On the Dijkgraaf-Vafa Conjecture

Cand. Scient. Thesis

by

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Abstract

This master's thesis gives a thorough and pedagogical introduction to the Dijkgraaf-Vafa conjecture which tells us how to calculate the exact effective glueball superpotential in a wide range of $\mathcal{N} = 1$ supersymmetric gauge theories in four space-time dimensions using a related matrix model. The introduction is purely field theoretical and reviews all the concepts needed to understand the conjecture. Furthermore, examples of the use of the conjecture are given. Especially, we find the one-cut solution of the matrix model and use this to obtain exact superpotentials in the case of unbroken gauge groups. Also the inclusion of the Veneziano-Yankielowicz superpotential and problems such as the nilpotency of the glueball superfield are discussed. Finally, we present the diagrammatic proof of the conjecture in detail, including the case where we take into account the abelian part of the supersymmetric gauge field strength. This master's thesis was handed in May 2004 and appears here with minor changes.

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Introduction

One of the main problems in supersymmetric gauge theories is to understand the low-energy dynamics. This is particularly important since the $\mathcal{N} = 1$ supersymmetric gauge theories in four space-time dimensions are believed to exhibit some of the same non-perturbative phenomena as QCD, e.g. confinement, mass gaps and chiral symmetry breaking. Investigation of the low-energy physics is, however, difficult due to the strong coupling of the gauge theory. But due to a remarkable conjecture by R. Dijkgraaf and C. Vafa it is now possible to systematically obtain the exact effective glueball superpotential in a wide range of $\mathcal{N} = 1$ supersymmetric gauge theories in four space-time dimensions. With the glueball superpotential we can e.g. calculate the values of the gaugino condensates and for the corresponding chiral symmetry breaking we can find the tension of the associated domain walls.

The Dijkgraaf-Vafa conjecture tells us that the full glueball superpotential is a sum of a potential which is perturbative in the glueball superfield and the Veneziano-Yankielowicz superpotential [1] for the supersymmetric Yang-Mills theory. Furthermore, the perturbative part is related in a simple way to the free energy of an associated matrix model. For a $\mathrm{U}(N)$ gauge group and matter in the form of an adjoint chiral superfield we should take the planar limit for the matrix model. More generally, for gauge groups and matter representations allowing a double line notation we should also consider the contribution to the free energy of the matrix model from diagrams with Euler characteristic $\chi = 1$, that is, with the topology of the projective plane or the disk. So we have a dramatic simplification in calculating the glueball superpotential both to the zero-momentum modes of the matrix model and further to planar and perhaps $\chi = 1$ diagrams. Also the gauge couplings are related to the free energy of the matrix model.

Originally, the conjecture arose from considerations in string theory. In a series of articles large N dualities in string theory were investigated and led to the relation between the glueball superpotential and the free energy of the matrix model in [2] and [3]. In [4] this was summarised and the general conjecture was stated entirely within gauge theory and further a sketch of a field theoretic proof was given. Proofs of the conjecture (for the form of the perturbative part of the effective superpotential) given entirely within gauge theory followed shortly. For the case of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory broken to $\mathcal{N} = 1$ by a tree-level superpotential a proof was given using factorisation of Seiberg-Witten curves in [5]. In the slightly more general case of an $\mathcal{N} = 1$ supersymmetric theory with $\mathrm{U}(N)$ gauge group and an adjoint chiral superfield a diagrammatic proof was given in [6], and a proof using generalised Konishi anomalies was given in [7]. Special cases have been proven using these methods in a large number of articles. A perturbative proof for the reduction to the zero-momentum modes of the matrix model for general gauge groups and (massive) matter representations was given in [8].

The aim of this thesis will be to give a thorough and pedagogical introduction to the Dijkgraaf-Vafa conjecture and the concepts needed to understand the conjecture. We will work entirely within supersymmetric gauge theories. Our prime example will be an $\mathcal{N}=1$ supersymmetric gauge theory with a $\mathrm{U}(N)$ gauge group and matter in the form of an adjoint chiral superfield. But we will also see how the Dijkgraaf-Vafa conjecture can be used to obtain exact superpotentials and we will see that the matrix model also captures the form of the non-perturbative Veneziano-Yankielowicz superpotential. Furthermore, we will in detail go through the diagrammatic proof of the conjecture.

The outline of the thesis is as follows. In the first chapter we will introduce supersymmetry. We will establish the supersymmetry algebra as the unique extension of the Poincaré algebra and consider its representations, the supermultiplets. The focus will then be on the $\mathcal{N}=1$ supersymmetric field theories using chiral and vector superfields, but we will also consider $\mathcal{N}=2$ supersymmetric theories.

The main chapter is the second chapter. We will start by introducing the Dijkgraaf-Vafa conjecture in our prime case of a $\mathrm{U}(N)$ gauge group with adjoint chiral matter. Since most of the concepts in the conjecture needs explaining, the rest of the chapter will be devoted to this. We will start by introducing the supersymmetric vacua and the vacuum moduli space and then consider these in our special case. Especially, we will explain the breaking of the gauge group and the non-zero masses needed in the conjecture. We will then consider the quantised theory and introduce the supergraphs. With these we will prove the perturbative non-renormalisation theorem. To understand the effective superpotential we will present the Wilsonian renormalisation and the integrating out procedure. In section 2.5 we will then investigate the Wilsonian effective superpotential. We will introduce the important concept of holomorphy and use it together with the symmetries of the theory to prove the non-renormalisation theorem more generally and to constrain the form of the non-perturbative corrections. We will here also need to learn about instantons and chiral anomalies. We will then go on and introduce the ILS linearity principle and the concept of "integrating in" which we will use to define the glueball superpotential. After explaining the lore of the low energy gauge dynamics including phases, confinement, mass gaps and chiral symmetry breaking, we will use the integrating in procedure to derive the form of the Veneziano-Yankielowicz superpotential and then we will consider the glueball superpotential in our special case.

After introducing the double line notation and the 't Hooft large N limit in section 2.6, we will consider the matrix model in section 2.7. Here we will first present the diagrammatic approach; also for the broken gauge group using ghosts. We will then see that the measure of the matrix model gives a term with the same form as the Veneziano-Yankielowicz superpotential under the Dijkgraaf-Vafa conjecture. The exact solution for the planar limit of the matrix model is then examined and we will obtain algebraic equations for the one-cut solution. In section 2.8 we will use these and the Dijkgraaf-Vafa conjecture to obtain the exact glueball superpotential for a cubic tree-level superpotential. We will here see that we get a term in the matrix model which exactly matches the Veneziano-Yankielowicz superpotential.

At the end of the chapter we will discuss the problem of the nilpotency of the glueball superfield and how to interpret the full effective superpotential obtained via the matrix model. Finally, we will state the Dijkgraaf-Vafa conjecture for classical gauge groups with adjoint and fundamental matter.

In the third and last chapter we will present the diagrammatic proof of the conjecture. The main focus we will be on the $\mathrm{U}(N)$ case, but we will also consider general gauge groups and matter allowing a double line notation. We will give a detailed proof for the case where we take into account the abelian part of the supersymmetric gauge field strength. At last we will show the reduction to zero-modes for general gauge groups and matter representations.

In appendix A we will present our notation. In appendix B we show that the Minkowski space can be seen as cosets in the Poincaré group. The spinors and the notation for these will be introduced in appendix C. The general Lagrangian for an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory is derived in Appendix D. Finally, in appendix E we will show how to calculate some integrals needed in the one-cut solution of the matrix model.

Chapter 1

Supersymmetry

In this chapter we will introduce the concept of supersymmetry. While being introduced in the early 1970s there has been no experimental evidence for this theory. But as shown in the next section it is natural to consider supersymmetric theories since supersymmetry is the unique extension of the Poincaré algebra. We will, however, give no other motivation than the simple beauty of the results derived in this thesis. Our introduction to supersymmetry will follow a theoretic stream of logic rather than a chronological and it is based on [9], [10], [11], [12], [13] and [14].

1.1 Supersymmetry Algebras

In this section we will introduce the supersymmetry algebra.

1.1.1 The Coleman-Mandula Theorem

Given a relativistic quantum field theory we can ask ourselves which symmetries we can have beyond the manifest Poincaré invariance. Supposing that the symmetry generators form a Lie algebra in the usual way, the answer was given in 1967 by Coleman and Mandula (here taken from [10]):

Coleman-Mandula. *Given that*

1. *For any mass M there are only a finite number of particle types with mass less than M.*

2. *Any two-particle state undergoes some reaction at all energies except perhaps an isolated set.*

3. *The amplitudes for elastic two-body scattering are analytic functions of the scattering angle at all energies and angles except perhaps an isolated set. (Analyticity of the S-matrix).*

then the most general Lie algebra of symmetry generators consists of $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$ (i.e. the Poincaré algebra) and possibly internal symmetry generators commuting with the Poincaré generators and being independent of spin and momentum.

Here symmetry generators are defined as hermitian operators that commute with the S-matrix, whose commutators are again symmetry generators, that take single particle states into single particle states, and that act on multiparticle states as the direct sum of their action on single particle states. For definition of the Poincaré algebra please see appendix B. It should be noted that in the massless case the Poincaré algebra can be extended to the algebra of the conformal group. We will not go through the rather lengthy proof of the Coleman-Mandula theorem here, but just remark that it is based on the fact that the Poincaré symmetry of a scattering process only leaves the scattering angle unknown. Extra symmetry would restrict this angle to a discrete set. Hence the scattering amplitude would be zero by analyticity in contradiction with assumption number 2.

1.1.2 Superalgebras

The way to get around the Coleman-Mandula theorem is to introduce fermionic symmetry generators which do not satisfy commutation relations like in Lie algebras, but rather anticommutation relations. Hence instead of searching for the most general Lie algebra, we search for the most general $\mathbb{Z}_2$-graded algebra also known as Lie *superalgebra*. This is defined as a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. The elements $\mathcal{A}$ of V_0 are called *bosonic* and are given a *grading* $|\mathcal{A}| = 0$. The elements of V_1 are called *fermionic* and are given grading $|\mathcal{A}| = 1$. The superalgebra has a bilinear form $[-,-\}$ fulfilling

$$\begin{aligned} &[-,-\} : V_i \times V_j \to V_{i+j \ (\mathrm{mod}\ 2)}, \\ &[\mathcal{A},\mathcal{B}\} = -(-1)^{|\mathcal{A}||\mathcal{B}|}[\mathcal{B},\mathcal{A}\}. \end{aligned} \tag{1.1}$$

The bilinear form must obey the generalised super-Jacobi identity:

$$(-1)^{|\mathcal{A}||\mathcal{C}|}[[\mathcal{A},\mathcal{B}\},\mathcal{C}\} + (-1)^{|\mathcal{B}||\mathcal{A}|}[[\mathcal{B},\mathcal{C}\},\mathcal{A}\} + (-1)^{|\mathcal{C}||\mathcal{B}|}[[\mathcal{C},\mathcal{A}\},\mathcal{B}\} = 0. \tag{1.2}$$

We note that the bosonic space with the restricted bilinear form is a Lie algebra. If it is possible to take products of operators we see that the brackets are realisable as:

$$[\mathcal{A},\mathcal{B}\} = \mathcal{A}\mathcal{B} - (-1)^{|\mathcal{A}||\mathcal{B}|}\mathcal{B}\mathcal{A}. \tag{1.3}$$

These brackets reduce to commutators on the bosonic space and anticommutators on the fermionic space. Thus in general we will use $[-,-]$ on bosonic operators or a mix of bosonic and fermionic operators, and we will use $\{-,-\}$ on fermionic operators. Now according to (1.1) the bracket of a bosonic operator with a fermionic operator is again a fermionic operator. The super-Jacobi identity tells us that this actually gives a representation: The fermionic space furnishes a representation of the bosonic Lie subalgebra.

1.1.3 The Haag-Sohnius-Lopuszanski Theorem

Let us now ask ourselves the question: What is the most general Lie *super*algebra of symmetries under the assumptions of the Coleman-Mandula theorem? The answer was given by Haag, Sohnius and Lopuszanski. In four space-time dimensions (which we will use throughout this thesis) there is one unique solution namely the (extended) supersymmetry algebra:

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0, \tag{1.4a}$$
$$[\mathcal{P}_\mu, \mathcal{Q}^A_\alpha] = [\mathcal{P}_\mu, \bar{\mathcal{Q}}_{B\dot{\alpha}}] = 0, \tag{1.4b}$$
$$\{\mathcal{Q}^A_\alpha, \bar{\mathcal{Q}}_{B\dot{\beta}}\} = 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\mathcal{P}_\mu \delta^A_B, \tag{1.4c}$$
$$\{\mathcal{Q}^A_\alpha, \mathcal{Q}^B_\beta\} = \varepsilon_{\alpha\beta}\mathcal{Z}^{AB}, \tag{1.4d}$$
$$\{\bar{\mathcal{Q}}_{A\dot{\alpha}}, \bar{\mathcal{Q}}_{B\dot{\beta}}\} = \varepsilon_{\dot{\alpha}\dot{\beta}}\mathcal{Z}^\dagger_{AB}. \tag{1.4e}$$

Here the fermionic generators are the *supercharges* $\mathcal{Q}^A_\alpha$ and $\bar{\mathcal{Q}}_{A\dot\alpha} = \left(\mathcal{Q}^A_\alpha\right)^\dagger$ where $A = 1,\ldots,\mathcal{N}$. As the indices α and $\dot\beta$ indicate, the supercharges transform under the homogenous Lorentz group in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations respectively (left/right Weyl spinors). For spinorial representations please see appendix C where also σ^μ is defined in (C.10). As shown in the appendix these two conjugate Weyl spinors can be put together to form a Majorana spinor. Hence the fermionic generators consist of $\mathcal{N}$ Majorana spinorial generators. The representation of the supercharges determines their commutation relations with $\mathcal{J}_{\mu\nu}$ and hence we have left these out. We have also left out commutation relations with the internal symmetries as we will not need them. $\mathcal{Z}^{AB}$ and its hermitian conjugate are central charges i.e. they commute with all other symmetry generators including the internal symmetries. Please note that they are antisymmetric by definition of the anticommutator.

The Haag-Sohnius-Lopuszanski theorem has three assumptions besides those given in the Coleman-Mandula theorem: The fermionic operators $\mathcal{Q}$ operate in a Hilbert space with positive definite metric. If $\mathcal{Q}$ is a fermionic generator then so is the hermitian conjugate $\mathcal{Q}^\dagger$. At last we will assume physical states with $-\mathcal{P}^2 \geq 0$ and $\mathcal{P}^0 > 0$. We will not prove the whole theorem here, but we will see why we only have Majorana spinors and how (1.4c) comes about.

As described above the bosonic part of the superalgebra constitute a Lie algebra and by the Coleman-Mandula theorem this Lie algebra is generated by $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$ and possibly internal symmetries. As we also saw the fermionic space is a representation of this Lie algebra. The Lorentz generators constitute a subalgebra of the bosonic algebra and hence the $\mathcal{Q}$'s furnish a representation of the homogenous Lorentz group. Referring to appendix C the representations of the proper orthochronous Lorentz group can be labelled by (j_+, j_-) corresponding to the product of conjugate spin $j_+ \in \mathbb{N}_0/2$ and spin $j_- \in \mathbb{N}_0/2$ representations. Now (following [10]) we can label the $\mathcal{Q}$'s according to this: $\mathcal{Q}^{j_+j_-}_{m_+m_-}$ with $m_\pm = -j_\pm,\ldots,j_\pm$. By the spin statistics theorem $j_+ + j_-$ must be a half integer since the $\mathcal{Q}$'s anticommute . But we can do better. Let us look at

$$\{\mathcal{Q}^{j_+j_-}_{m_+m_-}, \left(\mathcal{Q}^{j_+j_-}_{m_+m_-}\right)^\dagger\}, \tag{1.5}$$

where we used the assumption that the hermitian conjugate of a fermionic generator is again a fermionic generator. Since it is the conjugate, $\left(\mathcal{Q}^{j_+j_-}_{m_+m_-}\right)^\dagger$ must transform in the (j_-, j_+) representation. Actually $\left(\mathcal{Q}^{j_+j_-}_{m_+m_-}\right)^\dagger \sim \mathcal{Q}^{j_-j_+}_{-m_-,-m_+}$. By the usual addition of spin (1.5), which belong to the bosonic space by (1.1), can now be expanded in bosonic operators $\mathcal{X}^{CD}_{cd}$ transforming in the (C, D) representation where $C, D = |j_+ - j_-|,\ldots,j_+ + j_-$. Using the properties of Clebsch-Gordan coefficients one can see that the top component fulfils:

$$\mathcal{X}^{j_++j_-,j_++j_-}_{j_++j_-,-j_+-j_-} = \{\mathcal{Q}^{j_+j_-}_{j_+,-j_-}, \left(\mathcal{Q}^{j_+j_-}_{j_+,-j_-}\right)^\dagger\}.$$

The Coleman-Mandula theorem tells us that the right hand side, being a part of the bosonic Lie algebra, is a linear combination of internal symmetries, $\mathcal{P}_\mu$'s, and $\mathcal{J}_{\mu\nu}$'s. The internal symmetries belong to the $(0,0)$ representation since they are Lorentz invariants, the $\mathcal{P}_\mu$'s belong to the $(\frac{1}{2},\frac{1}{2})$ representation (appendix C) while $\mathcal{J}_{\mu\nu}$ transforms in the $(1,0)\oplus(0,1)$ representation.[1] Looking at the left hand side this leaves us with the possibilities $j_+ + j_- =$

[1] The last part is not shown in appendix C, but it is shown that on spinors the representation of $\mathcal{J}_{\mu\nu}$ namely $\Sigma_{\mu\nu}$ is a linear combination of $(\sigma_{\mu\nu})_\alpha{}^\beta$ and $(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$. By lowering the indices these are actually symmetric. Hence by spin addition $(\sigma_{\mu\nu})_{\alpha\beta}$ is the symmetric part of spin zero plus spin one. Since the antisymmetric

$0, \frac{1}{2}$. Since $j_+ + j_-$ should be half integer we must have $j_+ + j_- = \frac{1}{2}$. For $j_+ + j_- \neq \frac{1}{2}$ we conclude that $\mathcal{Q}^{j_+ j_-}_{j_+, -j_-} = 0$ since the Hilbert space by assumption has a positive definite metric. By raising and lowering we then have $\mathcal{Q}^{j_+ j_-}_{m_+ m_-} = 0$ in the case of $j_+ + j_- \neq \frac{1}{2}$ and for all $m_\pm$. Consequently we can choose a basis of fermionic generators as $\mathcal{Q}^A_\alpha$ and their hermitian adjoints $\bar{\mathcal{Q}}_{A\dot{\alpha}}$ as wanted. Now (1.5) reduces to:

$$\{\mathcal{Q}^A_\alpha, \bar{\mathcal{Q}}_{B\dot{\beta}}\} = 2N^A_B \, (\sigma^\mu)_{\alpha\dot{\beta}} \, \mathcal{P}_\mu, \tag{1.6}$$

where we have used the correspondence between 4-vectors and the $(\frac{1}{2}, \frac{1}{2})$ representation from appendix C. $(\sigma^\mu)_{\alpha\dot{\beta}} \mathcal{P}_\mu$ is (according to [10]) positive definite if (and only if) we use the assumption that the states fulfil $-P^2 \geq 0$ and $P^0 > 0$. Since the anticommutator is positive definite we see that the N-matrix is positive definite. By (1.6) N^A_B is also hermitian ($\mathcal{P}_\mu$ is hermitian) and hence we can redefine our fermionic operators such that $N^A_B \mapsto \delta^A_B$. We thus end up with (1.4c). The rest of the proof of the Haag-Sohnius-Lopuszanski theorem follows using spin addition as above and the super-Jacobi identity. It should be noted that in the massless case an extension to a superconformal algebra is allowed.

If $\mathcal{N} > 1$ the algebra (1.4) is referred to as the *extended* supersymmetry algebra. If $\mathcal{N} = 1$ it is just called the supersymmetry algebra. In this case the central charges are zero by antisymmetry. The $\mathcal{N} = 1$ algebra then simplifies to:

$$\begin{aligned}
[\mathcal{P}_\mu, \mathcal{Q}_\alpha] &= [\mathcal{P}_\mu, \bar{\mathcal{Q}}_{\dot{\alpha}}] = 0. \\
\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} &= 2\,(\sigma^\mu)_{\alpha\dot{\beta}} \, \mathcal{P}_\mu. \\
\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} &= \{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0.
\end{aligned} \tag{1.7}$$

This algebra can actually be written in a more compact form using the Majorana notation for the supercharges (C.28):

$$\mathcal{Q}_{\mathrm{M}} \sim \begin{pmatrix} \mathcal{Q}_\alpha \\ \bar{\mathcal{Q}}^{\dot{\alpha}} \end{pmatrix}. \tag{1.8}$$

Using this, the supersymmetry algebra becomes:

$$\begin{aligned}
[\mathcal{P}_\mu, \mathcal{Q}_a] &= 0. \\
[\mathcal{Q}_a, \mathcal{Q}_b] &= -2{\gamma^\mu}_{ab} \mathcal{P}_\mu.
\end{aligned} \tag{1.9}$$

Here we have used Latin indices for the Majorana spinors. The γ-matrices are defined in (C.14). Please note that the last index on the γ-matrices has been lowered using the charge conjugation matrix as defined in appendix C.

1.1.4 R-Symmetry

Without central charges the extended supersymmetry algebra (1.4a)–(1.4e) is invariant under the unitary transformations:

$$\mathcal{Q}^A_\alpha \mapsto U^{AB} \mathcal{Q}^B_\alpha, \qquad \boldsymbol{U} \in \mathrm{U}(\mathcal{N}).$$

These automorphisms are called R-symmetries and the group is denoted $\mathrm{U}(\mathcal{N})_R$. We can split the group in an abelian and a non-abelian part denoted $\mathrm{U}(1)_R$ and $\mathrm{SU}(\mathcal{N})_R$ respectively. In the case of $\mathcal{N} = 1$ only $\mathrm{U}(1)_R$ survives. The R-symmetries are not necessarily symmetries of the actions we define, but we will generally choose the actions in such a way that they are symmetric. However, quantum mechanically $\mathrm{U}(1)_R$ will often be broken by anomaly effects as we will see in section 2.5.4.

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tensor $\varepsilon_{\alpha\beta}$ is invariant (i.e. spin zero) $(\sigma_{\mu\nu})_{\alpha\beta}$ must be spin one or rather in the $(1,0)$ representation $(0 \oplus_s 1 = 1)$. Hence ${(\sigma_{\mu\nu})_\alpha}^\beta$ and ${(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}}_{\dot{\beta}}$ transform respectively in the $(1,0)$ and $(0,1)$ representations.

1.2 Supermultiplets

In this section we will find the 1-particle finite-dimensional unitary representations of the supersymmetry algebra.

1.2.1 The Wigner Method

Since the Poincaré algebra is non-compact it is a theoretical fact that it is not possible to find any finite-dimensional unitary representations. The way to find unitary representations is then to use the Wigner method of induced representations. The idea is simply to fix the non-compact transformations i.e. the translations and the boosts.

To be more precise let us first remark that[2] P^2 is a Casimir operator of the Poincaré algebra and since P^μ commutes with the supercharges it is a Casimir for the whole supersymmetry algebra. Thus P^2 is constant on the irreducible representations by Schur's lemma. Let us now diagonalise the commuting P^μ's such that each state is labelled by p^μ satisfying the equation of motion $p^2 = -M^2$. Consider now some conventional value of p^μ. The subgroup of the Poincaré group leaving this momentum invariant is called *the little group*. This group is independent of the choice of momentum on the mass shell since the invariance groups are adjointly related by Lorentz transformations and hence isomorphic. The idea is that the little group is either compact or, if it is not, we will compactify it. Hence it has unitary finite-dimensional representations. These induce unitary representations of the whole Poincaré algebra. This is done since the vector space, V, of the representation of the little group trivially defines a vector bundle on the mass shell. The states can now be seen as the sections, $\phi(p)$, of this vector bundle taking values in V. These sections are acted upon by the whole Poincaré group in a natural way by simply splitting a Poincaré transformation into the part belonging to the little group and the part changing p^μ. The first part simply works under the chosen representation on the vector-valued ϕ and the last part changes the momentum (as defined in (B.3)). The space of sections is of course infinite-dimensional because of the continuous parameter p^μ, however, V is finite dimensional.

Since the supercharges commute with P^μ and hence leave the momentum parameter invariant they can be included in the algebra of the little group thus defining the *little supergroup*. It is the irreducible representations of this group that we are looking for. Naturally, the representations of the little *super*group are reducible when we see these as representations of the little group. Thus different particle (Poincaré-) representations will be related by supersymmetry and the representations of the little supergroup are hence called *supermultiplets*.

Since P^2 is a Casimir all particles in a supermultiplet have the same mass. Yet another feature of the supermultiplets is that they have an equal number of bosonic and fermionic states. This is readily seen (following [9]) by introducing the fermion number operator N_F defined such that it commutes with the bosonic operators and $(-1)^{N_F}$ anticommutes with

[2] We use normal font for the operators since we now look at representations.

Q_α. We thus see:[3]

$$\begin{aligned}
\mathrm{Tr}\Big((-1)^{N_F}\{Q^A_\alpha,\bar{Q}_{B\dot{\beta}}\}\Big) &= \mathrm{Tr}\Big((-1)^{N_F}Q^A_\alpha\bar{Q}_{B\dot{\beta}}+(-1)^{N_F}\bar{Q}_{B\dot{\beta}}Q^A_\alpha\Big)\\
&= \mathrm{Tr}\Big(-Q^A_\alpha(-1)^{N_F}\bar{Q}_{B\dot{\beta}}+Q^A_\alpha(-1)^{N_F}\bar{Q}_{B\dot{\beta}}\Big)\\
&= 0,
\end{aligned}$$

where we have used the cyclicity of the trace for the last term in the second equality. Using (1.4c) the left hand side is equal to $2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\delta^A_B\,\mathrm{Tr}\big((-1)^{N_F}P_\mu\big)$. Choosing the momentum and the indices properly this gives $\mathrm{Tr}\big((-1)^{N_F}\big)=0$ proving that we have an equal number of fermionic and bosonic states. Hence each particle has a *superpartner* of opposite fermionic parity.

1.2.2 Massless Supermultiplets

The massless supermultiplets are phenomenologically the most interesting. This is because supersymmetry has not been observed in nature. Hence supersymmetry must be broken at the energies of modern accelerators. The masses of the particles are hence negligible compared to the high energy at which we (might) have supersymmetry.

In the case of massless multiplets, i.e. $P^2=0$, we boost to the momentum $p_\mu\sim(-E,0,0,E)$ where E is some conventional energy. The little group is in this case SO(2) or rather spin(2) since we represent the spin group rather than the Lorentz group. However, also the generators K^1-L^2 and K^2-L^1 are in the algebra of the little group (L^i and K^i are defined in (C.1)). But we remove these by hand since they render the little group non-compact and the corresponding continuous parameters are not seen in nature. The spin(2) group is generated by a single generator which with our choice of momentum is L^3. The representations of this are one-dimensional and indexed by the eigenvalue of L^3, the helicity λ. Because a rotation by 4π can continuously be deformed into the identity we have $\lambda\in\mathbb{Z}/2$.[4]

In the specified frame (1.4c) becomes:

$$\{Q^A_\alpha,\bar{Q}_{B\dot{\beta}}\}=2\begin{pmatrix}2E & 0\\ 0 & 0\end{pmatrix}_{\alpha\dot{\beta}}\delta^A_B. \tag{1.10}$$

Since we assumed a positive definite inner product, $\{Q^A_2,\bar{Q}_{A\dot{2}}\}=0$ (no sum) shows that $Q^A_2=\bar{Q}_{A\dot{2}}=0$. Inserting this into (1.4d) and (1.4e) further shows that all the central charges vanish. The only non-zero supercommutator left is:

$$\{Q^A_1,\bar{Q}_{B\dot{1}}\}=4E\delta^A_B. \tag{1.11}$$

After rescaling, this is simply the algebra of $\mathcal{N}$ fermionic creation and corresponding annihilation operators. Using that Q^A_α transforms as (using (C.27)):[5]

$$[J^{\mu\nu},Q^A_\alpha]=-i\,(\sigma^{\mu\nu})_\alpha{}^\beta\,Q^A_\beta, \tag{1.12}$$

[3]The trace is well-defined since we use a finite-dimensional representation. However, if supersymmetry is broken states can be mapped out of the Hilbert space as we will see in section 2.2 – hereby dismissing the proof.

[4]As explained in appendix C the first homotopy group of the proper orthochronous Lorentz group is $\mathbb{Z}_2$ so we need a double loop to get the identity: $\exp\big(i4\pi L^3\big)=1$

[5]Please note the minus sign on the right hand side. This sign is of course determined by the super-Jacobi identity. In the same way as we found in section B.3 that $\mathcal{P}^\mu$ transforms in the vector representation, this sign makes sure that Q_α transforms in the $(\frac{1}{2},0)$ representation.

Table 1.1:

$\mathcal{N}$	1	1	2	2	3	4
Name	Gauge	Chiral	Gauge	Hyper	Gauge	Gauge
$\lambda = 1$	1	0	1	0	1	1
$\lambda = 1/2$	1	1	2	1+1	3+1	4
$\lambda = 0$	0	1+1	1+1	2+2	3+3	6
$\lambda = -1/2$	1	1	2	1+1	1+3	4
$\lambda = -1$	1	0	1	0	1	1
Total number	4	4	8	8	16	16

The number of massless states in global supersymmetry-multiplets for each helicity λ. CPT-invariance of the supermultiplets is assumed. Plus-signs are used to indicate when states of the same helicity stem from CPT doubling. Based on table in [13].

and $L^3 = J^{12}$ we get:

$$[L^3, Q_1^A] = -\tfrac{1}{2} Q_1^A. \tag{1.13}$$

This shows that Q_1^A lowers the helicity by 1/2 and hence $\bar{Q}_{B\dot{1}}$ raises the helicity by 1/2. Now we can construct the multiplets. Since they should be finite-dimensional there must be some state $\Omega_{\lambda_{\min}}$ with lowest helicity $\lambda_{\min}$ defined by:

$$Q_1^A \Omega_{\lambda_{\min}} = 0, \qquad A = 1, \ldots, \mathcal{N}. \tag{1.14}$$

This state must be non-degenerate for the sake of irreducibility of the supermultiplet. All other states are obtained by raising:

$$\Omega_{\lambda_{\min}+1/2n, A_1,\ldots,A_n} = N \bar{Q}_{A_n \dot{1}} \cdots \bar{Q}_{A_1 \dot{1}} \Omega_{\lambda_{\min}}, \tag{1.15}$$

where N is a proper normalisation factor. These states transform as rank n antisymmetric tensors under the $\mathrm{SU}(\mathcal{N})_R$ symmetries and hence they are $\binom{\mathcal{N}}{n}$-fold degenerate as helicity eigenstates. We also see that we reach a highest helicity state with helicity $\lambda_{\min} + \mathcal{N}/2$ by raising with all the different $\bar{Q}_{A\dot{1}}$'s. Thus the total number of states is $\sum_{n=0}^{\mathcal{N}} \binom{\mathcal{N}}{n} = 2^{\mathcal{N}}$. However, CPT reverses the sign of helicity so if we want a CPT-invariant theory we must directly sum each multiplet with its CPT-conjugate antimultiplet having the opposite helicities.[6]

Now we can tabulate all massless multiplets. However, we are only interested in multiplets with helicities $|\lambda| < \frac{3}{2}$ since otherwise the only consistent couplings require gravitation – and we will only deal with *global* supersymmetries. The result is given in table 1.1 . We note that the spectra for $\mathcal{N} = 3$ and $\mathcal{N} = 4$ are the same – they are in fact equal. We also observe that the number of bosonic and fermionic states are the same as expected.

The reason that some of the supermultiplets are called gauge multiplets is of course that they contain two states with helicities 1 and -1 respectively i.e. making up a massless gauge boson. These are always followed by the superpartner – a fermion made out of helicity $\pm 1/2$ states i.e. a Weyl (or Majorana) fermion called the *gaugino*. Please note that only in special dimensions it is possible to make a gauge multiplet as shown in section C.6.

[6]In the case of $\mathcal{N} = 2$ we have a helicity-symmetric multiplet with 2 helicity zero particles transforming as a $\mathrm{SU}(2)_R$ doublet. We could ask if this is not its own antimultiplet. But this can not be true since the two particles would then be real and thus could not transform as a $\mathrm{SU}(2)_R$ doublet. However, for $\mathcal{N} = 4$ we have a multiplet that is its own antimultiplet. This is possible since here the 6 helicity zero particles transform as a rank 2 antisymmetric tensor under $\mathrm{SU}(4)_R$. This is actually the vector representation of SO(6) which is real.

The chiral multiplet is a matter multiplet consisting of one Majorana fermion, say a quark, along with its superpartner a complex scalar called the *squark*.

1.2.3 Massive Supermultiplets

In the case of massive supermultiplets we boost to the rest frame with 4-momentum $p_\mu \sim (-M, 0, 0, 0)$. Thus the little group is SO(3) or rather its spin cover SU(2). Hence we get states described by spin $j \in \mathbb{N}_0/2$ and $m = -j, \ldots, j$.

We will here assume that the central charges are all zero. Consequently the only non-zero supercommutator is (1.4c) and it takes the form:

$$\{Q^A_\alpha, \bar{Q}_{B\dot{\beta}}\} = 2M\delta_{\alpha\dot{\beta}}\delta^A_B. \tag{1.16}$$

This shows that we now have $2\mathcal{N}$ fermionic creation and corresponding annihilation operators. As before we start from a "vacuum", Ω, defined by

$$Q^A_\alpha \Omega = 0, \qquad \alpha = 1, 2, \; A = 1, \ldots, \mathcal{N}. \tag{1.17}$$

However, this time the state needs not be non-degenerate, but can be in some spin representation. Again all other states are built by making all possible raisings (N is a normalisation factor):

$$N\bar{Q}_{A_n\dot{\alpha}_n} \cdots \bar{Q}_{A_1\dot{\alpha}_1}\Omega. \tag{1.18}$$

This is a rank n tensor under the $\mathrm{SU}(\mathcal{N})_R$ symmetries. But from (1.16) we see that we also have a SU(2) symmetry in the spinorial α index.[7] The state (1.18) is also a rank n tensor under this symmetry group. However, it is only antisymmetric if we pairwise change the indices $A_i\alpha_i$. Thinking of this paired index as one index taking $2\mathcal{N}$ values, we see that we have $\binom{2\mathcal{N}}{n}$ states with n raisings. Thus the total number of states is $\sum_{n=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{n} = 2^{2\mathcal{N}}$ multiplied with the dimension of the spin representation of the vacuum.

The states (1.18) can be spin summed using Clebsch-Gordan coefficients to gain spin eigenstates. We can, however, easily determine the state with maximal spin. To gain this state we must symmetrise in as many spin-$\frac{1}{2}$ indices $\dot{\alpha}$ as possible (remembering that a pair of antisymmetric indices is spin zero since $\varepsilon_{\dot{\alpha}\dot{\beta}}$ is an invariant tensor). However, since the indices should be pairwise antisymmetric the A_i-indices must simultaneously be antisymmetrised. Hence we can maximally symmetrise in $\mathcal{N}$ spin indices adding spin $\frac{1}{2}\mathcal{N}$ to the vacuum Ω. Consequently, in order to avoid states with spin $3/2$ or more we can only have $\mathcal{N} = 1, 2$.

In the case of $\mathcal{N} = 1$ we can start from a spin 0 "vacuum" and get the chiral multiplet with a complex scalar and one Majorana fermion. We can also start from a spin $\frac{1}{2}$ vacuum getting a gauge multiplet with a scalar field, a Dirac fermion and a gauge boson.

In the case of $\mathcal{N} = 2$ we have to start from a spin 0 vacuum and we will have 5 scalars, 4 Majorana spinors and one gauge boson.

1.3 $\mathcal{N} = 1$ Supersymmetric Field Theories

Instead of venturing ahead combining fields to make supersymmetric Lagrangians it is possible in the case of $\mathcal{N} = 1$ supersymmetry to define the *superspace*. This is the analog of what Minkowski space is to the Poincaré algebra. Supersymmetry can then be realised as differentials of fields defined on superspace and it will be easy to make manifestly supersymmetric Lagrangians.

[7] Actually the full symmetry group of the massive algebra is SO(6).

1.3.1 Superspace

Let us define anticommuting Grassmann numbers ξ^α and $\bar{\xi}^{\dot{\alpha}}$. These anticommute with each other and with the fermionic operators while they commute with the bosonic operators. Please note that they have indices like Weyl spinors and we will use the same summing, raising and lowering conventions as for Weyl spinors. Using ξ and $\bar{\xi}$ we can turn the anti-commutators of the $\mathcal{N}=1$ supersymmetry algebra into commutators (here assuming that the super-bracket is realisable as an (anti)commutator):

$$\begin{aligned}[][\xi\mathcal{Q},\bar{\xi}\bar{\mathcal{Q}}] &= \xi\mathcal{Q}\bar{\xi}\bar{\mathcal{Q}}-\bar{\xi}\bar{\mathcal{Q}}\xi\mathcal{Q} = -\xi^\alpha\{\mathcal{Q}_\alpha,\bar{\mathcal{Q}}^{\dot{\beta}}\}\bar{\xi}_{\dot{\beta}} \\ &= 2\xi\sigma^\mu\bar{\xi}\mathcal{P}_\mu. \end{aligned} \tag{1.19}$$

The rest of the commutators with $\xi\mathcal{Q}$ and $\bar{\xi}\bar{\mathcal{Q}}$ are zero.

Since we now only have commutation relations, we can form the *Lie supergroup* by exponentiating the generators of the supersymmetry algebra, but with the supercharges having Grassmannian coefficients. The Baker-Campbell-Hausdorff formula then applies as usual. We note that the supersymmetry algebra forms a semi-direct product of the Lorentz generators with the momentum generators and supercharges.[8] This means that we are in the same situation as in appendix B where the Minkowski space is defined as cosets of the Lorentz group in the Poincaré group. Now we define *superspace* as cosets of the Lorentz group in the *Lie supergroup*. In analogy with (B.6) the cosets can uniquely be written as:

$$e^{i\left(-x^\mu\mathcal{P}_\mu+\theta\mathcal{Q}+\bar{\theta}\bar{\mathcal{Q}}\right)}G_{\text{Lorentz}}, \tag{1.20}$$

where we denoted the Grassmannian coefficients by θ^α and $\bar{\theta}_{\dot{\alpha}}$. This gives us a one-to-one correspondence between the cosets modulo the Lorentz group and superspace coordinates $(x,\theta,\bar{\theta})$.[9]

In analogy with section B.3 the action of the Lie supergroup on the superspace is determined by left multiplication of the group on the cosets. This means that translations work as usual: $x^\mu \mapsto x^\mu + \tau^\mu$. Multiplication with $\exp(i\omega^{\mu\nu}\mathcal{J}_{\mu\nu})$ shows that x^μ transforms as a vector and that θ and $\bar{\theta}$ transform as left and right Weyl spinors respectively. The *super-translations* with ξ and $\bar{\xi}$ are easily obtained using the Baker-Campbell-Hausdorff formula (since only the first commutator is non-zero):

$$\begin{aligned} e^{i\left(\xi\mathcal{Q}+\bar{\xi}\bar{\mathcal{Q}}\right)}e^{i\left(-x^\mu\mathcal{P}_\mu+\theta\mathcal{Q}+\bar{\theta}\bar{\mathcal{Q}}\right)} &= e^{i\left(-x^\mu\mathcal{P}_\mu+(\theta+\xi)\mathcal{Q}+\left(\bar{\theta}+\bar{\xi}\right)\bar{\mathcal{Q}}+i\frac{1}{2}[\xi\mathcal{Q}+\bar{\xi}\bar{\mathcal{Q}},\theta\mathcal{Q}+\bar{\theta}\bar{\mathcal{Q}}]\right)} \\ &= e^{i\left(-\left(x^\mu+i\theta\sigma^\mu\bar{\xi}-i\xi\sigma^\mu\bar{\theta}\right)^\mu\mathcal{P}_\mu+(\theta+\xi)\mathcal{Q}+\left(\bar{\theta}+\bar{\xi}\right)\bar{\mathcal{Q}}\right)}. \end{aligned} \tag{1.21}$$

This corresponds to the transformation

$$(x,\theta,\bar{\theta}) \mapsto (x',\theta',\bar{\theta}') = (x+i\theta\sigma^\mu\bar{\xi}-i\xi\sigma^\mu\bar{\theta},\theta+\xi,\bar{\theta}+\bar{\xi}). \tag{1.22}$$

This representation of the supercharges on superspace we now turn into a representation, Q and $\bar{Q}$, on the fields defined on superspace. However, to get a representation on fields we must remember that the coordinate should transform in the opposite way as in (1.22).[10]

[8] That is: The momentum generators and the supercharges constitute a subalgebra. Also the Lorentz generators form a subalgebra and the commutator of a Lorentz generator with a momentum generator or a supercharge is a sum of momentum generators and supercharges.

[9] $(x,\theta,\bar{\theta})$ should in the strict mathematical sense be seen as coordinate functions in a noncommutative geometry.

[10] Suppose the group G acts on M as $g.x$ with $g\in G$ and $x\in M$. This induces an action of G on $\mathbb{C}^M$ namely $g.f(x)=f(g^{-1}.x)$ where $f\in\mathbb{C}^M$.

But following [9] we do not take this into consideration so we will get an antirepresentation of the supersymmetry algebra. But at the same time [9] at this point tacitly changes the definition of the Q's such that the infinitesimal change of the field F is (note the missing i):

$$\begin{aligned}\delta_\xi F &= [i\xi\mathcal{Q}+i\bar{\xi}\bar{\mathcal{Q}},F] \equiv \left(\xi Q+\bar{\xi}\bar{Q}\right)F,\\ \delta_\xi F(x,\theta,\bar{\theta}) &= F(x',\theta',\bar{\theta}')-F(x,\theta,\bar{\theta}),\end{aligned} \tag{1.23}$$

where $(x',\theta',\bar{\theta}')$ is defined in (1.22). Thus the supercharges are realised by the differential operators:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{1.24}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\left(\sigma^\mu\right)_{\beta\dot{\alpha}}\partial_\mu. \tag{1.25}$$

These satisfy (note the change in sign compared to (1.7) and (A.1) due to the above mentioned redefinitions):

$$\{Q_\alpha,\bar{Q}_{\dot{\beta}}\} = 2i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\partial_\mu. \tag{1.26}$$

The rest of their mutual anticommutators are zero. However, please note that according to (C.39) we have (due to the missing i's):

$$\left(Q_\alpha\right)^\dagger = -\bar{Q}_{\dot{\alpha}}. \tag{1.27}$$

On the other hand according to (C.37) we now have

$$\left(Q_\alpha\right)^* = \bar{Q}_{\dot{\alpha}}. \tag{1.28}$$

We can also define multiplication from the right on the cosets. This gives in the same way as above differential operators D_α and $\bar{D}_{\dot{\alpha}}$ – the covariant derivatives – where:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \tag{1.29}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\beta\left(\sigma^\mu\right)_{\beta\dot{\alpha}}\partial_\mu. \tag{1.30}$$

Since left and right multiplications commute the covariant derivatives anticommute with the supercharges. They satisfy the opposite algebra of (1.26):

$$\{D_\alpha,\bar{D}_{\dot{\beta}}\} = -2i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\partial_\mu. \tag{1.31}$$

The fields on superspace transforming as (1.23) are called *superfields*. Given such a general complex superfield $F(x,\theta,\bar{\theta})$ we can expand it in component fields by looking at its power series in θ and $\bar{\theta}$. This power series must terminate since the θ's and the $\bar{\theta}$'s all anticommute. Using (C.40) and (C.46) we get (the notation of components follow [13]):

$$\begin{aligned}F(x,\theta,\bar{\theta}) = \phi(x) &+ \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}g^*(x) + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x)\\ &+ i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\rho(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x).\end{aligned} \tag{1.32}$$

Supposing $F(x,\theta,\bar{\theta})$ is a bosonic Lorentz scalar we see that ϕ, f, g, and D are complex scalars while ψ, χ, λ, and ρ are lefthanded Weyl spinors and A_μ is a gauge field. Hence we note that the number of bosonic degrees of freedom equals that of the fermionic. The action of supercharges on superfields induces an action on the components simply by defining the transformed components as the components of the transformed superfield.

The set of superfields are closed to addition and multiplication since the supercharges are differentials. Thus the superfields form a linear representation of the supersymmetry algebra. However, this representation is reducible as can be seen from the number of components in (1.32) since according to table 1.1 there should only be four degrees of freedom in an irreducible representation after imposing the equation of motion. Hence we must constrain the superfields.

Let us conclude this subsection by noting that for a set of superfields F^i any differentiable function of them is again a superfield. We also note that using (1.23)–(1.25) the supersymmetry variation of the top component $D(x)$ is a space-time derivative – hence it can be used as a manifestly supersymmetric Lagrangian (we assume that the boundary terms are zero). This makes it easy to build invariant Lagrangians.

1.3.2 Chiral Superfields

One way to constrain a (complex) superfield Φ is to demand

$$\bar{D}_{\dot{\alpha}}\Phi = 0, \qquad \dot{\alpha} = \dot{1}, \dot{2}. \tag{1.33}$$

This is a supersymmetric covariant constraint since the supercharges anticommute with $\bar{D}$. Complex superfields fulfilling (1.33) are called *chiral* superfields. Correspondingly an anti-chiral superfield is defined by $D_\alpha\Phi = 0$. We see that the complex conjugate of a chiral field is anti-chiral and vice-versa hence giving a one-to-one correspondence between the two sets. We also note that only constant fields can be both chiral and anti-chiral since then $\{D_\alpha, \bar{D}_{\dot{\beta}}\}\Phi = 0$ which by (1.31) can be used to show that $\partial_\mu\Phi = 0$. This is all in analogy with holomorphic functions.

The components of a chiral field are easily found by noting that both $x_+^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and θ are annihilated by $\bar{D}_{\dot{\alpha}}$. Expressed in these coordinates $D_\alpha = \partial/\partial\theta^\alpha + 2i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial/\partial x_+^\mu$ and $\bar{D}_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}}$. Hence the most general chiral field can be written as:

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x_+) + \sqrt{2}\theta\psi(x_+) + \theta\theta F(x_+)\\ &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) + \sqrt{2}\theta\psi(x)\\ &\quad -\frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x).\end{aligned} \tag{1.34}$$

Here we have used (C.41) and (C.42) to expand. As usual $\Box = \partial_\mu\partial^\mu$ and we have put in a $\sqrt{2}$ for standard normalisation. F will turn out to be an auxiliary field and hence after taking the equations of motion, we see that the components fit that of the chiral multiplet thus realising this off-shell.

The components can be obtained using the covariant derivatives (using (C.49)):

$$\phi(x) = \Phi|, \qquad \psi_\alpha(x) = \frac{1}{\sqrt{2}}D_\alpha\Phi|, \qquad F(x) = -\frac{1}{4}DD\Phi|, \tag{1.35}$$

where $|$ means setting $\theta = \bar{\theta} = 0$.

Since all components higher (i.e. with more θ's in front) than F are just derivatives, the supersymmetry variation of F is just a space-time derivative making it usable for constructing manifestly supersymmetric Lagrangians. Actually $\delta_\xi F = i\sqrt{2}\partial_\mu\left(\bar{\xi}\bar{\sigma}^\mu\psi\right)$.

Given N chiral multiplets represented by the (Lorentz invariant, bosonic) chiral fields Φ^i with $i = 1, \ldots, N$ we search for the most general supersymmetric Lagrangian including these. As we have seen above there are two ways to create Lagrangians: We can use the top component, $D/2$, of a general superfield or the F component of a chiral superfield. Such terms are called D-terms and F-terms respectively.

We can write these terms as "integrations" over superspace parameters. To do this we define Lorentz invariant differentials by $\mathrm{d}^2\theta \equiv \frac{1}{4}\mathrm{d}\theta_\alpha \mathrm{d}\theta^\alpha$, $\mathrm{d}^2\bar{\theta} \equiv \frac{1}{4}\mathrm{d}\bar{\theta}^{\dot{\alpha}}\mathrm{d}\bar{\theta}_{\dot{\alpha}}$ and $\mathrm{d}^4\theta \equiv \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$. As usual fermionic integration corresponds to differentiation such that e.g. $\mathrm{d}\theta^\alpha = \partial/\partial\theta^\alpha$. By the above normalisations the F component of a chiral field Φ is simply $F = \int \mathrm{d}^2\theta\Phi$ and the D-term, $D/2$, of a general superfield Ψ is $D/2 = \int \mathrm{d}^4\theta\Psi$. Here it is assumed that we put $\theta = \bar{\theta} = 0$ after differentiating. Since the space-time integral of a space-time derivative is assumed to be zero, we see that the covariant derivatives work as differentials under the space-time integration in the action. Hence:

$$
\begin{aligned}
\int \mathrm{d}^4x\mathrm{d}^2\theta\Phi &= -\frac{1}{4}\int \mathrm{d}^4x DD\Phi|. \\
\int \mathrm{d}^4x\mathrm{d}^4\theta\Psi &= \frac{1}{16}\int \mathrm{d}^4x DD\bar{D}\bar{D}\Psi.
\end{aligned}
\tag{1.36}
$$

There is no need for a restriction "|" in the last equation since all θ-dependence is removed by the covariant derivatives and the removal of total space-time derivative terms.

This shows us that we have a redundancy in our definition of F-terms. Actually any D-term can now be rewritten as:

$$
\int \mathrm{d}^4x\mathrm{d}^4\theta\Psi = -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\bar{D}\bar{D}\Psi, \tag{1.37}
$$

where there is also no need to set $\bar{\theta} = 0$ since these are again removed by the two $\bar{D}$'s along with the removal of total derivative terms. $\bar{D}\bar{D}\Psi$ is a chiral field since the product of three $\bar{D}_\alpha$'s vanish by anti-commutivity. A superfield that can be written in this way with Ψ being a local field is called a *chirally exact* superfield. In order to avoid redundancy we redefine F-terms as the $\theta\theta$-term of chiral fields that are not chirally exact.[11]

The most general D-term obtained from the chiral fields Φ^i and the corresponding anti-chiral complex conjugates, denoted $\bar{\Phi}^i$, is simply the D-term of a real differentiable function $K(\Phi^i, \bar{\Phi}^i)$ since this is again superfield. This is called the *Kähler potential*. We have here excluded the possibility of a dependence on space-time derivatives of Φ and $\bar{\Phi}$. This is because such terms, when expanded, give rise to more than two space-time derivatives on bosonic fields and more than one on fermionic fields. Such terms can be excluded when looking at low energy effective theories or renormalisable theories. Since the covariant derivatives anti-commute with the supercharges, the covariant derivative of a superfield is again a superfield. However, we will also assume that K does not depend on such fields.

The most general F-term can be found by noting that a differentiable function of chiral fields is again chiral. Hence we look at the holomorphic function $W(\Phi^i)$. There can be no dependence on $\bar{\Phi}$ because then W would not be chiral. However, as noted above $\bar{D}\bar{D}\bar{\Phi}^i$ is actually chiral and could contribute in a given term. But we can move the $\bar{D}$'s to the front of the term since they annihilate all the chiral fields thus showing that we are really dealing with a chirally exact superfield not contributing to the F-term. However, yet another possibility is the space-time derivative of a chiral field. Since ∂_μ commutes with the covariant derivatives this is again a chiral superfield. Even though we could have such contributions

[11] This definition is due to [15]. The material in this reference can now also be found in [16].

in W, we choose to assume that we do not have such terms. In this case $W(\Phi^i)$ is called the *superpotential*. Naturally, this contribution to the Lagrangian is complex so we have to add its complex conjugate (which is the analogue of the F-term for an anti-chiral field).

With these assumptions the most general (low energy effective) $\mathcal{N} = 1$ supersymmetric Lagrangian density of N chiral fields, Φ^i, is

$$\mathcal{L} = \int \mathrm{d}^2\theta W(\Phi^i) + \int \mathrm{d}^2\bar{\theta}\overline{W(\Phi^i)} + \int \mathrm{d}^4\theta K(\Phi^i, \bar{\Phi}^i). \tag{1.38}$$

In the next section we will expand the gauge version of this Lagrangian in components.

If we further constrain this Lagrangian to be renormalisable, we must require all coupling constants to have non-negative mass dimension. Since the lowest component of Φ is a complex scalar it must have mass dimension one: $[\Phi] = 1$. Since the supercharges obey (1.4c) they must have $[Q] = 1/2$. Hence by (1.24) and (1.25) we have $[\theta] = [\bar{\theta}] = -1/2$. Correspondingly $[\mathrm{d}\theta] = [\mathrm{d}\bar{\theta}] = 1/2$. Since $[\mathcal{L}] = 4$ we must have $[K(\Phi^i, \bar{\Phi}^i)] = 2$ and the only renormalisable possibility is $K(\Phi^i, \bar{\Phi}^i) = K_{ij}\Phi^i\bar{\Phi}^j$. Here K_{ij} is hermitian so we can diagonalise it by a change of the Φ^i's. The superpotential must have $[W(\Phi^i)] = 3$ so it can be at most cubic. Discarding its constant part we get the most general renormalisable Lagrangian of chiral fields:

$$\mathcal{L} = \int \mathrm{d}^4\theta \Phi^i\bar{\Phi}^i + \int \mathrm{d}^2\theta \left(a_i\Phi^i + \frac{1}{2}m_{ij}\Phi^i\Phi^j + g_{ijk}\Phi^i\Phi^j\Phi^k \right), \tag{1.39}$$

where m_{ij} and g_{ijk} are symmetric in their indices. Please note that we still have the freedom to perform a unitary rotation to diagonalise the mass matrix. Also the linear term in the superpotential can be removed (provided non-zero masses) by the transformation

$$\Phi^i \mapsto \Phi^i + b^i, \tag{1.40}$$

where b^i is constant (and hence chiral). This Lagrangian describes (for $N = 1$) the Wess-Zumino model.

1.3.3 R-Symmetry

The Lagrangians can also be restricted by employing the R-symmetries from section 1.1.4. Actually, using the Coleman-Mandula theorem one can prove that for $\mathcal{N} = 1$ we can only have a single generator, R, which does not commute with Q. R generates $\mathrm{U}(1)_R$ and we normalise it such that Q has charge -1 (denoted by $R(Q) = -1$). Correspondingly the complex conjugate $\bar{Q}$ must have charge $+1$. Since R does not commute with Q, the different components have different charge. Hence the coordinates θ have charge. By (1.24) and (1.25) we see that $R(\theta) = 1$ and $R(\bar{\theta}) = -1$. Consequently $R(\mathrm{d}\theta) = -1$ and $R(\mathrm{d}\bar{\theta}) = 1$. Assuming the Lagrangian is invariant under R-symmetry the overall charge of the Kähler potential must be zero and the charge of the superpotential must be 2. If the superfields are given R-charges $R(\Phi^i)$ we see that a renormalisable Kähler potential has charge zero since it is real. The superpotential is, however, strongly restricted by the R-symmetry.

1.3.4 Supersymmetric Gauge Theories

A second way (and the last necessary) to constrain a superfield, V, is to impose reality:

$$V = V^*. \tag{1.41}$$

Such a superfield is called a *vector superfield*. Please note that this constraint is also supersymmetric covariant since by (1.28) and (C.31) $\xi Q+\bar{\xi}\bar{Q}$ is real, and it is bosonic. When expanding V in components, it is customary to use the following notation:

$$\begin{aligned} V(x,\theta,\bar{\theta}) \;&=\; v(x)+\theta\chi(x)+\bar{\theta}\bar{\chi}(x)+\theta\theta f(x)+\bar{\theta}\bar{\theta}f^*(x)+\bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) \\ &\quad +i\theta\theta\bar{\theta}\left(\bar{\lambda}(x)+\tfrac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right)-i\bar{\theta}\bar{\theta}\theta\left(\lambda(x)+\tfrac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ &\quad +\tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x)+\tfrac{1}{2}\Box v(x)\right). \end{aligned} \tag{1.42}$$

Here v, A_μ and D must be real. The motivation for defining the components in this way is that we get nice gauge transformations. In the abelian case the gauge transformation of the vector field simply is:

$$V\mapsto V'=V+i\Lambda-i\bar{\Lambda}, \tag{1.43}$$

where Λ is a chiral superfield. According to (1.34) and (1.42) this means that $A_\mu\mapsto A_\mu+\partial_\mu(\phi+\phi^*)$. Hence the real field A_μ transforms exactly as a gauge field which, of course, is the reason that V is called a vector (or gauge) superfield. λ and D are gauge invariant in the abelian case which is the reason for the notation in (1.42).

To see how the non-abelian gauge transformations work, we must be a bit more careful. Let us assume that we have a compact gauge group G with corresponding Lie algebra $\mathcal{G}$. To obtain a unitary representation the generators T_a must be hermitian. We use roman indices a,b,c for the adjoint gauge indices. The gauge transformations must commute with the supersymmetry transformations because otherwise the commutator of a gauge transformation and a supersymmetry transformation would yield a new type of supersymmetry (since it exchanges bosons and fermions). However, it would be a local transformation since gauge transformations are local. But as noted above we will only consider global supersymmetries thus demanding gauge and supersymmetry transformations to commute.

In order to find the transformation of the vector superfield we first have to look at gauge transformations of chiral matter. Consider a representation $\mathbf{r}$ of G not necessarily irreducible. The representation is furnished by the components of $\dim(\mathbf{r})$ chiral superfields, Φ^i. Since gauge transformations commute with supersymmetry transformations each independent component must transform in the same way:

$$\phi^i\mapsto (e^{-i\Lambda^a(x)T_a^{(\mathbf{r})}})^i{}_j\phi^j, \tag{1.44}$$

and the same for ψ and F with reference to (1.34). Here $T_a^{(\mathbf{r})}$ are the generators of the gauge group in the representation $\mathbf{r}$ and $\Lambda^a(x)$ are real functions. However, according to (1.34) the chiral superfield does not transform in this way because some of the higher components involve derivatives. Instead (1.34) shows that we have the transformation:

$$\Phi\mapsto e^{-i\Lambda^a(x_+)T_a^{(\mathbf{r})}}\Phi. \tag{1.45}$$

Since x_+ is not real, this means that $\Phi^\dagger\Phi$ is not invariant, but:

$$\Phi^\dagger\mapsto\Phi^\dagger e^{i(\Lambda^a(x_+))^*T_a^{(\mathbf{r})}}. \tag{1.46}$$

Thus we need a superfield connection to make $\Phi^\dagger\Phi$ invariant. Let us choose this connection hermitian to keep the product real. We can then write it as e^{2V} where $V=V^aT_a^{(\mathbf{r})}$ and the V^a's must be vector superfields by hermiticity. The reason we have $2V$ instead of just V will become clear later. The gauge transformation of V must be:

$$e^{2V}\mapsto e^{2V'}=e^{-i(\Lambda^a(x_+))^*T_a^{(\mathbf{r})}}e^{2V}e^{i\Lambda^a(x_+)T_a^{(\mathbf{r})}}. \tag{1.47}$$

By the Baker-Campbell-Hausdorff formula the transformation of V can be written purely with commutators and is hence independent of the chosen representation. Thus we can think of V as being Lie algebra valued simply taking the proper representation when working on the chiral fields. Let us now note that we can preserve the invariance of $\Phi^\dagger e^{2V}\Phi$ when going to the larger group of *extended* gauge transformations where we replace the Λ^a's with chiral superfields. Hence (1.45) becomes:

$$\Phi \mapsto e^{-i\Lambda\left(x,\theta,\bar{\theta}\right)}\Phi, \tag{1.48}$$

where $\Lambda = \Lambda^a T_a^{(\mathbf{r})}$. This has the advantage that it preserves the chirality of the fields and hence makes it possible to build invariant Lagrangians. The transformation of the vector fields becomes:

$$e^{2V} \mapsto e^{2V'} = e^{-i\Lambda^\dagger} e^{2V} e^{i\Lambda}. \tag{1.49}$$

We note that hermiticity is preserved.

In the abelian case the transformation of the components v, χ and f in the vector superfield simply is $\delta v = i\phi - i\phi^*$, $\delta\chi = i\sqrt{2}\psi$ and $\delta f = iF$. This means that by choosing ϕ, ψ and F properly these components can be set to zero. This is called the Wess-Zumino gauge (WZ-gauge). In the non-abelian case the first two terms in V' (using the Baker-Campbell-Hausdorff formula) is $V' \sim V + \frac{1}{2}\left(i\Lambda - i\Lambda^\dagger\right)$ similar to the abelian transformation. This suggests that the Wess-Zumino gauge also is possible in the non-abelian case. This is in fact true, however, one has to consider all orders. In this gauge:

$$V(x,\theta,\bar{\theta}) = \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \qquad \text{(WZ-gauge)}. \tag{1.50}$$

Here the components are Lie algebra valued. Using (C.41) we see that this gauge has the nice property that

$$V^2 = -\tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}A_\mu A^\mu, \qquad V^3 = 0 \qquad \text{(WZ-gauge)}. \tag{1.51}$$

The Wess-Zumino gauge does not fix the gauge totally. For infinitesimal Λ, and V in Wess-Zumino gauge we get (using the Baker-Campbell-Hausdorff formula (B.7)):

$$V' = V + \frac{i}{2}(\Lambda - \Lambda^\dagger) + \frac{1}{2}[V, i(\Lambda + \Lambda^\dagger)] + \frac{1}{6}[V,[V,i(\Lambda - \Lambda^\dagger)]] \qquad \text{(WZ-gauge)}. \tag{1.52}$$

Using this we see that the infinitesimal gauge transformation that preserves Wess-Zumino gauge is

$$\Lambda = \omega(x_+) = \omega(x) - i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\omega(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\omega(x), \tag{1.53}$$

where ω is Lie algebra valued and hermitian. According to (1.52) (the last term is zero since it has to many θ's) the infinitesimal gauge transformations of the components in the Wess-Zumino gauge are:

$$\begin{aligned} \delta_\omega A_\mu &= \partial_\mu\omega - i[\omega, A_\mu], \\ \delta_\omega\lambda &= -i[\omega,\lambda], \\ \delta_\omega D &= -i[\omega, D]. \end{aligned} \tag{1.54}$$

This is exactly as usual for a gauge field A_μ, and λ and D in the adjoint representation supposing that the group elements are expressed as in (1.44).

We will see that D is an auxiliary field so we note that in the Wess-Zumino gauge the field content of the vector superfield, after imposing the equations of motion, is the same as in the vector supermultiplet hence realising this off-shell.

Although Wess-Zumino gauge allows a gauged realisation of the vector multiplet it is breaking supersymmetry (since some components are zero). This means that supersymmetry should be realised by adding a gauge transformation going back to the Wess-Zumino gauge. Thus on the components the anticommutator of the supersymmetries is no longer just proportional to the momentum generators. This makes sense since the local gauge transformations do not commute with the momentum generators, but as assumed in the beginning they commute with the supersymmetry transformations and hence also their commutator.

Let us now look for the supersymmetric version of the gauge field strength. In normal gauge theory this transforms in the adjoint representation. By (1.54) the lowest component of V that transforms in this way is λ. Let us therefore make a field with λ as its lowest component. The solution is[12]

$$\mathcal{W}_\alpha = -\frac{1}{8}\bar{D}\bar{D}e^{-2V}D_\alpha e^{2V}. \tag{1.55}$$

We note that the supersymmetric field strength is a fermionic spinor superfield which clearly is chiral – actually chirally exact.[13] It is Lie algebra valued since $e^{-2V}D_\alpha e^{2V}$ can be written as commutators. Actually, in Wess-Zumino gauge we get (using (1.51)):

$$e^{-2V}D_\alpha e^{2V} = 2D_\alpha V - 2[V, D_\alpha V] \qquad \text{(WZ-gauge)}. \tag{1.56}$$

After a bit of calculation we then get that in Wess-Zumino gauge:[14]

$$\mathcal{W}_\alpha(x,\theta,\bar{\theta}) = -i\lambda_\alpha(x_+) + \theta_\alpha D(x_+) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha{}^\beta\theta_\beta F_{\mu\nu}(x_+) + \theta\theta\sigma^\mu_{\alpha\dot{\beta}}D_\mu\bar{\lambda}^{\dot{\beta}}(x_+) \qquad \text{(WZ-gauge)}, \tag{1.57}$$

where the components are Lie algebra valued. Here $F_{\mu\nu}$ is the usual gauge field strength given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \tag{1.58}$$

and D_μ is the gauge covariant derivative:

$$D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + i[A_\mu, \bar{\lambda}]. \tag{1.59}$$

Thus $\mathcal{W}_\alpha$ neatly turns out to contain the usual gauge field strength thus justifying it to be the supersymmetric gauge field strength (this is the reason for the involved definition (1.55)).

The gauge transformation of $\mathcal{W}_\alpha$ is:

$$\mathcal{W}_\alpha \mapsto \mathcal{W}'_\alpha = e^{-i\Lambda}\mathcal{W}_\alpha e^{i\Lambda}. \tag{1.60}$$

This simply shows that $\mathcal{W}_\alpha$ transforms in the adjoint representation. To prove this one uses that Λ is chiral and hence $\Lambda^\dagger$ is anti-chiral along with the relation

$$[\bar{D}_{\dot{\alpha}}, \{\bar{D}_{\dot{\beta}}, D_\gamma\}] = 0, \tag{1.61}$$

which follows immediately from (1.31).

[12] We use calligraphic font to distinguish the supersymmetric field strength from the superpotential.
[13] $\mathcal{W}_\alpha$ is actually also constrained by Bianchi identities.
[14] We note that since $F_{\mu\nu}$ is antisymmetric, we could have rewritten $\frac{1}{2}\sigma^\mu\bar{\sigma}^\nu F_{\mu\nu} = \sigma^{\mu\nu}F_{\mu\nu}$.

Let us now look for the most general supersymmetric gauge invariant Lagrangian. If we only allow two space-time derivatives on bosonic fields and one on fermionic fields as in section 1.3.2 we can at most have terms of $\mathcal{W}$ squared. To obtain Lorentz invariance we must naturally look at $\mathcal{W}^\alpha \mathcal{W}_\alpha$ which is also chiral. Thus the most general gauge kinetic and self interaction term is:

$$\mathcal{L}_G = \int \mathrm{d}^2\theta \tau_{ab}(\Phi^i)\, \mathcal{W}^{\alpha a} \mathcal{W}^b_\alpha + \text{c.c.}, \tag{1.62}$$

where we have added the complex conjugate to make the Lagrangian real. We have here included functions τ_{ab} holomorphic in the chiral fields Φ^i in the spirit of section 1.3.2. τ_{ab} must be symmetric and transform as an invariant tensor in the adjoint representation of the gauge group hence putting restrictions on the dependence on the chiral fields. We note that since the gauge field strength by (1.55) is chirally exact, it can be written as a D-term using (1.37) (up to total space-time derivative terms):

$$\mathcal{L}_G = -\int \mathrm{d}^4\theta \tau_{ab}(\Phi^i)\, \frac{1}{16} \left(e^{-2V} D_\alpha e^{2V}\right)^a \left(\bar{D}\bar{D} e^{-2V} D_\alpha e^{2V}\right)^b + \text{c.c.} \tag{1.63}$$

However, the integrand of this full superspace integral is not gauge invariant. Thus it is natural to keep the Lagrangian in the form (1.62) and in the rest of this thesis we will think of it as a $\theta\theta$-term.

The Kähler term in (1.38) must now have the form

$$\mathcal{L}_K = \int \mathrm{d}^4\theta K\left(e^{2V^{(\mathbf{r})}}\Phi, \Phi^\dagger\right), \tag{1.64}$$

where $V^{(\mathbf{r})}$ is the vector field in the appropriate representation. The Kähler potential must be formed such that it is gauge invariant. The most simple example is $\Phi^\dagger e^{2V^{(\mathbf{r})}}\Phi$ as we saw above. This is renormalisable (polynomial) since in Wess-Zumino gauge $e^V = 1 + V + \frac{1}{2}V^2$. By the assumptions in section 1.3.2 the Kähler potential does not include derivatives. This means that it is enough to require K to be globally (but complex) gauge invariant.

We can also have a superpotential term $\mathcal{L}_W$ as above which now must be formed such that it is gauge invariant. However, here it is enough to require the Lagrangian to be globally gauge invariant since we have neither derivatives nor complex conjugates of the chiral fields:

$$\mathcal{L}_W = \int \mathrm{d}^2\theta W(\Phi^i) + \text{c.c.} \tag{1.65}$$

If the gauge group G has an abelian factor we can also include a *Fayet-Iliopoulos* term. Let κ be a non-zero element in the center of the algebra and as usual let Tr denote the invariant inner product of the Lie algebra (we can think of Tr as trace in the fundamental representation) then:

$$\mathcal{L}_{FI} = \int \mathrm{d}^4\theta\, \mathrm{Tr}(\kappa V) = \frac{1}{2}\,\mathrm{Tr}(\kappa D), \tag{1.66}$$

where the last part is in Wess-Zumino gauge. This is supersymmetric since it is the D-term of a superfield and it is gauge invariant since by (1.54) D transforms in the adjoint representation.

With the above constraints the most general Lagrangian is:

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_G + \mathcal{L}_W + \mathcal{L}_{FI}. \tag{1.67}$$

One could add a D-term m^2V^2 to gain a representation of the massive gauge supermultiplet, however, this term is not gauge invariant.

Let us expand these Lagrangians into components using (1.35) and (1.36). By simple differentiation we get:

$$\mathcal{L}_W = F^i \frac{\partial W(\phi^i)}{\partial \phi^i} - \frac{1}{2}\psi^i\psi^j \frac{\partial^2 W(\phi^i)}{\partial\phi^i\partial\phi^j} + \text{c.c.} \tag{1.68}$$

In order to expand $\mathcal{L}_G$ we calculate (in Wess-Zumino gauge):

$$\begin{aligned}\mathcal{W}^{\alpha a}\mathcal{W}^b_\alpha = e^{i\theta\sigma^\mu\bar\theta\partial_\mu}\Big[&-\lambda^a\lambda^b - i\theta\lambda^a D^b - i\theta\lambda^b D^a + \tfrac{1}{2}\theta\left(\sigma^\mu\bar\sigma^\nu\right)\left(\lambda^a F^b_{\mu\nu} + \lambda^b F^a_{\mu\nu}\right)\\ &-\theta\theta\left(i\lambda^a\sigma^\mu\left(D_\mu\bar\lambda\right)^b + i\lambda^b\sigma^\mu\left(D_\mu\bar\lambda\right)^a + \tfrac{1}{2}\left(F^{a\mu\nu}F^b_{\mu\nu} + iF^{a\mu\nu}\tilde F^b_{\mu\nu}\right) - D^aD^b\right)\Big]\end{aligned}$$
$$\text{(WZ-gauge)}, \tag{1.69}$$

where $\tilde F$ is the Poincaré dual defined by

$$\tilde F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\kappa}F^{\rho\kappa}. \tag{1.70}$$

In order to get (1.69) we have used (C.40), (C.45) and (C.48).[15] The differential operator in the beginning of the equation simply ensures that we are evaluating in x_+. Using this we get (independent of the choice of gauge):

$$\begin{aligned}\mathcal{L}_G \;=\;& -\lambda^a\lambda^b\left(F^i\frac{\partial\tau_{ab}}{\partial\phi^i} - \frac{1}{2}\psi^i\psi^j\frac{\partial^2\tau_{ab}}{\partial\phi^i\partial\phi^j}\right)\\ &-\frac{1}{2\sqrt{2}}\frac{\partial\tau_{ab}}{\partial\phi^i}\psi^i\left(-i\lambda^a D^b - i\lambda^b D^a + \frac{1}{2}\left(\sigma^\mu\bar\sigma^\nu\right)\left(\lambda^a F^b_{\mu\nu} + \lambda^b F^a_{\mu\nu}\right)\right)\\ &-\tau_{ab}\left(i\lambda^a\sigma^\mu\left(D_\mu\bar\lambda\right)^b + i\lambda^b\sigma^\mu\left(D_\mu\bar\lambda\right)^a + \tfrac{1}{2}\left(F^{a\mu\nu}F^b_{\mu\nu} + iF^{a\mu\nu}\tilde F^b_{\mu\nu}\right) - D^aD^b\right)\\ &+\text{c.c.}\end{aligned} \tag{1.71}$$

where τ_{ab} is seen as a function of ϕ^i. As noted above one could have used Majorana spinors instead of Weyl spinors and thereby have written all of this even more compactly. However, we shall not need this.

The Kähler term requires more computation. The result is (following, but correcting [13]):

$$\begin{aligned}\mathcal{L}_K \;=\;& -g_{i\bar\imath}D_\mu\phi^i\left(D^\mu\phi\right)^{*\bar\imath} - ig_{i\bar\imath}\bar\psi^{\bar\imath}\bar\sigma^\mu D_\mu\psi^i + \frac{1}{4}R_{i\bar k j\bar l}\psi^i\psi^j\bar\psi^{\bar k}\bar\psi^{\bar l}\\ &+g_{i\bar\imath}\left(F^i - \frac{1}{2}\Gamma^i_{jk}\psi^j\psi^k\right)\left(\bar F^{\bar\imath} - \frac{1}{2}\Gamma^{\bar\imath}_{\bar j\bar k}\bar\psi^{\bar j}\bar\psi^{\bar k}\right)\\ &+\left(\frac{1}{2}D^a(T^{(\mathbf{r})}_a)^j{}_i\phi^i\frac{\partial K}{\partial\phi^j} + i\sqrt{2}g_{i\bar\imath}(T^{(\mathbf{r})}_a)^i{}_k\phi^k\bar\lambda^a\bar\psi^{\bar\imath} + \text{c.c.}\right).\end{aligned} \tag{1.72}$$

The added complex conjugate in the last line is only for the last two terms. Here $g_{i\bar\imath}$ is the *Kähler metric* defined by

$$g_{i\bar\imath} = \frac{\partial^2 K(\phi^i,\bar\phi^{\bar\imath})}{\partial\phi^i\partial\bar\phi^{\bar\jmath}}. \tag{1.73}$$

[15] The sign on the $F\tilde F$ term is not standard in the literature. The reason ought to be the also unconventional sign on $\sigma^0 = -\mathbf{1}$.

Γ and R are the corresponding Levi-Civita connection and Riemann curvature tensor respectively. $D_\mu\phi^i$ is the usual gauge covariant derivative while $D_\mu\psi^i$ also contains the Levi-Civita connection:

$$D_\mu\psi^i = \partial_\mu\psi^i + iA^a_\mu (T^{(\mathbf{r})}_a)^i{}_j\psi^j + \Gamma^i_{jk}\left(D_\mu\phi^j\right)\psi^k. \tag{1.74}$$

We have used a notation where the indices of the complex conjugates are barred to stress that they are treated as independent coordinates.

We immediately see that the F^i's and D^a's are auxiliary fields with no derivatives as postulated above. Hence they can be replaced with their equations of motions. This is also true in the quantised theory: Because the auxiliary fields appear at most quadratically they can be integrated out.[16] However, the supersymmetry transformations on these auxiliary fields then give restrictions which exactly correspond to the equations of motion. Thus on the remaining fields supersymmetry is only realised on-shell. This agrees with the supermultiplets from section 1.2 being on-shell.

Let us finish this section by looking at the case where τ_{ab} from (1.62) is independent of the chiral fields. This is the case when we look at renormalisable Lagrangians because according to (1.57) the mass dimension of $\mathcal{W}_\alpha$ is $3/2$ (the lowest component is a spinor) and hence $\int \mathrm{d}^2\theta\mathcal{W}^\alpha\mathcal{W}_\alpha$ has mass dimension four. We can now split the gauge group in its abelian and simple parts. The Lagrangian (1.62) splits into a sum with one term for each part of the gauge group. There can be no mixed terms due to gauge invariance.

Let us first look at the abelian part. Here each $\mathcal{W}^a_\alpha$ is gauge invariant. This allows us to write the Lagrangian as:

$$\mathcal{L}_{G,\text{abelian}} = \int \mathrm{d}^2\theta \frac{1}{16\pi i}\tau_{ab}\mathcal{W}^{\alpha a}\mathcal{W}^b_\alpha + \text{c.c.}, \tag{1.75}$$

where τ_{ab} is the theta angles and gauge couplings:

$$\tau_{ab} = \frac{\vartheta_{ab}}{2\pi} + i\frac{4\pi}{g^2_{ab}}. \tag{1.76}$$

Plugging into (1.71) yields:

$$\mathcal{L}_{G,\text{abelian}} = -\frac{1}{2g^2_{ab}}\left(i\lambda^a\sigma^\mu\partial_\mu\bar\lambda^b + i\lambda^b\sigma^\mu\partial_\mu\bar\lambda^a\right) - \frac{1}{4g^2_{ab}}F^a_{\mu\nu}F^{b\mu\nu} - \frac{\vartheta_{ab}}{32\pi^2}F^a_{\mu\nu}\tilde F^{b\mu\nu} + \frac{1}{2g^2_{ab}}D^aD^b. \tag{1.77}$$

Here we have used (C.44) and allowed integration by parts to get the first term. (1.77) is the standard form of (supersymmetric) Yang-Mills theory and this is the reason for the chosen normalisation factor in (1.75).

For a simple factor τ_{ab} must be proportional to the Killing form. It can also be shown that we can choose the generators of the simple factor such that in any irreducible representation, $\mathbf{r}$, we have

$$\mathrm{Tr}_{(\mathbf{r})}\left(T^{(\mathbf{r})}_a T^{(\mathbf{r})}_b\right) = C(\mathbf{r})\delta_{ab}. \tag{1.78}$$

[16] Actually this is oversimplified. The coefficients of the squares of the auxiliary fields are functions possibly depending on x^μ. This means that when we do the functional integration of the auxiliary fields we encounter the determinants of the coefficient functions. Let us call such a coefficient function $f(x)$. The corresponding "matrix" is $A(x,y) = f(x)\delta^4(x-y)$ such that e.g. $\iint \mathrm{d}^4x\mathrm{d}^4y D^a(x)A(x,y)D^a(y) = \int \mathrm{d}^4x D^a(x)f(x)D^a(x)$. The determinant can be rewritten as the exponential of the trace. Hence we have to add a term to the action proportional to $\int \mathrm{d}^4x \ln A(x,x) = \int \mathrm{d}^4x \ln f(x)\delta^4(0)$. However, (following [10]) using dimensional regularisation of the determinant this term is eliminated since here $\delta^4(0) = \int \frac{\mathrm{d}^dq}{(2\pi)^d} = \frac{\Gamma(1-d/2)}{\Gamma(0)}\left(\frac{1}{4\pi}\right)^{d/2} = 0$ because the Γ-function of zero is infinite and d is the dimension close to, but not equal 4.

Here $C(\mathbf{r})$ is called the quadratic invariant. The gauge kinetic Lagrangian for the simple factor then becomes:

$$\begin{aligned}\mathcal{L}_{G,\text{simple}} &= \int \mathrm{d}^2\theta \frac{\tau}{16\pi i C(\mathbf{r})} \operatorname{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha \mathcal{W}_\alpha) + \text{c.c.} \\ &= \int \mathrm{d}^2\theta \frac{\tau}{16\pi i} \mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha + \text{c.c.},\end{aligned} \tag{1.79}$$

where the complex τ contains the theta angle and the gauge coupling constant:

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2}. \tag{1.80}$$

There is one coupling τ for each simple factor in the gauge group. Expanding in the same way as in (1.77) gives:

$$\mathcal{L}_{G,\text{simple}} = -\frac{i}{g^2}\lambda^a\sigma^\mu\left(D_\mu\bar{\lambda}\right)^a - \frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} - \frac{\vartheta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{1}{2g^2}D^aD^a. \tag{1.81}$$

We will see in section 2.5.3 that the theta terms are total derivative terms which in the non-abelian case can be non-zero due to instanton effects. The coupling constant g, which here enters by multiplying terms with $1/g^2$, can be put into the structure constants by rescaling $V \mapsto gV$. This removes the overall $1/g^2$ (but puts a factor g^2 on the ϑ-term) and the coupling g will then multiply the structure constants in the definition of $F_{\mu\nu}$ and D_μ in equations (1.58) and (1.59) – as we often see in non-supersymmetric gauge theory.

We also note that our Lagrangians are not invariant under rescaling of the generators T_a. Scaling $T_a \mapsto \alpha T_a$ will scale the structure constants as ${f_{ab}}^c \mapsto \alpha {f_{ab}}^c$. The vectors in the Lie algebra should be the same under this change of basis and hence the components of the vector field and correspondingly the supersymmetric gauge field strength scale as $V^a \mapsto \frac{1}{\alpha}V^a$ and $\mathcal{W}^a_\beta \mapsto \frac{1}{\alpha}\mathcal{W}^a_\beta$ (we have to remember the scaling of the structure constants in the last scaling). We conclude that to keep our Lagrangians invariant under the scaling we must require that the couplings scale as $g \mapsto g/\alpha$ and $\vartheta \mapsto \alpha^2\vartheta$. Accordingly we have to choose some normalisation of the generators for the theory to make sense.[17]

1.4 $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

In this section we will briefly discuss the $\mathcal{N}=2$ supersymmetric gauge field theories. These can be obtained from the results in the last section by noting that $\mathcal{N}=2$ supersymmetric field theories are special cases of $\mathcal{N}=1$ supersymmetric field theories.

1.4.1 $\mathcal{N}=2$ Supersymmetric Lagrangians from $\mathcal{N}=1$ Supermultiplets

Let us as above denote the supercharges of the $\mathcal{N}=2$ supersymmetry as $\mathcal{Q}^1$ and $\mathcal{Q}^2$ where we have suppressed the spinor indices. In section 1.2 we obtained the supermultiplets as representations of the little supergroup. The little supergroup was obtained by adding the supercharges to the algebra of the little group. However, looking at the supersymmetry algebra (1.4) we see that we also get a group if we only add $\mathcal{Q}^1$ and its hermitian conjugate $\bar{\mathcal{Q}}_1$ to the little group. This is exactly the $\mathcal{N}=1$ little supergroup which accordingly is a subgroup of the $\mathcal{N}=2$ little supergroup. Thus the $\mathcal{N}=2$ supermultiplets split into

[17] We could also have chosen to obtain an invariant theory by multiplying with e.g. the quadratic invariant in the fundamental representation, $C(\text{fund})$, in (1.79 since the quadratic invariant scales as $C(\mathbf{r}) \mapsto \alpha^2 C(\mathbf{r})$.

Figure 1.1:

$$\begin{tikzcd}
 & A_\mu & \\
\lambda \arrow[ur, "\bar{Q}_1"] \arrow[rr, dashed, leftrightarrow, "\mathrm{SU}(2)_R"] & & \psi \arrow[ul, "\bar{Q}_2"'] \\
 & \phi \arrow[ul, "\bar{Q}_2"] \arrow[ur, "\bar{Q}_1"'] &
\end{tikzcd}$$

The $\mathcal{N} = 2$ gauge supermultiplet with supersymmetry transformations. The spinor indices on the supercharges have been suppressed.

$\mathcal{N} = 1$ supermultiplets just as supermultiplets are multiplets of Poincaré representations. However, we could just as well had looked at the subgroup corresponding to adding $\mathcal{Q}^2$ and its hermitian conjugate to the little group thus giving another splitting in $\mathcal{N} = 1$ supermultiplets. This change corresponds to the R-symmetry:

$$\mathcal{Q}^1 \mapsto \mathcal{Q}^2, \qquad \mathcal{Q}^2 \mapsto -\mathcal{Q}^1. \tag{1.82}$$

Now the method to get an $\mathcal{N} = 2$ Lagrangian is simply to split the $\mathcal{N} = 2$ supermultiplets into $\mathcal{N} = 1$ supermultiplets corresponding to say $\mathcal{Q}^1$, then write the most general $\mathcal{N} = 1$ supersymmetric Lagrangian with these supermultiplets, and finally impose the discrete R-symmetry (1.82). The Lagrangian will then be $\mathcal{N} = 2$ supersymmetric since it is invariant under $\mathcal{Q}^1$ by construction and hence invariant under $\mathcal{Q}^2$ by the R-symmetry (1.82).

1.4.2 Renormalisable $\mathcal{N} = 2$ Supersymmetric Lagrangians

Let us now find the most general renormalisable Lagrangian for the fields of the $\mathcal{N} = 2$ massless gauge supermultiplet. For simplicity we will assume a simple gauge group G. The fields of this multiplet are according to table 1.1 a gauge field A_μ, two Weyl (or Majorana) fermions λ and ψ, and a complex scalar ϕ. In figure 1.1 it is shown how these fields are related by the supercharges $\bar{Q}_1$ and $\bar{Q}_2$ according to equation (1.15). Breaking the supermultiplet into the $\mathcal{N} = 1$ supermultiplets of the supercharge $\mathcal{Q}^1$ we get a gauge multiplet (λ, A_μ) and a chiral multiplet (ϕ, ψ) corresponding to the superfields V and Φ respectively.

A_μ transforms in the adjoint representation of the gauge group so the whole $\mathcal{N} = 2$ supermultiplet must transform in the adjoint representation since we argued above that supersymmetry transformations and gauge transformations must commute. Thus Φ is in the adjoint representation and both V and Φ can be seen as taking values in the gauge Lie algebra.[18] The most general Lagrangian involving these two fields is given in (1.67) where we demand renormalisability. This means that the gauge kinetic and self interaction term takes the form (1.79) since the gauge group is assumed simple. Using (1.39) and (1.64) the renormalisable gauge invariant Kähler term takes the form $\int \mathrm{d}^4\theta \, \mathrm{Tr}\big(\Phi^\dagger e^{2\,\mathrm{ad}V}\Phi\big)$ where $\mathrm{ad}V$ is the adjoint defined in (B.9). The trace can be taken in any representation, but we will now have to care about the normalisation of the terms in the Lagrangian since we have to

[18]This also means that the $\mathcal{N} = 2$ supersymmetric Lagrangian can not fulfil the standard model since that would require the matter to belong to a complex representation (the chiral) of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, but the adjoint representation is always real. Even when matter is added the representation will still be real.

impose the symmetry (1.82). On the components we see from figure 1.1 that this symmetry takes the form:

$$\psi^a \mapsto \lambda^a \qquad \lambda^a \mapsto -\psi^a. \tag{1.83}$$

Thus the kinetic term for ψ and λ must have the same normalisation. Using (1.68) we see that the superpotential term for the chiral field must be proportional to Φ since we have no mass or interaction terms for the gaugino field λ. A linear superpotential is trivial and hence we will set it to zero. There can be no Fayet-Iliopoulos term since we assumed a simple group. The $\mathcal{N} = 2$ supersymmetric Yang-Mills Lagrangian then takes the form (using (1.72) and (1.81)):

$$\mathcal{L}_{\mathcal{N}=2} = \frac{\tau}{16\pi i C(\mathbf{r})} \operatorname{Tr}_{(\mathbf{r})} \left(\int \mathrm{d}^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha + 2 \int \mathrm{d}^4\theta \Phi^\dagger e^{2\,\mathrm{ad}V} \Phi \right) + \text{c.c.} \tag{1.84}$$

Expanded in components this gives (again using (1.72) and (1.81)):

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=2} = \frac{1}{g^2 C(\mathbf{r})} \operatorname{Tr}_{(\mathbf{r})} \Big(& - i\lambda\sigma^\mu D_\mu \bar{\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{g^2 \vartheta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} DD - D_\mu \phi \, (D^\mu \phi)^\dagger \\ & - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + F\bar{F} + D[\phi^\dagger, \phi] - i\sqrt{2}[\bar{\lambda}, \bar{\psi}]\phi - i\sqrt{2}[\lambda, \psi]\phi^\dagger \Big), \end{aligned} \tag{1.85}$$

where we have used the cyclic property of the trace (properly signed for the anticommuting fields) to put the commutators in the above form. The commutator $[\lambda, \psi]$ simply means $\lambda\psi - \psi\lambda$ thus defining the spinor indices. Here the bars on the spinor fields mean hermitian conjugates in the gauge algebra. We immediately see that this Lagrangian is invariant under the symmetry (1.83).

As mentioned above we can now eliminate the auxiliary fields using their equations of motion $F = 0$ and $D = -[\phi^\dagger, \phi]$. This gives the scalar potential:

$$U = \frac{1}{2g^2 C(\mathbf{r})} \operatorname{Tr}_{(\mathbf{r})} \left([\phi^\dagger, \phi]^2 \right). \tag{1.86}$$

It is possible to add matter to the model in the form of the hypermultiplet from table 1.1. Such a hypermultiplet splits into two chiral multiplets. Adding this to the Lagrangian allows mixed superpotential terms.

For non-renormalisable Lagrangians one can use an $\mathcal{N} = 2$ superspace formulation to get the most general Lagrangian. As is shown in appendix D, the Lagrangian is then determined by a holomorphic function known as the prepotential.

Chapter 2

The Dijkgraaf-Vafa Conjecture

In [4] R. Dijkgraaf and C. Vafa formulated a conjecture telling us how to systematically compute the exact low energy effective superpotential of a wide range of $\mathcal{N}=1$ supersymmetric gauge theories in four space-time dimensions.

We will start by simply stating the conjecture in the case of a $\mathrm{U}(N)$ gauge group and adjoint matter. However, most of the concepts used in the Dijkgraaf-Vafa conjecture needs explaining so the rest of the chapter will be devoted to understanding the conjecture. Along the way we will understand some of its implications and put it into its right context. At the end of the chapter we will present the Dijkgraaf-Vafa conjecture for general gauge groups and matter representations.

2.1 The Dijkgraaf-Vafa Conjecture with U(N) Gauge Group and Adjoint Matter.

Following [4] we will state the Dijkgraaf-Vafa conjecture in the case of a $\mathrm{U}(N)$ gauge group and adjoint matter.[1]

2.1.1 The Traceless Case

We will study a four-dimensional supersymmetric field theory. The tree-level Lagrangian of the theory is obtained by first looking at a renormalisable $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $\mathrm{U}(N)$ and corresponding Lie algebra $\mathcal{U}(N)$. Assuming no Fayet-Iliopoulos term since this makes it more difficult to obtain supersymmetric vacua as we will see in section 2.2, the Lagrangian is simply (1.84) with an abelian part added (in section 2.1.4 we will write out the full Lagrangian). Then we add a tree-level superpotential for the adjoint $\mathcal{U}(N)$-valued chiral field Φ to the Lagrangian:

$$\int \mathrm{d}^2\theta W_{\mathrm{tree}}(\Phi) = \int \mathrm{d}^2\theta \,\mathrm{Tr}\, P_{n+1}(\Phi)\,, \tag{2.1}$$

where P_{n+1} is a complex polynomial of degree $n+1$ which we in general will think of as having the form:

$$P_{n+1}(\Phi) = \frac{1}{2}m\Phi^2 + \sum_{k=3}^{n+1}\frac{g_k}{k}\Phi^k, \tag{2.2}$$

[1] The notation will differ slightly from [4] and some points are taken from [7].

where m and the couplings g_k are complex.[2] Assuming $n \geq 1$ this breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry as we saw in section 1.4. We should think of $\Phi = \Phi^a T_a$ as taking values in the fundamental representation[3] since e.g. in the adjoint representation the abelian part would vanish. Thus here and in the following Tr will denote trace in the fundamental representation.

We will not restrict ourselves to renormalisable superpotentials and hence allow P_{n+1} to have a degree higher than three. In that case we can think of the superpotential as obtained from a superpotential at a higher energy scale by integrating out other fields using Wilsonian renormalisation (explained in section 2.4).

As we will explain in section 2.2 the supersymmetric classical vacua are obtained by diagonalising Φ and demanding the eigenvalues to be in the set of critical points of the polynomial P_{n+1}. Since P'_{n+1} has degree n, there must be n critical points which we denote $a_1, \ldots, a_n$. We will assume these to be isolated and – as we will show – this means that the vacua are massive. Since Φ is an $N \times N$-matrix, a vacua is obtained by choosing a partition:

$$N = N_1 + \ldots + N_n, \tag{2.3}$$

corresponding to distributing N_i of the eigenvalues at the critical point a_i. This furthermore breaks the gauge symmetry group as:

$$\mathrm{U}(N) \mapsto \mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_n). \tag{2.4}$$

If N_i is zero we will leave the corresponding factor out.

Looking at the corresponding quantised theory (section 2.3), the lore of the low energy dynamics (section 2.5.6) tells us that we have confinement and gaugino condensation in the $\mathrm{SU}(N_i)$ subgroups of the $\mathrm{U}(N_i)$ factors. The gauge coupling becomes strong at the (complex) dynamically generated scale Λ and a mass gap is generated. To describe this we introduce the *traceless glueball superfield*[4] for each factor of $\mathrm{SU}(N_i)$:

$$\hat{S}_i = -\frac{1}{16\pi^2}\,\mathrm{Tr}\left(\hat{\mathcal{W}}^{\alpha}_{(i)}\hat{\mathcal{W}}_{(i)\alpha}\right), \tag{2.5}$$

where the supersymmetric gauge field strength for $\mathrm{SU}(N_i)$ is denoted $\hat{\mathcal{W}}^{\alpha}_{(i)}$. Please note that here and in the following there is no sum over the factor indices in parenthesis. As noted above we use the fundamental representation for the Lie algebra valued fields. Here we use a normalisation such that the generators for the simple $\mathrm{SU}(N_i)$ part $T^{(\mathrm{fund})}_{(i)a}$ with $a = 1, \ldots, N_i^2 - 1$ fulfil:

$$\mathrm{Tr}\left(T^{(\mathrm{fund})}_{(i)a}T^{(\mathrm{fund})}_{(i)b}\right) = \frac{1}{2}\delta_{ab}, \qquad \mathrm{Tr}\left(T^{(\mathrm{fund})}_{(i)a}\right) = 0. \tag{2.6}$$

Thus from equation (1.69) we see that the lowest component of the traceless glueball superfield is $\frac{1}{32\pi^2}\hat{\lambda}^a_{(i)}\hat{\lambda}^a_{(i)}$ where, as before, the hat means restriction to the traceless part of the algebra. The condensation of the gauginos described by the field $\hat{\lambda}^{\alpha}_{(i)}$ thus corresponds to $\hat{S}_i$ getting a dynamical expectation value. We also see that the chiral half of the gauge Lagrangian for $\mathrm{SU}(N_i)$ (equation (1.79)) is given by the $\theta\theta$-component of $\hat{S}_i$.

[2]The reason that we have m and not m^2 in the tree-level superpotential is that Φ has mass dimension one and the whole superpotential has mass dimension three as we saw in section 1.3.2.

[3]This should not be confused with Φ *transforming* in the adjoint representation, but means that $\Phi = \Phi^a T^{(\mathrm{fund})}_a$ where $T^{(\mathrm{fund})}_a$ are the generators of the gauge group in the fundamental representation.

[4]This is – perhaps more properly – also called the *gaugino condensate chiral superfield.*

The physical quantities of the vacua are determined by the 1PI effective action. However, as this is too hard to find we focus on the Wilsonian effective action and we denote the corresponding Lagrangian $\mathcal{L}_{\text{eff}}$. It is the *generalised superpotential* W_{eff} of this effective Lagrangian which we want to determine (dealt with in detail in section 2.5):

$$\mathcal{L}_{\text{eff}} = \int \mathrm{d}^2\theta W_{\text{eff}} + \text{c.c.} + \int \mathrm{d}^4\theta \ldots, \tag{2.7}$$

where the dots denote some local gauge invariant superspace function which is not in the focus of interest here. Hence the generalised superpotential consists of the $\theta\theta$-terms that can not be written as local gauge invariant D-terms. Thus it includes terms like (1.79) i.e. $\hat{S}_i$ can contribute to W_{eff}.[5] Actually, it is the lore that the elementary fields at low energy exactly are the $\hat{S}_i$'s. $W_{\text{eff}}\big(\hat{S}_i, g_k\big)$ is thus called the effective glueball superpotential. The vacuum expectation value of $\hat{S}_i$ is then simply determined by:

$$\frac{\partial W_{\text{eff}}\big(\hat{S}_k, g_k\big)}{\partial \hat{S}_i} = 0. \tag{2.8}$$

As we will see in section 2.5.7, even without matter ($\Phi = 0$) we have an effective superpotential called the Veneziano-Yankielowicz superpotential. This is given by:

$$W_{\text{VY}} = \sum_i N_i \hat{S}_i \left(1 - \ln \frac{\hat{S}_i}{\Lambda_i^3}\right), \tag{2.9}$$

where we have one term for each gauge group factor and the scales Λ_i are described in section 2.5.8.

The Dijkgraaf-Vafa conjecture tells us that in order to determine W_{eff} we have to look at the related bosonic *one matrix model* (we will entertain ourselves with the details in section 2.7) with partition function given by:

$$Z_{\text{matrix}} = \int \mathcal{D}M e^{-\frac{1}{g_{\text{s}}} W_{\text{tree}}(M)}, \tag{2.10}$$

i.e. where the potential is the tree-level superpotential. Here M are $N' \times N'$ hermitian matrices (spanned by the generators of $\mathrm{U}(N')$) and g_{s} is a simple (dimensionful) scaling factor – the "s" simply refers to the stringy origin of the Dijkgraaf-Vafa conjecture, but here it, a priori, has nothing to do with string theory.

The vacua of the matrix model are, analogously to the gauge theory case, determined by diagonal matrices with the N' eigenvalues in the set of critical points, $a_1, \ldots, a_n$, of P_{n+1}. Thus, in analogy with (2.3) choosing a vacuum (modulo permutation of eigenvalues) corresponds to a partition:

$$N' = N'_1 + \ldots + N'_n. \tag{2.11}$$

To obtain the correspondence with the gauge theory we must here demand that $N'_i = 0$ if $N_i = 0$ in order to have the same gauge symmetry breaking pattern. Otherwise N'_i and N_i are completely independent. We can now obtain the free energy of the matrix model by a perturbative expansion around the chosen vacuum. However, in the case of broken gauge symmetry we must remember to take into account Faddeev-Popov ghosts in the

[5] Remember that $\mathcal{W}_\alpha \mathcal{W}^\alpha$ is chirally exact and hence can be written as a D-term as in (1.63), but not as a gauge invariant D-term!

matrix model. These actually take the same form as in the gauge theory in accordance with correspondence between the potential of the matrix model and the tree-level superpotential.[6]

We should now take the 't Hooft large N' limit (explained in section 2.6) where $N_i' \gg 1$, $g_\mathrm{s} \ll 1$ while keeping $g_\mathrm{s} N_i'$ fixed and finite. We will see how to topologically characterise diagrams such that the free energy[7] can be written as a topological expansion:

$$Z_\mathrm{matrix} = e^{-\sum_{g\geq 0} g_\mathrm{s}^{2g-2} \mathcal{F}_g\left(g_\mathrm{s} N_i'\right)}, \tag{2.12}$$

where g is the genus of the surface i.e. an integer number greater than or equal zero which corresponds to the number of handles that has been added to the sphere. The $\mathcal{F}_g$'s depend only on g_s and N_i' through the products $g_\mathrm{s} N_i'$. We see that the dominant contribution stems from $g = 0$ and we can thus restrict to $\mathcal{F}_{g=0}$ which is called the planar limit. The connection to the gauge theory is to identify:

$$\hat{S}_i \equiv g_\mathrm{s} N_i'. \tag{2.13}$$

We note that this is a formal identification that allows us to obtain the glueball superpotential from the planar limit of the free energy in the matrix model. We will see in section 2.7.3 that when the matrix model is solved exactly, $\hat{S}_i$ should be identified with the filling fractions in the multi-cut solution.

Now we can state the Dijkgraaf-Vafa conjecture in the case where we ignore the abelian part of the $\mathcal{W}_\alpha$'s such that the glueball superfield is traceless – but we do not disregard the abelian part of Φ:

Dijkgraaf-Vafa conjecture; Traceless case.

$$W_\mathrm{eff}\left(\hat{S}_i, g_k\right) = W_\mathrm{eff,pert}\left(\hat{S}_i, g_k\right) + W_\mathrm{VY}\left(\hat{S}_i\right), \tag{2.14a}$$

$$W_\mathrm{eff,pert}\left(\hat{S}_i, g_k\right) = \sum_i N_i \frac{\partial \mathcal{F}_{g=0}\left(\hat{S}_i, g_k\right)}{\partial \hat{S}_i}, \qquad \hat{S}_i \equiv g_\mathrm{s} N_i'. \tag{2.14b}$$

The dependence on the couplings in the tree-level superpotential, g_k, has been added for completeness. $W_\mathrm{eff,pert}$ is perturbative in $\hat{S}_i$ and the couplings g_k. It is obtained by integrating out the massive chiral field Φ (and $\bar{\Phi}$) while treating $\mathcal{W}_\alpha$ as a background field. We can write this as (we are in Minkowski space):

$$Z_\mathrm{holo} = \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi}\; e^{iS_\mathrm{tree}}\big|_\mathrm{holomorphic}, \tag{2.15}$$

$$Z_\mathrm{holo} = e^{i\int \mathrm{d}^4 x \mathrm{d}^2\theta W_\mathrm{eff,pert}}, \tag{2.16}$$

where Z_holo is the partition function where we only keep the contribution to the $\theta\theta$-term. That is, we only include the holomorphic contributions with no conjugates (section 2.5.1). This is what the restriction in the first line means. S_tree is the tree-level action including the superpotential (2.1). The Dijkgraaf-Vafa conjecture in this case is summarised in table 2.1.[8]

[6]This perturbative (diagrammatic) approach on the matrix model side in the formulation of the Dijkgraaf-Vafa conjecture can be found in [17].

[7]For now we ignore any contribution from the measure of the matrix model. Such a contribution could been seen as giving the Veneziano-Yankielowicz superpotential (section 2.7.2).

[8]It appears to be more suggestive to compare the matrix model with the gauge theory written in Euclidean space. However, when Wick rotating to Euclidean space the superpotentials receive a sign change which seems to give the wrong sign in the comparison. In section 3.2.2 we will see that signs cancel in the right way due to the minus sign in the definition of the glueball superfield (2.5).

Table 2.1:

Dijkgraaf-Vafa conjecture; Traceless case	
Gauge theory	Matrix model
$Z_{\text{holo}} = \int \mathcal{D}\Phi\mathcal{D}\bar{\Phi}\; e^{iS_{\text{tree}}}\Big\vert_{\text{holo}}$	$Z_{\text{matrix}} = \int \mathcal{D}M e^{-\frac{1}{g_{\text{s}}}W_{\text{tree}}(M)}$
$Z_{\text{holo}} = e^{i\int \mathrm{d}^4x\mathrm{d}^2\theta W_{\text{eff,pert}}}$	$Z_{\text{matrix}} = e^{-\sum_{g\geq 0} g_{\text{s}}^{2g-2}\mathcal{F}_g\left(g_{\text{s}}N_i'\right)}$
$W_{\text{eff}} = W_{\text{eff,pert}} + W_{\text{VY}}$	W_{VY} as a measure contribution
Vacua:	Vacua:
$N = N_1 + \ldots + N_n$	$N' = N_1' + \ldots + N_n'$
No limit	't Hooft large N_i' limit
$W_{\text{eff,pert}} = \sum_i N_i \frac{\partial \mathcal{F}_{g=0}(\hat{S}_i)}{\partial \hat{S}_i}$	
$\hat{S}_i = g_{\text{s}}N_i'$	

The Dijkgraaf-Vafa conjecture in the case of a traceless glueball superfield. Formulae are explained in the text.

2.1.2 The U(N_i)-Case

Let us now turn to the case where we also consider the abelian parts of the supersymmetric gauge field strengths. In this case we define the *glueball superfield* S_i and the field $w_{i\alpha}$ as (here we follow [7], but naturally with our normalisations):

$$\begin{aligned} S_i &= -\frac{1}{16\pi^2}\,\mathrm{Tr}\Big(\mathcal{W}_{(i)}^{\alpha}\mathcal{W}_{(i)\alpha}\Big)\,, && (2.17)\\ w_{i\alpha} &= \frac{\sqrt{2}}{4\pi}\,\mathrm{Tr}\big(\mathcal{W}_{(i)\alpha}\big)\,, && (2.18)\end{aligned}$$

where $\mathcal{W}_{(i)}^{\alpha}$ is the supersymmetric gauge field strength for the whole U(N_i) subgroup. As a standard we will use the $N_i \times N_i$ identity matrix (with trace N_i) to span the abelian part of the algebra, and the formerly introduced $T_{(i)a}^{(\text{fund})}$ to span the simple part. Using equation (2.6) we get the following relation between $\hat{S}$, S and $w_{i\alpha}$:

$$S_i = \hat{S}_i - \frac{1}{2N_i}w_i^{\alpha}w_{i\alpha} \qquad \text{(no sum over i).} \tag{2.19}$$

Even though $\hat{S}_i$ are the elementary fields at low energy (the $w_{i\alpha}$'s are IR-free) it is easier to write the Dijkgraaf-Vafa conjecture using S_i. The dependence on $w_{i\alpha}$ will then simply be quadratic. Now we have to identify $g_{\text{s}}N_i'$ from the matrix model with S_i. The conjecture for the form of $W_{\text{eff,pert}}$ is then:

Dijkgraaf-Vafa conjecture; U(N_i)-case.

$$\begin{aligned} W_{\text{eff,pert}}(S_i, w_{i\alpha}, g_k) &= \sum_i N_i \frac{\partial \mathcal{F}_{g=0}(S_i, g_k)}{\partial S_i} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 \mathcal{F}_{g=0}(S_i, g_k)}{\partial S_i \partial S_j}w_i^{\alpha}w_{j\alpha},\\ S_i &\equiv g_{\text{s}}N_i'. \end{aligned} \tag{2.20}$$

Here we have again included the dependence on the couplings g_k from the tree-level superpotential W_{tree}. We notice that by setting $w_{i\alpha} = 0$ this case reduces to the traceless case (2.14).[9]

We note that the coefficient function of $w_i^\alpha w_{j\alpha}$ in (2.20) is not the abelian complexified gauge coupling τ_{ij} as in equation (1.75). This is because we have to split the gauge algebra into its simple and abelian part. Thus S_i should be split in $\hat{S}_i$ and $w_{i\alpha}$ using (2.19). Expanding the functions according to this yields:

$$\tau_{ij} \propto \frac{\partial^2 \mathcal{F}_{g=0}\big(\hat{S}_i, g_k\big)}{\partial \hat{S}_i \partial \hat{S}_j} - \delta_{ij} \frac{1}{N_i} \sum_l N_l \frac{\partial^2 \mathcal{F}_{g=0}\big(\hat{S}_i, g_k\big)}{\partial \hat{S}_i \partial \hat{S}_l}, \tag{2.21}$$

where there is no sum over i. However, we note that we also have (mixed) terms of higher order than quadratic in the $w_{i\alpha}$'s when we split S_i into $\hat{S}_i$ and $w_{i\alpha}$. From (2.21) we see that $\sum_j \tau_{ij} N_j = 0$ which is a reflection of the fact that the overall $U(1)$ is decoupled since Φ is in the adjoint (see (2.26) below where there is no coupling between the abelian part of the vector field and Φ).

For the non-perturbative part we should again add the Veneziano-Yankielowicz superpotential by hand. However, the literature is unfortunately inconclusive on how to do this. Naturally, there should be a term like (2.9), but perhaps the traceless glueball superfields should here be replaced by the full glueball superfields S_i with some modification since the overall U(1) should be decoupled. We will discuss this briefly in section 2.8.

A couple of remarks are in order. First we note that the N_i dependence in the effective superpotential is extremely simple. Secondly, we emphasise that the reduction to planar diagrams on the gauge theory side is exact while we have to take the 't Hooft large N' limit on the matrix model side. Furthermore, we will see in section 2.7.1 that for a given diagram in the matrix model we get one factor of $S_i = g_s N_i'$ for each index loop indexed by i. Thus it is very simple to make an expansion of $W_{\text{eff,pert}}$ to a given order in S_i. However, this immediately poses a problem: Looking at the definition of S_i (or $\hat{S}_i$) it is a sum of products of a finite number of Grassmannian variables $\mathcal{W}_{(i)}^{a\alpha}$ where a is the adjoint index. Hence S_i must be nilpotent. But on the matrix model side we can go to any order in $g_s N_i'$ i.e. any number of loops. We will discuss this in section 2.9.

2.1.3 Proofs

The Dijkgraaf-Vafa conjecture actually consists of two parts:

First we have the statement that the total Wilsonian effective action is obtained as a sum of the Veneziano-Yankielowicz superpotential for the pure super-Yang Mills theory, W_{VY}, and the effective potential obtained by integrating out the massive chiral fields, $W_{\text{eff,pert}}$. This has not been proven, but one can argue that it is true [7] if we assume that the $\hat{S}_i$ fields are the elementary fields in the low energy effective theory as the lore says.

The second part of the conjecture is the exact form of $W_{\text{eff,pert}}$ using the related matrix model. This conjecture originates in topological string theory and is proven herein. However, the conjecture can be proven within supersymmetric gauge theory itself. This can be done using Feynman diagrams as we will see in chapter 3. One can also use Seiberg-Witten theory and the ILS linearity principle (section 2.5.5) to derive the effective superpotential [5]. A last and powerful way to prove the conjecture within supersymmetric gauge theory is to use the chiral ring and the generalised Konishi anomalies [7].

[9] In the case of $N_i = 1$ one should be careful since the field S_i should appear as a field in its own right as above – contrary to what one might think (see also section 2.9).

The more general form of the Dijkgraaf-Vafa conjecture that we will present in section 2.10 can also be proven using the diagrammatic (as we also will see in chapter 3), the Seiberg-Witten, and the generalised Konishi anomaly method.

2.1.4 The Lagrangian

Let us finish this section by writing out the Lagrangian that we have assumed in the conjecture – for completeness, further reference, and to see how general it is. The fundamental fields are the vector field V (and hence $\mathcal{W}_\alpha$) and the chiral field Φ. $\mathcal{W}_\alpha$ and Φ are both U(N)-adjoint fields and as noted above they should take values in the fundamental representation. It will be useful for us to split them in their abelian and simple parts. In accordance with the notation above we use a hat to designate the projection onto the simple part of the space and we choose to use a tilde to designate the projection onto the abelian part. Thus e.g. for Φ we choose to write:

$$\begin{aligned}
\Phi &= \tilde{\Phi} + \hat{\Phi}, \\
\tilde{\Phi} &= \tilde{\Phi}^0 \mathbf{1}_{N\times N}, \\
\hat{\Phi} &= \hat{\Phi}^a T_a^{(\mathrm{fund})}, \qquad a = 1, \ldots, N^2 - 1.
\end{aligned} \tag{2.22}$$

The generators $T_a^{(\mathrm{fund})}$ of the simple part was introduced above in (2.6). In the same way we expand $\mathcal{W}_\alpha$ and other fields taking values in the fundamental representation of $\mathcal{U}(N)$. Please note that we can set:

$$T_0^{(\mathrm{fund})} \equiv \mathbf{1}_{N\times N}, \tag{2.23}$$

and thus obtain a basis for the fundamental representation of $\mathcal{U}(N)$ as $T_a^{(\mathrm{fund})}$ with $a = 0, \ldots, N^2 - 1$. These fulfil:

$$\mathrm{Tr}\Big(T_a^{(\mathrm{fund})} T_b^{(\mathrm{fund})}\Big) = c(a)\, \delta_{ab}, \tag{2.24}$$

where $c(0) = N$ and $c(a) = 1/2$ for $a = 1, \ldots, N^2 - 1$.

We should find the most general renormalisable $\mathcal{N} = 2$ supersymmetric Lagrangian that contains the Φ and V superfields (making up the $\mathcal{N} = 2$ gauge multiplet). Actually, we also assumed in chapter 1 that the Lagrangians should be used for low energy effective actions as exactly is the case here. As noted above we also assume no Fayet-Iliopoulos term. Under these assumptions we found the most general Lagrangian in section 1.4 for a simple gauge group. So we simply have to consider how to include the abelian part. Let us first look at the (renormalisable) Kähler term which according to (1.39) and (1.64) has the form:

$$\mathcal{L}_K = \int \mathrm{d}^4\theta \Phi^\dagger e^{2V^{(\mathrm{adj})}} \Phi. \tag{2.25}$$

However, the adjoint representation of U(N) is not irreducible. In fact the adjoint representation of the abelian generator is, naturally, zero. So we can split the Kähler term into two terms corresponding to the simple and the abelian part respectively. These two parts can have different normalisation. Now, as we saw in section 1.4 the full $\mathcal{N} = 2$ Lagrangian is simply the sum of the properly normalised Kähler terms and $\mathcal{L}_G$ from (1.62). We can split $\mathcal{L}_G$ into its abelian part (1.75) and its simple part (1.79). These two parts can have different τ's (actually, we could normalise the abelian part of the gauge field strength such that the two couplings are equal, but we choose for generality to keep them different here). To fulfil the R-symmetry (1.83) (also for the abelian part) we must normalise the two Kähler terms properly. Thus we simply get (1.84) with a corresponding abelian term added as promised in section 2.1.1. To get the full Lagrangian used in the conjecture, $\mathcal{L}_{\mathrm{U}(N)}$, we simply have to add the superpotential term (2.1):[10]

$$\begin{aligned}\mathcal{L}_{\mathrm{U}(N)} &= \frac{\tau}{8\pi i}\operatorname{Tr}\Big(\int \mathrm{d}^2\theta \hat{\mathcal{W}}^\alpha\hat{\mathcal{W}}_\alpha + 2\int \mathrm{d}^4\theta\hat{\Phi}^\dagger e^{2\,\mathrm{ad}\hat{V}}\hat{\Phi}\Big)\\ &\quad +\frac{\tau_{\mathrm{abel}}}{16\pi i N}\operatorname{Tr}\Big(\int \mathrm{d}^2\theta\tilde{\mathcal{W}}^\alpha\tilde{\mathcal{W}}_\alpha + 2\int \mathrm{d}^4\theta\tilde{\Phi}^\dagger\tilde{\Phi}\Big) + \int \mathrm{d}^2\theta\operatorname{Tr}P_{n+1}(\Phi)\\ &\quad +\text{c.c.},\end{aligned}\tag{2.26}$$

where the traces are in the fundamental representation. We have put a hat on V in the Kähler term for the simple part to point out that only the simple part contributes. τ_{abel} is the gauge coupling for the abelian part of the Lagrangian. In the superpotential we can use (2.22) to split Φ into $\tilde{\Phi}$ and $\hat{\Phi}$ as in the rest of the Lagrangian. The complex conjugate added in the last line is for the whole equation.

Now we want to expand this Lagrangian in components. This has already been done for the simple part of the $\mathcal{N}=2$ Lagrangian in (1.85) and we can expand the abelian part in the same way. The superpotential is expanded using (1.68). We get:

$$\begin{aligned}\mathcal{L}_{\mathrm{U}(N)} &= \frac{2}{g^2}\operatorname{Tr}\Big(-i\hat{\lambda}\sigma^\mu\hat{D}_\mu\bar{\hat{\lambda}} - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{g^2\vartheta}{32\pi^2}\hat{F}_{\mu\nu}\frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}\hat{F}_{\rho\kappa} + \frac{1}{2}\hat{D}\hat{D} - \hat{D}_\mu\hat{\phi}(\hat{D}^\mu\hat{\phi})^\dagger\\ &\qquad -i\bar{\hat{\psi}}\bar{\sigma}^\mu\hat{D}_\mu\hat{\psi} + \hat{F}\bar{\hat{F}} + \hat{D}[\hat{\phi}^\dagger,\hat{\phi}] - i\sqrt{2}[\bar{\hat{\lambda}},\bar{\hat{\psi}}]\hat{\phi} - i\sqrt{2}[\hat{\lambda},\hat{\psi}]\hat{\phi}^\dagger\Big)\\ &\quad +\frac{1}{g^2_{\mathrm{abel}}N}\operatorname{Tr}\Big(-i\tilde{\lambda}\sigma^\mu\partial_\mu\bar{\tilde{\lambda}} - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{g^2_{\mathrm{abel}}\vartheta_{\mathrm{abel}}}{32\pi^2}\tilde{F}_{\mu\nu}\frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}\tilde{F}_{\rho\kappa} + \frac{1}{2}\tilde{D}\tilde{D}\\ &\qquad -\partial_\mu\tilde{\phi}(\partial^\mu\tilde{\phi})^\dagger - i\bar{\tilde{\psi}}\bar{\sigma}^\mu\partial_\mu\tilde{\psi} + \tilde{F}\bar{\tilde{F}}\Big)\\ &\quad +\Bigg(\operatorname{Tr}\big(FP'_{n+1}(\phi)\big) - \frac{1}{2}\operatorname{Tr}\Bigg(\psi\sum_{a=0}^{N^2-1}\psi^a\frac{\partial}{\partial\phi^a}P'_{n+1}(\phi)\Bigg) + \text{c.c.}\Bigg),\end{aligned}\tag{2.27}$$

where we emphasise that the tildes refer to the projection onto the abelian part of the algebra and not the Poincaré dual. Hence the Poincaré duals have been written out using definition (1.70). We have put hats on the gauge covariant derivatives for the simple part to emphasise that these depend only on the simple part of the gauge algebra. τ and τ_{abel} have been expanded as in (1.80) and (1.76):

$$\tau = \frac{\vartheta}{2\pi} + i\frac{4\pi}{g^2},\tag{2.28}$$

$$\tau_{\mathrm{abel}} = \frac{\vartheta_{\mathrm{abel}}}{2\pi} + i\frac{4\pi}{g^2_{\mathrm{abel}}}.\tag{2.29}$$

In order to obtain the contribution from the superpotential using (1.68) we had to differentiate $W_{\mathrm{tree}} = \operatorname{Tr}P_{n+1}$ as a function of the fields ϕ^a in $\phi = \phi^a T_a$. Since the trace is linear we

[10] It should here be noted that in this notation it is not possible to give the gauge couplings different units. This is because by multiplying the gauge couplings onto the Kähler term we would then give the Φ^a's "gauge-units" with Φ^0 and Φ^a, $a>0$ having different units. However, the superpotential can not be made dimensionless because of these different units since the same superpotential-couplings multiply Φ^0 as well as Φ^a, $a>0$. In order to fix this problem the superpotential should really be defined as (using g and g_{abel} from (2.28) and (2.29)): $\operatorname{Tr}P_{n+1}\Big(\frac{1}{g_{\mathrm{abel}}}\Phi^0T_a^{(\mathrm{fund})} + \frac{1}{g}\sum_{a=1}^{N^2-1}\Phi^aT_a^{(\mathrm{fund})}\Big)$. However, we will stick to the notation in (2.26) because the superpotential looks as in the Dijkgraaf-Vafa conjecture.

can move the differentiation inside the trace. We can then use

$$\frac{\partial \phi}{\partial \phi^a} = T_a^{(\mathrm{fund})}, \qquad a = 0, \dots, N^2-1, \tag{2.30}$$

to obtain the first term in (1.68) which contains only one derivative:

$$F^a \frac{\partial W(\phi^a)}{\partial \phi^a} = F^a \operatorname{Tr}\Big(\frac{\partial}{\partial \phi^a} P_{n+1}(\phi)\Big) = F^a \operatorname{Tr}\Big(T_a^{(\mathrm{fund})} P'_{n+1}(\phi)\Big) = \operatorname{Tr}\big(F P'_{n+1}(\phi)\big)\,, \tag{2.31}$$

where we used the cyclicity of trace in the second equality to put the generator $T_a^{(\mathrm{fund})}$ to the front. However, when we make two differentiations as is the case in the last term in (1.68) we can only fix the placement of one the generators that are generated. Thus we settle for only carrying out one of the derivatives explicitly in the last term of (2.27).

Since the F and D-fields are auxiliary, even in the quantised theory, we should replace them with their (algebraic) equations of motion. Using (2.27) we immediately get:

$$\begin{aligned}
\tilde{D}^0 &= 0. && (2.32)\\
\hat{D}^a &= -[\hat{\phi}^\dagger, \hat{\phi}]^a, \qquad a = 1, \dots, N^2-1. && (2.33)\\
\tilde{F}^0 &= -g_{\mathrm{abel}}^2 \overline{\operatorname{Tr}\Big(T_0^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)} = -g_{\mathrm{abel}}^2 \overline{\operatorname{Tr}\big(P'_{n+1}(\phi)\big)}. && (2.34)\\
\hat{F}^a &= -g^2 \overline{\operatorname{Tr}\Big(T_a^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)}, \qquad a = 1, \dots, N^2-1. && (2.35)
\end{aligned}$$

$\bar{F}$ obeys the complex conjugated equations as that of F.

Plugging these expectation values back into the Lagrangian gives a scalar potential:

$$U(\phi) = \frac{1}{g^2} \operatorname{Tr}\Big([\hat{\phi}^\dagger, \hat{\phi}]^2\Big) + g^2 \sum_{a=1}^{N^2-1} \Big|\operatorname{Tr}\Big(T_a^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)\Big|^2 + g_{\mathrm{abel}}^2 \Big|\operatorname{Tr}\Big(T_0^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)\Big|^2 . \tag{2.36}$$

The part stemming from the D field in this potential, naturally, is the same as we found in the $\mathcal{N}=2$ case (1.86).

Finally we can we get the full Lagrangian in components after elimination of the auxiliary fields:

$$\begin{aligned}
\mathcal{L}_{\mathrm{U}(N)} = {} & \frac{2}{g^2} \operatorname{Tr}\Big(-i\hat{\lambda}\sigma^\mu \hat{D}_\mu \bar{\hat{\lambda}} - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{g^2\vartheta}{32\pi^2}\hat{F}_{\mu\nu}\frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}\hat{F}_{\rho\kappa} - \hat{D}_\mu\hat{\phi}(\hat{D}^\mu\hat{\phi})^\dagger \\
& -i\bar{\hat{\psi}}\bar{\sigma}^\mu \hat{D}_\mu \hat{\psi} - i\sqrt{2}[\bar{\hat{\lambda}}, \bar{\hat{\psi}}]\hat{\phi} - i\sqrt{2}[\hat{\lambda}, \hat{\psi}]\hat{\phi}^\dagger \Big) \\
& + \frac{1}{g_{\mathrm{abel}}^2 N} \operatorname{Tr}\Big(-i\tilde{\lambda}\sigma^\mu \partial_\mu \bar{\tilde{\lambda}} - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{g_{\mathrm{abel}}^2\vartheta_{\mathrm{abel}}}{32\pi^2}\tilde{F}_{\mu\nu}\frac{1}{2}\varepsilon^{\mu\nu\rho\kappa}\tilde{F}_{\rho\kappa} \\
& -\partial_\mu\tilde{\phi}(\partial^\mu\tilde{\phi})^\dagger - i\bar{\tilde{\psi}}\bar{\sigma}^\mu \partial_\mu \tilde{\psi} \Big) \\
& + \Bigg(-\frac{1}{2} \operatorname{Tr}\Bigg(\psi \sum_{a=0}^{N^2-1} \psi^a \frac{\partial}{\partial \phi^a} P'_{n+1}(\phi) \Bigg) + \text{c.c.} \Bigg) - \frac{1}{g^2} \operatorname{Tr}\Big([\hat{\phi}^\dagger, \hat{\phi}]^2\Big) \\
& - g^2 \sum_{a=1}^{N^2-1} \Big|\operatorname{Tr}\Big(T_a^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)\Big|^2 - g_{\mathrm{abel}}^2 \Big|\operatorname{Tr}\Big(T_0^{(\mathrm{fund})} P'_{n+1}(\phi)\Big)\Big|^2 .
\end{aligned} \tag{2.37}$$

Let us end this chapter by discussing how general this Lagrangian is. We should compare it to the most general $\mathcal{N} = 1$ Lagrangian with $\mathrm{U}(N)$ gauge group and an adjoint chiral matter field for use in low-energy effective theories, i.e. (1.67). As noted above we have left out a Fayet-Iliopoulos term. The gauge part of the Lagrangian is the most general renormalisable that we can have, i.e. we have assumed that the τ's do not depend on Φ. The Kähler term is also assumed renormalisable, but its normalisation has also been constrained such that we obtained $\mathcal{N} = 2$ supersymmetric invariance when disregarding the superpotential. However, this normalisation is not important at all even in the first version of a diagrammatic proof of the Dijkgraaf-Vafa conjecture in [6]. Thus the only way the restriction to $\mathcal{N} = 2$ supersymmetry plays a role is that the matter field Φ is in the adjoint representation. However, the $\mathcal{N} = 2$ supersymmetry was important in the string theory from which the conjecture emerged. Naturally, it is not at all important when we introduce the generalised conjecture in section 2.10.

At last let us look at the superpotential (2.1). This we did not even constrain by renormalisability, however, it is not the most general superpotential that we a priori could think of. Rather, the most general superpotential is a multi-trace form:

$$\mathcal{L} = \int \mathrm{d}^2\theta \sum_k \sum_{n_1,\ldots,n_k} g^{(k)}_{n_1,\ldots,n_k} \operatorname{Tr}\left(\Phi^{n_1}\right) \cdots \operatorname{Tr}\left(\Phi^{n_k}\right). \tag{2.38}$$

The Dijkgraaf-Vafa conjecture does not apply immediately in this case. One has to linearise the superpotential to a single trace form by introducing auxiliary fields. The Dijkgraaf-Vafa conjecture can then be used to obtain the effective action including these auxiliary fields. One finally obtains the correct effective superpotential by integrating out the auxiliary fields [18], [19].

One might think that the superpotential (2.1) would include any superpotential of the form:

$$\mathcal{L} = \int \mathrm{d}^2\theta \sum_{m,n} g_{m,n} \left(\tilde{\Phi}^0\right)^m \operatorname{Tr}\left(\hat{\Phi}^n\right), \tag{2.39}$$

i.e. a product of a trace over the abelian part (proportional to $\tilde{\Phi}^0$) and a trace over the non-abelian part of $\mathrm{U}(N)$. But expanding (2.1) gives (expanding the polynomial as $\sum_n g_n \Phi^n$):

$$\begin{aligned} W_{\text{tree}}(\Phi) &= \operatorname{Tr}\left(\sum_n g_n \left(\tilde{\Phi} + \hat{\Phi}\right)^n\right) = \operatorname{Tr}\left(\sum_n g_n \sum_{m=0}^{n} \binom{n}{m} \tilde{\Phi}^m \hat{\Phi}^{n-m}\right) \\ &= \sum_n g_n \sum_{m=0}^{\infty} \binom{n}{m} \left(\tilde{\Phi}^0\right)^m \operatorname{Tr}\left(\hat{\Phi}^{n-m}\right) = \sum_{m,n=0}^{\infty} g_{n+m} \binom{n+m}{m} \left(\tilde{\Phi}^0\right)^m \operatorname{Tr}\left(\hat{\Phi}^n\right). \end{aligned} \tag{2.40}$$

Here we have used the binomial formula which applies since $\tilde{\Phi}$ is abelian. We have also used that $\binom{n}{m} = 0$ for integers with $m > n$. Thus we see that our superpotential is a constrained form of (2.39).

2.2 Supersymmetric Vacua

As we saw in section 1.2 the mass is a Casimir of the supersymmetry algebra and hence both the fermionic and bosonic particles in a supermultiplet have the same mass. This is not observed in nature so supersymmetry is broken at our energies. This makes the spontaneous breaking of supersymmetry to a very important issue. In spite of this, we will focus our interest on vacua which do not break supersymmetry spontaneously. The treatment builds on [9], [10], [11], [12], [13], and [14].

2.2.1 Supersymmetric Vacua

In section 1.1.3 we chose the Hamiltonian of the field theory realising the supersymmetry algebra to be positive. That this is true can also be seen directly from (1.4c) using that the Pauli matrices have zero trace:

$$H = \frac{1}{4}\left(\{\mathcal{Q}_1^A, \bar{\mathcal{Q}}_{A\dot{1}}\} + \{\mathcal{Q}_2^A, \bar{\mathcal{Q}}_{A\dot{2}}\}\right) \qquad \text{(no sum } A\text{)}, \tag{2.41}$$

for any A. Since the barred generators are the hermitian conjugates, the expectation value of H is clearly positive or zero in any state. Actually, we see that the expectation value in a vacuum (or any other state) is:

$$\langle \text{vac}|H|\text{vac}\rangle = \frac{1}{4}\left(\left\|\mathcal{Q}_1^A|\text{vac}\rangle\right\|^2 + \left\|\bar{\mathcal{Q}}_{A\dot{1}}|\text{vac}\rangle\right\|^2 + \left\|\mathcal{Q}_2^A|\text{vac}\rangle\right\|^2 + \left\|\bar{\mathcal{Q}}_{A\dot{2}}|\text{vac}\rangle\right\|^2\right). \tag{2.42}$$

Thus we clearly see that *the vacuum $|vac\rangle$ is supersymmetric if and only if the vacuum energy vanishes.* On the other hand, supersymmetry is spontaneously broken if and only if vacuum energy is strictly positive.[11].

Now let us assume that $\mathcal{N} = 1$. Another necessary and sufficient criterion for spontaneous breaking of supersymmetry is that there exist a field, ψ, with a non-zero supersymmetry variation in the vacuum:

$$\langle \text{vac}|\delta_\xi \psi|\text{vac}\rangle \neq 0. \tag{2.43}$$

Using that $\delta_\xi\psi = [i\xi\mathcal{Q} + i\bar{\xi}\bar{\mathcal{Q}}, \psi]$ we immediately see that this can never be fulfilled if the supercharges annihilate the vacuum which is the case if the vacuum energy is zero. As we are dealing with a relativistic quantum field theory, we assume that the Poincaré invariance is manifest, i.e. the Poincaré generators annihilate the vacuum. Thus only scalar fields that transform trivially under the Lorentz group can have a non-zero expectation. Consequently, $\delta_\xi\psi$ must be a scalar field and hence ψ is fermionic. This fermion is called the Goldstino since it plays the same role as the Goldstone boson. The Goldstino is also massless.

Let us look back at the superfield representation of the supersymmetry algebra. In the case of a chiral superfield the Goldstino must be the Weyl spinor ψ from the component expansion (1.34). The supersymmetry variation of ψ can be found using (1.23). Using again that only fields that transform trivially under the Lorentz group can have non-zero expectation values, one then gets that supersymmetry is spontaneously broken if the auxiliary field F gets a non-zero expectation value. This is known as F-term supersymmetry breaking. Looking at the vector multiplet the Goldstino is the gaugino and supersymmetry is spontaneously broken if the auxiliary field D gets a non-zero expectation value. This is known as D-term supersymmetry breaking.

We end this subsection by noting that when the supersymmetry is spontaneously broken we no longer have equality of bosonic and fermionic states. This is because the norm of $\mathcal{Q}|\text{vac}\rangle$ becomes infinite (with an infinite space and finite energy density). Thus the proof in section 1.2.1 no longer works.

2.2.2 Breaking and No-Breaking of Supersymmetry in $\mathcal{N} = 1$ Supersymmetric Field Theories

Let us look at an $\mathcal{N} = 1$ supersymmetric field theory with a compact gauge group and with chiral fields in some representation $\mathbf{r}$. What we want to do here and in the rest of

[11] It is not possible to simply shift the energy – and thus making this observation meaningless – because the Hamiltonian $\mathcal{P}^0$ is a part of the supersymmetry algebra (1.4).

this section is to consider the tree-level expansion of the theory i.e. the semi-classical limit. The vacuum expectation values of the component fields must be translationally invariant since the vacuum does not break the Poincaré invariance. Thus the expectation value of the scalar field ϕ (which we will denote ϕ_0) is independent of the space-time coordinate x.[12] The expectation values of the rest of the fields, not transforming trivially under the Poincaré group, vanish. Thus the only contribution to the vacuum energy comes from the scalar potential. The vacuum expectation value of ϕ in the semi-classical approximation is the value that minimises the scalar potential constrained to constant fields:

$$\left.\frac{\partial U|_{\text{constant fields}}}{\partial \phi}\right|_{\phi=\phi_0} = 0. \tag{2.44}$$

This is, naturally, the semi-classical limit of the quantum expectation value obtained by minimising the 1-PI effective potential:

$$\frac{\partial V_{\text{eff}}}{\partial \phi_{\text{cl}}} = 0. \tag{2.45}$$

We will now find the scalar potential for the theory. For simplicity we will assume that the gauge couplings τ_{ab} from equation (1.62) is independent of Φ and $\tau_{ab} = \frac{1}{4}\delta_{ab}$. Furthermore, we assume that the Kähler metric is invertible. We normalise the Fayet-Iliopoulos term such that it is equal to $\kappa_a D^a$ where κ_a only has non-zero values in the abelian directions to ensure gauge invariance. We have here also assumed that the inner-product on the Lie algebra is diagonal. Combining all the terms from (1.67) and expanding into components as in section 2.1.4 yields the equations of motion for the auxiliary fields:

$$F^i = -g^{i\bar{\imath}}\frac{\partial \overline{W(\phi)}}{\partial \bar{\phi}^{\bar{\imath}}}, \tag{2.46}$$

$$D^a = -\kappa_a - \left(\frac{1}{2}\frac{\partial K\big(\phi,\phi^\dagger\big)}{\partial \phi^i}(T_a^{(\mathbf{r})})^i{}_j\phi^j + \text{c.c.}\right). \tag{2.47}$$

Here the Kähler metric with upper indices is the inverse of the Kähler metric with lower indices defined in (1.73). The scalar potential is then:

$$\begin{aligned} U &= \frac{1}{2}D^a D^a + g_{i\bar{\imath}}F^i\bar{F}^{\bar{\imath}} \\ &= \frac{1}{2}\sum_a\left(\kappa_a + \left(\frac{1}{2}\frac{\partial K\big(\phi,\phi^\dagger\big)}{\partial \phi^i}(T_a^{(\mathbf{r})})^i{}_j\phi^j + \text{c.c.}\right)\right)^2 + g^{i\bar{\imath}}\frac{\partial W(\phi)}{\partial \phi^i}\frac{\partial \overline{W(\phi)}}{\partial \bar{\phi}^{\bar{\imath}}}, \end{aligned} \tag{2.48}$$

where the first line is evaluated using (2.46) and (2.47). The Kähler metric is positive for unitarity of the theory since it multiplies the kinetic energy in (1.72). Thus the potential energy is positive (or zero) as promised.

We now see that if we can find a solution ϕ_0 to $U = 0$ then we automatically have a global minimum and hence the vacuum expectation value of ϕ. But at the same time it is the condition that supersymmetry is unbroken in this vacuum. It is these supersymmetric vacua that we investigate in this thesis.

[12] The space-time independence can also be obtained by demanding the vacuum energy to vanish. After computing the Hamiltonian one sees that this implies that the covariant derivative of ϕ is zero; $D_\mu\phi = 0$ and that $F^a_{\mu\nu} = 0$. The last equation means that the gauge field A_μ is pure gauge and therefore we can set it to zero. Thus the first equation reduces to $\partial_\mu\phi = 0$.

From (2.48) we immediately see that the condition of finding vacua with unbroken supersymmetry translates into:

$$F^i = 0,\ D^a = 0 \text{ have solution } \phi_0 \text{ for all } i \text{ and } a \longleftrightarrow \text{Supersymmetric vacuum.} \tag{2.49}$$

This is the same result we found in the last subsection when giving the F-term and D-term condition for spontaneous supersymmetry breaking.

If we for any constant field ϕ can find a F^i that is non-zero, we have F-term breaking of supersymmetry. The order parameter will be $\langle F \rangle$. In the case where we do have solutions to $F^i = 0$ for all i we talk about F-flatness. From (2.46) these F-flatness equations can be rewritten as:

$$\frac{\partial W(\phi)}{\partial \phi^i} = 0, \qquad i = 1, \ldots, \dim(\mathbf{r})\,. \tag{2.50}$$

One can also obtain that F-term breaking generically only happens when an R-symmetry is broken.

When we always are able to find a $D^a \neq 0$ we have D-term breaking. The order parameter is here $\langle D \rangle$. When we do have solutions we talk about D-flatness. Using (2.47) the D-flatness equations can be rewritten as:

$$\kappa_a + \frac{1}{2}\frac{\partial K\big(\phi, \phi^\dagger\big)}{\partial \phi^i}(T_a^{(\mathbf{r})})^i{}_j\phi^j + \frac{1}{2}\left(\phi^i\right)^* (T_a^{(\mathbf{r})})^i{}_j \left(\frac{\partial K\big(\phi, \phi^\dagger\big)}{\partial \phi^j}\right)^* = 0, \tag{2.51}$$

where we have assumed a unitary representation such that the generators are hermitian. In many theories there will be no solutions to these equations when the Fayet-Iliopoulos term is non-zero.

On the other hand, we can always find solutions to the D-flatness equations in the form (2.51) if we assume that $\kappa_a = 0$ (the proof here is taken from [20]). Simply take an arbitrary vector ϕ_0. We can then look at the surface obtained by performing *complex* gauge transformations on this vector $\phi_0 \mapsto \phi(\Lambda) = \exp\Big(-i\Lambda^a(x)T_a^{(\mathbf{r})}\Big)\,\phi_0$ where $\Lambda^a \in \mathbb{C}$. The renormalisable Kähler potential, $\phi^\dagger\phi$, must have a minimum on this surface since it is real and positive definite when ϕ runs through $\phi(\Lambda)$. Thus the general Kähler potential, $K\big(\phi, \phi^\dagger\big)$, must have a local minimum if we assume that it is the renormalisable Kähler potential perturbed with some extra gauge invariant real terms. The only thing that could spoil this is if the renormalisable potential has some flat directions that the full Kähler potential does not share thus making it possible to break the minimum. However, the flat directions of $\phi^\dagger\phi$ correspond to performing the standard real gauge transformations under which the full Kähler potential is also invariant. The local minimum must be a stationary point when varying Λ:

$$0 = \frac{\partial K\big(\phi, \phi^\dagger\big)}{\partial \phi^{i*}} i\left(T_a^{(\mathbf{r})}\right)^j{}_i \phi^{j*}\delta\Lambda^{a*} - \frac{\partial K\big(\phi, \phi^\dagger\big)}{\partial \phi^i} i\left(T_a^{(\mathbf{r})}\right)^i{}_j \phi^j\delta\Lambda^a. \tag{2.52}$$

The terms multiplying the variations $\delta\Lambda^a$ and $\delta\Lambda^{a*}$ must be zero. Hence the minimum point fulfil the D-flatness equations (2.51) with $\kappa_a = 0$ thus finishing the proof. This is the reason that we do not allow a Fayet-Iliopoulos term in the Lagrangian of the Dijkgraaf-Vafa conjecture.

As a corollary in the case of $\kappa_a = 0$, we see that if there exist a solution to the F-flatness equations (2.50) then there must exist a solution ϕ_0 satisfying both F-flatness (2.50) and D-flatness (2.51) thus showing the existence of a supersymmetric vacuum. This result simply follows from the proof of the existence of a solution to the D-flatness equations using that the superpotential is holomorphic in ϕ and thus invariant under the whole complexified gauge group. This means that in order to determine if supersymmetry is unbroken (with no Fayet-Iliopoulos term) the necessary and sufficient condition is the F-flatness equations (2.50).

Please note that we have $\dim(\mathbf{r})$ F-flatness equations and $\dim(\mathcal{G})$ D-term equations in $\dim(\mathbf{r})$ variables ϕ^i. However, the F-flatness equations are constrained by gauge invariance of the superpotential:

$$\frac{\partial W(\phi)}{\partial \phi^i}\left(T_a^{(\mathbf{r})}\phi\right)^i = 0, \tag{2.53}$$

thus giving $\dim(\mathcal{G})$ constraints and hence we have $\dim(\mathbf{r})$ equations in $\dim(\mathbf{r})$ variables. Generically, we will thus always have solutions, however, this is not necessarily true when considering specific theories.

2.2.3 Classical Vacuum Moduli Space

The *vacuum moduli space* is defined as the space of supersymmetric inequivalent vacua. We know from the last section that in the semi-classical limit this space is parameterised by the solutions ϕ_0 to the F- and D-flatness equations (2.49), however, we should here keep in mind that the solutions should be inequivalent. This defines the vacuum moduli space as a complex manifold and we can endow it with a metric by pulling back the Kähler metric (1.73) from the target space of ϕ's.

Now we must remember that all observables are independent of the choice of gauge. Thus gauge transformations relate equivalent vacua. Hence the parametrisation of the moduli space is simply obtained by the solutions to the flatness equations modulo gauge transformations. We noticed in the last subsection that the orbit of the complexified gauge group, $G_{\mathbb{C}}$, through any vector ϕ contains a solution to the D-flatness equations. Thus the space of solutions to the D-flatness equations is simply $\{\phi^i\}/G_{\mathbb{C}}$ which can also be parameterised by holomorphic gauge invariants modulo algebraic relations. That is there exist a set of independent holomorphic gauge invariants $X_r(\phi)$ that parameterise the D-flatness space of solutions. The total moduli space is then simply obtained by restricting these gauge invariants by the F-flatness equations.

We will comment on the quantised vacuum moduli space in section 2.5.2

2.2.4 Classical Vacua for $\mathcal{L}_{\mathrm{U}(N)}$

Let us now turn to the Lagrangian of our interest namely $\mathcal{L}_{\mathrm{U}(N)}$. In (2.36) we found the scalar potential which clearly is positive and vanishes if and only if the F's and D's in (2.32)-(2.35) are equal to zero. These flatness equations actually have the same form as (2.50) and (2.51) and thus the results above also apply in this case.

Let us first focus on the D-flatness. From (2.32) we see that there is no restriction on the abelian part, however, as the abelian part of ϕ does not contribute to any commutator we can reformulate the D-flatness equations obtained from (2.32) and (2.33) as:

$$[\phi^\dagger, \phi] = 0. \tag{2.54}$$

This is true for an arbitrary gauge group with adjoint matter as can be seen from (2.51) by setting $K = \phi^\dagger\phi$, $\kappa^a = 0$, and taking the generators in the adjoint representation. We immediately see that we get a solution by requiring ϕ to be in a Cartan subalgebra of the gauge group. For the unitary and symplectic gauge groups this gives all possible solutions, however, for $\mathrm{O}(N)$ with $N \geq 7$ there can be more general solutions [21, Appendix I].

In our case the gauge group is U(N). Thus ϕ is $\mathcal{U}(N)$-valued, however, with complex coefficients since the ϕ^a's are complex fields. Let us assume that ϕ is a solution to the D-flatness equation (2.54). Splitting the ϕ^a's in their real and imaginary part splits the matrix ϕ into its hermitian and anti-hermitian part $\phi = \phi_1 + i\phi_2$ where both ϕ_1 and ϕ_2 are hermitian. Since ϕ_1 and ϕ_2 are hermitian they can be diagonalised by a unitary matrix. Using (2.54) we see that ϕ_1 and ϕ_2 commute and thus can be diagonalised by the same unitary matrix. Thus ϕ is diagonalised by this unitary matrix. Since the matrix is unitary and ϕ transforms adjointly, this precisely corresponds to a gauge transformation. Thus, by a suitable gauge transformation we can choose any solution to (2.54) to be in the Cartan subalgebra of the diagonal matrices (we could, naturally, have chosen any other Cartan subalgebra).

The choice of diagonal matrix is in general not unique. Naturally, adjoint transformations, $\phi \mapsto U\phi U^{-1}$, preserve the eigenvalue spectrum, but the eigenvalues in the diagonal form can be permuted. This is the action of the *Weyl group*. For a general group and a specific choice of a Cartan subalgebra, the Weyl group is the subgroup of the gauge group that permutes the generators of the Cartan subalgebra. It thus transforms an element in the Cartan subalgebra back into the Cartan subalgebra. This group is finite. In our U(N)-case of diagonal matrices the Weyl group simply permutes the axes. Since the Weyl transformations are gauge transformations, they relate physically equivalent vacua and the parametrisation of the vacuum moduli space should be independent hereof.

The gauge invariant parametrisation (and thus also Weyl group invariant) of the vacuum moduli space is easily obtained in this case. We simply note that the characteristic polynomial $\det(\lambda - \phi)$ is invariant under adjoint transformations. Hence the coefficients must be gauge invariant. We obtain these by expanding:

$$\det(\lambda - \phi) = \lambda^N \det\left(\mathbf{1} - \frac{\phi}{\lambda}\right) = \lambda^N e^{\operatorname{Tr}\ln\left(\mathbf{1} - \frac{\phi}{\lambda}\right)} = \lambda^N \exp\left(-\sum_{n=1}^{\infty} \frac{\operatorname{Tr}(\phi^n)}{n\lambda^n}\right)$$
$$= \lambda^N - \operatorname{Tr}\phi\,\lambda^{N-1} - \frac{1}{2}\left(\operatorname{Tr}(\phi^2) - \operatorname{Tr}(\phi)\operatorname{Tr}(\phi)\right)\lambda^{N-2} - \dots \quad (2.55)$$

As we can guess from the expansion (which only holds true for $\lambda \gg \phi$), the holomorphic independent gauge invariants are $\operatorname{Tr}\phi^n$ with $n = 1, \dots, N$. Please note that there can only be N parameters since we only have N eigenvalues of ϕ. This also holds true for SU(N) with the exception that $\operatorname{Tr}(\phi) = 0$.

Let us now turn to the F-flatness equations.[13] Using equations (2.34) and (2.35) these can be written as:

$$\operatorname{Tr}\left(T_a^{(\text{fund})} P'_{n+1}(\phi)\right) = 0, \qquad a = 0, \dots, N^2 - 1. \quad (2.56)$$

Generally for a Lie algebra, a product of generators can not be expressed as a sum of generators. Thus, generally, a polynomial of matrices ϕ can not be expanded in $T_a^{(\text{fund})}$'s. However, in our U(N) case the complexified span of the $T_a^{(\text{fund})}$'s gives the whole set of complex matrices i.e. the Lie algebra corresponding to the group of invertible complex matrices, GL($N, \mathbb{C}$). Thus $P'_{n+1}(\phi)$ can be expressed as a complex linear combination of $T_a^{(\text{fund})}$'s. Using (2.24) we then get:[14]

$$P'_{n+1}(\phi) = 0. \quad (2.57)$$

[13] In the rest of this section we will be very thorough since we have not found any of the following proofs in the literature.

[14] Another way to get this is to realise that by (2.56) the trace of P'_{n+1} with any matrix vanishes thus P'_{n+1} vanishes.

We note that this would not be true if we had only looked at SU(N). In that case we would get $P'_{n+1} - \frac{1}{N}\,\mathrm{Tr}\big(P'_{n+1}\big)\,\mathbf{1} = 0$.

If we now assume ϕ to be diagonal to solve the D-flatness equation, (2.57) simply splits into N equations for the eigenvalues ϕ_{ii}:

$$P'_{n+1}(\phi_{ii}) = 0, \qquad i = 1, \ldots, N. \tag{2.58}$$

Thus we obtain the classical vacuum expectation values, ϕ_0, by constraining the eigenvalues to be in the set of roots of P'_{n+1}, i.e. the critical points of P_{n+1} as promised in section 2.1. This gives the final constraints to obtain the vacuum moduli space.

Let us now think of a specific point in the vacuum moduli space. This is determined by a diagonal matrix ϕ_0. Each of the eigenvalues must be equal to one of the critical points a_i of P_{n+1}. However, the placement of the eigenvalues are unimportant since the Weyl group relate equivalent vacua. Hence the vacuum is simply specified by the partition of N in (2.3). We can simply think of putting the a_1-eigenvalues first in ϕ_0, then the a_2-eigenvalues and so on. The unbroken gauge group consist of the matrices of U(N) that have ϕ_0 as a fixed point under adjoint transformations. We note that this is a subgroup of the full gauge group. The generators of the unbroken group are determined as corresponding to the infinitesimal Λ^a's with $[\Lambda^a T_a, \phi_0] = 0$. Let us choose a basis for the fundamental representation of U(N) as $\mathbf{D}_i$, $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ where $i < j$ and:

$$\begin{aligned} (\mathbf{D}_i)_{kl} &= \delta_{ki}\delta_{li}, \\ (\mathbf{A}_{ij})_{kl} &= \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}, \\ (\mathbf{B}_{ij})_{kl} &= -i\delta_{ki}\delta_{lj} + i\delta_{kj}\delta_{li}. \end{aligned} \tag{2.59}$$

The $\mathbf{D}_i$'s span the Cartan subalgebra of the diagonal matrices and are all in the unbroken subgroup. Using that ϕ_0 is diagonal we get that

$$([a\mathbf{A}_{ij} + b\mathbf{B}_{ij}, \phi_0])_{kl} = (a-ib)\left((\phi_0)_{jj} - (\phi_0)_{ii}\right)\delta_{ki}\delta_{lj} + (a+ib)\left((\phi_0)_{ii} - (\phi_0)_{jj}\right)\delta_{kj}\delta_{li}, \tag{2.60}$$

where we have no sums. We thus see that the commutator on the left hand side only has non-zero indices at (i, j) and (j, i). Thus the condition $[\Lambda^a T_a, \phi_0] = 0$ splits into $[a\mathbf{A}_{ij} + b\mathbf{B}_{ij}, \phi_0] = 0$ for all $i < j$. For a or b non-zero we see that this is only possible if $(\phi_0)_{ii} = (\phi_0)_{jj}$ which in turn allow both a and b to be non-zero. This proves that the partition (2.3) has an unbroken gauge group $\mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_n)$ as in (2.4).

Given a vacuum determined by ϕ_0 and the rest of the fields having expectation value zero, we should expand the Lagrangian around these expectation values:

$$\mathcal{L}_\sigma(\sigma) \equiv \mathcal{L}_{\mathrm{U}(N)}(\sigma + \phi_0)\,, \tag{2.61}$$

where the rest of the fields are unchanged. We know that supersymmetry should be unbroken, but that some gauge symmetries, generally, are broken. Above we found a nice basis $\{T_a\}$ of generators for the fundamental representation of U(N) in which the generators of the unbroken subgroup merely is a subset – no need for taking linear combinations. We can easily obtain that in this basis, the metric defined by $\mathrm{Tr}(T_a T_b)$ is also diagonal and positive definite. Let us work in this basis. The expansion of the Lagrangian must be done by Taylor expansion and even in the simple case of getting the mass terms this is quite elaborate. We will not do the calculation of the masses here, but refer to [10] for such a calculation. The result is that we will have three types of masses: We will have zero masses corresponding to the unbroken gauge multiplets. Secondly, we will have strictly non-zero masses determined by the square-root of the non-zero eigenvalues of the matrix with the (a, b) entry given by:[15]

$$\phi_0^\dagger \{T_a^{(\mathrm{adj})}, T_b^{(\mathrm{adj})}\}\phi_0, \tag{2.62}$$

where $T_a^{(\mathrm{adj})}$ is the generator in the adjoint representation. We note that this entry is zero if $T_a^{(\mathrm{adj})}$ or $T_b^{(\mathrm{adj})}$ is unbroken and it is positive when restricted to the broken subspace, i.e. where a and b are the indices of broken generators. Thus the number of eigenvectors with positive eigenvalues equals the number of broken gauge symmetries. These will be the masses of the massive gauge multiplets that arise by the supersymmetric version of the Higgs mechanism. At last we will have masses corresponding to the eigenvalues of the mass matrix:

$$M_{ab} = \left.\frac{\partial^2 W(\phi)}{\partial\phi^a \partial\phi^b}\right|_{\phi=\phi_0}. \tag{2.63}$$

These complex masses are the masses of the chiral multiplets in a representation of the unbroken gauge group. Naturally, the masses of the components will be real positive. As we see from (2.36) by expanding around the vacuum, the masses of the scalars (and thus of the multiplet) are the absolute values of the eigenvalues of M_{ab} if we disregard the gauge coupling constants in (2.36) (which would not be there if we had defined the superpotential as in footnote 10).

To be a bit more precise, we have N_B broken gauge symmetries with $[T_a, \phi_0] \neq 0$. As we saw above these $[T_a, \phi_0]$ are linearly independent and must span the whole space of broken symmetries since for any unbroken symmetry $T_{b,\mathrm{unb}}$ we have:

$$\mathrm{Tr}(T_{b,\mathrm{unb}}[T_a, \phi_0]) = \mathrm{Tr}(T_a[\phi_0, T_{b,\mathrm{unb}}]) = 0, \tag{2.64}$$

since $[T_{b,\mathrm{unb}}, \phi_0] = 0$. Thus $[T_a, \phi_0]$ is orthogonal to the space of unbroken symmetries and thus in the N_B-dimensional space of broken symmetries. In this space we find N_B complex scalars of which we find N_B real scalars of zero mass. These are the Goldstone bosons which can be removed by choosing unitary gauge ("eaten"). The remaining N_B scalars will have the positive masses determined by (2.62). In the same directions we find two Weyl spinors with the same mass namely the gaugino and the spinor from the chiral multiplet. With this positive mass we also find a gauge boson. This makes up the massive gauge multiplet that we found in section 1.2.3.

As we saw above the orthogonal space to the non-zero $[T_a, \phi_0]$'s is the unbroken subspace. In these directions we find massless gauge multiplets and chiral multiplets. The masses of the chiral multiplets are determined by the mass matrix in (2.63) restricted to the unbroken subspace. We will end this section by proving that all the eigenvalues of this matrix, and hence the masses, are non-zero if the roots of P'_{n+1} are different from each other as was claimed above in section 2.1.1. As in (2.31) we get

$$M_{ab} = \left.\frac{\partial}{\partial\phi^b}\mathrm{Tr}\big(T_a P'_{n+1}(\phi)\big)\right|_{\phi=\phi_0} = \mathrm{Tr}\big(T_a T_b P''_{n+1}(\phi_0)\big)\,, \tag{2.65}$$

where we have used that T_b is unbroken and thus commutes with ϕ_0. This made it possible to move T_b to the front. We note that it here was crucial that we only look at the unbroken directions because, as noted above, this could not have been done using the cyclicity of the

[15] As this mass matrix arises from the term $-\frac{1}{g^2}\mathrm{Tr}\Big([\hat{\phi}^\dagger, \hat{\phi}]^2\Big)$ we really should divide by $1/g^2$ to get the masses.

trace. We immediately see that we must assume P_{n+1} to be non-trivial i.e. of degree two or greater. Now let v^a by an arbitrary vector in the unbroken subspace and let us assume that:

$$M_{ab}v^b = \mathrm{Tr}\big(T_a \phi P''_{n+1}(\phi_0)\big) = 0, \tag{2.66}$$

where $\phi = T_b v^b$, and a and b are in the unbroken space. Since ϕ_0 is diagonal, $P''_{n+1}(\phi_0)$ is diagonal and each diagonal entry is equal to $P''_{n+1}((\phi_0)_{ii})$. This is non-zero by the assumption that the roots of P'_{n+1} are all different and that $(\phi_0)_{ii}$ is one of the roots. This is easily seen by differentiating and evaluating in one of the roots. What we should now realise is that since T_a is in the unbroken subspace

$$T_a P''_{n+1}(\phi_0) = T_a P''_{n+1}((\phi_0)_{kk})\,, \tag{2.67}$$

for some suitable k depending on a. This we see by checking it for all the three types of generators given in (2.59). Since $P''_{n+1}(\phi_0)$ is diagonal it can be expanded in the $\mathbf{D}_i$'s. Thus equation (2.67) is obviously fulfilled for $T_a = \mathbf{D}_i$ which are all in the unbroken subspace; here $k = i$. If $T_a = \mathbf{A}_{ij}$ then by (2.60) $(\phi_0)_{ii} = (\phi_0)_{jj}$ and thus $\big(P''_{n+1}(\phi_0)\big)_{ii} = \big(P''_{n+1}(\phi_0)\big)_{jj}$. Using this we easily see by calculation that $\mathbf{A}_{ij}P''_{n+1}(\phi_0) = \mathbf{A}_{ij}\left(P''_{n+1}(\phi_0)\right)_{ii}$ thus realising (2.67) with $k = i$. The same holds true for $\mathbf{B}_{ij}$ thus finishing the proof (2.67). Using cyclicity of the trace and that T_a commutes with ϕ_0, we can rewrite (2.66) as

$$\mathrm{Tr}\big(\phi T_a P''_{n+1}(\phi_0)\big) = \mathrm{Tr}(\phi T_a)\, P''_{n+1}((\phi_0)_{kk}) = 0. \tag{2.68}$$

Using that $P''_{n+1}((\phi_0)_{kk}) \neq 0$, we conclude that $\mathrm{Tr}(\phi T_a) = 0$ for all generators T_a of the unbroken subgroup. Using the invertibility of the metric we get that $\phi = 0$ thus concluding that we have no zero-eigenvalue vector and hence no zero mass. This finishes the proof.

What we should do now is to quantise the theory we have obtained thus defining a perturbation theory for the fluctuations around this vacuum. The massive gauge multiplets should be integrated out since they are not in the focus of our interest. The final theory will have a very complex structure and thus we will mostly deal with the case of unbroken gauge symmetry. This happens when all the eigenvalues in ϕ_0 are chosen to be the same. If we can choose all the eigenvalues to be zero the Lagrangian $\mathcal{L}_\sigma$ from (2.61) simply is $\mathcal{L}_{\mathrm{U}(N)}$.

2.3 Quantised Theory

In the last section we obtained the Lagrangian, which we want to quantise, by expanding around the expectation values of the chosen vacuum. We can expand this Lagrangian into components i.e. scalars, spinors and gauge bosons. These we already know how to quantise using the path integral technique. However, instead of quantising the component fields it is possible directly to quantise the superfields thus making the supersymmetry manifest. We note, however, that even though our Lagrangian is classically supersymmetric since we chose a supersymmetric vacuum, we do not know a priori if supersymmetry is broken in the quantised case. We will discuss this issue later. We will not go into details in this section, but just give a short review of the subject.

2.3.1 Supergraphs

Our goal is to develop a perturbation theory for the quantum fluctuations around the chosen vacuum. As usual, the important objects to calculate are the n-point Green's functions. From a superspace point of view we want the n-point Green's functions of the superfields.

From these one can then obtain the usual Green's functions by expanding in components as in (1.32). The tool to calculate the Green's functions is as usual the Feynman graphs which in this case are called *supergraphs*. We will focus on the case where the Lagrangian only contains chiral fields. Assuming renormalisability, the Lagrangian is given by (1.39).

The first thing to do when developing the supergraphs is to find the propagators. But before finding these we should realise that we have a problem: The chiral superfield is a constrained superfield by the chirality condition (1.33) (and the analogous for the anti-chiral field $\bar{\Phi}$). Thus we have a problem in defining a path integral with integrations over unconstrained superfields.

Another problem is that when we deal with supersymmetry the points should be superspace points. But the fermionic integral in the superpotential term is only over half the superspace. So we have to do something to put this term in the same form as the Kähler term with a four-dimensional θ-integration.

The propagators that we are looking for are $\langle T\Phi(x,\theta,\bar{\theta})\,\bar{\Phi}(x',\theta',\bar{\theta}')\rangle_0$ and the corresponding with $\Phi\Phi$ or $\bar{\Phi}\bar{\Phi}$. Here the vacuum expectation values should be found using the non-interacting Lagrangian, $\mathcal{L}_0$, and T is the time-ordering operator. There are several ways to find these propagators. Firstly we could simply expand the Lagrangian in component fields where we know how to do the calculations thus evading all problems. A second supersymmetric approach used by [9] imposes the constraint of chirality using the projection operator onto chiral fields defined by:

$$P_{\mathrm{ch}} = \frac{1}{16}\frac{\bar{D}\bar{D}DD}{\Box}, \tag{2.69}$$

which is well-defined since $\Box = \partial_\mu\partial^\mu$ commutes with the covariant derivatives. We immediately see that $P_{\mathrm{ch}}\Psi$ is chiral for any superfield Ψ since $\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} = 0$. That it is really a projection onto the chiral fields then follows since for any chiral field Φ we get by simple calculation using (1.31), (1.61) and (C.47):

$$\frac{1}{16}\frac{\bar{D}\bar{D}DD}{\Box}\Phi = \Phi. \tag{2.70}$$

This we could also have stated as the usual projection condition $P_{\mathrm{ch}}^2 = P_{\mathrm{ch}}$. Naturally, the corresponding projection onto anti-chiral fields

$$P_{\mathrm{anti-ch}} = \frac{1}{16}\frac{DD\bar{D}\bar{D}}{\Box} \tag{2.71}$$

is also needed. These projections also solve the problem that the superpotential terms only have integrations over half the superspace. Using (1.37) and (2.70) e.g. the mass term can be rewritten as:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\frac{1}{2}m\Phi^2 = -\frac{m}{8}\int \mathrm{d}^4x\mathrm{d}^4\theta\Phi\frac{DD}{\Box}\Phi. \tag{2.72}$$

At last, the chirality of the fields also has to be imposed when varying a chiral field. Using that the chiral field Φ only can depend on x_+ and θ as in (1.34), this boils down to the rule:

$$\frac{\delta}{\delta\Phi(x,\theta,\bar{\theta})}\Phi(x',\theta',\bar{\theta}') = -\frac{1}{4}\bar{D}\bar{D}\,\delta^{(2)}(\theta-\theta')\,\delta^{(2)}(\bar{\theta}-\bar{\theta}')\,\delta^{(4)}(x-x'). \tag{2.73}$$

Now we are able to find the equations of motions and after some calculations the propagators will be (in Minkowski space):[16]

$$\langle T\begin{pmatrix}\Phi\left(x,\theta,\bar{\theta}\right)\Phi\left(x',\theta',\bar{\theta}'\right) & \Phi\left(x,\theta,\bar{\theta}\right)\bar{\Phi}\left(x',\theta',\bar{\theta}'\right)\\ \bar{\Phi}\left(x,\theta,\bar{\theta}\right)\Phi\left(x',\theta',\bar{\theta}'\right) & \bar{\Phi}\left(x,\theta,\bar{\theta}\right)\bar{\Phi}\left(x',\theta',\bar{\theta}'\right)\end{pmatrix}\rangle_0 =$$

$$\frac{i}{\Box-m^2}\begin{pmatrix}\frac{m}{4}\bar{D}\bar{D} & \frac{1}{16}\bar{D}\bar{D}DD\\ \frac{1}{16}DD\bar{D}\bar{D} & \frac{m}{4}DD\end{pmatrix}\delta^{(4)}\left(x-x'\right)\delta^{(2)}\left(\theta-\theta'\right)\delta^{(2)}\left(\bar{\theta}-\bar{\theta}'\right). \quad (2.74)$$

A third way (and the last that we will present) to obtain the propagators is to introduce potential superfields as is done in [10]. This is in analogy with the well-known problem of the gauge field strength that is constrained by the homogeneous Maxwell equations forcing us to introduce the unconstrained gauge potential. In this case we introduce the unconstrained superfields Π^i defined from the chiral fields Φ^i as:

$$\Phi^i = \bar{D}\bar{D}\Pi^i. \quad (2.75)$$

As we see from (2.70) we can always find such a field, but it need not be a local field due to the $\Box^{-1}$. This would in turn also have meant that any chiral field would be chirally exact as discussed below equation (1.37) – this is not the case. We note that analogous to (1.28) $\bar{\Phi}^i = DD\bar{\Pi}^i$. The Lagrangian (1.39) now becomes:

$$\mathcal{L} = \int \mathrm{d}^4\theta\bar{\Pi}^i DD\bar{D}\bar{D}\Pi^i - 4\int \mathrm{d}^4\theta\left(\tilde{W}\left(\Pi^i\right) + \text{c.c.}\right), \quad (2.76)$$

where we have used that we can always integrate D and $\bar{D}$ by parts under the four-dimensional superspace integral since the superspace integral of a super-covariant derivative is zero by (1.36). $\tilde{W}$ is defined as $W\left(\bar{D}\bar{D}\Pi^i\right)$ where one pair of $\bar{D}\bar{D}$ has been removed in each term when using (1.37) to change the half superspace integral to the full superspace integral. But we still have a problem in defining the propagator since the Lagrangian by the definition of Π^i clearly is invariant under the transformations:

$$\Pi^i \mapsto \Pi^i + \bar{D}F, \quad (2.77)$$

where F is any superfield. This is in analogy with the gauge transformation of the gauge potential. However, there is no need for introducing Faddeev-Popov ghosts here to define the path integral since all Green's functions that we want to determine are invariant under (2.77). The solution is simply to project onto the space orthogonal to the zero-eigenvalue vector in (2.77) when determining the propagator. This projection is simply the anti-chiral projection (2.71) since $P_{\text{anti-ch}}\bar{D}F = 0$. Thinking of the mass term as an interaction term, the propagator for $\Pi^i\bar{\Pi}^j$ is then by (2.76):

$$-iDD\bar{D}\bar{D}\Delta_{ij}\left(x,\theta,\bar{\theta};x',\theta',\bar{\theta}'\right) = P_{\text{anti-ch}}\,\delta^{(4)}\left(x-x'\right)\delta^{(2)}\left(\theta-\theta'\right)\delta^{(2)}\left(\bar{\theta}-\bar{\theta}'\right)\delta_{ij}. \quad (2.78)$$

Here there can be some unspecified terms of the form $\bar{D}F$ that adds to the right hand side. But these are, as explained, unimportant. Using the definition of the anti-chiral projection (2.78) immediately solves as

$$\Delta_{ij}\left(x,\theta,\bar{\theta};x',\theta',\bar{\theta}'\right) = \frac{i}{16\Box}\delta^{(4)}\left(x-x'\right)\delta^{(2)}\left(\theta-\theta'\right)\delta^{(2)}\left(\bar{\theta}-\bar{\theta}'\right)\delta_{ij}. \quad (2.79)$$

Inserting the $\bar{D}\bar{D}$ and DD from the definition of the potential superfields we get the propagator for the chiral fields. The result is the same as (2.74) with $m = 0$.

[16] Here taken from [9]. We note that it is tacitly assumed that the mass m is real.

2.3.2 Non-Renormalisation

Using the method of potential superfields the Feynman rules for the vertices can immediately be obtained since the interactions from (2.76) simply are given by $\tilde{W}$ and its complex conjugate. We can use the supergraphs to find the 1PI-effective action here following [9] and [10]. For a general diagram contributing to the 1-PI effective action we see from (2.79) that we get a four-dimensional delta function in θ from each propagator. These delta functions are acted on by super-covariant derivatives from the Feynman rules of the vertices. We also have full superspace integrations for each vertex which means that we can always integrate the super-covariant derivatives by parts. For a given delta function we can then remove the super-covariant derivatives using integration by parts and then perform the corresponding four-dimensional θ-integration. Using this method we can eliminate all θ-integrations but one by the delta functions. All super-covariant derivatives then act on the external fields and all delta functions have been removed.[17] A simple counting of vertices, internal and external lines shows that the super-covariant derivatives can be used to change the external Π^i's back to Φ's except in the case of tree-level diagrams. Thus the effective action can be written:

$$\int \mathrm{d}^4\theta \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n F_1\left(x_1,\theta,\bar{\theta}\right) \ldots F_n\left(x_n,\theta,\bar{\theta}\right) G(x_1,\ldots,x_n) + \text{tree-level diagrams.} \tag{2.80}$$

Here the F_i's only depend on the external chiral fields and the super-covariant derivatives, and G is just some translationally invariant function. We see that contributions to the superpotential term, which only has integration over half the superspace, can only come from the tree-level diagrams which in the 1-PI case are the simple vertices. We have thus reached the important result that the superpotential is not renormalised perturbatively and no new terms are introduced. Thus only the Kähler term is renormalised.

There is, however, one flaw in this proof and that is when we consider the $\Box^{-1}$ in the propagator (2.79). If we include the mass term in the propagator, the $\Box^{-1}$ changes into $1/\left(\Box - m_i^2\right)$ where we have assumed that the mass term is diagonal giving Φ^i mass m_i. Naturally, with the $\delta^{(4)}(x-x')$ this just gives the usual Feynman propagator. With the mass we also have $\Pi\Pi$ and $\bar{\Pi}\bar{\Pi}$ propagators that also have the Feynman propagator as a factor.[18] Now if we have a zero mass field, we could have a 1PI-diagram contributing to the first term in (2.80) of the form:

$$\int \mathrm{d}^4\theta \frac{DD}{\Box} f\left(\Phi^i\right) = -4 \int \mathrm{d}^2\theta f\left(\Phi^i\right), \tag{2.81}$$

where f is a function of the chiral fields Φ^i, and we used (1.37) and (2.70) as in (2.72). This clearly gives a change in the superpotential. The first explicit example of such a diagram was given in [22] as a two loop diagram. In the article they actually do the calculations in components, but it is noted that the corresponding supergraph yields the same result. It is important to realise that such a contribution arose because we had a massless field and we thus have an IR divergency in the D-term. The IR divergency stems from the propagator $\Box^{-1}$ which can bring about an integration $\int_0 \mathrm{d}k \frac{1}{k}$ which is logarithmically divergent. Actually, all loop contributions to the F-term from 1PI diagrams come from IR divergent D-terms [23].

[17] When considering the details one needs to use that $\delta(\theta) = \theta$. Thus $\delta(0) = 0$ and $\delta(\theta)\,\delta(\theta) = 0$.

[18] A remark is in order for consistency with the second way of obtaining the propagators that led to (2.74). When doing diagrams we should not use the propagators in (2.74). The reason is that we also have to use the formula (2.73) for varying the external chiral currents one introduces to develop the Feynman graphs. Taking this into consideration one gets the Grisaru-Roček-Siegel propagator which really just is the same as the propagator for the potential superfields.

We will return to the subject of such non-renormalisation theorems in section 2.5.2. However, there is another way to obtain effective actions without the above described problems. We will introduce this method in the next section.

2.4 Wilsonian Renormalisation

Suppose we are interested in the dynamics of the quantised theory at energies below some energy-momentum cut-off μ. All the physics is then captured by the Wilsonian effective action that we will introduce in this section. The treatment is based on [12], [13], [24] and [25].

2.4.1 Wilsonian Effective Action

Our starting point is a quantum field theory regularised by an UV cut-off Λ. Naturally, such a sharp momentum cut-off is not preserved by gauge symmetry, but let us not worry about that here. Another problem is that the condition $k^\mu k_\mu < \Lambda^2$ in Minkowski space does not ensure that each 4-momentum coordinate is bounded – thus we rotate to Euclidean space. For concreteness let us think of a scalar field theory with Lagrangian $\mathcal{L}_\Lambda$. The generating functional is then given by (setting the external current to zero for simplicity):

$$Z = \int [\mathcal{D}\phi]_\Lambda \, e^{-\int \mathrm{d}^4x \mathcal{L}_\Lambda(\phi)}, \tag{2.82}$$

where the scalar field only has non-zero momentum modes for $\|k\| < \Lambda$, and the functional integration is defined as $[\mathcal{D}\phi]_\Lambda = \prod_{\|k\|<\Lambda} \mathrm{d}\phi(k)$. To obtain an action that only depends on the energy-momentum below the cut-off μ, we want to integrate out $\phi(k)$ with $\mu \leq \|k\| < \Lambda$. To do this we split the scalar field as $\phi = \tilde{\phi} + \hat{\phi}$ where

$$\tilde{\phi}(k) = \begin{cases} \phi(k) & \text{if } \|k\| < \mu \\ 0 & \text{if } \mu \leq \|k\| < \Lambda \end{cases}, \qquad \hat{\phi}(k) = \begin{cases} 0 & \text{if } \|k\| < \mu \\ \phi(k) & \text{if } \mu \leq \|k\| < \Lambda \end{cases}. \tag{2.83}$$

Now we can split the Lagrangian into the original Lagrangian evaluated in $\tilde{\phi}$ and a mixed term:

$$\mathcal{L}_\Lambda(\tilde{\phi} + \hat{\phi}) = \mathcal{L}_\Lambda(\tilde{\phi}) + \mathcal{L}_{\text{mixed}}(\tilde{\phi}, \hat{\phi}), \tag{2.84}$$

where there are no terms only depending on $\tilde{\phi}$ in the mixed Lagrangian. Since $\tilde{\phi}$ and $\hat{\phi}$ are orthogonal in momentum space, quadratic terms of the form $\tilde{\phi}\hat{\phi}$ are zero. Thus the kinetic term and the mass term in $\mathcal{L}_{\text{mixed}}$ only depend on $\hat{\phi}$ making this a Lagrangian in $\hat{\phi}$ with $\tilde{\phi}$ as an external field. We can now integrate the $\hat{\phi}$ field out:

$$Z = \int \left[\mathcal{D}\tilde{\phi}\right]_\Lambda e^{-\int \mathrm{d}^4x \mathcal{L}_\Lambda(\tilde{\phi})} \int \left[\mathcal{D}\hat{\phi}\right]_\Lambda e^{-\int \mathrm{d}^4x \mathcal{L}_{\text{mixed}}(\tilde{\phi},\hat{\phi})} = \int \left[\mathcal{D}\tilde{\phi}\right]_\mu e^{-\int \mathrm{d}^4x \mathcal{L}_\mu(\tilde{\phi})}, \tag{2.85}$$

where the functional integrations only are over the non-zero modes. $\mathcal{L}_\mu$ is the *Wilsonian effective Lagrangian* at the energy μ. For calculations of processes with energies and momenta less than μ it gives precisely the same result as using the old Lagrangian, however, now it is only necessary to perform the momentum integrations up to μ.

For small couplings we can make Feynman diagrams representing the process of integrating out $\hat{\phi}$ in (2.85) with $\tilde{\phi}$ as an external field. It is natural to put this diagrammatic contribution in exponential form. As usual this corresponds to only taking into account connected diagrams. $\mathcal{L}_\mu$ is then simply the sum of $\mathcal{L}_\Lambda$ and these diagrammatic contributions:

$$\mathcal{L}_\mu(\tilde{\phi}) = \mathcal{L}_\Lambda(\tilde{\phi}) + \text{connected diagrams}. \tag{2.86}$$

We note that the propagators in the diagrams will only have to be integrated over the energy-momentum shell $\mu \leq \|k\| < \Lambda$. These diagrams thus suffer from neither UV nor IR divergencies which is an essential feature in the Wilsonian effective action. Usually the diagrams give an infinite number of non-zero correction terms – all those that are allowed by symmetries. These corrections are new (and renormalisations of the old) interactions terms that compensate for the degrees of freedom that has been integrated out. However, we immediately see that supersymmetry is an exception (actually the only known) since here there is no renormalisation of the superpotential as we saw in the last section for a theory with only chiral fields. The exceptions to the non-renormalisation theorem mentioned in the last section had their root in IR divergencies which are not a problem here.

2.4.2 Renormalisation Group Running of Couplings

We can use the above procedure for a general theory if we integrate out the high energy-momentum modes for all the particles simultaneously to obtain $\mathcal{L}_\mu$. We can think of μ as a continuous parameter. We note that this e.g. can be done by in steps integrating out infinitesimal energy-momentum shells. The couplings $g_i(\mu)$ are then continuous functions of μ. The renormalisation group running in the space of theories is then given by the Wilson equation:

$$\mu \frac{\partial g_i(\mu)}{\partial \mu} = \beta_i(g(\mu), \mu)\,. \tag{2.87}$$

We can constrain the μ-dependence in the β-function by introducing dimensionless couplings. If Δ_i is the mass dimension of g_i then we define the dimensionless couplings as $\mathcal{G}_i(\mu) = \mu^{-\Delta_i} g_i(\mu)$. However, $\mathcal{G}_i(\mu)$ can only depend on $\mathcal{G}(\Lambda)$ and μ/Λ since these are the only dimensionless parameters. Differentiating and setting $\Lambda = \mu$ then gives:

$$\mu \frac{\partial \mathcal{G}_i(\mu)}{\partial \mu} = \tilde{\beta}_i(\mathcal{G}(\mu))\,, \tag{2.88}$$

where we note that there is now no explicit dependence on μ on the right hand side. Using the dimensionless couplings one can show a theorem due to J. Polchinski saying (with some assumptions) that if the initial couplings $\mathcal{G}(\Lambda)$ lie on generic N-dimensional surface then for $\mu \ll \Lambda$ they will approach a fixed surface that is independent of the initial surface and Λ. This surface is also approximately stable under further running of the parameter μ. Here N is the number of renormalisable couplings. Using this we see that the physical observables are independent of Λ as they should be.[19]

A fixed point is as usual a point where the left hand side of (2.87) vanishes. We can linearise the β-function around the fixed point. The eigenvalues of the $\mu\frac{\partial}{\partial\mu}$ operator then determines whether the coupling is damped or grows along the flow. The corresponding operators are for the former called *irrelevant* and for the latter *relevant* operators. Zero eigenvalues correspond to *marginal* operators. The free theory is, naturally, a fixed point since $\mathcal{L}_{\text{mixed}}$ in (2.84) in this case is independent of $\tilde{\phi}$. In the vicinity of this point where we have weak coupling, we can determine the relevant operators by dimensional analysis. As above g_i has mass dimension Δ_i and thus its natural order of magnitude is Λ^{Δ_i}. Thus a coupling with positive mass dimension will become increasingly important at lower μ and the contrary for a coupling with negative mass dimension. This, naturally, only holds true as long as the couplings are not changed too much by the quantum corrections i.e.

[19] This can be used to justify the usual renormalisation scheme where we take Λ to infinity and express the couplings and masses in terms of physical quantities.

in the vicinity of the free fixed point. Thus we see that the relevant operators here are the super-renormalisable, the marginal are the renormalisable, and the irrelevant are the non-renormalisable operators.

One can now ask why we do not drop the irrelevant operators when letting μ flow towards low energies. However, when we want to examine a vacuum we expand the fields around the vacuum expectation values. In this way irrelevant and relevant terms are mixed. Thus irrelevant terms can contribute to relevant terms after the redefinition of the fields. However, by the above dimensional analysis we see that it is enough to keep two derivatives on scalar fields and one on the fermionic fields – just as we did in chapter 1 when developing the general supersymmetric Lagrangians. With these Lagrangians we can obtain the vacuum expectation values and investigate the relevant and marginal physics around the vacua. But if we do not want to let μ go all the way to zero, we are to some extent making an approximation here. Also note that we assumed that we were close to a free fixed point as $\mu \to 0$ so we could characterise the relevant operators as the renormalisable ones. This is actually true for a wide range of theories: The Coleman-Gross theorem tells us that for small couplings a theory of scalars, spinors and U(1) gauge bosons has an IR free fixed point.

In general when wanting to determine the effective action at some scale μ one has to guess which degrees of freedom are relevant (or marginal) at that scale.

2.4.3 Integrating Out Massive Fields

We note that both the kinetic terms and the masses are renormalised as we let μ float to lower energies. However, we can as usual renormalise the wavefunctions to keep the kinetic terms invariant. This will naturally also renormalise the couplings that are then called canonical couplings. Let us assume that we have a massive field ϕ. Normally the canonically renormalised mass will not decay as the energy-momentum scale μ runs to zero so at some point μ will be less than the canonically renormalised mass of ϕ. This means, especially if $\mu \ll m$, that the mass term dominates the kinetic term which we can then disregard. Let us here pause the flow of the rest of the fields and perform the remaining integration of the field ϕ. What we should think here is then:[20] When we want to integrate out the remaining momentum shell of thickness μ the propagator for the field ϕ is only given by the inverse mass (squared). A loop with this propagator then contains a momentum integration of the thickness μ. The loop must then scale as μ/m to some power and can thus be discarded. Thus we only have tree-level diagrams left i.e. the semi-classical approximation. This tells us that when we go below the mass of a field we can completely integrate it out by replacing it with its equation of motion:

$$\frac{\partial \mathcal{L}_\mu}{\partial \phi} = 0, \tag{2.89}$$

where we used that the dependence in the Lagrangian on $\partial\phi/\partial x^\mu$ can be removed since terms including such space-time derivatives give rise to tree-level diagrams that scales as μ/m to some power. The integrating out procedure will be important for us in the next section.

2.4.4 Wilsonian vs. 1PI Effective Action

In the limit $\mu \to 0$ we can compare the Wilsonian and the 1PI effective action. As we saw in the last subsection the massive fields will then be completely integrated out of the Wilsonian effective action. However, the massless fields are still degrees of freedom. This

[20] The literature is unfortunately a bit vague at this point.

is contrary to the 1PI effective action where all fields are integrated out. But if there are no massless fields the 1PI and the Wilsonian action coincide. As discussed above, when one integrates out the massless fields in the 1PI effective action one gets IR divergencies. We saw that because there were no IR divergencies in the Wilsonian effective action, it fulfilled the non-renormalisation theorem. In the next section we will discuss that this also ensures the intimately related concept of holomorphy of coupling constants in the Wilsonian effective action – this is not case for the 1PI effective potential.

It should be mentioned that the Wilsonian effective action actually suffers from IR divergencies when we are varying in the space of theories. Simply think of a $\mathcal{L}_\mu$ with some field ϕ with mass $m < \mu$. Here the field is integrated out. If we now vary m to zero we have accidentally integrated out a massless field and we get an IR divergency. It is a fact that all singularities in the Wilsonian effective action arise in this way as fields becoming massless.

Let us end this section by noting that the Lagrangian $\mathcal{L}_{\mathrm{U}(N)}$ in the Dijkgraaf-Vafa conjecture can be seen as an effective Lagrangian obtained by integrating out a very massive field in an underlying theory. Hence we can understand the inclusion of non-renormalisable powers in the tree-level superpotential. In the next section we will see how further renormalisation group running changes the effective superpotential.

2.5 The Wilsonian Effective Superpotential

In this section we will study the Wilsonian effective superpotential at some scale μ. Let us assume that supersymmetry is unbroken at this scale. Then we can write the effective Lagrangian in the general supersymmetric form (2.7). This allows us to define the Wilsonian effective generalised superpotential as the $\theta\theta$-term in (2.7). As also noted there we have here enlarged the definition of the superpotential to also include the supersymmetric gauge field strength hence the name “generalised superpotential”.[21] This made sense since the gauge kinetic term, even though it can be written as a local D-term, can not be written as a gauge invariant D-term. We will begin our investigation of this effective superpotential by introducing the concept of holomorphy which will be crucial to us not only in this section, but also when we give the diagrammatic proof of the Dijkgraaf-Vafa conjecture.

2.5.1 Holomorphy

Since the mid-eighties it has been known that the effective superpotential should be holomorphic not only in the chiral fields as demanded by supersymmetry, but also in the bare coupling constants. However, it was discovered that it here is essential that the effective superpotential is the Wilsonian one since the IR-divergencies from massless particles in the 1PI effective superpotential can violate this holomorphy - i.e. give an holomorphic anomaly (this was made clear in [26] and [27], and is reviewed in [23]).

In [28] N. Seiberg introduced[22] a trick well-known from string theory that allows us to prove the holomorphicity in the (complex!) coupling constants: We think of the bare couplings g_k in the (not generalised) superpotential as being the scalar components of background chiral superfields. E.g. they could be very massive chiral superfields integrated out at the energy at which we write our bare Lagrangian and with fine-tuned couplings such

[21] We will simple refer to the generalised superpotential as “superpotential” when no confusion should be possible.

[22] Actually, the trick is also mentioned in [27], however, it is not used in its full power as in [28]. It should also be mentioned that [28] is based on an unpublished argument by N. Seiberg and J. Polchinski.

that the expectation value of the scalar components exactly match the bare couplings g_k (and the rest of the components have zero expectation value). Now, as we flow down in energy the Wilsonian effective superpotential will still be holomorphic in the chiral fields due to supersymmetry which we assume to be unbroken. This now includes the auxiliary background fields which are just the bare constants. Thus the effective superpotential is holomorphic in the bare coupling constants, i.e. independent of g_k^*, as long as supersymmetry is unbroken.[23] This remains true also for non-perturbative corrections as it is based only on the assumption of unbroken supersymmetry. However, we note that it is important not to use the canonically renormalised couplings since the Kähler term is not holomorphic.

The holomorphicity also applies to the complex gauge coupling constants τ from (1.76) and (1.80). And, naturally, it also applies when we look at the generalised superpotential.

Using the full power of holomorphy and treating the couplings as background fields we can prove powerful non-renormalisation theorems as we will see already in the next section.

2.5.2 Non-Renormalisation Theorems

In this section we will use the idea from the last section that the couplings in the generalised superpotential can be seen as background chiral superfields. We will prove that the generalised superpotential is not renormalised perturbatively or to be more precise that the only renormalisation that takes place is one-loop renormalisation of the complex gauge coupling τ.

In the article [28] (reviewed in [29] and [30]) Seiberg introduced the following scheme for determining the effective superpotentials:

1. Holomorphy: As discussed in the previous section the dependence in the generalised superpotential on the bare couplings should be holomorphic due to supersymmetry.

2. Symmetries and selection rules: Setting the background fields (i.e. the couplings) to zero gives a large global symmetry group of the Lagrangian where we also include R-symmetries. This symmetry group is spontaneously broken for non-zero expectation values of the background fields (non-zero couplings). However, letting the background fields transform (assigning transformation rules to the couplings) the global symmetry group can be restored for the Lagrangian. If non-anomalous, these symmetries must be shared by the effective Lagrangian giving powerful restrictions that are further sharpened when using the above holomorphy.

3. Various limits: E.g. in the weak coupling limit it can be possible to put restrictions on the Lagrangian using perturbation theory. Sometimes one can require smoothness in the weak coupling limit or even, when setting some masses to zero, use that there must be singularities due to massless particles that have been integrated out as explained in section 2.4.4.

This scheme will often determine the effective superpotential since a holomorphic function is determined by its asymptotic behaviour and its singularities. We will now use it to prove the perturbative non-renormalisation theorem. This was formerly done using the supergraph method as in section 2.3.2 which generalises to the case where vector superfields are involved. Here we will present a proof based on [10] and [20] using the Seiberg scheme.

[23] Naturally, also the term superpotential no longer makes sense when supersymmetry is broken.

Let us assume that our bare Lagrangian at the UV energy μ_0 has the form:

$$\mathcal{L}_{\mu_0} = \int \mathrm{d}^4\theta \Phi^\dagger e^{2V^{(\mathbf{r})}} \Phi + \left(\int \mathrm{d}^2\theta \frac{\tau}{16\pi i C(\mathbf{r})} \mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha \mathcal{W}_\alpha) + \int \mathrm{d}^2\theta W_{\mu_0}(\Phi^i) + \text{c.c.} \right). \quad (2.90)$$

Here we have used (1.67) and assumed a renormalisable Kähler term. The gauge group is for simplicity assumed simple and the gauge kinetic term renormalisable so that we could use (1.79). The superpotential could be non-renormalisable.

Using the background field method we think of the bare complex gauge coupling τ as being a background chiral superfield. The same we do with the couplings g_k in the superpotential $W_{\mu_0}(\Phi^i)$ which we can write as:

$$W_{\mu_0}(\Phi^i) = \sum_k g_k O_k(\Phi^i), \quad (2.91)$$

where $O_k(\Phi^i)$ is a gauge invariant multiple of the chiral fields, i.e. a multiple of the independent holomorphic gauge invariants, X_r, parameterising the vacuum moduli space introduced in section 2.2.3.

Now we want to determine the effective Lagrangian $\mathcal{L}_\mu$ at the lower energy μ. We assume that the cut-off respects supersymmetry and gauge invariance. Thus $\mathcal{L}_\mu$ a priori takes the form of the most general gauge invariant and supersymmetric Lagrangian i.e.:

$$\mathcal{L}_\mu = \int \mathrm{d}^4\theta G_\mu(\Phi^i, \bar{\Phi}^i, V, \tau, \tau^*, g_k, g_k^*, D_\alpha \ldots) + \left(\int \mathrm{d}^2\theta F_\mu(\Phi^i, \mathcal{W}_\alpha, \tau, g_k) + \text{c.c.} \right). \quad (2.92)$$

Here G_μ and F_μ are very general gauge invariant functions since we do not know which operators are relevant at the scale μ. This means that we can not constrain ourselves to the Lagrangians given in section 1.3 where we assumed maximally two space-time derivatives on bosonic fields and one on fermionic fields as explained in section 2.4.2. However, we do note that the generalised superpotential F_μ is holomorphic in g_k and τ as determined by supersymmetry. Also, F_μ does not depend on the covariant derivatives D_α or space-time derivatives since according to [10] such terms can be reformulated as D-terms as we did with the chirally exact terms.

The next item on the Seiberg scheme is to constrain $\mathcal{L}_\mu$ by extended global symmetries. There are two of those. The first is a $\mathrm{U}(1)_R$ R-symmetry under which the generalised superpotential must have charge 2 as explained in section 1.3.3 . When all the couplings are zero we obtain an R-symmetry simply by choosing Φ^i and V to be R-neutral. In order for this to be an R-symmetry with non-zero couplings we must assign transformation rules to the couplings, i.e. the background fields. All couplings g_k must have charge 2 to give the tree-level superpotential an overall charge of 2. On the other hand $\mathcal{W}_\alpha$ given by (1.55) must have charge 1 since $R(\partial/\partial\theta^\alpha) = R(D_\alpha) = -1$. We then conclude that τ has zero charge. We will now assume that $\mathcal{L}_\mu$ is obtained perturbatively. Then the R-symmetry is non-anomalous as we will see in the next section. Thus $\mathcal{L}_\mu$ must be invariant under the R-symmetry and F_μ must have charge 2. Since it is holomorphic, it only depends on couplings and fields having non-negative charges and we conclude that it takes the form:

$$F_\mu(\Phi^i, \mathcal{W}_\alpha, \tau, g_k) = W_\mu(\Phi^i, g_k, \tau) + \frac{1}{16\pi i} \tau_{\mu,ab}(\Phi^i, \tau)\, \mathcal{W}^{\alpha a} \mathcal{W}^b_\alpha, \quad (2.93)$$

where W_μ is linear in the g_k's.

The second symmetry we will use is $\tau \mapsto \tau + \xi$ where ξ is a real number. This is a symmetry of the Lagrangian since as we will see in the next section the real part of τ i.e. ϑ

multiplies a total derivative term that does not contribute in the perturbative regime. Thus τ can only appear as multiplying a $\mathcal{W}^{\alpha a}\mathcal{W}^{b}_{\alpha}$ term as in the tree-level Lagrangian. Hence W_μ is independent of τ and demanding gauge invariance gives $\tau_{\mu,ab}(\Phi^i,\tau) = c_\mu\delta_{ab}\tau + d_{\mu,ab}(\Phi^i)$ since the gauge group is simple.

The last step in the Seiberg scheme is to consider limits. Setting g_k equal to zero we have a global symmetry of the bare Lagrangian: $\Phi \mapsto e^{-i\alpha}\Phi$. This constrains $d_{\mu,ab}(\Phi^i)$ to be independent of Φ^i and thus equal to $\delta_{ab}d_\mu$ for the sake of gauge invariance. Thus we can write F_μ as:

$$F_\mu(\Phi^i, \mathcal{W}_\alpha, \tau, g_k) = W_\mu(\Phi^i, g_k) + \frac{c_\mu\tau + d_\mu}{16\pi i C(\mathbf{r})}\,\mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha\mathcal{W}_\alpha)\,. \tag{2.94}$$

Let us now take the limit where the gauge coupling and the g_k's are small i.e. weak coupling. Since W_μ is linear in the g_k's, the only diagrams that can contribute to W_μ is the single vertex diagrams determined from the bare superpotential W. This shows us that the superpotential is not renormalised: $W_\mu = W$.

To find c_μ and d_μ we also use the weak coupling limit. We note that this for the gauge coupling amounts to taking τ to infinity. We can think of τ as being purely imaginary since, as noted above, ϑ does not contribute. We can then develop the supergraph rules for the vector field V. The propagator and self-interaction vertices are derived from the term $\int \mathrm{d}^2\theta \frac{\tau}{16\pi i C(\mathbf{r})}\,\mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha\mathcal{W}_\alpha)$. This means that the propagator goes as $1/\tau$ and the self-interaction vertices go as τ. The interactions with matter come from the Kähler term and are τ independent. We note that each of these interaction vertices has two Φ-lines attached since we assumed a renormalisable Kähler term in (2.90). There can be no Φ-Φ interaction since we have no dependence on the g_k's in c_μ and d_μ. Now, let there be given a diagram contributing to the last term of F_μ in (2.94). We will count the τ dependence in this diagram. Let V_V and I_V be the number of pure gauge boson vertices and internal gauge boson lines respectively. The power of τ in the diagram then is:

$$N_\tau = V_V - I_V. \tag{2.95}$$

Introducing V_Φ for the number of (not pure gauge) interaction vertices and I_Φ for the number of internal Φ-lines (or $\bar{\Phi}$-lines), the number of loops is given by (assuming a connected momentum space diagram):

$$L = I_V + I_\Phi - V_V - V_\Phi + 1. \tag{2.96}$$

However, since there are no external Φ-lines and each interaction vertex contains two Φ-lines we have $V_\Phi = I_\Phi$ so $N_\tau = 1 - L$. Thus only tree-level diagrams contributes to c_μ i.e. it takes the same value as in the bare Lagrangian: $c_\mu = 1$. And d_μ only receives one-loop renormalisation (we can include the tree-level which is zero). The renormalised superpotential then takes the form:

$$F_\mu = W(\Phi^i) + \frac{\tau(\mu)}{16\pi i C(\mathbf{r})}\,\mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha\mathcal{W}_\alpha)\,, \tag{2.97}$$

where $\tau(\mu)$ is the one-loop renormalised complex gauge coupling. This ends the proof of the perturbative non-renormalisation theorem. We could also have obtained this result from a general non-renormalisable bare Lagrangian, however, the counting of the powers of τ would have been a bit harder [20].

$\tau(\mu)$ can be determined from standard quantum gauge theory calculations; here taken from [10], [12] and [25]. The renormalisation group running (2.87) of the (real) gauge coupling g is found to be $\mu\frac{\partial}{\partial\mu}g = -\frac{b}{16\pi^2}g^3$ to one-loop order. Here b depends on the quadratic invariants, $C(\mathbf{r})$, defined in (1.78) of the representations of the complex bosons and the Weyl fermions. Using that we have one adjoint Weyl fermion for each vector superfield, and we have a Weyl fermion and a complex boson for each chiral superfield representation $\mathbf{r}_n$, one gets:[24]

$$b = 3C(\text{adj}) - \sum_n C(\mathbf{r}_n)\,, \tag{2.98}$$

where the sum is over the different representations of the chiral fields. We note that this is not invariant under scalings of the gauge group generators since these scale the gauge coupling as explained at the end of section 1.3.4. Solving the Wilson equation yields:

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\ln\left(\frac{|\Lambda|}{\mu}\right), \tag{2.99}$$

where $|\Lambda|$ is the *strong coupling scale* of the theory. It is simply a constant of integration that by definition is scale invariant and we can express it as:

$$|\Lambda| = \mu e^{-\frac{8\pi^2}{bg^2(\mu)}}. \tag{2.100}$$

The reason for the modulus is that we can now express the running complex gauge coupling $\tau(\mu)$ using (1.80) as:[25]

$$\tau(\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right), \tag{2.101}$$

where Λ is the (complex) holomorphic scale[26] given by:

$$\Lambda \equiv |\Lambda|\, e^{i\vartheta/b} = \mu e^{\frac{2\pi i}{b}\tau(\mu)}. \tag{2.102}$$

The strong coupling scale is naturally so called since when μ approaches $|\Lambda|$ the effective gauge coupling diverges as seen from (2.99). We also see that the sign of b determines whether the theory is UV or IR free. For b positive the theory is UV free and IR strongly coupled i.e. asymptotically free. From (2.98) we see that this happens for non-abelian gauge theories with not too much light[27] charged matter. On the other hand, for b negative the theory is weakly coupled in the IR and runs to strong coupling in the UV. This happens e.g. in the abelian case where $C(\text{adj}) = 0$ or for theories with enough charged matter.

For an asymptotically free theory we must demand that the scale μ in the non-renormalisation theorem is greater than the strong coupling scale – so the theorem does not solve the strong coupling problem of asymptotically free theories.

We should note that for a general (classical) gauge group we get one holomorphic scale for each simple factor. We also note that the 1PI complex gauge coupling does receive higher order loop contributions. However, the effective theory is not holomorphic in the 1PI

[24] This is often written with the quadratic Casimir $C_2(\text{adj})$ instead of $C(\text{adj})$ which one can do since the quadratic Casimir and the quadratic invariant are equal in the adjoint representation.

[25] We note that this tells us that $d_\mu = \frac{b}{2\pi i}\ln\left(\frac{\mu_0}{\mu}\right)$. Here μ_0 is the scale for the tree-level Lagrangian.

[26] Not to be mistaken with the UV cut-off in section 2.4.

[27] We note that the fields contributing to the counting in b are only those which have not been integrated out at the scale μ i.e. the light matter. This also means that Λ depends on how much matter that has been integrated out.

coupling and the relation between the Wilsonian and the 1PI complex gauge couplings is consequently non-holomorphic [27].

An important consequence of the non-renormalisation theorem is that if supersymmetry is unbroken classically, it is unbroken to any order in perturbation theory. This follows since the superpotential is not renormalised and thus we still have the classical solution to the F-flatness equations. But the Kähler term is now taking a general form since we can not constrain this term with holomorphy. However, the proof in section 2.2.2 still works for this general Kähler term and thus shows us that we can always find a simultaneous solution to both the D- and F-flatness equations. Consequently, supersymmetry in unbroken perturbatively. However, this does not mean that the quantum moduli space is the same as the classical. As noted in section 2.2.3 the classical moduli space is endowed with the Kähler metric pulled back from the target space. But this metric is, naturally, changed by the Kähler term renormalisation thus changing the moduli space. It should be mentioned that non-perturbatively it is possible to break supersymmetry since we here do not have a strict non-renormalisation theorem in the non-perturbative regime.

The Dijkgraaf-Vafa conjecture provides a systematic way to obtain the effective generalised superpotential. We have learned in this section that the non-trivial part of these effective superpotentials must be non-perturbative contributions. We will discuss such contributions shortly, but first we will have to take a look at the ϑ-angle and chiral anomalies.

2.5.3 The ϑ-Angle, Instantons and Chiral Anomalies

In this section we will rather briefly study the $F_{\mu\nu}\tilde{F}^{\mu\nu}$-term in the gauge kinetic and self-interaction Lagrangian (1.71). The presentation is based on [12], [31], [32] and [33]. Let us look at a non-abelian gauge group and concentrate on a simple factor. For simplicity we think of this subgroup as SU(2). The results will be true for general non-abelian gauge groups since they all contain SU(2) subgroups. From (1.81) we see that the term of our interest takes the form:

$$S_\vartheta = -\int \mathrm{d}^4x \frac{\vartheta}{16\pi^2} \,\mathrm{Tr}\Big(F_{\mu\nu}\tilde{F}^{\mu\nu}\Big) \,, \tag{2.103}$$

where the trace here and in the rest of this subsection is taken in the fundamental representation where, as before, the quadratic invariant from (1.78) is chosen to be 1/2. Using the definition of the field strength (1.58) we get:

$$S_\vartheta = -\int \mathrm{d}^4x \frac{\vartheta}{8\pi^2} \varepsilon^{\mu\nu\rho\sigma}\partial_\mu \,\mathrm{Tr}\bigg(A_\nu\partial_\rho A_\sigma + i\frac{2}{3}A_\nu A_\rho A_\sigma\bigg) \,, \tag{2.104}$$

which, as promised above, is a total space-time derivative term.

For the integral (2.103) to be finite we must require that $F_{\mu\nu}$ vanishes at infinity.[28] Hence, at infinity the gauge potential is pure gauge such that $A_\mu = i\,(\partial_\mu g)\,g^{-1}$ where g is an element in the gauge group.[29] Plugging into (2.104) gives:

$$S_\vartheta = \frac{\vartheta}{24\pi^2}\int_{S^3} \mathrm{d}^3\xi\,\varepsilon^{ijk}\,\mathrm{Tr}\big(g^{-1}(\partial_i g)g^{-1}(\partial_j g)g^{-1}(\partial_k g)\big) \,, \tag{2.105}$$

where we have rewritten the four-dimensional space-time integral over the total derivative as a surface integral over the 3-sphere S^3 at infinity.[30] However, this integral is known from

[28] Since $F_{\mu\nu}$ transforms adjointly, the condition $F_{\mu\nu} = 0$ at infinity is gauge invariant.

[29] We have here used that the gauge transformation of A_μ with g is $A'_\mu = gA_\mu g^{-1} + i\,(\partial_\mu g)\,g^{-1}$ consistent with the definition of D_μ in (1.59).

[30] We have here assumed a regular gauge with no divergencies.

homotopy theory. It counts (ignoring the ϑ-angle) the number of times $g(x)$ wraps S^3 around SU(2) which in turn is topologically equivalent to S^3. This is an integer – the winding number – and it determines to which homotopy (*Pontryagin-*) class the gauge potential belongs. This corresponds to the third homotopy group being $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$. The winding number is a topological invariant and is thus unchanged under continuous deformations of the fields. We can now write S_ϑ as:

$$S_\vartheta = \vartheta n, \qquad n \in \mathbb{Z}, \tag{2.106}$$

where n is the winding number. Since the contribution to the path integral is e^{iS_ϑ}, we see that

$$\vartheta \mapsto \vartheta + 2\pi \tag{2.107}$$

is a symmetry of the theory or, more correctly, it is an exact equivalence of theories. This shows us that we can think of ϑ as an angle. This symmetry can be expressed in the complex gauge coupling as:

$$\tau \mapsto \tau + 1. \tag{2.108}$$

To investigate the above field configuration further we switch to temporal gauge $A_t = 0$ and the surface in (2.105) is taken to be a cylinder parallel to the time axis. Then the only contribution to our surface integral comes from the caps at $t = \pm\infty$. The surface integral is thus the difference between the full space integrals over the pure gauge configurations at $t = \pm\infty$. The pure gauge configurations at the caps are determined by the space dependent group elements g_∞ and $g_{-\infty}$ respectively and the full space integrals are ϑ times integers say n_∞ and $n_{-\infty}$. The gauge field configuration then interpolates between two vacua configured by g_∞ and $g_{-\infty}$ and the winding number is the difference $n = n_\infty - n_{-\infty}$. Both vacua can, by definition, be gauge transformed into the classical vacua with zero gauge potential, i.e. $g_{\pm\infty} = \mathbf{1}$ and $n_{\pm\infty} = 0$. However, the gauge transformation can not be continuously deformed into the identity for $n_\infty, n_{-\infty} \neq 1$ since the configurations belong to different homotopy classes. This kind of gauge transformations are called *large* gauge transformations.

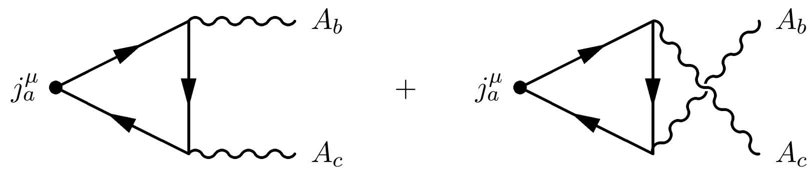
The homotopically different vacua are related by the above field configurations with non-zero winding number. They are not physically equivalent as we already see from the fact that the interpolating gauge configurations have different weights in the path integral due the $S_\vartheta = \vartheta n$ term. Also, since $F^{\mu\nu}$ can not vanish identically (this would give zero winding number) there is an energy barrier between the vacua. The corresponding quantum mechanical tunnelling amplitude is e^{-S_E} where S_E is the Euclidean action (focusing on the gauge part only) with $S_E = \frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})$.

We thus have an infinity of homotopically inequivalent vacua. This naturally assumes that the above interpolating gauge configurations exist and have finite action implying that the tunnelling amplitudes are not vanishing. Such solutions indeed exist. In fact, by rotating to Euclidean space we can for each winding number n find a unique gauge configuration in the homotopy class which minimises the action, i.e. satisfies the classical equation of motion $D_\mu F_{\mu\nu} = 0$ and has finite action. These are called instantons and the corresponding winding number is called the instanton number or charge. It is easy to show that the field strengths for the instantons are (anti-)selfdual $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$.[31] For instantons the tunnelling amplitude given by the FF term can then be calculated using (2.106) (without ϑ):

$$e^{-S_E} = \left(e^{-8\pi^2/g^2}\right)^{|n|} = \left(\frac{|\Lambda|}{\mu}\right)^{|n|b}, \tag{2.109}$$

[31] It is important that we are in Euclidean space because here $\tilde{\tilde{F}}_{\mu\nu} = F_{\mu\nu}$. This is contrary to Minkowski space where $\tilde{\tilde{F}}_{\mu\nu} = -F_{\mu\nu}$ and thus (anti-)selfduality only has trivial solutions.

Figure 2.1:



The diagrams contributing to the chiral anomaly in four space-time dimensions. Taken from [12].

where we have used the strong coupling scale from (2.100). Since the strong coupling scale is non-perturbative,[32] we see that instantons are non-perturbative effects given by powers of $|\Lambda|$ and thus become increasingly important in the strongly coupled regime.

Using the instantons one can calculate non-perturbative corrections to the effective Lagrangian, but we will not go into details of this vast field. However, let us just note that the instanton solutions only exist in non-abelian groups. Thus, as mentioned before, we do not have such contributions in abelian subgroups. However, the abelian ϑ-angle should not be neglected in the IR effective actions since it here couples to massive sources not described by the same IR physics. So the real part of the complex gauge couplings (2.21) in the Dijkgraaf-Vafa conjecture should not be neglected.

The $F_{\mu\nu}\tilde{F}^{\mu\nu}$-term also plays an important role when we look at anomalies in chiral symmetries. A chiral symmetry is defined as a symmetry in which the left and right handed part of the spinors transform differently. As in the rest of this thesis we here assume that the spinors are Majorana spinors. Now, if the left-handed Weyl spinor transforms in the $\mathbf{r}_{\text{chiral}}$ representation of some symmetry group G_{chiral} then due to the Majorana reality condition (C.18) the right-handed Weyl spinor transforms in the complex conjugate representation $\bar{\mathbf{r}}_{\text{chiral}}$. We thus conclude that for a chiral symmetry the spinor must transform in a complex representation. Corresponding to this symmetry we as usual have a classically conserved current, j_a^μ, for each generator $T_a^{(\text{chiral})}$ of G_{chiral}.

In four space-time dimensions this symmetry can only be anomalous, i.e. $\langle \partial_\mu j_a^\mu \rangle \neq 0$, if the fermions are coupled to gauge fields. We thus assume a gauged theory with gauge group G. The anomaly can be calculated perturbatively using diagrams and is actually a one-loop effect. The diagrams contributing to the calculation of $\langle \partial_\mu j_a^\mu \rangle$ in four dimensions are the triangle diagrams shown in figure 2.1. The result of the calculation in Minkowski space is (treating the gauge fields as external fields):[33]

$$\langle \partial_\mu j_a^\mu \rangle \propto \sum_{\mathbf{r}} \mathrm{Tr}_{\mathbf{r}}(T_a\{T_b, T_c\})\, F^b_{\mu\nu} \tilde{F}^{c\mu\nu}, \tag{2.110}$$

where the sum is over the different representations of the Weyl fermions. The right hand side should be seen as external fields. Setting T_a to be a generator of the chiral symmetry group, and T_b and T_c to be gauge generators we get the chiral anomaly. Let us suppose that the Weyl fermions transform in the representations $(\mathbf{r}_i, \mathbf{r}_{\text{chiral},i})$ of the symmetry group $G \times G_{\text{chiral}}$. The anomaly then depends on $\sum_i \mathrm{Tr}_{\mathbf{r}_{\text{chiral}}}\big(T_a^{(\text{chiral})}\big)\, \mathrm{Tr}_{\mathbf{r}_i}\big(T_b T_c\big)$. We conclude

[32] If we try to expand the exponential e^{-1/g^2} in a Taylor series around $g = 0$ each coefficient would be zero.

[33] In this expression all the generators could be generators of the gauge group. For consistency we must then demand that we have no gauge anomaly i.e. $\sum_{\mathbf{r}} \mathrm{Tr}_{\mathbf{r}}(T_a\{T_b, T_c\}) = 0$.

that the anomaly can only be in the abelian factors of the chiral symmetry group. We thus assume G_{chiral} to be a U(1) symmetry. The different fermion representations have charge q_i, i.e. $\psi_{(i)} \mapsto e^{i\alpha q_i}\psi_{(i)}$ where $\psi_{(i)}$ is the Weyl fermion in the i^{th} representation. The exact result of the calculation is then:[34]

$$\langle \partial_\mu j^\mu \rangle = -\frac{1}{16\pi^2}\sum_i q_i \operatorname{Tr}_{\mathbf{r}_i}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = -\frac{\sum_i q_i C(\mathbf{r}_i)}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}, \tag{2.111}$$

where we have used the quadratic invariant from (1.78). We see that the anomaly is proportional to $\operatorname{Tr}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)$ and thus, by the considerations above, the abelian chiral symmetry is anomalous in the non-perturbative regime with non-abelian gauge groups.[35]

One can also calculate the anomaly in the path integral formalism. The reason for the anomaly is simply that the measure has a non-trivial Jacobian under the chiral transformation. To see this we first promote the chiral symmetry to a local transformation with $\psi'_{(i)} = e^{i\alpha(x)q_i}\psi_{(i)}$. One finds that under this transformation the measure changes as:

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{-i\frac{\sum_i q_i C(\mathbf{r}_i)}{16\pi^2}\int \mathrm{d}^4x\,\alpha(x) F^a_{\mu\nu}\tilde{F}^{a\mu\nu}}. \tag{2.112}$$

Letting $S[\psi]$ denote the action we then get:

$$\begin{aligned}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ldots e^{iS[\psi]} &= \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\ldots e^{iS[\psi']} \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ldots e^{iS[\psi]+i\delta S+i\frac{\sum_i q_i C(\mathbf{r}_i)}{16\pi^2}\int \mathrm{d}^4x\,\alpha(x)F^a_{\mu\nu}\tilde{F}^{a\mu\nu}} \\ \approx \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ldots e^{iS[\psi]}&\left(1+i\delta S+i\frac{\sum_i q_i C(\mathbf{r}_i)}{16\pi^2}\int \mathrm{d}^4x\,\alpha(x)F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right),\end{aligned} \tag{2.113}$$

where the last line is to the first order in α. Using that under the localised chiral transformation the change in the action is

$$\delta S = \int \mathrm{d}^4x\,\alpha(x)\partial_\mu j^\mu, \tag{2.114}$$

we immediately get (2.111). Since the result agrees with the above one-loop calculation, we conclude that the anomaly is one-loop exact.

For an $\mathcal{N}=1$ supersymmetric theory with simple gauge group we see from (1.81) and (2.114) that the change in the Lagrangian under the anomalous chiral symmetry corresponds to changing the ϑ-angle as $\vartheta \mapsto \vartheta + 2\alpha\sum_i q_i C(\mathbf{r}_i)$. In this way the anomalous breaking of the chiral symmetry has been transformed into an explicit breaking by the ϑ-term. The effective Lagrangian then has the symmetry:

$$\begin{aligned}\psi_{(i)} &\mapsto e^{i\alpha q_i}\psi_{(i)}, \\ \vartheta &\mapsto \vartheta + 2\alpha\sum_i q_i C(\mathbf{r}_i).\end{aligned} \tag{2.115}$$

[34]We note that there here is a factor $\frac{1}{2}$ compared to most standard calculations since we look at Majorana fermions instead of Dirac fermions thus halving the degrees of freedom. Also, we have changed sign compared to standard texts (e.g. [25]) since the sign depends on $\operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5$ and our sign here is unconventional due to the definition of σ^0 in (C.10). This was also the reason for the unconventional sign on the $F\tilde{F}$-term (i.e. the ϑ-term) in (1.69) as we discussed there.

[35]Since we saw in (2.104) that $\operatorname{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = \partial_\mu K^\mu$ for some K^μ, we might think that we could define a non-anomalous current as $j^\mu - K^\mu$. However, this is not possible since K^μ is not gauge invariant under large gauge transformations as explained above.

Since the anomaly is related to the ϑ-angles, we conclude that there can only be one independent anomalous chiral symmetry for each simple factor in the gauge group.

We can use that the ϑ-angle is periodic to see that the anomalous chiral symmetry group is in fact not completely broken, but broken to a discrete group (as done in e.g. [34]). However, in our case we should be a bit careful since the Lagrangian is not invariant under rescalings of the gauge group generators as discussed at the end of section 1.3.4. We here appreciate that in (2.115) ϑ and $C(\mathbf{r}_i)$ transform in the same way under the rescalings. However, the instanton calculation that showed ϑ to be periodic was done for a specific normalisation chosen such that $C(\text{fund}) = 1/2$ for $\mathrm{SU}(N)$. Generalising this to any gauge group the ϑ-angle is 2π periodic if we choose a normalisation (following [10]) such that for T_a, T_b and T_c in the "standard" SU(2) subalgebra we have the structure constant:

$$f_{abc} = \varepsilon_{abc}. \tag{2.116}$$

Given this normalisation we see that the chiral symmetry is unbroken for the discrete set of α's obeying:

$$2\alpha \sum_i q_i C(\mathbf{r}_i) \in 2\pi\mathbb{Z}. \tag{2.117}$$

We can now use our knowledge of the $F^{\mu\nu}\tilde{F}_{\mu\nu}$-term to constrain the non-perturbative corrections to the superpotential.

2.5.4 Non-Perturbative Corrections

The perturbative non-renormalisation theorem proven in section 2.5.2 depended on two symmetries: Real translations of τ and an R-symmetry. With the knowledge from the last section we see that both of these symmetries can be broken non-perturbatively: τ can only be translated by integers $\tau \mapsto \tau + n$, $n \in \mathbb{Z}$ (using (2.108)) and the R-symmetry can be anomalous. However, we will now see that the Seiberg scheme used in 2.5.2 still constrains the form of the effective superpotential (based on [10], [12], [28] and [30]).

Let us assume the same setup as in section 2.5.2. This means that the Lagrangian at the UV energy μ_0 is given by (2.90) and we want to determine the effective generalised superpotential at the lower energy μ. The general form of the Lagrangian at energy μ is (using holomorphy) again given by (2.92).

According to the Seiberg scheme in section 2.5.2 we shall now use symmetries and limits to constrain the generalised superpotential. Firstly, the weak coupling limit of the couplings g_k should be smooth (we do not at this point integrate out fields so we should also have smoothness in the limit of masses going to zero). This means that we can rule out negative powers of g_k. But also terms like e^{-1/g_k^2} are ruled out [15, 16] since g_k is complex and the exponential diverges when e.g. g_k goes to zero from the imaginary direction in the complex plane. This means that we can simply expand the generalised superpotential in non-negative powers of the couplings g_k.

The dependence on τ is more complicated. Here we can have a non-perturbative dependence as in $e^{2\pi i\tau}$ since the imaginary part of τ given by (1.80) is $4\pi/g^2$ and thus positive. This means that in the limit of the gauge coupling $g \to 0$ we have $e^{2\pi i\tau} \to 0$. We can exchange $e^{2\pi i\tau}$ with Λ^b using the holomorphic scale from (2.102) since the quotient only depends on the scale. Λ^b also has the nice property that it is periodic in ϑ and as we saw in (2.109) integer powers of Λ^b can be obtained by instanton corrections. However, we can have contributions that depend on (not necessarily integer) powers of Λ. Below we will actually see an example of an expectation value with the wrong ϑ-periodicity. The contributions

Table 2.2:

	$\mathrm{U}(1)_i$	$\mathrm{U}(1)_R$
Φ^j	δ_{ij}	0
g_k	$-N_{k,\Phi^i}$	2
$\mathcal{W}_\alpha$	0	1
Λ^b	$2C(\mathbf{r}_i)$	$2C(\mathrm{adj}) - 2\sum_i C(\mathbf{r}_i)$

Charges for the symmetries constraining the effective generalised superpotential. N_{k,Φ^i} is the order of Φ^i in the term with coupling g_k in the bare superpotential. Based on [12].

of Λ^a must have $a > 0$ since $\Lambda \to 0$ corresponds to the smooth weak coupling limit $g \to 0$ assuming an asymptotically free theory with $b > 0$. We note that Λ depends on τ holomorphically. Thus Λ should be regarded as a background chiral superfield and the dependence on it should be holomorphic. Naturally, we can also have a perturbative dependence on τ through $\ln(\Lambda)$. We know how this looks in perturbation theory from equation (2.97) so let us extract it from the effective generalised superpotential (using (2.101)):

$$F_\mu\big(\Phi^i, \mathcal{W}_\alpha, \tau, g_k\big) = -\frac{b}{32\pi^2 C(\mathbf{r})} \ln\left(\frac{\Lambda}{\mu}\right) \mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha \mathcal{W}_\alpha) + F'_\mu\big(\Phi^i, \mathcal{W}_\alpha, \Lambda, g_k\big)\,. \tag{2.118}$$

Please note that we have not yet proven that the perturbative part should look like this, but simply extracted it in expectation of such a term and indeed one can show that it will look like this. We also note that F'_μ only depends on τ through powers of Λ. The reason is that there will be chiral anomalies. As we saw in (2.115) the anomaly is cancelled by the first term if we translate τ under the anomalous chiral symmetry and hence F'_μ should be invariant under the symmetry. This can only happen if the dependence of τ is through powers of Λ that under (2.115) has a definite charge.

Let us now turn to the symmetries that can restrict F'_μ. These are naturally the same as we used in section 2.5.2: We have a global symmetry, $\mathrm{U}(1)_i$, rotating Φ^i, but not the rest of the chiral fields: $\Phi^j \mapsto e^{i\alpha\delta_{ij}}\Phi^j$. Here Φ^i transforms in the representation $\mathbf{r}_i$ of the gauge group. To ensure the $\mathrm{U}(1)_i$ symmetry with non-zero couplings we must as before assign charges, $-N_{k,\Phi^i}$, to the couplings g_k where N_{k,Φ^i} is the order of Φ^i in the term with coupling g_k in the bare superpotential (2.91). Since this is a chiral symmetry, it can be anomalous and thus we should also assign a charge to Λ. The other symmetry that we used in section 2.5.2 is the R-symmetry, $\mathrm{U}(1)_R$. Here we found that $R\big(\Phi^i\big) = 0$ and $R(\mathcal{W}_\alpha) = 1$. Using (1.34), (1.57) and that $R(\theta) = 1$ this means that the spinor from the chiral multiplet has charge -1 and the gaugino has charge $+1$. Thus it is a chiral symmetry and it can be anomalous. Using (2.115) we can find the corresponding charge of Λ. The charges for the symmetries are given in table 2.2.

The effective superpotential now heavily depends on the sign of the charge of Λ^b under the anomalous R-symmetry. For a positive charge there is only one possible power of Λ whereas for a negative charge, Λ can be used to compensate the positive charges of g_k and $\mathcal{W}_\alpha$ thus allowing arbitrary powers of these.

Let us focus on the case where the charge is zero and the R-symmetry is not anomalous. This is exactly the case for the Lagrangian $\mathcal{L}_{\mathrm{U}(N)}$ presented in the Dijkgraaf-Vafa conjecture in section 2.1 where we ignore the abelian part of the gauge group. Here we only have one chiral field in the adjoint representation exactly compensating the contribution to the anomaly from the gaugino. Using that the effective generalised superpotential should have charge two under the R-symmetry we get two kinds of terms in F'_μ: One term linear in the g_k's and one term proportional to two $\mathcal{W}_\alpha$'s. This means that the effective generalised superpotential takes the form:

$$F_\mu\left(\Phi^i, \mathcal{W}_\alpha, \tau, g_k\right) = -\frac{b}{32\pi^2 C(\mathbf{r})} \ln\left(\frac{\Lambda}{\mu}\right) \mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha \mathcal{W}_\alpha) + \sum_k g_k O_{\mu,k}\left(\Phi^i, \Lambda\right) + h_{\mu,ab}\left(\Phi^i, \Lambda\right) \mathcal{W}^{\alpha a} \mathcal{W}^b_\alpha. \quad (2.119)$$

We can constrain the dependence on Φ^i and Λ using the $\mathrm{U}(1)_i$ symmetry. Let us for simplicity assume that we only have one chiral field Φ which is in the adjoint representation. Let N_Φ and N_Λ respectively denote the powers of Φ and Λ in a given term. Then we see from table 2.2 that for $h_{\mu,ab}$ we have $N_\Phi = -\frac{2C(\mathrm{adj})}{b} N_\Lambda$. Thus the term independent of Φ is also independent of Λ and will in fact simply give the one-loop running of τ that we have already taken out. The rest of the terms in $h_{\mu,ab}$ have negative powers of Φ thus giving new non-perturbative contributions. For $O_{\mu,k}$ we have

$$N_\Phi = N_{k,\Phi} - \frac{2C(\mathrm{adj})}{b} N_\Lambda. \quad (2.120)$$

Setting $N_\Lambda = 0$ we have $N_\Phi = N_{k,\Phi}$ and considering the weak coupling limit we see that this gives us the bare superpotential. For $N_\Lambda > 0$ we get corrections to terms with $N_\Phi < N_{k,\Phi}$. Thus a term of order N_Φ in Φ only receives new contributions arising from terms in the bare superpotential of higher order in Φ.

To be able to perform further analysis of the effective Lagrangian we will in the next subsection introduce the concept of integrating in matter.

Let us end this section with an example of the breaking of the $\mathrm{U}(1)_R$ symmetry to a finite group. We consider the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. In this case we only have one adjoint fermion, the gaugino, with charge $+1$. Thus (2.117) shows us that the R-symmetry is unbroken if $2\alpha C(\mathrm{adj}) \in 2\pi\mathbb{Z}$. It is here customary to introduce the dual Coxeter number, h. The precise definition of the dual Coxeter number is [35]:

$$\mathrm{Tr}_{\mathrm{adj}}\left(T_a^{(\mathrm{adj})} T_b^{(\mathrm{adj})}\right) = h\psi^2 \delta_{ab}, \quad (2.121)$$

where ψ^2 is the length of the highest root. This definition makes the dual Coxeter number invariant under rescaling of the generators. The dual Coxeter numbers for the classical and exceptional groups can be found in table 2.3 along with other group theoretical facts. We should now remember that (2.117) holds true for a normalisation such that (2.116) is fulfilled for the standard SU(2) subalgebra. With this normalisation we have $\psi^2 = 1$ such that $C(\mathrm{adj}) = h$. Thus by (2.117) the R-symmetry is broken to $\mathbb{Z}_{2h}$ i.e. the gaugino transforms as:[36]

$$\lambda_\beta \mapsto e^{i\alpha} \lambda_\beta, \qquad \alpha = \frac{2\pi}{2h} n, \qquad n \in \mathbb{Z}_{2h}. \quad (2.122)$$

[36]This result can also be obtained by an instanton calculation (following [14]). In the case of a $\mathrm{SU}(N)$ gauge group we have $2N = 2h$ zero-modes from the gaugino. The first non-vanishing correlator must have $2h$ gaugino insertions to soak the zero modes. Requiring this correlator to be invariant we see that $\mathrm{U}(1)_R$ breaks to $\mathbb{Z}_{2h}$.

Table 2.3:

	SU(N)	SO(N), $N \geq 4$	Sp(k)	E_6	E_7	E_8	F_4	G_2
dim(G)	$N^2 - 1$	$N(N-1)/2$	$2k^2 + k$	78	133	248	52	14
rank(G)	$N - 1$	$[N/2]$	k	6	7	8	4	2
C(fund)	1/2	1	1/2					
$h(G)$	N	$N - 2$	$k + 1$	12	18	30	9	4

Dimensions, dim(G), ranks, rank(G), and dual Coxeter numbers, $h(G)$, for the classical and exceptional groups. For the classical groups the quadratic invariant in the fundamental representation, C(fund), is also given – in a normalisation such that the highest root has length 1. Based on [12] and [35].

2.5.5 ILS Linearity Principle and Integrating In

In the last section we did not consider integrating out massive fields which, as we explained in section 2.4.3, is possible at energy scales below the masses of the fields. In this section we will do this and even see that we can integrate the fields back in.

The setup will be the same as in the last subsection and section 2.5.2 with an $\mathcal{N} = 1$ supersymmetric Lagrangian with simple gauge group. However, we will assume that the tree level (i.e. the bare) superpotential W_{tree} that was given by (2.91) now is linear in the basis for holomorphic gauge invariants $X_k(\Phi^i)$ introduced in section 2.2.3:[37]

$$W_{\text{tree}} = \sum_k g_k X_k. \tag{2.123}$$

Let us for now ignore the dependence on $\mathcal{W}_\alpha$ in the effective superpotential – we will return to this later. In [36] K. Intriligator, R. G. Leigh and N. Seiberg (ILS for short) conjectured that the effective (generalised) superpotential can be put into the form:[38]

$$W_{\text{eff}}(X_k, \Lambda, g_k) = W_{\text{dyn}}(X_k, \Lambda) + W_{\text{tree}}(X_k, g_k)\,. \tag{2.124}$$

This is called the ILS linearity principle since it states that the dependence on the couplings g_k is linear. However, we note that it further asserts that the term depending on the couplings is the tree-level superpotential. This can often be obtained using the Seiberg scheme from section 2.5.2, but actually it is not the form we found in (2.119) for the case of a single adjoint chiral superfield. Equation (2.119) is nicely linear in the couplings, but the term depending on the couplings is not the tree-level superpotential. However, the conjecture in [36] is that one can always redefine the fields X_k as a function of the couplings g_k to bring the effective superpotential into the form (2.124).

Since this is not proven directly let us see that it is true in a special case using (2.119). We assume that the gauge group is SU(N) and that the tree-level superpotential is given by $W_{\text{tree}} = \frac{1}{2}mX_2 + gX_3$ where, as we found in (2.55), the basic gauge invariants here are $X_k = \text{Tr}(\Phi^k)$, $k \leq N$. Using (2.119), (2.120) and assuming only non-negative integer powers

[37] The X_k can be constrained. This can be fixed by including Lagrange multipliers in the effective superpotential.

[38] In writing this we use that gauge invariance requires the Φ^i dependence to be expressed through the basic gauge invariants X_k.

of Φ we get (dropping constant terms and using $b = 2C(\text{adj}) = 2N$ in this case):

$$W_{\text{eff}}(X_k, \Lambda, g_k) = W_{\text{tree}} + cg\Lambda X_2, \tag{2.125}$$

where c is a constant and we have used that the only X_k of order less than three in Φ is X_2 ($X_1 = \operatorname{Tr}\Phi = 0$ in the non-abelian case). By redefining $X_3' = X_3$ and $X_2' = gX_2$ we get $W_{\text{tree}} = \frac{1}{2}\frac{m}{g}X_2' + gX_3'$ and $W_{\text{dyn}} = c\Lambda X_2'$. Here W_{dyn} is independent of the couplings $m' = m/g$ and $g' = g$ as wanted.

Let us now continue with the general case following [36] and [37]. We assume that we are at so low energies that we can integrate out some massive field say X_0 using (2.89) (or rather when working with the elementary fields Φ^i we integrate out all X_k that involve Φ^i). Ignoring the Kähler-term at these low energies (2.89) shows that we integrate X_0 out by solving the equation of motion

$$\frac{\partial W_{\text{eff}}}{\partial X_0}(\langle X_0\rangle) = 0 \tag{2.126}$$

for $\langle X_0\rangle(X_1, \ldots, X_n, \Lambda, g_k)$ (assuming $k = 0, \ldots, n$) and inserting this back into W_{eff} thus obtaining the effective action, $W_{\text{eff,L}}$, with X_0 integrated out:

$$\begin{aligned} W_{\text{eff,L}}(X_1, X_2, \ldots, X_n, g_k, \Lambda) &= W_{\text{eff}}(\langle X_0\rangle, X_1, \ldots, X_n, g_k, \Lambda) \\ &= W_{\text{dyn}}(\langle X_0\rangle, X_1, \ldots, X_n, \Lambda) + \sum_{k\neq 0} g_k X_k + g_0\langle X_0\rangle. \end{aligned} \tag{2.127}$$

The reason for the index "L" is that this is nothing but the Legendre transform of $W_{\text{dyn}}(X_k, \Lambda)$ in (2.124) as we see by rewriting (2.126) as $\frac{\partial W_{\text{dyn}}}{\partial X_0} = -g_0$. Using that the implicit dependence on g_0 through $\langle X_0\rangle$ is zero by the virtue of (2.126) we get that:[39]

$$\frac{\partial W_{\text{eff,L}}}{\partial g_0} = \langle X_0\rangle. \tag{2.128}$$

This is nothing but the inverse Legendre transform. If we know the g_0 dependence in the effective Lagrangian with X_0 integrated out, we can use (2.128) to solve for g_0 as a function of $\langle X_0\rangle$ and the other fields and couplings, and then use (2.127) to obtain W_{dyn}. W_{eff} is then obtained by adding W_{tree}. This is called *integrating in*. This means that we loose no information in integrating out a field and we can see X_k and g_k as being dual.

Whereas the dependence on g_0 in $W_{\text{eff,L}}$ is complicated, the linearity principle still applies for the rest of the fields since $\partial W_{\text{eff,L}}/\partial g_k = X_k$ for $k \neq 0$ by (2.126). This means that we can continue the integrating out (and in) for the rest of the fields. Without loss of information we can obtain an effective Lagrangian only depending on the couplings g_k and Λ.

The integrating in procedure through equation (2.128) should not be unfamiliar since this is just the equation we have for the 1PI effective superpotential. The couplings are here the external currents. Actually, we could have carried out our treatment in the 1PI formalism (if well-defined) and if the 1PI action and the Wilsonian action agrees, this would prove the linearity principle [30].

The integrating in method is powerful (following [37]) since it sometimes allows us to determine the effective superpotential for an "upstairs" theory with an extra massive field $\hat{\Phi}$ from a known effective superpotential of a "downstairs" theory simply by integrating in the $\hat{\Phi}$ field. Two points should, however, be borne in mind. Firstly, the holomorphic scales

[39] We note that if the tree-level superpotential was given as in (2.91) with the O_k's being – not necessarily linear – functions of the basic gauge invariants, then the right hand side in (2.128) would be $O_k(\langle X_0\rangle)$.

depend on the matter representation as noted in section 2.5.2. Let Λ_d denote the scale for the downstairs theory and let b_d be the corresponding constant in the β-function determined by (2.98). Correspondingly for the upstairs theory we have Λ_u and b_u. If the gauge group is not simple, we have one such pair for each simple factor. Assuming simple thresholds (or absorbing factors into the Λ's) we can simply compare the running gauge couplings $g(\mu)$ (for each factor) at the mass, m, of the field to be integrated in (using (2.99)):

$$\Lambda_d^{b_d} = \Lambda_u^{b_u} m^{C(\mathbf{r})}, \tag{2.129}$$

where $C(\mathbf{r})$ is the quadratic invariant for the representation of $\hat{\Phi}$. This follows from (2.98) yielding $b_u = b_d - C(\mathbf{r})$.

The second point to consider is that when flowing to the upstairs theory, adding a tree-level superpotential with couplings g_k, and then flowing down again by integrating out $\hat{\Phi}$ as in (2.127) does not give the downstairs effective superpotential $W_{\text{eff},d}$ back, but rather:

$$W_{\text{eff,L}} = W_{\text{eff},d}(X_k, \Lambda_d) + W_I(X_k, \Lambda_u, g_k)\,, \tag{2.130}$$

where W_I is the renormalisation group irrelevant term that the downstairs theory does not know about. W_I must vanish for $g_k = 0$ or $m \to \infty$. Only if such a W_I superpotential can be ruled out, we can use the integrating in procedure from the downstairs theory to the upstairs.

Let us now consider the dependence on $\mathcal{W}_\alpha$. We will assume that the dependence hereon is always[40] through the (traceless since we study a simple group) glueball superfield, $\hat{S}$, defined as in (2.5), but now, naturally, for any simple group. (2.5) was written for $C(\text{fund}) = 1/2$. Generally we define $\hat{S}$ as:

$$\hat{S} = -\frac{1}{32\pi^2 C(\mathbf{r})}\,\mathrm{Tr}_{(\mathbf{r})}(\mathcal{W}^\alpha \mathcal{W}_\alpha)\,. \tag{2.131}$$

This is a massive field so at low energies, which we will assume to be at, it is integrated out. Looking at the tree-level Lagrangian with the UV cut-off μ_0 the gauge kinetic term (1.79) takes the form (using (2.101) as we did in (2.118)):

$$W_G = 2\pi i \tau \hat{S} = \ln\left(\frac{\Lambda^b}{\mu_0^b}\right)\hat{S}. \tag{2.132}$$

Thus we can see $\ln\left(\Lambda^b/\mu_0^b\right)$ as the coupling for $\hat{S}$. Assuming that the principle of linearity[41] also holds for $\hat{S}$, we can integrate $\hat{S}$ back in analogously to (2.128) by solving:

$$\frac{\partial W_{\text{eff}}}{\partial \ln\left(\frac{\Lambda^b}{\mu_0^b}\right)} = \langle \hat{S} \rangle. \tag{2.133}$$

This gives Λ as a function of $\langle \hat{S}\rangle$ and X_r (or g_k if the fields have been integrated out) which we can substitute back in to obtain W_{dyn} as a function of $\hat{S}$ (in analogy with (2.127)):

$$W_{\text{dyn}}\big(X_k, \hat{S}\big) = W_{\text{eff}}\Big(X_k, \Lambda\big(X_k, \hat{S}\big)\Big) - \ln\left(\frac{\Lambda\big(X_k, \hat{S}\big)^b}{\mu_0^b}\right)\hat{S}. \tag{2.134}$$

[40] E.g. in (2.119) the dependence on $\mathcal{W}_\alpha$ is not necessarily through $\hat{S}$ so this is an assumption.

[41] As noted in [37] the assumption of simple thresholds above is actually a generalisation of the linearity principle for $\hat{S}$.

To obtain the effective superpotential we simply add the tree-level potential (2.132). Naturally, the μ_0 dependence then goes out so the low-energy effective superpotential is independent of this scale. Now we can see $\ln\left(\Lambda^b/\mu_0^b\right)$ and $\hat{S}$ as a canonical pair just as g_k and X_k was above. We can also extend this analysis to the case where we have a semi-simple gauge group with more than one holomorphic scale and corresponding glueball superfields. We should mention that we were able to perform the integrating in procedure since there can be no W_I-term as in (2.130) because the total Λ dependence is accounted for in W_{eff} [5].

We note that the meaning of the superpotential with $\hat{S}$ integrated in is unclear since it is a massive field that really should be integrated out at the low energies. However, it is this potential that we determine in the Dijkgraaf-Vafa conjecture – with all matter fields integrated out and the glueball superfields integrated in. This superpotential has the virtue that the expectation values of the glueball superfields are found by the equations of motion analogous to (2.126):

$$\frac{\partial W_{\text{eff}}}{\partial \hat{S}}\left(\langle \hat{S} \rangle\right) = 0. \tag{2.135}$$

Thus we have explained equation (2.8).

We should not think of the Dijkgraaf-Vafa conjecture as depending on the ILS linearity principle since we in the next chapter will prove the conjecture (i.e. obtain $W_{\text{eff,pert}}$) diagrammatically without using the linearity principle. The linearity principle can actually quite easily be proven [38] using the Dijkgraaf-Vafa conjecture and the techniques we develop in the diagrammatical proof. But one can also prove the Dijkgraaf-Vafa conjecture using the linearity principle along with Seiberg-Witten curves [5]. To complete the picture one can quickly obtain the Konishi anomaly using the linearity principle and the anomalous $\mathrm{U}(1)_i$ symmetry from table 2.2 – and the other main proof of the Dijkgraaf-Vafa conjecture [7] actually uses a generalised form of this Konishi anomaly.

We will use the integrating in procedure to obtain the Veneziano-Yankielowicz superpotential in section 2.5.7, but let us first review some of the lore of gauge dynamics.

2.5.6 The Lore of Gauge Dynamics

One of the aims in obtaining the low-energy effective action is to be able to determine what *phase* the theory is in. The phase of a theory depends on the parameters of the theory and the choice of vacuum state. It is characterised by the energy potential, $V_{\text{elec}}(R)$, between two electrical test charges separated by a large distance, R. By electric we here mean in the abstract gauge group sense. The presentation is based on [12], [23] and [30] and we are in four space-time dimensions.

It is here important for us that G. 't Hooft and A. Polyakov showed the possibility of magnetic monopoles for non-abelian gauge groups.[42] These 't Hooft-Polyakov monopoles were obtained as solitonic solutions in the Georgi-Glashow model, however, it was first with the Seiberg-Witten theory for $\mathcal{N} = 2$ supersymmetric theories that exact calculations could be done of e.g. monopole condensation.

Table 2.4 shows the different conjectured phases characterised by the behaviour of the electric potential, $V_{\text{elec}}(R)$. The behaviour is only determined up to an additive constant. In the first three phases (Coulomb, free electric and free magnetic) there are massless gauge fields and the potentials are of the form $e^2(R)/R$ where $e(R)$ is the renormalised charge. In the Coulomb phase the charge is constant. In the free electric phase the massless charged

[42] What is meant here is that the UV Lagrangian has a non-abelian gauge symmetry. Naturally, this gauge group could be spontaneously broken in the IR.

Table 2.4:

Phase:	$V_{\text{elec}}(R) \sim$	$V_{\text{magn}}(R) \sim$	E-M duality
Coulomb	$\frac{1}{R}$	$\frac{1}{R}$	
Free electric	$\frac{1}{R\ln(R\Lambda)}$	$\frac{\ln(R\Lambda)}{R}$	
Free magnetic	$\frac{\ln(R\Lambda)}{R}$	$\frac{1}{R\ln(R\Lambda)}$	
Higgs	constant	ρR	
Confining	σR	constant	

The table shows the characterising behaviour of the electric and magnetic potential for the different phases of gauge theories. Also shown is the electric-magnetic (E-M) duality.

particles renormalise the charge to zero as $R \to \infty$ while in the free magnetic phase massless monopoles renormalise the charge to infinity at large distances. In table 2.4 the behaviour of the magnetic potential, $V_{\text{magn}}(R)$, between two magnetic test charges is also shown. For the first three phases we directly get the magnetic potential from the electric by the Dirac quantisation condition $e(R)g(R) \sim 1$ where $g(R)$ is the renormalised magnetic charge.

The Higgs phase is characterised by condensation of an electrically charged particle. This gives a mass gap to the theory via the Higgs-mechanism. The potential is then of the Yukawa type and thus exponentially decays to zero at large R. This can be seen by the charges being screened by the condensate or equivalently by the gauge bosons acquiring mass. The flux between two magnetic sources, on the other hand, is confined into a thin flux-tube with constant tension ρ thus giving the linear potential. This is in analogy with the Meissner effect from superconductivity.

The confining phase is the phase of our interest. This phase is solely for non-abelian groups whereas we can find the above for both abelian as well as non-abelian groups. Empirically we know this phase from QCD: In the UV the degrees of freedom are the gluons and the coloured quarks while in the IR we have the colourless hadrons - i.e. colour confinement. The qualitative explanation of confinement can, as suggested by Mandelstam and 't Hooft, be seen as a dual Meissner effect where the confining phase is dual to the Higgs phase. The duality here is the electric-magnetic duality that exchanges the electric and magnetic charges. As is also indicated in table 2.4 this duality exchanges the free electric and free magnetic phase. The Coulomb phase is self-dual which one finds easily in the abelian case, but in the non-abelian case one again has to go to supersymmetric theories where it is part of the Montonen-Olive duality. Getting back to the confining phase this can then be seen as the electric-magnetic dual of the Higgs phase. Here it is now monopoles that form a condensate and the electric flux between the two electric test charges is confined to a thin tube with constant string tension σ. The corresponding linear potential $V_{\text{elec}}(R) \sim \sigma R$ shows that it requires an infinite amount of energy to separate two charged particles. This explains why we only see the gauge-invariant fields (hadrons in QCD) at low energy. As in the Higgs phase we also have a mass gap in the confining phase.

The confining phase has another characteristic: If we only have adjoint matter, the Wilson loop operator for large loops satisfies the area law $\langle \mathrm{Tr}\,\mathcal{P}e^{i\oint \mathrm{d}x^\mu A_\mu}\rangle \sim e^{-\sigma\cdot\mathrm{Area}}$ where "Area" is the area of the Wilson loop.[43] As opposed to this the Wilson loop operator in the Higgs phase will rather depend on the perimeter of the loop. However, when we have particles in the fundamental representation we can not distinguish the confining and the Higgs phase because virtual pairs can screen the sources. As a last remark we can also have dyons (particles with both electric and magnetic charge) if we have a non-zero ϑ-angle. If the dyons condensate, we get a phase called oblique confinement.

Let us now turn to supersymmetric gauge theories. Let us concentrate on supersymmetric Yang-Mills theory with a simple gauge group i.e. pure superglue. We want to know what happens at low energy i.e. below the strong coupling scale, $|\Lambda|$. Based on e.g. lattice simulations it is believed that we have confinement and a mass gap.

But the theory also shows another phenomenon characteristic for the strong coupling regime of gauge theories, namely spontaneous breaking of chiral symmetry. We look at the $\mathrm{U}(1)_R$ symmetry from table 2.2. This is simply measuring the gaugino number since we have no chiral matter. We have already seen in (2.122) that the symmetry is anomalous and broken to the discrete group $\mathbb{Z}_{2h}$ where h is the dual Coxeter number from table 2.3. As explained in footnote 36 the breaking to $\mathbb{Z}_{2h}$ could also be seen using that the first possibly non-zero correlator in an instanton background is $\langle(\lambda\lambda)^h\rangle \propto \Lambda^{3h}$ where λ^a_α is the gaugino field. The dependence on Λ has been found using dimensional analysis and by requiring a holomorphic dependence on τ as in section 2.5.1. However, at strong coupling the R-symmetry is spontaneously broken further down to $\mathbb{Z}_2$ by the gaugino bilinear getting a non-zero dynamical expectation value i.e. gaugino condensation:

$$\langle\lambda\lambda\rangle \neq 0 \quad \longleftrightarrow \quad \mathbb{Z}_{2h} \mapsto \mathbb{Z}_2. \tag{2.136}$$

Here the non-trivial element in $\mathbb{Z}_2$ simply works as a sign change on λ^a_α since referring to (2.122) the only symmetry in $\mathbb{Z}_{2h}$ leaving $\lambda\lambda$ invariant is the one with $n = h$ which simply gives a sign change. We note that the gaugino condensate can just as well be described by the traceless glueball superfield getting a non-zero expectation value since its lowest component is the gaugino bilinear as mentioned in section 2.1.1. The glueball superfield is further believed to be the relevant field for the low energy theory.

Associated with the breaking of $\mathbb{Z}_{2h} \mapsto \mathbb{Z}_2$ we get h inequivalent vacua. This is because states that are related by the generators of $\mathbb{Z}_{2h}/\mathbb{Z}_2$ are no longer treated as equivalent. The generators of $\mathbb{Z}_{2h}/\mathbb{Z}_2$ are $e^{ik\pi R/h}$ where $k = 0, 1, \ldots, h-1$ and R is the gaugino number. Working with these generators on a given vacuum state we get the h inequivalent states $|k\rangle$ with k as above. This also solves another problem: Using dimensional analysis and holomorphy as before, we get $\langle\lambda\lambda\rangle = \text{constant} \times \Lambda^3$. In the proper normalisation we know from (2.107) that ϑ should be 2π-periodic. But using (2.98) we see that $b = 3C(\mathrm{adj}) = 3h$ and hence by (2.102) we have $\Lambda^3 \sim e^{i\vartheta/h}$ which is only $2\pi h$-periodic. However, using the definition of the h vacua $|k\rangle$ the expectation value of the gaugino condensate depends on the vacua and is given by:

$$\langle\lambda\lambda\rangle_k = a\Lambda^3 e^{2\pi i k/h}, \tag{2.137}$$

where a is a constant independent of the chosen vacuum and the index k refers to in which vacuum the expectation value is taken. However, using the anomaly (2.115) we see that

[43]This is seen by choosing a rectangular loop with length T in the time direction and length R in a space direction. The interpretation of the Wilson loop is then that it measures the Euclidean action of a process where two heavy charged particles are created and then separated by a distance R for a time T before they are annihilated. We then get $\langle \mathrm{Tr}\,\mathcal{P}e^{i\oint \mathrm{d}x^\mu A_\mu}\rangle = e^{-TV(R)}$ and inserting $V(R) \sim \sigma R$ gives the result.

effect of $e^{ik\pi R/h} \in \mathbb{Z}_{2h}/\mathbb{Z}_2$, which gives a change in vacuum as $|n\rangle \mapsto |n+k\rangle$ (modulo h), is equivalent to translating $\vartheta \mapsto \vartheta + k2\pi$ which just gives the exponential in (2.137). In this way all observables of the theory are 2π periodic even though the gaugino condensate in a given vacuum is not. Each of these h vacua has a mass gap. We note that this is possible since the breaking of the chiral symmetry now allows a mass-term for the gaugino.

For any supersymmetric theory we have the gaugino number which now is not simply $\mathrm{U}(1)_R$ from table 2.2, but rather given by summing the generators of $\mathrm{U}(1)_R$ and the $\mathrm{U}(1)_i$'s. By gaugino condensation we will again have h vacua.

Let us end this subsection by noting that the existence of the h vacua are strongly suggested by the Witten index being $\mathrm{Tr}(-1)^{N_F} = h$. Here the trace is over the zero energy states and N_F is the fermion number operator introduced in section 1.2.1. This corresponds to each of the h vacua contributing with $(-1)^{N_F} = 1$ to the Witten index. That the Witten index for supersymmetric Yang-Mills theory is equal to the dual Coxeter number is true for both the classical and the exceptional groups. We note that this also means that supersymmetry is unbroken in the strongly coupled regime.

2.5.7 The Veneziano-Yankielowicz Superpotential

In this section we will consider the low energy effective action for the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the case of a simple gauge group. More precisely, we want to determine the glueball superpotential with $\hat{S}$ integrated in as discussed in the section 2.5.5. The superpotential was first found, prior to the integrating in method, by G. Veneziano and S. Yankielowicz in [1] as the unique form that fulfils the proper anomaly matching conditions. We will here derive the superpotential using the integrating procedure (as done for $\mathrm{SU}(N)$ in [7] and [12]). To do this we have to know the low energy effective superpotential, $W_{\text{eff}}(\Lambda)$, as a function of the holomorphic scale Λ. Assuming that we have chiral symmetry breaking by gaugino condensation as described in the last subsection, the expectation value of $\hat{S}$ must be non-zero as determined from (2.137):

$$\langle \hat{S} \rangle = a\Lambda^3, \tag{2.138}$$

where a is a constant (different from the one in (2.137)) and we look at a particular vacuum among the inequivalent vacua. Using (2.135) we can see this as a differential equation for the glueball superpotential which can then be determined. But let us take an extra step and first use (2.133) to obtain:

$$W_{\text{eff}}(\Lambda) = C(\text{adj})\, a\Lambda^3. \tag{2.139}$$

Where we, by (2.98), have used that $b = 3C(\text{adj})$. We can then use (2.134) to obtain $W_{\text{dyn}}\big(\hat{S}\big)$ noting that by (2.138) $\Lambda\big(\hat{S}\big) = \big(\hat{S}/a\big)^{1/3}$. Adding the tree-level superpotential (2.132) finally gives the effective superpotential – the Veneziano-Yankielowicz superpotential:

$$W_{\text{VY}}\big(\hat{S}\big) = C(\text{adj})\, \hat{S}\bigg(1 - \ln \frac{\hat{S}}{a\Lambda^3}\bigg). \tag{2.140}$$

The $\hat{S} \ln \hat{S}$ term in this expression is multi-valued so rotating $\hat{S} \mapsto e^{2\pi i}\hat{S}$ gives $W_{\text{VY}} \mapsto W_{\text{VY}} - 2\pi i C(\text{adj})\, \hat{S}$. But from (2.132) we see that this is exactly cancelled by rotating $\vartheta \mapsto \vartheta + 2\pi C(\text{adj})$ in perfect agreement with $\hat{S} \mapsto e^{2\pi i}\hat{S}$ corresponding to a chiral transformation of the gaugino as $\lambda \mapsto e^{\pi i}\lambda$ and the anomaly equation (2.115).

Determination of the correct constant a is non-trivial, but can be found for instance using instanton calculus,[44] monopoles in the theory compactified to three dimensions or Seiberg-Witten curves (a review can be found in [39] and the constant a for general classical and exceptional groups is given in [40]). Naturally, the factor a can be absorbed in Λ. However, using the standard normalisation $C(\text{fund}) = 1/2$ we actually find that $a = 1$ for a $\mathrm{SU}(N)$ gauge group and thus the Veneziano-Yankielowicz superpotential here is ($C(\text{adj}) = N$):

$$W_{\mathrm{VY},\,\mathrm{SU}(N)} = N\hat{S}\left(1 - \ln\frac{\hat{S}}{\Lambda^3}\right). \tag{2.141}$$

Extending this solution to semi-simple groups suggests the claimed form of the Veneziano-Yankielowicz superpotential in (2.9). However, as we discuss in the next subsection we can actually not use the integrating in technique in the case considered in section 2.1.

In the derivation of the Veneziano-Yankielowicz superpotential we have actually ignored the fact that for small N in the classical groups we have no non-abelian dynamics. Choosing the normalisation such that $C(\text{adj})$ is the dual Coxeter number given in table 2.3 and absorbing the normalisation a into the scale Λ we can then write (following [8]; "sgt" stands for standard gauge theory):

$$W_{\mathrm{VY}} = h_{\mathrm{sgt}}\hat{S}\left(1 - \ln\frac{\hat{S}}{\Lambda^3}\right), \qquad h_{\mathrm{sgt}} = \begin{cases} N - \delta_{N,1} & \text{for } \mathrm{SU}(N) \\ N + 1 - \delta_{N,0} & \text{for } \mathrm{Sp}(N) \\ N - 2 + \delta_{N,1} + 2\delta_{N,0} & \text{for } \mathrm{SO}(N) \end{cases}, \tag{2.142}$$

Let us emphasise that our derivation relied on the assumption of gaugino condensation. However, in [41] the Veneziano-Yankielowicz superpotential has been derived in the Dijkgraaf-Vafa context – but all things considered, the proof relies on the supersymmetric Ward identities in the same way as in the original proof in [1].[45] Finally, we will later see that the Veneziano-Yankielowicz superpotential can be seen as a contribution from the free energy of the matrix model in the Dijkgraaf-Vafa setup.

2.5.8 The Glueball Superpotential – Our Case

We will end this section by summing up what we have learnt about the Wilsonian low energy effective superpotential for the $\mathcal{N} = 1$ supersymmetric theory with gauge group $\mathrm{U}(N)$ and adjoint chiral matter which we used in the Dijkgraaf-Vafa conjecture in section 2.1. The Lagrangian is given by $\mathcal{L}_{\mathrm{U}(N)}$ from (2.26).[46] As we saw in section 2.2.4 we have to choose around which classical supersymmetric vacuum to expand, and that the classical expectation value of the chiral field breaks the gauge group as $\mathrm{U}(N) \mapsto \mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_n)$. The massive gauge multiplets corresponding to this breaking we simply integrate out. We also proved that if the critical points of the tree-level superpotential are isolated then the chiral multiplets in these vacua are massive. Integrating out these massive fields leaves us with an $\mathcal{N} = 1$ supersymmetric pure gauge theory. It is, however, important to realise that the holomorphic scale, Λ_u, before integrating out is not the same as the scale, Λ, in the low energy theory

[44] Naturally, this does not mean that the gaugino condensation is a semi-classical phenomenon like the instantons. Rather, the normalisation factor a can be determined by calculating $\langle S^{3h}\rangle$ which is saturated by a one-instanton and then use the cluster decomposition principle. Here h is the dual Coxeter number.

[45] Thanks to J. Wheater for a discussion on this subject.

[46] Actually, $\mathcal{L}_{\mathrm{U}(N)}$ is the Lagrangian for $\mathcal{N} = 2$ supersymmetry broken to $\mathcal{N} = 1$ by a tree-level superpotential, but as we discussed in section 2.1.4 this only affected the normalisation of the Kähler terms and is not important in the proof of the conjecture.

(here assuming an unbroken gauge group). Using the simple threshold relation (2.129) we can match the scales as:

$$\Lambda^3 = \Lambda_u^2 m, \tag{2.143}$$

where m is the mass of the chiral field. We have here used that $b_d = 3N$, $b_u = 2N$ and $C(\text{adj}) = N$ since we assume $C(\text{fund}) = 1/2$ (see (2.129) for notation). With a broken gauge group we should be more careful. The gauge fields are in this case not independently coupled since at the UV energy μ_0 for the tree-level Lagrangian the complex gauge couplings are all equal to the coupling for the unbroken SU(N). So the holomorphic scales $\Lambda_{i,u}$ corresponding to the factors SU(N_i) are related as (using (2.102)):

$$\left(\frac{\Lambda_{1,u}}{\mu_0}\right)^{2N_1} = \cdots = \left(\frac{\Lambda_{n,u}}{\mu_0}\right)^{2N_n}. \tag{2.144}$$

The relation to the low energy scales Λ_i is as in (2.143), but there can be different masses, m_i, for each factor: $\Lambda_i^3 = \Lambda_{i,u}^2 m_i$.

We saw in section 2.5.2 that the abelian U(1) part of U(N_i) described by the $w_{i\alpha}$ from (2.18) is (perturbatively) weakly coupled at low energy (IR free). On the other hand, for the non-abelian SU(N_i) subgroups we have strong coupling in the IR. Here we expect, as described in section 2.5.6, confinement and gaugino condensation breaking the gaugino number as $\mathbb{Z}_{2N_i} \mapsto \mathbb{Z}_2$ and giving N_i inequivalent vacua each with a mass gap. Since we have confinement, we know that the low energy theory should be described by singlet gauge fields. When focusing on the superpotential part of the effective Lagrangian the relevant, elementary fields are believed to be the traceless glueball superfields $\hat{S}_i$ defined in equation (2.5). The IR dynamics is described by the *glueball superpotential* for $\hat{S}_i$ and $w_{i\alpha}$, $W_{\text{eff}}(S_i, w_{i\alpha}, g_k)$. As noted in section 2.5.5 the glueball superfield is generally massive (mass of order $|\Lambda|$) so we should see the glueball superpotential as obtained by integrating out the chiral matter fields, but with the glueball superfields integrated in. Naturally, the result is not just the Veneziano-Yankielowicz superpotential obtained in the last section, but depends non-trivially on the bare superpotential couplings, g_k. Actually, as mentioned in section 2.1, the full superpotential, obtained by taking into account the full path integral, is conjectured to be the sum of the Veneziano-Yankielowicz superpotential, accounting for the gauge dynamics, and $W_{\text{eff,pert}}$ obtained by integrating out the chiral fields. The way we obtain $W_{\text{eff,pert}}$ is to perform the path integral over the chiral fields while treating $\mathcal{W}_\alpha$ as a (constant) background field thus allowing us to get the $\hat{S}$ dependence. We should also note that the Veneziano-Yankielowicz superpotential in the case of a broken gauge group can not be determined using the integrating in technique from the last subsection since the $\hat{S}_i$ fields are not independently coupled as explained above.

The full glueball superpotential then determines the IR dynamics of $\hat{S}_i$ and we can e.g. get the expectation values of the glueball superfields by the equation of motion (2.8). A non-zero expectation value gives gaugino condensation, chiral symmetry breaking and the corresponding inequivalent vacua. Also the tension in the domain walls connecting these vacua can be determined from the glueball superpotential. We can obtain $W_{\text{eff,pert}}$ to a certain order in $\hat{S}$ by going to the corresponding loop order for the matrix model Feynman diagrams using the Dijkgraaf-Vafa conjecture. This gives us the non-perturbative corrections to a corresponding fractional order in Λ (also called fractional instantons). This is what Dijkgraaf and Vafa refer to as "a perturbative window into non-perturbative physics" [4]. However, we note that this interplay between perturbative and non-perturbative physics happens through the Veneziano-Yankielowicz term. But all that we will prove in the Dijkgraaf-Vafa conjecture is simply the form of $W_{\text{eff,pert}}$. We do not prove confinement, mass gaps or $\hat{S}$ being the elementary field, and we have to add the Veneziano-Yankielowicz superpotential by hand.

In [42] F. Ferrari has investigated the above theory in the case of a cubic tree-level superpotential $W_{\text{tree}} = \text{Tr}\left(\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right)$. Using the Dijkgraaf-Vafa conjecture or rather, which amounts to the same, using the setup that he used to prove the Dijkgraaf-Vafa conjecture in [5], the low-energy effective superpotential can be found (see also section 2.8) and investigated. Assuming that the gauge group is unbroken it is found that there exist some critical points in the quantum parameter space given by $8g^2\Lambda^3/m^3 = e^{-2\pi ik/N}$ with $k = 0, \ldots, N-1$. In these points the glueball superfield actually is massless if one assumes the Kähler potential to be well-behaved. We note, however, that from the Dijkgraaf-Vafa conjecture we know nothing about the Kähler potential so this result should be seen with some reservation. Further, using the Seiberg-Witten theory it is shown that in these points, for N odd, there is monopole condensation and hence confinement. Thus it is claimed that we here have a phase with confinement without a mass gap in contradiction with the lore. For N even we do not have total confinement and for $N = 2$ there is no confinement at all in these points. Other interesting phenomena are tensionless domain walls and the ϑ angle not being periodic. The investigation of the phases and parameter spaces of the supersymmetric theories has been carried on in many articles e.g. [43], [44], [45], and [46]. For more references see [16].

With this example of the interesting results of the Dijkgraaf-Vafa conjecture we will end this section. To prove the Dijkgraaf-Vafa conjecture and understand the planar limit in the matrix model we will in the next section introduce the double line notation allowing a geometric interpretation of the Feynman diagrams.

2.6 Double Line Notation

In this section we will introduce the double line notation for theories with fields in the adjoint representation. This will lead us to a topological classification of the Feynman diagrams and the t' Hooft large N limit. We will use this for the matrix model in the next section and in the diagrammatic proof of the Dijkgraaf-Vafa conjecture in the next chapter.

2.6.1 Double Line Propagators

Let us consider a theory in which we have a real field Φ^a where a is an index in the adjoint representation of $\text{U}(N)$. Φ could e.g. be the gauge potential which was the case G. 't Hooft considered in his article introducing the double line notation and the large N limit [47] (a more general introduction can be found in [48]). However, we naturally think of Φ as a real version of the chiral adjoint field in the Dijkgraaf-Vafa conjecture. As we see below we can even use all of this for the hermitian matrix model also introduced in the conjecture (the double line notation for the matrix model is introduced in e.g. [49]).

The whole idea of the double line notation is based on the observation that we can use the generators of $\text{U}(N)$ in the fundamental representation – as we have done numerous times above – to write the adjoint field as a hermitian matrix:

$$\Phi^a, \quad a = 1, \ldots, N^2 \quad \longleftrightarrow \quad \Phi^i{}_j = \Phi^a (T_a^{(\text{fund})})^i{}_j, \quad i, j = 1, \ldots, N. \tag{2.145}$$

We here note that if Φ^a had been complex then $\Phi^i{}_j$ could be any complex matrix. In (2.145) the upper index transforms in the fundamental representation and the lower index in the anti-fundamental representation. To be precise: If $\Phi^a \mapsto (\exp\left(-i\alpha^c T_c^{(\mathrm{adj})}\right))^a{}_b \Phi^b$ then

$$\Phi^i{}_j \mapsto (e^{-i\alpha^a T_a^{(\mathrm{fund})}})^i{}_k \Phi^k{}_l (e^{i\alpha^a T_a^{(\mathrm{fund})}})^l{}_j = (e^{-i\alpha^a T_a^{(\mathrm{fund})}})^i{}_k (e^{-i\alpha^a T_a^{(\mathrm{anti\text{-}fund})}})^j{}_l \Phi^k{}_l, \tag{2.146}$$

where we have used the definition of the anti-fundamental, i.e. conjugate, representation $T_a^{(\mathrm{anti\text{-}fund})} = -\left(T_a^{(\mathrm{fund})}\right)^T$. (2.146) is proven using the definition of the adjoint representation $(T_a^{(\mathrm{adj})})^b{}_c \Phi^c = [T_a, \Phi^c T_c]^b$ which also gives $(T_a^{(\mathrm{adj})})^b{}_c = i f_{ac}{}^b$. We can state (2.146) as: The adjoint representation of $\mathrm{U}(N)$ is $\boldsymbol{N} \otimes \overline{\boldsymbol{N}}$.

As we have done before we can use the matrix notation to write the invariant Lagrangian using traces, e.g.:

$$\mathcal{L} = \frac{1}{g_{\mathrm{s}}} \left(\frac{1}{2} m \,\mathrm{Tr}\big(\Phi^2\big) + \sum_k \frac{g_k}{k} \,\mathrm{Tr}\Big(\Phi^k\Big) \right). \tag{2.147}$$

Naturally, we could also have a kinetic term, however, we are only interested in the index structure of the Feynman diagrams, not the space-time dependence. Actually, the theory could just as well be the matrix model used in the Dijkgraaf-Vafa conjecture with a global $\mathrm{U}(N)$ symmetry. We have put a coupling g_{s} in front of the Lagrangian (for the gauge theory g_{s} is the Yang-Mills coupling squared). This coupling is first needed in the large N expansion and is not necessary for the double line notation. We could also have multi-trace terms, but we will assume that we do not have such terms.

We will now develop the Feynman rules for the Lagrangian (2.147). To this end we can think of $\Phi^i{}_j$ as a complex particle if $i > j$ and a real particle if $i = j$. In the usual way we treat $\Phi^j{}_i = \Phi^{*i}{}_j$ and $\Phi^i{}_j$ as independent. Thus we can see the path integral as having an integration over each entry of the matrix Φ and all entries are independent. The propagator is determined from the quadratic term in the usual way. We quickly get (here in Euclidean notation and disregarding the space-time dependence):

$$\int \mathcal{D}\Phi \, e^{-\frac{m}{2g_{\mathrm{s}}}\mathrm{Tr}(\Phi^2)} \Phi^{i_1}{}_{j_1} \cdots \Phi^{i_n}{}_{j_n} = \frac{\partial}{\partial J^{j_1}{}_{i_1}} \cdots \frac{\partial}{\partial J^{j_n}{}_{i_n}} \, e^{\frac{g_{\mathrm{s}}}{2m}\mathrm{Tr}(J^2)} \Big|_{J=0}, \tag{2.148}$$

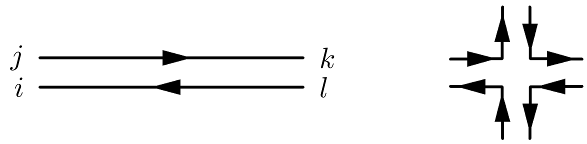
where J is the external current which here is a hermitian matrix. We can directly read off the propagator from $g_{\mathrm{s}}\mathrm{Tr}\big(J^2\big)/2m = g_{\mathrm{s}} J^i{}_j \delta^l_i \delta^j_k J^k{}_l / 2m$:[47]

$$\langle \Phi^i{}_j \Phi^k{}_l \rangle_0 = \frac{g_{\mathrm{s}}}{m} \delta^i_l \delta^k_j. \tag{2.149}$$

If Φ is the gauge field then we naturally have the usual propagator $1/k^2$ instead of $1/m$. The propagator is consistent with our view of $\Phi^i{}_j$ as complex particles. If $i > j$ we have $\langle \Phi^i{}_j \Phi^k{}_l \rangle_0 = \langle \Phi^i{}_j \Phi^{*l}{}_k \rangle_0$ which immediately gives us (2.149). As usual with complex particles we assign arrows to the propagators. Here we must use double lines for the propagators since we have two indices. The upper index, which transformed in the fundamental representation, is then associated with an incoming arrow and the anti-fundamental lower index with an outgoing arrow. The double line propagator corresponding to (2.149) is shown in figure 2.2. The single lines in the double line propagator are called *index lines* since they are indexed by $i = 1, \ldots, N$. The interaction vertices from (2.147) all have the same index structure consistent with the double line propagators and can be derived from (2.148). They are just proportional to Kronecker delta functions that connect the index lines and preserve the directions of these. As an example the quartic vertex is also shown in figure 2.2. Let us finally note that if Φ^a was complex, and hence $\Phi^i{}_j$ a general complex matrix, then we would have to assign an extra overall arrow to the whole double line propagator.

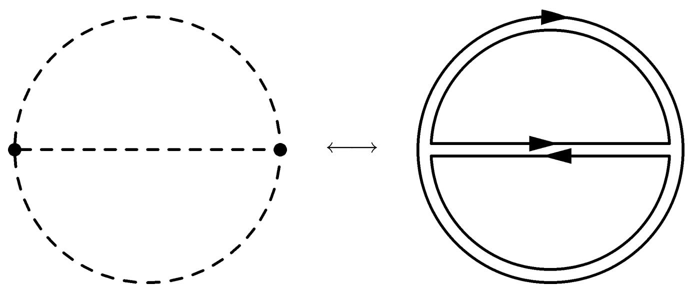
[47] In the case of $\mathrm{SU}(N)$ we have to subtract $\delta^i_j \delta^k_l / N$ on the right hand side since in this case Φ is traceless.

Figure 2.2:



On the left the double line propagator and on the right the quartic vertex.

Figure 2.3:



A planar diagram. On the left in single line adjoint representation with interaction vertices denoted by dots. On the right the same diagram in double line notation.

2.6.2 Topological Classification of Double Line Diagrams

Let us now look at the connected vacuum diagrams in the double line notation. These "fat" graphs have enough structure to associate each of them with a Riemann surface or rather a topological class of these. To do this we view each index line as a perimeter of a face in a simplicial decomposition of a surface. To this end we compactify the space by adding a point at infinity so we also can see the outer index loop as a face on a compact surface. In our U(N) case the surface is further oriented since we have directions on the index lines. The compact oriented surfaces in $\mathbb{R}^3$ are topologically classified by the Euler characteristic, χ, which is a topologically invariant integer. The surfaces are simply given by the sphere, S^2, with g handles added. g is called the genus of the surface. The Euler characteristic is then given by:

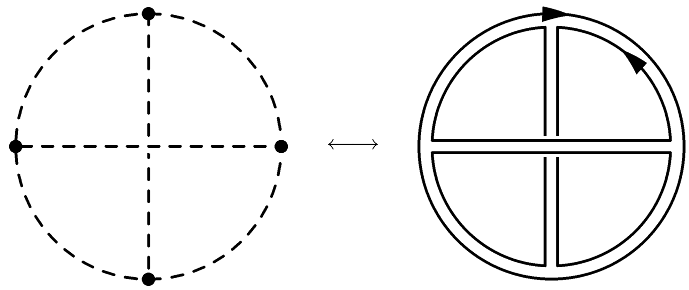
$$\chi = 2 - 2g. \tag{2.150}$$

We can determine the Euler characteristic of the surface corresponding to a given diagram by simply counting the number of vertices, V, edges i.e. double line propagators, E, and faces i.e. index loops, F. Then:

$$\chi = V - E + F. \tag{2.151}$$

If the topology of the diagram is that of a sphere, $g = 0$, the diagram is called planar. An example of a planar diagram with cubic vertices is given in figure 2.3. We can simply count $V = 2$, $E = 3$ and $F = 3$ so $\chi = 2 - 3 + 3 = 2$. An example of a diagram with the topology of a torus, $g = 1$, is given in figure 2.4. Here we count $V = 4$, $E = 6$ and $F = 2$ so $\chi = 4 - 6 + 2 = 0$. We can obtain surfaces with boundaries if we add fundamental matter

Figure 2.4:



A diagram with the topology of a torus. On the left in single line adjoint representation with interaction vertices denoted by dots. On the right the same diagram in double line notation.

Figure 2.5:

The cross-over part of the SO(N) propagator.

to our Lagrangian. Such matter only has one index so it will only correspond to a single index line which we can interpret as a boundary. If we look at SO(N) or USp(N) instead of U(N) we must also include non-orientable surfaces. E.g. in the case of SO(N) we can also use (2.145), however, we must here remember that the fundamental representation is real and Φ in this case is antisymmetric. We thus see that the adjoint of SO(N) is $\boldsymbol{N} \otimes_{as} \boldsymbol{N}$. Since we can not distinguish the upper from the lower indices we can not orient the index lines. Further we have to constrain the path integral to only include antisymmetric matrices. We could do this by replacing the antisymmetric matrices with $(\Phi - \Phi^T)/2$ where Φ is a hermitian matrix and then integrate over the unconstrained Φ. Expanding the quadratic terms shows that the propagator must contain a cross-over term as shown in figure 2.5 which indeed gives rise to non-orientable surfaces. In these more general cases the surfaces are characterised by starting from the sphere S^2 and adding g handles, b boundaries and c cross-caps. The Euler characteristic is then given by [50]:

$$\chi = 2 - 2g - b - c. \tag{2.152}$$

We obtain the boundaries by adding fundamental matter single index lines and the cross-caps from the cross-over part of the SO(N) or USp(N) propagator. We note that $\chi \leq 2$ and only odd with fundamental matter or another group than U(N).

2.6.3 't Hooft Large N Limit

Using the topological classification obtained in the last subsection we can now explain the 't Hooft large N limit as introduced in [47]. We will assume a U(N) group and no fundamental matter. For a given connected vacuum diagram we count the dependence on N and the coupling g_s from (2.147). For each vertex we have a factor $1/g_\mathrm{s}$, for each double line propagator we have a factor g_s as seen from (2.149), and finally we have a factor N for each index loop since we here sum $\sum_i \delta_i^i = N$. Thus we get an overall factor of:

$$N^F g_s^{E-V} = N^{V-E+F} (g_s N)^{E-V} = N^\chi (g_s N)^{E-V} = N^{2-2g} (g_s N)^{E-V} , \tag{2.153}$$

where we have used (2.150) and (2.151). We thus see that by taking $N \to \infty$ while keeping $g_s N$ fixed the planar diagrams with $g = 0$ are the dominant ones. This is the 't Hooft large N limit also called the planar limit. To connect with the Dijkgraaf-Vafa conjecture we rewrite the factor (2.153) and include the dependence on the couplings g_k from (2.147) (with a minus for Euclidean space) and the mass m for completeness:

$$g_s^{2g-2} (g_s N)^F m^{-E} \prod_k (-g_k)^{V_k}, \tag{2.154}$$

where V_k is the number of vertices of order k so that $V = \sum_k V_k$. What we want to calculate is the free energy, W_{free}, which exactly is the sum over (minus) the connected vacuum diagrams. We can now group the diagrams topologically and write (using (2.154)):

$$W_{\text{free}} = \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g(g_s N) , \tag{2.155}$$

where $\mathcal{F}_g$ is the contribution to the free energy (modulo factors of g_s^{2g-2}) from diagrams with genus g and whose dependence on g_s and N only is through $g_s N$. In the 't Hooft large N limit we see that the planar contribution is dominant since $g_s \to 0$ and thus $W_{\text{free}} \approx g_s^{-2} \mathcal{F}_{g=0}$. This explains (2.12) and the planar limit taken on the matrix side in the Dijkgraaf-Vafa conjecture in the case of an unbroken gauge group.

2.7 The Matrix Model

In this section we will investigate the matrix model side of the Dijkgraaf-Vafa conjecture.

2.7.1 The Matrix Model and the Dijkgraaf-Vafa Conjecture

In the Dijkgraaf-Vafa conjecture for a U(N) gauge group we are instructed to calculate the free energy in the planar limit for a bosonic matrix model with partition function (2.10):

$$Z_{\text{matrix}} = \int \mathcal{D}M e^{-\frac{1}{g_s} W_{\text{tree}}(M)},$$

where M are $N' \times N'$ hermitian matrices and $W_{\text{tree}} = \operatorname{Tr} P_{n+1}(M)$. The constraint to hermitian matrices requires the couplings in W_{tree} to be real contrary to the gauge theory side where Φ is a general complex matrix. Thus the matrix model is real (real eigenvalues, real couplings) and we should perform an analytic continuation after obtaining the wanted free energy to compare with the gauge theory side in the Dijkgraaf-Vafa conjecture. There should be no ambiguity in this continuation since we restrict to the planar limit [5]. Naturally, this is of no concern in calculations where one simply formally uses g_k as couplings with no reference to whether they are real or complex. From the partition function we also see that M must have mass dimension one and g_s mass dimension three to fit the dimensions of the couplings. Of course, one can scale these dimensions away using the holomorphic scale Λ which has mass dimension one.

The matrix model does not have any supersymmetry as on the gauge theory side, but it is gauged in the sense that the potential is (globally) $\mathrm{U}(N)$ invariant under $M \mapsto UMU^{-1}$ for $U \in \mathrm{U}(N)$. We will treat this symmetry as an equivalence like the gauge symmetry. We can e.g. use this to diagonalise the matrices. Using the same manipulations that led to (2.57) the classical equation of motion for M can be shown to be $P'_{n+1}(M) = 0$. Choosing M to be diagonal we conclude that the "vacua" for the matrix model are obtained analogously to the supersymmetric vacua by distributing the N' eigenvalues of M over the critical points $a_1, \dots, a_n$ of P_{n+1}. This corresponds to the partition of N' in (2.11). As with the supersymmetric vacua we should think modulo permutation of the eigenvalues and the vacuum then breaks the gauge symmetry as:

$$\mathrm{U}(N') \mapsto \mathrm{U}(N'_1) \times \cdots \times \mathrm{U}(N'_n). \tag{2.156}$$

In the Dijkgraaf-Vafa conjecture we should, in the terminology of section 2.1.1, here choose $N'_i = 0$ if $N_i = 0$ to obtain the same symmetry breaking pattern as on the gauge theory side. However, due to the realness of the matrix model we here have a problem since the real form of the polynomial P'_{n+1} does not necessarily have n real roots and the eigenvalues of the hermitian matrix M must be real. The solution is probably to constrain the couplings g_k to certain real intervals and then analytically continue to all real numbers. E.g. for the quartic potential $P_{n+1} = \frac{1}{2}mM^2 + \frac{1}{4}gM^4$ we have the set of critical points $\{0, \pm\sqrt{-m/g}\}$. We can thus choose m positive and g negative to have the same set of critical points to distribute the eigenvalues over as in the gauge theory.[48]

What we should do now is to formulate a perturbation theory for the fluctuations around the chosen vacuum. Assuming, as in section 2.1.1, that the critical points are distinct, the vacua will be massive in the sense that the parts corresponding to unbroken gauge group factors have non-zero masses. This can be proven in the same way as done for the gauge theory side in section 2.2.4. In the case of an unbroken gauge group, i.e. if all eigenvalues are equal, the perturbative expansion is simple. As an example consider a cubic interaction which has two critical points a_1 and a_2. Expanding around $a_1 \mathbf{1}_{N' \times N'}$ gives (disregarding a constant term):

$$W_{\text{tree}} = \tfrac{1}{2}\Delta\,\mathrm{Tr}\big(M^2\big) + \tfrac{g}{3}\,\mathrm{Tr}\big(M^3\big)\,, \tag{2.157}$$

where $\Delta = g(a_1 - a_2)$ which we note is non-zero for distinct critical points. We can then simply expand the partition function as:

$$\int \mathcal{D}M e^{-\frac{1}{g_\text{s}} W_{\text{tree}}} = \int \mathcal{D}M e^{-\frac{1}{2g_\text{s}}\Delta\,\mathrm{Tr}\left(M^2\right)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1}{g_\text{s}}\frac{g}{3}\,\mathrm{Tr}\big(M^3\big)\right)^n. \tag{2.158}$$

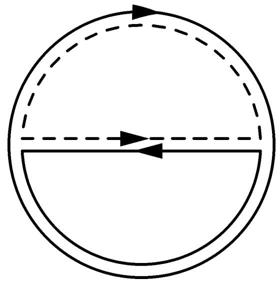
In the case of broken gauge symmetry we have to be more careful and take into account Faddeev-Popov ghosts in the matrix model [17]. These ghosts are described by two Grassmannian matrices B and C which contribute to the potential as:

$$W_{\text{ghost}} = \mathrm{Tr}(B[M, C])\,. \tag{2.159}$$

This ghost term can be derived from the Vandermonde determinant that, as we will see in the next section, arises from choosing a gauge such that M is diagonal. From this term we immediately see that the ghosts are only propagating if we expand around a vacuum with broken symmetry since a matrix proportional to the identity will disappear from the commutator. On the gauge theory side a ghost term of precisely the same form arises with

[48] Unfortunately, this would in turn mean that $W_{\text{tree}}(M)$ is not bounded from below.

Figure 2.6:



Example of a planar diagram for the matrix model with ghosts. The solid index lines carry indices running over $1, \ldots, N_1'$ while the dashed index lines carry indices running over $1, \ldots, N_2'$. We have two ghost double line propagators and one double line propagator associated with M_{11}.

B and C being anticommuting chiral superfields. This is in line with the Dijkgraaf-Vafa conjecture assuming that the superpotential in the gauge theory and the potential in the matrix model should be the same.

To be specific, let us consider the cubic model as we did above (following [17]). We now consider the broken case where the vacuum expectation value M_0 has N_1' eigenvalues equal to a_1 and N_2' eigenvalues equal to a_2. We should expand around this vacuum:

$$M \mapsto M_0 + M = \begin{pmatrix} a_1 \mathbf{1}_{N_1' \times N_1'} & 0 \\ 0 & a_2 \mathbf{1}_{N_2' \times N_2'} \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{2.160}$$

where we have written the matrix in block notation (M_{ij} is a $N_i' \times N_j'$ matrix). Using this block notation, also for the ghost term, one finds the quadratic terms in the potential to be:

$$\frac{1}{2}\Delta \operatorname{Tr}\bigl(M_{11}^2\bigr) - \frac{1}{2}\Delta \operatorname{Tr}\bigl(M_{22}^2\bigr) + \Delta \operatorname{Tr}(B_{21}C_{12}) - \Delta \operatorname{Tr}(B_{12}C_{21}), \tag{2.161}$$

where $\Delta = (a_1 - a_2)$ and we have set $g = 1$. We thus see that the off-diagonal blocks for M do not propagate which is clear since they are pure gauge terms. On the other hand, only the off-diagonal blocks for the ghost fields propagate. Using (2.161) it is trivial to write down the double line propagators using the same method as in last section. We see that we have two types of propagators. Firstly the usual double line propagators with the same type of index running in both index lines. We have two propagators of this type corresponding to respectively M_{11} and M_{22} and with indices running over respectively $1, \ldots, N_1'$ and $1, \ldots, N_2'$. The second type of double line propagators stems from the ghost terms. In these propagators the index in one index line runs from $1, \ldots, N_1'$ while the index in the other runs from $1, \ldots, N_2'$. We have two propagators of this type corresponding to the two ghost terms in (2.161). As can be derived from (2.159) we also get vertices connecting a standard double line propagator with two ghost propagators. An example of a diagram with ghosts can be seen in figure 2.6.

Generalising these results we see that in the case of broken symmetry we have diagrams in the double line notation in which the index loops, i.e. the faces, are indexed according to which broken part they are associated with. With partition $N' = \sum_i N_i'$ the index loops are indexed by i and the index loop gives a contribution N_i'. As an example the index loops in the diagram in figure 2.6 give a contribution of $N_1'^2 N_2'$. As in section 2.6.2 the connected vacuum diagrams can be topologically classified. Let F_i be the number of faces indexed with i then the total number of faces is $F = \sum_i F_i$. We simply perform the topological classification "blind" to the index i i.e. only depending on F and also we do not distinguish between the types of double line propagators when counting edges. Finally, to connect with the Dijkgraaf-Vafa conjecture we should take the large N' limit, or more precisely, we should let $g_\mathrm{s} \ll 1$ while keeping $g_\mathrm{s} N_i'$ fixed and finite.[49] We can count the g_s dependence in a given diagram and we find, in analogy with (2.154), a factor:

$$g_\mathrm{s}^{2g-2} \prod_i \left(g_\mathrm{s} N_i'\right)^{F_i} . \tag{2.162}$$

We can make a topological expansion of the free energy as in (2.155). The leading contribution is again from genus $g = 0$ and we get the planar limit of the free energy $g_\mathrm{s}^{-2} \mathcal{F}_{g=0}(g_\mathrm{s} N_i)$. Identifying $S_i = g_\mathrm{s} N_i'$ we see from (2.162) that we get one factor of S_i for every index loop indexed by i as stated at the end of section 2.1.2. In section 2.8 we will give an explicit example of this (in the unbroken case).

2.7.2 The Measure

So far we have not discussed the measure of the matrix model. However, when we obtained the Feynman rules for double line propagators, we implicitly assumed in (2.148) that the normalisation of the measure was such that for the free theory $\langle 1 \rangle_0 = 1$. This means that in the measure we have to divide with the volume of the gauge group vol(G). In [51] it is found that in the planar limit (here for a broken gauge group):

$$\frac{1}{\mathrm{vol}(\mathrm{U}(N_1') \times \cdots \times \mathrm{U}(N_n'))} \sim e^{\frac{1}{2}\sum_i N_i'^2 \ln N_i' + \cdots}, \tag{2.163}$$

where we have only kept the part of the planar contribution that involves $\ln N_i'$. We can here substitute $N_i' = \hat{S}_i / g_\mathrm{s}$ and compare with the planar free energy $e^{-g_\mathrm{s}^{-2} \mathcal{F}_{g=0}(\hat{S}_i)}$. This gives us a contribution to $\mathcal{F}_{g=0}$ of the form $-\frac{1}{2}\sum_i \hat{S}_i^2 \ln \hat{S}_i$. As noted by Dijkgraaf and Vafa, if we here extend the connection (2.14b) between W_eff and $\mathcal{F}_{g=0}$ in the Dijkgraaf-Vafa conjecture to also apply for this term, we see that the essential $-\sum_i N_i \hat{S}_i \ln\left(\hat{S}_i\right)$ part of the Veneziano-Yankielowicz superpotential is reproduced. We emphasise that there is no field theoretic proof for this, but it suggests a tighter relation between the matrix model and the gauge theory than we can actually prove.

Let us also note that in the case of unbroken symmetry (following [7]) we can obtain this coupling independent part of the planar free energy simply by setting the couplings in the potential to zero and use Gaussian integration:

$$\mu^{-N'^2} \int \mathcal{D}M e^{-\frac{1}{2g_\mathrm{s}} m \,\mathrm{Tr}\left(M^2\right)} = \left(\frac{2\pi g_\mathrm{s}}{m\mu^2}\right)^{N'^2/2}, \tag{2.164}$$

where μ is a scale of mass-dimension one introduced to make the measure dimensionless. We thus get a contribution to $\mathcal{F}_{g=0}$ given by:

$$\Delta\mathcal{F}_{g=0} = -\frac{1}{2}\hat{S}^2 \ln\left(\frac{2\pi\hat{S}}{N' m\mu^2}\right) = -\frac{1}{2}\hat{S}^2 \ln\left(\frac{\hat{S}}{e^{3/2} m\Lambda_u^2}\right), \tag{2.165}$$

[49] We have not been careful with the normalisation of the ghost term. To take the 't Hooft limit we should also give this term a $1/g_\mathrm{s}$ normalisation.

where we in the last line identify $N'\mu^2/2\pi$ with $e^{3/2}\Lambda_u$ where Λ_u is the holomorphic scale before integrating out the chiral field in the gauge field theory. Extending the Dijkgraaf-Vafa conjecture as above this exactly gives the Veneziano-Yankielowicz superpotential if we use the matching of scales (2.143): $\Lambda^3 = m\Lambda_u^2$.

2.7.3 Exact Solution of the Matrix Model

Above we have solved the matrix model perturbatively using diagrams. However, as we now show, it is also possible to use non-perturbative techniques to obtain the exact solution of the matrix model in the planar limit. This was first done in [52]. Here we will also use [49].

It is customary to redefine the matrix model coupling to:

$$g_{\mathrm{m}} \equiv g_{\mathrm{s}} N', \tag{2.166}$$

such that the potential takes the form $\frac{N'}{g_{\mathrm{m}}} W_{\mathrm{tree}}(M)$. Noting that this potential only depends on the eigenvalues of M due to the cyclicity of the trace, we can diagonalise the hermitian matrix as $M = U^\dagger \Lambda U$ where U is a unitary matrix and Λ is a diagonal matrix consisting of the eigenvalues λ_i, $i = 1, \ldots, N'$, of M. The integration over M then splits into an integration over the eigenvalues and a trivial integration over the unitary matrices. Assuming unit measure for the unitary matrices we get:

$$Z_{\mathrm{matrix}} = \int \mathcal{D}M e^{-\frac{N'}{g_{\mathrm{m}}} W_{\mathrm{tree}}(M)} = \int \prod_{i=1}^{N'} \mathrm{d}\lambda_i \, \Delta^2(\lambda) \, e^{-\frac{N'}{g_{\mathrm{m}}} \sum_i P_{n+1}(\lambda_i)}, \tag{2.167}$$

where $\Delta(\lambda) = \prod_{i<j} (\lambda_j - \lambda_i)$ is the Vandermonde determinant which can be derived using the Faddeev-Popov method.[50] Exponentiating the Vandermonde determinants then gives us an effective potential:

$$Z_{\mathrm{matrix}} = \int \prod_i \mathrm{d}\lambda_i \, e^{-\frac{N'}{g_{\mathrm{m}}} \sum_i P_{n+1}(\lambda_i) + \sum_{i\neq j} \ln|\lambda_j - \lambda_i|}. \tag{2.168}$$

In the large N' limit the exact solution for the free energy $g_{\mathrm{s}}^{-2} \mathcal{F}_{g=0} = \frac{N'^2}{g_{\mathrm{m}}^2} \mathcal{F}_{g=0}$ is found using the steepest descent method: We simply evaluate the effective potential in its critical point (the saddle-point), i.e.:

$$\begin{aligned} \mathcal{F}_{g=0} &= \frac{g_{\mathrm{m}}}{N'} \sum_i P_{n+1}(\lambda_i) - \frac{g_{\mathrm{m}}^2}{N'^2} \sum_{i\neq j} \ln|\lambda_j - \lambda_i| \,, && (2.169) \\ 0 &= -\frac{N'}{g_{\mathrm{m}}} P'_{n+1}(\lambda_i) + 2 \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}, \quad i = 1, \ldots, N'. && (2.170) \end{aligned}$$

We can describe the distribution of the eigenvalues with the density of eigenvalues:

$$\rho(\lambda) = \frac{1}{N'} \sum_i \delta(\lambda - \lambda_i) \,, \qquad \int \mathrm{d}\lambda \, \rho(\lambda) = 1. \tag{2.171}$$

[50] Simply write one as $1 = \int \prod_i \mathrm{d}\lambda_i \mathrm{d}U' \, \delta^{(N'^2)} \big(U' M U'^\dagger - \mathrm{diag}(\lambda_i) \big) \, \Delta^2(\lambda)$. The λ_i integrations constrain λ_i to be the eigenvalues of M and $\mathrm{diag}(\lambda_i) = \Lambda$. The δ-function constrain U' to be in a neighbourhood of the matrix U that diagonalised M: $U' = (1+T)U$ where T is infinitesimal. Thus $\left(U' M U'^\dagger - \Lambda \right)_{ij} \simeq [T, \Lambda]_{ij} = T_{ij} \left(\lambda_j - \lambda_i \right)$. The result then follows from performing the integration over the real and complex part of T_{ij}, $i < j$.

Due to the term from the Vandermonde determinants we see that there is a Coulomb repulsion between the eigenvalues. This means that the eigenvalues will spread out evenly and in the large N' limit we get a continuous distribution of eigenvalues and a smooth density, ρ. In general (as we will show below) the support of ρ is disconnected consisting of maximally n intervals (cuts), $\mathrm{supp}(\rho) = \bigcup_k C_k$. There is one cut C_k for each critical point, a_k, of P_{n+1} and $a_k \in C_k$. From (2.170) we see that in the classical limit $g_{\mathrm{m}} \to 0$ the eigenvalues are all in the critical points and the cuts C_k simply shrink to the points a_k. Using ρ we can rewrite the sums over λ as integrals, and (2.169) and (2.170) become:

$$\begin{aligned} \mathcal{F}_{g=0} &= g_{\mathrm{m}} \int \mathrm{d}\lambda\, \rho(\lambda)\, P_{n+1}(\lambda) - g_{\mathrm{m}}^2 \,⨍\, \mathrm{d}\lambda\, \mathrm{d}\lambda'\, \rho(\lambda)\, \rho(\lambda') \ln\left|\lambda - \lambda'\right|, \qquad (2.172)\\ 0 &= -P'_{n+1}(\lambda) + 2g_{\mathrm{m}} ⨍ \mathrm{d}\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}, \quad \lambda \in \mathrm{supp}(\rho). \qquad (2.173) \end{aligned}$$

Here we have principal value diagrams since $j \neq i$ in the sums over λ.

To solve the matrix model it is convenient to introduce the resolvent:

$$R(z) \equiv \frac{1}{N'} \mathrm{Tr}\left(\frac{1}{M - z}\right) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{\rho(\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\rho). \qquad (2.174)$$

The resolvent is analytic with branch cuts at $\mathrm{supp}(\rho)$. The point is that we can determine $R(z)$ which in turn determines ρ since by redrawing contours of integration, (2.174) gives:

$$\rho(\lambda) = \frac{1}{2\pi i}\big(R(\lambda + i\epsilon) - R(\lambda - i\epsilon)\big), \quad \lambda \in \mathrm{supp}(\rho), \qquad (2.175)$$

where ϵ is infinitesimal. Indeed we can determine $R(z)$ using the saddle-point equation which from (2.173) takes the form:

$$R(\lambda + i\epsilon) + R(\lambda - i\epsilon) = -\frac{1}{g_{\mathrm{m}}} P'_{n+1}(\lambda)\,, \quad \lambda \in \mathrm{supp}(\rho). \qquad (2.176)$$

However, we can also multiply (2.173) with $\frac{1}{g_{\mathrm{m}}} \frac{\rho(\lambda)}{\lambda - z}$ and integrate over λ to obtain (after a couple of rewritings):[51]

$$R^2(z) - \frac{1}{N'} R'(z) + \frac{1}{g_{\mathrm{m}}} P'_{n+1}(z) R(z) + f(z) = 0, \qquad (2.177)$$

where $f(z)$ is a polynomial of degree $n-1$ given by:

$$f(z) = \frac{1}{g_{\mathrm{m}}} \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \rho(\lambda) \frac{P'_{n+1}(z) - P'_{n+1}(\lambda)}{z - \lambda}. \qquad (2.178)$$

Since we are in the large N' limit, we can disregard the R'/N' term in (2.177) and thus the equation is purely algebraic. We solve this most easily by splitting R in its regular and singular part $R(z) = R_{\mathrm{reg}}(z) + R_{\mathrm{sing}}(z)$. The regular part is $R_{\mathrm{reg}}(z) = -P'_{n+1}(z)\,/2g_{\mathrm{m}}$ which is also directly seen from (2.176). The singular part is then determined from (2.177) as:

$$R_{\mathrm{sing}}(z) = \sqrt{\tfrac{1}{4g_{\mathrm{m}}^2} P'^2_{n+1}(z) - f(z)}. \qquad (2.179)$$

[51] This equation is also found in [7] with R being the expectation value of $\frac{1}{N'}\mathrm{Tr}\Big(\frac{1}{M-z}\Big)$. The result is obtained using the loop equations and that the correlation functions factor in the large N' limit. This tells us, as we have already noted, that the saddle-point approximation is exact in the large N' limit.

Since $P_{n+1}'^2$ is of order $2n$, we see that R has at most $2n$ different branch points and hence at most n branch cuts C_k giving the promised support of ρ. However, we have not really solved the problem yet since f depends on ρ. But the n coefficients in f are actually determined by the n filling fractions [2]:[52]

$$\frac{N_i'}{N'} = -\frac{1}{2\pi i}\oint_{C_i} \mathrm{d}z\, R(z) = \int_{C_i} \mathrm{d}\lambda\, \rho(\lambda). \tag{2.180}$$

We see that the filling fraction N_i'/N' is the relative number of eigenvalues in the i^{th} cut. Finally we have solved for R and hence ρ as a function of g_{m}, N_i'/N' and the couplings g_k. We can then obtain the free energy using (2.172).

In this exact case the relation between the gauge theory side and the matrix model side in the Dijkgraaf-Vafa conjecture is through the filling fractions:[53]

$$\hat{S}_i \equiv g_{\mathrm{m}}\frac{N_i'}{N'} = g_{\mathrm{s}} N_i'. \tag{2.181}$$

This definition, naturally, gives the same result as the one we have in the diagrammatic case (2.13). We can see this from the classical limit $g_{\mathrm{m}} \to 0$ where the filling fraction N_i'/N' simply becomes the relative number of eigenvalues in the i^{th} critical point. The form of the exact solution is used to prove the Dijkgraaf-Vafa conjecture in the original superstring proof, in the proof using the generalised Konishi anomaly, and in the proof using the Seiberg-Witten curves.

2.7.4 One-Cut Solution

Let us consider the exact solution of the matrix model in the case of a single cut. This corresponds to the case of unbroken gauge symmetry where one N_i' equals N' and the rest are zero. We write the cut as $C_i = [a, b]$. To determine R we use equation (2.176). We note, following [49], that the homogeneous version of this equation (i.e. setting the right hand side to zero) has the solution $g(z) = \sqrt{(z-b)(z-a)} = \exp\big(\frac{1}{2}\ln(z-b) + \frac{1}{2}\ln(z-a)\big)$, where the logarithms are defined with the usual branch cut along the negative real axis. Thus $g(\lambda \pm i\epsilon) = \pm i\sqrt{(b-\lambda)(\lambda-a)}$ for $\lambda \in [a, b]$. We can then rewrite (2.176) as:

$$r(\lambda + i\epsilon) - r(\lambda - i\epsilon) = -\frac{1}{g_{\mathrm{m}}}\frac{P_{n+1}'(\lambda)}{i\sqrt{(b-\lambda)(\lambda-a)}}, \quad \lambda \in [a, b], \tag{2.182}$$

where we have defined $r(z) = R(z)/g(z)$. In analogy with the relation between equation (2.174) and (2.175) we get:[54]

$$R(z) = g(z)r(z) = \frac{\sqrt{(z-b)(z-a)}}{2\pi g_{\mathrm{m}}}\int_a^b \mathrm{d}\lambda\, \frac{1}{\lambda - z}\frac{P_{n+1}'(\lambda)}{\sqrt{(b-\lambda)(\lambda-a)}}, \quad z \in \mathbb{C}\setminus[a, b]. \tag{2.183}$$

To obtain the solution we should determine a and b using the constraint (2.180) which simply tells us that $R(z)$ behaves as $-1/z$ for $|z| \to \infty$. This condition also rules out the possibility

[52] Actually, if we sum all these conditions we simply get the normalisation condition (2.171) for ρ which in turn is equivalent to require $R(z)$ to behave like $-1/z$ for $|z|$ large.

[53] From (2.172) and (2.180) we see that the dependence on g_{m} in the free energy is always through $g_{\mathrm{m}}N_i'/N' = g_{\mathrm{s}}N_i'$ as it should be.

[54] We can also prove this using $1/(\lambda - \lambda' - i\epsilon) - 1/(\lambda - \lambda' + i\epsilon) = 2i\epsilon/((\lambda-\lambda')^2 - \epsilon^2) = 2\pi i\,\delta(\lambda - \lambda')$.

of a regular term in (2.183). Using that $\sqrt{(z-b)(z-a)}/(\lambda-z) = -1-(\lambda-(b+a)/2)/z + \mathcal{O}(1/z^2)$ for large $|z|$, we get the two constraints:

$$\int_a^b \mathrm{d}\lambda \, \frac{P'_{n+1}(\lambda)}{\sqrt{(b-\lambda)(\lambda-a)}} = 0, \tag{2.184}$$

$$\int_a^b \mathrm{d}\lambda \, \frac{\lambda P'_{n+1}(\lambda)}{\sqrt{(b-\lambda)(\lambda-a)}} = 2\pi g_{\mathrm{m}}. \tag{2.185}$$

These two equations determine a and b as functions of g_{m} and the couplings g_k. Naturally, we should here choose the solution such that for the critical point a_i with $N'_i = N'$ we have $a_i \in [a,b]$.

In principle we have solved the problem and we can obtain the free energy (2.172) using ρ determined from R using (2.175). However, what we really need in the Dijkgraaf-Vafa conjecture (2.14b) is the derivative of the free energy $\partial \mathcal{F}_{g=0}/\partial g_{\mathrm{m}}$ (where $\hat{S} \equiv g_{\mathrm{m}}$). To obtain this we first note that (as we show in appendix E):

$$\frac{\partial}{\partial g_{\mathrm{m}}}\big(g_{\mathrm{m}}\rho(\lambda, g_{\mathrm{m}})\big) = \frac{1}{\pi\sqrt{(b-\lambda)(\lambda-a)}}. \tag{2.186}$$

Following [5] we can use this and (as we also show in appendix E)

$$⨍_a^b \mathrm{d}\lambda \, \frac{\ln|\lambda-\lambda'|}{\sqrt{(b-\lambda)(\lambda-a)}} = \pi \ln\left(\frac{b-a}{4}\right), \quad \forall \lambda' \in [a,b], \tag{2.187}$$

to rewrite (2.172) as:

$$\frac{\partial}{\partial g_{\mathrm{m}}}\mathcal{F}_{g=0} = \int_a^b \mathrm{d}\lambda \, \frac{P_{n+1}(\lambda)}{\pi\sqrt{(b-\lambda)(\lambda-a)}} - 2g_{\mathrm{m}} \ln\left(\frac{b-a}{4}\right). \tag{2.188}$$

Thus (2.184), (2.185) and (2.188) determines the effective superpotential $W_{\mathrm{eff,pert}}$ using the Dijkgraaf-Vafa conjecture (2.14) for an unbroken gauge group. But here we should be careful. In the solution of the matrix model we have actually been working with dimensionless variables i.e. where the dimensions have been scaled away using the only mass-scale at hand namely Λ_u – the scale for the gauge theory before integrating out the chiral fields. At the point where we wish to use the Dijkgraaf-Vafa conjecture we should restore the dimensions such that g_{m} and $P_{n+1}(\lambda)$ have dimensions 3, and thus a and b dimension 1, and $\mathcal{F}_{g=0}$ dimension 6. The only place where we will be able to see this, is in the logarithm in the last term of (2.188) which will take the form $\ln\big((b-a)/4\Lambda_u\big)$. We can then identify $\hat{S} \equiv g_{\mathrm{m}}$ and $W_{\mathrm{eff,pert}} = N\partial\mathcal{F}_{g=0}/\partial g_{\mathrm{m}}$.

However, as we show in appendix E, we can go one step further and actually perform the integrations if we expand P_{n+1} as in (2.2). We then get the following algebraic equations for the solution (with $m = g_2$):

$$\begin{aligned}
0 &= \sum_{p=2}^{n+1} g_p \sum_{q=0}^{[(p-1)/2]} \frac{p-2q}{p} \binom{p}{2q} \binom{2q}{q} \left(\frac{a+b}{2}\right)^{p-2q-1} \left(\frac{b-a}{4}\right)^{2q}, &(2.189a)\\
\hat{S} &= \sum_{p=2}^{n+1} g_p \sum_{q=1}^{[p/2]} \frac{q}{p} \binom{p}{2q} \binom{2q}{q} \left(\frac{a+b}{2}\right)^{p-2q} \left(\frac{b-a}{4}\right)^{2q}, &(2.189b)\\
W_{\text{eff,pert}} &= N \sum_{p=2}^{n+1} g_p \sum_{q=0}^{[p/2]} \frac{1}{p} \binom{p}{2q} \binom{2q}{q} \left(\frac{a+b}{2}\right)^{p-2q} \left(\frac{b-a}{4}\right)^{2q}\\
&\quad -2N\hat{S} \ln\left(\frac{b-a}{4\Lambda_u}\right). &(2.189c)
\end{aligned}$$

These are the same equations as obtained on the gauge theory side in [5] using factorisation of Seiberg-Witten curves and the ILS linearity principle (for other examples of Seiberg-Witten theory in the matrix model framework inspired by the Dijkgraaf-Vafa conjecture see e.g. [53], [54], [55], [56], [57], [58], [59], and [60]). Here we should identify $(a+b)/2$ with $z = \text{Tr}(\Phi)/N$ and $(b-a)/4$ with $\Lambda(z, \hat{S}, g_{p\neq 2})$. In this way (2.189a) corresponds to integrating out z (using (2.126)) and (2.189b) corresponds to integrating in $\hat{S}$ (i.e. (2.133)). This gives the relation between the matrix model and the gauge theory in this proof.

2.8 Exact Superpotentials

In this section we will use the Dijkgraaf-Vafa conjecture and the exact solution of the matrix model obtained in the last section to find the exact effective glueball superpotential in the case of a cubic tree-level superpotential. We will also briefly discuss the even tree-level superpotential.

2.8.1 Cubic Tree-Level Superpotential

We consider the case of a cubic tree-level superpotential:

$$W_{\text{tree}} = \text{Tr}\left(\frac{1}{2}m\Phi^2 + \frac{g}{3}\Phi^3\right). \tag{2.190}$$

The planar free energy in the matrix model was first obtained in [52]. The effective superpotential in the gauge theory has been obtained already in [61], and later in e.g. [17] and [15, 16] using the Dijkgraaf-Vafa conjecture.

The critical points for the cubic potential are 0 and $-m/g$. Let us choose the vacuum where all eigenvalues are 0 and the gauge symmetry thus is unbroken. Plugging into the single-cut solution (2.189) gives:

$$\begin{aligned}
0 &= mz + gz^2 + 2g\Delta, &(2.191a)\\
\hat{S} &= m\Delta + 2gz\Delta, &(2.191b)\\
W_{\text{eff,pert}} &= N\left(m\left(\frac{1}{2}z^2 + \Delta\right) + g\left(\frac{1}{3}z^3 + 2z\Delta\right)\right) - N\hat{S}\ln\left(\frac{\Delta}{\Lambda_u^2}\right). &(2.191c)
\end{aligned}$$

Here we have defined $z = (a+b)/2$ and $\Delta = ((b-a)/4)^2$ as in [42]. We can use (2.191a) to solve for Δ. After a couple of rewritings the result is (as also obtained in [42] using

Seiberg-Witten curves):

$$\hat{S} = -\frac{m^3}{2g^2}\frac{gz}{m}\left(1+\frac{gz}{m}\right)\left(1+2\frac{gz}{m}\right), \tag{2.192a}$$

$$W_{\text{eff}} = N\frac{m^3}{g^2}\left(\frac{1}{2}\left(\frac{gz}{m}\right)^2+\frac{1}{3}\left(\frac{gz}{m}\right)^3\right)+N\hat{S}\ln\left(1+2\frac{gz}{m}\right) + N\hat{S}\left(1-\ln\left(\frac{\hat{S}}{m\Lambda_u^2}\right)\right). \tag{2.192b}$$

In (2.192) we have actually found the Veneziano-Yankielowicz superpotential by solving the matrix model exactly! This result has been pointed out in [62]. It is in accordance with the result we obtained in section 2.7.2. That we get the exact right form also tells us that our normalisation of the measure (essential in equation (2.169)) is well-chosen. Naturally, in this exact solution of the matrix model there is now no need to add the Veneziano-Yankielowicz superpotential by hand as in (2.14a) and hence we have exchanged $W_{\text{eff,pert}}$ with the more suitable W_{eff} in (2.192b). We emphasise that the Dijkgraaf-Vafa conjecture for the relation between the matrix model and the gauge theory has not been proven for the $-N\hat{S}\ln\hat{S}$ term. In the diagrammatic proof that we will give in chapter 3 only the perturbative behaviour of $\hat{S}$ is captured. And in the generalised Konishi anomaly proof [7], which in fact relates the exact superpotential to the exact free energy in the matrix model, the relation is only proven for terms depending on the couplings g_k in the tree-level superpotential.[55] Thus the Dijkgraaf-Vafa conjecture does not give a derivation of the Veneziano-Yankielowicz superpotential as also discussed above.

Let us briefly look at the case where we also consider the abelian part of the supersymmetric gauge field strength (section 2.1.2). We should here remember the extra double derivative term in (2.20). Let us assume that (2.20) can be used for the exact free energy of the matrix model found above, and let us consider the non-perturbative term. Due to the double derivative term in (2.20) we should then have an extra term which is the derivative of the Veneziano-Yankielowicz superpotential (since (2.192b) is the matrix model free energy differentiated once). Further, we should remember to replace $\hat{S}$ with the full glueball superfield S. However, when expanding S in $\hat{S}$ and w_α as in (2.19) we will have cancellations such that the two non-perturbative terms simply gives the Veneziano-Yankielowicz superpotential expressed in the traceless glueball superfield $\hat{S}$ as in (2.192b). That is, we have no dependence on the abelian part w_α. This is as expected since this part is decoupled, as discussed in section 2.1.2, and IR free. The cancellation is just the same as we saw for the gauge coupling in (2.21) which is zero in the unbroken case. As we noted in section 2.1.2 it is not clear how to add the non-perturbative part by hand in the unbroken case when we consider the abelian parts. However, in the light of the above we could guess that the full superpotential is obtained by the extension of the Dijkgraaf-Vafa conjecture where we use (2.20) to give the full effective superpotential using the exact (non-perturbative) solution of the matrix model.

Since (2.192a) is a cubic equation we can solve it exactly as:

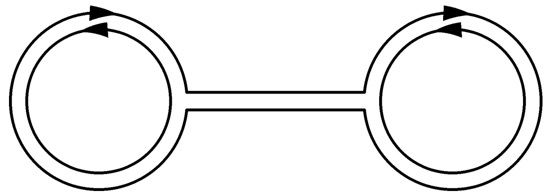
$$\frac{gz}{m} = A + \frac{1}{12A} - \frac{1}{2}, \qquad A \equiv \sqrt[3]{-\frac{1}{2}\frac{g^2}{m^3}\hat{S}+\sqrt{-\frac{1}{12^3}+\frac{1}{4}\left(\frac{g^2}{m^3}\hat{S}\right)^2}}. \tag{2.193}$$

Here we have chosen the solution that satisfies[56] $\lim_{\hat{S}\to 0} z = 0$ thus ensuring $0 \in [a, b]$ i.e.

[55] Naturally, it is captured by the non-perturbative methods in the proof using Seiberg-Witten curves [5].

[56] Since $\hat{S} \equiv g_{\text{m}}$, the matrix model classical limit, $g_{\text{m}} \to 0$, corresponds to $\hat{S} \to 0$.

Figure 2.7:



The dumbbell diagram contributing to the second order term in $W_{\text{eff,pert}}(\hat{S})$.

we have chosen the cut around the critical point 0. Naturally, there is another solution corresponding to $\lim_{\hat{S}\to 0} z = -m/g$. Inserting (2.193) into (2.192b) thus explicitly gives us the exact effective superpotential.

Using (2.193) we can expand the perturbative part of W_{eff} in a power series in $\hat{S}$. To third order the result is:

$$W_{\text{eff,pert}} = -N\frac{m^3}{g^2}\left(2\left(\frac{g^2}{m^3}\hat{S}\right)^2 + \frac{32}{3}\left(\frac{g^2}{m^3}\hat{S}\right)^3 + \mathcal{O}\left(\hat{S}^4\right)\right). \tag{2.194}$$

We can compare this with the results that we get from the matrix model diagrams. The contribution from a given diagram is given by (2.154) and a combinatorial factor. Since there is no diagram with less than three index loops the first non-zero term must be of order 3 in the matrix model free energy and hence order 2 in $W_{\text{eff,pert}}$. There are two types of diagrams with three index loops. Firstly we find the diagram given in figure 2.3 which with our choice of tree-level potential has a combinatorial weight of 1/6.[57] Secondly, we have the diagram shown in figure 2.7 which has the combinatorial weight 1/2. The total contribution from the two diagrams is then $-\frac{2}{3}m^{-3}g^2(g_{\text{s}}N')^3$ where the minus sign stems from the definition of the free energy. This gives the contribution $-2Nm^{-3}g^2\hat{S}^2$ to $W_{\text{eff,pert}}$ using the Dijkgraaf-Vafa conjecture (2.14b) – in agreement with (2.194).

We note that the form of the expansion of $W_{\text{eff,pert}}(\hat{S})$ is completely determined by the analysis we made in section 2.5.4. Remembering the results obtained there we must demand that W_{eff} has a power series expansion in g. Further we can use the symmetries in table 2.2. This tells us that we have a global U(1) symmetry under which W_{eff} and $\hat{S}$ are invariant, m has charge -2, and g has charge -3. Secondly we have a $\text{U}(1)_R$ symmetry under which m, g, $\hat{S}$ and W_{eff} has charge 2. This constrains the form of $W_{\text{eff,pert}}$ to be a power series expansion of the form:

$$W_{\text{eff,pert}} = \frac{m^3}{g^2}\sum_k c_k\left(\frac{g^2}{m^3}\hat{S}\right)^k. \tag{2.195}$$

This is exactly the form we get as we can see from (2.192b) and (2.193). We note that $W_{\text{eff,pert}}$ is singular in $m \to 0$ which also is expected since the chiral field has been integrated out. The possibility of inverse powers of m is also the reason that we do not have to restrict

[57] When counting the number of diagrams one should remember that only the planar diagrams contribute. If the counting instead is done in the adjoint single-line notation one should remember that the vertices are proportional to $\text{Tr}(T_aT_bT_c)$. Thus the legs in the vertices can not be permuted arbitrarily, but only cyclically, and we should only count the subgroup of diagrams that corresponds to planar diagrams in the double line notation.

to linear terms in the R-charged variables as in (2.119). We thus see that the essence of the Dijkgraaf-Vafa conjecture for $W_{\text{eff,pert}}$ is that it determines the coefficients in (2.195): The coefficients are obtained by combinatorial counting of planar diagrams in the related matrix model.

2.8.2 Even Tree-Level Superpotential

Let us briefly examine the case of an even tree-level superpotential. We consider the unbroken case where we expand around the zero critical point. Here we have a $\lambda \mapsto -\lambda$ symmetry in the eigenvalues. Thus $a = -b$ in (2.189) and the solution reduces to:

$$\hat{S} = \frac{1}{2}\sum_{p\geq 1} g_{2p}\binom{2p}{p}\left(\frac{b-a}{4}\right)^{2p}, \tag{2.196a}$$

$$W_{\text{eff,pert}} = N\sum_{p\geq 1}\frac{g_{2p}}{2p}\binom{2p}{p}\left(\frac{b-a}{4}\right)^{2p} - 2N\hat{S}\ln\left(\frac{b-a}{4\Lambda_u}\right). \tag{2.196b}$$

This is actually the general solution one finds for a SU(N) gauge group as also obtained already in [61].

2.9 Nilpotency of the Glueball Superfield

As noted in section 2.1.2 the glueball superfield defined in (2.5) or (2.17) is a sum of terms proportional to $\mathcal{W}^{\alpha a}\mathcal{W}^a_\alpha$ for $a = 1, \ldots, \dim(G)$. Here G is SU(N) for $\hat{S}$, U(N) for S or we can think of the glueball superfield for a general gauge group G. Since $\mathcal{W}^{\alpha a}$ is Grassmannian, we see that the glueball superfield classically is nilpotent:

$$S^{\dim(G)+1} = 0. \tag{2.197}$$

This even holds true in perturbation theory as can be seen using R-symmetries and dimensional analysis.

However, we can say something even more powerful if we consider the *chiral ring*. First we define chiral operators as gauge invariant operators annihilated by the supercharge $\bar{Q}_{\dot{\alpha}}$. One can show that the products of chiral operators are again chiral operators. By considering chiral operators modulo terms like $\{\bar{Q}_{\dot{\alpha}}, \ldots\}$ (where the dots indicate some gauge invariant operator) the equivalence classes form a ring which, per definition, is the chiral ring. The point is that one can show that in a supersymmetric vacuum, which we will assume in the following, the expectation value of a product of chiral operators does not depend on the chosen representatives. In fact, the expectation values factorise and are space-time independent. There is a one-to-one correspondence to chiral superfields by noting that the lowest component of a chiral superfield is a chiral operator. The chiral ring is then obtained by considering the chiral fields modulo chirally exact terms of the form $\bar{D}\bar{D}F$ where F is a superfield for which $\bar{D}^{\dot{\alpha}}F$ is gauge invariant. Now, in the chiral ring we have classically and in perturbation theory [7] (conjectured for all groups, but certainly true for the classical groups):

$$S^h = 0 \qquad \text{(Chiral ring, perturbation theory)}, \tag{2.198}$$

where h is the dual Coxeter number given in table 2.3. Actually, we also have $S^{h-1} \neq 0$ (anyway for SU(N)) in the chiral ring. The relation (2.198) is changed by non-perturbative effects (as we have seen by instantons) into:

$$S^h = a(G)\Lambda^{3h} \qquad \text{(Chiral ring, exact)}, \tag{2.199}$$

where $a(G)$ is a normalisation depending on the group. As mentioned in section 2.5.7 we have $a = 1$ for $\mathrm{SU}(N)$. Naturally, also the relation (2.197) is changed non-perturbatively, see also [63].

We thus see that we should be very careful when employing the Dijkgraaf-Vafa conjecture for the terms of order h and higher in S. Actually, in [64] (and using the generalised Konishi anomaly method in [65] and [66]) in the case of a $\mathrm{Sp}(k)$ gauge group with matter in the form of a two-index antisymmetric tensor chiral superfield the superpotential was obtained from the matrix model using the general Dijkgraaf-Vafa conjecture (as we will introduce in the next section). The results were compared to superpotentials obtained earlier in gauge theory using the power of holomorphy. It was found that the results agreed up to order k and discrepancies were found at order $h = k + 1$ and higher orders. An even more basic example of discrepancies is found by considering a $\mathrm{U}(1)$ gauge group with adjoint chiral matter. Here we can think of the dual Coxeter number as being zero and indeed we have immediate discrepancy. Using the Dijkgraaf-Vafa conjecture we really can define a superpotential in S via the matrix model whereas there should be no glueball superpotential in ordinary gauge theory (the theory is IR free).

The point here is [8] that the terms in the superpotential of order h or higher depend on the UV completion of the theory. To see this let us in the UV turn on the term:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta \sum_{k\geq h} a_k S^k, \tag{2.200}$$

where the glueball superfields here should be considered quantum mechanically smeared. Due to (2.198) this will not change the action classically (nor perturbatively), however, in the IR the term is relevant due to the non-perturbatively corrected chiral ring relation (2.199). Thus two theories which agree classically can differ quantum mechanically. Thus we can split the superpotential as $W_{\mathrm{eff}}(S) = W_R(S) + W_A(S)$ where W_R consists of the terms of order less than h, and W_A of the terms of order h and greater. W_R is determined unambiguously from the tree-level superpotential by integrating out the matter fields, and the coefficients are given by the Dijkgraaf-Vafa conjecture. W_A, on the other hand, is ambiguous and should really be seen as a part of the definition of the quantum theory – i.e. it depends on the choice of F-term completion for the theory.

In the case of a $\mathrm{U}(N)$ gauge group with adjoint matter as considered in section 2.1 there is a natural way of determining the F-term completion. Simply consider $\mathrm{U}(Nk)$ where k is a (large) positive integer. Here the coefficients of S^n are determined unambiguously up to order Nk by the Dijkgraaf-Vafa conjecture. And actually we see from (2.14) that the dependence on k is a simple multiplicative factor. The F-term completion for a $\mathrm{U}(N)$ theory is then simply defined by noting that the coefficient of S^n is determined by the coefficient in the $\mathrm{U}(Nk)$ theory with $Nk > n$ divided by k. Thus the effective superpotential to any order is obtained from the matrix model using the Dijkgraaf-Vafa conjecture. This can also be done for the other classical groups, but the dependence on N is generally not multiplicative.

There is, however, another approach to F-term completion applicable for any classical group, denoted $\mathrm{G}(N)$, and any matter representation. Instead of $\mathrm{G}(N)$ we consider the larger supergroup $\mathrm{G}(N+k|k)$ (which corresponds to adding k brane/anti-brane pairs). With this group the terms in the superpotential are unambiguous up to order $N + 2k$ and the coefficients are actually independent of k.[58] We can thus take $k \to \infty$ to determine the full superpotential – the F-term completion. In the end we then use that all of the $\mathrm{G}(N + k|k)$ theories have a Higgs branch where in the IR the supergroup is Higgsed down to $\mathrm{G}(N)$. The

[58] This is due to the supertrace in an index loop giving $(N + k) - k = N$.

completion obtained in this way corresponds, in essence, to treating the S's in the different index loops (in the gauge theory) as being distinct.

The $G(N+k|k)$-completion also tells us when we have discrepancies compared to the standard gauge completion. This happens when there are residual instanton effects in the broken part of the group in the Higgsing $\mathrm{G}(N+k|k) \mapsto \mathrm{G}(N)$. By the knowledge of section 2.5.3 a necessary condition for this is that the third homotopy group $\pi_3(\mathrm{G}(N+k|k)/\mathrm{G}(N)) \neq 0$. In the case of a $\mathrm{U}(N)$ gauge group with adjoint matter we only have residual instanton effects for $\mathrm{U}(1+1|1) \mapsto \mathrm{U}(1)$ and this explains the discrepancy we found for the $\mathrm{U}(1)$ theory. The residual instanton effects also explain the discrepancy in the $\mathrm{Sp}(k)$ case considered above. The ambiguity is also investigated in e.g. [67], [68], [69], [70], [71], [72] and [73].

Thus, in conclusion, we can trust the glueball superpotential obtained by the Dijkgraaf-Vafa conjecture up to order h unambiguously, whereas the terms of order h and greater correspond to a choice of F-term completion. In the case of a $\mathrm{U}(N)$ gauge group ($N > 1$) with adjoint matter we can use the Dijkgraaf-Vafa conjecture to any order since the choice of completion here is the natural one. We should then think of S as an unconstrained elementary field and the relation (2.199) as obtained on-shell using the equations of motion.

2.10 The General Dijkgraaf-Vafa Conjecture

Now that we have understood the Dijkgraaf-Vafa conjecture in details in the case of a $\mathrm{U}(N)$ gauge group with a single adjoint chiral field, let us end this chapter by presenting the Dijkgraaf-Vafa conjecture in the case of more general gauge groups and matter representations following [4]. We assume an $\mathcal{N}=1$ supersymmetric theory with a classical gauge group G i.e. a product with factors of $\mathrm{U}(N)$, $\mathrm{SO}(N)$ and $\mathrm{Sp}(k)$. We further assume that the matter content of the theory in the form of chiral fields, Φ^a, allows a double line notation as in section 2.6. Also, we demand that it is possible to add mass terms to the matter fields. We will here think of a single adjoint chiral field and flavours, i.e. chiral fields in the fundamental/anti-fundamental representation, with Yukawa couplings. One could consider more exotic matter, but in general the comparison between the gauge theory and the matrix model should be done diagram by diagram and we do not have a nice relation to the total free energy of the matrix model as below [64]. The matter is described by a tree-level superpotential $W_{\text{tree}}(\Phi^a)$.[59]

The classical supersymmetric vacua for our system are by (2.50) again determined by the critical points of W_{tree}. We will assume a massive supersymmetric vacuum where the gauge group is broken to $\prod_i G_i$ where each of the G_i's is one of the classical groups $\mathrm{U}(N_i)$, $\mathrm{SO}(N_i)$ or $\mathrm{Sp}(k_i) = \mathrm{USp}(N_i = 2k_i)$. All fields should be massive in the vacuum except the $\mathcal{N}=1$ super Yang-Mills part for the gauge group $\prod_i G_i$. Corresponding to each of the G_i's we have a glueball superfield S_i defined analogously to (2.5). We will here ignore the abelian part as in section 2.1.1 – the result when considering this is as in section 2.1.2.

The Dijkgraaf-Vafa conjecture once again tells us that the effective glueball superpotential is determined as a sum of a superpotential $W_{\text{eff,pert}}(S_i)$, perturbative in S_i, and the Veneziano-Yankielowicz superpotential (extending (2.140)):

$$W_{\text{VY}} = \sum_i C(\text{adj}(G_i))\, S_i \left(1 - \ln \frac{S_i}{a_i \Lambda_i^3}\right). \tag{2.201}$$

Here a_i is the normalisation from (2.138) depending on G_i, and $C(\text{adj}(G_i))$ is the quadratic

[59] This should only include single traces as we discussed at the end of section 2.1.4.

invariant for the adjoint representation of G_i which, in a proper normalisation, equals the dual Coxeter number for G_i given in table 2.3.[60]

Furthermore, $W_{\text{eff,pert}}$ is determined by the related matrix model with potential given by $\frac{1}{g_s}W_{\text{tree}}$. Here the matrices can now correspond to all of the classical groups $\mathrm{U}(N')$, $\mathrm{SO}(N')$ and $\mathrm{USp}(N')$, so e.g. for $\mathrm{SO}(N')$ we have real antisymmetric matrices. If we have fundamental matter, we should, naturally, also include integrations for this. In the matrix model we expand around the saddle-point corresponding to the breaking of $G \mapsto \prod_i G_i$ where each G_i has a corresponding N'_i. As explained in section 2.6.2 we can classify the matrix model diagrams topologically. Analogously to section 2.6.3 we can make an expansion of the free energy in powers of the matrix model coupling g_s using the Euler characteristic χ:

$$W_{\text{free}} = \sum_{\chi} g_s^{-\chi}\mathcal{F}_{\chi}\big(g_s N'_i\big) = \frac{1}{g_s^2}\mathcal{F}_{\chi=2}\big(g_s N'_i\big) + \frac{1}{g_s}\mathcal{F}_{\chi=1}\big(g_s N'_i\big) + \mathcal{O}\big(g_s^0\big)\,. \tag{2.202}$$

As we will see in the proof of the Dijkgraaf-Vafa conjecture, it is only diagrams with Euler characteristic $\chi \geq 1$ that contribute. In the case of a $\mathrm{U}(N)$ gauge group with a single adjoint chiral field this means that only the $\chi = 2$ diagrams contribute corresponding to genus $g = 0$. However, in the general case we can also have a $\chi = 1$ contribution. The contribution to $\mathcal{F}_{\chi=1}$ comes from diagrams with two different topologies as we see from (2.152). We have the diagrams with one boundary (stemming from the fundamental matter) which have the topology of the disk D^2 and whose contribution to $\mathcal{F}_{\chi=1}$ we denote $\mathcal{F}_{D^2}$. Secondly, we have the diagrams with one cross-cap with topology of the projective plane $\mathbb{RP}^2$ stemming from the cross-over in the double line propagators of $\mathrm{SO}(N)$ and $\mathrm{USp}(N)$, see figure 2.5. The contribution from these diagrams to $\mathcal{F}_{\chi=1}$ is denoted $\mathcal{F}_{\mathbb{RP}^2}$. We thus have:

$$\mathcal{F}_{\chi=1} \equiv \mathcal{F}_{D^2} + \mathcal{F}_{\mathbb{RP}^2}. \tag{2.203}$$

We have to distinguish between these two contributions since they have different weights in the Dijkgraaf-Vafa conjecture, even though the Euler characteristic is the same. To be consistent with the notation depending on the topology we also define $\mathcal{F}_{S^2} \equiv \mathcal{F}_{\chi=2} = \mathcal{F}_{g=0}$.

The relation to the gauge theory is similarly to (2.13) through $2C(\text{fund})S_i \equiv g_s N'_i$ and we can state the Dijkgraaf-Vafa conjecture as (for the traceless glueball superfields):

Dijkgraaf-Vafa conjecture; General case.

$$W_{\text{eff}}(S_i) = W_{\text{eff,pert}}(S_i) + W_{\text{VY}}(S_i), \tag{2.204a}$$

$$W_{\text{eff,pert}}(S_i) = \sum_i N_i \frac{\partial \mathcal{F}_{S^2}(S'_i)}{\partial S'_i} + 4\mathcal{F}_{\mathbb{RP}^2}(S'_i) + \mathcal{F}_{D^2}(S'_i), \tag{2.204b}$$

$$S'_i \equiv g_s N'_i, \quad S'_i = 2C(\text{fund})S_i. \tag{2.204c}$$

As we will see in section 3.2.2 the reason for including $C(\text{fund})$ is that it appears in the definition of S in (2.131).[61] For a $\mathrm{U}(N)$ gauge with adjoint matter and $C(\text{fund}) = 1/2$ this, naturally, reduces to (2.14).

This form of the conjecture is corrected from the original proposal in [4] in the case of $\mathrm{SO}(N)$ and $\mathrm{USp}(N)$ gauge groups as was first found in [74] and [75] (also investigated in [76] and [77]). There it is also shown (using the loop equations) that $\mathcal{F}_{\mathbb{RP}^2} = \mp\frac{1}{2}\partial\mathcal{F}_{S^2}/\partial S$

[60] We do not include the small N exceptions as in (2.142) since our $\mathrm{G}(N+k|k)$-completion discussed in section 2.9 has no small N exceptions because we take k large.

[61] We note that S'_i is invariant under the scaling of the gauge group generators discussed at the end of section 1.3.4. This means that $W_{\text{eff,pert}}$ is invariant under such scalings as it should be.

with minus for SO(N) and plus for USp(N). The addition of fundamental matter was first done in [78] (see also [55], [56], [60], [79], [80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90], [91], [92], [93], [94], [95], and [96] for the U(N)-case and [97], [98], [99], [100], [101], and [102] for the SO(N) and USp(N)-case). Let us finally mention that the reduction to zero-momentum modes holds true for any gauge group and any (massive) representation as shown in [8].

Chapter 3

Diagrammatic Proof of the Dijkgraaf-Vafa Conjecture

In this chapter we will present the diagrammatic proof for the form of $W_{\text{eff,pert}}$ in the Dijkgraaf-Vafa conjecture. Following [6] the strategy will be as follows: Using holomorphy we will choose the anti-chiral part of the action suitably and then integrate it out. This gives us an effective chiral action for which we can develop perturbative methods. In the case of a classical gauge group and a single adjoint chiral field we then show that the path integral reduces to zero-momentum modes and that we can relate the diagrams to matrix model diagrams in the conjectured way.

At the end of the chapter we will show the localisation to zero-momentum modes for general gauge groups and matter representations.

3.1 Setup for Perturbation Theory

3.1.1 Gauge Covariant Notation

In the following it will be convenient to use a gauge covariant notation for the superspace derivatives and fields. We will use the so-called gauge chiral representation (described in [103]). The point is to define derivatives ∇_A transforming under gauge transformations with the chiral superfield $\Lambda\big(x,\theta,\bar{\theta}\big)$ (section 1.3.4) as:

$$\nabla_A \mapsto e^{-i\Lambda}\nabla_A e^{i\Lambda}. \tag{3.1}$$

Using the transformation property (1.49) of the vector superfield $e^{2V} \mapsto e^{-i\Lambda^\dagger}e^{2V}e^{i\Lambda}$, Λ being chiral, and $\Lambda^\dagger$ being anti-chiral we get the covariant derivatives:

$$\nabla_\alpha \equiv e^{-2V}D_\alpha e^{2V}, \quad \bar{\nabla}_{\dot{\alpha}} \equiv \bar{D}_{\dot{\alpha}}, \quad \nabla_{\alpha\dot{\alpha}} \equiv -i\{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\}. \tag{3.2}$$

We note that $\bar{D}_{\dot{\alpha}}$ is the conjugate of D_α whereas the same is not true for $\bar{\nabla}_{\dot{\alpha}}$ and ∇_α. Furthermore, we see that:

$$\nabla_\alpha = D_\alpha + e^{-2V}\left(D_\alpha e^{2V}\right), \tag{3.3}$$

where the last term gives the gauge connections.

Analogous to the chiral fields we define the *covariantly* chiral fields as being annihilated by the covariant derivative $\bar{\nabla}_{\dot{\alpha}}$. The covariantly chiral fields are, naturally, simply the chiral fields. Correspondingly we define the covariantly anti-chiral fields $\Phi^\sharp$ as being annihilated by $\bar{\nabla}_{\dot{\alpha}}$, however, acting from the right. Given a chiral field Φ these take the form:

$$\Phi^\sharp \equiv \Phi^\dagger e^{2V}, \quad \Phi^\sharp \overleftarrow{\bar{\nabla}}_\alpha = \Phi^\dagger \overleftarrow{\bar{D}}_\alpha e^{2V} = 0. \tag{3.4}$$

This also means that $\Phi^\sharp$ transforms covariantly as $\Phi^\sharp \mapsto \Phi^\sharp e^{i\Lambda}$.

As an analogue of (2.70) we define for a chiral field Φ:

$$\Box_+ \Phi \equiv \frac{1}{16}\bar{\nabla}\bar{\nabla}\nabla\nabla\Phi. \tag{3.5}$$

Using that (by simple calculation)

$$[\bar{\nabla}_{\dot{\alpha}}, \nabla_\alpha{}^{\dot{\alpha}}] = i8\mathcal{W}_\alpha, \tag{3.6}$$

we easily get:

$$\Box_+ \Phi = \frac{1}{8}\nabla^\alpha{}_{\dot{\alpha}}\nabla_\alpha{}^{\dot{\alpha}}\Phi - \frac{1}{2}\mathcal{W}^\alpha\nabla_\alpha\Phi - \frac{1}{2}\nabla^\alpha\mathcal{W}_\alpha\Phi. \tag{3.7}$$

Defining

$$\Box_{\text{cov}} = \frac{1}{8}\nabla^\alpha{}_{\dot{\alpha}}\nabla_\alpha{}^{\dot{\alpha}}, \tag{3.8}$$

which for $V = 0$ simply is $\Box$, and rewriting the last differentiation in (3.7), where ∇^α works on both $\mathcal{W}_\alpha$ and Φ, we get:

$$\Box_+ \Phi = \Big(\Box_{\text{cov}} - \mathcal{W}^\alpha\nabla_\alpha - \frac{1}{2}\left(\nabla^\alpha\mathcal{W}_\alpha\right)\Big)\Phi. \tag{3.9}$$

Here we have used that ∇^α obeys the Leibnitz rule, and we should remember that when $\nabla^\alpha = e^{-2V}D^\alpha e^{2V}$ works on $\mathcal{W}_\alpha$, then V should be in the adjoint representation.

For a real representation, e.g. the adjoint representation which is the case of our interest, we have $V^T = -V$ and hence $(\Phi^\sharp)^T = e^{-2V}\bar{\Phi}$. $\Phi^\sharp$ is then annihilated from the left: $\nabla_\alpha(\Phi^\sharp)^T = 0$. Analogously to (3.5) we then define in a real representation:

$$\Box_- \left(\Phi^\sharp\right)^T \equiv \frac{1}{16}\nabla\nabla\bar{\nabla}\bar{\nabla}\left(\Phi^\sharp\right)^T. \tag{3.10}$$

On covariantly chiral fields we then see:[1]

$$\nabla\nabla\Box_+\Phi = \Box_-\nabla\nabla\Phi, \tag{3.11}$$

since $\nabla\nabla\Phi$ is covariantly anti-chiral. We note that, by definition, $\Box_+$ maps chiral fields to chiral fields and $\Box_-$ maps covariantly anti-chiral fields to covariantly anti-chiral fields.

3.1.2 Anti-Chiral Part

Let us consider an $\mathcal{N} = 1$ supersymmetric gauge theory with arbitrary chiral matter given by the chiral field Φ. We will assume that the Lagrangian is renormalisable with the exception that we allow arbitrary tree-level superpotentials, W_{tree}. We want to determine the perturbative part of the effective glueball superpotential, $W_{\text{eff,pert}}$, (discussed in section 2.5.8) obtained by integrating out the chiral superfield while treating the vector superfield as an external background field. Thus the relevant part of the action is given by:

$$S(\Phi, \bar{\Phi}) = A\int \mathrm{d}^4x\mathrm{d}^4\theta\Phi^\dagger e^{2V}\Phi + \left(\int \mathrm{d}^4x\mathrm{d}^2\theta W_{\text{tree}}\left(\Phi\right) + \text{c.c.}\right). \tag{3.12}$$

[1] There is a typo in this equation in [6].

Here we have allowed for a normalisation A of the Kähler term as we e.g. have in the case of $\mathcal{L}_{\mathrm{U}(N)}$ in (2.26). We will demand that the supersymmetric vacuum is massive and for now consider the case of an unbroken gauge group. For convenience, we will assume that Φ transforms in a real representation thus allowing the mass term $\frac{1}{2}m\Phi^T\Phi$. However, our results until section 3.2 are valid in any representation.

The main point in this section is that $W_{\mathrm{eff,pert}}$ can only depend holomorphically on the couplings in W_{tree} as explained in section 2.5.1. In the notation of (2.15) what we want to determine is then Z_{holo}. We can thus treat both Φ and $\bar{\Phi}$, and $W_{\mathrm{tree}}(\Phi)$ and $\overline{W_{\mathrm{tree}}}(\bar{\Phi})$ independently and we can choose the form of $\overline{W_{\mathrm{tree}}}(\bar{\Phi})$ freely. We choose:[2]

$$\overline{W_{\mathrm{tree}}}(\bar{\Phi}) = \frac{1}{2}\bar{m}\Phi^\dagger\bar{\Phi} = \frac{1}{2}\bar{m}\bar{\Phi}^T\bar{\Phi}. \tag{3.13}$$

Thus the action is quadratic in $\bar{\Phi}$ and we can simply integrate it out by completing the square. However, before we do this let us remark that we can scale the normalisation A in (3.12) away by taking $\bar{\Phi} \mapsto \bar{\Phi}/A$ and $\bar{m} \mapsto A^2\bar{m}$. Since the final result does not depend on $\bar{m}$, we can choose $\bar{m}$ freely and thus absorb the normalisation A in this way. This was used in section 2.1.4 to note that there is no need for the $\mathcal{N} = 2$ supersymmetry of the Lagrangian without the tree-level superpotential in section 2.1.1 and 2.1.2.

Switching to the gauge covariant notation introduced in the last subsection, the part of the action depending on $\bar{\Phi}$ can then be written:

$$\Delta S = \int \mathrm{d}^4x\mathrm{d}^4\theta\Phi^\sharp\Phi + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\frac{\bar{m}}{2}\Phi^\sharp\big(\Phi^\sharp\big)^T, \tag{3.14}$$

where we have used that $\Phi^\sharp\big(\Phi^\sharp\big)^T = \Phi^\dagger e^{2V}e^{2V^T}\bar{\Phi} = \Phi^\dagger\bar{\Phi}$ in our real representation. We could change the functional integration over $\bar{\Phi}$ to an integration over $\Phi^\sharp$. In the next section we will change variables from Φ to $e^{2V}\Phi$ so the Jacobian under $\mathcal{D}\Phi\mathcal{D}\bar{\Phi} \mapsto \mathcal{D}\big(e^{2V}\Phi\big)\mathcal{D}\big(e^{-2V}\bar{\Phi}\big)$ should be independent of V in analogy with (2.112) since there should be no anomalies in a real representation.

We can rewrite the last term of (3.14) as:

$$\int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\frac{\bar{m}}{2}\Phi^\sharp\big(\Phi^\sharp\big)^T = -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^4\theta\frac{\bar{m}}{2}\Phi^\sharp\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T, \tag{3.15}$$

where $\Box_+^{-1}$, naturally, should be defined modulo its kernel. To prove (3.15) we use the conjugated analogue of (1.37) to replace $\int \mathrm{d}^2\theta$ with $-\frac{1}{4}DD$. Then by the Leibnitz rule for D_α:

$$\frac{\bar{m}}{2\cdot 16}DD\Phi^\sharp\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T = \frac{\bar{m}}{2\cdot 16}\Phi^\dagger DDe^{2V}\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T = \frac{\bar{m}}{2\cdot 16}\Phi^\sharp\nabla\nabla\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T. \tag{3.16}$$

Finally using (3.11) to write $(\Box_+)^{-1} = (\nabla\nabla)^{-1}(\Box_-)^{-1}\nabla\nabla$ and using (3.10) to note that $(\Box_-)^{-1}\nabla\nabla\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T = 16\big(\Phi^\sharp\big)^T$, we get the wanted result.

We can then complete the square as:

$$\Delta S = \int \mathrm{d}^4x\mathrm{d}^4\theta\bigg\{-\frac{\bar{m}}{8}\left(\Phi^\sharp - \frac{1}{4\bar{m}}(\nabla\nabla\Phi)^T\right)\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\left(\Phi^\sharp - \frac{1}{4\bar{m}}(\nabla\nabla\Phi)^T\right)^T + \frac{1}{8\cdot 16\bar{m}}(\nabla\nabla\Phi)^T\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\nabla\nabla\Phi\bigg\}. \tag{3.17}$$

[2]One might think that it would easier to take $\overline{W_{\mathrm{tree}}}(\bar{\Phi}) = 0$ and then calculate the partition function. However, the diagrammatic method that we will use turns out to be easiest [6].

The only non-trivial part in completing the square is the term:

$$\frac{1}{4\cdot 8}(\nabla\nabla\Phi)^T\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T = \frac{1}{32}\big(DD(e^{2V}\Phi)^T\big)e^{2V}\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T$$
$$= \frac{1}{32}\Phi^T\nabla\nabla\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T, \quad (3.18)$$

where we in the last line have integrated DD by parts which is allowed since $\mathrm{d}^2\theta D \sim DDD = 0$. As above, $\nabla\nabla(\Box_+)^{-1}\bar{\nabla}\bar{\nabla}$ can be replaced with 16 to show that the term reduces to the wanted $\frac{1}{2}\Phi^\sharp\Phi$.

Using (3.17) we can integrate $\Phi^\sharp$ out by translating the integration variable to $\Phi^\sharp - \frac{1}{4\bar{m}}(\nabla\nabla\Phi)^T$. The Gaussian integration then gives a factor depending on the determinant of $(\Box_+)^{-1}\bar{\nabla}\bar{\nabla}$ which seems to depend on V. However, we do note that according to the method of unconstrained superfields introduced in section 2.3.1 we should really integrate over $\bar{\Pi}$ defined by (the complex conjugate of) equation (2.75) i.e. $\bar{\Phi} = DD\bar{\Pi}$. Analogous to the derivations in equations (3.16) and (3.18) we can write:

$$-\frac{\bar{m}}{8}\Phi^\sharp\frac{1}{\Box_+}\bar{\nabla}\bar{\nabla}\big(\Phi^\sharp\big)^T = -2\bar{m}\Pi^\dagger DD\bar{\Pi}. \quad (3.19)$$

Here the Gaussian integration just contributes with a constant independent of the background field V which we can disregard.

The integrating out procedure leaves us with the last term in (3.17) as a contribution to the action for Φ. In this term we can immediately use the definition (3.5) of $\Box_+$ to see that it cancels out. Using (1.37) we can then rewrite the term to a $\int \mathrm{d}^2\theta$-term:

$$\frac{1}{8\bar{m}}\int \mathrm{d}^4x\mathrm{d}^2\theta\left(-\frac{1}{4}\bar{D}\bar{D}\right)\Phi^T\nabla\nabla\Phi = -\frac{1}{2\bar{m}}\int \mathrm{d}^4x\mathrm{d}^2\theta\Phi^T\Box_+\Phi, \quad (3.20)$$

where we again used the definition of $\Box_+$. Thus the end result of this subsection is (using (3.9)):

$$Z_{\text{holo}} = \int \mathcal{D}\Phi e^{iS(\Phi)}, \quad (3.21)$$

with

$$S(\Phi) = \int \mathrm{d}^4x\mathrm{d}^2\theta\left\{\frac{-1}{2\bar{m}}\Phi^T\left(\Box_{\text{cov}} - \mathcal{W}^\alpha\nabla_\alpha - \frac{1}{2}\left(\nabla^\alpha\mathcal{W}_\alpha\right)\right)\Phi + W_{\text{tree}}(\Phi)\right\}. \quad (3.22)$$

3.1.3 Simplifications

In this section we will simplify the result (3.22) by taking into consideration the form of the background and which terms that can contribute.

In order to obtain the glueball superpotential (which has no derivatives) we can think of the background field $\mathcal{W}_\alpha$ as being constant. Following [6] we will further make the following simplifications:

- We choose the background such that $\mathcal{W}_\alpha$ is also covariantly constant i.e.:

$$\nabla_{\alpha\dot{\alpha}}\mathcal{W}_\beta = 0. \quad (3.23)$$

  By the definition of $\Box_{\text{cov}}$ in (3.8) we then conclude that $\mathcal{W}_\alpha$ commutes with $\Box_{\text{cov}}$. From (3.22) we see that we can choose the propagator in perturbation theory to be like $\Box_{\text{cov}}^{-1}$ and we can treat the $\mathcal{W}^\alpha\nabla_\alpha$ term as an interaction giving $\mathcal{W}_\alpha$ insertions. With the assumption (3.23) we can thus move such insertions around in the loops when only considering the space-time part.

- We drop the term $\nabla^\alpha \mathcal{W}_\alpha$ in (3.22) since we do not consider such contributions. This naturally calls for a redefinition of Z_{holo} not to include these terms, however, we will see below that in a very simple background this term is not present at all.

- Analogous to the gauge chiral representation in section 3.1.1 we have a gauge anti-chiral representation transforming covariantly with respect to anti-chiral $\bar{\Lambda}$ gauge transformations. Here the covariantly chiral field is $\Phi' = e^{2V}\Phi$ (and thus $\Phi'^T = \Phi^T e^{-2V}$) and it is annihilated by $\bar{\nabla}'_{\dot\alpha} = e^{2V}\bar{D}_{\dot\alpha}e^{-2V}$. Switching to this basis we can rewrite (3.22) as:
$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta \Big\{ \frac{-1}{2\bar{m}}\Phi'^T \left(\Box'_{\text{cov}} - \mathcal{W}'^\alpha D_\alpha\right)\Phi' + W_{\text{tree}}(\Phi') \Big\}, \tag{3.24}$$
where the point is that we have D_α instead of ∇_α. Here $\Box'_{\text{cov}} = e^{2V}\Box_{\text{cov}}e^{-2V}$, $\mathcal{W}'_\alpha = e^{2V}\mathcal{W}_\alpha e^{-2V}$ and we have used the gauge invariance of the tree-level superpotential to write $W_{\text{tree}}(\Phi) = W_{\text{tree}}(\Phi')$. We can then change the functional integral to Φ' and drop the primes since the glueball superfield depends on $\mathrm{Tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha) = \mathrm{Tr}(\mathcal{W}'^\alpha\mathcal{W}'_\alpha)$. However, let us note that we also can obtain this reduction to D_α simply using (3.3) and dropping the term $\mathcal{W}^\alpha e^{-2V}\left(D_\alpha e^{2V}\right)$ since by (1.50) and (1.56) we see that (in Wess-Zumino gauge) this term always contains at least one $\bar\theta$ and hence does not contribute to the $\int \mathrm{d}^2\theta$-integral.

- Finally we can replace $\Box_{\text{cov}}$ with the usual $\Box = \partial_\mu\partial^\mu$. This is because the connection terms in $\Box_{\text{cov}}$ generally only appear in the effective action in order to covariantise derivative terms – but we have none of these. However, to see that we can drop the connection terms in general requires a detailed covariant supergraph analysis or the assumption of a very simple background as below.

Thus the action we will use for our calculation of Z_{holo} is:

$$S(\Phi) = \int \mathrm{d}^4x\mathrm{d}^2\theta \Big\{ \frac{-1}{2\bar{m}}\Phi^T \left(\Box - \mathcal{W}^\alpha D_\alpha\right)\Phi + W_{\text{tree}}(\Phi) \Big\}. \tag{3.25}$$

Here we actually can replace D_α with $\partial/\partial\theta^\alpha$ using the definition (1.29) since we again can drop terms depending on $\bar\theta$.

The above reduction can also be seen by choosing the simplest possible non-trivial background as done in [15, 16]. Simply set $A_\mu = 0$, take the gaugino field λ_α to be constant, choose $\bar\lambda_{\dot\alpha} = 0$, and disregard the auxiliary field D. Then by (1.50) and (1.57) we have (in Wess-Zumino gauge):
$$V = -i\bar\theta\bar\theta\theta\lambda, \quad \mathcal{W}_\alpha = -i\lambda_\alpha. \tag{3.26}$$
By (3.3) $\nabla_\alpha = D_\alpha + 2\bar\theta\bar\theta\mathcal{W}_\alpha$ and after a short calculation (3.8) then gives:

$$\Box_{\text{cov}} = \Box + 2i\mathcal{W}\sigma^\mu\bar\theta\partial_\mu + 2\bar\theta\bar\theta\mathcal{W}\mathcal{W}. \tag{3.27}$$

Inserting into (3.9) yields:

$$\Box_+ = \Box - \mathcal{W}^\alpha D_\alpha + 2i\mathcal{W}\sigma^\mu\bar\theta\partial_\mu. \tag{3.28}$$

If we drop the last term since it depends on $\bar\theta$, we get (3.25) as wanted.

3.1.4 Perturbation Theory Setup

In this subsection we will develop the perturbation theory based on the action (3.25). The crucial point is that we have been able to integrate $\bar{\Phi}$ out since we are only interested in the F-term. In this way we are only left with $\Phi\Phi$-propagators contrary to the usual supergraphs developed in section 2.3.1. The new Feynman rules are also the reason that we escape the perturbative non-renormalisation theorems from section 2.3.2 and 2.5.2.[3] Since $\bar{\Phi}$ has been integrated out and there is no longer any dependence on $\bar{\theta}$, we can also restrict the superspace to the half-superspace generated by θ and think of Φ as a general superfield in this space. Thus we do not have to worry about imposing chirality by e.g. introducing unconstrained superfields as in (2.75).

The effective action $W_{\text{eff,pert}}$ is obtained from Z_{holo} as in equation (2.16). However, we will here Wick rotate into Euclidean space:

$$Z_{\text{holo}} = \int \mathcal{D}\Phi e^{-S^{(\text{E})}(\Phi)} = e^{-\int \mathrm{d}^4x\mathrm{d}^2\theta W^{(\text{E})}_{\text{eff,pert}}}, \tag{3.29}$$

where the label E denotes Euclidian space. We note that there is a simple sign change between the superpotential in Euclidean and Minkowski space-time:

$$W^{(\text{E})}_{\text{eff,pert}}(S) = -W_{\text{eff,pert}}(S)\,. \tag{3.30}$$

Equation (3.29) tells us that we should consider (minus) the sum of connected Feynman diagrams. In developing the Feynman rules for these diagrams it will be convenient to use the momentum space formulation not only for the space-time part, but also for the fermionic parameters θ_α. To obtain the fermionic Fourier transformation we note that by (C.42):

$$-4\int \mathrm{d}^2\pi e^{\theta\pi} = \theta\theta = \delta^{(2)}(\theta)\,, \tag{3.31}$$

where π_α is the fermionic momentum and we have defined $\int \mathrm{d}^2\pi\, \pi\pi = 1$. Thus the Fourier transformation is given by:

$$f(\theta) \;=\; -4\int \mathrm{d}^2\pi e^{\theta\pi}\tilde{f}(\pi)\,, \tag{3.32}$$

$$\tilde{f}(\pi) \;\equiv\; \int \mathrm{d}^2\theta e^{-\theta\pi} f(\theta)\,. \tag{3.33}$$

In this way the spinorial derivative is replaced by a fermionic momentum:

$$\frac{\partial}{\partial\theta^\alpha} \mapsto \pi_\alpha, \tag{3.34}$$

which is consistent with the hermitian adjoint of the spinorial derivative given in (C.39).

When we construct the propagator we have two choices: Either to see the $\mathcal{W}^\alpha\partial/\partial\theta^\alpha$ term as giving a vertex or to include it as a part of the propagator. We will use the latter and assume that $\mathcal{W}_\alpha$ is not only constant in space-time, but also independent of θ like in the case of the simple background (3.26). This is sufficient to determine the form of $W_{\text{eff,pert}}$. Now Z_{holo} takes the form:

$$Z_{\text{holo}} = e^{-\int \mathrm{d}^4x\mathrm{d}^2\theta\left\{\frac{-1}{2\bar{m}}\Phi^T(\Box - \mathcal{W}^\alpha\partial/\partial\theta^\alpha - m_{\text{E}}\bar{m})\Phi + \text{interactions}\right\}}, \tag{3.35}$$

[3] This should not be misunderstood as the perturbative non-renormalisation theorems being wrong. Rather, the perturbation theory for the field Φ after integrating out $\bar{\Phi}$ captures the form of the non-perturbative corrections from section 2.5.4.

where we have included the mass term in the inverse propagator term. Here $m_{\mathrm{E}} = -m$ is the mass in the Euclidean superpotential where there is a sign change as in (3.30). The sign change in the two other terms in the inverse propagator has been absorbed into $\bar{m}$. The interaction terms are then the remaining terms in $W^{(\mathrm{E})}_{\mathrm{tree}}$. The propagator $\Delta(x,\theta;x',\theta')$ is then determined by:

$$\frac{-1}{\bar{m}}\left(\Box - \mathcal{W}^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} - m_{\mathrm{E}}\bar{m}\right)\Delta\big(x,\theta;x',\theta'\big) = \delta^{(4)}\big(x-x'\big)\,\delta^{(2)}\big(\theta-\theta'\big)\,. \tag{3.36}$$

Using (3.31) and expanding $\delta^{(4)}(x-x')$ similarly we get the propagator:

$$\Delta\big(x,\theta;x',\theta'\big) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}(-4)\int \mathrm{d}^2\pi \frac{\bar{m}}{p^2 + \mathcal{W}^{\alpha}\pi_{\alpha} + m_{\mathrm{E}}\bar{m}} e^{i(x-x')p} e^{(\theta-\theta')\pi}, \tag{3.37}$$

where we have repressed the gauge group representation indices. From this propagator we see that the $\bar{m}$ dependence explicitly cancels out since we can rescale $p^2 \mapsto \bar{m}p^2$ and $\pi_\alpha \mapsto \bar{m}\pi_\alpha$. $\bar{m}$ then cancels in the fraction and since the fermionic Jacobian is the inverse of the bosonic Jacobian, the dependence also cancels here. In the following we can therefore set $\bar{m} = 1$.

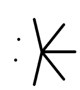
The Feynman rule for the vertices is to multiply with (minus) a coupling times a gauge group invariant tensor and then integrate over x and θ. Since $\mathcal{W}_\alpha$ is merely a (fermionic) constant, such an integration is over the exponentials from the propagators (3.37) and as usual this gives a delta function $(2\pi)^4\,\delta^{(4)}(\sum_i p_i)\,(-\frac{1}{4})\,\delta^{(2)}(\sum_i \pi_i)$ giving bosonic and fermionic momentum conservation at the vertices. However, since we consider connected diagrams one of these delta functions simply is $(2\pi)^4\,\delta^{(4)}(0)\,(-\frac{1}{4})\,\delta^{(2)}(0)$ which expresses the fact that the total incoming momentum in the diagram is zero. This delta function thus equals an integration $\int \mathrm{d}^4x\mathrm{d}^2\theta$ and this is our overall integration in the effective action. We should not worry about the integrand being x and θ independent since what we want is the form of the effective Lagrangian and here it suffices to take $\mathcal{W}^\alpha$ constant. We can now use the remaining delta functions to remove some of the momentum integrations leaving us with L integrations and thus $4L$ independent bosonic loop momenta and $2L$ fermionic loop momenta where:

$$L = E - V + 1. \tag{3.38}$$

Here E is the number of propagators and V is the number of vertices. In conclusion, we are left with the momentum space Feynman rules:


p^μ, π_α

$$= \frac{1}{p^2 + \mathcal{W}^{\alpha}\pi_{\alpha} + m_{\mathrm{E}}}, \tag{3.39}$$



$$= -g_{\mathrm{E},k} f_{a_1\cdots a_k}, \tag{3.40}$$

$$W^{(\mathrm{E})}_{\mathrm{eff,pert}} = -\sum \text{connected diagrams}. \tag{3.41}$$

Here $g_{\mathrm{E},k} = -g_k$ is the Euclidean coupling (with a sign change as above) for the interaction term which has the form $g_{\mathrm{E},k} f_{a_1\cdots a_k}\Phi^{a_1}\ldots\Phi^{a_k}$.

The trick is now to write the propagators, indexed by $i = 1, \ldots, E$, as:

$$\int_0^\infty \mathrm{d}s_i e^{-s_i\left(p_i^2 + \mathcal{W}^{\alpha}\pi_{i\alpha} + m_{\mathrm{E}}\right)} = \frac{1}{p_i^2 + \mathcal{W}^{\alpha}\pi_{i\alpha} + m_{\mathrm{E}}}, \tag{3.42}$$

where s_i is the so-called Schwinger time variable, and p_i and π_i are the momenta flowing through the i^{th} propagator.[4] The point is that we can now factorise the propagator in the contribution coming from respectively p_i^2, $\mathcal{W}^\alpha \pi_{i\alpha}$ and m_{E}. This gives us a corresponding factorisation of the amplitude, A, of a given diagram:

$$A = n_{\text{symm}} \int_0^\infty \prod_i \mathrm{d}s_i A_{\text{bosonic}}(s_i) A_{\text{fermionic}}(s_i) e^{-\sum_i s_i m_{\mathrm{E}}} \prod_k (-g_{\mathrm{E},k})^{V_k}, \tag{3.43}$$

where n_{symm} is a symmetry factor, A_{bosonic} is the contribution from the space-time momentum integrations, $A_{\text{fermionic}}$ is the contribution from the fermionic momenta which also holds all the gauge group index contractions, and V_k is the number of vertices of order k.

It is easy to calculate A_{bosonic}. To this end we introduce the L independent loop momenta p'_a which are related by a matrix L_{ia} to the momenta flowing through the propagators, p_i, as:

$$p_i^\mu = \sum_{a=1}^{L} L_{ia} p_a'^\mu, \quad i = 1, \ldots, E. \tag{3.44}$$

We can then write A_{bosonic} as:

$$\begin{aligned} A_{\text{bosonic}} &= \int \prod_{a=1}^{L} \frac{\mathrm{d}^4 p'_a}{(2\pi)^4} \exp\Big(-\sum\nolimits_i s_i p_i^2\Big) = \int \prod_{a=1}^{L} \frac{\mathrm{d}^4 p'_a}{(2\pi)^4} \exp\Big(-\sum\nolimits_i s_i \Big(\sum\nolimits_a L_{ia} p'_a\Big)^2\Big) \\ &= \int \prod_{a=1}^{L} \frac{\mathrm{d}^4 p'_a}{(2\pi)^4} \exp\Big(-\sum\nolimits_{a,b} p_a'^\mu M_{ab}(s_i) p'_{b\mu}\Big) = \frac{1}{(4\pi)^{2L}} \frac{1}{(\det M(s_i))^2}, \end{aligned} \tag{3.45}$$

where we in the last line have used Gaussian integration and we have introduced the real symmetric matrix:

$$M_{ab}(s_i) = \sum_i s_i L_{ia} L_{ib}. \tag{3.46}$$

What we want to prove is that the dependence on the Schwinger time variables s_i in $A_{\text{fermionic}}(s_i)$ cancels the s_i dependence in A_{bosonic} thus giving us a localisation to zero-momentum modes. Modulo gauge group factors $A_{\text{fermionic}}$ takes the form:

$$A_{\text{fermionic}}(s_i) \sim (-4)^L \int \prod_{a=1}^{L} \mathrm{d}^2 \pi'_a e^{-\sum_i \sum_a s_i \mathcal{W}^\alpha_{(\mathbf{r}_i)} L_{ia} \pi'_{a\alpha}}, \tag{3.47}$$

where we have introduced the fermionic loop momenta π'_a which are related to the propagator momenta like in (3.44):

$$\pi_{i\alpha} = \sum_{a=1}^{L} L_{ia} \pi'_{a\alpha}, \quad i = 1, \ldots, E. \tag{3.48}$$

In (3.47) we have explicitly shown that $\mathcal{W}^\alpha$ depends on the representation. Here $\mathbf{r}_i$ is the representation that flows through the i^{th} propagator, where we allow the representation to depend on the propagator as we saw was the case for a broken gauge group. We see that the cancellation of the dependence on s_i in $A_{\text{fermionic}}$ and A_{bosonic} depends non-trivially on

[4]This is inspired by string theory where we have a cancellation between the factors obtained from integration over the space-time momenta and the integration over fermionic momenta. The Schwinger time variables can be seen as the length of an edge and are thus the field theory limit of the world-sheet moduli from string theory.

Figure 3.1:

The inverse double line propagator from the $\mathcal{W}^\alpha_{(\mathrm{adj})}\pi_\alpha$ term. Crosses denote $\mathcal{W}^\alpha$ insertions.

the gauge group representation. We will show this cancellation for general gauge groups and matter representations in section 3.3. However, we will start by considering the case of classical gauge group with an adjoint chiral field for which we will prove the direct relation to the matrix model given by the Dijkgraaf-Vafa conjecture.

3.2 Reduction to the Matrix Model

3.2.1 Double Line Notation and $\mathcal{W}^\alpha$ Insertions

In this section we will continue the evaluation of equation (3.43) for the amplitude A of a given diagram. But here we restrict the gauge group to be one of the classical gauge groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$ or $\mathrm{Sp}(k) = \mathrm{USp}(N = 2k)$, and the matter is in the form of a single adjoint chiral superfield, Φ. Since we have integrated $\bar{\Phi}$ out, we can think of Φ as a real field and use the double line notation for diagrams introduced in section 2.6. Here the difference is that we have a $\Phi^T \mathcal{W}^\alpha_{(\mathrm{adj})}\pi_{i\alpha}\Phi$ term. To determine the corresponding inverse double line propagator we write the adjoint field as a hermitian matrix $\Phi^a T_a^{(\mathrm{fund})}$ as in (2.145). The term now takes the form $\mathrm{Tr}(\Phi[\mathcal{W}^\alpha \pi_{i\alpha}, \Phi])$ where $\mathcal{W}^\alpha = \mathcal{W}^{a\alpha} T_a^{(\mathrm{fund})}$.[5] The contribution to the inverse propagator $\delta\Delta^{-1} \equiv \Gamma^\alpha \pi_{i\alpha}$ is then determined by:

$$\Phi^i{}_j \left(\Gamma^\alpha\right)^{lj}_{ik} \Phi^k{}_l \equiv \mathrm{Tr}(\Phi[\mathcal{W}^\alpha, \Phi]) = \Phi^i{}_j \left(\left(\mathcal{W}^\alpha\right)^j{}_k \Phi^k{}_i - \Phi^j{}_k \left(\mathcal{W}^\alpha\right)^k{}_i \right), \tag{3.49}$$

giving us the result:

$$\left(\Gamma^\alpha\right)^{lj}_{ik} = \delta^l_i \left(\mathcal{W}^\alpha\right)^j{}_k - \left(\mathcal{W}^\alpha\right)^l{}_i \, \delta^j_k. \tag{3.50}$$

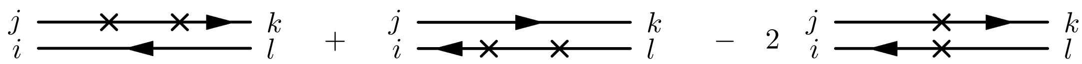
This inverse propagator corresponds in the double line diagrams to insertions of $\mathcal{W}^\alpha$ in the single index lines (figure 3.1) with a sign which is correlated with the single index lines having opposite directions (and hence being parallel or anti-parallel with the direction of the fermionic momentum). By expanding $\mathrm{Tr}(\Phi[\mathcal{W}^\alpha \pi_{i\alpha}, [\mathcal{W}^\alpha \pi_{i\alpha}, \Phi]])$ we see that two of these inverse double line propagators multiply as:

$$\left(\Gamma^\alpha \pi_{i\alpha} \Gamma^\alpha \pi_{i\alpha}\right)^{lj}_{ik} = \left(\Gamma^\alpha \pi_{i\alpha}\right)^{mj}_{in} \left(\Gamma^\alpha \pi_{i\alpha}\right)^{ln}_{mk}. \tag{3.51}$$

This simply tells us to join the double lines in the obvious way and defines the exponentials in the Schwinger representation of the propagators in $A_{\mathrm{fermionic}}$ (equation (3.47)). Thus the second order term, shown in figure 3.2, involves double line propagators with two $\mathcal{W}^\alpha$ insertions. Since the exponential involves the fermionic propagator momentum $\pi_{i\alpha}$, the

[5] Please note that when going to the trace formulation we actually get a factor of $C(\mathrm{fund})$ as in $\Phi^a\Phi^a = \mathrm{Tr}(\Phi\Phi)\,/C(\mathrm{fund})$. For the p^2 and $\mathcal{W}^\alpha\pi_\alpha$ terms this factor can be absorbed in $\bar{m}$. For the mass term we think of the factor as redefining the mass. E.g. to have a mass term $\frac{1}{2}m\,\mathrm{Tr}(\Phi^2)$ in W_{tree} as in (2.2) the mass term in (3.35) should really have been multiplied with $C(\mathrm{fund})$ – in this way when we go to the trace formulation we will get the wanted mass term.

Figure 3.2:



Double line notation for the second order term of the exponential in the Schwinger representation of the propagator. Crosses denote $\mathcal{W}^\alpha$ insertions.

expansion terminates at second order and thus figure 3.1 and 3.2 show all the possible double line propagators with insertions; especially we note that there can be maximally two insertions per double line propagator.

With the double line notation we see that we should define Feynman rules giving diagrams where we can have $\mathcal{W}^\alpha$ insertions in the index lines. However, the important point is that we should have exactly $2L$ insertions since the integrations $\int \prod_{a=1}^{L} \mathrm{d}^2\pi_a$ over the $2L$ fermionic loop momenta in (3.47) bring down exactly $2L$ factors of $\mathcal{W}^\alpha$. On the other hand, there can be at most two W^α insertions in an index loop. This is because $\mathrm{Tr}(\mathcal{W}^{\alpha_1}\dots\mathcal{W}^{\alpha_n}) = 0$ for $n \geq 3$ as shown in [7]. We can prove this following [15, 16] by noting that by the relation (3.6), the definition (3.2) and the fact that $\mathcal{W}_\beta$ is chiral we get:

$$\{\mathcal{W}_\alpha, \mathcal{W}_\beta\} = \frac{1}{8i}[\bar{\nabla}_{\dot{\alpha}}, \nabla_\alpha{}^{\dot{\alpha}}]\mathcal{W}_\beta = -\frac{1}{8}\bar{\nabla}_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}\nabla_\alpha\mathcal{W}_\beta. \tag{3.52}$$

where the vector field V in the definition of ∇^α is in the adjoint. Thus

$$\begin{aligned}\mathrm{Tr}(\mathcal{W}^{\alpha_1}\dots\mathcal{W}^{\alpha_i}\mathcal{W}^{\alpha_{i+1}}\dots\mathcal{W}^{\alpha_n}) = &-\mathrm{Tr}(\mathcal{W}^{\alpha_1}\dots\mathcal{W}^{\alpha_{i+1}}\mathcal{W}^{\alpha_i}\dots\mathcal{W}^{\alpha_n}) \\ &-\frac{1}{8}\bar{D}\bar{D}\,\mathrm{Tr}(\mathcal{W}^{\alpha_1}\dots\nabla^{\alpha_i}\mathcal{W}^{\alpha_{i+1}}\dots\mathcal{W}^{\alpha_n})\,.\end{aligned} \tag{3.53}$$

Due to the transformation of the gauge covariant derivative the last trace is gauge invariant and we conclude that we can anticommute the $\mathcal{W}^{\alpha_i}$'s under a trace in the chiral ring (section 2.9). Since the Weyl index α can only take two values, traces of more than two $\mathcal{W}^\alpha$'s are zero (classically) in the chiral ring and can thus be disregarded.

Suppose the diagram under consideration has F index loops (and E propagators, L momentum loops, and V vertices). We can thus have maximally $2F$ $\mathcal{W}^\alpha$ insertions and since we know that we have $2L$ insertions, we conclude that $F \geq L$. Since the Euler characteristic $\chi = V - E + F$ (2.151) and $L = E - V + 1$ (3.38) we conclude:

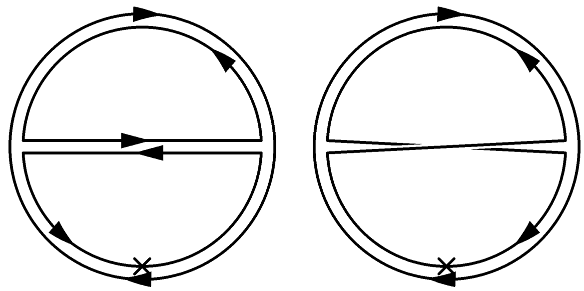
$$\chi = F - L + 1 \geq 1. \tag{3.54}$$

Thus for a $\mathrm{U}(N)$ gauge group and an adjoint chiral superfield only planar diagrams contribute and we conclude that the planar limit is exact. We note that this result came about simply by considering the number of fermionic integrations. For the $\mathrm{SO}(N)$ and $\mathrm{USp}(N)$ gauge groups we can also have diagrams with $\chi = 1$ and topology $\mathbb{RP}^2$ if we include a single cross-cap using the cross-over double line propagator from figure 2.5. If we take into consideration fundamental matter, we further have the possibility of diagrams with one boundary (and the topology of a disk) and hence $\chi = 1$.

3.2.2 Reduction to the Matrix Model

We now want to show the reduction to the matrix model using the results from the last subsection.

Figure 3.3:



On the left a planar diagram with an insertion. On the right a non-orientable diagram obtained from the diagram on the left by replacing a double line propagator with a cross-over.

Let us first note that an index loop with a single insertion contributes with a factor $\mathrm{Tr}(\mathcal{W}^\alpha)$. An index loop with two insertions gives a factor proportional to the glueball superfield from (2.131):

$$\mathrm{Tr}\Big(\mathcal{W}^\alpha \pi_{i\alpha} \mathcal{W}^\beta \pi_{j\beta}\Big) = -\frac{1}{2}\,\mathrm{Tr}(WW)\,\pi_i\pi_j = 16\pi^2 C(\mathrm{fund}) S \pi_i \pi_j, \tag{3.55}$$

where we have used the fact that the matrices $\mathcal{W}^\alpha$ and $\mathcal{W}^\beta$ anticommute under the trace and hence behave as spinors allowing us to use (C.42).

However, we have to care about the sign of an insertion since, as we found above (figure 3.1), the sign depends on whether the direction of the single index line with the insertion is parallel or anti-parallel to the direction of the momentum. This actually also holds true in the case of non-orientable diagrams. Take the case of a $\mathrm{SO}(N)$ gauge group. If we replace a double line with a cross-cap[6] as in figure 3.3, we will join two index loops to one and flip the arrows in one of the index loops i.e. change the order of how to contract the indices. But for $\mathrm{SO}(N)$ $(\mathcal{W}_\alpha)^i{}_j$ is antisymmetric and hence this change in the direction of arrows on the single lines exactly gives a sign change. Since $\mathcal{W}_\alpha$ does not depend on where in the index loop it is inserted, this means that we can keep track of the signs and the insertions by introducing an auxiliary set of fermionic variables $\mathcal{W}'^\alpha_m$, $m = 1, \ldots, F$, corresponding to the index loop insertions such that $\mathcal{W}^\alpha_{(\mathbf{r}_i)}$ in (3.47) is given by:

$$\mathcal{W}^\alpha_{(\mathbf{r}_i)} = \sum_{m=1}^{F} K_{im} \mathcal{W}'^\alpha_m. \tag{3.56}$$

Here K_{im} is an $E \times F$ matrix defined to be $+1$ if the direction of the single index line on the side of the m^{th} index loop in the i^{th} double line propagator is parallel to the momentum π_i, -1 if it is anti-parallel, and zero if the i^{th} double line propagator is not part of the m^{th} index loop at all. If we have the same face on both sides of the propagator, we should sum the contributions. Thus in this case if the index lines have opposite directions K_{im} is zero

[6] To obtain a cross-cap we replace a normal double line propagator with the cross-over part of the $\mathrm{SO}(N)$ propagator from figure 2.5. But the double line propagator should not have the same face on both sides. E.g. if we draw the middle propagator in the dumbbell diagram in figure 2.7 as a cross-over, we essentially get the same diagram.

and if they have the same direction it is ± 2. If K_{im} is zero (3.56) implies that we will not have any insertions in that propagator. For $\chi \geq 1$ this is consistent with the fact that in such a propagator the momentum must be zero and we really do not have any insertion since $\mathcal{W}^\alpha \pi_{i\alpha} = 0$. This happens e.g. in the dumbbell diagram in figure 2.7. However, if $\chi < 1$ we can have propagators with $K_{im} = 0$ and non-zero momentum, see figure 2.4. Thus this method only works for $\chi \geq 1$, but, actually, this is all we need by (3.54). The middle propagator in the non-orientable diagram in figure 3.3 gives an example where $K_{im} = 2$. This is also consistent since we only have one variable $\mathcal{W}_m'^\alpha$ which can give insertions, but we can have insertions in both sides with the same sign – hence we need the factor 2. In conclusion $K_{im} \in \{0, \pm 1, \pm 2\}$ and we only get ± 2 for $\chi = 1$. The advantage is further, as we will see below, that the contraction of the gauge group indices reduces to thinking of $\mathcal{W}_m'^\alpha$ as a simple fermionic variable (not a matrix) and performing fermionic integrations over these variables. Using these auxiliary variables we can now prove the reduction to the matrix model.

Let us start by considering a $\mathrm{U}(N)$ gauge group and the traceless glueball superfield. In this case we have seen that only planar diagrams contribute to $W_{\text{eff,pert}}$. Consider such a planar diagram with amplitude A as in (3.43). Using the new double line Feynman rules this amplitude splits into amplitudes from diagrams with the different possible insertions. For a planar diagram we have $F = L + 1$ using (3.54). Thus we have two possibilities of distributing the $2L$ insertions: Either we have 2 index loops with just one insertion and the remaining $L - 1$ index loops have two insertions or we have 1 index loop without any insertions and the remaining L index loops with two insertions. In the traceless case we should disregard the first option because it gives contributions depending on $\operatorname{Tr} \mathcal{W}^\alpha$, which is zero since we choose a background such that the abelian part of $\mathcal{W}^\alpha$ is zero. So let us consider the last case. Let us fix the index loop without any insertions, e.g. as the outer loop. The remaining index loops have two insertions thus giving a factor of S^{F-1}. However, to calculate the precise value we should consider all the different ways to distribute the insertions. In figure 3.4 all the possible insertions have been shown in the case of a stop sign diagram. We can now calculate the dependence on the Schwinger parameters s_i since an insertion in the i^{th} propagator gives a factor s_i as seen from (3.47). Remembering that a propagator with a double insertion comes with a factor 1/2 from expanding the exponential, we then see that the contribution from the diagrams is proportional to:

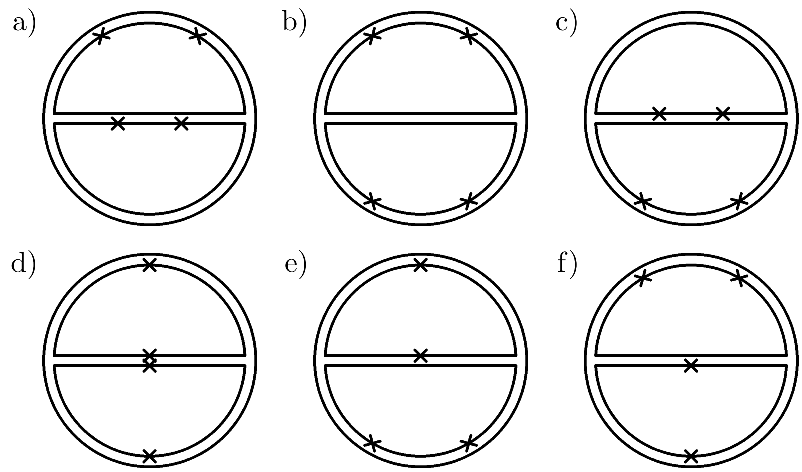
$$\frac{1}{4}\left(s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2\right) + \frac{1}{2}\left(s_1 s_2^2 s_3 + s_1 s_2 s_3^2 + s_1^2 s_2 s_3\right), \tag{3.57}$$

One can check that this actually is $(\det M(s_i))^2/4$ thus cancelling the s_i dependence in A_{bosonic} from (3.45) and proving the reduction to zero-modes for this diagram. Actually, one has to be a little more careful in (3.57). When calculating diagram d in figure 3.4 we get an extra factor $\frac{1}{2}$ compared to the other diagrams which is, however, cancelled by the factor of 2 coming with the double line propagators that have insertions in both sides – see figure 3.2.

However, using the auxiliary fermionic variables there is no need to explicitly do this elaborate expansion into diagrams with insertions. We simply constrain the matrix K_{im} by removing the column corresponding to the index loop without insertions thus ensuring no insertions in this index loop. But the resulting matrix is, by definition, exactly the matrix L_{ia} from (3.44) if we choose the L loop momenta to run as the L index loops with insertions and, especially, with the same orientation, see figure 3.5. This is possible since we have a planar, oriented diagram. It is here a point that the Jacobian for the transformation from the independent propagator momenta to these loop momenta simply is one.[7] We

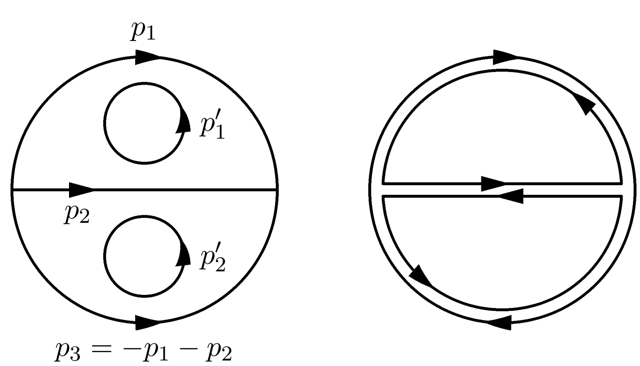
[7] We can sketch a proof of this as follows: Consider the matrix relating the loop momenta p'_a to the inde-

Figure 3.4:



The stop sign diagram with all the different possible distributions of the four insertions. The outer loop has been chosen to be without insertions.

Figure 3.5:



On the left the stop sign diagram in single line notation with the loop momenta p'_1 and p'_2 chosen to run like the inner index loops in the corresponding double line diagram (displayed to the right).

can then rewrite (3.56) as $\mathcal{W}^{\alpha}_{(\mathbf{r}_i)} = \sum_a L_{ia}\mathcal{W}'^{\alpha}_a$ where we have used the same label for the (remaining) auxiliary fermionic variables as for the momentum loops. The point is that the auxiliary variables also captures the index contraction structure of the insertions: By (3.55) we can identify $\mathcal{W}'^{\alpha}\mathcal{W}'_{\alpha}$ with $-32\pi^2 C(\text{fund})\, S$ where $\mathcal{W}'^{\alpha}$ is a simple fermionic variable, not a matrix. Thus, to obtain $A_{\text{fermionic}}(s_i)$ in (3.47) we simply have to include the $2L$ integrations $\prod_a(-32)\pi^2 C(\text{fund})\, S \int \mathrm{d}^2\mathcal{W}'_a$ with $\int \mathrm{d}^2\mathcal{W}'\, \mathcal{W}'^{\alpha}\mathcal{W}'_{\alpha} = 1$. In this way we get exactly two insertions in the index loops with insertions and no insertion in the remaining index loop. Furthermore, this takes care of all gauge index contractions and we will get the right signs both from the insertions and from the ordering of the fermionic variables. Also, we are ensured that we have the right Weyl index contractions. We then get:

$$\begin{aligned} A_{\text{fermionic}}(s_i) &= \left(\prod_{a=1}^{L}(-32)\pi^2 C(\text{fund})\, S \int \mathrm{d}^2\mathcal{W}'_a\,(-4)\int \mathrm{d}^2\pi'_a\right) e^{-\sum_i\sum_{a,b} s_i L_{ia}\mathcal{W}'^{\alpha}_a L_{ib}\pi'_{b\alpha}} \\ &= \left(-2C(\text{fund})\, S\right)^{F-1}(4\pi)^{2L}\int\Big(\prod_a \mathrm{d}^2\mathcal{W}'_a(-4)\mathrm{d}^2\pi'_a\Big) e^{-\sum_{a,b}\mathcal{W}'^{\alpha}_a M_{ab}(s_i)\pi'_{b\alpha}} \\ &= (-1)^{F-1}\left(2C(\text{fund})\, S\right)^{F-1}(4\pi)^{2L}(\det M(s_i))^2, \end{aligned} \tag{3.58}$$

where we have used the definition of $M_{ab}(s_i)$ from (3.46) and in the last line we have made a fermionic Gaussian integration.[8] We see that the s_i dependence in $A_{\text{fermionic}}(s_i)$ cancels that of A_{bosonic} in (3.45) thus giving us the wanted localisation to zero-modes.

The total amplitude (3.43) now reduces to:

$$A = NF(-1)^{F-1}\left(2C(\text{fund})\, S\right)^{F-1} n_{\text{symm}}\int_0^{\infty}\prod_{i=1}^{E}\mathrm{d}s_i e^{-\sum_i s_i m_{\text{E}}}\prod_k(-g_{\text{E},k})^{V_k}, \tag{3.59}$$

where we have multiplied with NF since we have F possibilities in choosing the loop without insertions and this loop contributes with a factor N. The integrations over the Schwinger variables s_i are now trivial and gives m_{E}^{-E}. To compare with the matrix model we should use $m_{\text{E}} = -m$ and $g_{\text{E},k} = -g_k$ where m is the mass and g_k the couplings in W_{tree} in Minkowski space. We thus get a factor $(-1)^{V-E}$ which by (2.151) can be written as $(-1)^{\chi-F}$. With the definition

$$S' = 2C(\text{fund})\, S \tag{3.60}$$

we can thus write

$$A = (-1)^{\chi-F}(-1)^{F-1}N\frac{\partial}{\partial S'}S'^F n_{\text{symm}} m^{-E}\prod_k(-g_k)^{V_k}. \tag{3.61}$$

From (2.154) we see that the amplitude of the corresponding planar double line diagram in the matrix model is given by:

$$A_{\text{matrix}}\left(g_{\text{s}}N'\right) = n_{\text{symm}} g_{\text{s}}^{-2}\left(g_{\text{s}}N'\right)^F m^{-E}\prod_k(-g_k)^{V_k}. \tag{3.62}$$

pendent propagator momenta p_i. Take a double line propagator at the boundary. Here the loop momentum, denoted p'_1, and the propagator momentum, denoted p_1, is the same up to a sign – see e.g. figure 3.5. Thus the top row in the transformation matrix is ± 1 in the first entry and zero in the rest. Adjacent to the inner face of this double line propagator we have another propagator whose propagator momentum, denoted p_2, thus depends on p'_1 and some other loop momenta p'_2. Thus the second row in the matrix has something in the two first entries and zeroes in the rest. Continuing in this way we get a triangle matrix whose determinant simply is the multiple of the entries in the diagonal. Since these are ± 1, the Jacobian is 1.

[8] Naturally, the sign and normalisation from the fermionic Gaussian integration is important for us. Going through the proof of Gaussian integration, one finds that the fermionic integrations gives the determinant squared and L factors of $\int \mathrm{d}^2\mathcal{W}'\,(-4)\int \mathrm{d}^2\pi'\mathcal{W}'^1\pi'_1\mathcal{W}'^2\pi'_2$ which simply is 1 by (C.40).

Thus, if we identify S' with $g_{\rm s}N'$, we can write:

$$A = -N\frac{\partial}{\partial S'} g_{\rm s}^2 A_{\rm matrix}\big(g_{\rm s}N' \equiv S'\big)\,, \tag{3.63}$$

where we have used that $\chi = 2$ so $(-1)^{\chi-F}(-1)^{F-1} = -1$. From (3.41) we know that the Euclidian effective superpotential is obtained as minus the sum of such amplitudes. Thus by (3.30) the effective glueball superpotential in Minkowski space is (plus) the sum of these amplitudes of connected planar diagrams. On the other hand, by (2.12) such a sum over $g_{\rm s}^2 A_{\rm matrix}$ gives $-\mathcal{F}_{g=0}$. Thus we finally get:

$$W_{\rm eff,pert} = N\frac{\partial}{\partial S'}\mathcal{F}_{g=0}\big(g_{\rm s}N' \equiv S'\big)\,, \tag{3.64}$$

which simply is (2.14b) in the unbroken case with $C(\text{fund}) = 1/2$ and $\hat{S} = S$. For this case we have thus proven the Dijkgraaf-Vafa conjecture for the relation between the perturbative part of the effective glueball superpotential and the planar limit of the matrix model.

Now, in much the same way we can prove the conjecture for other gauge groups and matter representations that allow a double line notation:

D^2 **diagrams:** If we have a classical group with an adjoint chiral field and include fundamental matter then, as mentioned above, we should add amplitudes from diagrams with a single boundary and hence topology D^2. Here $\chi = 1$ and hence $F = L$ by (3.54) which, naturally, corresponds to one of the index loops being replaced by a boundary. Since we only have L index loops, all of these must have two insertions. This gives us an overall factor of $(-1)^F S'^F$. So in this case we should keep all the auxiliary variables $\mathcal{W}'^{\alpha}_m$ and integrate over them. By choosing the loop momenta to run as the L index loops, the matrix L_{ia} is again equal to K_{im}. Thus the fermionic integration is the same as in the case above and we get a factor that cancels $A_{\rm bosonic}$. In this case we have no overall NF factor so the amplitude, in analogy with (3.61), becomes:

$$A = (-1)^{\chi-F}(-1)^F S'^F n_{\rm symm} m^{-E}\prod_k(-g_k)^{V_k}, \tag{3.65}$$

where $(-1)^{\chi-F}$ as before comes from changing $m_{\rm E}$ and $g_{{\rm E},k}$ to m and g_k. Together with $(-1)^F$ this gives an overall minus sign since $\chi = 1$. Using (2.154) (with $2-2g$ replaced by $\chi = 1$) we can identify the last part of (3.65) with $g_{\rm s}$ times the amplitude from the corresponding diagram in the matrix model under the identification $S' \equiv g_{\rm s}N'$. The contribution to $W_{\rm eff,pert}$ from these disk diagrams are again the sum of the amplitudes. The sum over the corresponding amplitudes in the matrix model times $-g_{\rm s}$ gives $\mathcal{F}_{D^2}$ from (2.203). Thus we get:

$$\Delta W_{\rm eff,pert}(S) = \mathcal{F}_{D^2}\big(S' \equiv g_{\rm s}N'\big)\,, \tag{3.66}$$

as claimed in (2.204).

$\mathbb{RP}^2$ **diagrams:** In the case of a SO(N) or USp(N) gauge group with adjoint matter we should also consider the contribution from diagrams with the topology of the projective plane $\mathbb{RP}^2$ and hence $\chi = 1$. Since the Euler characteristic is the same as for the D^2 diagrams above, this case is very similar, especially with two insertions in all of the $F = L$ loops. The only difference is when we choose the matrix L_{ia}. We can once again choose the loop momenta to run as the index loops and in this way L_{ia} equals

K_{im}. But we should here pay special attention to the cross-cap in the form of a cross-over propagator that joins two oriented loops to one. For this cross-over propagator, indexed by i, and the corresponding index loop, indexed by a, we have $L_{ia} = K_{ia} = \pm 2$ corresponding to $p_i = \pm 2p'_a$ (see figure 3.3). This means that the Jacobian for the change from the independent propagator momenta to the loop momenta is not 1 as above, but 2. Since we have 4 sets of the L bosonic momenta (p^μ) we get a factor of 2^4 from these. However, we also have 2 sets of the fermionic momenta (π^α) and since the Grassmannian Jacobian is the inverse of the bosonic, this gives a factor of $(\pm 2)^{-2}$. Thus in this case we get a factor 4 compared to the contribution from the disk diagrams above:

$$\Delta W_{\text{eff,pert}}(S) = 4\mathcal{F}_{\mathbb{RP}^2}\left(S' \equiv g_{\text{s}} N'\right), \tag{3.67}$$

which finishes the proof of (2.204) in the unbroken case. A proof for this case can also be found in [74].

Disconnected diagrams: Suppose that we have matter which specifically needs a projection to be traceless. This is e.g. the case for a SU(N) gauge group with adjoint matter as mentioned in footnote 47 on page 72. In these cases we should also consider disconnected double line propagators. One can then prove [64] that the number of disconnected components is one higher than the number of disconnected propagators. Thus, in the case of an even tree-level superpotential these disconnected diagrams will not contribute at all. However, in general the disconnected diagrams should be considered say for a USp(N) gauge group with traceless antisymmetric matter. Since a given amplitude here is a multiple of factors from the disconnected pieces, we can not give a relation to the total free energy of the matrix model and the comparison should be done diagram by diagram.

Broken gauge group: We now consider what happens when the vacuum breaks the gauge group. We will assume that we can make diagrams for the gauge theory just as we did for the matrix model in the broken case (section 2.7.1) which is plausible since the ghost term has the same form. We should then consider diagrams where the index loops have an extra label telling to which broken part, G_i, they belong. The $\mathcal{W}^\alpha$ insertion in a loop with the label i should then be the gauge field strength, $\mathcal{W}^\alpha_{(i)}$, for G_i. Two insertions in an index loop then correspond to the glueball superfield S_i for G_i as in (2.5) and (2.17). For the contribution corresponding to diagrams of topology D^2 and $\mathbb{RP}^2$ there is no change in the form of the relation to the matrix model, except that S now has a label i and we should identify $S'_i \equiv g_{\text{s}} N'_i$. For the planar diagrams (and a traceless glueball superfield) we should split the contribution from a given diagram according to which type of index loop we choose to be without insertions. Let us assume that we have chosen the type i and let $F = \sum_j F_j$ where F_j is the number of index loops of type j. Since the loop without insertion now gives a factor N_i and we have F_i possibilities in choosing this loop, we get a factor of $N_i F_i S_i^{F_i - 1}$ which should be multiplied with factors of $S_j^{F_j}$ from the remaining types of index loops. We can write this as $N_i \partial/\partial S_i \prod_i S_i^{F_i}$. Finally, we should sum over the choice of type of index loop without insertions, i. Thus (3.64) becomes:

$$W_{\text{eff,pert}} = \sum_i N_i \frac{\partial}{\partial S'_i} \mathcal{F}_{g=0}\left(S'_i \equiv g_{\text{s}} N'_i\right), \tag{3.68}$$

as in (2.14b).

In the next section we will finish the proof of the Dijkgraaf-Vafa conjecture for the form of $W_{\text{eff,pert}}$ by considering the abelian part of $\mathcal{W}_\alpha$. But let us end this section by noting that it is essential in the localisation to zero-modes that we are in four space-time dimensions. This is because A_{bosonic} in (3.45) is determined by the space-time dimension whereas $A_{\text{fermionic}}$ in (3.47) depends on the dimension of the Weyl spinors. Thus, the cancellation between A_{bosonic} and $A_{\text{fermionic}}$ only takes place in four space-time dimensions.

3.2.3 Abelian Part

In this section we will finish the proof from the last subsection by taking into consideration the abelian part of $\mathcal{W}^\alpha$. We will think of a $\mathrm{U}(N)$ gauge group with adjoint matter. A sketch of the proof can be found in [6], but since we have not found a detailed diagrammatic proof in the literature, we will be thorough here.[9]

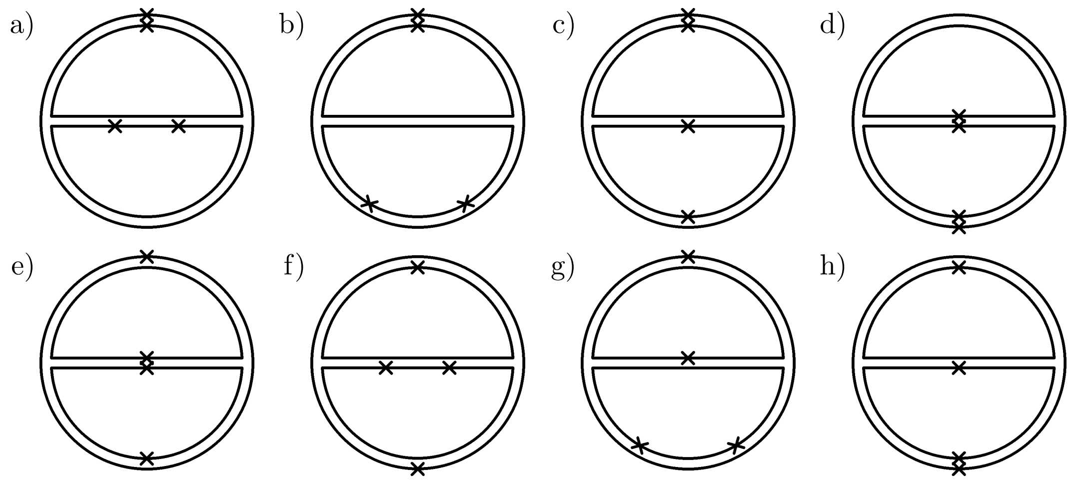
With a non-zero abelian part of $\mathcal{W}^\alpha$ we should allow index loops with a single $\mathcal{W}^\alpha$ insertion giving a contribution of $\mathrm{Tr}(\mathcal{W}^\alpha)$. Since we have $2L$ insertions, only planar diagrams have enough index loops to allow single insertions. We thus get an extra contribution to $W_{\text{eff,pert}}$ obtained in (3.64) from the planar diagrams in which 2 of the $F = L + 1$ loops have one insertion and the rest have two insertions. We will consider the case of a broken gauge group since for adjoint matter the overall $\mathrm{U}(1)$ in $\mathrm{U}(N)$ is completely decoupled as mentioned in section 2.1.2.

Let us first point out that the cancellation between A_{bosonic} and $A_{\text{fermionic}}$ in this case does not happen in exactly the same way as for the planar diagrams in the last subsection. Consider as an example the stop sign diagram. In the last section we considered the insertion pattern where one of the index loops has no insertions and the rest have two insertions and we obtained the possible diagrams in figure 3.4. Furthermore, we saw that the dependence on the Schwinger variables s_i exactly corresponded to the six terms in the square of the determinant of M_{ab} from (3.46) thus cancelling A_{bosonic}. However, in this case we do not have six, but eight possible insertion distributions. These are shown in figure 3.6 where the outer index loop and the upper of the inner index loops have been chosen to only have one insertion. We should compare the s_i dependence in these diagrams to the ones in figure 3.4. If we factor out the dependence on the traces of the $\mathcal{W}_\alpha$'s which, naturally, is different in the two set of diagrams, we will actually get that diagrams a, b, c and d in figure 3.6 gives -2 times respectively the diagrams a, b, f and c in figure 3.4. For diagrams a, b and c in figure 3.6 this can be seen by moving the single insertion in the outer loop to the opposite side of the double line propagator and thus to the index line in the corresponding inner loop. This gives us the same s_i dependence, but we get a minus sign since the insertion change sign and a factor of 2 from changing a propagator with insertions in both sides to one with insertions in only one side (remember the factor 2 in figure 3.2). For diagram d in figure 3.6 one should be more careful and remember which insertions should be Weyl contracted with each other. For the remaining diagrams in figure 3.6 one finds that e and f are equal and each corresponds to minus diagram d in figure 3.4. In the same way g and h in figure 3.6 are equal and each corresponds to minus diagram e in figure 3.4. Thus, all in all, we again get the determinant of M_{ab} squared and a cancellation between A_{bosonic} and $A_{\text{fermionic}}$, but also an extra factor -2.

For a general diagram we can use the auxiliary variables to see the reduction to the matrix model. Let $F = \sum_i F_i$ where F_i is the number of index loops of type i. Consider the

[9] In the generalised Konishi anomaly proof in [7], the proof for the abelian part is included in a natural way.

Figure 3.6:



The stop sign diagram with all the different possible distributions of the four insertions in the case where we choose the outer index loop and the upper of the inner index loops to only have one insertion.

case where we choose the single insertions to be in the loops of type i and j respectively. Thus, from these two index loops we get a contribution of $\mathrm{Tr}\left(\mathcal{W}^\alpha_{(i)}\pi_{l\alpha}\right)\mathrm{Tr}\left(\mathcal{W}^\beta_{(j)}\pi_{k\beta}\right)$ and after performing the fermionic momentum integrations we will, by Lorentz invariance, end up with a factor proportional to:

$$\mathrm{Tr}\left(\mathcal{W}^\alpha_{(i)}\right)\mathrm{Tr}\left(\mathcal{W}_{(j)\alpha}\right) = \frac{(4\pi)^2}{2} w_i^\alpha w_{j\alpha}, \tag{3.69}$$

where we have introduced w_i^α from (2.18). Let us now also introduce the auxiliary variables $\mathcal{W}'^\alpha_m$. We choose the labelling such that $m = 0$ corresponds to the index loop of type i with a single insertion and $m = 1$ to the index loop of type j with the other single insertion. We can thus identify (3.69) with $\mathcal{W}'^\alpha_0\mathcal{W}'_{1\alpha}$ and to calculate $A_{\text{fermionic}}$ we should include an integration

$$\frac{(4\pi)^2}{2} w_i^\alpha w_{j\alpha} \frac{1}{2}\int \mathrm{d}\mathcal{W}'_{0\alpha}\mathrm{d}\mathcal{W}'^\alpha_1\Big|. \tag{3.70}$$

Here $\frac{1}{2}\int \mathrm{d}\mathcal{W}'_{0\alpha}\mathrm{d}\mathcal{W}'^\alpha_1\big|\,\mathcal{W}'^\alpha_0\mathcal{W}'_{1\alpha} = 1$ and the restriction means that we should set the parts of $\mathcal{W}'^\alpha_0$ and $\mathcal{W}'^\alpha_1$ that are not integrated over to zero, just as above where we removed the auxiliary variables corresponding to the index loop without insertions, i.e.:

$$\mathrm{d}\mathcal{W}'_{0\alpha}\mathrm{d}\mathcal{W}'^\alpha_1\big| = \mathrm{d}\mathcal{W}'_{0\alpha=1}\mathrm{d}\mathcal{W}'^{\alpha=1}_1\Big|_{\mathcal{W}'_{0\,\alpha=2}=\mathcal{W}'^{\alpha=2}_1=0} + \mathrm{d}\mathcal{W}'_{0\alpha=2}\mathrm{d}\mathcal{W}'^{\alpha=2}_1\Big|_{\mathcal{W}'_{0\,\alpha=1}=\mathcal{W}'^{\alpha=1}_1=0} \tag{3.71}$$

For the remaining index loops, numbered by $a = 2, \dots, L$, with two insertions we get glueball superfield contributions and similar to (3.58) we should include the $2(L-1) = 2(F-2)$ integrations

$$(-1)^{F-2}(4\pi)^{2(F-2)} S_i^{F_i-1} S_j^{F_j-1}\Big(\prod_{k\neq i,j} S_k^{F_k}\Big)\prod_{a=2}^{L}\int \mathrm{d}^2\mathcal{W}'_a, \tag{3.72}$$

where we have chosen $C(\text{fund}) = \frac{1}{2}$ as in section 2.1. The fermionic integrations in $A_{\text{fermionic}}$ from (3.47) thus take the form:

$$\frac{1}{2}\int \mathrm{d}\mathcal{W}'_{0\alpha}\mathrm{d}\mathcal{W}'^{\alpha}_{1}\Big|\Big(\prod_{a=2}^{L}\int \mathrm{d}^2\mathcal{W}'_a\Big)\Big(\prod_{a=1}^{L}(-4)\int \mathrm{d}^2\pi'_a\Big)e^{-\sum_i\sum_{mb}s_iK_{im}\mathcal{W}'^{\alpha}_m L_{ib}\pi'_{b\alpha}} \tag{3.73}$$

This splits into two contributions by (3.71) (which actually give the same by Lorentz invariance). Let us consider the last contribution. Using (C.22) and (C.23) we can write the second integration as $\int \mathrm{d}\mathcal{W}'_{0\alpha=2}\mathrm{d}\mathcal{W}'^{\alpha=2}_1 = \int \mathrm{d}\mathcal{W}'^{\alpha=1}_0\mathrm{d}\mathcal{W}'^{\alpha=2}_1$. We can factorise the exponential in (3.73) by writing out the sum over α into two contributions (remembering the restriction in (3.71)):

$$e^{-\sum_i\sum_{mb}s_iK_{im}\mathcal{W}'^{\alpha=1}_m L_{ib}\pi'_{b\,\alpha=1}}\Big|_{\mathcal{W}'^{\alpha=1}_1=0}\; e^{-\sum_i\sum_{mb}s_iK_{im}\mathcal{W}'^{\alpha=2}_m L_{ib}\pi'_{b\,\alpha=2}}\Big|_{\mathcal{W}'^{\alpha=2}_0=0}. \tag{3.74}$$

By choosing the loop momenta to run as the index loops with $m = 1,\dots,L$ we have, as above, $K_{ia} = L_{ia}$, $a = 1,\dots,L$. The second exponential in (3.74) then has the form:

$$e^{-\sum_i\sum_{a,b=1}^{L}s_iL_{ia}\mathcal{W}'^{\alpha=2}_a L_{ib}\pi'_{b\,\alpha=2}}, \tag{3.75}$$

whereas the first exponential in (3.74) is:

$$\exp\Bigg(-\sum_i s_i\Big(\sum_{a=2}^{L}L_{ia}\mathcal{W}'^{\alpha=1}_a + K_{i0}\mathcal{W}'^{\alpha=1}_0\Big)\Big(\sum_b L_{ib}\pi'_{b\,\alpha=1}\Big)\Bigg). \tag{3.76}$$

Now, the trick is that since we have an oriented diagram, the single index lines in the double line propagators have opposite directions and by definition of K_{im} we conclude that:

$$\sum_m K_{im} = 0, \tag{3.77}$$

for all propagators labelled by i. Thus in our case $K_{i0} = -\sum_{a=1}^{L}L_{ia}$ and we can rewrite (3.76) as:

$$\exp\Bigg(-\sum_i s_i\Big(\sum_{a=2}^{L}L_{ia}(\mathcal{W}'^{\alpha=1}_a - \mathcal{W}'^{\alpha=1}_0) - L_{i1}\mathcal{W}'^{\alpha=1}_0\Big)\Big(\sum_b L_{ib}\pi'_{b\,\alpha=1}\Big)\Bigg). \tag{3.78}$$

Since the integrations over $\mathcal{W}'^{\alpha=1}_a$, $a = 2,\dots,L$, only work on this exponential, we can safely translate the integrations over these with $\mathcal{W}'^{\alpha=1}_0$. The integration over $\mathcal{W}'^{\alpha=1}_0$ also only works on this exponential and we can thus substitute $\mathcal{W}'^{\alpha=1}_0$ with $-\mathcal{W}'^{\alpha=1}_1$ and get:

$$\exp\Bigg(-\sum_i s_i\Big(\sum_{a=1}^{L}L_{ia}\mathcal{W}'^{\alpha=1}_a\Big)\Big(\sum_b L_{ib}\pi'_{b\,\alpha=1}\Big)\Bigg). \tag{3.79}$$

This is the same form as we got for the second exponential in (3.75). We can now do the same for the first term in (3.71) and thus (3.73) becomes:

$$-2\Big(\prod_{a=1}^{L}\int \mathrm{d}^2\mathcal{W}'_a\Big)\Big(\prod_{a=1}^{L}(-4)\int \mathrm{d}^2\pi'_a\Big)e^{-\sum_i\sum_{ab=1}^{L}s_iL_{ia}\mathcal{W}'^{\alpha}_a L_{ib}\pi'_{b\alpha}}, \tag{3.80}$$

where the minus sign is from the substitution of $\mathcal{W}_0'$ with $-\mathcal{W}_1'$ and the factor 2 comes from:

$$\frac{1}{2}\int \mathrm{d}\mathcal{W}'_{1\alpha}\mathrm{d}\mathcal{W}'^{\alpha}_{1} = 2\int \mathrm{d}^2\mathcal{W}'_1. \tag{3.81}$$

The integration in (3.80) is the same as we had in (3.58) which gave $(\det M(s_i))^2$, but now with an extra factor of -2 that we also found when comparing the diagrams in figure 3.6 to the diagrams in figure 3.4. We have thus obtained the localisation to zero-modes.

Collecting the terms from (3.70) and (3.72) and using (3.45) we get:

$$A_{\text{bosonic}}A_{\text{fermionic}} = -2\frac{1}{2}w_i^\alpha w_{j\alpha}(-1)^{F-2}S_i^{F_i-1}S_j^{F_j-1}\Big(\prod_{k\neq i,j}S_k^{F_k}\Big). \tag{3.82}$$

Thus, with this choice of in which type of index loops we should have only one insertion, the contribution to the total amplitude (3.43) is:

$$-a_{i,j}(-1)^{F-2}(-1)^{\chi-F}w_i^\alpha w_{j\alpha}S_i^{F_i-1}S_j^{F_j-1}\Big(\prod_{k\neq i,j}S_k^{F_k}\Big)n_{\text{symm}}m^{-E}\prod_k(-g_k)^{V_k}, \tag{3.83}$$

where $(-1)^{\chi-F}$ as before comes from the change of m_{E} and $g_{\mathrm{E},k}$ to m and g_k. $a_{i,j}$ is a combinatorial factor telling in how many ways we can choose the index loops with single insertions. If $i\neq j$ we have $a_{i,j}=F_iF_j$, whereas for $i=j$ we have $a_{i,i}=\binom{F_i}{2}=F_i(F_i-1)/2$. To obtain the total amplitude we should sum over the choices of i and j i.e. a sum as $\sum_{i\leq j}$. Using that $\sum_{i<j}=\frac{1}{2}\sum_{i\neq j}$ we get the total amplitude for the planar diagrams having two index loops with single insertions:

$$A = -\frac{1}{2}\sum_{i,j}w_i^\alpha w_{j\alpha}\frac{\partial^2}{\partial S_i\partial S_j}\Big(\prod_k S_k^{F_k}\Big)n_{\text{symm}}m^{-E}\prod_k(-g_k)^{V_k}, \tag{3.84}$$

since $(-1)^{F-2}(-1)^{\chi-F}=1$ for $\chi=2$. The same analysis that led from (3.61) to (3.64) gives us the contribution to $W_{\text{eff,pert}}$ from the planar diagrams with two single insertions:

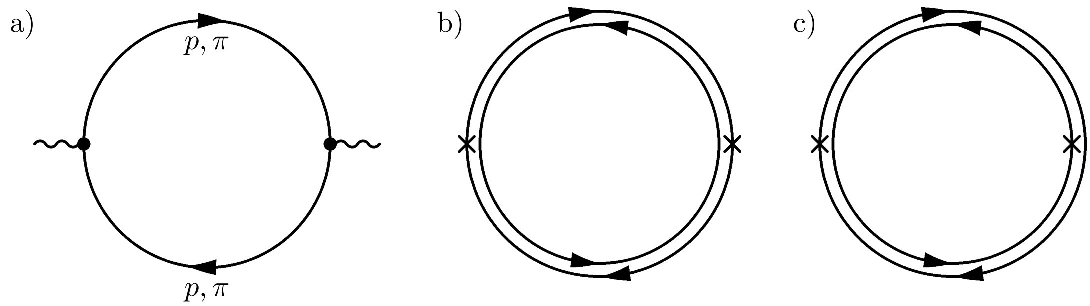
$$\Delta W_{\text{eff,pert}} = \frac{1}{2}\sum_{i,j}w_i^\alpha w_{j\alpha}\frac{\partial^2}{\partial S_i\partial S_j}\mathcal{F}_{g=0}\big(g_{\mathrm{s}}N_i'\equiv S_i\big)\,, \tag{3.85}$$

thus finishing the proof of (2.20).

We have now finished the proof for the form of $W_{\text{eff,pert}}$ in the Dijkgraaf-Vafa conjecture. However, we have actually missed one type of diagram in the proof. If we think of the $\mathcal{W}^\alpha\pi_\alpha$ term as an interaction term, the one-loop diagram shown in figure 3.7 should also be taken into consideration (as done in [7]). The diagram has two propagators and two $\mathcal{W}^\alpha$ interaction vertices as shown in diagram a) in figure 3.7. Since $\mathcal{W}^\alpha$ is a constant, there is no external momentum and the same momentum runs through both propagators.

Let us use the double line notation for the diagram. The $\mathcal{W}^\alpha$ interaction vertices takes the same form as in figure 3.1. Since the diagram has one loop, it should have two $\mathcal{W}^\alpha$ insertions. The two types of insertion patterns are shown in b) and c) in figure 3.7. In the broken case the diagrams in which the index loops are of different types should cancel. Let us therefore consider a diagram in which both index lines are of the same type i. As seen from (3.55) the insertions of type b) give a term proportional to $-2N_i\cdot 8\pi^2 S_i\pi\pi$. Here we have a minus sign since the insertions sit in different sides of the double lines (but both in the same index loop), the factor two comes from the choice of index loop without insertions,

Figure 3.7:



The one-loop diagram with two $\mathcal{W}^\alpha$ interaction vertices. a) shows the diagram in single line notation while b) and c) show the diagram in double line notation with the two possible types of insertion patterns.

and the factor N_i comes from the trace in that index loop. Correspondingly, the second insertion pattern in diagram c) in figure 3.7 gives a contribution of $-8\pi^2 w_i^\alpha w_{i\alpha}\pi\pi$. Since one of the interaction vertices gives an overall chiral superspace integration, the contribution to the Euclidean effective superpotential is given by:

$$2\frac{1}{2!}\left(-\frac{1}{2}\right)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}(-4)\int \mathrm{d}^2\pi \frac{1}{(p^2+m_{\mathrm{E}})^2}(-1)(4\pi)^2\left(N_i S_i + \frac{1}{2}w_i^\alpha w_{i\alpha}\right)\pi\pi. \tag{3.86}$$

Here the factors in front are a factor 2 from the symmetry of the diagram, a factor 1/2! from going to second order in the interaction, and a factor $(-1/2)^2$ from the normalisation of the interaction term as seen from (3.35). Introducing a cut-off, Λ_0, the four-dimensional bosonic momentum integration gives standardly:

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{(p^2+m_{\mathrm{E}})^2} = \frac{1}{(4\pi)^2}\ln\left(\frac{\Lambda_0 e^{-1}}{|m_{\mathrm{E}}|}\right). \tag{3.87}$$

Inserting into (3.86), summing over the type of index loop i, and rotating into Minkowski space (giving a sign change) gives the contribution to the glueball superpotential:

$$\sum_i \left(N_i S_i + \frac{1}{2}w_i^\alpha w_{i\alpha}\right)\ln\left(\frac{m}{\Lambda_0 e^{-1}}\right) = \sum_i N_i \hat{S}_i \ln\left(\frac{m}{\Lambda_0 e^{-1}}\right), \tag{3.88}$$

where we have used (2.19) to write the result in terms of the traceless glueball superfield, $\hat{S}_i$. We see that this is nothing but the m dependent part of the Veneziano-Yankielowicz superpotential (2.9) if we use the matching of scales (2.143): $\Lambda^3 = \Lambda_u^2 m$. This is the same m-dependence we obtained in the matrix model in (2.165).

3.3 General Gauge Groups and Matter Representations

In the last section we have proven diagrammatically the localisation to zero-modes (and the form of $W_{\mathrm{eff,pert}}$) for the gauge groups and matter representations that allowed a double line notation. Following [8] let us end this chapter by briefly showing the localisation to zero-modes for general gauge groups and representations.

We will use the same setup as in section 3.1.4, but we will assume that the constant field strength $\mathcal{W}^\alpha$ is abelian, i.e. lies in a Cartan subalgebra. Let us denote the generators of the Cartan subalgebra by H_a so $\mathcal{W}^\alpha = \mathcal{W}^{\alpha a} H_a$. For the given representation of the chiral field $\mathbf{r}$ we can diagonalise the $H_a^{(\mathbf{r})}$'s simultaneously since they commute:

$$\left(H_a^{(\mathbf{r})}\right)^i_{\ j} \Phi^j = \lambda_a^i \Phi^i. \tag{3.89}$$

We can thus define the $\dim(\mathbf{r})$ vectors $\vec{\lambda}^i = (\lambda_1^i, \dots, \lambda_{\mathrm{rank}(G)}^i)$ where $\mathrm{rank}(G)$ is the rank of the gauge group. These are called the *weights* of the representation.[10] Using these we can write:

$$\left(\mathcal{W}^\alpha_{(\mathbf{r})}\right)^i_{\ j} = \mathcal{W}^{\alpha a} \lambda_a^i \delta_j^i \equiv \vec{\mathcal{W}}^\alpha \cdot \vec{\lambda}^i \delta_j^i. \tag{3.90}$$

Since $\mathcal{W}^\alpha_{(\mathbf{r})}$ is diagonal, we can use a notation where the propagators just carry a single gauge index sitting on $\vec{\lambda}$ instead of two indices.

Let us think of a given diagram. With $\vec{\lambda}_i$ we denote the weight propagating through the i^{th} propagator which is then equal to $\vec{\lambda}^j$ where j is the gauge index for the propagator. The $\vec{\lambda}_i$'s are like charges for the Cartan generators and especially we have conservation of charges at the vertices (due to gauge invariance of the interaction terms). This means that we can introduce loop Cartan charges, $\vec{\lambda}'_a$,[11] analogous to the bosonic and fermionic momenta in (3.44) and (3.48):

$$\vec{\lambda}_i = \sum_a L_{ia} \vec{\lambda}'_a. \tag{3.91}$$

For a given choice of $\vec{\lambda}'_a$ we see that $A_{\text{fermionic}}(s_i)$ from (3.47) takes the form of a factor from the vertices and:

$$\begin{aligned}(-4)^L \int \prod_{a=1}^L \mathrm{d}^2\pi'_a e^{-\sum_i s_i \vec{\mathcal{W}}^\alpha \cdot \vec{\lambda}_i \pi_{i\alpha}} &= (-4)^L \int \prod_{a=1}^L \mathrm{d}^2\pi'_a e^{-\sum_i s_i \sum_{ab} L_{ia} \vec{\mathcal{W}}^\alpha \cdot \vec{\lambda}'_a L_{ib} \pi'_{b\alpha}} \\ &= (-4)^L \int \prod_{a=1}^L \mathrm{d}^2\pi'_a e^{-\sum_{ab} \vec{\mathcal{W}}^\alpha \cdot \vec{\lambda}'_a M_{ab} \pi'_{b\alpha}},\end{aligned} \tag{3.92}$$

where we have introduced the matrix M_{ab} from (3.46). Exactly as in (3.58) (the integrations over $\mathcal{W}'$ there was just a bookkeeping device) we see that this gives $(\det M(s_i))^2$ times a multiple of factors of $\vec{\mathcal{W}}^\alpha \cdot \vec{\lambda}'_a$. The last factor cancels the s_i dependence in A_{bosonic} from (3.45). We are then left with the trivial s_i dependence in the mass part of the propagators (see (3.43)) and we have thus obtained the wanted localisation to zero-modes.

One can pursue the matter and obtain the total amplitude by remembering the factor from the vertices, which depends on the choice of $\vec{\lambda}'_a$, and then sum over the $\vec{\lambda}'_a$'s (and there is also an overall sum analogous to the overall superspace integration). In this way the relation to the matrix model can be found e.g. for the U(N) gauge group with adjoint matter as done in [8].

[10] The precise mathematical definition of the weights (as given in [104]) is that they are the linear functionals, M^i, on the Cartan subalgebra given by $M^i(\sum_a c^a H_a) = \sum_a c^a \lambda_a^i$.

[11] The loop index a should not be confused with the adjoint gauge index.

Conclusions

The prime goal of this thesis has been to give a thorough introduction to the Dijkgraaf-Vafa conjecture, its diagrammatic proof and the concepts needed to understand the conjecture.

We started in chapter 2 by stating the conjecture and then used the rest of the chapter to understand the concepts used in the conjecture. Along the way we got a better understanding of the conjecture and its context. Using the Seiberg scheme of holomorphy, symmetries and various limits we proved the non-renormalisation theorem telling us that the low energy superpotential, which is what the Dijkgraaf-Vafa conjecture determines, essentially consists of non-perturbative corrections. And in the case of a cubic tree-level superpotential we saw that the Seiberg scheme determined the form of the perturbative part of the glueball superpotential and thus the essence of the Dijkgraaf-Vafa conjecture is to determine the coefficients in the power series for the glueball superfield. Furthermore, we briefly pointed out that there is an interplay between the ILS linearity principle and the Dijkgraaf-Vafa conjecture.

We have also seen some of the limitations in the conjecture. We still need to assume that the glueball superfield is the fundamental field (confinement). And we can only prove the relation to the matrix model for the perturbative part of the glueball superpotential, not that the full glueball superpotential simply is given by adding the Veneziano-Yankielowicz superpotential. Furthermore, we have discussed the nilpotency of the glueball superfield. We have seen that the terms in the glueball superpotential up to the power of the dual Coxeter number are determined unambiguously. And we saw how to interpret the full solution for the glueball superpotential given by the Dijkgraaf-Vafa conjecture as an F-term completion using supergroups, but also mentioned the discrepancies that arise here when comparing to standard gauge theory. We have also pointed out the difficulties in stating the conjecture in a general form when considering multi-trace tree-level superpotentials and in the case of general matter where the relation to the matrix model should be done diagram by diagram.

In the diagrammatic proof of the conjecture we were able to integrate out the conjugated chiral field using holomorphy. In this way we escaped the non-renormalisation theorems and were able to derive the strong results for the non-perturbative corrections in the glueball superpotential purely diagrammatically. In the proof we have been careful and kept track of all the details. In this way we found the $2C(\mathrm{fund})$ factor in the identification of the glueball superfield with $g_{\mathrm{s}}N_i'$ in the matrix model for a general classical gauge group. We have also seen the cancellation of the intricate sign from rotating to Euclidean space with the minus sign in the definition of the glueball superfield. And we have given a thorough diagrammatic proof in the case where one takes into account the abelian part of supersymmetric gauge field strength. All in all, we have understood very basically how the different terms in the Dijkgraaf-Vafa conjecture arise by considering insertions in diagrams. We have seen the projection to planar and Euler characteristic $\chi = 1$ diagrams by counting of these insertions. With the localisation to zero-momentum modes for general gauge groups and (massive) matter representations we have seen the generality of the Dijkgraaf-Vafa conjecture.

It is natural to ask if there is a meaning with the diagrams in the matrix model of higher genera. Indeed one finds that with gravitational couplings one should take these non-planar diagrams into account and much of recent research has been done in this area.

The use of the Dijkgraaf-Vafa conjecture has also been discussed. We have only briefly mentioned the important results obtained for the parameter spaces of different theories using the conjecture. But we have seen how to solve the one-cut case in the matrix model exactly and with this solution we have used the Dijkgraaf-Vafa conjecture to obtain the exact effective glueball superpotential for a cubic tree-level superpotential.

At last we have seen that we can recover the Veneziano-Yankielowicz superpotential from the matrix model free energy by considering the measure of the matrix model or by evaluating the partition function for zero couplings in the tree-level superpotential. But perhaps most strikingly the term emerged when we solved the matrix model exactly. Thus there were no need for adding the Veneziano-Yankielowicz superpotential by hand. This calls for a proof of this deeper relationship between the gauge theory and the matrix model.

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Appendix A

Notation

The topic of supersymmetry suffers under an abundance of notations. We choose to follow the notation of [9] as strict as possible, however, we shall make deviations.

We choose to use the "mostly plus" metric for the Minkowski space:

$$\eta_{\mu\nu} \sim \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

This also defines the similarity symbol for use when we do not have a formal equality (as here for the same matrix/tensor in index notation and written out with all entries in bracket notation). Also note that we use Greek letters $\mu, \nu, \ldots$ for Lorentz indices contrary to [9]. Latin indices i, j, k will be used for spatial coordinates.

This choice of metric then demands that we choose to represent the four-momentum operator as:

$$p_\mu = -i\partial_\mu, \tag{A.1}$$

such that p^0 is the Hamiltonian and p^i the usual momentum operators $-i\partial_i$.

The orientation in Minkowski space is given by the totally antisymmetric Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$ defined by:

$$\varepsilon^{0123} = -\varepsilon_{0123} \equiv +1.$$

$[-,-]$ will be used for commutators and $\{-,-\}$ for anticommutators.

In order to simplify expressions we choose units such that $\hbar = c = 1$.

Matrices can be in bold font while operators are in normal font – except abstract Lie algebra operators which can be in calligraphic font.

The term Lagrangian will also be used for the Lagrangian density as long as no confusion should be possible.

For notation on spinors please see appendix C.

Appendix B

Minkowski Space as Cosets in the Poincaré Group

In this appendix we describe how the Minkowski space emerges as the cosets of the Poincaré group modulo the Lorentz group. The treatment builds on [11].

The idea is based on the fact that the Poincaré algebra is the semidirect product of the translation group and the Lorentz group and hence the same for the corresponding algebras. Any point in Minkowski space can be reached as a translation of the origin – but not uniquely since any Lorentz transformation keeps the origin fixed. Hence we must identify Minkowski space with the translations modulo the Lorentz transformations.

B.1 Poincaré Algebra

To be precise let $\mathcal{P}_\mu$ be the generators of the translations and let $\mathcal{J}_{\mu\nu} = -\mathcal{J}_{\nu\mu}$ be the antisymmetric generators of the Lorentz group such that the total Poincaré algebra is given by (taken from [24]):

$$\begin{aligned}
[\mathcal{P}_\mu, \mathcal{P}_\nu] &= 0, \\
[\mathcal{P}_\mu, \mathcal{J}_{\nu\rho}] &= -i\left(\eta_{\mu\nu}\mathcal{P}_\rho - \eta_{\mu\rho}\mathcal{P}_\nu\right), \\
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= -i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right).
\end{aligned} \tag{B.1}$$

An element in the Poincaré group then has the form:

$$e^{-i\tau^\mu \mathcal{P}_\mu + i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}}, \tag{B.2}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Please note the signs which have been chosen carefully. In a representation where this is the (active) coordinate transformation of fields, we want:

$$e^{-i\tau^\mu P_\mu + i\frac{1}{2}\omega^{\mu\nu}J_{\mu\nu}}\psi(x) = \psi'(x) = \psi(x_{\text{Passive}}), \tag{B.3}$$

where P and J is the representations of $\mathcal{P}$ and $\mathcal{J}$ respectively, and x_{Passive} to first order in τ and ω is given by

$$x^\mu_{\text{Passive}} \simeq x^\mu - \omega^\mu{}_\nu x^\nu - \tau^\mu. \tag{B.4}$$

With the signs given in (B.2) this is exactly realised by (A.1) i.e. $\mathcal{P}_\mu \rightsquigarrow P_\mu = -i\partial_\mu$ and

$$\mathcal{J}_{\mu\nu} \rightsquigarrow J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right). \tag{B.5}$$

It checks easily that we indeed get the right transformation (to first order in τ and ω):

$$\begin{aligned} e^{-i\tau^\mu P_\mu + i\frac{1}{2}\omega^{\mu\nu}J_{\mu\nu}}\psi(x^\mu) &= (1-\tau^\mu\partial_\mu + \omega^{\mu\nu}x_\mu\partial_\nu)\,\psi(x^\mu) \\ &= \psi\big(x^\mu - \tau^\mu - \omega^\mu{}_\nu x^\nu\big)\,. \end{aligned}$$

We also note that P and J satisfy the Poincaré algebra (B.1) as they should.

B.2 Minkowski Space as Right Cosets

Now let us consider the cosets ([11] call them right cosets) given by the Poincaré group, $G_{\text{Poincaré}}$, modulo the Lorentz group, G_{Lorentz}. They have the form gG_{Lorentz}, $g \in G_{\text{Poincaré}}$. However, we can always use a pure translation as a representative of the coset and this translation is unique. Hence we have the unique form of a given coset:

$$\exp(-ix^\mu\mathcal{P}_\mu)G_{\text{Lorentz}}. \tag{B.6}$$

Proof. Since [11] does not prove this let us do it here. Take an arbitrary element in the Poincaré group $\exp\big(-iy^\mu\mathcal{P}_\mu + i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}\big)$. We now want to use the Baker-Campbell-Hausdorff formula (here from [10]), which says:

$$e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{A}+\mathcal{B}+\frac{1}{2}[\mathcal{A},\mathcal{B}]+\frac{1}{12}[\mathcal{A},[\mathcal{A},\mathcal{B}]]+\frac{1}{12}[\mathcal{B},[\mathcal{B},\mathcal{A}]]+\cdots}, \tag{B.7}$$

where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary elements in a given Lie algebra. Now let us try to set $\mathcal{A} = i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}$ and $\mathcal{B} = -ix^\mu\mathcal{P}_\mu$. From (B.1) we get that $[\mathcal{B},\mathcal{A}] \sim \mathcal{P}$ and hence $[\mathcal{B},[\mathcal{B},\mathcal{A}]] = 0$. Thus one of the series in the Baker-Campbell-Hausdorff formula truncates and the resulting series is linear in $\mathcal{B}$. The formula then takes the form (from [11]):

$$e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{C}}, \qquad C = \mathcal{A} + \left(\frac{-\operatorname{ad}\mathcal{A}}{e^{-\operatorname{ad}\mathcal{A}} - \mathbf{1}}\right)\mathcal{B}, \tag{B.8}$$

where $\mathbf{1}$ is the identity and the adjoint $\operatorname{ad}\mathcal{A}$ is defined, as usual, as the operator

$$\operatorname{ad}\mathcal{A}\cdot\mathcal{B} = [\mathcal{A},\mathcal{B}]. \tag{B.9}$$

Now let us work out how the adjoint works in our case. (B.1) gives:

$$\begin{aligned} \operatorname{ad}\big(i\tfrac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}\big)\cdot(-ix^\mu\mathcal{P}_\mu) &= [\tfrac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}, x^\mu\mathcal{P}_\mu] \\ &= ix^\mu\omega_\mu{}^\nu P_\nu. \end{aligned} \tag{B.10}$$

By induction:

$$\big(\operatorname{ad}\big(i\tfrac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}\big)\big)^k\cdot(-ix^\mu\mathcal{P}_\mu) = -(-1)^k i\mathbf{x}\cdot\boldsymbol{\omega}^k\cdot\mathcal{P}, \tag{B.11}$$

where $\mathbf{x}$ is the row vector made out of x^μ, $\boldsymbol{\omega}$ is the matrix made from $\omega_\mu{}^\nu$ and $\mathcal{P}$ is the column vector made out of $\mathcal{P}_\nu$. Now (B.8) becomes:

$$e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}}e^{-ix^\mu\mathcal{P}_\mu} = e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu} - i\mathbf{x}\cdot\left(\frac{\boldsymbol{\omega}}{e^{\boldsymbol{\omega}}-\mathbf{1}}\right)\cdot\mathcal{P}}. \tag{B.12}$$

Noting that as a real function $\frac{\boldsymbol{\omega}}{e^{\boldsymbol{\omega}}-\mathbf{1}}$ is non-zero, and that it and its inverse are well defined as series expansions in $\boldsymbol{\omega}$ we get:

$$e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}}e^{-i\mathbf{y}\cdot\left(\frac{e^{\boldsymbol{\omega}}-\mathbf{1}}{\boldsymbol{\omega}}\right)\cdot\mathcal{P}} = e^{-iy^\mu\mathcal{P}_\mu + i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}}.$$

Now taking the inverse and substituting $\mathbf{y}$ and $\boldsymbol{\omega}$ for $-\mathbf{y}$ and $-\boldsymbol{\omega}$ respectively we get the wanted result:

$$e^{-iy^\mu \mathcal{P}_\mu + i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}} = e^{-i\mathbf{y}\cdot\left(\frac{e^{-\boldsymbol{\omega}}-\mathbf{1}}{-\boldsymbol{\omega}}\right)\cdot\mathcal{P}} e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}}. \tag{B.13}$$

The uniqueness of the translation representative can easily be seen. Suppose that we had two pure translation representatives of the same coset (determined by x and x'). Then for suitable ω and ω' we have:

$$e^{-ix^\mu \mathcal{P}_\mu} e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}} = e^{-ix'^\mu \mathcal{P}_\mu} e^{i\frac{1}{2}\omega'^{\mu\nu}\mathcal{J}_{\mu\nu}}.$$

Hence

$$e^{-i(x-x')^\mu \mathcal{P}_\mu} = e^{i\frac{1}{2}\omega'^{\mu\nu}\mathcal{J}_{\mu\nu}} e^{-i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}} \in G_{\text{Lorentz}},$$

which is only possible for $x = x'$ □

Now we see that the well defined relation (bijection) between Minkowski space and the cosets in the Poincaré group modulo the Lorentz group is given by:

$$x^\mu \in \text{Minkowski space} \longleftrightarrow \exp(-ix^\mu \mathcal{P}_\mu) G_{\text{Lorentz}} \in G_{\text{Poincaré}}/G_{\text{Lorentz}}. \tag{B.14}$$

B.3 Action on Minkowski Space

Now arises the natural question of how the action of the Poincaré group is expressed in the coset space. The answer is the simple action of left multiplication of the Poincaré group on the cosets. This is easily checked. Left multiplication with a pure translation determined by τ gives:

$$e^{-i\tau^\mu \mathcal{P}_\mu} e^{-ix^\mu \mathcal{P}_\mu} G_{\text{Lorentz}} = e^{-i(x+\tau)^\mu \mathcal{P}_\mu} G_{\text{Lorentz}}, \tag{B.15}$$

hence sending $x^\mu \mapsto x^\mu + \tau^\mu$ which is the active translation, that we want. Now let us check how a Lorentz transformation determined by ω works. By the use of (B.12) and (B.13) we get

$$\begin{aligned}
e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}} e^{-ix^\mu \mathcal{P}_\mu} G_{\text{Lorentz}} &= e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu} - i\mathbf{x}\cdot\left(\frac{\boldsymbol{\omega}}{e^{\boldsymbol{\omega}}-\mathbf{1}}\right)\cdot\mathcal{P}} G_{\text{Lorentz}} \\
&= e^{-i\mathbf{x}\cdot\left(\frac{\boldsymbol{\omega}}{e^{\boldsymbol{\omega}}-\mathbf{1}}\right)\cdot\left(\frac{e^{-\boldsymbol{\omega}}-\mathbf{1}}{-\boldsymbol{\omega}}\right)\cdot\mathcal{P}} e^{i\frac{1}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}} G_{\text{Lorentz}} \\
&= e^{-i\mathbf{x}\cdot e^{-\boldsymbol{\omega}}\cdot\mathcal{P}} G_{\text{Lorentz}} \\
&= e^{-i(e^{\boldsymbol{\omega}})^\mu{}_\nu x^\nu \mathcal{P}_\mu} G_{\text{Lorentz}},
\end{aligned} \tag{B.16}$$

where we in the last line have used that ω is antisymmetric. Hence this generates the transformation $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ with

$$\Lambda^\mu{}_\nu = (e^{\boldsymbol{\omega}})^\mu{}_\nu\,, \tag{B.17}$$

as expected for an active Lorentz transformation.

Please note that the fact that this is a representation of the Lorentz group, is tightly bound to the generator $\mathcal{P}^\mu$ being multiplied with Λ^{-1} and not Λ as one might expect. $\mathcal{P}^\mu$ is also a representation of the Lorentz group, but the transformation matrices are not just multiplied on from the right like on x^μ, but are multiplied on right next to $\mathcal{P}^\mu$. I.e. first transforming with Λ_1 and then with Λ_2 yields $\Lambda_1^{-1}\Lambda_2^{-1}\mathcal{P}$, but because of the inverses this is a representation.

In conclusion, we have seen that the Minkowski space can be identified with the Poincaré group modulo the Lorentz group and the Poincaré transformations are given by left multiplication under this identification.

Appendix C

Spinors

In this chapter we will give an introduction to spinors with emphasis on what we will need in this thesis namely Majorana spinors, Weyl spinors and the conventions and notation that follows.[1]

C.1 Spinorial Representations

In relativistic quantum mechanics Lorentz invariance is the first principle. Hence our objects must be representations of the Lorentz group. The spinorial representations are a special case since they rather are representations of a double cover of the Lorentz group – the spin group.

Let us start by looking at the proper orthochronous Lorentz group with the six generators $\mathcal{J}_{\mu\nu} = -\mathcal{J}_{\nu\mu}$ with commutation relations given by (B.1). There is a very nice and easy way to list all finite-dimensional representations of the Lorentz group (here inspired by [25]). First we define the usual generators of rotations and boosts as:

$$L^i = \tfrac{1}{2}\varepsilon^{ijk}\mathcal{J}^{jk}, \qquad K^i = \mathcal{J}^{0i}. \tag{C.1}$$

In this basis we get from (B.2) that a Lorentz transformation takes the form:

$$e^{-i\boldsymbol{\theta}\cdot\mathbf{L}-i\boldsymbol{\beta}\cdot\mathbf{K}}, \tag{C.2}$$

with $\theta^i = -\frac{1}{2}\varepsilon^{ijk}\omega^{jk}$ and $\beta_i = \omega_{i0}$ i.e. the turning angle and the rapidity respectively. The signs of θ and β are chosen to comply with (B.17). However, we now want to make a second change of basis to the six generators defined by:

$$\mathbf{J}_\pm = \tfrac{1}{2}\left(\mathbf{L} \pm i\mathbf{K}\right). \tag{C.3}$$

One can show that $\mathbf{J}_+$ and $\mathbf{J}_-$ commute and both satisfy the commutation relations of angular momentum (i.e. the SU(2) algebra):

$$[J_\pm^i, J_\pm^j] = i\varepsilon^{ijk}J_\pm^k.$$

Consequently, we can write (C.2) as:

$$e^{i(-\boldsymbol{\theta}+i\boldsymbol{\beta})\cdot\mathbf{J}_+}e^{i(-\boldsymbol{\theta}-i\boldsymbol{\beta})\cdot\mathbf{J}_-}. \tag{C.4}$$

[1] This appendix is based on [9], [11] and [13], but with all notation and definitions altered to coincide with [9]. However, where [9] has not defined objects we will try to define everything logically and consistent with [9].

This shows that the representations of the Lorentz group can be seen as the complexified representations of SU(2) × SU(2) where the last part is the conjugate of the first. Since the irreducible finite-dimensional representations of SU(2) are characterised by an integer or half integer j (corresponding to the dimension of the representation being $2j+1$), we see that the finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers (j_+, j_-).

Let us investigate the representations with the lowest dimensions. (0,0) is of course the trivial representation. The two dimensional representation $(\frac{1}{2}, 0)$ is the complexified SU(2) spin $\frac{1}{2}$ representation. Let us use that $\mathbf{J}_+ = \frac{\boldsymbol{\sigma}}{2}$ and $\mathbf{J}_- = 0$ where the Pauli matrices, i.e. the generators of SU(2), as usual are given by:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.5}$$

Hence (C.4) becomes:

$$e^{i(-\boldsymbol{\theta}+i\boldsymbol{\beta})\cdot\frac{\boldsymbol{\sigma}}{2}} = e^{(-i\boldsymbol{\theta}-\boldsymbol{\beta})\cdot\frac{\boldsymbol{\sigma}}{2}}. \tag{C.6}$$

Since $-i\boldsymbol{\theta}-\boldsymbol{\beta}$ can be any complex number $(-i\boldsymbol{\theta}-\boldsymbol{\beta})\cdot\frac{\boldsymbol{\sigma}}{2}$ can by any complex traceless matrix and hence the represented group is $\mathrm{SL}(2,\mathbb{C})$. If we look at the representation $(0, \frac{1}{2})$ we get the same result with a simple change in the sign of $\boldsymbol{\beta}$ i.e. the group elements are:

$$e^{(-i\boldsymbol{\theta}+\boldsymbol{\beta})\cdot\frac{\boldsymbol{\sigma}}{2}}. \tag{C.7}$$

Now in order to investigate the representation $(\frac{1}{2}, \frac{1}{2})$ we note that if (C.7) works on ψ, the action on $\tilde{\psi} = \psi^T\sigma^2$ is (by the use of $\boldsymbol{\sigma} = -\sigma^2\boldsymbol{\sigma}^T\sigma^2$ and $\left(\sigma^2\right)^2 = \mathbf{1}$):

$$\tilde{\psi} \mapsto \tilde{\psi}e^{(i\boldsymbol{\theta}-\boldsymbol{\beta})\cdot\frac{\boldsymbol{\sigma}}{2}}. \tag{C.8}$$

This is the hermitian adjoint of the $(\frac{1}{2}, 0)$ representation working from the right. Thus we can see a $(\frac{1}{2}, \frac{1}{2})$ object as a matrix, $\mathbf{A}$, transforming like:

$$\mathbf{A} \mapsto \mathbf{M}\mathbf{A}\mathbf{M}^\dagger, \tag{C.9}$$

for $\mathbf{M} \in \mathrm{SL}(2,\mathbb{C})$. Let us now assume that $\mathbf{A}$ is hermitian. Then by defining

$$\sigma^0 = -\mathbf{1}, \qquad \sigma^\mu \sim (-\mathbf{1}, \boldsymbol{\sigma}), \tag{C.10}$$

we get a one to one correspondence between hermitian matrices and real 4-vectors by $V_\mu \leftrightarrow V_\mu\sigma^\mu$. Since (C.9) preserves hermiticity we can define a transformation of a 4-vector like

$$V_\mu\sigma^\mu \mapsto \mathbf{M}V_\mu\sigma^\mu\mathbf{M}^\dagger \equiv V'_\mu\sigma^\mu. \tag{C.11}$$

Now by calculation one gets that $\det(V_\mu\sigma^\mu) = -V_\mu V^\mu$. Using this and the fact that (C.11) preserves the determinant since $\det(\mathbf{M}) = 1$, we see that this transformation is a (proper orthochronous by further calculations) Lorentz transformation. Hence the real part of the representation $(\frac{1}{2}, \frac{1}{2})$ is the same as the defining vector representation of the Lorentz group. Or actually almost. Since both $\mathbf{M}$ and $-\mathbf{M}$ correspond to the same Lorentz transformation, we are actually dealing with a double cover of the Lorentz group. The group that we really are representing, $\mathrm{SL}(2,\mathbb{C})$, is called the spin group. Actually, it is the simply connected extension of the Lorentz group.[2] The spin representations are, of course, representations of the spin group $\mathrm{SL}(2,\mathbb{C})$.

[2]Following [24] a matrix in $\mathrm{SL}(2,\mathbb{C})$ can be written as a matrix from SU(2) times the exponential of a traceless hermitian matrix by the polar decomposition theorem (like in (C.6)). The topology of SU(2) is the same as the three dimensional ball S^3 and the topology of traceless hermitian matrices is the same as $\mathbb{R}^3$. Hence $\mathrm{SL}(2,\mathbb{C})$ is topologically equivalent to $S^3 \times \mathbb{R}^3$. The Lorentz group, which now can be seen as $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, is topologically equivalent to $S^3 \times \mathbb{R}^3/\mathbb{Z}_2$ and is not simply connected – actually, the first homotopy group is $\mathbb{Z}_2$.

C.2 The Clifford Algebra

The spinorial representations can be investigated using the Dirac γ-matrices obeying the Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}\mathbf{1}. \tag{C.12}$$

The connection to the spin group is through

$$\Sigma_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{C.13}$$

which one can check obeys the Lorentz algebra (B.1) and hence generates a spinorial representation.

Until now we have not said anything about the number of space-time dimensions. In most of this thesis, however, we shall use four space-time dimensions. In this case the dimension of the Dirac matrices must be (at least) four and all representations are unitarily equivalent. We will use the Weyl basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{C.14}$$

where $\bar{\sigma}$ is defined as

$$\bar{\sigma}^\mu \sim (-\mathbf{1}, -\boldsymbol{\sigma}). \tag{C.15}$$

Please note that γ^0 is hermitian while γ^i is antihermitian. In the index notation we give the γ-matrices indices $(\gamma^\mu)_a{}^b$ and spinors index down, ψ_a.

Taking the full Clifford algebra as a real algebra it is isomorphic to the algebra of real 4×4 matrices and thus has a natural real four-dimensional irreducible representation – the Majorana spinor. However, looking at the complexified Clifford algebra we get a complex four-dimensional irreducible representation – the Dirac spinors. Both are of course transforming with $\Sigma_{\mu\nu}$ as generators under Lorentz transformations. Connecting back to the (j_+, j_-)-notation the Dirac spinors correspond to $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and the Majorana spinors are the subrepresentation fixed by complex conjugation.

Another way to obtain the Majorana spinors is to look at how to define the conjugate of a spinor, ψ. There are two definitions – the Dirac conjugate, $\bar{\psi}_{\mathrm{D}}$, and the Majorana conjugate, $\bar{\psi}_{\mathrm{M}}$, defined as:

$$\bar{\psi}_{\mathrm{D}} = \psi^\dagger \gamma_0 \tag{C.16}$$
$$\bar{\psi}_{\mathrm{M}} = \psi^T C, \tag{C.17}$$

where C is the charge conjugation matrix defined by $C\gamma_\mu C^{-1} = -\gamma_\mu^T$. In our case $C = i\gamma^2\gamma^0$. Now the Majorana spinors are defined as those Dirac spinors that obey $\bar{\psi}_{\mathrm{D}} = \bar{\psi}_{\mathrm{M}}$. This can be rewritten as the Majorana reality condition:

$$\psi^* = -i\gamma_2\psi. \tag{C.18}$$

Actually, C is not a matrix like the γ-matrices, but rather it transforms like a bilinear form. We define it to have indices up, C^{ab}. Hence by the use of (C.17) it can be used to raise indices on Majorana spinors, but one should be careful since it is antisymmetric. We define raising by $\psi^a = C^{ab}\psi_b$ and hence lowering by $\psi_a = C_{ab}\psi^a$ where $C_{ab} = \left(C^{-1}\right)^{ab}$.

We will not proceed any further in this direction, but rather note that the Dirac and Majorana representations are not irreducible as representations of the spin group. To see this define

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{C.19}$$

Table C.1:

Name	Matrix	Notation	Transformation
Fundamental	$\mathbf{M}$	ψ_α	$\psi'_\alpha = {M_\alpha}^\beta \psi_\beta$
Conjugate	$\mathbf{M}^*$	$\bar{\psi}_{\dot{\alpha}}$	$\bar{\psi}'_{\dot{\alpha}} = {(M^*)_{\dot{\alpha}}}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}$
Dual	$\left(\mathbf{M}^T\right)^{-1}$	ψ^α	$\psi'^\alpha = {M^{-1}}^{\alpha}{}_{\beta} \psi^\beta$
Conjugate dual	$\left(\mathbf{M}^\dagger\right)^{-1}$	$\bar{\psi}^{\dot{\alpha}}$	$\bar{\psi}'^{\dot{\alpha}} = {(M^*)^{-1}}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}$

Representations of $\mathrm{SL}(2,\mathbb{C})$ and their notation.

This matrix has the property that it anticommutes with all other γ-matrices. Hence it commutes with $\Sigma_{\mu\nu}$. By Schur's lemma the spinorial representation is reducible and reduces corresponding to the eigenvalues of γ^5. Actually:

$$\gamma^5 = \begin{pmatrix} -i\mathbf{1} & 0 \\ 0 & i\mathbf{1} \end{pmatrix}, \tag{C.20}$$

showing that the four dimensional representations split into two (irreducible) representations – the Weyl spinors. The two representations are conjugate and are exactly the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ that we have already encountered.

C.3 Weyl Spinors

The two-dimensional representations we uncovered in last section are of course linked with the fact that $\mathrm{SL}(2,\mathbb{C})$ has a natural two-dimensional complex representation acting as a matrix. A two dimensional spinor hence transforms as $\psi_\alpha \mapsto {M_\alpha}^\beta \psi_\beta$ where $\mathbf{M}$ belongs to $\mathrm{SL}(2,\mathbb{C})$. We will use Greek indices (running from 1 to 2) for Weyl spinors while Dirac and Majorana spinors have Latin indices. However, we also get a representation (i.e. we keep the group multiplication structure) if we work with the conjugated, the transposed inverse or the hermitian inverse matrix. This is summarised in table C.1 where we also see the notation for the representations (we are here following [9]). Please note that we use dots on indices that transform in the conjugate representation.

But not all of these representations are inequivalent. Since we are dealing with 2×2 matrices with unit determinant, we have the following relations:

$$\begin{aligned} \varepsilon_{\alpha\beta} &= {M_\alpha}^\gamma {M_\beta}^\delta \varepsilon_{\gamma\delta}, \\ \varepsilon^{\alpha\beta} &= \varepsilon^{\gamma\delta} {M_\gamma}^\alpha {M_\delta}^\beta, \end{aligned} \tag{C.21}$$

where the antisymmetric tensors are defined by (as usual the tensor with indices up is the inverse of the tensor with indices down)

$$\varepsilon^{12} = -\varepsilon^{21} = -\varepsilon_{12} = \varepsilon_{21} = 1. \tag{C.22}$$

Thus we see that the antisymmetric tensors are invariant. Consequently, they can be used to raise and lower indices to define invariant inner products. But one has to be careful since we are dealing with antisymmetric tensors. We define raising and lowering as:

$$\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha \equiv \varepsilon_{\alpha\beta}\psi^\beta. \tag{C.23}$$

This is a consistent definition in itself since $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta = \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma}\psi^\gamma \stackrel{!}{=} \psi^\alpha$ by the use of $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$. But the definition is also consistent with table C.1 since:

$$\begin{aligned}\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \quad \mapsto \quad & \varepsilon^{\alpha\beta} {M_\beta}^\gamma \psi_\gamma = {M_\delta}^\epsilon {M^{-1}}_\epsilon{}^\alpha \varepsilon^{\delta\beta} {M_\beta}^\gamma \psi_\gamma \\ & = {M^{-1}}_\epsilon{}^\alpha \varepsilon^{\epsilon\gamma}\psi_\gamma = {M^{-1}}_\beta{}^\alpha \psi^\beta .\end{aligned}$$

Consequently, the fundamental representation and the dual representation are equivalent. The same holds true for the conjugate and the conjugate dual representations – raising and lowering are defined in exactly the same way with the same definition of the ε-tensor with dotted indices. The invariant products are defined as:

$$\psi\chi \equiv \psi^\alpha \chi_\alpha, \qquad \bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}. \tag{C.24}$$

The reason for these contraction conventions will become clear later. But please note that the ordering is important since e.g. $\psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\psi_\alpha\chi^\alpha$. Keeping in mind that half-integer spinors must be represented by anticommuting Grassmann numbers we actually see that these contractions commute since e.g. $\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\varepsilon^{\alpha\beta}\psi_\beta = \chi^\beta\psi_\beta = \chi\psi$. Looking back at the transformation (C.11) we see that the index structure of the σ-matrices is:

$$(\sigma^\mu)_{\alpha\dot\beta} \, . \tag{C.25}$$

Hence we get yet another Lorentz scalar by the contraction $\psi^\alpha\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\chi^{\dot\beta}$. We can also use the ε-matrices to raise and lower the indices on the σ-matrices using the definition (C.23). Using this we actually get:

$$\bar\sigma^{\dot\alpha\beta} = \sigma^{\beta\dot\alpha} = \varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\beta\delta}\sigma_{\delta\dot\gamma}. \tag{C.26}$$

Inserting this index structure in (C.13) we get:

$$\begin{aligned}\Sigma_{\mu\nu} &= \frac{i}{4}\begin{pmatrix} (\sigma_\mu)_{\alpha\dot\gamma}(\bar\sigma_\nu)^{\dot\gamma\beta} - (\sigma_\nu)_{\alpha\dot\gamma}(\bar\sigma_\mu)^{\dot\gamma\beta} & 0 \\ 0 & (\bar\sigma_\mu)^{\dot\alpha\gamma}(\sigma_\nu)_{\gamma\dot\beta} - (\bar\sigma_\nu)^{\dot\alpha\gamma}(\sigma_\mu)_{\gamma\dot\beta}\end{pmatrix} \\ &\equiv i\begin{pmatrix} {(\sigma_{\mu\nu})_\alpha}^\beta & 0 \\ 0 & {(\bar\sigma_{\mu\nu})^{\dot\alpha}}_{\dot\beta}\end{pmatrix},\end{aligned} \tag{C.27}$$

where we also defined $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$. The diagonal Lorentz generators (the benchmark of the Weyl basis for the γ-matrices) directly show us how the Dirac spinor, Ψ_{D}, and the Majorana spinor, Ψ_{M}, split into two Weyl spinors:

$$\Psi_{\mathrm{D}} \sim \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix}, \qquad \Psi_{\mathrm{M}} \sim \begin{pmatrix}\psi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix}. \tag{C.28}$$

Here we have used the Majorana reality condition (C.18) in writing the Majorana spinor. We have also used how to complex conjugate the Weyl spinor as will be defined in the next subsection. Rewriting the $\Sigma_{\mu\nu}$ generators to $\mathbf{K}$ and $\mathbf{L}$ we can easily see that the upper spinor transforms according to (C.6), i.e. like $(\frac{1}{2}, 0)$, while the lower spinor transforms according to (C.7), i.e. like $(0, \frac{1}{2})$.[3]

[3]We note that the transformation (C.7) is not exactly the conjugate of (C.6). In order to get the exact conjugate we have to do, as we did when going from (C.7) to the exactly conjugate representation in (C.8). Since the conjugate representation (that we used in table C.1) and (C.7) are equivalent, it does not matter which one we choose the dotted indices to transform in – as long as we consistently choose the same everywhere.

C.4 Complex Conjugation

We define complex conjugation on Weyl spinors as the involution:

$$(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}}, \qquad (\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}, \tag{C.29}$$

where the last part simply followed since we chose the ε-tensor with and without dotted indices to be the same. This definition fits with table C.1. However, when complex conjugation works on products of anticommuting spinors it reverses the order (or equivalently it adds an appropriate sign), e.g.:

$$(\psi_\alpha \chi_\beta)^* = \bar{\chi}_{\dot{\beta}} \bar{\psi}_{\dot{\alpha}}. \tag{C.30}$$

Now we see the reason for the placement of the indices in (C.24) because with these definitions we get the nice equation:

$$(\psi\chi)^* = (\psi^\alpha \chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi}. \tag{C.31}$$

Using the definition of complex conjugation we can show that the inner product of two Majorana spinors is real. First we find the Majorana conjugate in the Weyl spinor formalism: (using (C.17)):

$$\bar{\Psi}_{\mathrm{M}} = \left(\psi^\alpha, \bar{\psi}_{\dot{\alpha}}\right) \tag{C.32}$$

Hence the product of two Majorana spinors (Ψ and $\Phi = \begin{pmatrix} \phi_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix}$) becomes:

$$\bar{\Psi}_{\mathrm{M}} \Phi = \psi\phi + \bar{\psi}\bar{\phi}. \tag{C.33}$$

This is real by (C.31).

C.5 Differentiation of Spinors

The last thing we have to settle is the differentiation of Weyl spinors. As usual for Grassmannian differentiation it should fulfil:

$$\{\frac{\partial}{\partial \psi^\alpha}, \psi^\beta\} = \delta_\alpha^\beta, \tag{C.34}$$

and the same with indices lowered. But now we have to be careful when defining raising, lowering and conjugation of the differential. Using the definition of raising we see:

$$\{-\varepsilon^{\alpha\beta} \frac{\partial}{\partial \psi^\beta}, \psi_\gamma\} = -\varepsilon^{\alpha\beta} \varepsilon_{\gamma\delta} \{\frac{\partial}{\partial \psi^\beta}, \psi^\delta\} = -\varepsilon^{\alpha\beta} \varepsilon_{\gamma\delta} \delta_\beta^\delta = \delta_\gamma^\alpha.$$

Consequently we have:

$$-\varepsilon^{\alpha\beta} \frac{\partial}{\partial \psi^\beta} = \frac{\partial}{\partial \psi_\alpha}. \tag{C.35}$$

Also when complex conjugating spinorial derivatives we have to be a bit careful. When conjugating (C.34) we have to reverse the order of the products, however, conjugation should not change on which part the derivative acts. So instead of changing order we preserve the order and add a compensating sign. For a general field A we then have:

$$\{\frac{\partial}{\partial \psi^\alpha}, A\}^* = (-1)^{|A|} \{\left(\frac{\partial}{\partial \psi^\alpha}\right)^*, (A)^*\}, \tag{C.36}$$

where $|A|$ is 0 if A is bosonic, and 1 if it is fermionic. Using this on (C.34) gives

$$\left(\frac{\partial}{\partial\psi^\alpha}\right)^* = -\frac{\partial}{\partial\bar{\psi}^{\dot{\alpha}}}. \tag{C.37}$$

So complex conjugation on differentials does not work in the same way as on spinors.[4]

The hermitian adjoint is defined by:

$$\int \mathrm{d}x\mathrm{d}\psi\mathrm{d}\bar{\psi}B^*\frac{\partial}{\partial\psi^\alpha}A \equiv \int \mathrm{d}x\mathrm{d}\psi\mathrm{d}\bar{\psi}\left(\left(\frac{\partial}{\partial\psi^\alpha}\right)^\dagger B\right)^* A, \tag{C.38}$$

where $\int \mathrm{d}\psi\mathrm{d}\bar{\psi}$ is the integration over the spinorial degrees of freedom while A and B are two arbitrary fields. Taking care of all signs, also when doing the integration by parts, this gives

$$\left(\frac{\partial}{\partial\psi^\alpha}\right)^\dagger = \frac{\partial}{\partial\bar{\psi}^{\dot{\alpha}}}, \tag{C.39}$$

contrary to the usual derivative.

C.6 Space-time Dimensions

Even though this thesis deals with four space-time dimensions let us appreciate that it is only in a few space-time dimensions that it is possible to construct an $\mathcal{N}=1$ supersymmetric Yang-Mills theory. Following [35] let us look at which spinorial representations it is possible to make in a general dimension. We will assume that the signature of the metric is Lorentzian, i.e. we are looking at the group $\mathrm{SO}(D-1,1)$ where D is the space-time dimension.

In every space-time dimension it is possible to make the Dirac representation which after imposing the Dirac equation has dimension (i.e. number of real degrees of freedom) $2^{[D/2]}$. The brackets means the least whole number closest to $D/2$.

However, in even dimensions this representation is not irreducible since we can make a chirality matrix like γ^5. This cuts the Dirac representation into two inequivalent Weyl representations both of dimension $2^{[D/2-1]}$. The two representations are conjugate if $D \equiv 0 \pmod 4$ and they are self-conjugate if $D \equiv 2 \pmod 4$.

The Majorana reality condition defines the Majorana representation and cuts the dimension to the half: $2^{[D/2-1]}$. However, if the Majorana condition is to be consistent one must require $D \equiv 0,1,2,3,4 \pmod 8$.

A last representation can be obtained if we impose both the Majorana condition and the Weyl condition. But this requires that the Weyl representation is self-conjugate. Hence this Majorana-Weyl representation is only possible if $D \equiv 2 \pmod 8$. The dimension is then $2^{[D/2-2]}$.

When we look at supersymmetry we find that the number of bosonic and fermionic degrees of freedom must be the same. The number of bosonic degrees of freedom in the Yang-Mills model is $D-2$ (the transverse directions in the gauge field A_μ). This must fit with one of the above irreducible representations. Simple counting now tells us in which space-time dimensions and with which representations it is possible to make a supersymmetric Yang-Mills theory. The result can be seen in table C.2. We see that four space-time dimensions is one of the few dimensions in which it is possible. Please note that due to (C.28) the Majorana and the Weyl representations are equivalent.

[4] This could have been solved if we had defined the ε-tensor with dotted indices with a minus sign contrary to the ε-tensor without dotted indices.

Table C.2:

D	Spinor representation
3	Majorana
4	Majorana or Weyl
6	Weyl
10	Majorana-Weyl

The possible space-time dimensions, D, for which supersymmetric Yang-Mills theory can be realised and the corresponding representations. The table is taken from [11].

C.7 Weyl Spinor Algebra

Using our definitions we can find a lot of useful identities for Weyl spinors. Without proof we give the following relations for ψ, χ and θ being anticommuting Weyl spinors. The contractions follow the logic of (C.24), σ having indices (C.25) and $\bar{\sigma}$ having indices (C.26) such that e.g. $\chi\sigma^\mu\bar{\psi} = \chi^\alpha\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}$ (the relations are taken from [9]):

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\tfrac{1}{2}\varepsilon^{\alpha\beta}\theta\theta, \\ \theta_\alpha\theta_\beta &= \tfrac{1}{2}\varepsilon_{\alpha\beta}\theta\theta, \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \\ \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\tfrac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}. \end{aligned} \tag{C.40}$$

$$\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = -\tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}. \tag{C.41}$$

$$\left(\theta\psi\right)\left(\theta\chi\right) = -\tfrac{1}{2}\left(\psi\chi\right)\left(\theta\theta\right). \tag{C.42}$$

$$\left(\bar{\theta}\bar{\psi}\right)\left(\bar{\theta}\bar{\chi}\right) = -\tfrac{1}{2}\left(\bar{\psi}\bar{\chi}\right)\left(\bar{\theta}\bar{\theta}\right). \tag{C.43}$$

$$\begin{aligned} \chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi, \\ \left(\chi\sigma^\mu\bar{\psi}\right)^\dagger &= \psi\sigma^\mu\bar{\chi}. \end{aligned} \tag{C.44}$$

$$\begin{aligned} \chi\sigma^\mu\bar{\sigma}^\nu\psi &= \psi\sigma^\nu\bar{\sigma}^\mu\chi, \\ \left(\chi\sigma^\mu\bar{\sigma}^\nu\psi\right)^\dagger &= \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi}. \end{aligned} \tag{C.45}$$

$$\left(\psi\theta\right)\bar{\chi}_{\dot{\beta}} = -\tfrac{1}{2}\left(\theta\sigma^\mu\bar{\chi}\right)\left(\psi\sigma_\mu\right)_{\dot{\beta}}. \tag{C.46}$$

$$\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = -2\eta^{\mu\nu}. \tag{C.47}$$

$$\mathrm{Tr}\Big(\sigma^{\mu\nu}\sigma^{\rho\delta}\Big) = -\frac{1}{2}\Big(\eta^{\mu\rho}\eta^{\nu\delta} - \eta^{\mu\delta}\eta^{\nu\rho}\Big) - \frac{i}{2}\varepsilon^{\mu\nu\rho\delta}. \tag{C.48}$$

$$\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\theta\theta = 4. \tag{C.49}$$

Appendix D

$\mathcal{N} = 2$ Superspace and The Prepotential

In this appendix we will continue the analysis of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory from section 1.4. But here we will not assume renormalisability of the Lagrangian.

In order to get the most general $\mathcal{N} = 2$ supersymmetric pure Yang-Mills Lagrangian with no constriction of renormalisability it is an ease to use the $\mathcal{N} = 2$ superspace formulation (following [13] and [14]). This is a simple extension of the $\mathcal{N} = 1$ superspace from section 1.3.1 where the θ^α-coordinates now carry an extra $\mathrm{SU}(2)_R$ index: θ_i^α with $i = 1, 2$. Thus we have one θ^α-coordinate for each supercharge. The new index can be raised and lowered as the spinor index using the SU(2) invariant antisymmetric tensor ε_{ij}.

The supercharges are linearly realised on superfields in the same way as in the $\mathcal{N} = 1$ case – we just have to put the index i on the θ's and the Q's. We also get the covariant derivatives D_α^i and $\bar{D}_{\dot{\alpha}i}$ where as an example:

$$\bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}} - i\theta_i^\beta \left(\sigma^\mu\right)_{\beta\dot{\alpha}} \partial_\mu. \tag{D.1}$$

A chiral superfield is then defined as:

$$\bar{D}_{\dot{\alpha}i}\Phi = 0, \qquad \dot{\alpha} = \dot{1}, \dot{2},\ i = 1, 2. \tag{D.2}$$

In analogy with the $\mathcal{N} = 1$ case a differentiable function of chiral superfields is again a chiral superfield. Also the variation under supersymmetry of the component with four θ's and no $\bar{\theta}$'s is a total derivative. Thus a possible Lagrangian is:

$$\mathcal{L} = \int \mathrm{d}^2\theta^1 \mathrm{d}^2\theta^2 \Phi, \tag{D.3}$$

where the indices on the differentials are the $\mathrm{SU}(2)_R$ indices.

The chiral field is not irreducible as in the $\mathcal{N} = 1$ case. This means that we have to impose further constraints. To do this we introduce the $\mathcal{G}$-valued supergauge fields $A_{\alpha i}$ and $\bar{A}_{\dot{\alpha}i}$. With these we define the gauge-covariant version of the supersymmetric covariant derivatives as:

$$\tilde{D}_{\alpha i} = D_{\alpha i} + iA_{\alpha i} \qquad \tilde{\bar{D}}_{\dot{\alpha}i} = \bar{D}_{\dot{\alpha}i} + i\bar{A}_{\dot{\alpha}i}. \tag{D.4}$$

The constraint needed to generate an $\mathcal{N} = 2$ gauge field out of a chiral $\mathcal{G}$-valued field W (not to be confused with the $\mathcal{N} = 1$ superpotential) turns out to be:

$$\tilde{D}^{\alpha i}\tilde{D}_\alpha^j W = \tilde{\bar{D}}_{\dot{\alpha}}^i \tilde{\bar{D}}^{\dot{\alpha}j} W^\dagger. \tag{D.5}$$

Expanding this field as a power series in θ^2 gives the $\mathcal{N}$=1 superfield components:

$$W(\tilde{x}_+, \theta^1, \theta^2) = \Phi\big(\tilde{x}_+, \theta^1\big) + \sqrt{2}\theta^{\alpha 2}\mathcal{W}_\alpha\big(\tilde{x}_+, \theta^1\big) + \theta^2\theta^2 G\big(\tilde{x}_+, \theta^1\big)\,, \tag{D.6}$$

where $\tilde{x}_+^\mu = x^\mu + i\theta_i\sigma^\mu\bar{\theta}^i$ is the $\mathcal{N} = 2$ version of x_+. Φ and $\mathcal{W}$ are the $\mathcal{N} = 1$ chiral field and gauge field strength respectively corresponding to the splitting of the $\mathcal{N} = 2$ gauge supermultiplet into the $\mathcal{N} = 1$ chiral and gauge supermultiplet. $\mathcal{W}$ is based on the vector superfield V. G may be expressed as:

$$G\big(\tilde{x}_+, \theta^1\big) = -\int \mathrm{d}^2\bar{\theta}^1 \Phi\big(\tilde{x}_+ - i\theta_1\sigma\bar{\theta}^1, \theta^1, \bar{\theta}^1\big)^\dagger\, e^{2[-,V(\tilde{x}_+ - i\theta_1\sigma\bar{\theta}^1, \theta^1, \bar{\theta}^1)]}, \tag{D.7}$$

where $[-, V]$ simply means that the adjoint is working to the left with V to the right in the commutators. We see that the field W has the component fields of the $\mathcal{N} = 2$ gauge multiplet (after eliminating the auxiliary fields). The most general gauge invariant supersymmetric Lagrangian involving W is now:

$$\mathcal{L}_{\mathcal{N}=2} = -\frac{1}{8\pi i}\int \mathrm{d}^2\theta^1\mathrm{d}^2\theta^2\, \mathrm{Tr}\,\mathcal{F}(W) + \text{c.c.}, \tag{D.8}$$

where $\mathcal{F}$ is any holomorphic function. $\mathrm{Tr}\,\mathcal{F}$ is called the *prepotential*. The trace ensures gauge invariance since the fields transform in the adjoint representation. We can expand this in components as we did in (1.68) using (1.35) and (1.36):

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{16\pi i}\left(\int \mathrm{d}^2\theta^1 \mathcal{F}_{ab}(\Phi)\,\mathcal{W}^{\alpha a}\mathcal{W}^b_\alpha + 2\int \mathrm{d}^2\theta^1\mathrm{d}^2\bar{\theta}^1 \left(\Phi^\dagger e^{2[-,V]}\right)^a \mathcal{F}_a(\Phi)\right) + \text{c.c.} \tag{D.9}$$

Here $\mathcal{F}_a$ and $\mathcal{F}_{ab}$ are simply the derivatives $\mathcal{F}_a = \partial\mathcal{F}/\partial\Phi^a$ and $\mathcal{F}_{ab} = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ respectively where we have made an abuse of notation and used the name $\mathcal{F}$ for $\mathrm{Tr}\,\mathcal{F}$ as a function of Φ^a. Setting $\mathcal{F} = \frac{1}{2}\tau\Phi^a\Phi^a$ we get the renormalisable Lagrangian (1.84).[1] We can also see that the Kähler potential (without $1/4\pi$) is $\mathrm{Im}\big(\Phi^{\dagger a}\mathcal{F}_a(\Phi)\big)$ and thus the Kähler metric according to (1.73) is $g_{ab} = \mathrm{Im}(\mathcal{F}_{ab})$.

[1] To see that $[-, V]$ (working to the left) is the same as ad$V$ (working to the right) we have to rewrite the sum over the adjoint indices as a trace and use the cyclic properties of the trace.

Appendix E

Calculation of Integrals in the Matrix Model

In this appendix we calculate the integrals needed in section 2.7.4.

Let us first derive equation (2.186):

$$\frac{\partial}{\partial g_{\mathrm{m}}}\big(g_{\mathrm{m}}\rho(\lambda, g_{\mathrm{m}})\big) = \frac{1}{\pi\sqrt{(b-\lambda)(\lambda-a)}}.$$

In principle we could obtain this by differentiating the equation (2.183) we have obtained for $R(z)$ – remembering that a and b depend on g_{m}. However, it is more easy to note that (following [49]):

$$\Omega(z) \equiv \frac{\partial g_{\mathrm{m}} R(z)}{\partial g_{\mathrm{m}}} = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{\lambda - z}\frac{\partial}{\partial g_{\mathrm{m}}}\big(g_{\mathrm{m}}\rho(\lambda, g_{\mathrm{m}})\big), \tag{E.1}$$

is an analytic function with branch cut $[a, b]$ and fulfils: It has no regular part due to (2.176), like $R(z)$ it must behave like $-1/z$ for large $|z|$, and since $R(z)$ behaves like $\sqrt{z-a}$ close to the branch point a, the derivative $\Omega(z)$ can behave at most as $1/\sqrt{z-a}$ (and the same for b). This determines Ω uniquely to be:

$$\Omega(z) = -\frac{1}{\sqrt{(z-b)(z-a)}}. \tag{E.2}$$

Using (E.1) we then obtain in analogy with (2.175):

$$\frac{\partial}{\partial g_{\mathrm{m}}}\big(g_{\mathrm{m}}\rho(\lambda, g_{\mathrm{m}})\big) = \frac{1}{2\pi i}\left(\Omega(\lambda+i\epsilon) - \Omega(\lambda - i\epsilon)\right) = \frac{1}{\pi\sqrt{(b-\lambda)(\lambda-a)}}, \tag{E.3}$$

as wanted.

Let us now derive (2.187) following [5]:

$$I(\lambda') \equiv \int_a^b \mathrm{d}\lambda\, \frac{\ln|\lambda-\lambda'|}{\sqrt{(b-\lambda)(\lambda-a)}} = \pi \ln\left(\frac{b-a}{4}\right), \quad \forall \lambda' \in [a, b].$$

In analogy with equation (2.176) we have:

$$\frac{\partial I}{\partial \lambda'} = \frac{1}{2}\big(h(\lambda'+i\epsilon) + h(\lambda'-i\epsilon)\big), \quad \lambda' \in \mathbb{R}, \tag{E.4}$$

where

$$h(z) = \int_a^b \mathrm{d}\lambda \, \frac{1}{(z-\lambda)\sqrt{(b-\lambda)(\lambda-a)}}. \tag{E.5}$$

h is analytic in $\mathbb{C} \setminus [a,b]$ and in analogy with (2.175) we have:

$$h(\lambda' + i\epsilon) - h(\lambda' - i\epsilon) = -\frac{2\pi i}{\sqrt{(b-\lambda')(\lambda'-a)}}, \quad \lambda' \in [a,b]. \tag{E.6}$$

From the definition (E.5) we also see (using equation (E.15) below) that $h \sim \pi/z$ for large $|z|$. This determines h uniquely as:

$$h(z) = \frac{\pi}{\sqrt{(z-b)(z-a)}}, \quad z \in \mathbb{C} \setminus [a,b]. \tag{E.7}$$

Thus using (E.4) we get:

$$\frac{\partial I}{\partial \lambda'}(\lambda') = \begin{cases} 0 & \text{for } \lambda' \in [a,b] \\ h(\lambda') & \text{for } \lambda' \in \mathbb{R} \setminus [a,b] \end{cases}. \tag{E.8}$$

For $\lambda' > b$ we then get:

$$I(\lambda') = \pi \ln\Big(2\sqrt{(\lambda'-b)(\lambda'-a)} + 2\lambda' - a - b\Big) - \pi \ln 4, \quad \lambda' > b, \tag{E.9}$$

where the constant of integration has been fixed using that $I(\lambda') \sim \pi \ln(\lambda')$ for large λ'. Using the continuity of $I(\lambda')$ we then get the wanted result by plugging $\lambda' = b$ into (E.9).

Let us finally show how to derive equations (2.189). From equations (2.184), (2.185) and (2.188) we see that basically all we need is the integral:

$$I(p) \equiv \int_a^b \mathrm{d}\lambda \frac{\lambda^p}{\sqrt{(b-\lambda)(\lambda-a)}}. \tag{E.10}$$

Defining $\lambda' = \lambda - (a+b)/2$ we get:

$$\begin{aligned} I(p) &= \int_{-(b-a)/2}^{(b-a)/2} \mathrm{d}\lambda' \frac{\left(\lambda' + \frac{b+a}{2}\right)^p}{\sqrt{\left(\frac{b-a}{2}\right)^2 - \lambda'^2}} = \sum_{k=0}^{p} \binom{p}{k} \int_{-(b-a)/2}^{(b-a)/2} \mathrm{d}\lambda' \frac{\lambda'^k \left(\frac{b+a}{2}\right)^{p-k}}{\sqrt{\left(\frac{b-a}{2}\right)^2 - \lambda'^2}} \\ &= \sum_{k=0}^{p} \binom{p}{k} \left(\frac{b+a}{2}\right)^{p-k} \left(\frac{b-a}{2}\right)^k \int_{-1}^{1} \mathrm{d}\lambda' \frac{\lambda'^k}{\sqrt{1-\lambda'^2}} \\ &= \sum_{q=0}^{[p/2]} \binom{p}{2q} \left(\frac{b+a}{2}\right)^{p-2q} \left(\frac{b-a}{2}\right)^{2q} 2\int_0^1 \mathrm{d}\lambda' \frac{\lambda'^{2q}}{\sqrt{1-\lambda'^2}}, \end{aligned} \tag{E.11}$$

where we have used that the integral of an odd function is zero in the last line. Substituting $\lambda' = \sin y$ and using integrating by parts we see that:

$$\int_0^1 \mathrm{d}\lambda' \frac{\lambda'^k}{\sqrt{1-\lambda'^2}} = \int_0^{\pi/2} \mathrm{d}y \, \sin^k y = \frac{k-1}{k} \int_0^{\pi/2} \mathrm{d}y \, \sin^{k-2} y. \tag{E.12}$$

Thus:

$$\int_0^1 \mathrm{d}\lambda' \frac{1}{\sqrt{1-\lambda'^2}} = \frac{\pi}{2}, \tag{E.13}$$

and hence

$$\int_0^1 \mathrm{d}\lambda' \frac{\lambda'^{2q}}{\sqrt{1-\lambda'^2}} = \frac{1\cdot 3\cdots(2q-1)}{2\cdot 4\cdots(2q)}\frac{\pi}{2}. \tag{E.14}$$

Using that $\binom{2q}{q} = 2^{2q}\frac{1\cdot 3\cdots(2q-1)}{2\cdot 4\cdots(2q)}$ we finally get:

$$I(p) = \pi \sum_{q=0}^{[p/2]} \binom{p}{2q}\binom{2q}{q}\left(\frac{b+a}{2}\right)^{p-2q}\left(\frac{b-a}{4}\right)^{2q}. \tag{E.15}$$

We then obtain (2.189a) and (2.189c) simply by expanding $P_{n+1}(\lambda)$ in respectively (2.184) and (2.188). (2.189b) is obtained by expanding

$$\int_a^b \mathrm{d}\lambda\, \frac{(\lambda-(b+a)/2)P'_{n+1}(\lambda)}{\sqrt{(b-\lambda)(\lambda-a)}} = 2\pi g_{\mathrm{m}}, \tag{E.16}$$

which is (2.185) with the addition of (2.184) times $-(b+a)/2$. (E.16) is the original condition that $R(z) \sim -1/z$ for large $|z|$ in equation (2.183). The reason we write equation (2.189b) in this way is for comparison with the result obtained on the gauge theory side using Seiberg-Witten theory in [5].

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