p_l\cdot\varepsilon_j)(p_k\cdot\varepsilon_l)(\tilde{p}_i\cdot\tilde{p}_j)\Big] \\
&\qquad + (\varepsilon_i\cdot\varepsilon_k)(\varepsilon_j\cdot\varepsilon_l)\Big[(p_i\cdot p_j)(\tilde{p}_k\cdot\tilde{p}_l) + (p_i\cdot p_l)(\tilde{p}_j\cdot\tilde{p}_k) + (p_j\cdot p_k)(\tilde{p}_i\cdot\tilde{p}_l) + (p_k\cdot p_l)(\tilde{p}_i\cdot\tilde{p}_j)\Big]\Big\} \ ,
\end{aligned} \tag{4.3.71}$$

with the four different permutations given as

$$\begin{aligned}
\sigma(i,j,k,l) &= \sigma(1,2,3,4) \ , \quad \sigma(i,j,k,l) = \sigma(1,2,4,3) \ , \\
\sigma(i,j,k,l) &= \sigma(2,1,3,4) \ , \quad \sigma(i,j,k,l) = \sigma(2,1,4,3) \ .
\end{aligned} \tag{4.3.72}$$

The dot products can be expressed as spinor products such that the right hand side of (4.3.71) reads

$$\begin{aligned}
&\sum_{\sigma(i,i,k,l)} \frac{[\zeta i]\,\langle\zeta k\rangle}{\langle\zeta i\rangle\,[\zeta k]\,\langle\zeta j\rangle\,[\zeta l]}\Big[\Big([ij]\,\langle kl\rangle\langle\zeta i\rangle\,[\zeta k] + [kj]\,\langle il\rangle\langle\zeta k\rangle\,[\zeta i]\Big)(\tilde{p}_j\cdot\tilde{p}_l) \\
&\qquad - [ij]\,\langle jl\rangle\langle\zeta i\rangle\,[\zeta j]\,(\tilde{p}_k\cdot\tilde{p}_l) - [lj]\,\langle il\rangle\langle\zeta l\rangle\,[\zeta i]\,(\tilde{p}_j\cdot\tilde{p}_k) \\
&\qquad\quad - [kj]\,\langle jl\rangle\langle\zeta k\rangle\,[\zeta j]\,(\tilde{p}_i\cdot\tilde{p}_l) - [lj]\,\langle kl\rangle\langle\zeta l\rangle\,[\zeta k]\,(\tilde{p}_i\cdot\tilde{p}_j) \\
&\quad + [\zeta j]\,\langle\zeta l\rangle\Big((p_i\cdot p_j)(\tilde{p}_k\cdot\tilde{p}_l) + (p_k\cdot p_l)(\tilde{p}_i\cdot\tilde{p}_j) + (p_i\cdot p_l)(\tilde{p}_j\cdot\tilde{p}_k) + (p_j\cdot p_k)(\tilde{p}_i\cdot\tilde{p}_l)\Big)\Big] \ ,
\end{aligned} \tag{4.3.73}$$

with the same permutations (4.3.72) appearing in the sum.

Collecting results

Collecting the previous results from expressions (4.3.57), (4.3.64), (4.3.69) and (4.3.73) and introducing the appropriate numerical factors provides the final result for the symmetric amplitude

$$
\begin{aligned}
\mathcal{A}\big(i^- j^- k^+ l^+\big) = &-\frac{\pi^4 \alpha'^4}{8} \times \\
&\sum_{\sigma(i,j,k,l)} \Bigg\{ [ij]^2 \langle kl\rangle^2 + 4\, \tilde{p}_i^2\, \tilde{p}_j^2 \frac{[\zeta j]^2 \langle \zeta k\rangle^2}{\langle \zeta i\rangle^2 [\zeta l]^2} \\
&\quad + \frac{[\zeta i]\,\langle \zeta k\rangle}{\langle \zeta i\rangle\,[\zeta k]\,\langle \zeta j\rangle\,[\zeta l]} \Bigg[2\langle \zeta j\rangle\,[\zeta l]\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) \left(\tilde{p}_j^2\, \frac{\langle \zeta l\rangle^2}{\langle \zeta j\rangle^2} + \tilde{p}_l^2\, \frac{[\zeta j]^2}{[\zeta l]^2} \right) \\
&\quad + \Big([ij]\,\langle kl\rangle \langle \zeta i\rangle\,[\zeta k] + [kj]\,\langle il\rangle\langle \zeta k\rangle\,[\zeta i] \Big)\,(\tilde{p}_j\!\cdot\!\tilde{p}_l) \\
&\quad - [ij]\,\langle jl\rangle\langle \zeta i\rangle\,[\zeta j]\,(\tilde{p}_k\!\cdot\!\tilde{p}_l) - [lj]\,\langle il\rangle\langle \zeta l\rangle\,[\zeta i]\,(\tilde{p}_j\!\cdot\!\tilde{p}_k) \\
&\quad - [kj]\,\langle jl\rangle\langle \zeta k\rangle\,[\zeta j]\,(\tilde{p}_i\!\cdot\!\tilde{p}_l) - [lj]\,\langle kl\rangle\langle \zeta l\rangle\,[\zeta k]\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) \\
&\quad + [\zeta j]\,\langle \zeta l\rangle \Big(2\,(p_i\!\cdot\! p_j)\,(\tilde{p}_k\!\cdot\!\tilde{p}_l) + (p_k\!\cdot\! p_l)\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) + (p_i\!\cdot\! p_l)\,(\tilde{p}_j\!\cdot\!\tilde{p}_k) + (p_j\!\cdot\! p_k)\,(\tilde{p}_i\!\cdot\!\tilde{p}_l) \Big) \Bigg] \Bigg\} ,
\end{aligned}
\tag{4.3.74}
$$

with the same permutations as in (4.3.72). The origin for each of the terms in the expression above is rather clear except for the term with the factor of "2" in the last bracket. This particular term originates from (4.3.64).

It should be mentioned that since the respective momenta p_i^μ, are massive when considered from a four-dimensional perspective, the dot products $(p_i\!\cdot\! p_j)$ cannot be written as spinor products. In particular

$$
p_i^M p_{j,M} = p_i^\mu p_{j,\mu} - p_i^m k_{j,m} = p_i \cdot p_j - \tilde{p}_i \cdot \tilde{p}_j \ , \tag{4.3.75}
$$

so that

$$
p_i \cdot p_j \neq \tilde{p}_i \cdot \tilde{p}_j \ . \tag{4.3.76}
$$

This fact can be elucidated from a consideration of the two massive four-dimensional vectors a^μ and b^μ. These vectors can be written according to the massive decomposition (2.1.107) in Lorentz indices

$$
a_\mu = a^\flat_\mu + \frac{a^2}{2\, a^\flat \cdot q}\, q_\mu \ , \quad b_\mu = b^\flat_\mu + \frac{b^2}{2\, b^\flat \cdot q}\, q_\mu \ , \tag{4.3.77}
$$

where the vector q_μ is massless. The dot product between a and b is then

$$
a \cdot b = a^\flat \cdot b^\flat + \frac{a^2 \left(b^\flat \cdot q\right)^2 + b^2 \left(a^\flat \cdot q\right)^2}{2\left(a^\flat \cdot q\right)\left(b^\flat \cdot q\right)} \ . \tag{4.3.78}
$$

As was the case for the amplitudes $\mathcal{A}(++++)$ and $\mathcal{A}(-+++)$, it is natural to consider the four-dimensional limit of (4.3.74). Again this corresponds to (4.3.52) and it follows that

$$
\mathcal{A}(--++)\Big|_{d=4} = -\frac{\pi^4 \alpha'^4}{8} \sum_{\sigma(i,j,k,l)} [ij]^2 \langle kl\rangle^2 = -\frac{\pi^4 \alpha'^4}{2}\ [12]^2 \langle 34\rangle^4 \ , \tag{4.3.79}
$$

which is in agreement with (3.2.5) apart from the interchange $(1,2) \leftrightarrow (3,4)$ of particles. This is simply because the two amplitudes have opposite ordering of particles. It is reassuring that the result in the four-dimensional limit reduces properly to the result obtained from the calculations in four dimensions. The result for the amplitude (4.3.74) is as well written in appendix B.1 where the results for the amplitudes which have previously not been calculated are collected.

4.4 Amplitudes with momentum constraints

In section 4.3 four-point scattering amplitudes in six dimensions have been studied with a constraint on the gauge field in the auxiliary dimensions. The topic of this section is scattering amplitudes where instead the momentum components have been constrained in the auxiliary dimensions. This is the approach discussed in section 4.1.3. It follows from (4.1.13) that the constraint (4.1.5) leads to a four-dimensional gauge field and two complex scalars in four dimensions. The amplitudes for the scattering of theses scalars is exactly what will be studied in this section. The study will be limited to the amplitudes which only involve scalars and no gauge fields.

4.4.1 Constructing the Lagrangian for scalar interactions

The starting point is to consider the DBI-Lagrangian in six dimensions

$$\mathcal{L} = \frac{1}{\pi^2 \alpha'^2} \sqrt{-\det\left(\eta_{MN} + \pi\alpha' F_{MN}\right)} \ . \tag{4.4.1}$$

Mathematica is used to explicitly construct the field strength tensor and the metric as matrices and the determinant can be evaluated in full generality. This yields a lot of interaction terms controlling different amplitudes

$$\mathcal{A}(A_i, A_j \cdots) \ , \quad \mathcal{A}\big(A_i, \cdots, \phi, \bar{\phi}, \cdots\big) \ , \quad \mathcal{A}\big(\phi, \bar{\phi}, \cdots\big) \ . \tag{4.4.2}$$

The choice is made to consider only pure scalar interactions. This means that all cross terms in the Lagrangian will be neglected and only terms with the structure

$$\left(\partial^\mu \phi \partial_\mu \bar{\phi}\right) \ , \tag{4.4.3}$$

will be considered. The sum of the (6×6) matrices η_{MN} and $\pi\alpha' F_{MN}$ is

$$\eta_{MN} + \pi\alpha' F_{MN} =$$
$$\pi\alpha' \begin{bmatrix} (\pi\alpha')^{-1} & F_{01} & F_{02} & F_{03} & \partial_0 A_4 & \partial_0 A_5 \\ -F_{01} & -(\pi\alpha')^{-1} & F_{12} & F_{13} & \partial_1 A_4 & \partial_1 A_5 \\ -F_{02} & -F_{12} & -(\pi\alpha')^{-1} & F_{23} & \partial_2 A_4 & \partial_2 A_5 \\ -F_{03} & -F_{13} & -F_{23} & -(\pi\alpha')^{-1} & \partial_3 A_4 & \partial_3 A_5 \\ -\partial_0 A_4 & -\partial_1 A_4 & -\partial_2 A_4 & -\partial_3 A_4 & -(\pi\alpha')^{-1} & 0 \\ -\partial_0 A_5 & -\partial_1 A_5 & -\partial_2 A_5 & -\partial_3 A_5 & 0 & -(\pi\alpha')^{-1} \end{bmatrix} , \tag{4.4.4}$$

with the use of (4.1.5). Neglecting the cross terms in the evaluation of the determinant is equivalent to putting every entry of the (4×4) matrix

$$F_{\mu\nu} = 0 \ , \tag{4.4.5}$$

and hence (4.4.4) becomes

$$\eta_{MN} + \pi\alpha' F_{MN} =$$
$$\pi\alpha' \begin{bmatrix} (\pi\alpha')^{-1} & 0 & 0 & 0 & \partial_0 A_4 & \partial_0 A_5 \\ 0 & -(\pi\alpha')^{-1} & 0 & 0 & \partial_1 A_4 & \partial_1 A_5 \\ 0 & 0 & -(\pi\alpha')^{-1} & 0 & \partial_2 A_4 & \partial_2 A_5 \\ 0 & 0 & 0 & -(\pi\alpha')^{-1} & \partial_3 A_4 & \partial_3 A_5 \\ -\partial_0 A_4 & -\partial_1 A_4 & -\partial_2 A_4 & -\partial_3 A_4 & -(\pi\alpha')^{-1} & 0 \\ -\partial_0 A_5 & -\partial_1 A_5 & -\partial_2 A_5 & -\partial_3 A_5 & 0 & -(\pi\alpha')^{-1} \end{bmatrix} . \qquad (4.4.6)$$

The determinant of this expression is evaluated by Mathematica with the result

$$\begin{aligned} -\det\left(\eta_{MN} + \pi\alpha' F_{MN}\right) = \\ & 1 - 2\pi^2\alpha'^2\left(\partial\phi_0\partial_0\bar\phi - \partial_1\phi\partial_1\bar\phi - \partial_2\phi\partial_2\bar\phi - \partial_3\phi\partial_3\bar\phi\right) \\ & + \pi^4\alpha'^4\big(- 2\partial_0\phi\partial_1\phi\partial_0\bar\phi\partial_1\bar\phi - 2\partial_0\phi\partial_2\phi\partial_0\bar\phi\partial_2\bar\phi - 2\partial_0\phi\partial_3\phi\partial_0\bar\phi\partial_3\bar\phi \\ & \qquad + 2\partial_1\phi\partial_2\phi\partial_1\bar\phi\partial_2\bar\phi + 2\partial_1\phi\partial_3\phi\partial_1\bar\phi\partial_3\bar\phi + 2\partial_2\phi\partial_3\phi\partial_2\bar\phi\partial_3\bar\phi\big) \\ & - \pi^4\alpha'^4\big(- \partial_0\phi\partial_0\phi\partial_1\bar\phi\partial_1\bar\phi - \partial_0\phi\partial_0\phi\partial_2\bar\phi\partial_2\bar\phi - \partial_0\phi\partial_0\phi\partial_3\bar\phi\partial_3\bar\phi \\ & \qquad - \partial_1\phi\partial_1\phi\partial_0\bar\phi\partial_0\bar\phi + \partial_1\phi\partial_1\phi\partial_2\bar\phi\partial_2\bar\phi + \partial_1\phi\partial_1\phi\partial_3\bar\phi\partial_3\bar\phi \\ & \qquad - \partial_2\phi\partial_2\phi\partial_0\bar\phi\partial_0\bar\phi + \partial_2\phi\partial_2\phi\partial_1\bar\phi\partial_1\bar\phi + \partial_2\phi\partial_2\phi\partial_3\bar\phi\partial_3\bar\phi \\ & \qquad - \partial_3\phi\partial_3\phi\partial_0\bar\phi\partial_0\bar\phi + \partial_3\phi\partial_3\phi\partial_1\bar\phi\partial_1\bar\phi + \partial_3\phi\partial_3\phi\partial_2\bar\phi\partial_2\bar\phi\big) \ . \end{aligned} \qquad (4.4.7)$$

By experimentation it is then obtained that

$$\begin{aligned} -\det\left(\eta_{MN} + \pi\alpha' F_{\mu N}\right) = {} & 1 - 2\pi^2\alpha'^2\partial_\mu\phi\partial^\mu\bar\phi \\ & + \pi^4\alpha'^4\,\partial_\mu\phi\partial^\mu\bar\phi\partial_\nu\phi\partial^\nu\bar\phi - \pi^4\alpha'^4\partial_\mu\phi\partial^\mu\phi\partial_\nu\bar\phi\partial^\nu\bar\phi \ , \end{aligned} \qquad (4.4.8)$$

holds since cancellations between the two terms in the second line of (4.4.8) take place. Upon introducing the abbreviations

$$a \equiv \partial_\mu\phi\partial^\mu\bar\phi \ , \qquad b \equiv \partial_\mu\phi\partial^\mu\bar\phi\partial_\nu\phi\partial^\nu\bar\phi \ , \qquad c \equiv \partial_\mu\phi\partial^\mu\phi\partial_\nu\bar\phi\partial^\nu\bar\phi \ , \qquad (4.4.9)$$

the Lagrangian $\left(-\det\left(\eta_{MN} + \pi\alpha' F_{\mu N}\right)\right)^{1/2}$ can be expanded as a Taylor series in α' as

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= \frac{1}{\pi^2\alpha'^2}\sqrt{-\det\left(\eta_{MN} + \pi\alpha' F_{\mu N}\right)} \\ &= \frac{1}{\pi^2\alpha'^2}\left[1 - \pi^2\alpha'^2 a + \frac{1}{2}\pi^4\alpha'^4\left(-a^2 + b - c\right) + \frac{1}{2}\pi^6\alpha'^6\left(-a^3 + ab - ac\right) + \mathcal{O}\left(\alpha'^8\right)\right] , \end{aligned} \qquad (4.4.10)$$

with the use of Mathematica. From (4.4.9) it is apparent that $a^2 = b$ and $a^3 = ab$ such that (4.4.10) simplifies and is given to order α'^6 as

$$\begin{aligned} \mathcal{L}_{\text{scalar}} = {} & \frac{1}{\pi^2\alpha'^2} - \partial_\mu\phi\partial^\mu\bar\phi - \frac{1}{2}\pi^2\alpha'^2\partial_\mu\phi\partial^\mu\phi\partial_\nu\bar\phi\partial^\nu\bar\phi \\ & - \frac{1}{2}\pi^4\alpha'^4\partial_\mu\phi\partial^\mu\bar\phi\partial_\nu\phi\partial^\nu\phi\partial_\lambda\bar\phi\partial^\lambda\bar\phi + \mathcal{O}\left(\alpha'^8\right) \ , \end{aligned} \qquad (4.4.11)$$

where the expressions for a, b and c have been substituted back. From (4.4.11) the kinetic term $\partial_\mu\phi\partial^\mu\bar\phi$ is recognized. As well is the interaction term for the four point amplitude and the term contributing directly to the the six point amplitude. These terms will be discussed separately below.

4.4.2 Scalar four-point amplitudes

The expanded Lagrangian (4.4.11) provides the scalar amplitudes for four particle and six particle scattering. The four-point amplitude is provided directly by the Lagrangian and will be calculated below while only the direct contribution to the six-point amplitude is present. In order to evaluate the full six-point amplitude one has also to take into account the contribution from the second order expansion of the action.

In the Wick contractions of scalar fields, a particular scalar field must be contracted into a conjugate field. This requirement limits the number of possible contractions in the computation of amplitudes. The scalar four-point amplitude

$$\mathcal{A}\big(\phi_1\phi_2\bar{\phi}_3\bar{\phi}_4\big) = \frac{1}{2}\pi^2\alpha'^2\Big\langle \phi_1\phi_2\bar{\phi}_3\bar{\phi}_4\Big|\partial_\mu\phi\partial^\mu\phi\partial_\nu\bar{\phi}\partial^\nu\bar{\phi}\Big\rangle \ , \tag{4.4.12}$$

is evaluated directly by the full contraction

$$\phi_1\phi_2\bar{\phi}_3\bar{\phi}_4\Big|\partial_\mu\phi\partial^\mu\phi\partial_\nu\bar{\phi}\partial^\nu\bar{\phi} = i\,(p_1\cdot p_2)\,(p_3\cdot p_4) = i\,(p_1\cdot p_2)^2 \ , \tag{4.4.13}$$

since this is the only contribution because of the requirement from above. The full contraction (4.4.13) has a symmetry factor of four and the four point amplitude is therefore given by

$$\mathcal{A}\big(\phi_1\phi_2\bar{\phi}_3\bar{\phi}_4\big) = -\frac{i}{2}\,\pi^2\alpha'^2\Big\langle \phi_1\phi_2\bar{\phi}_3\bar{\phi}_4\Big|\partial_\mu\phi\partial^\mu\phi\partial_\nu\bar{\phi}\partial^\nu\bar{\phi}\Big\rangle = -\frac{i}{2}\,\pi^2\alpha'^2 s^2 \ , \tag{4.4.14}$$

in terms of Mandelstam variables. This result is also found in appendix B.2.

4.4.3 Scalar six-point amplitudes

At tree level, the six-point scalar amplitude

$$\mathcal{A}\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big) = \mathcal{A}_0\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big) + \mathcal{A}_{\text{pole}}\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big) \ , \tag{4.4.15}$$

has two contributions as indicated above. The first contribution $\mathcal{A}_0$ is the direct one from the term in (4.4.11) consisting of six scalar fields. This contribution is calculated directly from the action and the diagram for this interaction is shown in figure 4.1. The second contribution $\mathcal{A}_{\text{pole}}$ is from the square of the term in (4.4.11) consisting of four scalar fields. This squared contribution originates from the second order term in the Taylor expansion of the exponentiated action. This contribution is a contraction of two four point vertices and hence it involves a pole in the propagating momentum. The diagram for the interaction is shown for one of the possible permutations of particles in figure 4.2.

The direct contribution to the amplitude

The direct contribution $\mathcal{A}_0\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big)$ is evaluated by the systematic study of all possible Wick contractions of the object

$$\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\Big|\partial_\mu\phi\partial^\mu\bar{\phi}\partial_\nu\phi\partial^\nu\phi\partial_\lambda\bar{\phi}\partial^\lambda\bar{\phi} \ , \tag{4.4.16}$$

with one particular full contraction given as

$$\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\Big|\partial_\mu\phi\partial^\mu\bar{\phi}\partial_\nu\phi\partial^\nu\phi\partial_\lambda\bar{\phi}\partial^\lambda\bar{\phi} = i\,(p_1\!\cdot\! p_4)\,(p_2\!\cdot\! p_3)\,(p_5\!\cdot\! p_6)\ \ . \tag{4.4.17}$$

Interchanging the respective contractions of fields ϕ_2 and ϕ_3 does not alter the right hand side of (4.4.16) and neither does an interchange of the contractions of $\bar{\phi}_5$ and $\bar{\phi}_6$. The right hand side is consequently the result of four different full contractions.

The remaining eight full contractions are obtained by interchanging contractions of fields $(\phi_1 \leftrightarrow \phi_2)$,$(\phi_1 \leftrightarrow \phi_3)$,$(\bar{\phi}_4 \leftrightarrow \bar{\phi}_5)$,$(\bar{\phi}_4 \leftrightarrow \bar{\phi}_6)$ and the result is given by

$$\begin{aligned}
&\mathcal{A}_0\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big)\\
&\quad = -2i\,\pi^4\alpha'^4\Big\{(p_1\!\cdot\! p_2)\,[(p_3\!\cdot\! p_4)(p_5\!\cdot\! p_6)+(p_3\!\cdot\! p_5)(p_4\!\cdot\! p_6)+(p_3\!\cdot\! p_6)(p_4\!\cdot\! p_5)]\\
&\qquad\qquad + (p_1\!\cdot\! p_3)\,[(p_2\!\cdot\! p_4)(p_5\!\cdot\! p_6)+(p_2\!\cdot\! p_5)(p_4\!\cdot\! p_6)+(p_2\!\cdot\! p_6)(p_4\!\cdot\! p_5)]\\
&\qquad\qquad + (p_2\!\cdot\! p_3)\,[(p_1\!\cdot\! p_4)(p_5\!\cdot\! p_6)+(p_1\!\cdot\! p_5)(p_4\!\cdot\! p_6)+(p_1\!\cdot\! p_6)(p_4\!\cdot\! p_5)]\Big\}\ ,
\end{aligned} \tag{4.4.18}$$

where the symmetry factor has been included.

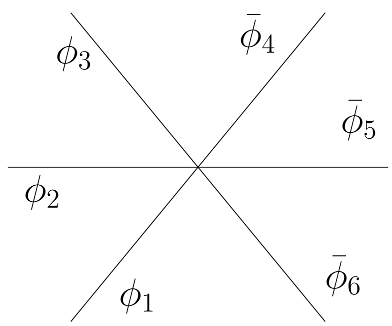

Figure 4.1: The direct contribution to the six-point scalar amplitude.

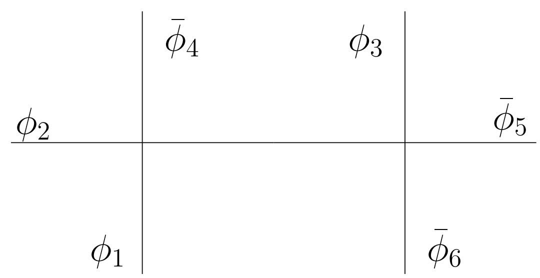

Figure 4.2: A contraction of to four-point scalar vertices with a specific configuration of particles. The contraction is calculated in (4.4.23).

The indirect contribution to the amplitude

The pole contribution to the six point amplitude is controlled by the second order term in the expansion

$$e^{iS} = \exp\left[i\int \mathrm{d}^d x\,\mathcal{L}\right] = 1 + i\int \mathrm{d}^d x\,\mathcal{L} - \frac{1}{2}\left(\int \mathrm{d}^d x\,\mathcal{L}\right)^2 + \mathcal{O}\left((iS)^3\right)\ , \tag{4.4.19}$$

such that the external fields are contracted into the term given explicitly by

$$-\frac{1}{2}\Big(\partial^\mu\phi(x)\,\partial_\mu\phi(x)\,\partial^\nu\bar{\phi}(x)\,\partial_\nu\bar{\phi}(x)\Big)\Big(\partial^\lambda\phi(y)\,\partial_\lambda\phi(y)\,\partial^\kappa\bar{\phi}(y)\,\partial_\kappa\bar{\phi}(y)\Big)\ . \tag{4.4.20}$$

Since this six particle vertex is a contraction of two four particle vertices, the particles have dependence on two distinct space time points x and y. This dependence plays a role for the internal contractions of fields and has been emphasized above. With the abbreviation

$$\phi(x) = \phi_x\ , \tag{4.4.21}$$

the pole contribution can be written formally as

$$\mathcal{A}_{\text{pole}}\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big) \to \sum \Big\langle \phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6 \,\Big|\, \partial^\mu\phi_x\partial_\mu\phi_x\partial^\nu\bar{\phi}_x\partial_\nu\bar{\phi}_x\partial^\lambda\phi_y\partial_\lambda\phi_y\partial^\kappa\bar{\phi}_y\partial_\kappa\bar{\phi}_y \Big\rangle \ , \tag{4.4.22}$$

where the summation is performed over all possible permutations of full contractions between external and internal fields in combination with all possible ways of performing one single internal contraction without generating a loop.

Loops are generated from internal contractions between a ϕ_x and a $\bar{\phi}_x$ and hence only internal contractions between a ϕ_x and a $\bar{\phi}_y$ are considered in order to constrain the calculations to non-loop level. There exist sixteen of these internal contractions and since they all have the same structure, a symmetry factor of 16 is obtained.

In order to illustrate the structure of the contractions in (4.4.22), one particular term in the sum is evaluated explicitly as

$$\begin{aligned} &\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6 \Big| \partial^\mu\phi_x\partial_\mu\phi_x\partial^\nu\bar{\phi}_x\partial_\nu\bar{\phi}_x\partial^\lambda\phi_y\partial_\lambda\phi_y\partial^\kappa\bar{\phi}_y\partial_\kappa\bar{\phi}_y \\ &\qquad = (p_1\cdot p_2)\,(p_5\cdot p_6)\,p_4\cdot(p_1+p_2)\,p_3\cdot(p_5+p_6)\,\frac{i}{(p_1+p_2+p_4)^2} \ , \end{aligned} \tag{4.4.23}$$

with a pole in the propagating momentum. As is the case for the contractions in (4.4.16), the diagram for the contractions above has a symmetry factor of four such that the total symmetry factor is 64. This comes about when the symmetry factors for the internal and external contractions are combined.

Summing up the contributions

When the remaining eight full contractions in (4.4.22) are calculated and all the nine contributions are summed, the pole part of the amplitude takes the form

$$\begin{aligned} &\mathcal{A}_{\text{pole}}\big(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\big) = \\ &\quad + 8i\,\pi^4\alpha'^4 \Bigg\{ (p_1\cdot p_2)\,(p_1+p_2)^\mu\,p_{3,\nu}\left[\frac{(p_5\cdot p_6)\,p_{4,\mu}\,(p_5+p_6)^\nu}{(p_1+p_2+p_4)^2}\right. \\ &\qquad\qquad \left. + \frac{(p_4\cdot p_6)\,p_{5,\mu}\,(p_4+p_6)^\nu}{(p_1+p_2+p_5)^2} + \frac{(p_4\cdot p_5)\,p_{6,\mu}\,(p_4+p_5)^\nu}{(p_1+p_2+p_6)^2}\right] \\ &\qquad + (p_1\cdot p_3)\,(p_1+p_3)^\mu\,p_{2,\nu}\left[\frac{(p_5\cdot p_6)\,p_{4,\mu}\,(p_5+p_6)^\nu}{(p_1+p_3+p_4)^2}\right. \\ &\qquad\qquad \left. + \frac{(p_4\cdot p_6)\,p_{5,\mu}\,(p_4+p_6)^\nu}{(p_1+p_3+p_5)^2} + \frac{(p_4\cdot p_5)\,p_{6,\mu}\,(p_4+p_5)^\nu}{(p_1+p_3+p_6)^2}\right] \\ &\qquad + (p_2\cdot p_3)\,(p_2+p_3)^\mu\,p_{1,\nu}\left[\frac{(p_5\cdot p_6)\,p_{4,\mu}\,(p_5+p_6)^\nu}{(p_2+p_3+p_4)^2}\right. \\ &\qquad\qquad \left. + \frac{(p_4\cdot p_6)\,p_{5,\mu}\,(p_4+p_6)^\nu}{(p_2+p_3+p_5)^2} + \frac{(p_4\cdot p_5)\,p_{6,\mu}\,(p_4+p_5)^\nu}{(p_2+p_3+p_6)^2}\right]\Bigg\} \ . \end{aligned} \tag{4.4.24}$$

In order to write the the full amplitude in a more compact form, the results from (4.4.18) and (4.4.24) are respectively expressed as a sum of three terms such that the full six point amplitude reads

$$\begin{aligned}
&\mathcal{A}(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6) = \\
&\quad \mathcal{A}_0\left(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\right) + \mathcal{A}_{\text{pole}}\left(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6\right) = \\
&\quad -2i\,\pi^4\alpha'^4 \sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)} (p_i\cdot p_j)\,(p_m\cdot p_n)\,p_{k,\mu}\,p_l^\mu \\
&\qquad + 8i\pi^4\alpha'^4 \sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)} (p_i\cdot p_j)\,(p_m\cdot p_n)\,\frac{1}{(p_i+p_j+p_l)^2}\,(p_i+p_j)^\mu\,p_{l,\mu}\,(p_m+p_n)^\nu\,p_{k,\nu}\ ,
\end{aligned} \tag{4.4.25}$$

with the permutations

$$\sigma(i,j,k) = \sigma(1,2,3)\ ,\quad \sigma(i,j,k) = \sigma(2,3,1)\ ,\quad \sigma(i,j,k) = \sigma(3,1,2)\ , \tag{4.4.26}$$

$$\sigma(l,m,n) = \sigma(1,2,3)\ ,\quad \sigma(l,m,n) = \sigma(2,3,1)\ ,\quad \sigma(l,m,n) = \sigma(3,1,2)\ . \tag{4.4.27}$$

As should be the case, it is noticed that the amplitude has an apparent symmetry under any permutation of momenta $(1,2,3)$ as well as the momenta $(4,5,6)$. As the last step, the expression (4.4.25) can be factorized further and the final result for the full six-point amplitude takes the more compact form

$$\begin{aligned}
&\mathcal{A}(\phi_1\phi_2\phi_3\bar{\phi}_4\bar{\phi}_5\bar{\phi}_6) = \\
&\qquad -2i\,\pi^2\alpha'^4 \sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)} (p_i\cdot p_j)\,(p_m\cdot p_n)\left[p_l^\mu - 4\,(p_i+p_j)^\nu\,p_{l,\nu}\frac{(p_m+p_n)^\mu}{(p_i+p_j+p_l)^2}\right]p_{k,\mu}\ ,
\end{aligned} \tag{4.4.28}$$

with the same permutations (4.4.26) and (4.4.27). This result is also part of the summary in appendix B.2.

4.4.4 Manipulations of pole terms

Although the expression (4.4.28) provides the six point scalar amplitude in a compact form, it is interesting to study a little bit further the result (4.4.25). Since all particles are massless, the contractions of momenta satisfy

$$(p_i+p_j)^\mu\,p_{k,\mu} = (p_i+p_j+p_k)^\mu\,p_{k,\mu}\ , \tag{4.4.29}$$

which suggests that (4.4.24) can be manipulated in a way that allows some pole free terms to be extracted. The motivation for this is to obtain some cancellation of pole free terms when (4.4.18) and (4.4.24) are added.

From (4.4.29) it follows that

$$\begin{aligned}
(p_1+p_2)^\mu\,p_{4,\mu}\,p_{3,\nu}\,(p_5+p_6)^\nu = &-(p_3+p_5+p_6)^\mu\,p_{4,\mu}\,(p_1+p_2+p_4)^2 \\
&-(p_3+p_5+p_6)^\mu\,p_{4,\mu}\,(p_5+p_6)_\nu\,(p_1+p_2+p_4)^\nu\ ,
\end{aligned} \tag{4.4.30}$$

where momentum conservation has been used. When (4.4.30) is substituted in the leftmost term in the first square bracket of (4.4.24) the result is two terms where the pole is cancelled in the first. The middle and the rightmost term can as well be rewritten by a substitution of the same relation (4.4.30) with suitable momentum vectors. The resultant expression for the contents of the first square bracket upon these three substitutions becomes

$$\begin{aligned}
(p_1\cdot p_2)\,(p_1+p_2)^\mu\,p_{3,\nu}\Bigg[&\frac{(p_5\cdot p_6)\,p_{4,\mu}\,(p_5+p_6)^\nu}{(p_1+p_2+p_4)^2}+\frac{(p_4\cdot p_6)\,p_{5,\mu}\,(p_4+p_6)^\nu}{(p_1+p_2+p_5)^2}\\
&+\frac{(p_4\cdot p_5)\,p_{6,\mu}\,(p_4+p_5)^\nu}{(p_1+p_2+p_6)^2}\Bigg]\\
=(p_1\cdot p_2)\Big[&-(p_5\cdot p_6)(p_3+p_5+p_6)\cdot p_4-(p_4\cdot p_6)(p_3+p_4+p_6)\cdot p_5\\
&-(p_4\cdot p_5)(p_3+p_4+p_5)\cdot p_6\Big]\\
+(p_1\cdot p_2)\,(p_1+p_2)^\mu\Bigg[&\frac{(p_5\cdot p_6)\,(p_5+p_6)_\nu\,(p_1+p_2+p_4)^\nu\,p_{4,\mu}}{(p_1+p_2+p_4)^2}\\
&+\frac{(p_4\cdot p_6)\,(p_4+p_6)_\nu\,(p_1+p_2+p_5)^\nu\,p_{5,\mu}}{(p_1+p_2+p_5)^2}\\
&+\frac{(p_4\cdot p_5)\,(p_4+p_5)_\nu\,(p_1+p_2+p_6)^\nu\,p_{6,\mu}}{(p_1+p_2+p_6)^2}\Bigg].
\end{aligned}\tag{4.4.31}$$

When the prefactor $8\pi^4\alpha'^4$ is included and (4.4.31) and (4.4.18) are added together it is found that the first line of (4.4.18) combines with the pole free terms of (4.4.31). The procedure of rewritings as described above is now employed on the remaining six terms of (4.4.24). This yields the expressions (A.1.28) and (A.1.29) which are equivalent to (4.4.31) with permutations in momentum vectors p^1, p^2 and p^3. The expression for the amplitude when the pole terms and the pole free terms have been combined is

$$\begin{aligned}
&\mathcal{A}(\phi_1\phi_2\phi_3\bar\phi_4\bar\phi_5\bar\phi_6)=\\
&\quad+8\pi^4\alpha'^4\sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)}(p_i\cdot p_j)\,(p_m\cdot p_n)\left(p_i+p_j-\frac{2}{8}\,p_k\right)^\mu p_{l,\mu}\\
&\quad+8\pi^4\alpha'^4\sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)}(p_i\cdot p_j)\,(p_m\cdot p_n)\,(p_m+p_n)_\nu\,\frac{(p_i+p_j+p_l)^\nu}{(p_i+p_j+p_l)^2}\,(p_i+p_j)^\mu\,p_{l,\mu},
\end{aligned}\tag{4.4.32}$$

which can then be expressed in a slightly more compact way as

$$\begin{aligned}
&\mathcal{A}(\phi_1\phi_2\phi_3\bar\phi_4\bar\phi_5\bar\phi_6)=\\
&\quad+8\pi^4\alpha'^4\sum_{\sigma(i,j,k)}\sum_{\sigma(l,m,n)}(p_i\cdot p_j)\,(p_m\cdot p_n)\\
&\quad\times\left[\left(p_i+p_j-\frac{2}{8}\,p_k\right)^\mu+\frac{(p_m+p_n)_\nu\,(p_i+p_j+p_l)^\nu}{(p_i+p_j+p_l)^2}\,(p_i+p_j)^\mu\right]p_{l,\mu},
\end{aligned}\tag{4.4.33}$$

where the sums are again over permutations (4.4.26) and (4.4.27). The expressions above contain the rather awkward factor of 2/8 which occurs as a consequence of the different front factors of $\mathcal{A}_0$ and $\mathcal{A}_{\text{pole}}$. From the calculations leading to (4.4.32) and (4.4.33) it is apparent that, if the respective front factors of (4.4.18) and (4.4.24) had been identical, instead a

"1" would be the factor of p_k in the two expressions above. This comes about because the contribution $\mathcal{A}_{\text{pole}}$ in case of identical front factors is entirely cancelled against the pole free part of (4.4.24). It is most easily realized by a comparison of the expression (4.4.18) with (4.4.31), (A.1.28) and (A.1.29) from where the cancellation of (4.4.18) in case of identical front factors is manifest.

4.5 Cross section for scalars

The results obtained in section 4.4.2 can be used in an estimate of the differential cross section for the scattering of four scalar particles. The computations are based on the discussion of the cross section in section 1.3.2.

The differential cross section for the scattering of four massless scalars is calculated by substituting (4.4.14) in (1.3.5). The result is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi^2}{256}\,\alpha'^4 s^3 \ , \tag{4.5.1}$$

where the s-variable has been used for the center of mass energy

$$s = E_{\text{cm}}^2 \ . \tag{4.5.2}$$

It is noted that (4.5.1) has the correct dimension of area. Because of the α'^4 dependence, the differential cross section is incredibly small. The center of mass energy in (4.5.1) can be controlled in an experiment and in principle, the center of mass energy can be increased sufficiently in order to compare with the α'^4 factor. It is reassuring to see that the differential cross section involves only the string parameter and the center of mass energy in the scattering since this was addressed in section 1.3.1.

The numerical value of the differential cross section can be determined from a very rough estimate. In this estimate the maximal energy $E_{\text{max}} \sim 14$ Tev at the LHC will be used as the center of mass energy. In natural units the second and the metre is related by

$$\text{s} \sim 3 \cdot 10^8 \ \text{m} \ , \tag{4.5.3}$$

which yields

$$\text{eV} \sim \frac{1}{2} \cdot 10^7 \ \text{m} \ , \tag{4.5.4}$$

from the value of Planck's reduced constant. Using

$$E_{\text{cm}} \sim 14 \cdot 10^{12} \ \text{eV} \ , \tag{4.5.5}$$

as discussed in section 1.3.1 gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \alpha'^4 s^3 \sim \left(10^{-35} \ \text{m}\right)^8 \left(10^7 \cdot 10^{12} \ \text{m}\right)^6 = 10^{-166} \ \text{m}^2 \ , \tag{4.5.6}$$

as a very rough estimate for the scalar differential cross section with a center of mass energy equal to the maximal energy at the LHC. This is indeed astronomically small as could be expected from the discussion of energy estimates in section 1.3.1.

Chapter 5

Discussion and conclusions

The goal of the thesis was to calculate amplitudes in six dimensions as a continuation of the work in [1]. On one hand, the motivation was to compute the amplitudes simply to see what they look like and in order to compare with the four-dimensional amplitudes. On the other hand, the purpose was to study whether the spinor-helicity formalism is suited for calculations in six dimensions and, at least naïvely, to gain simplifications in six-dimensional amplitudes similar to the simplifications in four-dimensional amplitudes due to the spinor-helicity formalism.

The results obtained in section 3.2 were calculated along the lines of [1] with diligent use of the spinor-helicity formalism as a very important ingredient. Due to the formalism, it can be read off directly from the action that the only non-vanishing four-point amplitude is the symmetric one $\mathcal{A}(++--)$. One interesting result from this section is that the amplitude $\mathcal{A}(++----)$ vanishes. The amplitude has a direct contribution from a six-point vertex and a contribution from a contraction of two four-point vertices and these two contributions turn out to be exactly equal and with opposite signs. It is not a priori obvious that this should be the case. The most important conclusion however from this section is to note how simple the calculations turn out due to the use of the spinor-helicity formalism. This is also the invitation to approach six-dimensional calculations.

The generic amplitude $\mathcal{A}\left(A_i A_j A_j A_k\right)$ was calculated as an intermediate step between the explicit four-dimensional and six-dimensional calculations and it was calculated without use of the spinor-helicity formalism. The calculation is in principle simple but it involves a large amount of terms which makes it complex in practice. However, the final result turns out to be simple and this might be an indication that an overall simplification should be possible also in higher dimensions than four. Furthermore it is reassuring to see that this amplitude vanishes for the case of four identical polarizations because this is consistent with the results for the specific calculations of amplitudes in sections 3.2 and 4.3.

The latter section contains the six-dimensional results for the amplitudes $\mathcal{A}(++++)$, $\mathcal{A}(-+++)$ and $\mathcal{A}\left(--++\right)$ which have not previously been calculated in this way. This makes these amplitudes interesting by themselves. Moreover it is reassuring that all three amplitudes reduce to the results obtained in section 3.2 in the four-dimensional limit. All the new amplitudes have been collected in appendix B as a summary of the results in the thesis which have not previously been calculated. It is important to stress that the amplitudes in six dimensions have been calculated with the use of the constraint $A_4 = A_5 = 0$ on the gauge field in the auxiliary dimensions and also with restrictions on the number of considered polarizations. Although these constraints simplify the calculations considerably, the results for the amplitudes are still complicated. Especially the result for the symmetric amplitude $\mathcal{A}(--++)$ is complicated and involves comprehensive calculations. However, had these simplifications not been included, the results for the amplitudes would have been even more complicated.

It is interesting to note the apparent existence of a certain hierarchy of amplitudes. The amplitude $\mathcal{A}(++--)$ is the only non-zero four-point amplitude in four dimensions whereas both $\mathcal{A}(-+++)$ and $\mathcal{A}(--++)$ are non-zero in six dimensions. This illustrates a certain ordering in complexity and indicates that there might be some deeper structure of simplifications to be found. However, the difference in complexity when going from four to six dimensions in this form is manifest and the results in the thesis indicate strongly that the used approach is not really the way to go for the purpose of six-dimensional calculations. The amplitude hierarchy suggests that simplifications as in four dimensions exist also in higher dimensions but at this point it is not at all obvious how these simplifications should be implemented. The most important lesson learned in this thesis is therefore that the generalization to higher dimensions is not so straightforward. However, the results in the thesis indicate that there exist a deeper structure in the amplitudes from which simplifications might be discovered. One approach is to construct a six-dimensional spinor-helicity formalism as in [16, 27] and implement this formalism in six-dimensional computations. This can be considered as just one of many invitations for further studies in the interesting field of scattering amplitude calculations in the frame of effective theories.

Appendix A

Computational details

A.1 Results of computations

Lorentz invariants from the field strength

In section 1.4.2 it is discussed that

$$C_1 = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \ , \quad C_2 = -\frac{1}{4}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \ , \tag{A.1.1}$$

are the only Lorentz invariants objects that can be constructed from $F_{\mu\nu}$. Equations (2.1.99) and (2.1.100) can be used to show that this is the case. It is observed that

$$^{+}F_{\dot{\alpha}}{}^{\dot{\beta}}\, {}^{+}F_{\dot{\beta}\dot{\gamma}} = {}^{+}F_{\dot{\alpha}\dot{\delta}}\, {}^{+}F_{\dot{\beta}\dot{\gamma}}\, \varepsilon^{\dot{\beta}\dot{\delta}} = -{}^{+}F_{\dot{\gamma}}{}^{\dot{\beta}}\, {}^{+}F_{\dot{\beta}\dot{\alpha}} \ , \tag{A.1.2}$$

and hence

$$^{+}F_{\dot{\alpha}}{}^{\dot{\beta}}\, {}^{+}F_{\dot{\beta}\dot{\gamma}} = \kappa \varepsilon_{\dot{\alpha}\dot{\gamma}} \ . \tag{A.1.3}$$

The constant κ is determined by

$$\varepsilon^{\dot{\gamma}\dot{\alpha}}\, {}^{+}F_{\dot{\alpha}}{}^{\dot{\beta}}\, {}^{+}F_{\dot{\beta}\dot{\gamma}} = \kappa\, \varepsilon^{\dot{\gamma}\dot{\alpha}}\, \varepsilon_{\dot{\alpha}\dot{\gamma}} \ , \tag{A.1.4}$$

and it follows that

$$\kappa = -\,\frac{1}{2}\, {}^{+}F^{2} \ , \tag{A.1.5}$$

and therefore

$$^{+}F_{\dot{\alpha}}{}^{\dot{\beta}}\, {}^{+}F_{\dot{\beta}\dot{\gamma}} = -\,\frac{1}{2}\, {}^{+}F^{2}\, \varepsilon_{\dot{\alpha}\dot{\gamma}} \ , \quad {}^{-}F_{\alpha}{}^{\beta}\, {}^{-}F_{\beta\gamma} = -\,\frac{1}{2}\, {}^{-}F^{2}\, \varepsilon_{\alpha\gamma} \ . \tag{A.1.6}$$

A Lorentz invariant object must have all indices contracted and it can be deduced from (A.1.6) that everything Lorentz invariant which can be constructed from $F_{\mu\nu}$ must therefore be proportional to ^{+}F and ^{-}F. This shows that (A.1.1) are the only possible Lorentz invariants up to constants.

Manipulations of the DBI-Lagrangian

The following is an explicit calculation inspired by [22]. The purpose is to calculate the relevant determinant in the Dirac-Born-Infeld Lagrangian,

$$\mathcal{L}_{\mathrm{DBI}} = \sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right)}\ , \tag{A.1.7}$$

and check the agreement with the expression

$$\mathcal{L}_{\mathrm{DBI}} = I_2 + I_4\left[1 + \mathcal{O}\left(F^2\right)\right]\ , \tag{A.1.8}$$

as in (3.3.1) with I_2 and I_4 given in (3.3.2) and (3.3.3). In order to evaluate the determinant of (A.1.7), the metric $\eta_{\mu\nu}$ and the field strength tensor $F_{\mu\nu}$ are constructed explicitly in Mathematica. In particular the field strength is constructed as the antisymmetric matrix

$$F_{\mu\nu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}\ , \tag{A.1.9}$$

with lower indices. The dual is constructed as a table in Mathematica according to the definition

$$\tilde{F}^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\ , \tag{A.1.10}$$

by using the previously constructed $F_{\mu\nu}$. Hence the dual is constructed with upper indices as

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & if & -ie & id \\ -if & 0 & ic & -ib \\ ie & -ic & 0 & ia \\ -id & ib & -ia & 0 \end{bmatrix}. \tag{A.1.11}$$

It has then been checked explicitly that

$$-\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right) = 1 + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{16}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\ , \tag{A.1.12}$$

with a different sign in front of the rightmost term compared to [22]. This is due to the definition of the dual which includes "i" in the definition. This calculation from Mathematica appears in figure A.1 from which it is apparent that (A.1.12) holds. Since the field strength

```
In[114]:= -Det[η + F]
Out[114]= 1 - a^2 - b^2 - c^2 + d^2 - c^2 d^2 + 2 b c d e + e^2 - b^2 e^2 - 2 a c d f + 2 a b e f + f^2 - a^2 f^2
In[131]:= Expand[1 - 1 / 2 * Tr[F.η.F.η] + 1 / 16 (Tr[F.Fdual])^2]
Out[131]= 1 - a^2 - b^2 - c^2 + d^2 - c^2 d^2 + 2 b c d e + e^2 - b^2 e^2 - 2 a c d f + 2 a b e f + f^2 - a^2 f^2
In[132]:= Expand[(1 - 1 / 4 Tr[F.Fdual])^2 + 1 / 4 (-Tr[F.η.F.η] - Tr[Fdual.η.Fdual.η] + 2 Tr[F.Fdual])]
Out[132]= 1 - a^2 - b^2 - c^2 + d^2 - c^2 d^2 + 2 b c d e + e^2 - b^2 e^2 - 2 a c d f + 2 a b e f + f^2 - a^2 f^2
```

Figure A.1: The Mathematica calculation which shows that (A.1.12) holds.

tensor constructed in (A.1.9) has been chosen to have lower indices, the evaluation in Mathematica is performed as

$$F_{\mu\nu}F^{\mu\nu} = -F_{\mu\nu}\eta^{\nu\kappa}F_{\kappa\lambda}\eta^{\lambda\mu} = -\operatorname{Tr} F\eta F\eta \ , \tag{A.1.13}$$

with the rightmost expression written in matrix notation using the matrices as defined in Mathematica. In the same way the trace of the field strength and its dual is just

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\operatorname{Tr} F\tilde{F} \ , \tag{A.1.14}$$

since the dual as defined in (A.1.11) has been chosen to have upper indices. Furthermore, it has been checked explicitly that

$$\left(1 + \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + \frac{1}{4}\left(F_{\mu\nu} - \tilde{F}_{\mu\nu}\right)^2 = -\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right) \ , \tag{A.1.15}$$

where the calculation of the left hand side appears in figure A.1. With the definitions

$$I_2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \quad I_4 = -\frac{1}{8}\left[F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right] \ , \tag{A.1.16}$$

it has been checked explicitly that

$$-\det\left(\eta_{\mu\nu} + F_{\mu\nu}\right) = \left(1 + I_2\right)^2 + 2I_4 \ , \tag{A.1.17}$$

with calculations and definitions of I_2 and I_4 in Mathematica appearing in figure A.2. For the

```
In[154]:= i2 = -1 / 4 Tr[F.η.F.η];

In[155]:= i4 = -1 / 8 ( Tr[F.η.F.η.F.η.F.η] - 4 i2^2 );

In[157]:= S = Expand[(1 + i2)^2 + 2 * i4]

Out[157]= 1 - a^2 - b^2 - c^2 + d^2 - c^2 d^2 + 2 b c d e + e^2 - b^2 e^2 - 2 a c d f + 2 a b e f + f^2 - a^2 f^2

In[158]:= -Det[η + F] - S

Out[158]= 0
```

Figure A.2: The Mathematica calculation which shows that (A.1.17) holds.

leftmost term in the definition of I_4, the Mathematica input is written in matrix notation as

$$F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} = +\operatorname{Tr} F\eta F\eta F\eta F\eta \ . \tag{A.1.18}$$

An expansion of the square root of (A.1.17) in α' yields

$$\sqrt{\left(1 + I_2\right)^2 + 2I_4} - 1 = I_2 + I_4\left[1 + \mathcal{O}\left(F^2\right)\right] \ , \tag{A.1.19}$$

which is used in (3.3.1).

Internal field strength contractions

The following is a more detailed calculation of the internal contractions of field strength spinors leading to (3.1.22). By writing

$$\partial_{\alpha\dot{\alpha}} = -ip_{\alpha\dot{\alpha}} \ , \tag{A.1.20}$$

it follows that

$$
\begin{aligned}
&{}^{+}F_{\dot\alpha\dot\beta}\,{}^{-}F_{\alpha\beta} = \\
&\quad \frac{1}{4}(-i)^2\left(p_{\gamma\dot\alpha}\,p_{\alpha\dot\gamma}\,A^{\gamma}{}_{\dot\beta}\,A_{\beta}{}^{\dot\gamma} + p_{\gamma\dot\alpha}\,p_{\beta\dot\gamma}\,A^{\gamma}{}_{\dot\beta}\,A_{\alpha}{}^{\dot\gamma} + p_{\gamma\dot\beta}\,p_{\alpha\dot\gamma}\,A^{\gamma}{}_{\dot\alpha}\,A_{\beta}{}^{\dot\gamma} + p_{\gamma\dot\beta}\,p_{\beta\dot\gamma}\,A^{\gamma}{}_{\dot\alpha}\,A_{\alpha}{}^{\dot\gamma}\right)\ .
\end{aligned} \tag{A.1.21}
$$

The Wick contractions yield

$$
\begin{aligned}
\overbrace{{}^{+}F_{\dot\alpha\dot\beta}\,{}^{-}F}_{\alpha\beta} &= -\frac{1}{4}\frac{(-2i)}{p^2}\left(p_{\gamma\dot\alpha}\,p_{\alpha\dot\gamma}\,\varepsilon^{\gamma}{}_{\beta}\,\varepsilon_{\dot\beta}{}^{\dot\gamma} + p_{\gamma\dot\alpha}\,p_{\beta\dot\gamma}\,\varepsilon^{\gamma}{}_{\alpha}\,\varepsilon_{\dot\beta}{}^{\dot\gamma} + p_{\gamma\dot\beta}\,p_{\alpha\dot\gamma}\,\varepsilon^{\gamma}{}_{\beta}\,\varepsilon_{\dot\alpha}{}^{\dot\gamma} + p_{\gamma\dot\beta}\,p_{\beta\dot\gamma}\,\varepsilon^{\gamma}{}_{\alpha}\,\varepsilon_{\dot\alpha}{}^{\dot\gamma}\right) \\
&= +\frac{i}{2p^2}\left(-p_{\beta\dot\alpha}\,p_{\alpha\dot\beta} - p_{\alpha\dot\alpha}\,p_{\beta\dot\beta} - p_{\beta\dot\beta}\,p_{\alpha\dot\alpha} - p_{\alpha\dot\beta}\,p_{\beta\dot\alpha}\right) \\
&= -\frac{i}{p^2}\left(p_{\alpha\dot\beta}\,p_{\beta\dot\alpha} + p_{\alpha\dot\alpha}\,p_{\beta\dot\beta}\right)\ ,
\end{aligned} \tag{A.1.22}
$$

which is the result (3.1.22).

Detailed calculations with the Schouten identity

The result (4.3.49) has been simplified by use of the Schouten identity. Before the simplification, the term reads

$$
\begin{aligned}
&\mathcal{A}(-+++) = \frac{\pi^4\alpha'^4}{8}\times \\
&4\frac{[\zeta 1]}{\langle\zeta 1\rangle}\sum_{\sigma(i,j,k)}\frac{1}{[\zeta i]}\Bigg\{\langle jk\rangle\,(p_1\cdot p_i)\,[\langle jk\rangle\langle\zeta i\rangle + \langle ij\rangle\langle\zeta k\rangle + \langle ki\rangle\langle\zeta j\rangle] + 4\,\langle jk\rangle^2\frac{[\zeta 1]\,\langle\zeta 1\rangle}{[\zeta i]}\,\tilde p_i^2 \\
&\qquad + \frac{[\zeta i]}{[\zeta j]}\Big[-\langle\zeta k\rangle\,(\,\langle ik\rangle\langle j1\rangle + \langle jk\rangle\langle i1\rangle\,) + \langle\zeta j\rangle\langle ik\rangle\langle k1\rangle + \langle\zeta i\rangle\langle jk\rangle\langle k1\rangle\Big]\,(p_i\cdot p_j)\Bigg\}\,,
\end{aligned} \tag{A.1.23}
$$

where the sum contains three cyclic permutations of indices given by

$$
\sigma(i,j,k) = \sigma(2,3,4)\ ,\quad \sigma(i,j,k) = \sigma(4,2,3)\ ,\quad \sigma(i,j,k) = \sigma(3,4,2)\ . \tag{A.1.24}
$$

Equation (2.1.76) has been used in (A.1.23) as

$$
\begin{aligned}
\langle jk\rangle\langle\zeta i\rangle + \langle ij\rangle\langle\zeta k\rangle + \langle ki\rangle\langle\zeta j\rangle &= j^{\dot\alpha}k^{\dot\gamma}\zeta^{\dot\beta}i^{\dot\delta}\varepsilon_{\dot\gamma\dot\alpha}\varepsilon_{\dot\delta\dot\beta} + j^{\dot\gamma}i^{\dot\alpha}\zeta^{\dot\beta}k^{\dot\delta}\varepsilon_{\dot\gamma\dot\alpha}\varepsilon_{\dot\delta\dot\beta} + j^{\dot\delta}k^{\dot\alpha}\zeta^{\dot\beta}i^{\dot\gamma}\varepsilon_{\dot\delta\dot\beta}\varepsilon_{\dot\gamma\dot\alpha} \\
&= j^{\dot\alpha}k^{\dot\gamma}\zeta^{\dot\beta}i^{\dot\delta}\left(\varepsilon_{\dot\gamma\dot\alpha}\varepsilon_{\dot\delta\dot\beta} + \varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\gamma\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\dot\delta\dot\gamma}\right) \\
&= 0\ ,
\end{aligned} \tag{A.1.25}
$$

where the bracket vanishes. The expression (A.1.23) is hence reduced to (4.3.49) which is the result for the six-dimensional amplitude $\mathcal{A}(-+++)$.

An explicit term in the amplitude $\mathcal{A}(--++)$

The following is the expanded form of (4.3.71) for all Wick contractions into the $\tilde\chi_3$ term

$$
F^{\mu\nu}F_{\nu\rho}\left(\mathcal{B}\mathcal{B}^T\right)^{\rho}{}_{\mu}\ , \tag{A.1.26}
$$

for the six-dimensional amplitude $\mathcal{A}(--++)$. The expression is

$$
\begin{aligned}
&\left\langle A_1^- A_2^- A_3^+ A_4^+ \middle| F^{\mu\nu} F_{\nu\rho} \left(\mathcal{B}\mathcal{B}^T\right)^\rho{}_\mu \right\rangle = \\
&2\Big\{ (\varepsilon_1\cdot\varepsilon_3) \Big[((p_1\cdot\varepsilon_2)(p_3\cdot\varepsilon_4) + (p_1\cdot\varepsilon_4)(p_3\cdot\varepsilon_2))(\tilde{p}_2\cdot\tilde{p}_4) - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(\tilde{p}_3\cdot\tilde{p}_4) \\
&\quad - (p_1\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)(\tilde{p}_2\cdot\tilde{p}_3) - (p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_4)(\tilde{p}_1\cdot\tilde{p}_4) - (p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_2)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \\
&\quad + (\varepsilon_1\cdot\varepsilon_4) \Big[((p_1\cdot\varepsilon_2)(p_4\cdot\varepsilon_3) + (p_4\cdot\varepsilon_2)(p_1\cdot\varepsilon_3))(\tilde{p}_2\cdot\tilde{p}_3) - (p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(\tilde{p}_3\cdot\tilde{p}_4) \\
&\quad - (p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(\tilde{p}_2\cdot\tilde{p}_4) - (p_4\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(\tilde{p}_1\cdot\tilde{p}_3) - (p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \\
&\quad + (\varepsilon_2\cdot\varepsilon_3) \Big[((p_2\cdot\varepsilon_4)(p_3\cdot\varepsilon_1) + (p_3\cdot\varepsilon_4)(p_2\cdot\varepsilon_1))(\tilde{p}_1\cdot\tilde{p}_4) - (p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_4)(\tilde{p}_3\cdot\tilde{p}_4) \\
&\quad - (p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)(\tilde{p}_1\cdot\tilde{p}_3) - (p_3\cdot\varepsilon_1)(p_1\cdot\varepsilon_4)(\tilde{p}_2\cdot\tilde{p}_4) - (p_3\cdot\varepsilon_4)(p_4\cdot\varepsilon_1)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \\
&\quad + (\varepsilon_2\cdot\varepsilon_4) \Big[((p_2\cdot\varepsilon_1)(p_4\cdot\varepsilon_3) + (p_2\cdot\varepsilon_3)(p_4\cdot\varepsilon_1))(\tilde{p}_1\cdot\tilde{p}_3) - (p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)(\tilde{p}_3\cdot\tilde{p}_4) \\
&\quad - (p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)(\tilde{p}_1\cdot\tilde{p}_4) - (p_4\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)(\tilde{p}_2\cdot\tilde{p}_3) - (p_4\cdot\varepsilon_3)(p_3\cdot\varepsilon_1)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \\
&\quad + (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) \Big[(p_1\cdot p_2)(\tilde{p}_3\cdot\tilde{p}_4) + (p_1\cdot p_4)(\tilde{p}_2\cdot\tilde{p}_3) + (p_2\cdot p_3)(\tilde{p}_1\cdot\tilde{p}_4) + (p_3\cdot p_4)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \\
&\quad + (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3) \Big[(p_1\cdot p_2)(\tilde{p}_3\cdot\tilde{p}_4) + (p_1\cdot p_3)(\tilde{p}_2\cdot\tilde{p}_4) + (p_2\cdot p_4)(\tilde{p}_1\cdot\tilde{p}_3) + (p_3\cdot p_4)(\tilde{p}_1\cdot\tilde{p}_2) \Big] \Big\}.
\end{aligned}
\tag{A.1.27}
$$

Rewriting scalar terms

This section contains the remaining pole terms from section 4.4.4. The terms are the middle and last term from (4.4.24) which are rewritten as in (4.4.31). The first term becomes

$$
\begin{aligned}
&(p_1\cdot p_3)(p_1+p_3)^\mu p_{2,\nu} \left[\frac{(p_5\cdot p_6)\, p_{4,\mu}\, (p_5+p_6)^\nu}{(p_1+p_3+p_4)^2} + \frac{(p_4\cdot p_6)\, p_{5,\mu}\, (p_4+p_6)^\nu}{(p_1+p_3+p_5)^2} \right. \\
&\qquad \left. + \frac{(p_4\cdot p_5)\, p_{6,\mu}\, (p_4+p_5)^\nu}{(p_1+p_3+p_6)^2} \right] \\
&= (p_1\cdot p_3) \Big[-(p_5\cdot p_6)(p_2+p_5+p_6)\cdot p_4 - (p_4\cdot p_6)(p_2+p_4+p_6)\cdot p_5 \\
&\qquad - (p_4\cdot p_5)(p_2+p_4+p_5)\cdot p_6 \Big] \\
&\quad + (p_1\cdot p_3)(p_1+p_3)^\mu \left[\frac{(p_5\cdot p_6)(p_5+p_6)_\nu (p_1+p_3+p_4)^\nu p_{4,\mu}}{(p_1+p_3+p_4)^2} \right. \\
&\qquad + \frac{(p_4\cdot p_6)(p_4+p_6)_\nu (p_1+p_3+p_5)^\nu p_{5,\mu}}{(p_1+p_3+p_5)^2} \\
&\qquad \left. + \frac{(p_4\cdot p_5)(p_4+p_5)_\nu (p_1+p_3+p_6)^\nu p_{6,\mu}}{(p_1+p_3+p_6)^2} \right],
\end{aligned}
\tag{A.1.28}
$$

and the second term reads

$$
\begin{aligned}
(p_2 \cdot p_3)\,(p_2+p_3)^\mu\, p_{1,\nu} &\left[\frac{(p_5 \cdot p_6)\, p_{4,\mu}\,(p_5+p_6)^\nu}{(p_2+p_3+p_4)^2} + \frac{(p_4 \cdot p_6)\, p_{5,\mu}\,(p_4+p_6)^\nu}{(p_2+p_3+p_5)^2} \right. \\
&\left. + \frac{(p_4 \cdot p_5)\, p_{6,\mu}\,(p_4+p_5)^\nu}{(p_2+p-3+p_6)^2}\right] \\
= (p_2 \cdot p_3) &\Big[- (p_5 \cdot p_6)(p_1+p_5+p_6)\cdot p_4 - (p_4 \cdot p_6)(p_1+p_4+p_6)\cdot p_5 \\
&\quad - (p_4 \cdot p_5)(p_1+p_4+p_5)\cdot p_6 \Big] \\
+ (p_2 \cdot p_3)\,(p_2+p_3)^\mu &\left[\frac{(p_5 \cdot p_6)\,(p_5+p_6)_\nu\,(p_2+p_3+p_4)^\nu\, p_{4,\mu}}{(p_2+p_3+p_4)^2} \right. \\
&+ \frac{(p_4 \cdot p_6)\,(p_4+p_6)_\nu\,(p_2+p_3+p_5)^\nu\, p_{5,\mu}}{(p_2+p_3+p_5)^2} \\
&\left. + \frac{(p_4 \cdot p_5)\,(p_4+p_5)_\nu\,(p_2+p_3+p_6)^\nu\, p_{6,\mu}}{(p_2+p_3+p_6)^2} \right], \qquad \text{(A.1.29)}
\end{aligned}
$$

after the rewriting.

Mathematica output from calculations in section 3.3.2

Figure A.3 shows the output from Mathematica for the full sum $\mathcal{S}$ in (3.3.15) for the calculation of the generic amplitude $\mathcal{A}(A_iA_jA_kA_l)$. Figure A.4 shows the output from Mathematica after $\mathcal{S}$ has been simplified at the end of section 3.3.2 by employing momentum conservation.The content in figure A.4 is the starting point for the simplifications of $\mathcal{S}$ discussed in section 3.3.3 that ends with (3.3.40) as the final result.

$$-8 pe_{1,4} pe_{2,3} pe_{3,2} pe_{4,1} + 8 pe_{1,3} pe_{2,4} pe_{3,2} pe_{4,1} + 8 pe_{1,2} pe_{2,3} pe_{3,4} pe_{4,1} + 8 pe_{1,4} pe_{2,3} pe_{3,1} pe_{4,2} - 8 pe_{1,3} pe_{2,4} pe_{3,1} pe_{4,2} +$$
$$8 pe_{1,3} pe_{2,1} pe_{3,4} pe_{4,2} + 8 pe_{1,2} pe_{2,4} pe_{3,1} pe_{4,3} + 8 pe_{1,4} pe_{2,1} pe_{3,2} pe_{4,3} - 8 pe_{1,2} pe_{2,1} pe_{3,4} pe_{4,3} + 2 ee_{3,4} pe_{3,2} pe_{4,1} pp_{1,2} +$$
$$2 ee_{4,3} pe_{3,2} pe_{4,1} pp_{1,2} - ee_{2,3} pe_{3,4} pe_{4,1} pp_{1,2} - 3 ee_{3,2} pe_{3,4} pe_{4,1} pp_{1,2} + 2 ee_{3,4} pe_{3,1} pe_{4,2} pp_{1,2} + 2 ee_{4,3} pe_{3,1} pe_{4,2} pp_{1,2} -$$
$$3 ee_{1,3} pe_{3,4} pe_{4,2} pp_{1,2} - ee_{3,1} pe_{3,4} pe_{4,2} pp_{1,2} - ee_{2,4} pe_{3,1} pe_{4,3} pp_{1,2} - 3 ee_{4,2} pe_{3,1} pe_{4,3} pp_{1,2} - 3 ee_{1,4} pe_{3,2} pe_{4,3} pp_{1,2} - ee_{4,1} pe_{3,2} pe_{4,3} pp_{1,2} +$$
$$4 ee_{1,2} pe_{3,4} pe_{4,3} pp_{1,2} + 2 ee_{2,4} pe_{2,3} pe_{4,1} pp_{1,3} + 2 ee_{4,2} pe_{2,3} pe_{4,1} pp_{1,3} - 3 ee_{2,3} pe_{2,4} pe_{4,1} pp_{1,3} - ee_{3,2} pe_{2,4} pe_{4,1} pp_{1,3} -$$
$$ee_{3,4} pe_{2,1} pe_{4,2} pp_{1,3} - 3 ee_{4,3} pe_{2,1} pe_{4,2} pp_{1,3} - 3 ee_{1,4} pe_{2,3} pe_{4,2} pp_{1,3} - ee_{4,1} pe_{2,3} pe_{4,2} pp_{1,3} + 4 ee_{1,3} pe_{2,4} pe_{4,2} pp_{1,3} +$$
$$2 ee_{2,4} pe_{2,1} pe_{4,3} pp_{1,3} + 2 ee_{4,2} pe_{2,1} pe_{4,3} pp_{1,3} - 3 ee_{1,2} pe_{2,4} pe_{4,3} pp_{1,3} - ee_{2,1} pe_{2,4} pe_{4,3} pp_{1,3} - 3 ee_{2,4} pe_{2,3} pe_{3,1} pp_{1,4} -$$
$$ee_{4,2} pe_{2,3} pe_{3,1} pp_{1,4} + 2 ee_{2,3} pe_{2,4} pe_{3,1} pp_{1,4} + 2 ee_{3,2} pe_{2,4} pe_{3,1} pp_{1,4} - 3 ee_{3,4} pe_{2,1} pe_{3,2} pp_{1,4} - ee_{4,3} pe_{2,1} pe_{3,2} pp_{1,4} +$$
$$4 ee_{1,4} pe_{2,3} pe_{3,2} pp_{1,4} - 3 ee_{1,3} pe_{2,4} pe_{3,2} pp_{1,4} - ee_{3,1} pe_{2,4} pe_{3,2} pp_{1,4} + 2 ee_{2,3} pe_{2,1} pe_{3,4} pp_{1,4} + 2 ee_{3,2} pe_{2,1} pe_{3,4} pp_{1,4} -$$
$$3 ee_{1,2} pe_{2,3} pe_{3,4} pp_{1,4} - ee_{2,1} pe_{2,3} pe_{3,4} pp_{1,4} + 2 ee_{3,4} pe_{3,2} pe_{4,1} pp_{2,1} + 2 ee_{4,3} pe_{3,2} pe_{4,1} pp_{2,1} - 3 ee_{2,3} pe_{3,4} pe_{4,1} pp_{2,1} -$$
$$ee_{3,2} pe_{3,4} pe_{4,1} pp_{2,1} + 2 ee_{3,4} pe_{3,1} pe_{4,2} pp_{2,1} + 2 ee_{4,3} pe_{3,1} pe_{4,2} pp_{2,1} - ee_{1,3} pe_{3,4} pe_{4,2} pp_{2,1} - 3 ee_{3,1} pe_{3,4} pe_{4,2} pp_{2,1} -$$
$$3 ee_{2,4} pe_{3,1} pe_{4,3} pp_{2,1} - ee_{4,2} pe_{3,1} pe_{4,3} pp_{2,1} - ee_{1,4} pe_{3,2} pe_{4,3} pp_{2,1} - 3 ee_{4,1} pe_{3,2} pe_{4,3} pp_{2,1} + 4 ee_{2,1} pe_{3,4} pe_{4,3} pp_{2,1} - ee_{3,4} pe_{1,2} pe_{4,1} pp_{2,3} -$$
$$3 ee_{4,3} pe_{1,2} pe_{4,1} pp_{2,3} - 3 ee_{2,4} pe_{1,3} pe_{4,1} pp_{2,3} - ee_{4,2} pe_{1,3} pe_{4,1} pp_{2,3} + 4 ee_{2,3} pe_{1,4} pe_{4,1} pp_{2,3} + 2 ee_{1,4} pe_{1,3} pe_{4,2} pp_{2,3} +$$
$$2 ee_{4,1} pe_{1,3} pe_{4,2} pp_{2,3} - 3 ee_{1,3} pe_{1,4} pe_{4,2} pp_{2,3} - ee_{3,1} pe_{1,4} pe_{4,2} pp_{2,3} + 2 ee_{1,4} pe_{1,2} pe_{4,3} pp_{2,3} + 2 ee_{4,1} pe_{1,2} pe_{4,3} pp_{2,3} -$$
$$ee_{1,2} pe_{1,4} pe_{4,3} pp_{2,3} - 3 ee_{2,1} pe_{1,4} pe_{4,3} pp_{2,3} - 2 ee_{1,4} ee_{2,3} pp_{1,4} pp_{2,3} + 2 ee_{1,3} ee_{2,4} pp_{1,4} pp_{2,3} + ee_{1,2} ee_{3,4} pp_{1,4} pp_{2,3} + ee_{2,1} ee_{4,3} pp_{1,4} pp_{2,3} -$$
$$3 ee_{3,4} pe_{1,2} pe_{3,1} pp_{2,4} - ee_{4,3} pe_{1,2} pe_{3,1} pp_{2,4} + 4 ee_{2,4} pe_{1,3} pe_{3,1} pp_{2,4} - 3 ee_{2,3} pe_{1,4} pe_{3,1} pp_{2,4} - ee_{3,2} pe_{1,4} pe_{3,1} pp_{2,4} -$$
$$3 ee_{1,4} pe_{1,3} pe_{3,2} pp_{2,4} - ee_{4,1} pe_{1,3} pe_{3,2} pp_{2,4} + 2 ee_{1,3} pe_{1,4} pe_{3,2} pp_{2,4} + 2 ee_{3,1} pe_{1,4} pe_{3,2} pp_{2,4} + 2 ee_{1,3} pe_{1,2} pe_{3,4} pp_{2,4} +$$
$$2 ee_{3,1} pe_{1,2} pe_{3,4} pp_{2,4} - ee_{1,2} pe_{1,3} pe_{3,4} pp_{2,4} - 3 ee_{2,1} pe_{1,3} pe_{3,4} pp_{2,4} + 2 ee_{1,4} ee_{2,3} pp_{1,3} pp_{2,4} - 2 ee_{1,3} ee_{2,4} pp_{1,3} pp_{2,4} +$$
$$ee_{2,1} ee_{3,4} pp_{1,3} pp_{2,4} + ee_{1,2} ee_{4,3} pp_{1,3} pp_{2,4} + 2 ee_{2,4} pe_{2,3} pe_{4,1} pp_{3,1} + 2 ee_{4,2} pe_{2,3} pe_{4,1} pp_{3,1} - ee_{2,3} pe_{2,4} pe_{4,1} pp_{3,1} - 3 ee_{3,2} pe_{2,4} pe_{4,1} pp_{3,1} -$$
$$3 ee_{3,4} pe_{2,1} pe_{4,2} pp_{3,1} - ee_{4,3} pe_{2,1} pe_{4,2} pp_{3,1} - ee_{1,4} pe_{2,3} pe_{4,2} pp_{3,1} - 3 ee_{4,1} pe_{2,3} pe_{4,2} pp_{3,1} + 4 ee_{3,1} pe_{2,4} pe_{4,2} pp_{3,1} +$$
$$2 ee_{2,4} pe_{2,1} pe_{4,3} pp_{3,1} + 2 ee_{4,2} pe_{2,1} pe_{4,3} pp_{3,1} - ee_{1,2} pe_{2,4} pe_{4,3} pp_{3,1} - 3 ee_{2,1} pe_{2,4} pe_{4,3} pp_{3,1} + ee_{1,4} ee_{2,3} pp_{2,4} pp_{3,1} -$$
$$2 ee_{2,4} ee_{3,1} pp_{2,4} pp_{3,1} + 2 ee_{2,1} ee_{3,4} pp_{2,4} pp_{3,1} + ee_{3,2} ee_{4,1} pp_{2,4} pp_{3,1} - 3 ee_{3,4} pe_{1,2} pe_{4,1} pp_{3,2} - ee_{4,3} pe_{1,2} pe_{4,1} pp_{3,2} - ee_{2,4} pe_{1,3} pe_{4,1} pp_{3,2} -$$
$$3 ee_{4,2} pe_{1,3} pe_{4,1} pp_{3,2} + 4 ee_{3,2} pe_{1,4} pe_{4,1} pp_{3,2} + 2 ee_{1,4} pe_{1,3} pe_{4,2} pp_{3,2} + 2 ee_{4,1} pe_{1,3} pe_{4,2} pp_{3,2} - ee_{1,3} pe_{1,4} pe_{4,2} pp_{3,2} -$$
$$3 ee_{3,1} pe_{1,4} pe_{4,2} pp_{3,2} + 2 ee_{1,4} pe_{1,2} pe_{4,3} pp_{3,2} + 2 ee_{4,1} pe_{1,2} pe_{4,3} pp_{3,2} - 3 ee_{1,2} pe_{1,4} pe_{4,3} pp_{3,2} - ee_{2,1} pe_{1,4} pe_{4,3} pp_{3,2} +$$
$$ee_{1,3} ee_{2,4} pp_{1,4} pp_{3,2} - 2 ee_{1,4} ee_{3,2} pp_{1,4} pp_{3,2} + 2 ee_{1,2} ee_{3,4} pp_{1,4} pp_{3,2} + ee_{3,1} ee_{4,2} pp_{1,4} pp_{3,2} + 4 ee_{3,4} pe_{1,2} pe_{2,1} pp_{3,4} -$$
$$3 ee_{2,4} pe_{1,3} pe_{2,1} pp_{3,4} - ee_{4,2} pe_{1,3} pe_{2,1} pp_{3,4} - ee_{2,3} pe_{1,4} pe_{2,1} pp_{3,4} - 3 ee_{3,2} pe_{1,4} pe_{2,1} pp_{3,4} - 3 ee_{1,4} pe_{1,2} pe_{2,3} pp_{3,4} -$$
$$ee_{4,1} pe_{1,2} pe_{2,3} pp_{3,4} + 2 ee_{1,2} pe_{1,4} pe_{2,3} pp_{3,4} + 2 ee_{2,1} pe_{1,4} pe_{2,3} pp_{3,4} - ee_{1,3} pe_{1,2} pe_{2,4} pp_{3,4} - 3 ee_{3,1} pe_{1,2} pe_{2,4} pp_{3,4} +$$
$$2 ee_{1,2} pe_{1,3} pe_{2,4} pp_{3,4} + 2 ee_{2,1} pe_{1,3} pe_{2,4} pp_{3,4} + ee_{2,4} ee_{3,1} pp_{1,2} pp_{3,4} + 2 ee_{1,4} ee_{3,2} pp_{1,2} pp_{3,4} - 2 ee_{1,2} ee_{3,4} pp_{1,2} pp_{3,4} +$$
$$ee_{1,3} ee_{4,2} pp_{1,2} pp_{3,4} + 2 ee_{2,4} ee_{3,1} pp_{2,1} pp_{3,4} + ee_{1,4} ee_{3,2} pp_{2,1} pp_{3,4} - 2 ee_{2,1} ee_{3,4} pp_{2,1} pp_{3,4} + ee_{2,3} ee_{4,1} pp_{2,1} pp_{3,4} - ee_{2,4} pe_{2,3} pe_{3,1} pp_{4,1} -$$
$$3 ee_{4,2} pe_{2,3} pe_{3,1} pp_{4,1} + 2 ee_{2,3} pe_{2,4} pe_{3,1} pp_{4,1} + 2 ee_{3,2} pe_{2,4} pe_{3,1} pp_{4,1} - ee_{3,4} pe_{2,1} pe_{3,2} pp_{4,1} - 3 ee_{4,3} pe_{2,1} pe_{3,2} pp_{4,1} +$$
$$4 ee_{4,1} pe_{2,3} pe_{3,2} pp_{4,1} - ee_{1,3} pe_{2,4} pe_{3,2} pp_{4,1} - 3 ee_{3,1} pe_{2,4} pe_{3,2} pp_{4,1} + 2 ee_{2,3} pe_{2,1} pe_{3,4} pp_{4,1} + 2 ee_{3,2} pe_{2,1} pe_{3,4} pp_{4,1} -$$
$$ee_{1,2} pe_{2,3} pe_{3,4} pp_{4,1} - 3 ee_{2,1} pe_{2,3} pe_{3,4} pp_{4,1} + ee_{1,3} ee_{2,4} pp_{2,3} pp_{4,1} - 2 ee_{2,3} ee_{4,1} pp_{2,3} pp_{4,1} + ee_{3,1} ee_{4,2} pp_{2,3} pp_{4,1} + 2 ee_{2,1} ee_{4,3} pp_{2,3} pp_{4,1} +$$
$$ee_{1,2} ee_{3,4} pp_{3,2} pp_{4,1} - 2 ee_{3,2} ee_{4,1} pp_{3,2} pp_{4,1} + 2 ee_{3,1} ee_{4,2} pp_{3,2} pp_{4,1} + ee_{2,1} ee_{4,3} pp_{3,2} pp_{4,1} - ee_{3,4} pe_{1,2} pe_{3,1} pp_{4,2} - 3 ee_{4,3} pe_{1,2} pe_{3,1} pp_{4,2} +$$
$$4 ee_{4,2} pe_{1,3} pe_{3,1} pp_{4,2} - ee_{2,3} pe_{1,4} pe_{3,1} pp_{4,2} - 3 ee_{3,2} pe_{1,4} pe_{3,1} pp_{4,2} - ee_{1,4} pe_{1,3} pe_{3,2} pp_{4,2} - 3 ee_{4,1} pe_{1,3} pe_{3,2} pp_{4,2} +$$
$$2 ee_{1,3} pe_{1,4} pe_{3,2} pp_{4,2} + 2 ee_{3,1} pe_{1,4} pe_{3,2} pp_{4,2} + 2 ee_{1,3} pe_{1,2} pe_{3,4} pp_{4,2} + 2 ee_{3,1} pe_{1,2} pe_{3,4} pp_{4,2} - 3 ee_{1,2} pe_{1,3} pe_{3,4} pp_{4,2} -$$
$$ee_{2,1} pe_{1,3} pe_{3,4} pp_{4,2} + ee_{1,4} ee_{2,3} pp_{1,3} pp_{4,2} + ee_{3,2} ee_{4,1} pp_{1,3} pp_{4,2} - 2 ee_{1,3} ee_{4,2} pp_{1,3} pp_{4,2} + 2 ee_{1,2} ee_{4,3} pp_{1,3} pp_{4,2} + ee_{2,1} ee_{3,4} pp_{3,1} pp_{4,2} +$$
$$2 ee_{3,2} ee_{4,1} pp_{3,1} pp_{4,2} - 2 ee_{3,1} ee_{4,2} pp_{3,1} pp_{4,2} + ee_{1,2} ee_{4,3} pp_{3,1} pp_{4,2} + 4 ee_{4,3} pe_{1,2} pe_{2,1} pp_{4,3} - ee_{2,4} pe_{1,3} pe_{2,1} pp_{4,3} - 3 ee_{4,2} pe_{1,3} pe_{2,1} pp_{4,3} -$$
$$3 ee_{2,3} pe_{1,4} pe_{2,1} pp_{4,3} - ee_{3,2} pe_{1,4} pe_{2,1} pp_{4,3} - ee_{1,4} pe_{1,2} pe_{2,3} pp_{4,3} - 3 ee_{4,1} pe_{1,2} pe_{2,3} pp_{4,3} + 2 ee_{1,2} pe_{1,4} pe_{2,3} pp_{4,3} + 2 ee_{2,1} pe_{1,4} pe_{2,3} pp_{4,3} -$$
$$3 ee_{1,3} pe_{1,2} pe_{2,4} pp_{4,3} - ee_{3,1} pe_{1,2} pe_{2,4} pp_{4,3} + 2 ee_{1,2} pe_{1,3} pe_{2,4} pp_{4,3} + 2 ee_{2,1} pe_{1,3} pe_{2,4} pp_{4,3} + ee_{1,4} ee_{3,2} pp_{1,2} pp_{4,3} + ee_{2,3} ee_{4,1} pp_{1,2} pp_{4,3} +$$
$$2 ee_{1,3} ee_{4,2} pp_{1,2} pp_{4,3} - 2 ee_{1,2} ee_{4,3} pp_{1,2} pp_{4,3} + ee_{2,4} ee_{3,1} pp_{2,1} pp_{4,3} + 2 ee_{2,3} ee_{4,1} pp_{2,1} pp_{4,3} + ee_{1,3} ee_{4,2} pp_{2,1} pp_{4,3} - 2 ee_{2,1} ee_{4,3} pp_{2,1} pp_{4,3}$$

Figure A.3: The output from Mathematica of the full result for the sum $\mathcal{S}$.

$$2 s^2 ee_{1,3} ee_{2,4} + 2 t^2 ee_{1,3} ee_{2,4} - 2 u^2 ee_{1,3} ee_{2,4} + 2 s^2 ee_{1,4} ee_{3,2} - 2 t^2 ee_{1,4} ee_{3,2} + 2 u^2 ee_{1,4} ee_{3,2} - 2 s^2 ee_{1,2} ee_{3,4} +$$
$$2 t^2 ee_{1,2} ee_{3,4} + 2 u^2 ee_{1,2} ee_{3,4} - 4 s ee_{3,4} pe_{1,2} pe_{2,1} + 4 s ee_{2,4} pe_{1,3} pe_{2,1} + 4 s ee_{3,2} pe_{1,4} pe_{2,1} + 4 s ee_{1,4} pe_{1,2} pe_{2,3} -$$
$$4 s ee_{1,2} pe_{1,4} pe_{2,3} + 4 s ee_{1,3} pe_{1,2} pe_{2,4} - 4 s ee_{1,2} pe_{1,3} pe_{2,4} + 4 u ee_{3,4} pe_{1,2} pe_{3,1} - 4 u ee_{2,4} pe_{1,3} pe_{3,1} + 4 u ee_{3,2} pe_{1,4} pe_{3,1} +$$
$$4 t ee_{2,4} pe_{2,3} pe_{3,1} - 4 t ee_{3,2} pe_{2,4} pe_{3,1} + 4 u ee_{1,4} pe_{1,3} pe_{3,2} - 4 u ee_{1,3} pe_{1,4} pe_{3,2} + 4 t ee_{3,4} pe_{2,1} pe_{3,2} - 4 t ee_{1,4} pe_{2,3} pe_{3,2} +$$
$$4 t ee_{1,3} pe_{2,4} pe_{3,2} - 4 u ee_{1,3} pe_{1,2} pe_{3,4} + 4 u ee_{1,2} pe_{1,3} pe_{3,4} - 4 t ee_{3,2} pe_{2,1} pe_{3,4} + 4 t ee_{1,2} pe_{2,3} pe_{3,4} + 4 t ee_{3,4} pe_{1,2} pe_{4,1} +$$
$$4 t ee_{2,4} pe_{1,3} pe_{4,1} - 4 t ee_{3,2} pe_{1,4} pe_{4,1} - 4 u ee_{2,4} pe_{2,3} pe_{4,1} + 4 u ee_{3,2} pe_{2,4} pe_{4,1} - 4 s ee_{3,4} pe_{3,2} pe_{4,1} - 8 pe_{1,4} pe_{2,3} pe_{3,2} pe_{4,1} +$$
$$8 pe_{1,3} pe_{2,4} pe_{3,2} pe_{4,1} + 4 s ee_{3,2} pe_{3,4} pe_{4,1} + 8 pe_{1,2} pe_{2,3} pe_{3,4} pe_{4,1} - 4 t ee_{1,4} pe_{1,3} pe_{4,2} + 4 t ee_{1,3} pe_{1,4} pe_{4,2} + 4 u ee_{3,4} pe_{2,1} pe_{4,2} +$$
$$4 u ee_{1,4} pe_{2,3} pe_{4,2} - 4 u ee_{1,3} pe_{2,4} pe_{4,2} - 4 s ee_{3,4} pe_{3,1} pe_{4,2} + 8 pe_{1,4} pe_{2,3} pe_{3,1} pe_{4,2} - 8 pe_{1,3} pe_{2,4} pe_{3,1} pe_{4,2} + 4 s ee_{1,3} pe_{3,4} pe_{4,2} +$$
$$8 pe_{1,3} pe_{2,1} pe_{3,4} pe_{4,2} - 4 t ee_{1,4} pe_{1,2} pe_{4,3} + 4 t ee_{1,2} pe_{1,4} pe_{4,3} - 4 u ee_{2,4} pe_{2,1} pe_{4,3} + 4 u ee_{1,2} pe_{2,4} pe_{4,3} + 4 s ee_{2,4} pe_{3,1} pe_{4,3} +$$
$$8 pe_{1,2} pe_{2,4} pe_{3,1} pe_{4,3} + 4 s ee_{1,4} pe_{3,2} pe_{4,3} + 8 pe_{1,4} pe_{2,1} pe_{3,2} pe_{4,3} - 4 s ee_{1,2} pe_{3,4} pe_{4,3} - 8 pe_{1,2} pe_{2,1} pe_{3,4} pe_{4,3}$$

Figure A.4: The output from Mathematica after simplifying the expression in figure A.3.

Appendix B

Summary of calculated amplitudes

This appendix is a summary of the results in the thesis for amplitudes which have not previously been calculated. Section B.1 summarizes results for photon amplitudes in six dimensions while section B.2 summarizes results for scalar amplitudes in four dimensions. The results for the respective amplitudes are given without any comments but the necessary definitions.

B.1 Summary of photon amplitudes in six dimensions

The results for the six-dimensional amplitudes $\mathcal{A}(-+++)$ and $\mathcal{A}(--++)$ for four gauge bosons obtained in section 4.3 are given respectively as

$$\mathcal{A}\big(i^- j^+ k^+ l^+\big) = \frac{\pi^4 \alpha'^4}{8} \frac{[\zeta 1]}{\langle \zeta 1\rangle} \sum_{\sigma(i,j,k)} \frac{1}{[\zeta i]} \Bigg\{ +4\, \langle jk\rangle^2 \frac{[\zeta 1]\,\langle \zeta 1\rangle}{[\zeta i]}\, \tilde{p}_i^2 + \frac{[\zeta i]}{[\zeta j]} \Big[-\langle \zeta k\rangle \left(\langle ik\rangle\langle j1\rangle + \langle jk\rangle\langle i1\rangle \right) + \langle \zeta j\rangle\langle ik\rangle\langle k1\rangle + \langle \zeta i\rangle\langle jk\rangle\langle k1\rangle \Big] (\tilde{p}_i\!\cdot\!\tilde{p}_j) \Bigg\}, \tag{B.1.1}$$

and

$$\begin{aligned}
\mathcal{A}\big(i^- j^- k^+ l^+\big) = -\frac{\pi^4\alpha'^4}{8} \times & \\
\sum_{\sigma(i,j,k,l)} \Bigg\{ & [ij]^2\,\langle kl\rangle^2 + 4\,\tilde{p}_i^2\,\tilde{p}_j^2 \frac{[\zeta j]^2\,\langle \zeta k\rangle^2}{\langle \zeta i\rangle^2\,[\zeta l]^2} \\
& + \frac{[\zeta i]\,\langle \zeta k\rangle}{\langle \zeta i\rangle\,[\zeta k]\,\langle \zeta j\rangle\,[\zeta l]} \Bigg[2\langle \zeta j\rangle\,[\zeta l]\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) \left(\tilde{p}_j^2\,\frac{\langle \zeta l\rangle^2}{\langle \zeta j\rangle^2} + \tilde{p}_l^2\,\frac{[\zeta j]^2}{[\zeta l]^2} \right) \\
& + \Big([ij]\,\langle kl\rangle\langle \zeta i\rangle\,[\zeta k] + [kj]\,\langle il\rangle\langle \zeta k\rangle\,[\zeta i] \Big) (\tilde{p}_j\!\cdot\!\tilde{p}_l) \\
& - [ij]\,\langle jl\rangle\langle \zeta i\rangle\,[\zeta j]\,(\tilde{p}_k\!\cdot\!\tilde{p}_l) - [lj]\,\langle il\rangle\langle \zeta l\rangle\,[\zeta i]\,(\tilde{p}_j\!\cdot\!\tilde{p}_k) \\
& - [kj]\,\langle jl\rangle\langle \zeta k\rangle\,[\zeta j]\,(\tilde{p}_i\!\cdot\!\tilde{p}_l) - [lj]\,\langle kl\rangle\langle \zeta l\rangle\,[\zeta k]\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) \\
& + [\zeta j]\,\langle \zeta l\rangle \Big(2\,(p_i\!\cdot\!p_j)\,(\tilde{p}_k\!\cdot\!\tilde{p}_l) + (p_k\!\cdot\!p_l)\,(\tilde{p}_i\!\cdot\!\tilde{p}_j) + (p_i\!\cdot\!p_l)\,(\tilde{p}_j\!\cdot\!\tilde{p}_k) + (p_j\!\cdot\!p_k)\,(\tilde{p}_i\!\cdot\!\tilde{p}_l) \Big) \Bigg] \Bigg\},
\end{aligned} \tag{B.1.2}$$

where the sums contain the cyclic permutations of indices given by

$$\sigma\,(i,j,k) = \sigma\,(2,3,4)\ \ ,\quad \sigma\,(i,j,k) = \sigma\,(4,2,3)\ \ ,\quad \sigma\,(i,j,k) = \sigma\,(3,4,2)\ \ . \tag{B.1.3}$$

The tilde symbol above the momenta indicates that the momentum is in the auxiliary dimensions

$$\tilde{p}_i \cdot \tilde{p}_j = p_i^n\, p_{j,n}\ , \tag{B.1.4}$$

where n is a Lorentz index for the two auxiliary dimensions taking values $(4,5)$. The dot product

$$p_i \cdot p_j = p_i^\mu\, p_{j,\mu}\ , \tag{B.1.5}$$

is just a contraction of ordinary four-dimensional Lorentz indices. The momentum bilinears are defined in terms of spinor indices as

$$\langle ij\rangle = i^{\dot\alpha} j_{\dot\alpha} = i^{\dot\alpha} j^{\dot\beta} \varepsilon_{\dot\beta\dot\alpha}\ ,\quad [ij] = i_\alpha j^\alpha = i^\alpha j^\beta \varepsilon_{\alpha\beta}\ . \tag{B.1.6}$$

B.2 Summary of scalar amplitudes in four dimensions

The result for the four-dimensional amplitude for four massless scalars as calculated in section 4.4.2 is

$$\mathcal{A}\big(\phi_1\phi_2\bar\phi_3\bar\phi_4\big) = -\frac{i}{2}\,\pi^2\alpha'^2 s^2\ , \tag{B.2.1}$$

where

$$s = 2\,p_1\cdot p_2\ , \tag{B.2.2}$$

in terms of the four-momenta for the scalars. The scalars are defined as

$$\phi \equiv \frac{1}{\sqrt{2}}\,(A_4 + iA_5)\ \ ,\quad \bar\phi \equiv \frac{1}{\sqrt{2}}\,(A_4 - iA_5)\ \ , \tag{B.2.3}$$

in terms of gauge field components in the auxiliary dimensions. The result for the four-dimensional amplitude for six massless scalars as calculated in section 4.4.3 is

$$\mathcal{A}\big(\phi_1\phi_2\phi_3\bar\phi_4\bar\phi_5\bar\phi_6\big) = \\ -2i\,\pi^2\alpha'^4 \sum_{\sigma(i,j,k)}\ \sum_{\sigma(l,m,n)} (p_i\!\cdot\! p_j)\,(p_m\!\cdot\! p_n) \left[p_l^\mu - 4\ (p_i+p_j)^\nu\, p_{l,\nu} \frac{(p_m+p_n)^\mu}{(p_i+p_j+p_l)^2}\right] p_{k,\mu}\ , \tag{B.2.4}$$

with the permutations of indices in the sums

$$\sigma\,(i,j,k) = \sigma\,(1,2,3)\ \ ,\quad \sigma\,(i,j,k) = \sigma\,(2,3,1)\ \ ,\quad \sigma\,(i,j,k) = \sigma\,(3,1,2)\ \ , \tag{B.2.5}$$

$$\sigma\,(l,m,n) = \sigma\,(1,2,3)\ \ ,\quad \sigma\,(l,m,n) = \sigma\,(2,3,1)\ \ ,\quad \sigma\,(l,m,n) = \sigma\,(3,1,2)\ \ . \tag{B.2.6}$$

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