Black Hole Entropy and 2D Conformal Field Theory - Towards Quantum Gravity

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Abstract

General relativity, despite providing an elegant and very satisfactory classical description of the gravitational interaction, has left us with several conceptual physical issues. One of them is related to the existence of black hole solutions, and the fact that these seem to be naturally endowed, through the laws of black hole mechanics, with a macroscopic entropy equal to the quarter of the area of their event horizon. It has therefore since then been a challenge for candidates to a quantum theory of gravity to reproduce, from first-principles, this entropy by a counting of microstates. In this thesis we give a review of one such proposal known as the Kerr/CFT correspondence. We also extend the hidden conformal symmetry approach to the case of the five dimensional boosted black string. Furthermore, we conjecture the existence of hidden conformal symmetry in D-dimensional boosted black strings. Finally we make a rather raw calculation on the supersymmetric black ring case where we show that admits a hidden conformal symmetry and compute the microscopic entropy.

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1 Introduction

The main goal of physics is to understand the totality and wholeness of the Cosmos, that is with our current understanding, to comprehend the various forces which govern it and shape it. Gravity was the first force to be understood scientifically by Isaac Newton, published in his most famous book, Principia in 1687, while three more would eventually follow called electromagnetism, weak and strong force. Newton's equations, despite being some hundreds of years old, make such accurate predictions that we still use them today in several fields, but while these equations described the force of gravity with great accuracy, there was nothing in them explaining how gravity works, which posed a rather difficult unsolvable problem to scientists. Almost 250 years passed until our perspective of how we look gravity underwent a fundamental change. In the early 1900's Albert Einstein proposed a new way to look at gravity. He came to think of the three dimensions of space and a single dimension of time as bound together in a single fabric of spacetime. Einstein hoped that by understanding the geometry of this four dimensional spacetime one could easily speak about things moving in this spacetime fabric. With this new approach the curving and stretching of the fabric of spacetime caused by heavy objects such as stars and planets that creates what we feel as gravity. So a new classical theory of gravity came to overthrow the notions of the past, in what we today know as the theory of general relativity.

The theory is governed by the Einstein fields equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

these equations describe how spacetime is being curved due to the presence of matter. The left hand side describes the spacetime geometry which is given by $G_{\mu\nu}$, called the Einstein tensor, while the right hand side describes the associated energy-momentum responsible for the spacetime curvature and is given by $T_{\mu\nu}$, called the Energy-Momentum tensor. G is Newton's gravitational constant.

The Einstein field equations are really difficult and complex to solve without doing suitable approximations, as they are non-linear. However, there have been cases where solutions to these equations have been provided completely, and are called exact solutions. These solutions [1] in four dimensions are fully characterized by only three parameters, the mass, the angular momentum and the charge (electric and magnetic) which is known as the *no-hair theorem*, which parametrize the solutions into a total four. The first exact solution to Einstein field equations was the Schwarzschild metric named after Karl Schwarzschild who found it in 1916. It is the simplest metric in general relativity and describes the space-time geometry outside a non-charged, perfectly spherical, non-rotating mass in asymptotically flat empty space. Soon afterwards (1916–1918) an extension to the solution was found including charge, discovered independently by Hans Reissner in 1916, Hermann Weyl in 1917 and Gunnar Nordström in 1918, named the Reissner–Nordström metric. The geometry of a stationary, rotating, axisymmetric, uncharged black hole in asymptotically flat empty, space was discovered unexpectedly by Roy Kerr in 1963 [1], called the Kerr metric. An own account of Kerr's discovery can be found in [2]. The extension to a rotating black hole with charge was found shortly after by Ezra Newman [3] in 1965, called the Kerr–Newman metric. The above solutions were generated in empty space (vacuum) [2]. For these vacuum field equations we set $T_{\mu\nu} = 0$, because the measurement of spacetime geometry is done only outside the mass in question. These solutions are called black hole solutions since they contain an event horizon. Black holes are the fascinating objects predicted by the theory, these objects are so massive that they curve spacetime in such a degree that they devour everything and even light cannot escape from them.

There is evidence that these black holes do actually form when very massive stars collapse under their own weight. Observational evidence for black holes also exist but are not straightforward to obtain. Since radiation cannot escape the extreme gravitational pull of a black hole, we cannot detect them directly. Instead we infer their existence by observing high-energy phenomena such as X-ray emission and jets, and the motions of nearby objects in orbit around the hidden mass. An added complication is that similar phenomena are observed around less massive neutron stars and pulsars. Therefore, identification as a black hole requires astronomers to make an estimate of the mass of the object and its size. A black hole is confirmed if no other object or group of objects could be so massive and compact.

With the advent of quantum theory, which was the result of a long and successful effort of physicists to account correctly for an extremely wide range of experimental results which the classical theory could not even begin to explain, physicists came to believe that a full theory of nature should incorporate classical physics, quantum mechanics and general relativity which would be a theory of quantum gravity. The task proved more difficult and complex than expected. The earliest attempts to incorporate Einstein's gravity with quantum mechanics failed due to infinite answers that do not made sense. The quest for a quantum theory of gravity is still an ongoing process today which meets lots of difficulties

[1] We are referring to solutions that possess a horizon.

[2] To avoid confusion we should mention that vacuum solutions that include charge are found by solving the so called Einstein-Maxwell equations where the energy-momentum tensor $T_{\mu\nu}$ is that of an electromagnetic field in free space.

and obstacles, even after the advent of string theory which offers some hope. The truth is that we have not yet understand how to quantise gravity, we have models to do it but what we actually do is to push these models using classical gravity down to quantum scale, something we should not be doing if we want a proper quantum theory of gravity.

Black holes came to be very important objects in the quest of formulating a theory of quantum gravity. The reason is that, on the one hand quantum mechanics is important for small scales and it is negligible for large things, most objects that we deal with are small and light like elementary particles, atoms etc, so that one can neglect gravity. On the other hand general relativity is important for big and heavy things for example planets, stars, so that we can neglect quantum mechanics. Black holes incorporate both regimes they are small and heavy so in order to fully understand them we will need a theory of quantum gravity. This make them the perfect objects for theoretical physicists to study.

Bekenstein [4] and Hawking [5] showed us that black holes are thermodynamical objects. Introductory reviews of black hole thermodynamics can be found in, [6], [7], [8], [9], [10], [11]. Bekenstein conjectured that the black hole entropy was proportional to the area of its event horizon divided by the Planck area and Hawking calculated the temperature of a black hole. He did that by quantising the matter on a fixed classical gravitational background and showed that a black hole would radiate thermally via microscopic processes that occur just outside the horizon. The net effect is to remove energy from the black hole, at a very, very slow rate, which results to the evaporation of the black holes! In reality, a solar mass black hole will take many many times the lifetime of the Universe to evaporate, but the main point is that this process gives rise to two related fundamental theoretical problems, the problem of information loss and the mysterious source of black hole entropy. What we are interested and going to discuss in this thesis concerns the black hole entropy.

As we know from thermodynamics in less exotic objects a statistical entropy can be assigned to them. This make us believe that these quantities should reflect some kind of underlying statistical interpretation of black holes. The Bekenstein-Hawking entropy, for example, should count the number of microscopic states of the black hole. But “a black hole has no hair”, a classical, equilibrium black hole is determined completely by its mass, charge, and angular momentum, with no room for additional microscopic states to account for thermal behavior. If black hole thermodynamics has a statistical mechanical origin, the relevant states must therefore be nonclassical. Indeed, they should be quantum gravitational — the Hawking temperature and Bekenstein-Hawking entropy depend upon both Planck’s constant and Newton’s constant G. Thus the problem of black hole statistical mechanics is not just a technical question about some particular configurations of matter and gravitational fields and we hope to give us new insight into the profound mysteries of quantum gravity. Such an interpretation has been achieved with what is known as *holographic dualities* that is relating a theory of gravity, to a quantum theory without gravity in fewer dimensions.

The first successful statistical counting of black hole entropy was performed in string theory for a five dimensional supersymmetric black hole by Andrew Strominger and Cumrun Vafa [12], by using string theory to identify their microstates with those of a dual two-dimensional conformal field theory (2D CFT). The essential strategy for the microscopic entropy counting was to construct a black hole geometry by D-branes and then to map the problem of the black hole to a dual boundary field theory on the D-branes. Then by using the BPS properties, they were able to count the microscopic entropy of the black hole in the weak coupling field theory. A generalized solution with the same charges and equal angular momenta in two orthogonal planes by Breckenridge, Myers, Peet and Vafa (BMPV) can be found in [13]. When the near-horizon limit has been taken, their work can be viewed as a typical example of the AdS/CFT correspondence [14], [15], [16], which shows that there exists a duality between the higher dimensional gravity and the CFT living on the boundary in less dimensions, providing that way a powerful tool to study the microscopic statistical mechanics of the black holes. An introductory review on black holes and black hole thermodynamics in string theory which heavily relies on supersymmetry (BPS-branes) can be found in [17].

Soon after it was realized by Strominger [18] that notable features of the construction of [12] and in particular the matching the universality of the 2D CFT Cardy formula to the universality of the Bekenstein-Hawking area law, followed largely from a careful analysis performed by Brown and Henneaux in [19]. That allowed him to compute the microscopic black hole entropy from the asymptotic growth of states without the use of string theory or supersymmetry, by using the fact that quantum gravity on AdS_3 is a conformal field theory. Brown and Henneaux investigated the asymptotic symmetries of the three-dimensional anti-de Sitter space (AdS_3) and found out that it consist of two copies of the Virasoro algebra with finite central charge, which is the symmetry group of the two-dimensional conformal field theories (CFTs). Actually when the central charge in the symmetry algebra of (2+1)-dimensional asymptotically AdS gravity was first discovered, it was considered to be mainly a mathematical curiosity, that changed when Strominger [18] and Birmingham, Sachs, and Sen [20] independently pointed out that this result could be used to compute the asymptotic density of states. The key to this computation is the Cardy formula [21], [22].

This approach was generalized to four dimensional Kerr black holes by Guica, Hartman, Song and Strominger [23], who investigated the near-horizon geometry of the extremal Kerr (NHEK) black hole which has $SL(2,\mathbb{R}) \times U(1)$ isometries [24] and considered the asymptotic symmetry following the work by Brown and Henneaux [19]. Under an appropriate choice of the boundary condition on the fall-off of the metric they found one copy (chiral) of a Virasoro algebra as its asymptotic symmetry which was realized from the enhancement of the $U(1)$ rotational symmetry group. This is actually in contrast to the case of Godel black hole [25] in which the $SL(2,\mathbb{R})$ isometry is enchanced to a Virasoro algebra. The importance of Kerr/CFT correspondence stands in the fact that the microscopic entropy of a black hole is being calculated for the first time without the need of string theory or supersymmetry and is actually quite physical as is in Einstein gravity. We review the Kerr/CFT correspondence in section 4. Since the conformal algebra was generated out of the $U(1)$ symmetry, the question was posed if another conformal algebra could be realized from the $SL(2,\mathbb{R})$ symmetry that was also found in the near horizon geometry of the extremal Kerr black hole. It turned out that it could by imposing new asymptotic condition which is stronger than that the Kerr/CFT correspondence [23] they enchanced the $SL(2,\mathbb{R})$ symmetry which admitted a Virasoro algebra. This investigation was done in [26] and also in [27]. However the boundary conditions were exclusive to each other. The chosen boundary conditions on one didn't allow the symmetry on the other. This issue was investigated in [28] were they proposed new asymptotic boundary conditions which allow both symmetries (left and right mover). Perturbations of the near-horizon extreme Kerr spacetime were examined in [29]. Higher derivative corrections to the asymptotic Virasoro symmetry of 4d extremal black holes can be found in [30]. A lot of work has been done since then to extend this procedure to other examples of extremal spinning black holes such as to Kerr-Newman-AdS-dS black hole [31]. In the same paper it was proposed that the Frolov-Thorne temperature may be of the general form $T_L = 1/2\pi k$ in four dimensions. This was then generalized to higher dimensions in [32] to be $T_L^a = 1/2\pi k_a$ based on all the examples that have been studied. To 4D [33] as well as in 5D [34] Kaluza-Klein black holes, to Kerr-Bolt spacetimes [35], to linear dilaton black hole in Einstein-Maxwell-Dilaton-Axion Gravity [36], to black hole solutions in gauged and ungauged supergravities [32], to the Kerr/AdS metrics in diverse dimensions [37]. The Kerr/CFT correspondence has also been extended to five dimensional black holes such as, to the (charged) Kerr black hole embedded in the five-dimensional Gödel universe [38], to 5D black holes [39], to minimal supergravity [40]. The static extremal AdS black hole in diverse dimensions is examined in [41]. The CFT description for extremal non rotating black holes has been also studied for the extremal Reissner-Nordstrøm (RN) black hole [42]. Furthermore, the correspondence has been used in string theory to extremal Kerr-Sen black hole that appears as solutions in the low energy limit of heterotic string theory in 5D [43] and to D1-D5-P and the BMPV black holes [44], [45]. Finally, it was shown in [46], that any solutions to vacuum general relativity whose asymptotic behavior agrees with that of NHEK geometry, also called the extremal Kerr throat are as well diffeomorphic to the NHEK geometry as well. We have to note here that the Kerr/CFT correspondence originally proposed in [23] which afterwards was generalized to other extremal black holes, was done by assuming that the central charges from non-gravitational fields vanish. The validity of the above assumption was checked in [47].

It has been expected that the Kerr/CFT correspondence should be true for general J and M, which could be far away from extreme limit. However, for generic non-extremal Kerr black holes, the NHEK geometry disappears and it was not clear how to associate a CFT from the near horizon geometry. In a paper by Castro, Maloney and Strominger [48] it was argued that the existence of conformal invariance in a near horizon geometry is not a necessary condition, instead the existence of a local conformal invariance in the solution space of the wave equation for the propagating field is sufficient to ensure a dual CFT description. We review this new approach in section 5. This approach has been extended to other kinds of black holes such as to the Schwarzschild black hole [49], the Kerr-Newman [50] and Kerr-Newman-AdS-dS [51] black holes, the Dyonic Reissner-Nordstrøm black bole [52], the uplifted 5D Reissner-Nordstrøm black hole [53], to extremal black holes (Kerr-Newman,RN,warped AdS3,null warped) [54] while the AdS_3 black hole has been examined in [55], to 4D extremal [56] and non extremal [33] Kaluza-Klein black holes, to extremal [57] non extremal [58] and AdS-dS [59] Kerr-Bolt spacetimes , to extreme and non-extreme Einstein-Maxwell-Dilaton-Axion black holes [60], to Kerr-Sen black hole (coming from heterotic string theory) and Kerr-Newman-Kasuya black hole (rotating, with both electric+magnetic charge) [61], to doubly-spinning 5D Myers-Perry [62], to the Cvetic-Youm solution (4D BH with 4 charges coming from string theory) [63], to 5D non-extremal charged rotating black holes in minimal gauged+ungauged supergravity [64], the BTZ black hole solution of cosmological topological massive gravity [65] where they also consider the non relativistic limit, while a different approach to the analysis of symmetries in the near-horizon region of black holes can be found in [66].

A new aspect of the hidden conformal symmetry was found in the study of Kerr-Newman black hole [67], in this paper the method based on the original paper [48] was used to investigate the wave equation which involved three charges, i.e., mass M, angular momentum J, and charge Q. However, the authors observed that when the quantum number of angular momentum J was taken to zero and a new vector operator was introduced for the charge Q, another hidden conformal symmetry emerged, which called the Q-picture description. In view of this, the former description was named the J-picture. Investigation has been done with this viewpoint for the Q-picture of Kerr-Newman-AdS-dS black Hole [68], as well as for Q-picture for the charged+rotating black hole solutions in the five dimensional minimal supergravity [69]. Finally the hidden conformal symmetry has been applied to compute the real-time correlators in Kerr/CFT, [70], [71]. A short introduction to the two above Kerr/CFT approaches can be found in the Cargèse Lectures [72] while an investigation of these two approaches and more specific what happens to the hidden conformal symmetry as the black hole goes extremal can be found in [73].

In the same spirit as the above we find it interesting to try to extend the hidden conformal symmetry approach to the case of black rings hoping to give a more complete picture of hidden conformal symmetry in black holes. Black rings are actually higher dimensional solutions to the Einstein's fields equations found by Emparan and Reall [74]. A comprehensive review on black rings can be found in [75]. The solution actually contains one angular momentum setting the other equal to zero (remember that in 4 dimensional space an object can rotate in two mutually orthogonal planes and thus have two independent angular momenta). A more general solution containing two angular momenta was presented in [76]. These objects carry electric charges [77], [78] and magnetic dipoles [79], the latter usually referred to as dipole charges. A generalized black ring solution containing the previous solutions as limiting cases was constructed in [80]. It is a 7 parameter family of supergravity solutions that describe non-supersymmetric black rings with three charges, three dipoles and two angular momenta. The existence of black rings raised the question of whether there are any supersymmetric black holes in five dimensions other than BMPV, the supersymmetric black ring solutions was found in [81].

The thesis is organized as follows. In section 2 we review some basic elements of two dimensional conformal field theory and make a derivation of the Cardy formula which is the basic tool we use later on in order to calculate the statistical entropy of black holes. In section 3 we present a small introduction on black holes and black hole thermodynamics and focus on the Kerr black hole as it is the main object under investigation. In section 4 we review the Kerr/CFT correspondence and in section 5 we review the Hidden Conformal Symmetry in Kerr Black Hole. In section 6 we discuss black rings and how they are connected with boosted black strings. We review the construction of a boosted black string and show that it admits a hidden conformal symmetry. We also take a look at the supersymmetric black ring case and show that it admits a hidden conformal symmetry as well and calculate the microscopic entropy. In section 7 we examine higher dimensional boosted black strings where we conjecture that hidden conformal symmetry in any dimension of boosted black string should exist.

2 Conformal Field Theory (CFT)

In this section we try to give a brief review on some elements of conformal field theory that will cover the essentials in order to understand the concept of the Kerr/CFT correspondence that will follow. Most of the material can be found in standard reviews and lectures i.e. Ginsparg lectures [82], Polchinski lectures [83] and books such as Di Francesco [84], Kaku [85] and Ketov [86].

2.1 Two dimensional conformal field theory

Conformal Field Theory is a field theory which is invariant under conformal transformations. A conformal transformation is a change in coordinates $x^\mu \rightarrow \tilde{x}^\mu$ such that the metric tensor is invariant up to a scale change:

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega(x) g_{\mu\nu} \tag{2.2}$$

Or to put it in other words, a conformal transformation is a transformation between two spaces M with metric element ds and M' with metric ds' such that $ds'^2 = \Omega ds^2 \leftrightarrow g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega(x) g_{\mu\nu}$. Here Ω is a local rescaling of distances independent of their direction, local because as we see the Ω factor is position dependent. These are consequently the coordinate transformations that preserve the angle $cos(\theta) = u \cdot w/(u^2 w^2)^{1/2}$ between two vectors u, w (where $u \cdot w = g_{\mu\nu} u^\mu w^\nu$) and as a consequence conformal field theories care about angles and not distances. The epithet conformal derives from that fact, meaning that the transformation does not affect the angle between two arbitrary curves crossing each other at some point in space, despite a local dilation.

These transformations form a group, known as *the conformal group*. The infinitesimal generators of the conformal group can be determined by considering an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, under which

$$g_{\mu\nu} \Longrightarrow g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.3}$$

To satisfy (2.2) we must require that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{2.4}$$

Let us now see how we get to the results (2.3) and (2.4). The metric $g_{\mu\nu}$ defines an invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and under a change of coordinates from $x^\mu \to x'^\mu$ the metric tensor changes as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x), \tag{2.5}$$

so under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$ we will have

$$\frac{\partial x'^\mu}{\partial x^\rho} = \delta^\mu_\rho + \partial_\rho \epsilon^\mu.$$

So we get this but looking at how our metric tensor transforms we actually need their inverses and then plug them in.

We can proceed now to find the inverses of $\frac{\partial x'^\mu}{\partial x^\rho}$ which must satisfy

$$\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\mu_\nu \Longrightarrow (\delta^\mu_\rho + \partial_\rho \epsilon^\mu)(\delta^\rho_\nu - \partial_\nu \epsilon^\rho) = \delta^\mu_\nu + \partial_\nu \epsilon^\mu - \partial_\nu \epsilon^\mu = \delta^\mu_\nu. \tag{2.6}$$

It has to be noted here that in the above calculation we considered terms up to first order. We can now replace our results meaning $(\delta^\rho_\nu - \partial_\nu \epsilon^\rho)$ and $(\delta^\sigma_\mu - \partial_\mu \epsilon^\sigma)$ back and get

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu), \tag{2.7}$$

where $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$ is the infinitesimal change of the metric $\delta g_{\mu\nu}$ after the transformation.

Thus,

$$\begin{aligned} g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} = \Omega(x) g_{\mu\nu}(x) \quad &\Longleftrightarrow \quad \delta g_{\mu\nu} = (\Omega - 1) g_{\mu\nu} \\ &\Longleftrightarrow \quad \delta g_{\mu\nu} = f(x) g_{\mu\nu}(x) \\ &\Longleftrightarrow \quad \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}(x). \end{aligned} \tag{2.8}$$

Here $\Omega(x) = e^{f(x)}$, which for very small $f(x) \to \Omega(x) = 1 + f(x)$.

The factor $f(x)$ is determined by taking the trace on both sides as can be seen below,

$$\begin{aligned} g^{\mu\nu}\partial_\mu\epsilon_\nu + g^{\mu\nu}\partial_\nu\epsilon_\mu = f(x)g_{\mu\nu}(x)g^{\mu\nu}(x) \quad &\Longleftrightarrow \quad \partial\epsilon + \partial\epsilon = f(x)g^\mu_\nu(x) \\ &\Longleftrightarrow \quad 2\partial\epsilon = f(x)d \\ &\Longleftrightarrow \quad f(x) = \frac{2\partial_\rho\epsilon^\rho}{d}. \end{aligned} \tag{2.9}$$

It is only left now to replace (2.9) to (2.8) and get (2.4).

Considering now two dimensions ($d = 2$) and setting for simplicity $g_{\mu\nu} = \eta_{\mu\nu}$ [3] we get from (2.8):

• For $\mu, \nu = 1$

$$\begin{aligned} \partial_1\epsilon_1 + \partial_1\epsilon_1 = (\partial_1\epsilon_1 + \partial_2\epsilon_2)\eta_{11} \quad &\Longleftrightarrow \quad 2\partial_1\epsilon_1 = \partial_1\epsilon_1 + \partial_2\epsilon_2 \\ &\Longleftrightarrow \quad \partial_1\epsilon_1 = \partial_2\epsilon_2 \end{aligned} \tag{2.10}$$

• For $\mu = 1, \nu = 2$

$$\begin{aligned} \partial_1\epsilon_2 + \partial_2\epsilon_1 = (\partial_1\epsilon_1 + \partial_2\epsilon_2)\eta_{12} \quad &\Longleftrightarrow \quad \partial_1\epsilon_2 + \partial_2\epsilon_1 = 0 \\ &\Longleftrightarrow \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \end{aligned} \tag{2.11}$$

In equations (2.10) and (2.11) we recognize the Cauchy-Riemann equations, something which motivates us to use complex coordinates,

$$z = \sigma^1 + i\sigma^2, \qquad \bar{z} = \sigma^1 - i\sigma^2$$

and write

$$\epsilon(z) = \epsilon^1 + i\epsilon^2, \qquad \bar{\epsilon}(z) = \epsilon^1 - i\epsilon^2$$

Thus two dimensional conformal transformations coincide with analytic coordinate transformations

$$z \longrightarrow z' = f(z), \quad \bar{z} \longrightarrow \bar{z}' = \bar{f}(\bar{z}),$$

the local conformal algebra of which is infinite dimensional.

The conformal transformation now becomes

$$z \rightarrow z' = z + \epsilon(z), \quad \epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \quad \bar{z} \longrightarrow \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}),$$

where by hypothesis $\epsilon(z)$ is an arbitrary analytical function and thus can be represented by an infinite Laurent series. If we consider an infinitesimal change given by $\epsilon_n(z) = -z^{n+1}$ and $\bar{\epsilon}_n(\bar{z}) = -\bar{z}^{m+1}$ we can compute the generators of that particular transformation which are

$$L_n = -z^{n+1}\partial_z, \quad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{2.12}$$

[3] We are using the Euclidean metric.

Doing a check to see if our generators generate correctly our transformation (that is the infinitesimal change) we have

$$L_n \cdot z = -z^{n+1}\partial_z \cdot z = -z^{n+1} = \epsilon_n(z)$$

and

$$\bar{L}_n \cdot z = -\bar{z}^{n+1}\partial_{\bar{z}} \cdot z = -\bar{z}^{n+1}\partial_{\bar{z}} = \bar{\epsilon}_n(\bar{z})$$

The commutation relations of these generators form an algebra

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, \quad [L_m, \bar{L}_n] = 0 \quad (2.13)$$

Lets see the computation in more detail before we comment on the result,

$$\begin{aligned}
[L_n, L_m] &= L_n L_m - L_m L_n = \\
&= (-z^{n+1}\partial_z)(-z^{m+1}\partial_z) - (-z^{m+1}\partial_z)(-z^{n+1}\partial_z) = \\
&= z^{n+1}\partial_z(z^{m+1}\partial_z) - z^{m+1}\partial_z(z^{n+1}\partial_z) = \\
&= z^{n+1}[(\partial_z z^{m+1})\partial_z + z^{m+1}\partial_z^2] - z^{m+1}[(\partial_z z^{n+1})\partial_z + z^{n+1}\partial_z^2] = \\
&= z^{n+1}(m+1)z^m\partial_z - z^{m+1}(n+1)z^n\partial_z = \\
&= (m+1)z^{m+n+1}\partial_z - (n+1)z^{m+n+1}\partial_z = \\
&= (m-n)z^{m+n+1}\partial_z = (n-m)L_{m+n},
\end{aligned}$$

where in the last line we used $L_{m+n} = z^{m+n+1}\partial_z$, which is justified by looking at (2.12).

This is the two dimensional local conformal algebra, also called the *Witt algebra*. We also notice that the two algebras $\{L_n\}$ and $\{\bar{L}_n\}$ commute meaning that they are independent something which justifies the use of z and $\bar{z}$ as independent coordinates. We have to note at this point that we are still in the classical case. In the quantum case the algebra (2.13) will be corrected by including an extra term proportional to a central charge and will become the *Virasoro algebra*. So as we see in two dimensions something quite remarkable happens in the conformal group, because the number of generators has become infinite, which corresponds to an infinite number of conserved charges something which in general simplifies a lot our life. Each of these two infinite-dimensional algebras contain a finite subalgebra generated by L_{-1}, L_0, L_1. This is the subalgebra associated with the global conformal group. Indeed from the definition (2.12) it is manifest that $L_{-1} = -\partial_z$ generates translations on the complex plane, that $L_0 = -z\partial_z$ generates scale transformations and rotations and that $L_1 = -z^2\partial_z$ generates special conformal transformations. The generators that preserve the real surface $\sigma^1, \sigma^2 \in \mathbb{R}$ are the linear combinations:

$$L_n + \bar{L}_n \quad \textit{and} \quad i(L_n - \bar{L}_n)$$

In particular $L_0 + \bar{L}_0$ generates dilations on the real surface, and $i(L_0 - \bar{L}_0)$ generates rotations.

To the question why is all that useful, we can answer that since the number of generators in the two dimensional local conformal algebra is infinite, thus having an infinite number of symmetries, it should imply/impose severe restrictions on the conformally invariant field theories in two dimensions, thus simplifying them to a great degree, and that is what our story is all about.

2.2 Energy-Momentum Tensor

The energy momentum tensor is of particular importance in conformal field theory. Considering the CFT to be coupled to the 2d metric $g_{\mu\nu}$ so its action has the form $S(\phi_i, g)$ we can use the definition suggested by general relativity

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\mu\nu}}, \tag{2.14}$$

which satisfies

$$\partial^\nu T_{\mu\nu} = 0, \quad TrT_{\mu\nu} = 0.$$

The first statement states that the energy-momentum tensor is conserved while the second statement states that is traceless which implies that $S(\phi_i, g)$ is scale invariant. So it is symmetric and covariantly conserved and traceless in any d of CFT. The tracelessness of the energy momentum tensor implies the invariance of the action under conformal transformations.
• Traceleness of $T_{\mu\nu}$ in 2D
On an euclidean plane parametrized by complex coordinates $z, \bar{z}$ with the line element $ds^2 = dzd\bar{z}$ the conservation of the energy-momentum tensor takes the form

$$\partial_\nu T^{\mu\nu} \Longrightarrow \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0, \qquad \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0,$$

while the scale invariance condition $T^\mu_\mu = 0$ in complex coordinates becomes

$$T_{z\bar{z}} = T_{\bar{z}z} = 0.$$

Lets do a small check why $T^\mu_\mu = 0$,

$$\begin{aligned} T^\mu_\mu = g^{\mu\nu}T_{\nu\mu} &= g^{zz}T_{zz} + g^{z\bar{z}}T_{z\bar{z}} + g^{\bar{z}z}T_{\bar{z}z} + g^{\bar{z}\bar{z}}T_{\bar{z}\bar{z}} \\ &= 0T_{zz} + 2T_{z\bar{z}} + 2T_{\bar{z}z} + 0T_{\bar{z}\bar{z}} \\ &= 4T_{z\bar{z}} = 0, \end{aligned}$$

where

$$g_{z\bar{z}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{2.15}$$

hence

$$g^{z\bar{z}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{2.16}$$

with this convention the measure of the factor is $dzd\bar{z} = 2d\sigma^1 d\sigma^2$. Therefore the energy-momentum tensor can be split into a holomorphic and antiholomorphic part in any two dimensional CFT

$$T_{zz} \equiv T(z), \quad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}). \tag{2.17}$$

The holomorphic and anti-holomorphic parts of a field are related by complex conjugation in Euclidean space, whereas in Minkowski space they correspond to left-moving modes and right-moving modes which are independent from each other. We give the definition of such functions in appendix A.

2.3 Quantum conformal invariance

So far our discussion had been entirely classical. We can continue now to the quantum aspects of the theory.

2.3.1 OPE and Fields in a CFT

Instead of trying to solve the problem piecemeal with pertubative (or even non-peturbative) methods based on a local action, some people proposed a program designed to solve the whole problem at once — that is to calculate all the correlations between all the fields — based on criteria of self consistency and symmetry, by carrying out the so called *bootstrap program.* The key ingredient to this approach is the assumption that the product of local quantum operators can always be expressed as a linear combination of well-defined local operators. The bootstrap approach based on the operator product expansion (or operator algebra) was proposed by Polyakov in [87] which was based on previous work done by Belavin, Polyakov and Zamolodchikov [88]. The operator product expansion (OPE) is actually a statement about what happens as two local operators approach each other. Then the OPE is:

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum C_{ij}^k\left[(z-w),(\bar{z}-\bar{w})\right]\mathcal{O}_k(w,\bar{w}), \tag{2.18}$$

where k runs over the set of all operators and C_{ij}^k are a set of functions which depend only on the separation between the two operators (in a way they encode the information of how the operators approach each other).

Let us take now as a local operator the energy-momentum tensor. For reasons of analyticity and dimensionality the corresponding TT OPE has the general form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{2.19}$$

and similarly for $\bar{T}\bar{T}$ OPE,

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \cdots \tag{2.20}$$

The constants c and $\tilde{c}$ are called the central charges and usually referred as left-moving and right-moving central charges.

We can take as an example the free scalar field and its corresponding energy-momentum tensor and work out the corresponding OPE. The energy-momentum tensor is given by the below relation, which is derived in detail in the appendix B and it is

$$T = -\frac{1}{\alpha'}\partial X \partial X. \tag{2.21}$$

In what follows we take the product of two energy-momentum tensors evaluated at different points:

$$T(z)T(w) = \frac{1}{\alpha'^2} : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \tag{2.22}$$

and try to expand it over a basis of local operator. By using the Wick theorem we get that the above expression is equivalent to

$$= \frac{1}{\alpha'^2}\Big(: \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : + : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) :$$

$$+ : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : + : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) :$$

$$+ : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : + : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \Big)$$

Thus we conclude that

$$\begin{aligned} T(z)T(w) &= \frac{2}{\alpha'^2}\left(\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{:\partial X(z)\partial X(w):}{(z-w)^2} + \cdots \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'}\frac{\partial^2 X(z)\partial X(w):}{(z-w)} + \cdots \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \end{aligned} \tag{2.23}$$

The dots are for the non-singular terms which include the totally normal ordered term, we discuss normal ordering in appendix C. We will be back to see what this result is telling us, when we first talk about fields in conformal field theory, something we do in the next subsection.

2.3.2 Fields

By field in conformal field theories we are referring to local operators. Actually the term fields corresponds to any local expression that we can write down. Taking a single boson ϕ, its derivative $\partial_\mu \phi$ or a composite operator such as the energy-momentum tensor or $e^{i\phi}$, are all called fields in the context of conformal field theory. These fields can be distinguished depending on their transformation properties.

Considering that a conformal field theory has a set of fields $\{A_i\}$ which in general are infinite in number and contain all the derivatives of the fields $A_i(x)$, there exists a subset of fields $\{\phi_j\} = \{A_i\}$ which can think of as a basis, from where we can generate all other fields. These fields are called *quasi-primary* and are all the fields which under a conformal map $z \to z' = w(z)$, $\bar{z} \to \bar{z}' = \bar{w}(\bar{z})$ transform as:

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) \tag{2.24}$$

where h and $\tilde{h}$ are called conformal weights and are real-valued numbers. They show how operators transform under rotations and scalings. For a given field of scaling dimension Δ and spin s

$$\Delta = h + \tilde{h}, \qquad s = h - \tilde{h}, \tag{2.25}$$

we define the holomorphic h and its antiholomorphic counterpart $\tilde{h}$ as:

$$h = \frac{1}{2}(\Delta + s), \quad \tilde{h} = \frac{1}{2}(\Delta - s).$$

Here spin s is not to be related to quantum mechanical spin. For example, under rotation $z \leftarrow ze^{i\theta}$, $\phi_j(z, \bar{z}) = e^{is_j\theta}\phi_j$, we say that ϕ_j has conformal spin s_j. The scaling dimension is the familiar dimension that we usually associate to fields and operators by dimensional analysis.

A field with the above transformation in two dimensions is also called *primary field* when in addition its OPE truncates at order $(z-w)^{-2}$ or similarly $(\bar{z}-\bar{w})^{-2}$ there are no higher singularities. Considering this last statement it holds that all primary fields are quasi-primary but not the reverse. An example of this, is the energy-momentum tensor which is a quasi primary field of weight $(h, \tilde{h}) = (2, 0)$ but not a primary field because it fails the primary test due to the $(z-w)^{-4}$ term as can be seen from (2.23). The rest of the conformal field theory fields are called *secondary* or *descendant* fields.

2.4 Radial Quantization and Virasoro algebra

We have to keep in mind that the theories we are considering are defined in Euclidean space, and usually obtained after a Wick rotation ($t = -i\tau$) from Minkowski space. We can remind here that in Euclidean geometry there is no real-time, Wick rotation provides us with a trick to get from Minkowski space to Euclidean by going to an imaginary time. Then we go to the complex plane to use some of its properties and for computational simplicity one can separate into holomorphic and antiholomorphic parts. One more thing that we do to make our life easier is to make another map, by a conformal transformation of the complex plane itself. This will serve as way to quantize the CFT. This procedure for defining a quantum theory on the plane is known as radial quantization.

Begin with Euclidean space in two dimensions

$$(x^1, x^2).$$

We compactify the coordinate, $x^1 = x^1 + 2\pi$ (which means it is convenient to make the space direction finite by imposing periodic boundary conditions in the x^1 direction). This defines a cylinder in the x^1, x^2 coordinates. The cylinder can be described by a single complex coordinate $w = x^1 + ix^2$. We then map the cylinder onto the complex plane $w \rightarrow z = e^{-iw}$. If we consider x^2 to be a time coordinate τ we see now that time runs radially out from the origin. The remote past $\tau = -\infty$ is situated at the origin $z = 0$ whereas the remote future $\tau = \infty$ lies at the point of infinity at the Riemann sphere, Furthermore, constant time slices on the cylinder are mapped to circles of constant radius on the z-plane.

Radial quantization is quite handy in order to discuss states in CFT. It is particularly useful for two dimensional conformal field theory in the Euclidean regime since it facilitates use of the full power of contour integrals and complex analysis to analyze short distance expansions, conserved charges, etc. When discussing about states we usually need to know where they live and how they evolve. For example in quantum field theory we quantize a theory by parametrizing the plane by Cartesian coordinates (t, x), The states live on spatial slices while their evolution is being governed by the Hamiltonian operator which generates time translations.

Now by conformal mapping the cylinder into the plane once can say that we get a different/alternate configuration. On the cylinder the states live on spatial slices of constant x^2 and evolve by the Hamiltonian

$$H = \partial_{x^2} \tag{2.26}$$

While on the plane the states live on circles of constant radius x^2 and their evolution is governed by the dilation operator

$$D = z\partial + \bar{z}\bar{\partial} \tag{2.27}$$

So actually the analogue of the Hamiltonian is the dilation operator (the dilation operator is the generator of scale transformations) on the plane. One more difference that we need to mention in this method of quantizing a theory is that time-ordering on the cylinder becomes radial ordering on the plane. Operators in correlation functions are ordered so that those at larger radial distance are moved to the left.

2.4.1 Virasoro generators

The generators of infinitesimal conformal transformations can be defined in terms of $T(z)$, as:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \quad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z}) \tag{2.28}$$

Lets see how this comes about. Under the transformation of T to the plane we get:

$$T_{cyl}(w) = -z^2 T_{plane}(z) + \frac{c}{24} \tag{2.29}$$

where the Schwarzian calculated to be $S(z, w) = \frac{1}{2}$.

The Schwarzian is defined to be

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right) \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \tag{2.30}$$

Its key property is that it preserves the group structure of successive conformal transformations. A conformal transformation incorporates both a coordinate transformation and a Weyl transformation (local rescaling of the metric tensor). In our case the mapping of T from the cylinder to the plane is a conformal one and such transformation is preserved by the Schwarzian.

We can Fourier expand T so that:

$$T_{cyl}(w) = -\sum_{m=-\infty}^{\infty} L_m e^{imw} + \frac{c}{24} \tag{2.31}$$

and by plugging in (2.31) to (2.29) we have the relation

$$T_{plane}(z) \quad = \quad \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}. \tag{2.32}$$

An analytical derivation of the above can be found in the Appendix D.

We can now invert (2.32) so that

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \tag{2.33}$$

which is relation (2.28). The antiholomorphic part is obtained in the same way except that now we replace with bars wherever needed.

Since we have the generators we would like to compute their algebra as usual given by $[L_m, L_n]$. We can think of L_m as a contour integral over $\oint dz$ and L_n as a contour integral over $\oint dw$, where both contours are evaluated around zero and z and w are coordinates on the complex plane.

$$\begin{aligned}
[L_m, L_n] &= L_m L_n - L_n L_m \\
&= \oint \frac{dz}{2\pi i} z^{m+1} T(z) \oint \frac{dz}{2\pi i} w^{n+1} T(w) \\
&\quad - \oint \frac{dw}{2\pi i} z^{n+1} T(w) \oint \frac{dz}{2\pi i} w^{m+1} T(z) = \\
&= \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}) - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} T(z) T(w) =
\end{aligned}$$

we write now the two integrals as one, by doing the z-integration around a fixed point w, so

$$\begin{aligned}
&= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) = \\
&= \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \right)
\end{aligned}$$

using the residue theorem

$$= \oint \frac{dw}{2\pi i} w^{n+1} \mathrm{Res} \left[z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \right) \right]$$

To compute the residue at the pole $z = w$, we first need to Taylor expand z^{m+1} around the point w, so,

$$z^{m+1} = w^{m+1} + (m+1)w^m(z-w) + \frac{1}{2}m(m+1)w^{m-1}(z-w)^2 + \frac{1}{6}m(m^2-1)w^{m-2}(z-w)^3 + \cdots$$

by substituting our Taylor expanded z^{m+1} and doing the multiplications we get

$$w^{m+1}\left[\frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] + w^m(m+1)\left[\frac{c}{2(z-w)^3} + \frac{2T(w)}{z-w}\right] + \\ + \frac{1}{2}m(m+1)w^{m-1}\left[\frac{c}{2(z-w)^2}\right] + \frac{1}{6}m(m^2-1)w^{m-2}\left[\frac{c}{2(z-w)}\right]$$

where we have neglected all terms without a pole. In order to compute the residue we will use the formulas for simple and for higher order poles, given by:

$$\mathrm{Res}(f,w) = \lim_{z\to w}(z-w)f(z)$$

and

$$\mathrm{Res}(f,w) = \frac{1}{(m-1)!}\lim_{z\to w}\frac{d^{m-1}}{dz^{m-1}}(z-w)^m f(z),$$

m denotes the order of the pole. We can proceed now to compute the residue which picks up contribution from each of the three terms,

$$= \oint \frac{dw}{2\pi i} w^{n+1}\left[w^{m+1}\partial T(w) + 2w^m(m+1)T(w) + \frac{c}{12}m(m^2-1)w^{m-2}\right]$$

what is left now is to do the integration over w which will give:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.34}$$

also similarly for

$$[L_n, \bar{L}_m] = 0 \tag{2.35}$$

and

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.36}$$

The algebra in (2.34) is known as the *Virasoro algebra*. Where as mentioned above c is a real valued number, called the *central charge* or the *conformal anomaly* which also has its antiholomorphic countepart $\bar{c}$. The central charge is actually determined by the short distance behaviour of the theory and not from symmetry considerations and actually depends on which CFT the T is computed for. For example, the central charge for the free boson field equals to $c = 1$ which can be seen by looking at (2.23), while for the free fermion field $c = 1/2$. Thus, it can be thought as a measure of the number of degrees of freedom. Looking at (2.34) one can notice that if this term is absent, the algebra is identical to the classical one. Strictly speaking such a constant term should not be allowed in the algebra for the fact that the commutator between two elements of the algebra must be an element of the algebra. So we cannot actually see c as a number but as an operator which commutes with any element of the algebra. But it follows that on any representation of the algebra this operator has a constant value.

2.4.2 c For Free Fermion Field

In this section we are going to show that the value of the central charge for a free fermion field is $c = \frac{1}{2}$, as we have already stated at the end of section 2.4.1. Free fermions are described by the two-dimensional action

$$S = \frac{1}{8\pi} \int d^2 z (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}), \tag{2.37}$$

where we are in Euclidean space and are using complex coordinates. The energy-momentum tensor components for the above action is

$$T(z) = \frac{1}{2} : \psi(z) \partial \psi(z) : \tag{2.38}$$

$$\bar{T}(\bar{z}) = \frac{1}{2} : \bar{\psi}(\bar{z}) \bar{\partial} \bar{\psi}(\bar{z}) : \tag{2.39}$$

It is straightforward to derive the OPEs

$$\psi(z)\psi(w) = -\frac{1}{z-w} \tag{2.40}$$

$$\bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) = -\frac{1}{\bar{z}-\bar{w}} \tag{2.41}$$

and by following the same steps as we did for the free boson field, but instead replacing now with the fermion field we find

$$T(w)T(z) = \frac{\frac{1}{4}}{(w-z)^4} + \cdots \tag{2.42}$$

From which we can see that the $c = \frac{1}{2}$. Also compute the corresponding OPE for

$$T(w)\psi(z) = \frac{\frac{1}{2}}{(w-z)^2} \psi(z) + \cdots \tag{2.43}$$

from which can see that the fermion field has conformal weight equal to $\frac{1}{2}$.

2.5 The Cardy Formula

The Cardy formula [21, 22], gives us the asymptotic density of states in a two-dimensional conformal field theory which is determined by only a few features of the symmetry algebra, independent of any details of the dynamics. That is, the central charge c and the ground state conformal weight Δ_0.

In our case the formula is used in order to compute the entropy of the black hole by counting microstates. The advantage of the Cardy formula is that while it let us count states, it does not require detailed knowledge of the states being counted. Actually the above statement which is the strong point of Cardy formula, is also its main weakness, although we can count microstates without a full quantum theory of gravity the actual states remain disguised. Nevertheless these results suggest an interesting effective description of a black hole entropy. In what follows we represent a derivation of the Cardy formula.

We begin with a conformal field theory (CFT) with central charge c, with the standard Virasoro algebra (2.34).

Now the partition function on the torus of modulus $\tau = \tau_1 + i\tau_2$ is

$$Z(\tau, \overline{\tau}) = Tre^{2\pi i \tau L_0} e^{-2\pi i \overline{\tau}\overline{L}_0} = \sum \rho(\Delta, \overline{\Delta}) e^{2\pi i \Delta \tau} e^{-2\pi i \overline{\Delta}\overline{\tau}}. \tag{2.44}$$

The trace over Hilbert space , means the addition of all the states of the theory, where $\rho(\Delta, \overline{\Delta})$ stands for the density of states. Cardy's basic result is that the below quantity is modular invariant

$$Z_0(\tau, \overline{\tau}) = \mathrm{Tr} e^{2\pi i (L_0 - \frac{c}{24})\tau} e^{-2\pi i (\overline{L}_0 - \frac{c}{24})\overline{\tau}}. \tag{2.45}$$

. We remind that c is a real number, that is the reason for no over-line. In particular, Z_0 is invariant under the transformation $\tau \to -1/\tau$. We can extract now $\rho(\Delta, \overline{\Delta})$ from Z by contour integration. Treat τ and $\overline{\tau}$ as independent complex variables (this is not necessary but it simplifies the computation), and let $q = e^{2\pi i \tau}$ and $\overline{q} = e^{2\pi i \overline{\tau}}$, so

$$\rho(\Delta, \overline{\Delta}) = \frac{1}{(2\pi i)^2} \int \frac{dq}{q^{\Delta+1}} \frac{d\overline{q}}{\overline{q}^{\overline{\Delta}+1}} Z(q, \overline{q}), \tag{2.46}$$

where the integrals are along contours that enclose $q = 0$ and $\bar{q} = 0$.

The basic trick is to note that (compare (2.44) and (2.45) and write the one in terms of the other)

$$Z_0(\tau, \overline{\tau}) = \mathrm{Tr} e^{2\pi i (L_0 - \frac{c}{24})\tau} e^{-2\pi i (\overline{L}_0 - \frac{c}{24})\overline{\tau}} = Z(\tau, \overline{\tau}) q^{-\frac{c}{24}} \overline{q}^{\frac{c}{24}}. \tag{2.47}$$

Using (2.47) we can rewrite (2.46) as

$$\rho(\Delta, \overline{\Delta}) = \frac{1}{(2\pi i)^2} \int \frac{dq}{q^{\Delta+1}} \frac{d\overline{q}}{\overline{q}^{\overline{\Delta}+1}} Z_0(\tau, \overline{\tau}) q^{\frac{c}{24}} \overline{q}^{-\frac{c}{24}}. \tag{2.48}$$

Our next step is to use the modular invariance under $\tau \to -1/\tau$ in order to rewrite the contour integral in a form suitable for a saddle point approximation. Our expressions for q and $\overline{q}$ become:

$$q = e^{2\pi i \tau} \longrightarrow \tilde{q} = e^{-\frac{2\pi i}{\tau}} \tag{2.49}$$

$$\overline{q} = e^{2\pi i\overline{\tau}} \longrightarrow \overline{\tilde{q}} = e^{-\frac{2\pi i}{\overline{\tau}}} \tag{2.50}$$

For simplicity let us continue our calculations with one complex variable τ. So we can write

$$\rho(\Delta) = \frac{1}{2\pi i}\int \frac{dq}{q^{\Delta+1}} Z_0(-1/\tau) q^{\frac{c}{24}} \tag{2.51}$$

and according to expression (2.47)

$$Z_0(-1/\tau) = \tilde{q}^{-\frac{c}{24}} Z(-1/\tau) \tag{2.52}$$

so we can substitute to (2.51)

$$\rho(\Delta) = \frac{1}{2\pi i}\int \frac{dq}{q^{\Delta+1}} \tilde{q}^{-\frac{c}{24}} Z(-1/\tau) q^{\frac{c}{24}}. \tag{2.53}$$

Substituting the expression we have for q and $\tilde{q}$ (also for $\overline{q}$ and $\overline{\tilde{q}}$ which, as stated before, we disregard for simplicity) we get

$$\begin{aligned}
\rho(\Delta) &= \frac{1}{2\pi i}\int e^{-2\pi i\tau(\Delta+1)} e^{-\frac{2\pi i}{\tau}(-\frac{c}{24})} e^{2\pi i\tau(\frac{c}{24})} Z(-1/\tau) 2\pi i e^{2\pi i\tau} d\tau \\
&= \frac{1}{2\pi i}\int e^{-2\pi i\tau(\Delta+1)} e^{\frac{2\pi i c}{24\tau}} e^{\frac{2\pi i c\tau}{24}} Z(-1/\tau) 2\pi i e^{2\pi i\tau} d\tau \\
&= \int e^{-2\pi i\tau\Delta} e^{\frac{2\pi i c}{24\tau}} e^{\frac{2\pi i c\tau}{24}} Z(-1/\tau) d\tau
\end{aligned} \tag{2.54}$$

where we changed variables as $dq = 2\pi i e^{2\pi i\tau} d\tau$. By construction, $Z(-1/\tau)$ approaches a constant, $\rho(\Delta_0)$, for large τ_2 , so the integral (2.54) can safely be evaluated by steepest descents provided that the imaginary part of τ is large at the saddle point. The key to a saddle point approximation is to separate the integrand into a rapidly varying phase and a slowly varying prefactor. We have to find the extremum of the exponent, which we do by taking the derivative $df/d\tau = 0$ where $f(\tau)$ we define as:

$$f(\tau) = -\tau\Delta + \frac{c}{24\tau} + \tau\left(\frac{c}{24}\right) \tag{2.55}$$

(for simplicity we have omitted the $(2\pi i)$'s from our above expression) $df/d\tau = 0 \rightarrow df/d\tau = -\Delta - \frac{1}{\tau^2}(\frac{c}{24}) + \frac{c}{24} = 0$

$$\Longrightarrow \tau = \sqrt{\frac{c}{-24\Delta + c}} \rightarrow \tau \approx i\sqrt{\frac{c}{24\Delta}}. \tag{2.56}$$

Where c on the denominator can be neglected as Δ is taken to be large.

Substituting (2.56) back into the integral (2.44) we get:

$$
\begin{aligned}
\rho(\Delta) &\approx \exp\left\{-2\pi i\Delta i\sqrt{\frac{c}{24\Delta}}\right\}\exp\left\{\frac{2\pi ic}{24i\sqrt{\frac{c}{24\Delta}}}\right\}\exp\left\{\frac{2\pi ici\sqrt{\frac{c}{24\Delta}}}{24}\right\} \\
&\cdot \quad Z_{\left(-1/i\sqrt{\frac{c}{24\Delta}}\right)} \\
&\approx \exp\left\{2\pi\sqrt{\frac{c\Delta}{24}}\right\}\exp\left\{\frac{2\pi c}{24}\sqrt{\frac{24\Delta}{c}}\right\}\exp\left\{-\frac{2\pi c}{24}\sqrt{\frac{c}{24\Delta}}\right\}Z(i\infty) \\
&\approx \exp\left\{2\pi\sqrt{\frac{c\Delta}{24}}\right\}\exp\left\{2\pi\sqrt{\frac{c\Delta}{24}}\right\}Z(i\infty) \\
&\approx \exp\left\{4\pi\sqrt{\frac{c\Delta}{24}}\right\}\longrightarrow\rho(\Delta)\approx\exp\left\{2\pi\sqrt{\frac{c\Delta}{6}}\right\} \qquad (2.57)
\end{aligned}
$$

Where the third exponent of our expression becomes one as the power goes to zero due to the large Δ. Relation (2.57) is giving us the asymptotic growth of states in a two-dimensional conformal field theory for large Δ.

In order now to compute the statistical entropy of a system we take the logarithm of the expression (2.57) so:

$$
\log\rho(\Delta) = 2\pi\sqrt{\frac{c\Delta}{6}} \Longrightarrow S = 2\pi\sqrt{\frac{c\Delta}{6}}, \qquad (2.58)
$$

where from statistical mechanics we used the relationship that in the microcanonical ensemble the logarithm of the number of microstates (Ω) of a system gives us the entropy (S) of the system, $S = \log\Omega$.

Lets try to sum up the above in a few lines. We considered a two-dimensional conformal field theory, which really, is a theory invariant under diffeormophisms and Weyl transformations, and chose complex coordinates z and $\bar{z}$. Such a theory is characterized by a pair of Virasoro algebras, one for left moving modes and one for right moving modes, and states which fall into representations of these algebras. Since the plane is conformal we transformed our theory to one on a cylinder, the central term is a conformal anomaly, but its effect on such transformation is simply to shift the stress-energy tensor (2.29). To count states we use a trick, we first compute the partition function, and then obtain the density of states from Legendre transformation. Therefore continue our theory to imaginary time and compactify the cylinder to a torus of modulus of modulus τ relation (2.44), from there we determined Z which allowed us to extract the density of states $\rho(\Delta,\bar{\Delta})$ by means of a contour integral. The derivation of the Cardy formula starts with observation (2.45) and in particular that Z_0 is invariant under the transformation $\tau \to -1/\tau$. Using this invariance we wrote $Z(\tau)$ in terms of $Z(-1/\tau)$ and a rapidly varying phase, and used the method of steepest descents to extract $\rho(\Delta,\bar{\Delta})$. It should be stressed that the transformation from the plane to the cylinder and the continuation to imaginary time are merely tricks to obtain the density of states, we are not assuming any fundamental role for compact spaces or Euclidean signature.

Logarithmic corrections to the entropy by the same method can be found in [89], while a similar proof using the method of steepest descents can be found in [90].

3 Black Holes

In this section we are going to discuss the fascinating objects we call *black holes*, which are the essential object of our study in this thesis. There is a plethora of ways that one can learn about black holes. There is an extensive literature which discusses black holes, internet lectures and introductory reviews that one can look into. Here a small sample of the things that I mostly looked into in order to create this section [91], [92], [93], [94].

3.1 What is a black hole

Black holes are extremely dense objects that have such an intense gravitational pull that even light cannot escape from them. One can imagine a point like object with infinite mass density thus infinite gravitational pull. They are believed to reside in the center of many galaxies and actually these objects lead to conflicts and paradoxes, between our two separate theories of nature, one for the very small, quantum mechanics, and one for the very large, general relativity. Established principles crash at their presence but it is there where progress is made. Since black holes are objects that belong to both regimes, physicists have no alternative but to combine these two different theories together.

A black hole has two important spacetime features. The first is what we call the *singularity*, a spacetime point with infinite curvature (where curvature is measured by the Riemann tensor), and the second the *event horizon*, which is considered as the critical limit from which whatever crosses it ends up at the singularity. That is because within the event horizon the escape velocity exceeds the speed of light ($v_{esc} > c$) and since special relativity taught us that nothing can move faster than the speed of light an object is trapped forever. At the event horizon the escape velocity is equal to the speed of light ($v_{esc} = c$) while outside the event horizon $v_{esc} < c$ and an object is able to escape. One can already establish this by using Newtonian physics, the escape velocity from a spherical mass M of radius R satisfies $\frac{1}{2}v_{esc}^2 = GM/R$, or $v_{esc} = \sqrt{2GM/R}$ (independent of the mass of the escaping object, by equivalence of inertial and gravitational masses). v_{esc} exceeds the speed of light if $R < R_s := 2GM/c^2$. The radius R_s is called the "Schwarzschild radius" for the mass M.

Black holes are actually solutions to Einstein's equations of motion. These black holes solutions in $(3+1)$-dimensional Einstein-Maxwell theory, are fully characterized by a very small number of parameters, rather than the potentially infinite set of parameters characterizing, say, a planet. They are characterized by, *mass*, as measured by the black hole's effect on orbiting bodies etc, *charge* (electric and magnetic), as measured by the strength of the electric force, and *angular momentum*(spin) meaning how fast the black hole is spinning (rotating).

It is common to say that a black hole has 'no-hair', which expresses the above statements, known as the *no hair theorem.*

We can have a better picture of the above statement if we think that once matter disappears behind a horizon, an exterior observer sees almost nothing of its individual properties. One can no longer for example make distinctions between protons, electrons etc. A huge amount of information is thus lost and the mass, angular momentum and charge completely determine the external field. These parameters play a role in changing the structure of these black objects. The mass M develops an event horizon and the charge Q allows a black object to interact with gauge field. The angular momentum ($J \neq 0$) induces a strong frame dragging and develops an ergo-region around its event horizon.

Below we classify black holes according to the parameters that distinguish them:

Table 1: Types of black holes in 4D

	Non-rotating ($J = 0$)	Rotating ($J \neq 0$)
Uncharged ($Q = 0$)	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner-Nordstrom	Kerr-Newman

What we are going to discuss about in this thesis will also concern Kerr black holes, which are black holes with angular momentum and no-charge and as all black holes will be completely determined by their mass (M) and angular momentum (J).

Exact black hole solutions in higher dimensions have also been obtained, these solutions are more exotic in comparison with the four dimensional ones, in the sense that they come with more complicated structure. The idea was actually first entertained by Kaluza [95] and Klein [96]. One may ask why we should spend time and energy to find these solutions. In response to that we can say that string theory which is a strong candidate for quantum gravity and a unified theory of nature requires various dimensions in spacetimes in order to study gravity. Furthermore these solutions might give us insights and information so to understand better the solutions that we consider more physical, the four dimensional ones. As an example to our second argument we actually found out that the universality characteristics that Einstein gravity solutions share, that is the spherical topology of their horizons and that their properties are uniquely determined by their conserved charges, can not be extend to higher dimensions, except in static case [97]. To be more precise in five dimensions, the Einstein theory allows not only the existence of Kerr-like black holes with topology S^3 , the Myers-Perry black hole [98], but also black holes with topology $S^1 \times S^2$, found by Emparan and Reall [74] and called rotating black rings. Which really means that for same J and M we can get different solutions which corresponds to different black holes. We will talk about black rings in section 6. The uniqueness theorems in four dimensional spacetimes for the vacuum black hole solutions have been established in [99], [100], [101], [102], [103] while for a comprehensive review one can look at [104].

3.2 Black hole thermodynamics

An object with *entropy* is microscopically, random, like a hot gas. A known configuration of classical fields has zero entropy, there is nothing random about electric or magnetic fields, or gravitational waves. Since black holes are exact solutions of Einstein equations, they were thought not to have any entropy either.

Let us consider first Hawking's area theorem [105] and the relation to the second law of thermodynamics. The area theorem states: If spacetime on and outside the future event horizon is a regular predictable space, and the stress tensor satisfies the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ for arbitrary null k^μ, then the area of the event horizon is non-decreasing. Bekenstein [4] observed that there was a close analogy between this result and the second law of thermodynamics and argued that black holes should be assigned an entropy proportional to the area of the event horizon. Black Holes are in fact thermodynamic objects, the close mathematical analogy between black holes and thermodynamics can be seen in the below table [4].

Table 2: Relation between thermodynamics and black holes

	Thermodynamic	Black Hole
Zeroth Law	T constant throughout body in equilibrium	κ constant over horizon of stationary black holes
First Law	$dE = TdS$ + work terms	$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ$
Second Law	$\delta S \geq 0$	$dA \geq 0$
Third Law	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

Here T denotes the temperature, E is the energy, S is the entropy, κ is the surface gravity, M is the mass, A is the horizon area(surface area), J is the angular momentum, Ω is the angular velocity and Φ is the electric potential. The subscript H, on Ω and Φ stands for the horizon. Actually κ, Ω and Φ are defined locally on the horizon, but they are always constant over the horizon of a stationary black hole. Identifying the thermodynamic quantities energy, entropy, temperature with the black hole mass, area, surface gravity one can write:

$$E \leftrightarrow M, \quad S \leftrightarrow \frac{A}{8\pi a}, \quad T \leftrightarrow \kappa a \tag{3.1}$$

where a is a constant number, the value of which we are going to determine in a bit. We should note that, we take the area of the black hole as a measure of its entropy, entropy in the sense of inaccessibility of information about its internal configuration.

[4]Table 2 can originally be found in [91].

Let us see now what table 2 is telling us: The zeroth law of thermodynamics states that in thermal equilibrium the temperature is constant throughout the system. The analogous statement for black holes is that stationary black holes have constant surface gravity on the entire horizon. The first law of thermodynamics says that the two equations at table 2 are equivalent, where $\Omega_H dJ$ and $\Phi_H dQ$ can be thought of as work done to the black hole, for instance throwing rocks into it. The second law of thermodynamics, states that, for a closed system the entropy never decreases, the analogous statement for black holes is that the area of the horizon never decreases. The third Law of thermodynamics, states that it is impossible to achieve $T = 0$ in any physical process or that entropy must go to zero $S \to 0$ as the temperature goes to zero $T \to 0$, for black holes this does not quite work, it turns out that $\kappa = 0$ corresponds to extremal black holes, which do not necessarily have a vanishing area A (but this does not lead to any contradictions since the third law is not truly fundamental it only applies to some situations/systems).

The thermodynamic temperature of a black hole in classical general relativity is absolute zero, since a black hole is a perfect absorber and thus does not emit anything. A black hole of given mass, angular momentum, and charge can have a large number of different unobservable internal configurations which reflect the possible different initial configurations of the matter which collapsed to produce the hole. The logarithm of this number can be regarded as the entropy of the black hole and is a measure of the amount of information about the initial state which was lost in the formation of the black hole. If one makes the hypothesis that the entropy is finite, one can deduce that the black holes must emit thermal radiation at some non-zero temperature. Hawking showed that quantum fields in a fixed classical gravitational background (a black hole background) allow the hole to radiate at temperature:

$$T_H = \frac{\hbar\kappa}{2\pi}. \tag{3.2}$$

From this expression we can also conjecture that κ does physically represent the thermodynamic temperature of a black hole. The above result is quite fascinating, it tells us that black holes glow, they give off radiation, they give off energy and in that process they evaporate. The implications of this are quite staggering as after a period of time a black hole will evaporate through a quantum mechanical effect. Here is where there also exists a big conflict, a paradox, that we do not quite yet understand, that is. General relativity says that things that fall into a black hole they end up at the singularity where they are destroyed, quantum mechanics says that things are eventually being radiated out.

By combining now the first law from table 2 with the Hawking temperature (3.2) one gets the expression for the black hole entropy or also called the Bekestein-Hawking entropy

$$S = \frac{A}{4\hbar G}, \tag{3.3}$$

where A is the surface area of the event horizon of the black hole. Thus the amount of information that a black hole can hold is analogue to the surface area of the black hole. Lets see the calculation in more detail:

$$
\begin{aligned}
TdS = \frac{\kappa}{8\pi G} dA \quad &\Longrightarrow \quad \frac{\hbar\kappa}{2\pi} dS = \frac{\kappa}{8\pi G} dA \\
&\Longrightarrow \quad S = \frac{A}{4G\hbar}
\end{aligned}
$$

a result which also fixes the value of our constant a in (3.1). As we see the Bekestein-Hawking entropy depends on both Planck's constant and Newton's constant, it is thus inherently quantum gravitational.

3.3 The Kerr Black Hole

A Kerr black hole is a rotating black hole (its angular momentum $J \neq 0$). A rotating body bulges along its equatorial plane due to the centrifugal forces generated by the rotation, so it cannot be described by the Schwarzschild solution which assumes spherical symmetry. What we actually need is a stationary (time independent — for a time independent rotation of the star — but not invariant under time-reversal) vacuum solution to the field equations that has axial symmetry about the z-axis which at great distances from the star resumes flat space (asymptotically flat). The Kerr metric is actually an exact solution of the Einstein field equations of general relativity, which is quite remarkable given that these equations are highly non-linear which makes exact solutions very difficult to find.

3.3.1 The Kerr metric

As we already have stated the gravitational field of a rotating black hole is given by the Kerr metric which has properties of being axially symmetric and stationary. This metric in Boyer-Lindquist coordinates takes the form:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - \alpha \sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2+\alpha^2)d\phi - \alpha dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (3.4)$$

where

$$\Delta \equiv r^2 - 2Mr + \alpha^2, \qquad \rho^2 \equiv r^2 + \alpha^2\cos^2\theta$$

and

$$J = M\alpha.$$

The two constants M and α parametrize the possible solutions, α is the angular momentum per unit mass and M is the geometric mass. What we notice from (3.4) is as $\alpha \to 0$ the Kerr metric reduces to the Schwarzschild metric. And if we keep α fixed and let $M \to 0$ we recover flat spacetime in ellipsoidal coordinates. The cross-term $d\phi dt$ has the effect of 'dragging' this inertial space along with the rotating body, just as water is being dragged along by the surface of a spinning ball. The effect is called 'dragging of inertial frames' and the gyroscopic precession it produces is called 'Lense-Thirring effect' [106], [107], [108]. The

Kerr metric has two Killing vectors, since the metric coefficients are independent of t and ϕ, both $K = \partial_t$ and $R = \partial_\phi$ are killing vectors. Although the Kerr metric can be written in a number of different ways [5] a particular reason that why we use these coordinates could be that they minimize the number of off-diagonal terms of the metric, there is now only one off diagonal component which particularly helps in analyzing the asymptotic behavior and in trying to understand the key difference between an event horizon and an ergosphere. We can also notice that the form of the metric in (3.4) exhibits symmetry under time reversal, the transformation ($t \longrightarrow -t$) also changes the rotation direction for example the sign of the angular momentum ($\alpha \longrightarrow -\alpha$). The horizon radius is located at

$$r_+ = M + \sqrt{M^2 - \alpha^2} \tag{3.5}$$

The horizon radius is found by letting $\Delta = 0$ and solving quadratically, from where we acquire two values for r, which are $r_\pm = M \pm \sqrt{M^2 - \alpha^2}$, from which we ignore the r_- as is found inside the 'real' horizon. For $\alpha = 0$, r_+ becomes $r = 2M$ which is the Schwarzschild radius while r_- becomes $r = 0$.

As we see there are three possible cases. One case is where $M^2 > a^2$, which is considered to be the realistic case. Another case is where $M = \alpha$ or $J = M^2$, in which case we impose an enormous angular momentum and it is what we call the extremal case in which the event horizons coincide at $r = M$. Finally the case where $M^2 < a^2$ which features a naked singularity as the horizons disappear (the metric is regular except at $\rho = 0$ where there is the ring singularity) something that we do not believe it exist in nature, thus unphysical.

There exists a true curvature singularity at $\rho = 0$, which is possible for $r = 0$ and $\theta = \frac{\pi}{2}$. This singularity[6] is not a point in space but rather a disc. We can think of it as the rotation stretched the Schwarzschild singularity into spreading it out over a ring.

The angular velocity at the horizon of a Kerr black hole is given by

$$\Omega_H = \frac{\alpha}{r_+^2 + \alpha^2}, \tag{3.6}$$

It can be found by calculating the quantity $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$ which plays the role of 'generalized angular velocity of rotation of the ergosphere' relative to the external reference system.

The area is

$$A = 8\pi[M[M + (M^2 - \alpha^2)^{1/2}], \tag{3.7}$$

which is computed by integrating the surface of the Kerr horizon $r_+ = M + \sqrt{M^2 - a^2}$.

[5] A brief introduction to the mathematics and physics of the Kerr spacetime and rotating black holes, touching on the most common coordinate representations of the spacetime metric and the key features of the geometry can be found in [109].

[6] To distinguish between coordinate and curvature singlarities, we can examine the square Riemann tensor ($R_{abcd}R^{abcd}$) of the Kerr spacetime.

The surface gravity for a Kerr black hole takes the form

$$\kappa = \frac{(M^2 - \alpha^2)^{1/2}}{2M[M + (M^2 - \alpha^2)^{1/2}]}. \tag{3.8}$$

A definition for κ is the magnitude of acceleration with respect to Killing time, of a stationary zero angular momentum test particle just outside the horizon. It is the same as the force per unit mass that must be applied at infinity in order to hold the particle on its path.

Finally by making use of (3.2) can calculate the temperature of the Kerr black hole

$$T_H = \frac{\hbar\sqrt{(M^2 - \alpha^2)}}{4\pi M r_+}. \tag{3.9}$$

3.3.2 The extreme Kerr black hole

The extremal case is recovered when mass and angular momentum are related by $M^2 = \alpha^2$, imposing this on (3.6), (3.7), (3.8) and (3.9) our values become:

The angular velocity at the horizon becomes

$$\Omega_H = \frac{1}{2M}, \tag{3.10}$$

the area is

$$A = 8\pi M^2, \tag{3.11}$$

while the surface gravity is now zero

$$\kappa = 0. \tag{3.12}$$

Finally the temperature of the extremal Kerr solution is

$$T_H = 0, \tag{3.13}$$

as can easily be read from (3.12).

4 The Kerr/CFT correspondence

The Kerr/CFT correspondence proposed in [23] claims that there exists a correspondence between four dimensional Kerr black holes and conformal field theory in two dimensions. In other words we have two theories, the first general relativity and the second conformal field theory which we try to relate. This concept is known as *holographic dualities*, relating a theory of gravity, to a quantum theory without gravity in fewer dimensions. Now in order to do that we have to find a 'place' where these two theories correlate and that is happening in our case in the near horizon area of an extremal Kerr black hole. Now where do we base this conjecture, meaning that an extremal Kerr black hole is holographically dual to a two dimesional CFT? As we will see, it is based on the fact that the Cardy formula reproduces the gravitational entropy by counting microstates (microscopic degrees of freedom) which agrees with the Bekestein-Hawking entropy. It is important to note here that, it is an assumption that the underlying theory is unitary, else we could not use the generic results in two dimensional conformal field theory [110]. The approach would not be possible without the work of Brown and Henneaux [19]. They analyzed the asymptotic symmetries of general relativity in (2+1)-dimensional asymptotically anti-de Sitter space, they found a pair of commuting Virasoro algebras—the symmetry group of a two-dimensional conformal field theory—with central charges

$$c^{\pm} = \frac{3R}{2G}.$$

Here R is the AdS curvature. To obtain this result, Brown and Henneaux imposed boundary conditions

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} = -\frac{r^2}{R^2} + \mathcal{O}(1) & g_{tr} = \mathcal{O}(\frac{1}{r^3}) & g_{t\phi} = \mathcal{O}(1) \\ & g_{rr} = \frac{R^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) & g_{r\phi} = \mathcal{O}\left(\frac{1}{r^3}\right) \\ & & g_{\phi\phi} = r^2 + \mathcal{O}(1) \end{pmatrix},$$

at spatial infinity.

The behavior is preserved under the coordinate transformations

$$t \to t + \xi^t, \quad r \to r + \xi^r, \quad \phi \to \phi + \xi^\phi$$

and the Killing vector fields ξ_n^a, where $a = t, r, \phi$ satisfied the commutation relations of

$$[\xi_m, \xi_n] = -i(m-n)\xi_{m+n}.$$

The result actually showed that the AdS_3 was endowed with the 2D conformal symmetry on the boundary. After that they evaluated the central extension of the Virasoro algebras by using the Hamiltonian formalism, calculated the variation of the Hamiltonian, and added a surface term to obtain correct equations of motion and from that surface term they obtained a global charge, possible to evaluate the central charge. The generator of isometry (Hamiltonian) consists of the constraint part together with appropriate surface term

$$\mathcal{H}[\xi] = \int d^2x(\xi^0\mathcal{H}_0 + \xi^i\mathcal{H}_i) + Q[\xi],$$

the algebraic structure of the symmetric transformation group is given by the generators Poisson bracket

$$\{\mathcal{H}[\xi], \mathcal{H}[\eta]\}_P = \mathcal{H}\big[[\xi,\eta]\big] + K[\xi,\eta],$$

from which the evaluation of the last term $K[\xi,\eta]$ will give the central extension to the algebra.

The Kerr/CFT correspondence begins by approximating the extremal Kerr metric near the horizon as the "near-horizon extremal Kerr" (NHEK) metric, which appeared in the work of Bardeen and Horowitz [24], where a warped AdS_3 is found. A warped AdS_3 is similar to AdS_3 and breaks $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometry group of AdS_3 down to $SL(2,\mathbb{R}) \times U(1)$. Applying the analysis of Brown and Henneaux [19] to the case of warped AdS_3, the asymptotic symmetry group (ASG)[7] of the spacetime is examined. It is shown that the $U(1)$ isometry can be enhanced to a copy of Virasoro algebra, whose quantum version gives rise to a central charge $c_L = 12J$ with J being the angular momentum of the black hole. From the Frolov-Thorne vacuum for extreme Kerr, it turns out that the dual CFT has a nonvanishing left temperature $T_L = \frac{1}{2\pi}$. Then it is shown that the Bekenstein-Hawking entropy can be reproduced exactly by using the Cardy formula, which motivated the conjecture that the extreme Kerr black hole is dual to a two-dimensional CFT.

4.1 Near horizon of an extremal black hole

As mentioned above, since we are interested in the near horizon limit of an extremal Kerr black hole, (following the work of Bardeen and Horowitz [24]), we define new (dimensionless) coordinates for metric (3.4)

$$t = \frac{\lambda\hat{t}}{2M}, \quad y = \frac{\lambda M}{\hat{r} - M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M} \tag{4.1}$$

and zoom into the region $r = M$ (where horizon is located) by taking the limit $\lambda \longrightarrow 0$ keeping (t, y, ϕ, θ) fixed (λ is a scaling parameter). The result is the near-horizon extreme Kerr (NHEK) geometry in Poincaré-type coordinates (also called the *throat geometry*)

$$ds^2 = 2GJ\Omega^2 \left(\frac{-dt^2 + dy^2}{y^2} + d\theta^2 + \Lambda^2 (d\phi + \frac{dt}{y})^2 \right), \tag{4.2}$$

where

$$\Omega^2 \equiv \frac{1 + \cos^2\theta}{2}, \quad \Lambda \equiv \frac{2\sin\theta}{1 + \cos^2\theta}. \tag{4.3}$$

Here ϕ and θ take values $\phi \sim \phi + 2\pi$ and $0 \leq \theta \leq \pi$, respectively.

The coordinates (4.2) cover only part of the NHEK geometry. Global coordinates are given by

$$\begin{aligned} y &= (\cos\tau\sqrt{1+r^2} + r)^{-1}, && (4.4)\\ t &= y\sin\tau\sqrt{1+r^2}, && (4.5)\\ \phi &= \phi + \left(\frac{\cos\tau + r\sin\tau}{1 + \sin\tau\sqrt{1+r^2}} \right). && (4.6) \end{aligned}$$

[7] Asymptotic symmetries are generated by the diffeomorphisms whose action on the metric generates metric fluctuations compatible with the chosen boundary conditions. To be a well-defined charge in the asymptotic limit, the underlying integral must be finite as $r \to \infty$. If the charge vanishes, the asymptotic symmetry is rendered trivial. The asymptotic symmetry group is generated by the diffeomorphisms whose charges are well-defined meaning allowed by the boundary conditions and non-vanishing.

The metric (4.2) is then

$$d\bar{s}^2 = 2GJ\Omega^2 \left(-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Lambda^2 (d\phi + r d\tau)^2 \right). \tag{4.7}$$

The NHEK geometry has a $SL(2,\mathbb{R}) \times U(1)$ isometry group as remarked by [24]. The rotational $U(1)$ isometry is generated by the Killing vector

$$\zeta_0 = -\partial_\phi, \tag{4.8}$$

while the $SL(2,\mathbb{R})$ isometry group is generated by the Killing vectors

$$J_1 = 2\sin\tau \frac{r}{\sqrt{1+r^2}}\partial_\tau - 2\cos\tau\sqrt{1+r^2}\partial_r + \frac{2\sin\tau}{\sqrt{1+r^2}}\partial_\phi, \tag{4.9}$$

$$J_2 = -2\cos\tau \frac{r}{\sqrt{1+r^2}}\partial_\tau - 2\sin\tau\sqrt{1+r^2}\partial_r - \frac{2\cos\tau}{\sqrt{1+r^2}}\partial_\phi, \tag{4.10}$$

$$J_3 = 2\partial_\tau. \tag{4.11}$$

Let us summarize what we have done so far. We took the Kerr metric and from there we went to the extremal Kerr metric by setting $M^2 = \alpha^2$. After that we searched for the near horizon limit of that metric which was obtained in (4.2) and considering global coordinates we ended up at metric (4.7), which has $SL(2,\mathbb{R}) \times U(1)$ isometries, generated by the Killing vectors (4.8)-(4.11). As a next step, we will impose boundary conditions on the fall off metric which will allow us to enhance the $U(1)$ symmetry group, which will allow us to start using two dimensional conformal field theory tools, as we will see in the next section.

4.2 Boundary conditions

Following the work by Brown and Henneaux [19] we need to impose some boundary condition on the asymptotic variation of the metric and single out the desired ASG which is the one that preserves this boundary condition nontrivially. Supposing that the metric is perturbed as $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ is (4.7) and $h_{\mu\nu}$ stands for the deviation from the background metric (4.7). The boundary conditions are specified for the fluctuations at $r = \infty$.

The chosen boundary conditions in the basis (τ, r, θ, ϕ) are

$$h_{\mu\nu} = \begin{pmatrix} h_{\tau\tau} = \mathcal{O}(r^2) & h_{\tau r} = \mathcal{O}(\frac{1}{r^2}) & h_{\tau\theta} = \mathcal{O}\left(\frac{1}{r}\right) & h_{\tau\phi} = \mathcal{O}(1) \\ h_{r\tau} = h_{\tau r} & h_{rr} = \mathcal{O}\left(\frac{1}{r^3}\right) & h_{r\theta} = \mathcal{O}\left(\frac{1}{r^2}\right) & h_{r\phi} = \left(\frac{1}{r}\right) \\ h_{\theta\tau} = h_{\tau\theta} & h_{\theta r} = h_{r\theta} & h_{\theta\theta} = \mathcal{O}\left(\frac{1}{r}\right) & h_{\theta\phi} = \mathcal{O}\left(\frac{1}{r}\right) \\ h_{\phi\tau} = h_{\tau\phi} & h_{\phi r} = h_{r\phi} & h_{\phi\theta} = h_{\theta\phi} & h_{\phi\phi} = \mathcal{O}(1) \end{pmatrix}, \tag{4.12}$$

We may note here that the deviations $h_{\tau\tau}$ and $h_{\phi\phi}$ are of the same order as the leading terms in the background metric (4.7), which is a bit unusual in the sense that is quite different from the analysis made in the AdS_3 case studied in [19] where all the deviations are subleading. We also have to note here that the boundary conditions are purely a postulate and are not derived from somewhere. A physical approach for deriving the boundary conditions, rather than postulating them a priori, was proposed by Porfyriadis and Wilczek [111], where by requiring finiteness of the boundary effective action(s) for certain asymptotic transformations, they are able to derive the Virasoro algebra and central charge associated with the boundary of AdS_3 .

The diffeomorphisms which preserve the boundary conditions (4.12) require that

$$\mathcal{L}_\xi g_{\mu\nu} \sim h_{\mu\nu} \tag{4.13}$$

and are of the form

$$\xi = [-r\epsilon'(\phi)+\mathcal{O}(1)]\partial_r+[C+\mathcal{O}\left(\frac{1}{r^3}\right)]\partial_\tau+[\epsilon(\phi)+\mathcal{O}\left(\frac{1}{r^2}\right)]\partial_\phi+\mathcal{O}\left(\frac{1}{r}\right)\partial_\theta \tag{4.14}$$

where $\epsilon(\phi)$ is an arbitrary smooth function of ϕ and C is an arbitrary constant. The above does not contain the $SL(2,R)$ isometry subgroup of the background NHEK geometry, but still contains a $U(1)$ isometry subgroup generated by

$$\zeta_\epsilon = \epsilon(\phi)\partial_\phi - r\epsilon'(\phi)\partial_r. \tag{4.15}$$

These diffeomorphism generators preserve the chosen boundary conditions and since $\epsilon(\phi)$ is arbitrary function and ϕ is periodic $\phi \sim \phi+2\pi$ we can mode expand $\epsilon_n(\phi) = -e^{-in\phi}$, where n is an integer. Thus we have an infinite number of generators generated by $\zeta(n) = \zeta(\epsilon_n)$, which under the Lie brackets generate the Virasoro algebra

$$i[\zeta_m,\zeta_n]_{L.B} = (m-n)\zeta_{m+n}. \tag{4.16}$$

So as we see the appropriate choice of boundary conditions on the asymptotic behaviour of the metric allowed to enhance the $U(1)$ symmetry of the $SL(2,R)_R$ x $U(1)_L$ isometry group into a Virasoro algebra without central charge.

Finally the NHEK metric (4.7) transforms under (4.15) as

$$\delta_\epsilon d\bar{s}^2 = 4JG\Omega^2\left(r^2(1-\Lambda^2)\partial_\phi\epsilon d\tau^2 - \frac{r\partial_\phi^2\epsilon}{1+r^2}d\phi dr + \Lambda^2\partial_\phi\epsilon d\phi^2 - \frac{\partial_\phi\epsilon}{(1+r^2)^2}dr^2\right). \tag{4.17}$$

A detailed calculation can be found in the Appendix E. We should remind that one can calculate such transformations (infinitesimal changes on the metric) by acting with the Lie derivative $\mathcal{L}_\xi$ [8] on the corresponding metric by using

$$\mathcal{L}_\xi g_{\mu\nu}(x) = \xi^\lambda\partial_\lambda g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu\xi^\alpha + g_{\mu\beta}(x)\partial_\nu\xi^\beta = 0, \tag{4.18}$$

a relation known as the *Killing equation*. Or similarly

$$\mathcal{L}_\xi g_{\mu\nu}(x) = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0, \tag{4.19}$$

[8] the Lie derivative, $\mathcal{L}_\xi$, generalizes the simple derivative. It is a generalization of the differentiation procedure in curved space. The Lie derivative in flat space is trivial and reduces to the simple derivative. In the case where we hit a metric with a Lie derivative and we get zero, $\mathcal{L}_\xi g_{\mu\nu} = 0$, then the vector is a *Killing vector*, while in any other case is just a vector.

where ∇_μ is the covariant derivative and ξ is the vector. In our case the vector (4.15), is

$$\xi = \begin{pmatrix} 0 \\ -r\epsilon'(\phi) \\ 0 \\ \epsilon(\phi) \end{pmatrix}. \tag{4.20}$$

The next step, is to obtain asymptotic charges associated with asymptotic symmetries, and then a Dirac algebra for these asymptotic charges. This is the subject of our next section.

4.3 Central Charge in The Virasoro Algebra

In this section we are going to calculate the central charge using the formalism developed in [112], [113] which covariantize the calculation done by Brown and Henneaux [19]. To such an asymptotic symmetry generator ζ, one can associate the conserved charge

$$\delta Q_\zeta[g] = \frac{1}{8\pi G}\int_{\partial\Sigma} k_\zeta[h,g], \tag{4.21}$$

where the integral is over the boundary of a spatial slice $\partial\Sigma$ that extends to infinity and the two-form k_ζ is defined for a perturbation $h_{\mu\nu}$ around the background metric $g_{\mu\nu}$ by

$$\begin{aligned} k_\zeta[h,g] = \ & - \ \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}[\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta\sigma D^\nu h^{\mu\sigma} + \frac{1}{2}hD^\nu\zeta^\mu \\ & - \ h^{\nu\sigma}D_\sigma\zeta^\mu + \frac{1}{2}h^{\mu\nu}(D^\mu\zeta_\sigma + D_\sigma\zeta^\mu)]dx^a \wedge dx^\beta \end{aligned} \tag{4.22}$$

where D_μ is a covariant derivative on the background geometry, $g = \det g_{\mu\nu}$ and $h = g^{\mu\nu}h_{\mu\nu.}$ The Dirac bracket algebra of the asymptotic symmetry group is computed by varying the charges and includes a central term.

$$\{Q_{\zeta_m}, Q_{\zeta_n}\}_{D.B} = Q_{[\zeta_m,\zeta_n]} + \frac{1}{8\pi G}\int_{\partial\Sigma} k_{\zeta_m}[\mathcal{L}_{\zeta_n}\bar{g},\bar{g}], \tag{4.23}$$

where $\mathcal{L}_\zeta$ is the Lie derivative of $g_{\mu\nu}$ with respect to ζ. The finiteness of Q_n is being ensured by the boundary conditions (4.12). All trivial asymptotic symmetries are associated with vanishing charges as it should. The central extensions are non-trivial because they cannot be absorbed into the normalizations of the generators.

Having the near-horizon metric (4.7) and the Asymptotic Group Generators (ASG) (4.15) we compute

$$\mathcal{L}_{\zeta_n}\bar{g}_{\tau\tau} = 4GJ\Omega^2(1-\Lambda^2)r^2 ine^{-in\phi} \tag{4.24}$$

$$\mathcal{L}_{\zeta_n}\bar{g}_{r\phi} = -\frac{2GJ\Omega^2 r}{1+r^2}n^2e^{-in\phi} \tag{4.25}$$

$$\mathcal{L}_{\zeta_n}\bar{g}_{\phi\phi} = 4GJ\Lambda^2\Omega^2 ine^{-in\phi} \tag{4.26}$$

$$\mathcal{L}_{\zeta_n}\bar{g}_{rr} = -\frac{4GJ\Omega^2}{(1+r^2)^2}ine^{-in\phi}. \tag{4.27}$$

Substituting back to (4.22) it follows from (4.21) that

$$\frac{1}{8\pi G}\int_{\partial\Sigma} k_{\zeta_m}[\mathcal{L}_{\zeta_n}\bar{g},\bar{g}] = -iJ(m^3+2m)\delta_{m+n}. \tag{4.28}$$

By defining the dimensionless quantum versions of the conserved charges by

$$\hbar L_n \equiv Q_{\zeta_n} + \frac{3J}{2}\delta_n \tag{4.29}$$

and replacing the Dirac brackets by commutators as $\{.,.\}_{D.B} \to -\frac{1}{\hbar}[.,.]$. The quantum charge algebra is then

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{J}{\hbar}m(m^2-1)\delta_{m+n}. \tag{4.30}$$

From this we can read off the central charge for extreme Kerr

$$c_L = \frac{12J}{\hbar}. \tag{4.31}$$

One can see that c depends on J, which implies that for each value of J we get a different conformal field theory. This may not be important for extremal rotating black holes, which have a vanishing Hawking temperature and do not undergo Hawking radiation. But considering as for instance, in [48] which we are going to discuss in the next section —that a single conformal field theory also describes nonextremal black holes with the same angular momentum. Then Hawking radiation, which will typically change the value of J, will take us from one conformal dual to another. This suggests that a full treatment of Hawking radiation, including its back-reaction on the black hole, may involve flows between conformal field theories.

4.4 Conformal Temperature

In order to be able to use the Cardy formula to calculate the statistical entropy we need to assign a temperature to the extremal Kerr black hole. As has already been stated the temperature for an extremal Kerr black hole is zero $T_H = 0$. However this is not the relevant temperature for our conformal algebra and we need to adapt the ideas of Frolov and Thorne [114, 115, 116], in order to define a quantum theory in the extremal black hole geometry, and to associate a non-zero temperature with the vacuum state. Now near the horizon, the analogue of the Hartle-Hawking vacuum is the Frolov-Thorne vacuum. For a scalar field we can mode expand and write

$$\Phi = e^{-i\omega\hat{t}+im\hat{\phi}}f(r,\theta), \tag{4.32}$$

where ω and m stand for the asymptotic energy and angular momentum respectively. The vacuum is given by a density matrix with Boltzmann weighting factor

$$e^{-(\omega-m\Omega_H)/T_H}. \tag{4.33}$$

We can notice that by setting $\Omega_H = 0$ we recover the Hartle-Hawking vacuum.

However this expression for the density matrix is not suitable since $T_H = 0$. The problem can be solved by re-writing the above in terms of NHEK coordinates, defining the frequencies n_L, n_R and temperatures T_L, T_R and taking the extremal limit (in which $T_H \to 0$) after the procedure. Using the relation between Boyer-Lindquist coordinates and NHEK coordinates

$$t = \frac{\lambda \hat{t}}{2M} \quad \text{and} \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \tag{4.34}$$

we can write

$$e^{-i\omega\hat{t}+im\hat{\phi}} = e^{-\frac{i}{\lambda}(2M\omega-m)t+im\phi} = e^{-in_R+in_L\phi}, \tag{4.35}$$

where we set

$$n_L \equiv m, \quad \text{and} \quad n_R \equiv \frac{1}{\lambda}(2M\omega - m). \tag{4.36}$$

In terms of these variables the Boltzmann factor (4.33) is

$$e^{-(\omega-m\Omega_H)/T_H} = e^{-\frac{n_L}{T_L}-\frac{n_R}{T_R}}, \tag{4.37}$$

where the dimensionless left and right temperatures are

$$T_L = \frac{r_+ - M}{2\pi(r_+ - \alpha)}, \quad T_R = \frac{r_+ - M}{2\pi\lambda r_+}. \tag{4.38}$$

By looking at the temperature in the extremal limit $M^2 = GJ$ the above becomes

$$T_L = \frac{1}{2\pi}, \quad T_R = 0. \tag{4.39}$$

As we see it turns out that the dual CFT has a nonvanishing left temperature $T_L = 1/2\pi$. The right temperature measures deviations from extremality which justifies why $T_R = 0$ at the extreme limit.

4.5 Bekenstein-Hawking entropy and Cardy formula for extremal Kerr

Having found the temperature of the extremal Kerr black hole

$$T_L = \frac{1}{2\pi} \tag{4.40}$$

and shown that the central charge of the CFT is

$$c_L = \frac{12J}{\hbar}, \tag{4.41}$$

we have all the ingredients we need to compute the statistical entropy by using the Cardy formula

$$S_{\text{cardy}} = \frac{\pi^2}{3} c_L T_L. \tag{4.42}$$

The relation (4.42) is not exactly of the form of (2.58) we computed at subsection 2.5, we show how they are connected in the Appendix F.

By plugging in (4.40) and (4.41) we acquire the microscopic entropy for the extreme Kerr black hole

$$S_{\text{cardy}} = \frac{2\pi J}{\hbar}, \tag{4.43}$$

this exactly matches with the Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{\text{Area}}{4G} = \frac{2\pi J}{\hbar} = \frac{\pi^2}{3} c_L T_L = S_{\text{CFT}} \tag{4.44}$$

As we see in the above, the main goal, which was to compute the black hole entropy by counting microstates has been accomplished. The Kerr/CFT correspondence, has been expanded to many other black holes, which for all the microscopic entropy was successfully compute. While the details may differ a bit, most derivations of black hole/CFT duality are based on finding an appropriate boundary and imposing boundary conditions that specify properties of the black hole, then determining how these boundary conditions affect the symmetries of general relativity meaning the algebra of diffeomorphisms, subsequently look for a preferred subalgebra of diffeomorphism and finally use standard methods from conformal field theory.

Although this method is quite successful it actually tell us nothing about quantum gravitational states at all, which make us to conjecture, that the approach is not yet complete, but in its own domain of validity seems to work quite well. Lastly we should note that since the central charge is fixed by the angular momentum, each extremal Kerr black hole corresponds to an unique CFT, thus the extremal Kerr solution should be regarded as the vacuum state of the CFT.

5 Hidden Conformal Symmetry

In this section we are going to calculate the microscopic entropy for a nonextremal Kerr black hole via a different approach which makes use of the existence of a hidden conformal symmetry which is essential to set up a CFT dual to nonextremal Kerr black hole.

5.1 Hidden conformal symmetry of the Kerr B.H

In the Kerr/CFT correspondence we saw that an extreme Kerr black hole in the near horizon limit is holographically dual to a two-dimensional conformal field theory. We would like to show in this section that the same correspondence can be obtained away from the extreme Kerr limit. Away from extremality though the near horizon limit is now just Rindler space thus we do not get the necessary AdS_3 geometry which will allow us to set up the mechanism (the tools) of conformal symmetry (conformal invariance) to compute the statistical entropy of the Kerr black hole as we did in section 4.

Another approach was proposed by Castro, Maloney and Strominger in [48]. The conformal invariance is being obtained by looking at the near region of the Kerr black hole and how a scalar field is propagating in that region which is giving us the conformal (symmetry) invariance. That is why the term *hidden symmetry* is used. The conformal symmetry does not show up from the beginning at the Kerr metric but actually is being recovered by looking at the equation of motion of a scalar field in a Kerr background where the equation is separable and by taking the near region limit we see the existence of the conformal symmetry.
Let us see how this is achieved in steps:

• For the Kerr metric we use (3.4) where

$$\Delta \equiv r^2 - 2Mr + \alpha^2, \quad \rho^2 \equiv r^2 + \alpha^2 \cos^2 \theta.$$

The inner and outer horizons are located as already mentioned in section 3.3 at $r_{\pm} = M \pm \sqrt{M^2 - a^2}$

• The Klein-Gordon equation for a massless scalar field is given by:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0. \tag{5.1}$$

One can get to the above equation by taking the Lagrangian for the free massless scalar field and plugging it in the Euler-Lagrange equations of motion which in their turn can be obtained by variation of the action. We show this in appendix G.

Expanding the massless scalar field in eigenmodes we have:

$$\Phi(t, r, \theta, \phi) = e^{-i\omega t + im\phi} \Phi(r, \theta) \tag{5.2}$$

The reason why we choose to expand in these eigenmodes (t and ϕ) and not for example in r and or θ is connected with the fact that our Kerr metric possesses two Killing vectors, ∂_t and ∂_ϕ (no t or ϕ dependence).

• The next step is to plug (3.4) into (5.1) using (5.2) relation for the field Φ. After some computations and manipulations we can bring the result to a form looking like

$$\begin{aligned} \partial_r(\Delta \partial_r \Phi(r,\theta)) + \frac{(2Mr_+\omega - \alpha m)^2}{(r - r_+)(r_+ - r_-)} \Phi(r,\theta) - \frac{(2Mr_-\omega - \alpha m)^2}{(r - r_-)(r_+ - r_-)} \Phi(r,\theta) \\ +(r^2 + \alpha^2 cos^2\theta + 2M(r + 2M))\omega^2 \Phi(r,\theta) + \frac{1}{sin\theta} \partial_\theta (sin\theta \partial_\theta \Phi(r,\theta)) \\ - \frac{m^2}{sin^2\theta} \Phi(r,\theta) + a^2 cos^2\theta \omega^2 \Phi(r,\theta) = 0. \end{aligned} \tag{5.3}$$

The above equation is separable

$$\left[\frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) - \frac{m^2}{\sin^2\theta} + \omega^2 \alpha^2 \cos^2\theta \right] S(\theta) = -K_l S(\theta) \tag{5.4}$$

and

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - \alpha m)^2}{(r-r_+)(r_+-r_-)}\Phi - \frac{(2Mr_-\omega - \alpha m)^2}{(r-r_-)(r_+-r_-)} \right.$$
$$\left. +(r^2+2M(r+2M))\omega^2 \right] R(r) = K_l R(r) \tag{5.5}$$

The separation of variables is being done in two steps in equation (5.3). In step one we set: $\Phi(r,\theta) = R(r)S(\theta)$ and at step two we divide by: $R(r)S(\theta)$. The result is to take two separate equations, meaning one that depends only on variable r eq.(5.5) and one that depends only on variable θ eq.(5.4).

Both equations are solved by Heun functions which are not among the usual special functions and the separation constants K_l are eigenvalues on the sphere and are known only numerically. So actually we do not get something that we can work on from there. But looking at (5.4) we see that if we are able somehow to neglect the ω^2 term we see that we will have the Laplacian on a 2-sphere from which we will know what our separation constants are and then we can take a second look at the radial equation (5.5).

• The ω^2 term can be neglected by taking what we call the near region limit which is not the same as taking the near horizon limit (except in some particular cases i.e. an extremal kerr black hole).

Looking at (5.3) this can only happen if:

$$\omega M \ll 1,$$

meaning that the wavelength of the scalar excitation is large compared to the radius of the curvature and then study its behavior at the near region limit which is dictated by

$$r \ll \frac{1}{\omega}.$$

In that region the angular equation (5.4) is being reduced to the Laplacian on a 2-sphere:

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2}{\sin^2\theta}\right]S(\theta) = -K_l S(\theta), \quad l = -m, ..., m. \tag{5.6}$$

Where the m^2 term is really the coefficient that comes from our expanded field (6.20) upon differentiation over ϕ. The solutions $e^{im\phi}$ are spherical harmonics and the separation constants are

$$K_l = l(l+1).$$

It has to be noted here that for spherical harmonics in higher dimensions the eigenvalues follow, $K_l = -l(l+n-2)$. The radial wave equation (5.5) in that limit reduces to:

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - \alpha m)^2}{(r-r_+)(r_+-r_-)}\Phi - \frac{(2Mr_-\omega - \alpha m)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = K_l R(r), \tag{5.7}$$

which is solved by hypergeometric functions. As hypergeometric functions transform in representations of $SL(2,R)$, this suggests the existence of a hidden conformal symmetry.

What we need to stress here is that the conformal invariance (symmetry) is not coming from the spacetime geometry as for example in the case of [23], but rather emerges from the scalar field as it propagates in the near region of the Kerr black hole. Now the above equation can be rewritten as the quadratic Casimir of an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ algebra. This is being accomplished by introducing a set of "conformal coordinates"[9] and defining some vector fields as the generators of the algebra, which is the subject of the next section.

5.2 $\mathbf{SL}(2,\mathbb{R}) \times \mathbf{SL}(2,\mathbb{R})$

In this section, we will show that for the massless scalar particle, there exists a hidden $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ conformal symmetry acting on the solution space.

For this purpose it is convenient to adopt conformal coordinates

$$\begin{aligned} \omega^+ &= \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_R \phi} \\ \omega^- &= \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_L \phi + \frac{t}{2M}} \\ y &= \sqrt{\frac{r_+-r_-}{r-r_-}}e^{\pi(T_L+T_R)\phi + \frac{t}{4M}} \end{aligned} \tag{5.8}$$

where

$$T_R \equiv \frac{r_+-r_-}{4\pi\alpha}, \quad T_R \equiv \frac{r_++r_-}{4\pi\alpha}, \tag{5.9}$$

here r_+, r_- are the horizon of the Kerr black hole. We have to note here that the conformal coordinates would not work for an extremal Kerr black hole, because in that case $r_- = r_+$ and the coordinate $y = 0$ and not well defined. This issue of hidden conformal symmetry in extremal black holes is examined in [54].

Next with $(\omega^\pm, y)$ we define locally two sets of vector fields

$$H_1 = i\partial_+, \tag{5.10}$$

$$H_0 = i(\omega^+\partial_+ + \frac{1}{2}y\partial_y), \tag{5.11}$$

$$H_{-1} = i(\omega^{+2}\partial_+ + \omega^+ y\partial_y - y^2\partial_-) \tag{5.12}$$

and

$$\bar{H}_1 = i\partial_-, \tag{5.13}$$

$$\bar{H}_0 = i(\omega^-\partial_- + \frac{1}{2}y\partial_y), \tag{5.14}$$

$$\bar{H}_{-1} = i(\omega^{-2}\partial_- + \omega^- y\partial_y - y^2\partial_+) \tag{5.15}$$

[9]Conformal coordinates are used such us for example the metric can be written as $ds^2 = e^u(dx^2+dy^2)$.

These vector fields are constructed by brute force so as to satisfy the $SL(2,\mathbb{R})$ Lie Bracket algebra,

$$[H_0, H_\pm] = \mp i H_{\pm 1}, \quad [H_{-1}, H_1] = -2iH_0 \tag{5.16}$$

and similarly

$$[\bar{H}_0, \bar{H}_\pm] = \mp i \bar{H}_{\pm 1}, \quad [\bar{H}_{-1}, \bar{H}_1] = -2i\bar{H}_0. \tag{5.17}$$

The SL$(2,\mathbb{R})$ quadratic Casimir [10] is defined to be

$$\begin{aligned} \mathcal{H}^2 &= -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) \\ &= \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2 \partial_+ \partial_- \end{aligned} \tag{5.18}$$

and similarly for $\bar{\mathcal{H}}^2 = -\bar{H}_0^2 + \frac{1}{2}(\bar{H}_1 \bar{H}_{-1} + \bar{H}_{-1}\bar{H}_1)$.

In terms of (t, r, ϕ) coordinates, the vector fields are

$$\begin{aligned} H_1 &= ie^{-2\pi T_R \phi}\left(\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr - \alpha^2}{\Delta^{1/2}}\partial_t\right) \\ H_0 &= \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t \\ H_{-1} &= ie^{2\pi T_R \phi}\left(-\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr - \alpha^2}{\Delta^{1/2}}\partial_t\right) \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \bar{H}_1 &= ie^{-2\pi T_L \phi + \frac{t}{2M}}\left(\Delta^{1/2}\partial_r - \frac{\alpha}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right) \\ \bar{H}_0 &= -2iM\partial_t \\ \bar{H}_{-1} &= ie^{2\pi T_L \phi - \frac{t}{2M}}\left(-\Delta^{1/2}\partial_r - \frac{\alpha}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right). \end{aligned} \tag{5.20}$$

After computing the Casimir we find that

$$\mathcal{H}^2 = \partial_r \Delta \partial_r + \frac{(2Mr_+\partial_t + \alpha\partial_\phi)^2}{(r - r_+)(r_+ - r_-)} + \frac{(2Mr_-\partial_t + \alpha\partial_\phi)^2}{(r - r_-)(r_+ - r_-)}, \tag{5.21}$$

which matches the radial wave equation (5.5) and so we can write

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = l(l+1)\Phi. \tag{5.22}$$

So what is done here is that the scalar Laplacian has reduced to the SL$(2,\mathbb{R})$ Casimir. Therefore, the solution of the scalar field in the Kerr geometry in the near region forms representations of $SL(2,\mathbb{R})$. The SL$(2,\mathbb{R})_L \times$ SL$(2,\mathbb{R})_R$ conformal weights of the field Φ are

$$(h_L, h_R) = (l, l). \tag{5.23}$$

It has to noted here that these $SL(2,\mathbb{R})$ are not globally defined because they are broken under the periodic angular identification $\phi \sim \phi + 2\pi$. As a consequence these symmetries cannot be used to generate new global solutions from old ones.

[10] We define the quadratic Casimir in Appendix (H).

5.3 Kerr/CFT entropy

In order to be able to calculate the microscopic entropy of the Kerr black hole we make use of the Cardy formula which we assume it for the dual two dimensional conformal field theory. Knowing the values of the conformal temperatures T_L and T_R (5.9) and the central charges whose derivation has been completed for the extreme Kerr in [23, 26, 117] and are

$$c_L = c_R = 12J. \tag{5.24}$$

we can then plug in to the Cardy formula given by the relation

$$S_{cardy} = \frac{\pi^2}{3}(c_L T_L + c_R T_R). \tag{5.25}$$

so to obtain the microscopic entropy. So plugging (5.24) and (5.9)we get

$$S_{cardy} = \frac{\pi^2}{3}\left(\frac{6JM}{4\pi\alpha} + \frac{6J\sqrt{M^2-\alpha^2}}{4\pi\alpha}\right) \Longleftrightarrow S_{cardy} = 2\pi M r_+, \tag{5.26}$$

a result which exactly matches with the Bekenstein-Hawking entropy

$$S_{BH} = \frac{\text{Area}}{4} = \frac{8\pi M r_+}{4} \Longleftrightarrow S_{BH} = 2\pi M r_+. \tag{5.27}$$

Bare in mind that we have set $\hbar = 1$.

6 Hidden conformal symmetry of black rings

In this section we examine the five dimensional black rings. It would be quite interesting to see if a black ring/CFT correspondence is possible. We hope that by extending the Kerr/CFT will give a more complete picture of this black hole/CFT correspondence. We do that by following the hidden conformal symmetry approach originally introduced in [48] and discussed in section 5.

It happens that by limiting the black ring solution and following the hidden conformal symmetry method we again obtain a hypergeometric solution to the radial equation. This suggests a hidden conformal symmetry and it is our hope to show that this stands, by defining vector fields and computing the quadratic Casimir. Furthemore we take a look on the supersymmetric case of black ring examined in [118] where we calculate the Laplacian for a massless scalar field and perform a calculation of the quadratic Casimir by using the Killing vectors (we actually manipulate them a bit to serve our purpose) that the authors give. We are not discussing supersymmetric solutions here or give any information how to obtain them, we rather make a raw calculation on an already given metric. We show that the quadratic Casimir agrees with the radial equation, thus admitting a conformal symmetry.

6.1 What is a black ring

A black ring is a five-dimensional black hole with an event horizon of topology $S^1 \times S^2$. This is prevented in four-dimensions where the black hole's horizon can only have the topology of S^2 [105]. A black ring was obtained in general relativity as a solution to Einstein's fields equations by first doing a suitable ansatz by Emparan and Reall [74]. Heuristically a black ring can be the result of bending a black string into the shape of a circle and spinning it up to balance forces.

The metric for a black ring is given in [75] and is

$$\begin{aligned} ds^2 &= -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi\right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}F(x)\left[\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right] \end{aligned} \tag{6.1}$$

where

$$F(\xi) = 1 + \lambda\xi, \quad G(\xi) = (1-\xi^2)(1+\nu\xi) \tag{6.2}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}. \tag{6.3}$$

The coordinate system that is used is a somewhat unfamiliar, but in this system the black ring takes its simplest form. The construction can be found in [75].

Here λ and ν are dimensionless parameters and must lie in the range

$$0 < \nu \leq \lambda < 1, \tag{6.4}$$

ν characterizes the shape while λ the rotation velocity of the ring. For $\nu \to 0$ the ring becomes increasingly thin, for $\nu \to 1$ the ring flattens along the plane of rotation while for $\nu = 1$ the solution is same as for a naked singularity. In the case where both λ and ν vanish we recover flat spacetime (λ and ν are interconnected with relation (6.9) for regular solution as we will see below). R is the radius of the black ring and sets the scale for the solution, when this radius goes to zero the black ring reduces to the Myers-Perry black hole [98], while in the infinite radius limit it yields a boosted black string. The black ring has a curvature singularity at $y = 1/\lambda$, the regular event horizon is at $y = 1/\nu$ while the ergosphere is located at $y \pm \infty$. The solution is asymptotically flat with the spatial infinity being located at $x = -1$ and $y = -1$.

The above metric can be re-written in (r, θ) coordinates as

$$\begin{aligned} ds^2 &= -\frac{\hat{f}}{\hat{g}}\left(dt - r_H \sinh\sigma\cosh\sigma\sqrt{\frac{R + r_H\cosh^2\sigma}{R - r_H\cosh^2\sigma}}\frac{\frac{r}{R}-1}{r\hat{f}}Rd\psi\right)^2 \\ &\quad + \frac{\hat{g}}{(1+\frac{r\cos\theta}{R})^2}\left[\frac{f}{\hat{f}}\left(1-\frac{r^2}{R^2}\right)R^2d\psi^2 \right. \\ &\quad \left. + \frac{dr^2}{(1-\frac{r^2}{R^2})f} + \frac{r^2}{g}d\theta^2 + \frac{g}{\hat{g}}r^2\sin^2\theta d\phi^2\right] \end{aligned} \tag{6.5}$$

where

$$f = 1 - \frac{r_H}{r}, \quad \hat{f} = 1 - \frac{r_H \cosh^2 \sigma}{r} \tag{6.6}$$

and

$$g = 1 + \frac{r_H}{R} \cos\theta, \quad \hat{g} = 1 + \frac{r_H \cosh^2 \sigma}{R} \cos\theta. \tag{6.7}$$

Here σ is called the boost parameter and R is identified with the radius (S^1 radius) of the ring when the geometry described by the metric (6.5) is free of conical singularities. The curvature singularity lies at $r = 0$ while the regular event horizon is located at $r = r_H$ where r_H is identified with the S^2 radius. In the above there is a redefinition of parameters $(\nu, \lambda) \to (r_H, \sigma)$ as

$$\nu = \frac{r_H}{R}, \quad \lambda = \frac{r_H \cosh^2 \sigma}{R}. \tag{6.8}$$

Both metrics (6.1),(6.5) are not regular, they contain conical singularities. In order to make the metrics regular, in the first case we have to set the value

$$\lambda = \frac{2\nu}{1 + \nu^2}, \tag{6.9}$$

whereas in the second metric

$$|\sinh\sigma| \to 1, \tag{6.10}$$

or equivalently the velocity $|\upsilon| \to 1/\sqrt{2}$. By using the trigonometric relation $\cosh^2 \sigma - \sinh^2 \sigma = 1$, one can find that for the regular metric $\cosh^2 \sigma = 2$.

6.2 What is a black string

A neutral black string in five dimensions is constructed as the direct product of the Schwarzschild solution and a line, so the geometry of the horizon is $R \times S^2$ The metric of the Schwarzschild black hole solution in (3+1) dimensions is given by

$$ds^2 = -\left(1 - \frac{r_H}{r}\right) dt^2 + \frac{1}{1 - \frac{r_H}{r}} dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \tag{6.11}$$

where r_H stands for the horizon radius. We can go to the Schwarzschild black string (static black string) by adding a dimension in the above metric and have (4+1) dimensions, then we get

$$ds^2 = -\left(1 - \frac{r_H}{r}\right) dt^2 + \frac{1}{1 - \frac{r_H}{r}} dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2) + d\psi^2. \tag{6.12}$$

From here we can obtain the boosted black string solution by boosting (6.12) in the ψ-direction with the velocity $\upsilon = \tanh\xi$ with the coordinate transform

$$\xi = \begin{pmatrix} t' \\ \psi' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} t \\ \psi \end{pmatrix} \tag{6.13}$$

which will give us the metric for the boosted black string in five dimensions

$$
\begin{aligned}
ds^2 = &-\left(1-\cosh^2\sigma\frac{r_H}{r}\right)dt^2 + 2\frac{r_H}{r}\cosh\sigma\sinh\sigma dtd\psi \\
&+\left(1+\sinh^2\sigma\frac{r_H}{r}\right)d\psi^2 + \left(1-\frac{r_H}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2 d\phi^2).
\end{aligned}
\tag{6.14}
$$

A generalized version of (6.14) in D-dimensions is given by

$$
\begin{aligned}
ds^2 = &-\left(1-\cosh^2\sigma\frac{r_H^{d-3}}{r^{d-3}}\right)dt^2 + 2\frac{r_H^{d-3}}{r^{d-3}}\cosh\sigma\sinh\sigma dtd\psi \\
&+\left(1+\sinh^2\sigma\frac{r_H^{d-3}}{r^{d-3}}\right)d\psi^2 + \left(1-\frac{r_H^{d-3}}{r^{d-3}}\right)^{-1}dr^2 + r^2 d\Omega^2_{d-3}
\end{aligned}
\tag{6.15}
$$

which can be found in [119]. Here $D = d+1$ where D states the number of dimensions.

6.3 The black ring as a boosted black string

The rotating black ring becomes a boosted black string in the ultraspinning limit. At that limit the black ring becomes very thin. What we mean by that is to take the black ring metric as is given in (6.5) and impose some limits. Considering the large radius R limit

$$
r, r_H, r_H\cosh^2\sigma \ll R. \tag{6.16}
$$

The expressions we have for g and $\hat{g}$ (6.7) become,

$$
g \equiv 1, \quad \hat{g} \equiv 1 \tag{6.17}
$$

and by redefining $\psi \to \psi/R$, the metric (6.5) becomes

$$
\begin{aligned}
ds^2 = &-\hat{f}\left(dt - r_H\sinh\sigma\cosh\sigma\frac{(-1)}{r\hat{f}}d\psi\right)^2 \\
&+\frac{\hat{f}}{f}d\psi^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.
\end{aligned}
\tag{6.18}
$$

Which is the metric for a boosted black ring in the ψ-direction with boost parameter σ. What we actually do by taking (6.16) is to make the black ring really thin and then we zoom in ($r \ll R$) to the near horizon area. As a result we do not see any bending (curvature) anymore and because the ring is spinning the result is a black string with momenta, thus a boosted black string.

The metric (6.18) can be brought to a form that exactly matches[11] the metric in [120],

$$
ds^2 = -\hat{f}\left(dt - \frac{r_H\sinh 2\sigma}{2r\hat{f}}d\psi\right)^2 + \frac{\hat{f}}{f}d\psi^2 + \frac{1}{f}dr^2 + r^2 d\Omega_2^2. \tag{6.19}
$$

[11] In order to exactly match the 'signs' between the two metrics we set $\sigma = -\sigma$ on (6.18). That way the (sinh) changes sign but not the (cosh). It is perfectly fine to do that as it only changes the direction of the boost.

Actually the metric given in (6.14) is the same to the above (6.19), one can see that by using trigonometric identities. How these two are connected is shown in the Appendix I.

6.4 Hidden conformal symmetry for a boosted black string

In this section we are going to calculate the Laplacian of a massless scalar for a 5D boosted black string in the same spirit as done in section 5. As we have already seen the metric for a boosted black string can be obtained either by adding one dimension to the Schwarzschild metric (say $\hat{z}$) and boosting it along that direction or by taking certain limits on a 5D black ring.

The boosted black ring metric is given by (6.14) while the Klein-Gordon equation for a massless scalar field Φ is given by (5.1).

Expanding now our massless scalar field in eigenmodes we have

$$\Phi(t,r,\theta,\phi,\psi) = e^{-i\omega t+im\phi+i\beta\psi}\Phi(r,\theta). \tag{6.20}$$

Following the same steps as in section 5, we get the Laplacian,

$$\begin{aligned}
&\left[\frac{r^2(r-r_H+r_H\cosh^2\sigma)\omega^2}{r-r_H}+\frac{2r^2 r_H\cosh\sigma\sinh\sigma\beta\omega}{r-r_H}\right.\\
&\quad\left.+\frac{r^2(-r+r_H\cosh^2\sigma)\beta^2}{r-r_H}\right]\Phi(r,\theta)+(r-r_H)\partial_r\Phi(r,\theta)\\
&\quad+r\partial_r\Phi(r,\theta,\phi)+r(r-r_H)\partial_r^2\Phi(r,\theta)+\frac{\cos\theta}{\sin\theta}\partial_\theta\Phi(r,\theta)+\partial_\theta^2\Phi(r,\theta)\\
&\quad-\frac{m^2}{\sin^2\theta}\Phi(r,\theta)=0,
\end{aligned} \tag{6.21}$$

which can write slightly more compactly as

$$\begin{aligned}
&\left[\frac{r^2(r-r_H+r_H\cosh^2\sigma)\omega^2}{r-r_H}+\frac{2r^2 r_H\cosh\sigma\sinh\sigma\beta\omega}{r-r_H}\right.\\
&\quad\left.+\frac{r^2(-r+r_H\cosh^2\sigma)\beta^2}{r-r_H}\right]\Phi(r,\theta)+\partial_r(\Delta\partial_r\Phi(r,\theta))\\
&\quad+\nabla_{S^2}\Phi(r,\theta)=0,
\end{aligned} \tag{6.22}$$

where $\Delta = r^2 - rr_H$ and ∇_{S^2} is the Laplacian on a 2-sphere. Equation (6.22) is separable and by using the separation of variables method, can be separated into a radial and an angular part $\Phi(r,\theta) = R(r)S(\theta)$,

$$\begin{aligned}
&\left[\frac{r^2(r-r_H+r_H\cosh^2\sigma)\omega^2}{r-r_H}+\frac{2r^2 r_H\cosh\sigma\sinh\sigma\beta\omega}{r-r_H}\right.\\
&\quad+\frac{r^2(-r+r_H\cosh^2\sigma)\beta^2}{r-r_H}+(r-r_H)\partial_r\\
&\quad\left.+r\partial_r+r(r-r_H)\partial_r^2\right]R(r)=K_l R(r)
\end{aligned} \tag{6.23}$$

and

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2}{\sin^2\theta}\right]S(\theta) = -K_l S(\theta). \tag{6.24}$$

The separation constants are $K_l = l(l+1)$. The radial equation is being solved by Heun functions.

6.4.1 The near region solution

Now we will look in the near region of the boosted black string which is achieved by taking $r - r_H \ll \frac{1}{\omega}$. In this computation we will use the radial equation (18) from [120] to take the limit as we want to be consistent with that paper at this point, which comes from computing the Laplacian for a massless scalar field on (6.19).

We assume $\omega r_H \ll 1$ and define a new variable

$$z = 1 - \frac{r_H}{r}. \tag{6.25}$$

The radial wave equation (6.24) is written as

$$z(1-z)\partial_z^2\Phi + (1-z)\partial_z\Phi + \left[-\frac{l(l+1)}{1-z} + \frac{1-z}{z}Y^2\right]\Phi = 0, \tag{6.26}$$

where $Y \equiv r_H \cosh\sigma(\omega - \beta\tanh\sigma)$. The above equation is being found in [120]. If we instead use equation (6.23) to take the limit we end up with

$$z(1-z)\partial_z^2\Phi + (1-z)\partial_z\Phi + \left[-\frac{l(l+1)}{1-z} + \frac{1}{z}Y^2\right]\Phi = 0, \tag{6.27}$$

Both equations are solved by hypergeometric functions. It is rather interesting that we get two slightly different radial equations, it poses some questions as, is one radial more correct than the other? We are going to discuss a bit about this in the conclusion section 8.

Let us continue this section by using (6.26). As hypergeometric functions transform in representations of $SL(2,\mathbb{R})$, this suggests the existence of a hidden conformal symmetry, which we will examine below. The whole concept is to try to find three vector fields that will obey the $SL(2,\mathbb{R})$ Lie bracket algebra (5.16) of which the quadratic Casimir given by (5.18) will agree with the Laplacian (6.26).

In order to achieve this we first multiply (6.26) with, $(1-z)$, so as

$$z(1-z)^2\partial_z^2\Phi + (1-z)^2\partial_z\Phi + \left(\frac{(1-z)^2}{z}Y^2\right)\Phi = l(l+1)\Phi. \tag{6.28}$$

In order to manufacture our vector fields we try to simplify a bit the term $z(1-z)^2\partial_z^2\Phi + (1-z)^2\partial_z\Phi$ and thus we try to define a new variable. To do that we do the following, first we set

$$z(1-z)^2\partial_z^2\Phi + (1-z)^2\partial_z\Phi = \partial_y(\Delta(y)\partial_y)\Phi \tag{6.29}$$

where

$$\partial_y = \frac{\partial z}{\partial y}\frac{\partial}{\partial z}$$

and

$$\partial_y^2 = \partial_y \partial_y = \partial_y \left(\frac{\partial z}{\partial y}\frac{\partial}{\partial z}\right) = \frac{\partial^2 z}{\partial y^2}\partial_z + \left(\frac{\partial z}{\partial y}\right)^2 \partial_z^2.$$

Working the right side of (6.29) we get,

$$\left[\Delta' \frac{\partial z}{\partial y} + \Delta \frac{\partial^2 z}{\partial y^2}\right]\partial_z + \Delta \left(\frac{\partial z}{\partial y}\right)^2 \partial_z^2,$$

we equalize with (6.29) and we get a set of equations

$$\Delta \frac{\partial^2 z}{\partial y^2} + \Delta' \frac{\partial z}{\partial y} = (1-z)^2$$

$$\Delta \left(\frac{\partial z}{\partial y}\right)^2 = z(1-z)^2.$$

Solving this system, we obtain

$$z = \frac{-1 + C_1 y + C_2}{C_1 y + C_2}, \quad \Delta = \frac{z - 2z^2 + z^3}{(\partial_y z)^2},$$

where C_1 and C_2 are coefficients which we set $C_1 = 1$ and $C_2 = 0$ and by replacing we get

$$z = 1 - \frac{1}{y}, \quad \Delta = y^2 - y. \tag{6.30}$$

Thus equation (6.28) can be written now as

$$\left[\partial_y(\Delta \partial_y) + \frac{1}{\Delta} r_H^2 \cosh^2\sigma(\omega - \beta \tanh\sigma)^2\right]\Phi = l(l+1)\Phi, \tag{6.31}$$

where $\Delta = y^2 - y$. A verification that the above coordinate transformation is correct, can be done by working backwards, by using $z = 1 - \frac{1}{y}$ and $\partial_y = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} = \frac{1}{y^2}\partial_z$. The form of (6.31) is more convenient, in order for us to construct the vector fields. This is the subject of the next section.

6.4.2 Constructing the $SL(2,\mathbb{R})$

In this section we describe the $SL(2,\mathbb{R})$ symmetry of the boosted black string.

In terms of (y, ψ, t) we define the vector fields

$$\begin{aligned}
H_1 &= ie^{-\frac{1}{2}\frac{t}{r_H \cosh\sigma}}\left(\Delta^{1/2}\partial_y + \frac{(1-2y)r_H \sinh\sigma}{\Delta^{1/2}}\partial_\psi + \frac{(2y-1)r_H\cosh\sigma}{\Delta^{1/2}}\partial_t\right) \\
H_0 &= -2ir_H \sinh\sigma \partial_\psi + 2ir_H \cosh\sigma \partial_t \qquad (6.32) \\
H_{-1} &= ie^{\frac{1}{2}\frac{t}{r_H \cosh\sigma}}\left(-\Delta^{1/2}\partial_y + \frac{(1-2y)r_H \sinh\sigma}{\Delta^{1/2}}\partial_\psi + \frac{(2y-1)r_H\cosh\sigma}{\Delta^{1/2}}\partial_t\right)
\end{aligned}$$

These obey the $SL(2,\mathbb{R})$ Lie bracket algebra (5.16). The $SL(2,\mathbb{R})$ quadratic casimir is given by (5.18).

Computing it become

$$\mathcal{H} = \partial_y \Delta \partial_y - \frac{1}{\Delta} r_H^2 \cosh^2 \sigma (\partial_t - \tanh \sigma \partial_\psi)^2. \tag{6.33}$$

The scalar wave equation (6.31) can be written as

$$\mathcal{H}^2 \Phi = l(l+1)\Phi. \tag{6.34}$$

From the periodicity of the euclideanized time ($t \to i\tau$) direction $\tau \to \tau + \beta$ where $\beta = 1/T$ we can read the temperature $T = (4\pi r_H \cosh \sigma)^{-1}$ which agrees with the boosted black string temperature given in [120].

Here we expect to have one sector of the $SL(2,\mathbb{R})$ similarly to the Schwarzschild case [49]. At this point it would also be interesting to see if we can construct the vector fields corresponding to equation (6.27). We believe it is possible and will probably present it in future work.

6.5 Near the horizon of a supersymmetric black ring

Taking the metric of a supersymmetric black ring in the near horizon limit as given by [118], we try to evaluate the Laplacian for a free massless scalar field in the same spirit as in [48].

The near horizon limit of the black ring solution is:

$$ds^2 = -p^2 \left(\frac{dr^2}{4r^2} + \frac{L^2}{p^2} d\psi^2 + \frac{Lr}{p} dt d\psi \right) - \frac{p^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2), \tag{6.35}$$

which is the product of a locally AdS_3 with radius p and an S^2 with radius $\frac{p}{2}$. The range of coordinates are

$$0 \le r \le \infty, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi, \qquad 0 \le \psi \le 2\pi. \tag{6.36}$$

The Klein-Gordon equation for a massless scalar field Φ is given by (5.1). Expanding the massless scalar field in eigenmodes we have

$$\Phi(t, r, \theta, \psi, \phi) = e^{-i\omega t + im\phi + i\beta\psi} \Phi(r, \theta).$$

Calculating the above we get an equation looking like this

$$\begin{aligned} &\frac{\omega^2}{r^2}\Phi(r,\theta) + \frac{p\beta\omega}{rL}\Phi(r,\theta) + 2r\partial_r\Phi(r,\theta) + r^2\partial_r^2\Phi(r,\theta) \\ &\qquad + \frac{\cos\theta}{\sin\theta}\partial_\theta\Phi(r,\theta) + \partial_\theta^2 - \frac{m^2}{\sin^2\theta}\Phi(r,\theta) = 0, \end{aligned}$$

which can re-write a bit more nicely as

$$\frac{\omega^2}{r^2}\Phi(r,\theta) + \frac{p\beta\omega}{rL}\Phi(r,\theta) + \partial_r(\Delta\partial_r\Phi(r,\theta)) + \nabla_{S^2}\Phi(r,\theta) = 0,$$

where $\Delta = r^2$ and ∇_{S_2} is the Laplacian on a 2-sphere. The above equation can be separated into a radial part and an angular part, by using the *separation of variables* [12]method. By doing the necessary manipulations we get two equations, a radial and an angular part,

$$\frac{\omega^2}{r^2}R(r) + \frac{p\beta\omega}{rL}R(r) + \partial_r(\Delta\partial_r R(r)) = K_l R(r) \tag{6.37}$$

and

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S(\theta)) - \frac{m^2}{\sin^2\theta}S(\theta) = -K_l S(\theta) \tag{6.38}$$

The solutions are spherical harmonics and the separation constants are $K_l = l(l+1)$. The radial equation is solved by Whittaker functions, which are a modified form of the confluent hypergeometric functions to make the formulas involving the solutions more symmetric. This makes good sense as we are in the supersymmetric case of black ring. Would like to make two comments as opposed to [48], in that case was used the Kerr metric and after computing the Laplacian, the near-region limit was taken from where the hypergeometic functions emerged which led us to the conjecture of an $SL(2,\mathbb{R})$. In this case is not needed to take the near-region limit as from start we took the near horizon limit of the supersymetric black ring. Furthermore we find Whittaker functions which can be explained by the fact that there is more symmetry now. The question here is does such a function admits an $SL(2,\mathbb{R})$ symmetry?

Let us find out by computing the commutation relations of the Killing vectors given in [118] and the Casimir operator. The commutation relations of which may show us if we are on the right track. These Killing vectors are the result of the $AdS_2 \times S^1$ geometry which appears in the near horizon limit of the five dimensional black ring solution of $N = 2$ supergravity. Recall that the Killing vectors can be found by using the Killing equation (4.18) which should give zero as a result $\mathcal{L}_\xi g_{\mu\nu}(x) = 0$.

The vectors are

$$\begin{aligned} H_1 &= i\sqrt{2}\partial_t \\ H_0 &= it\partial_t - ir\partial_r \\ H_{-1} &= \frac{i\sqrt{2}}{2}\left((t^2 + r^{-2})\partial_t - rt\partial_r - \frac{p}{2Lr}\partial_\psi\right) \end{aligned} \tag{6.39}$$

and

$$\begin{aligned} \bar{H}_1 &= ie^{-\frac{2L}{p}\psi}\left(\frac{1}{r}\partial_t - r\partial_r - \frac{p}{2L}\partial_\psi\right) \\ \bar{H}_0 &= i\frac{p}{2L}\partial_\psi \\ \bar{H}_{-1} &= ie^{\frac{2L}{p}\psi}\left(\frac{1}{r}\partial_t + r\partial_r - \frac{p}{2L}\partial_\psi\right). \end{aligned} \tag{6.40}$$

[12]A more detailed view of this method can be found in section 5.

The commutation relations are satisfying

$$[H_0, H_1] = -iH_1, \qquad [H_0, H_{-1}] = iH_{-1}, \qquad [H_{-1}, H_1] = -2iH_0$$

and similarly for $(\bar{H}_0, \bar{H}_1, \bar{H}_{-1})$, which is an $SL(2,\mathbb{R})$ Lie bracket algebra.

The quadratic Casimir of which is given by the relation (5.18) After computing it we find

$$\mathcal{H} = -\frac{\partial_t^2}{r^2} + \frac{p\partial_\psi \partial_t}{rL} + \partial_r(\Delta\partial_r), \tag{6.41}$$

which matches the radial equation (6.37). Which now can be written as

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = l(l+1)\Phi. \tag{6.42}$$

It has to be mentioned that some minor changes were needed in order to get the correct result in respect with the original Killing vectors [118]. In (6.39) an 'i' was added to each vector and a normalization constant $\sqrt{2}$ to H_1 and H_{-1} while in (6.40) we again added an 'i' to each vector plus a term $\frac{p}{2L}$ to $\bar{H}_0$.

Since we are dealing with a sypersymmetric black hole which demands for the black hole to be extremal, the dual CFT is chiral in the sense that only one sector has nonvanishing temperature and the other sector is completely cooled down without excitation. Furthermore the generators are globally defined under the angular identification $\psi \sim \psi + 2\pi$, something which was not true in the hidden Kerr 5 case which under the angular identification $\phi \sim \phi + 2\pi$ breaks $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$.

We can go on to calculate the statistical entropy now by using the Cardy formula (4.42). The temperature can be read from the vector fields (6.40) and it is $T_L = L/\pi p$[13] and the central charge is given in [118] and it is $c = 6p^3$. Thus the statistical entropy

$$S_{\text{mic}} = 2\pi L p^2 = S_{\text{mac}}, \tag{6.43}$$

which exactly matches the macroscopic entropy.

7 Higher Dimensional Boosted Black Strings

In this section we examine boosted black strings in higher dimensions. Since the 5D boosted black ring admits a conformal symmetry it is interesting to see if higher dimensional versions share this symmetry. We start by using the metric for a D-dimensional boosted black string along the ψ-direction, which is given in [119] and calculate the Laplacian of a scalar field for boosted black strings in 6,7 and 8 dimensions. The equations are separable and by taking the near region limit where we find solutions as hypergeometric functions. From this we conjecture that the same results would apply to any dimension and we construct generalized versions of our results. This is done in section 7.1 while the raw calculations can be seen in subsections 7.2, 7.3 and 7.4.

[13] In [118] the temperature is given by $T = 1/2\pi e_0$ where e_0 must take the value $e_0 = p/2L$ so to compute the correct entropy.

7.1 Boosted black string in D-dimensions—Generalizing the procedure

The metric for a boosted black string in any dimension is given by (6.15). The expression for the n-sphere metric can be found in the appendix J.

As we will show below a generalized expression for the radial equation can be written for the D-dimensional boosted black string for $D \geq 6$ and is

$$\begin{aligned}\Bigg[&\frac{r^2(r^{d-3}-r_H^{d-3}+r_H^{d-3}\cosh^2\sigma)\omega^2}{r^{d-3}-r_H^{d-3}}+\frac{2r^2r_H^{d-3}\cosh\sigma\sinh\sigma\beta\omega}{r^{d-3}-r_H^{d-3}}\\&+\frac{r^2(-r^{d-3}+r_H^{d-3}\cosh^2\sigma)\beta^2}{r^{d-3}-r_H^{d-3}}+\frac{r^{d-3}-r_H^{d-3}}{r^{d-4}}\partial_r+(d-3)r\partial_r\\&+\frac{(r^{d-3}-r_H^{d-3})}{r^{d-5}}\partial_r^2\Bigg]R(r)=K_lR(r).\end{aligned} \tag{7.1}$$

The above generalization follows from the pattern that the radial equations show in the below calculations for 6,7 and 8 dimensions and is rooted in the initial metric (6.15) we used.

7.1.1 The near region limit

The near region limit is recovered by focusing in the vinicity of the horizon, $r - r_H \ll 1/\omega$. We assume $\omega r_H \ll 1$ and define a new variable

$$z = 1 - \frac{r_H^{d-3}}{r^{d-3}} \tag{7.2}$$

The radial wave equation (7.1) is now written as

$$(D-4)^2 z(1-z)\partial_z^2\Psi + (D-4)^2(1-z)\partial_z\Psi + \left[-\frac{l(l+n-2)}{1-z} + \frac{1}{z}Y^2\right]\Psi = 0 \tag{7.3}$$

where we defined $Y \equiv r_H \cosh\sigma(\omega + \beta\tanh\sigma)$. Equation (7.3) is hypergeometric and its solutions are hypergeometric functions. As we have already stated this functions admit an $SL(2,\mathbb{R})$ symmetry group which suggests the existense of conformal symmetry. This is the starting point in the hidden conformal symmetry approach in order to compute the microscopic entropy for these objects. In the sections below we provide the detailed calculations that eventually led us to conjecture the generalized expressions (7.1) and (7.3).

7.2 Six-dimensional boosted black string

For $D = 6 \Longrightarrow d = 5$, so from (6.15) we have

$$\begin{aligned}ds^2 = &-\left(1-\cosh^2\sigma\frac{r_H^2}{r^2}\right)dt^2 + 2\frac{r_H^2}{r^2}\cosh\sigma\sinh\sigma dt d\psi\\&+\left(1+\sinh^2\sigma\frac{r_H^2}{r^2}\right)d\psi^2+\left(1-\frac{r_H^2}{r^2}\right)^{-1}dr^2+r^2d\Omega_3^2,\end{aligned} \tag{7.4}$$

where $d\Omega_3^2$ is the volume element on a 3-sphere. Plugging in the above to the expression for the equation of motion for a massless scalar field Φ (5.1) which we have expanded into its eigenmodes

$$\Phi(t,r,\theta,\phi,\zeta,\psi) = e^{-i\omega t + im\zeta + i\beta\psi}\Phi(r,\theta,\phi),$$

we come up with the expression,

$$\begin{aligned}
&\left[\frac{r^2(r^2 - r_H^2 + r_H^2\cosh^2\sigma)\omega^2}{r^2 - r_H^2} + \frac{2r^2 r_H^2 \cosh\sigma\sinh\sigma\beta\omega}{r^2 - r_H^2} + \right. \\
&\left. + \frac{r^2(-r^2 + r_H^2\cosh^2\sigma)\beta^2}{r^2 - r_H^2}\right]\Phi(r,\theta,\phi) + \frac{r^2 - r_H^2}{r}\partial_r\Phi(r,\theta,\phi) \\
&+ 2r\partial_r\Phi(r,\theta,\phi) + (r^2 - r_H^2)\partial_r^2\Phi(r,\theta,\phi) + \nabla_{S^3}\Phi(r,\theta,\phi) = 0,
\end{aligned} \tag{7.5}$$

where ∇_{S^3} is the Laplacian on a 3-sphere. Equation (7.5) is separable, and can be separated by using the separation of variables method into a radial and an angular part $\Phi(r,\theta,\phi) = R(r)S(\theta,\phi)$

$$\begin{aligned}
&\left[\frac{r^2(r^2 - r_H^2 + r_H^2\cosh^2\sigma)\omega^2}{r^2 - r_H^2} + \frac{2r^2 r_H^2\cosh\sigma\sinh\sigma\beta\omega}{r^2 - r_H^2}\right. \\
&+ \frac{r^2(-r^2 + r_H^2\cosh^2\sigma)\beta^2}{r^2 - r_H^2} + \frac{r^2 - r_H^2}{r}\partial_r + 2r\partial_r \\
&\left. + (r^2 - r_H^2)\partial_r^2\right]R(r) = K_l R(r)
\end{aligned} \tag{7.6}$$

and

$$\left[\frac{1}{\sin^2\theta}\partial_\theta(\sin^2\theta\partial_\theta) + \frac{1}{\sin^2\theta\sin\phi}\partial_\phi(\sin\phi\partial_\phi) - \frac{m^2}{\sin^2\theta\sin^2\phi}\right]S(\theta,\phi) = -K_l S(\theta,\phi). \tag{7.7}$$

The separation constants are $K_l = l(l+2)$. The radial equation is being solved by Heun functions as in the case of 5D black rings.

The near region limit is taken as in (7.1.1). We define $z = 1 - \frac{r_H^2}{r^2}$ and the radial equation (7.6) is reduced to

$$4z(1-z)\partial_z^2\Psi + 4(1-z)\partial_z\Psi + \left[-\frac{l(l+2)}{1-z} + \frac{1}{z}Y^2\right]\Psi = 0 \tag{7.8}$$

7.3 Seven-dimensional boosted black string

For $D = 7 \Longrightarrow d = 6$, so from (6.15) we have

$$\begin{aligned}
ds^2 = &-\left(1 - \cosh^2\sigma\frac{r_H^3}{r^3}\right)dt^2 + 2\frac{r_H^3}{r^3}\cosh\sigma\sinh\sigma dt d\psi + \\
&\left(1 + \sinh^2\sigma\frac{r_H^3}{r^3}\right)d\psi^2 + \left(1 - \frac{r_H^3}{r^3}\right)^{-1}dr^2 + r^2 d\Omega_4^2,
\end{aligned} \tag{7.9}$$

where $d\Omega_4^2$ is the metric on a 4-sphere. Plugging in the above to the expression for the equation of motion for a massless scalar field Φ (5.1), which we have expanded into its eigenmodes

$$\Phi(t,r,\theta,\phi,\zeta,\kappa,\psi) = e^{-i\omega t+im\kappa+i\beta\psi}\Phi(r,\theta,\phi,\zeta),$$

we get the equation

$$\begin{aligned}&\Bigg[\frac{r^2(r^3-r_H^3+r_H^3\cosh^2\sigma)\omega^2}{r^3-r_H^3}+\frac{2r^2r_H^3\cosh\sigma\sinh\sigma\beta\omega}{r^3-r_H^3}+\\ &\frac{r^2(-r^3+r_H^3\cosh^2\sigma)\beta^2}{r^3-r_H^3}\Bigg]\Phi+\frac{r^3-r_H^3}{r^2}\partial_r\Phi+3r\partial_r\Phi+\\ &\frac{(r^3-r_H^3)}{r}\partial_r^2+\nabla_{S^4}\Phi=0,\end{aligned}\tag{7.10}$$

where ∇_{S^4} is the Laplacian on a 4-sphere. Equation (7.10) is separable, and can be separated by using the separation of variables method into a radial and an angular part $\Phi(r,\theta,\phi,\zeta)=R(r)S(\theta,\phi,\zeta)$

$$\begin{aligned}&\Bigg[\frac{r^2(r^3-r_H^3+r_H^3\cosh^2\sigma)\omega^2}{r^3-r_H^3}+\frac{2r^2r_H^3\cosh\sigma\sinh\sigma\beta\omega}{r^3-r_H^3}\\ &+\frac{r^2(-r^3+r_H^3\cosh^2\sigma)\beta^2}{r^3-r_H^3}+\frac{r^3-r_H^3}{r^2}\partial_r+3r\partial_r\\ &+\frac{(r^3-r_H^3)}{r}\partial_r^2\Bigg]R(r)=K_lR(r)\end{aligned}\tag{7.11}$$

and

$$\begin{aligned}&\Bigg[\frac{1}{\sin^3\theta}\partial_\theta(\sin^3\theta\partial_\theta)+\frac{1}{\sin^2\theta\sin^2\phi}\partial_\phi(\sin^2\phi\partial_\phi)+\frac{1}{\sin^2\theta\sin^2\phi\sin\zeta}\partial_\zeta(\sin\zeta\partial_z)\\ &-\frac{m^2}{\sin^2\theta\sin^2\phi\sin^2\zeta}\Bigg]S(\theta,\phi,\zeta)=-K_lS(\theta,\phi,\zeta)\end{aligned}\tag{7.12}$$

The separation constants are $K_l=l(l+3)$.

The near region limit is taken as in (7.1.1). We define $z=1-\frac{r_H^3}{r^3}$ and the radial equation (7.11) is reduced to

$$9z(1-z)\partial_z^2\Psi+9(1-z)\partial_z\Psi+\left[-\frac{l(l+3)}{1-z}+\frac{1}{z}Y^2\right]\Psi=0\tag{7.13}$$

7.4 Eight-dimensional boosted black string

For $D=8\Longrightarrow d=7$, so from (6.15) we have

$$\begin{aligned}ds^2=&-\left(1-\cosh^2\sigma\frac{r_H^4}{r^4}\right)dt^2+2\frac{r_H^4}{r^4}\cosh\sigma\sinh\sigma dtd\psi\\ &+\left(1+\sinh^2\sigma\frac{r_H^4}{r^4}\right)d\psi^2+\left(1-\frac{r_H^4}{r^4}\right)^{-1}dr^2+r^2d\Omega_5^2,\end{aligned}\tag{7.14}$$

where $d\Omega_5^2$ is the metric on a 5-sphere. Plugging in the above to the expression for the equation of motion for a massless scalar field Φ (5.1), which we have expanded into its eigenmodes

$$\Phi(t,r,\theta,\phi,\zeta,\kappa,\eta,\psi) = e^{-i\omega t+im\eta+i\beta\psi}\Phi(r,\theta,\phi,\zeta,\kappa),$$

we get the equation

$$\begin{aligned}&\left[\frac{r^2(r^4-r_H^4+r_H^4\cosh^2\sigma)\omega^2}{r^4-r_H^4}+\frac{2r^2r_H^4\cosh\sigma\sinh\sigma\beta\omega}{r^4-r_H^4}\right.\\ &\left.+\frac{r^2(-r^4+r_H^4\cosh^2\sigma)\beta^2}{r^4-r_H^4}\right]\Phi+\frac{r^4-r_H^4}{r^3}\partial_r\Phi+4r\partial_r\Phi\\ &+\frac{(r^4-r_H^4)}{r^2}\partial_r^2+\nabla_{S^5}\Phi=0,\end{aligned} \tag{7.15}$$

where ∇_{S^5} is the Laplacian on a 5-sphere. Equation (7.15) is separable, and can be separated by using the separation of variables method into a radial and an angular part $\Phi(r,\theta,\phi,\zeta,\kappa) = R(r)S(\theta,\phi,\zeta,\kappa)$

$$\begin{aligned}&\left[\frac{r^2(r^4-r_H^4+r_H^4\cosh^2\sigma)\omega^2}{r^4-r_H^4}+\frac{2r^2r_H^4\cosh\sigma\sinh\sigma\beta\omega}{r^4-r_H^4}\right.\\ &+\frac{r^2(-r^4+r_H^4\cosh^2\sigma)\beta^2}{r^4-r_H^4}+\frac{r^4-r_H^4}{r^3}\partial_r+4r\partial_r\\ &\left.+\frac{(r^4-r_H^4)}{r^2}\partial_r^2\right]R(r)=K_lR(r)\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}&\left[\frac{1}{\sin^4\theta}\partial_\theta(\sin^4\theta\partial_\theta)+\frac{1}{\sin^2\theta\sin^3\phi}\partial_\phi(\sin^3\phi\partial_\phi)\right.\\ &+\frac{1}{\sin^2\theta\sin^2\phi\sin^2\zeta}\partial_\zeta(\sin^2\zeta\partial_z)+\frac{1}{\sin^2\theta\sin^2\phi\sin^2\zeta\sin\kappa}\partial_\kappa(\sin\kappa)\\ &\left.-\frac{m^2}{\sin^2\theta\sin^2\phi\sin^2\zeta\sin^2\kappa}\right]S(\theta,\phi,\zeta,\kappa)=-K_lS(\theta,\phi,\zeta,\kappa).\end{aligned} \tag{7.17}$$

The separation constants are $K_l = l(l+4)$.

The near region limit is taken as in (7.1.1). We define $z = 1 - \frac{r_H^4}{r^4}$ and the radial equation (7.16) is reduced to

$$16z(1-z)\partial_z^2\Psi+16(1-z)\partial_z\Psi+\left[-\frac{l(l+4)}{1-z}+\frac{1}{z}Y^2\right]\Psi=0 \tag{7.18}$$

8 Conclusions and Open Questions

Although gravity was the first force to be understood and formulated, it still keeps its secrets well hidden from the human mind. General relativity is viewed as an effective field theory which mainly follows from its lack of renormalizability and the existence of singularities. This suggests that a proper understanding of gravity requires the identification of the relevant degrees of freedom in the ultraviolet (UV). Quantum gravity though is a poorly understood subject. A way to motivate it is to appeal to the universal link between statistical mechanics and thermodynamics when studying the black hole-thermodynamics relation.

Thermodynamics is a branch of physics relatively independent of the microscopic details of the system under consideration. The birth of statistical mechanics, initiated with Boltzmann's work explaining the properties of macroscopic systems in thermal equilibrium in terms of the statistical averages of their microscopic degrees of freedom. In black hole physics, it has long been believed that the information loss about the true microscopic state of the system, responsible for the existence of entropy, is fully localised at the curvature singularity lying deep in the interior of the black hole. This expectation is challenged by the holographic principle. Information takes space, and for a black hole, it involves a classical scale, the horizon scale. This suggests that information about the state of the black hole, even if typically encoded in Planck scale physics, may be spread over macroscopic scales, such as the horizon scale, instead of being merely localised at the singularity. The description of the microscopics and semi classical methods have recently been developed for extremal and non extremal black holes in an attempt to understand more realistic black holes by explaining their macroscopic entropy given by the Bekenstein-Hawking formula

$$S_{BH} = \frac{A}{4\hbar G},$$

in terms of degrees of freedom on a two dimensional CFT living on the boundary of the black hole horizon, whose number of independent quantum states is controlled at large temperatures by the cardy formula

$$S_{CFT} = \frac{\pi^2}{3}(c_L T_L + c_R T_R).$$

The main purpose of this thesis was to examine these methods by reviewing the original papers [23] and [48] in section 4 and 5. Furthermore we found interesting to extend the procedure and check if it is also true for black rings in the limit $R \to \infty$ and in the supersymmetric black ring case. Furthermore we continued and conjectured that the boosted black ring admits conformal invariance in any dimension. Our thought was that maybe by contributing a bit to the Kerr/CFT [14] correspondence idea will eventually give us more information about the deeper mysteries of such a correspondence and understand it better. What we did not do in the above is to calculate the microscopical entropy which naturally would be the next step after showing like we did that a conformal invariance is present at the solution space of the massless scalar field on the background under consideration (except in the supersymmetric case of black ring where S_{CFT} was computed and matched S_{BH}).

[14] Actually would be more precise at present to rename Kerr/CFT correspondence into NHEK/CFT correspondence.

The assignment of entropy to a classical spacetime raises the question as for what the microscopic degrees of freedom responsible for the entropy are, for example what the analogue of the molecules in a gas is for spacetime!

Considering this is a great step that gives us more evidence that a formulation of quantum theory of gravity should exist, the Kerr/CFT correspondence has not addressed more information about the dual CFT in addition to the central charge. Therefore, new schemes should be developed to understand the properties of the CFT and dynamics behind this duality. We get no information of what is the CFT and what/where are the microstates, these are open questions that beg for an answer. Another question arises from the use of Cardy formula and how legitimate it is under these circumstances. In order to use the Cardy formula modular invariance and $T \gg c$ is required. In the Kerr/CFT case we assume the modular invariance part but there is a conflict arising from the second requirement since $\frac{1}{2\pi} \ll c$. But the Cardy formula works fine and thus there must exist a deep reason for it. An argument to the applicability of Cardy formula in the Kerr/CFT regime was tried in [121] where they showed that the comparison of the entropy of the extreme Kerr black hole and the entropy in the CFT can be understood within the Cardy regime by considering a D0-D6 system with the same entropic properties. That way they justified the calculation of the entropy and provided a string theory embedding of the Kerr/CFT correspondence. Considering the above one can ask how solid is this matching between gravity and the CFT (the extreme Kerr as a CFT). Several supporting evidence exist towards that direction. One comes from the extension of the procedure to other black holes as we have already mentioned in the introduction. Also it has been shown in a variety of contexts [122,123,124,125] that the Kerr scattering amplitudes which have been computed in [126,127,128,129,130] correspond exactly to the finite temperature CFT correlators.

Considering the hidden symmetry approach several questions are looking for an answer. Where did the hidden conformal symmetry come from and why does the Cardy formula work? Is there a generalization of the standard ASG analysis which can be applied to the $r \ll \frac{1}{\omega}$ near-region to explain the hidden conformal symmetry?

One of the fantastic features of the Kerr/CFT correspondence though, is that it applies to real world. Kerr black holes have been observed in the universe, data is being continuously collected and analyzed. The observational data may as well give us the push we need to complete and have a deeper understanding of the theoretical structure.

In the boosted black string case we showed the existence of a conformal symmetry. It would be interesting to continue and compute the statistical entropy by using the Cardy formula and see if it matches the macroscopic entropy, but the central charge is missing from the picture. It has not been calculated until now for this particular case. One way maybe to do this is if we knew how to define the stress tensor of the dual CFT in similar manner to that of the AdS/CFT correspondence [131]. In the case of the generalized expressions we presented in this thesis for the boosted black string in section 7, we believe that a construction of the Casimir as a next step is possible in each dimension and maybe a generalized version of the $SL(2,\mathbb{R})$ vectors in D-dimensions should also be possible.

Another very interesting question that came out from this thesis, is that we obtained two slightly different radial equations when took the near region limit of the boosted black string. That is from radial equation (18) from [120] we got (6.26) and from radial equation (6.23) we got (6.27). This is the outcome of the different terms ignored in each case. Both equations though are solved by hypergeometric functions. Which is really more 'correct' between the two? We think it is something that should be investigated further. Another case that we find that should be investigated is the neutral black ring, given by the metric (6.1) or in (r,θ) coordinates by (6.5). It will be the case of future work.

Lastly we would like to mention the universality problem. The microscopic origin of the Bekenstein-Hawking entropy of black holes is one of the most intriguing challenges for modern theoretical physics. Its solution is not only important for delivering a microscopic basis for black hole thermodynamics but it also represents one crucial test, that any quantum theory of gravity has to pass. It has been tackled using many different frameworks and approaches such as String theory, AdS/CFT correspondence, asymptotic symmetries, D-branes, induced gravity and entanglement entropy, loop quantum gravity. Many of these approaches reproduce the Bekestein-Hawking black hole entropy, some exactly, some up to some numerical constant in such a good way that this success is considered by some physicists almost as a problem [132]. This universality may suggest that some underlying feature of the classical theory may control the quantum density of states.

A On Holomorphic and anti-Holomorphic functions

A *holomorphic* function, is actually a synonym for analytic function. Is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of everypoint in its domain. The existence of complex derivative is a very strong condition, for it implies that any holomorphic function is actually infinite differentiable and equal to its own Taylor series.

Definition Let $f : \Omega \to \mathbb{C}$ and $a \in \Omega$. We say f is holomorphic at a if

$$\lim_{z\to a} \frac{f(z) - f(a)}{z - a}, \quad \text{exists.}$$

This is the same as the definition of the derivative for *real* functions, except that all of the quantities are complex. Since if limit exists they are unique, we can give this one the name $f'(a)$, called the derivative of f at a. If f is holomorphic at every point of Ω, we say it is holomorphic on Ω. In this case we denote the derivative by f'. If we wish to take several derivatives we denote by $f^{(k)}$, the k^{th} derivative of f. The *antiholomorphic* function can be seen as the reflection of a holomorphic function for instance,

Holomorphic	–	antiholomorphic
left moving	–	right moving
x-axis	–	reflection of x-axis

Lastly, if a function is both holomorphic and antiholomorphic, then it is constant on any connected component(space) of its domain.

B Energy-momentum Tensor for the Free Scalar Field

In this section we are going to show how the energy-momentum tensor for the 2D free scalar field is derived. The action for a free scalar field is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \partial^\alpha X = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g} g^{\alpha\beta}\partial_\alpha X \partial_\beta X \tag{B.1}$$

where $\frac{1}{4\pi\alpha'}$ is a normalization term which contains conventions originating from string theory, α' is the *Regge-slope parameter* related to the string tension and string length via $T_s = \frac{1}{2\pi\alpha'}$ and $l_s = \sqrt{\alpha'}$, finally $\sqrt{g} = \sqrt{detg}$ is added because we went into curved space.

Now by definition we have

$$\begin{aligned}
T_{\alpha\beta} &= -\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\alpha\beta}} = -\frac{4\pi}{\sqrt{g}}\frac{\partial}{\partial g^{\alpha\beta}}\left(\frac{1}{4\pi\alpha'}\sqrt{g}g^{\gamma\delta}\partial_\gamma X\partial_\delta X\right) \\
&= -\frac{4\pi}{\sqrt{g}}\frac{1}{4\pi\alpha'}\frac{\partial}{\partial g^{\alpha\beta}}\left(\sqrt{g}g^{\gamma\delta}\partial_\gamma X\partial_\delta X\right) = -\frac{1}{\sqrt{g}\alpha'}\frac{\partial}{\partial g^{\alpha\beta}}\left(\sqrt{g}g^{\gamma\delta}\partial_\gamma X\partial_\delta X\right) \\
&= -\frac{1}{\sqrt{g}\alpha'}\left(\frac{\partial\sqrt{g}}{\partial g^{\alpha\beta}}\right)g^{\gamma\delta}\partial_\gamma X\partial_\delta X - \frac{1}{\alpha'}\left(\frac{\partial g^{\gamma\delta}}{\partial g^{\alpha\beta}}\right)\partial_\gamma X\partial_\delta X \\
&= -\frac{1}{\sqrt{g}\alpha'}\left(-\frac{1}{2}\sqrt{g}g_{\alpha\beta}\right)\partial_\gamma X\partial^\gamma X - \frac{1}{\alpha'}\partial_\alpha X\partial_\beta X \\
&= \frac{1}{2\alpha'}g_{\alpha\beta}\partial_\gamma\partial^\gamma - \frac{1}{\alpha'}\partial_\alpha X\partial_\beta X = \\
&= -\frac{1}{\alpha'}\left(\partial_\alpha X\partial_\beta X - \frac{1}{2}\delta_{\alpha\beta}(\partial X)^2\right).
\end{aligned} \tag{B.2}$$

In the first line of the calculation we have changed some dummy indices so not to conflict with indices of the term $\left(\frac{\partial}{\partial g^{\alpha\beta}}\right)$. For the first parentheses in the third line we vary the formula $g = detg_{\mu\nu} = e^{tr\ln g_{\mu\nu}}$ which gives

$$\delta g = \delta(e^{tr\ln g_{\mu\nu}}) = e^{tr\ln g_{\mu\nu}} tr\delta\ln g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu}$$

so

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \quad \text{thus} \quad \delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} \Rightarrow \frac{\delta\sqrt{g}}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}$$

the minus sign $(-)$ is being picked up because we need to raise indices $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}$. For the second parentheses in the same line we use the relation $\frac{\partial g^{\gamma\delta}}{\partial g^{\alpha\beta}} = \frac{1}{2}(\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} + \delta^{\delta}_{\beta}\delta^{\gamma}_{\alpha})$.

The energy momentum tensor (B.2) takes a simpler form in complex coordinates

$$T = -\frac{1}{\alpha'}\partial X\partial X \quad \text{and} \quad \bar{T} = -\frac{1}{\alpha'}\bar{\partial}X\bar{\partial}X \tag{B.3}$$

The equation of motion for X is $\partial\bar{\partial}X = 0$. The general solution decomposes as,

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}). \tag{B.4}$$

When evaluated on this solution, T and $\bar{T}$ become holomorphic and anti-holomorphic functions respectively.

C Operator Ordering

In the Hamiltonian formalism, the transition from classical mechanics to quantum mechanics is achieved by promoting observables to operators which are not necessarily commuting. Consequently, the Hamiltonian of a classical system is supposed to go over the quantum operator

$$H(x,p) \to H(x_{op}, p_{op})$$

classically we know that

$$xp = px$$

Therefore the order of these operators does not matter. Quantum mechanically however these terms do not commute so the order of these operators is quite crucial and it is not clear what such a term ought to correspond in the quantum theory. This is the operator ordering problem and one perscription to deal with is the Normal Ordering or Wick Ordering in which in the case of x's and p's, orders the products such that the momenta stand to the left of the positions or using another example, one might think a product of quantum fields and equivalently their creation and annihilation operators, where all creation operators are placed to the left of the annihilation operators in the product.

We denote a Normal ordered operator by : :, for example

$$:\hat{\alpha}\hat{\alpha}^{\dagger}:=\hat{\alpha}^{\dagger}\hat{\alpha}$$

In quantum field theory normal ordering is the trick used in order to get rid of the infinite energy of a field's grounds energy (the expectation value of the fields ground state). Because of the fact that any normal ordered operator has a vacuum expectation value of zero

$$\langle 0| : \hat{O} : |0\rangle = 0 \tag{C.1}$$

D Conformal transformation of T — from the cylinder to the plane

In this section we provide a derivation of (2.32)

$$\begin{aligned}
-z^2 T_{plane} + \frac{c}{24} &= -\sum_{m=-\infty}^{\infty} L_m e^{imW} + \frac{c}{24} \Longleftrightarrow \\
T_{plane}(z) &= \frac{1}{z^2} \sum_{m=-\infty}^{\infty} L_m e^{imw} \Longleftrightarrow \\
T_{plane}(z) &= \frac{1}{e^{-2iw}} \sum_{m=-\infty}^{\infty} L_m e^{imw} \Longleftrightarrow \\
T_{plane}(z) &= \sum_{m=-\infty}^{\infty} \frac{1}{z^2} L_m z^{-m} \Longleftrightarrow T_{plane}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^2 z^m} \\
T_{plane}(z) &= \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \qquad \text{(D.1)}
\end{aligned}$$

Where we used $z = e^{-iw}$ in order to go from the 3rd to the 4th line, this relation is due to the conformal transformation from the cylinder to the plane.

E Transformation on NHEK Metric (4.7) Under (4.15)

For the $\tau\tau$ component of the metric, which will be the only one to write down explicitly we have

$$\begin{aligned}
\mathcal{L}_\xi g_{\tau\tau} &= \xi^r \partial_r g_{\tau\tau}(x) \\
&= -r\partial_\phi \epsilon(\phi) \partial_r \left[2GJ\Omega^2(-1 - r^2 + \Lambda^2 r^2)\right] \\
&= -r\partial_\phi \epsilon(\phi) \left[2GJ\Omega^2(-2r - 2\Lambda^2 r)\right] \\
&= 4GJ\Omega^2 \left[r^2(1 - \Lambda^2)\right] \partial_\phi \epsilon(\phi)
\end{aligned}$$

For the rr component of the metric we have

$$\begin{aligned}
\mathcal{L}_\xi g_{rr} &= \xi^r \partial_r g_{rr}(x) + g_{rr}(x) \partial_r \xi^r + g_{rr}(x) \partial_r \xi^r \\
&= 2GJ\Omega^2 \left(-r\partial_\phi \epsilon(\phi) \partial_r \left(\frac{1}{1+r^2}\right) + \frac{1}{1+r^2} \partial_r(-r\partial_\phi \epsilon(\phi)) + \frac{1}{1+r^2}(-r\partial_\phi \epsilon(\phi))\right) \\
&= 2GJ\Omega^2 \left(\frac{2r^2}{(1+r^2)^2} \partial_\phi \epsilon(\phi) - \frac{2}{1+r^2} \partial_\phi \epsilon(\phi)\right) \\
&= 4GJ\Omega^2 \left(\frac{r^2}{(1+r^2)^2} - \frac{1+r^2}{(1+r^2)^2}\right) \partial_\phi \epsilon(\phi) \\
&= -4GJ\Omega^2 \frac{\partial_\phi \epsilon(\phi)}{(1+r^2)^2}
\end{aligned}$$

For the $\phi\phi$ component of the metric we have

$$\begin{aligned}\mathcal{L}_\xi g_{\phi\phi} &= g_{\phi\phi}(x)\partial_\phi\xi^\phi + g_{\phi\phi}(x)\partial_\phi\xi^\phi \\ &= 2GJ\Omega^2\Lambda^2\partial_\phi\epsilon(\phi) + 2GJ\Omega^2\Lambda^2\partial_\phi\epsilon(\phi) \\ &= 4GJ\Omega^2\left(\Lambda^2\partial_\phi\epsilon(\phi)\right)\end{aligned}$$

For the ϕr component of the metric we have

$$\begin{aligned}\mathcal{L}_\xi g_{\phi r} &= g_{rr}\partial_\phi\xi^r \\ &= 2GJ\Omega^2\frac{1}{1+r^2}\partial_\phi(-r\partial_\phi\epsilon(\phi)) \\ &= -2GJ\Omega^2\frac{r\partial_\phi^2\epsilon(\phi)}{1+r^2}\end{aligned}$$

While the rest of the components give us zero when computed, for instance the $\phi\tau$ component

$$\begin{aligned}\mathcal{L}_\xi g_{\phi\tau} &= \xi^r\partial_r g_{\phi\tau}(x) + g_{\phi\tau}(x)\partial_\phi\xi^\phi \\ &= -r\partial_\phi\epsilon(\phi)\partial_r(2r\Lambda^2) + 2r\Lambda^2\partial_\phi\epsilon(\phi) \\ &= -2r\Lambda^2\partial_\phi\epsilon(\phi) + 2r\Lambda^2\partial_\phi\epsilon(\phi) \\ &= 0\end{aligned}$$

or, the $\theta\theta$ component

$$\mathcal{L}_\xi g_{\theta\theta} = 0.$$

We also have to point out, that the metric is symmetric $dx^\mu dx^\nu = dx^\nu dx^\mu$, which is why a factor of two is missing from the cross-terms here, with respect to (4.17).

F The Cardy formula in Kerr/CFT

The Cardy formula gives the entropy of the two-dimensional CFT as found in section 2.5 as

$$S = 2\pi\sqrt{\frac{c\Delta}{6}}, \tag{F.1}$$

where c is the central charge and Δ is the energy. The temperature of the CFT is then given by

$$d\Delta = TdS. \tag{F.2}$$

We proceed by finding the derivative of S with respect to Δ

$$\begin{aligned}\frac{dS}{d\Delta} =&\pi\left(\frac{c\Delta}{6}\right)^{-\frac{1}{2}}\frac{c}{6} = \pi\left(\frac{c}{6}\right)^{-\frac{1}{2}}\frac{c}{6}\frac{1}{\sqrt{\Delta}} \\ =&\pi\left(\frac{c}{6}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\Delta}} \Longleftrightarrow \\ \frac{1}{T} =&\pi\left(\frac{c}{6}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\Delta}} \Longleftrightarrow \sqrt{\Delta} = \pi\sqrt{\frac{c}{6}}T\end{aligned}$$

Substituting back to (F.1) we get

$$
\begin{aligned}
S =& 2\pi\sqrt{\frac{c}{6}}\sqrt{\Delta} = 2\pi^2\frac{c}{6}T \\
\Longleftrightarrow S =& \frac{\pi^2}{3}cT,
\end{aligned}
$$

which is the form that the authors in the Kerr/CFT correspondence paper [23] are using.

G Derivation of Klein-Gordon equation of motion for a massless scalar field

The standard way to specify specify a particle theory is via its Lagrangian.

$$L = K - V, \tag{G.1}$$

here K and V are the kinetic energy and the potential energy of the system. The time integral of the Lagrangian is called the action and is given by,

$$S = \int L dt \tag{G.2}$$

In field theory, a distinction is occasionally made between the Lagrangian L, of which the action is the time integral and the Lagrangian density, which is the spatial integral of the Lagrangian density

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \tag{G.3}$$

and is a function of the fields and its first derivatives, of which the action is given by integrating over all space-time,

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{G.4}$$

Considering now the Lagrangian density for a massless scalar field ϕ which is given by,

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\partial_\mu \phi \partial^\mu \phi \tag{G.5}$$

and using the Euler-Lagrange equation of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{G.6}$$

we get to the Klein-Gordon equation of motion for a massless scalar field,

$$
\begin{aligned}
&\frac{\partial}{\partial \phi}\left(\frac{1}{2}\sqrt{g}g^{\mu\nu}(\partial\phi)^2\right) - \partial_\mu \frac{\partial}{\partial(\partial_\mu \phi)}\left(\frac{1}{2}\sqrt{g}g^{\mu\nu}(\partial\phi)^2\right) = 0 \Longrightarrow \\
&\Longrightarrow -\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = 0 \Longrightarrow \partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = 0
\end{aligned} \tag{G.7}
$$

where $g_{\mu\nu}$ is the metric tensor and $\partial^2 = \partial_\mu \partial^\mu$.

H What is the Casimir Operator

The Casimir operator or a Casimir invariance has the property that it commutes with all elements of the algebra essentially by construction or can say it is maximally commuting. The 'maximally commuting subalgebra' is called the centre. It is the set of all objects that commute with everything, hence the word maximally. The casimir is an element of the centre.

For example in the case of rotational group, the operator

$$J^2 = \sum_i J_i^2, \quad i = 1, 2, 3 \tag{H.1}$$

commutes with all generators

$$[J^2, J_i] = 0. \tag{H.2}$$

Thus J^2 is the Casimir operator of the rotation group. Here J_i are the 3x3 matrices of the SO(3) rotational group.

The three commutators of the generators define the algebra of rotation group and that of $SO(3)$. It should be noted that the commutators of the generators define the multiplication rules of the group. Essentially, the generators and commutators define the 'local' properties of the group. However the 'global properties' are not determined by the generators and commutators of the group.

I Equivalence of boosted black string metrics (6.14) and (6.19)

In this section we will show that the boosted black string metric given in [119], and is

$$\begin{aligned} ds^2 =& -\left(1 - \cosh^2\sigma \frac{r_H}{r}\right) dt^2 + 2\frac{r_H}{r}\cosh\sigma\sinh\sigma dt d\psi \\ &+ \left(1 + \sinh^2\sigma \frac{r_H}{r}\right) d\psi^2 + \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \end{aligned} \tag{I.1}$$

is equal to the boosted black string metric given in [120]

$$ds^2 = -\bar{f}\left(dt - \frac{r_H \sinh 2\sigma}{2r\bar{f}} d\psi\right)^2 + \frac{f}{\bar{f}} d\psi^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2, \tag{I.2}$$

where $d\Omega_2^2 = d\theta^2 + \sin\theta^2 d\phi^2$, $f = 1 - \frac{r_H}{r}$, $\bar{f} = 1 - \frac{r_H \cosh^2\sigma}{r}$ and $\sinh 2\sigma = 2\sinh\sigma\cosh\sigma$.

The metric (I.1) seems pretty straight forward (note that is being computed for D=5 where D is the number of dimensions). Performing some algebra to

(I.2) we get

$$\begin{aligned} ds^2 = &-\left(1-\frac{r_H\cosh^2\sigma}{r}\right)dt^2 + 2\frac{r_H}{r}\cosh\sigma\sinh\sigma dt d\psi \\ &-\frac{r_H^2\sinh^2\sigma\cosh^2\sigma}{r^2(1-\frac{r_H\cosh^2\sigma}{r})}d\psi^2 + \frac{1-\frac{r_H}{r}}{1-\frac{r_H\cosh^2\sigma}{r}}d\psi^2 + \left(1-\frac{r_H}{r}\right)^{-1}dr^2 \\ &+ r^2 d\Omega_2^2. \end{aligned} \tag{I.3}$$

The two metrics seem pretty similar. The only thing that is left is to equal the third and fourth term from (I.3) with the third term on (I.1). Doing some algebra on the third and fourth term we have:

$$\begin{aligned} &-\frac{r_H^2\sinh^2\sigma\cosh^2\sigma}{r^2(1-\frac{r_H\cosh^2\sigma}{r})}d\psi^2 + \frac{1-\frac{r_H}{r}}{1-\frac{r_H\cosh^2\sigma}{r}}d\psi^2 = \left(\frac{r_H\cosh^2\sigma + r - r_H}{r}\right)d\psi^2 = \\ &= \left(\frac{r_H(\sinh^2\sigma+1)+r-r_H}{r}\right)d\psi^2 = \left(\frac{r_H}{r}\sinh^2\sigma + \frac{r_H}{r} + \frac{r}{r} - \frac{r_H}{r}\right)d\psi^2 = \\ &= \left(1+\frac{r_H}{r}\sinh^2\sigma\right)d\psi^2 \end{aligned}$$

Where we used the relation, $\cosh^2\sigma - \sinh^2\sigma = 1$. With this last result we see that (I.1) = (I.2).

J The n-sphere metric

The S^n volume element is given by the relation

$$d\Omega_{n-2}^2 = d\alpha_1^2 + \sin^2\alpha_1\left(d\alpha_2^2 + \sin^2\alpha_2\left(\cdots + \sin^2\alpha_{n-3}d\alpha_{n-2}^2\right)\right). \tag{J.1}$$

Some examples follow to get maybe a better feeling of (J.1). For

$$\begin{aligned} &\text{2-sphere} \Longrightarrow ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \\ &\text{3-sphere} \Longrightarrow ds^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\zeta^2) \\ &\text{4-sphere} \Longrightarrow ds^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi(d\zeta^2 + \sin^2\zeta d\kappa^2)) \\ &\text{5-sphere} \Longrightarrow ds^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi(d\zeta^2 + sin^2\zeta(d\kappa^2 + sin^2\kappa d\eta^2))) \end{aligned}$$

From the above we can see that, in each increase in the dimension of a sphere an angle is being added.

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Until next time . . . !

Afterthought...

I would like to close this thesis with a thought.

— It seems to me that theoretical scientists around the world, people with high education and very smart in their own field what are really trying to do is to see nature as a whole. That is to bring about a unification of the several theories into one undivided and inseparable unified theory. One wonders if that is possible. Going into that question, we can see that these theories were created by thought. Thought is the accumulation of knowledge and past experiences. Since thought is limited in its own perspective, as we seem to need more and more knowledge, one can say that through thought we will not be able to achieve that which is total-complete. Thought brings about a fragmentation as the theories we have are fragmented. and a mind bind to thought is limited. It seems to me that only a mind capable free from the limitation of thought will be able to perceive this unity and wholeness. Considering the above statements mathematics will not be able to do that because they are a creation of thought, thus limited. The question that arises then is. Does such a mind exist, a mind not limited by thought and what is the state of that mind? I am not saying in any way that a scientific mind which is solely based on knowledge and discovery is its mission should not act. On the contrary a scientific approach is most wise, unprejudiced and precise. —–

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