Aspects of the correspondence between string theory on orbifolds and quiver gauge theories

Master's Thesis

by

Rafal Mostowy

Under supervision of Niels Obers

Niels Bohr Institute,
University of Copenhagen,
August 2007

"Nobody ever promised you a rose garden"
- Joe Polchinski

Contents

Introduction **vii**

1 The gauge/string duality **1**

1.1 Anti-de Sitter space . . . 1

1.1.1 The global coordinates . . . 2

1.1.2 Conformal compactification of AdS_{d+1} . . . 2

1.1.3 Polar/stereographic coordinates . . . 4

1.1.4 Poincaré coordinates . . . 6

1.2 Conformal field theory . . . 7

1.2.1 Conformal group . . . 7

1.2.2 Correlation function in conformal field theory . . . 9

1.2.3 Superconformal invariance of $\mathcal{N} = 4$ SYM . . . 11

1.2.4 Chiral primaries . . . 13

1.2.5 Conformal symmetry of $\mathcal{N} = 4$ SYM is maintained after quantisation 15

1.3 The AdS/CFT correspondence . . . 16

1.3.1 Type IIB string theory and gauge theories . . . 16

1.3.2 The near-horizon approximation . . . 17

1.3.3 Comparison of the string theory and the gauge theory . . . 18

1.3.4 Implications of the correspondence . . . 19

1.3.5 The dictionary of Maldacena's conjecture . . . 20

1.3.6 Two-point correlation function in AdS/CFT . . . 22

1.4 BMN operators and the double scaling limit . . . 25

2 The Dilatation Operator **29**

2.1 Correlation function of $\mathcal{N} = 4$ SYM . . . 29

2.2 Anomalous dimensions from the two-point function . . . 31

2.3 Anomalous dimensions from the dilatation operator . . . 32

2.4 A short manual for the dilatation operator . . . 38

2.5 Splitting up $SO(6)$ . . . 40

2.5.1 Splitting up the dilatation operator . . . 41

2.5.2 Splitting up the basis . . . 42

2.6 $SU(3)$ and $SU(2)$ bosonic subsectors . . . 44

Summary . . . 45

3 Quantisation of orbifolded string theory in the pp-wave limit 47
3.1 The orbifold/quiver gauge correspondence 47
3.2 $\mathcal{N}=2$ quiver gauge theory from $\mathcal{N}=4$ SYM 49
3.2.1 The set-up 50
3.2.2 $\mathcal{N}=2$ QGT bosonic action 52
3.3 Double scaling limit and the DLCQ pp-wave 54
3.4 DLCQ quantisation of the type IIB plane-wave string 57
3.5 Identification of charges 60
3.6 Matching the string states with the gauge operators 61
3.7 Rewriting $\mathcal{N}=2$ MRV states in $\mathcal{N}=4$ notation 64
3.8 Spectrum comparison 66
3.8.1 $\mathcal{N}=2$ MRV states 66
3.8.2 $\mathcal{N}=4$ MRV states 73
Summary 76

4 Semiclassical spinning strings in $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ 79
4.1 Strings rotating in $\mathbb{R}_t \times \mathbf{S}^5/\mathbb{Z}_M$ 79
4.1.1 Geometry and symmetries 79
4.1.2 Semiclassical solutions 81
4.1.3 Remark on integrability 84
4.1.4 Semiclassical solutions continued 86
4.1.5 Specification to $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$ 88
4.2 The dual $\mathcal{N}=2$ gauge theory side 89
4.2.1 Bethe equations for the full $SU(2)$ subsector 90
4.2.2 Solution 93
Summary 97

Outlook 99

A One-loop integrals 101

B Proof of the equivalence of MRV states 103

Acknowledgements 107

Bibliography 109

Introduction

Physics has certainly one of the most surprising and fascinating histories of all sciences. Nowadays, it is hard to find a field of natural science that has not been influenced in some way by rapid development of physics in the twentieth century. Ironically, in the end of 1900's, namely just before the beginning of the scientific revolution, people started to suspect that all that had been found at that time (electrodynamics, thermodynamics, some aspects of nuclear physics, and others) would never be unified, and physics is at its dead end. And then, a mere two decades later came Einstein with his special, and then general, theory of relativity, then Bohr, Dirac, Heisenberg, Schrödinger, et al. with probably the most mysterious theory of all, the quantum mechanics, then the idea of quantum fields by Dyson, Schwinger, and others, and astonishing amount of other discoveries in all fields of physics, solid state physics, atomic physics, cosmology, particle physics, and so forth.

But although so many different, astounding papers have been published in the last century, all theoretical physicists have had the same goal in their minds - a unification. Even though now we have the theory, called Standard Model, that has managed to combine three of the four fundamental interactions in nature, as of today gravity remains dissociated from other forces, and Einstein's theory of relativity resides out of the context of Standard Model.

String theory appeared as an attempt to unify gravity with the other three forces, and so far it is the only serious candidate for a theory of everything. The idea of a string as a fundamental object of nature goes back to the 1960's, however until 1984, when Green and Schwartz proved that a theory based on it is free of anomalies, nobody really had been taking it seriously. In the 1990's Witten showed that all five, different superstring theories that had been known can in fact be unified to one theory which he called M-theory. This all, followed by Polchinski's discovery in 1995 of higher-dimensional objects in string theory, called D-branes, has to a large extent built a picture of string theory which we know today - a mathematically consistent, unified, beautiful theory which is extremely difficult (if not impossible) to test experimentally, and thus not very well established in order to describe the real world. For that reason, one can say that while in a mathematical sense superstring theory is indeed a theory (although still not everything is understood), from the physical point of view it is rather a conjecture.

Soon after the discovery of D-branes a connection between string theory and gauge theories was noticed, subsequently followed by a significant number of publications on that subject. This has led to a full establishment of the string/gauge duality in 1997 by Maldacena [1], and this discovery, together with further details worked out by Gubser, Klebanov, Polyakov, and Witten [2, 3], has been known as the holographic principle, or the AdS/CFT duality. It claims that the ten dimensional type IIB superstring theory compactified on the $AdS_5 \times \mathbf{S}^5$ manifold[1] is equivalent to the maximally supersymmetric Yang-Mills theory in four dimensions, which is the $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory with the $SU(N)$ gauge group. The main assumption is that in order for the conjecture to hold, one has to take the limit of large N by keeping the product $\lambda = 4\pi g_s N$ fixed. This idea is based on an older observation by 't Hooft [4] that the perturbative expansion in $1/N$ is very similar in nature to the perturbative genus expansion in a generic interacting string theory, and hence the name of this limit: the 't Hooft limit. Ironically, the most exciting feature of the AdS/CFT duality is also its biggest problem, and that is because it relates two entirely different theories with two completely different properties, which itself is phenomenal but also makes the duality very difficult to test. One can understand this by looking at its basic dictionary

$$g_{\mathrm{YM}}^2 = \frac{\lambda}{N} = 4\pi g_s; \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda}, \tag{1}$$

where λ is the 't Hooft coupling, g_s is the string coupling constant, g_{YM}^2 is the Yang-Mills coupling constant, $\sqrt{\alpha'}$ is the fundamental string length, and R is the radius of AdS_5 and $\mathbf{S}^5$. If one now takes the string theory in a highly curved background, namely where the string length is of the order of the radius of manifolds it lives on ($\alpha' \sim R^2$), then the coupling gauge constant will be relatively small. On the other hand, if the curvature of the background is small in string units ($\alpha' \ll R^2$), then g_{YM}^2 will be large. And since the tests are being performed pertubatively, the allowance for an expansion on the one side (for example the flat string background) usually makes the expansion on the other side unfeasible (strongly coupled gauge theory), and therefore one can see that the gauge/string duality is a weak/strong type of duality.

In spite of great interest in the subject of many theorists and already some confirmations of Maldacena's conjecture (successfully) performed soon after his publication, the weak/strong duality problem remained. For example, the AdS/CFT correspondence predicts that the scaling dimensions Δ of local operators in gauge theory should scale as

$$\Delta(\lambda, \tfrac{1}{N}) \equiv E(\tfrac{R^2}{\alpha'}, g_s), \tag{2}$$

where E is the energy spectrum of dual string theory states. If we want to quantise the string theory, in order to obtain the string states, we need $R^2 \gg \alpha'$ but then how can we expand the two point function of local gauge states from which we can read off the scaling dimensions? One of the ideas to get around this was the Berenstein, Maldacena, and Nastase's suggestion [5] to quantise the string theory in the pp-wave limit, and consider the gauge operators with large J charge on $\mathbf{S}^5$. This would allows to expand the spectra on both sides by taking

$$\lambda, J \to \infty, \qquad \frac{\lambda}{J^2} = \text{finite}, \qquad \Delta - J = \text{finite}, \tag{3}$$

and indeed the spectra do match up to two loops, however there is a disagreement at three loops.

A different idea, yet based on the same limit, is due to Frolov and Tseytlin [6]. They have considered the semiclassical picture of spinning strings in a submanifold of $AdS_5 \times$

[1] The AdS_5 is the five dimensional, hyperbolic anti-de Sitter space, and $\mathbf{S}^5$ is the five sphere.

$\mathbf{S}^5$, and due to the limit (3), which in the semiclassical picture is merely a limit of large conserved charges, quantum effects become suppressed. One can then calculate the energy of spinning strings as a function of other charges, and compare it to the gauge states, where the calculation of scaling dimensions as functions of conserved charges is performed by the Bethe ansatz procedure[2]. The hope of integrability in $\mathcal{N}=4$ appears because of Minahan and Zarembo's observation [9], followed by other tests [41, 42, 43, 44], that the $\mathcal{N}=4$ anomalous dimension matrix can be mapped to an integrable, Heisenberg system of spin chains. However, here the same problem occurs as in the case of the pp-wave limit, namely the discrepancy at three loops appears again. It is believed that the reason for this is that even though the ratio $\frac{\lambda}{J^2}$ is kept finite, it is still large enough so that the expansion breaks down at some point, and one has to introduce the so-called dressing factor [7, 8].

A particularly interesting aspect of the AdS/CFT correspondence is what happens to it when one acts on both sides of the duality with a discrete group. Manifolds modded out by discrete groups have been known for some time now under the name of orbifolds, and it has also been known since mid-1980's that string theories on orbifolds might have their supersymmetry reduced with respect to their "parent" theories (the same string theory on the smooth manifold in the covering space), and that is because strings propagating on orbifolds might encounter singular points which break the supersymmetry of the theory. Therefore one might expect that the gauge/string duality modded out by some discrete group could yield a new kind of duality, and the properties of the original holographic principle as we know might help us determine if the gauge theories with less supersymmetry are integrable, like we suspect the $\mathcal{N}=4$ SYM theory is. In this thesis we perform some tests of the AdS/CFT duality, orbifolded by the discrete group $\mathcal{G}=\mathbb{Z}_M$ which becomes the type IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$, presumably dual to the $\mathcal{N}=2$ quiver gauge theory [54, 55].

The thesis is organised as follows. In chapter 1 we give a brief introduction to the gauge/string duality, its constituents, and show that the two-point correlation function is a well defined object according to the postulates of the holographic principle; we mention in the end a few words about the Berenstein-Maldacena-Nastase paper [5]. In chapter 2 we discuss the formalism of obtaining the one-loop anomalous dimensions of generic, unprotected, local gauge operators by the procedure of the dilatation operator [41], and briefly explain the idea of spin-chains. In chapter 3 we introduce the orbifolding procedure, quantise the resulting string theory in the pp-wave limit, and then formulate the gauge dual states in the $SU(2)$ subsector. Afterward, we compare their energy spectra up to one loop. In chapter 4 we consider the approach of semiclassical strings spinning in $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$, calculate their energy, and compare it to the one-loop anomalous dimension of the gauge states, solved by the method of rational solutions. We conclude with a short remark on algebraic curve, and in outlook of the thesis we suggest some directions in which one might proceed afterwards.

[2]To be precise, it is performed by calculating the anomalous dimension matrix by the use of the dilatation operator for $\mathcal{N}=4$ SYM, and then its diagonalisation through the Bethe ansatz procedure [51, 20].

Chapter 1

The gauge/string duality

We start by giving a brief introduction to the AdS/CFT duality. We begin with the presentation of various descriptions of anti-de Sitter space, followed by short discussions of conformal group, conformal field theory, and then of the $\mathcal{N} = 4$ superconformal Yang-Mills theory. The emphasis of this introduction is put on the two-point correlation functions as well-defined objects from the point of view of the duality. Therefore, after explaining main arguments confirming why the AdS/CFT conjecture should hold in the first place, we show that its prediction of the form of the two point function agree with the conformal field theory postulates. We end this chapter by discussing a very specific limit in which the tests of AdS/CFT duality will be performed throughout this thesis.

1.1 Anti-de Sitter space

In order to understand the gauge/string duality better it is crucial to discuss the properties of anti-de Sitter space extensively. Its geometry, spacetime structure, symmetries are very different from de Sitter space, whose properties our world seems to possess, and therefore I will try to be as clear as possible, in spite of rather extensive use of mathematics and topology.

Anti-de Sitter space is the negative curvature analogue of de Sitter space named after the Dutch physicist Willem de Sitter; in the language of general relativity it is a maximally symmetric vacuum solution of Einstein's field equation with a negative cosmological constant. From the mathematical point of view, it is useful to consider a $(d+1)$-dimensional AdS_{d+1} space as a submanifold of a pseudo-Euclidean $(d+2)$-dimensional embedding space $\mathbb{R}^{2,d}$, with coordinates and metric

$$X^a = (X^0, X^1, \ldots, X^d, X^{d+1}), \qquad \eta_{ab} = \mathrm{diag}(-1, +1, +1, \ldots, +1, -1), \tag{1.1}$$

defined such that

$$-R^2 = -(X^0)^2 + (X^1)^2 + (X^2)^2 + \ldots + (X^d)^2 - (X^{d+1})^2, \tag{1.2}$$

preserved by the Lorentz-like group $SO(2,d)$, acting as

$$X^a \to X'^a = \Lambda^a{}_b X^b, \qquad \Lambda^a{}_b \in SO(2,d), \tag{1.3}$$

and the metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d}(dX^i)^2 - (dX^{d+1})^2. \tag{1.4}$$

Note that, although we see two times X^0 and X^{d+1} in the definition (1.2), we have to realise that it is the pseudo-Euclidean space which possesses them, and not the anti-de Sitter space; the AdS space, as a submanifold, has just one proper time which we will show below.

This kind of description, however, is a starting point for more than one geometric interpretation; while working in such space people customarily choose various parametrisations, depending on the type of problems they are facing. It is a commonly known fact that certain coordinates for certain mathematical puzzles might become very handy, and in fact we will see that explicitly when endeavouring to solve a few of them. For that reason we proceed to presenting the most popular descriptions of AdS_{d+1} space.

1.1.1 The global coordinates

They are introduced by the following parametrisation

$$\begin{aligned} X^0 &= R\cosh\rho\sin\tau \qquad && \rho \in [0,\infty) \\ X^i &= R\sinh\rho\,\Omega_i && i = 1,\dots,d \\ X^{d+1} &= R\cosh\rho\cos\tau && \tau\in[0,2\pi), \end{aligned} \tag{1.5}$$

where

$$\sum_{i=1}^{d}\Omega_i^2 = 1 \tag{1.6}$$

denotes the $\mathbf{S}^d$ unit sphere. The metric then reads

$$ds^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2). \tag{1.7}$$

Now we can identify the proper time τ, since we can see by looking at (1.5) that translations in τ correspond to rotations of X^0's and X^{d+1}'s. For that reason we say that this representation of AdS_{d+1} space has closed time-like curves, and the hyperboloid (1.2) has the topology of $\mathbf{S}^1\times\mathbb{R}^d$, with $\mathbf{S}^1$ representing these curves (see figure 1.1 (a)). One can always, however, "unwrap" the circle $\mathbf{S}^1$ by taking $\tau\to t\in\mathbb{R}$, obtaining the universal covering space (CAdS), which has topology of $\mathbb{R}^{1,d}$ and contains no time-like curves (see figure 1.1 (b)). For our purposes though, we will hereafter use only CAdS, and refer to it as AdS.

1.1.2 Conformal compactification of AdS_{d+1}

Another parametrisation can be obtained from (1.5) by introducing new coordinate ζ, and setting

$$\zeta := \arctan(\sinh\rho), \qquad \zeta\in\left[0,\tfrac{\pi}{2}\right) \tag{1.8}$$

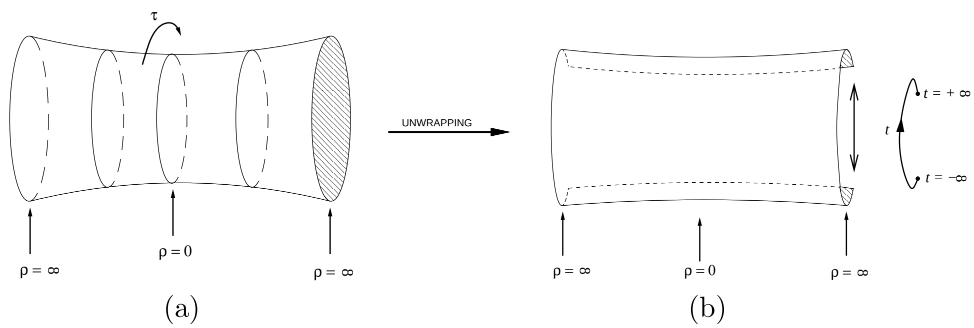

Figure 1.1: The figure (a) is the AdS_{d+1} space visualised in terms of the $\mathbf{S}^1 \times \mathbb{R}^d$ manifold with time-like curves, whereas the figure (b) is the universal covering AdS_{d+1} space (CAdS), represented by the $\mathbb{R}^{1,d}$ manifold.

due to which the metric takes the form

$$ds^2 = \frac{R^2}{\cos^2\zeta}\left(-dt^2 + d\zeta^2 + \sin^2\zeta \, d\Omega_{d-1}^2\right), \tag{1.9}$$

which has the topology of $\mathbb{R} \times \mathbf{S}^d$, $\mathbb{R}$ being the time direction.

It is crucial to note that the casual structure of the spacetime is invariant under the (conformal) rescaling

$$ds^2 \longrightarrow \left(\frac{R^2}{\cos^2\zeta}\right)^{-1} ds^2, \tag{1.10}$$

however this time, unlike in the case of (1.5) when the global coordinates covered the whole spacetime, here they in fact cover only one half of it, as the coordinate ζ takes the value $0 \leqslant \zeta < \pi/2$, rather than $0 \leqslant \zeta < \pi$. The point at $\zeta = \pi/2$ is added, being the boundary of the space with the topology of $\mathbf{S}^{d-1}$. As a result, the AdS boundary $\partial(AdS_{d+1})$ can be seen as having topology of $\mathbb{R} \times \mathbf{S}^{d-1}$ (see figure 1.2) The whole of anti-de Sitter space

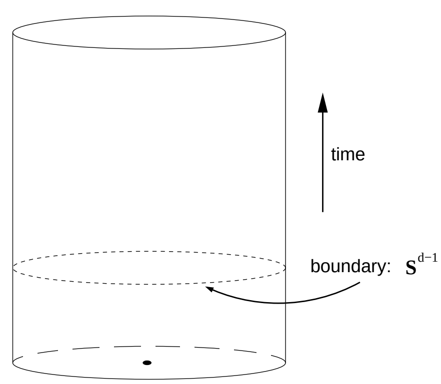

Figure 1.2: The AdS_{d+1} space can be visualised as a higher-dimensional "cylinder" of topology $\mathbb{R} \times \mathbf{S}^d$, where each circle at a constant time t is a $(d-1)$-sphere. Hence, the AdS boundary $\partial(AdS_{d+1})$ has the topology $\mathbb{R} \times \mathbf{S}^{d-1}$.

is conformal to the region $0 \leqslant \zeta < \pi/2$. Since for this coordinate system spatial infinity (boundary that is) has a finite value, the AdS spacetime can be conveniently represented by the use of Penrose diagrams (see figure 1.3), where coordinates of $\mathbf{S}^{d-1}$ unit sphere are suppressed, becoming a 2D plot of proper time t versus the "azimuth" angle ζ in (1.8). In

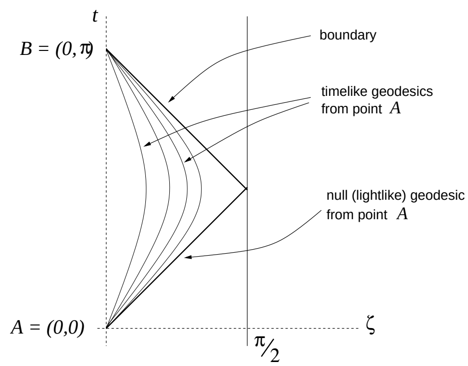

Figure 1.3: The Penrose diagram of AdS space shows that the massive objects move along the time-like geodesics which never reach the boundary, while the massless ones can reach it and come back in a finite amount of time.

this diagram, the null lines at 45° are drawn to clarify the conformal structure. Massless particles, moving along the null geodesics, will reach the spatial infinity of AdS space in finite amount of time, whereas massive particles moving along the time-like geodesics will never get there. (For detailed calculations of this phenomenon, please refer to [10].) This is a crucial property of AdS space.

The set of coordinates (1.9) allows us to imagine how to view AdS space, and for that please refer to figure 1.4. The $\mathbf{S}^d$ part of AdS space can be seen as a higher-dimensional version of a two dimensional disk. Each bat is actually the same size, and the circular boundary is infinitely far from the center of the disk. The projection from true, hyperbolic space to this representation of it (which is merely the conformal scaling) squashes the distant bat to fit the infinite space inside a finite circle. On the right-hand side we see that these circles, which in fact are $\mathbf{S}^d$-spheres, are layered inside the cylinder as each one of them represents the AdS space in a certain time "snapshot". In order to include the time dimension, the bat-hyperplanes have to move also along the time direction. However, the bats are made of massive particles and thus they can never reach the boundary of the "cylinder", unlike the (massless) light beam shot from the disc, which can reach it, reflect, and come back in the finite amount of time (provided appropriate boundary conditions are set, see figure 1.3).

1.1.3 Polar/stereographic coordinates

These coordinates, known also as the Poincaré-disc coordinates, are introduced by the following map

$$(X^0, X^\mu) \longrightarrow (X^0(\rho, y^\mu), X^\mu(\rho, y^\mu)), \qquad \mu = 1, ..., d+1 \tag{1.11}$$

such that

$$X^0(\rho, y^\mu) = \rho\frac{1+y^2}{1-y^2} \qquad X^\mu(\rho, y^\mu) = \rho\frac{2y^\mu}{1-y^2} \tag{1.12}$$

$$y^2 = \sum_{i=1}^{d}(y^i)^2 - (y^{d+1})^2 \qquad dy^2 = \sum_{i=1}^{d}(dy^i)^2 - (dy^{d+1})^2 \tag{1.13}$$

with the metric

$$ds^2 = -d\rho^2 + \frac{4\rho^2}{(1-y^2)^2}dy^2. \tag{1.14}$$

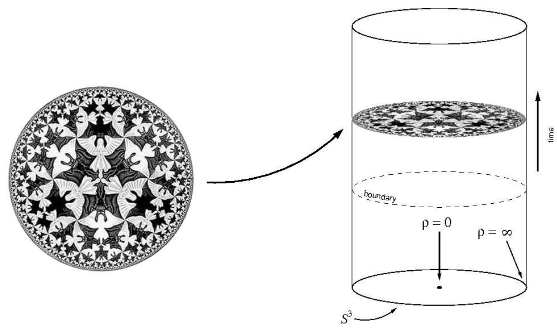

Figure 1.4: The AdS space as the cylinder from figure 1.2, where all the bats in the disk, which is an $\mathbf{S}^d$-sphere, are of the same size but since the boundary of the circle is infinitely far from the centre of the disk the bats, which are made of massive particles, visually seem to decrease towards the boundary.

One of the ways to describe the AdS_{d+1} space is to consider an Euclidean space $\mathbb{R}^{d+1}$, and let $\mathcal{B}_{d+1}$ be the open unit-ball

$$0 < y^2 = \sum_{i=1}^{d+1}(y^\mu)^2 < 1, \tag{1.15}$$

where we Wick rotated to Euclidean space by setting $y^{d+1} \to -iy^{d+1}$. Then, AdS_{d+1} can be identified as $\mathcal{B}_{d+1}$ with the metric

$$ds^2 = \frac{4R^2}{(1-y^2)^2}\sum_{i=1}^{d+1}(dy^\mu)^2, \tag{1.16}$$

which can be obtained from (1.14) by setting $\rho \equiv R$. We can compactify $\mathcal{B}_{d+1}$ to get a unit-ball $\widetilde{\mathcal{B}}_{d+1}$, defined by

$$\sum_{i=1}^{d+1}(y^\mu)^2 \leqslant 1, \tag{1.17}$$

the boundary of which is the $\mathbf{S}^d$ unit sphere at $y^2 = 1$. The $\mathbf{S}^d$ is the Euclidean version of the conformal compactification of Minkowski space, and the fact that the $\mathbf{S}^d$ is the boundary of $\widetilde{\mathcal{B}}_{d+1}$ is the Euclidean version of the statement that Minkowski space is the boundary of AdS_{d+1}. We clearly see that the metric of $\widetilde{\mathcal{B}}_{d+1}$ is singular at $|y| = 1$ as the overall scale factor blows up there. This scale factor can be removed by a Weyl rescaling of the metric

$$ds^2 \to d\tilde{s}^2 = \mathsf{f}^2 ds^2, \qquad \mathsf{f} = 1 - y^2, \tag{1.18}$$

and then $d\tilde{s}^2$ restricts to a metric on $\mathbf{S}^d$, however such rescaling is not unique. That is because one could equally well replace f by

$$\mathsf{f} \to e^w \mathsf{f}, \tag{1.19}$$

with w any real function on $\widetilde{\mathcal{B}}_{d+1}$, which would induce the conformal transformation

$$d\tilde{s}^2 \to e^{2w} d\tilde{s}^2 \tag{1.20}$$

in the metric of $\mathbf{S}^d$. Therefore, we conclude that a unique, well-defined limit to the boundary of AdS_{d+1} can exist only if the appropriate boundary field theory is scale invariant.

A closely descendant parametrisation to the one considered above is obtained by the following change of variables in (1.16)

$$y^2 \equiv r^2, \qquad r = \tanh\left(\frac{\tilde{y}}{2}\right), \qquad \tilde{y} \in [0, \infty) \tag{1.21}$$

due to which

$$r^2 = dr^2 + r^2 d\Omega_d^2, \tag{1.22}$$

hence

$$ds^2 = d\tilde{y}^2 + \sinh^2\tilde{y}\, d\Omega_d^2, \tag{1.23}$$

where the overall factor R^2 was dropped for convenience. In this representation boundary lies at $\tilde{y} = \infty$.

1.1.4 Poincaré coordinates

Let us go back to Minkowski space again. If we parametrise the metric (1.5) in the following way

$$\begin{aligned} X^0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)), \qquad & X^d &= \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)) \\ X^i &= Rux^i, \quad i = 1, ..., d-1, \qquad & X^{d+1} &= Rut, \end{aligned} \tag{1.24}$$

where $u \in (0, \infty)$, and $\vec{x} \in \mathbb{R}^{d-1}$, then we will receive the following metric

$$ds^2 = \frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2). \tag{1.25}$$

(Again, R^2 dropped.) The coordinates $(u, t, \vec{x})$ are called the Poincaré coordinates, and are called such because the metric (1.25) is invariant after such a group transformation that consists of two subgroups, namely the Poincaré transformations (or inhomogeneous Lorentz group) group $ISO(1, d-1)$ on $(t, \vec{x})$, and $SO(1,1)$ conformal group which acts on the full set of coordinates as

$$(u, t, \vec{x}) \to (\mathsf{c}^{-1}u, \mathsf{c}t, \mathsf{c}\vec{x}), \qquad \mathsf{c} > 0. \tag{1.26}$$

In the AdS/CFT correspondence this is identified with the dilatation $\mathfrak{D}$ in the conformal symmetry group of $\mathbb{R}^{1,d-1}$.

Again, one can always rotate to Euclidean space by Wick-rotating the time coordinate $X^{d+1} \to -iX^{d+1}$, or in this case the Poincaré time coordinate $t \to -it_E$, getting the Euclidean version of (1.25)

$$ds^2 = \frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2). \tag{1.27}$$

In these coordinates, at $u = \infty$ we have a sphere $\mathbf{S}^d$ at the boundary, with one point removed. The full boundary sphere is recovered by adding a point corresponding to $u = 0$ (or, equivalently, $\vec{x} = \infty$).

Finally, one could view AdS_{d+1} as the upper half space $x^0 \geqslant 0$ in a space with coordinates x^μ, for $\mu = 0, 1, ..., d$, and the metric

$$ds^2 = \frac{1}{(x^0)^2}\sum_{\mu=0}^{d}(dx^\mu)^2. \tag{1.28}$$

It can be simply obtained from (1.27) by setting $x^0 = 1/u$, and identifying $t_E \equiv x^d$. Likewise, in this representation the boundary consists of a copy of $\mathbb{R}^d$ at $x^0 = 0$, together with a single point P at $x^0 = \infty$. Thus, from this point of view the boundary is a conformal compactification of $\mathbb{R}^d$, obtained by adding a point P at infinity, which gives a sphere $\mathbf{S}^d$.

As the description of AdS space presented above is a necessary minimum, I refer to the literature for more details [10, 11, 12, 13, 14, 15, 16].

1.2 Conformal field theory

1.2.1 Conformal group

From an abstract point of view, conformal field theories are Euclidean quantum field theories that are characterised by the property that their symmetry group contains, in addition to Euclidean symmetries, local conformal transformations, i.e. such symmetry which preserves angles. Given the space $\mathbb{R}^{p,q}$, $p + q = d$, with flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, where

$$\eta_{\mu\nu} = \mathrm{diag}(\underbrace{-, \ldots, -}_{p \text{ entries}}, \underbrace{+, \ldots, +}_{q \text{ entries}}), \tag{1.29}$$

of signature (p, q), the conformal transformations are defined as the general coordinate transformations, the parameters of which define a conformal Killing vector $\xi^\mu(x)$. The defining equation for this conformal Killing vector is given by

$$\delta_\xi g_{\mu\nu}(x) \equiv \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) = w(x) g_{\mu\nu}(x), \tag{1.30}$$

where $w(x)$ is an arbitrary function of x^μ, $\xi_\mu = g_{\mu\nu}\xi^\nu$, and the covariant derivative is given by $\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^{\,\rho}\,\xi_\rho$. In flat, d-dimensional Minkowski space (1.30) implies

$$\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) - \frac{1}{d}\eta_{\mu\nu}\partial_\rho \xi^\rho(x) = 0. \tag{1.31}$$

Equation (1.31), which determines conformal symmetries, is often referred to as the conformal Killing equation. In $d = 2$ it leads to an infinite-dimensional conformal algebra (known as the Virasoro algebra), while for $d > 2$ the conformal algebra is finite-dimensional. Indeed, the solutions are

$$\xi^\mu(x) = a^\mu + \omega^{\mu\nu} x_\nu + \lambda x^\mu + (x^2 \beta^\mu - 2x^\mu x \cdot \beta). \tag{1.32}$$

Corresponding to the parameters a^μ are the translations P_μ, to $\omega^{\mu\nu}$ correspond the Lorentz rotations $M_{\mu\nu}$, to λ are associated dilatations D, and β^μ are parameters of special conformal transformations K_μ. This is expressed as follows for the full set of conformal transformations δ_C (up to an overall constant factor):

$$\delta_C = a^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu} + \lambda D + \beta^\mu K_\mu, \tag{1.33}$$

These are the most general infinitesimal spacetime transformations which leave the light-cone invariant, and the algebra generated by these transformations is isomorphic to $SO(p+1, q+1)$.

One can always integrate (1.33) to finite conformal transformations; we find first of all the Poincaré group (μ index suppressed)

$$x \to x' = x + a \tag{1.34}$$
$$x \to x' = \Lambda x \qquad ({\Lambda^\mu}_\nu \in SO(p, q)), \tag{1.35}$$

which corresponds to $w = 1$ in (1.30). On top of that we have the dilatations

$$x \to x' = \lambda x, \tag{1.36}$$

corresponding to $w = \lambda^{-2}$, and also the special conformal transformations (SCT)

$$x \to x' = \frac{x + \beta x^2}{1 + 2\beta \cdot x + \beta^2 x^2}, \tag{1.37}$$

where $w(x) = (1 + 2\beta \cdot x + \beta^2 x^2)^2$.

And so, for $d > 2$ the total number of parameters in the conformal group is equal to $\frac{1}{2}(d+1)(d+2)$. The conformal group is thus isomorphic to the orthogonal group on $(d+2) \times (d+2)$ matrices. For the special case of four dimensions, the conformal group is easily constructed from the six generators of the Lorentz group, four generators of translations, four generators for the proper conformal transformations, and one generator for scale transformations, giving a total of 15 generators. The conformal group in four dimensions is therefore

$$SO(2, 4) \sim SU(2, 2). \tag{1.38}$$

Working out the Lie algebra of the conformal group is rather straightforward. We can choose the simplest possible representation, namely a scalar field $\phi(x)$. Then we will find out that

$$\begin{aligned} P_\mu &= i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\ \mathfrak{D} &= ix^\mu\partial_\mu, \\ K_\mu &= i(2x_\mu x\cdot\partial - x^2\partial_\mu), \end{aligned} \tag{1.39}$$

which satisfy the following Lie algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} + g_{\mu\sigma}M_{\nu\rho}), \tag{1.40}$$

$$[P_\mu, M_{\rho\sigma}] = i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), \tag{1.41}$$

$$[K_\mu, M_{\rho\sigma}] = i(g_{\mu\rho}K_\sigma - g_{\mu\sigma}K_\rho), \tag{1.42}$$

$$[\mathfrak{D}, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0, \tag{1.43}$$

$$[P_\mu, \mathfrak{D}] = iP_\mu, \qquad [K_\mu, \mathfrak{D}] = -iK_\mu, \tag{1.44}$$

$$[P_\mu, K_\nu] = 2i(g_{\mu\nu}\mathfrak{D} - M_{\mu\nu}). \tag{1.45}$$

It can be proved that this algebra is the same for more general fields as well. For more on the conformal group refer to [17].

1.2.2 Correlation function in conformal field theory

Let us consider a local gauge invariant field $\mathcal{O}(x)$. The Jacobian of conformal transformation will be

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\sqrt{\det g'_{\mu\nu}}} = w^{-d/2}. \tag{1.46}$$

For dilatations (1.36) and SCT (1.37) it will be given respectively by

$$\left|\frac{\partial x'}{\partial x}\right| = \lambda^d, \tag{1.47}$$

and

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1+2\beta\cdot x+\beta^2x^2)^d} \equiv \gamma^{-d}(x). \tag{1.48}$$

Following [17] we define a theory with conformal invariance as such which satisfies the following properties:

(a) There is a set of fields $\{\mathfrak{F}_i\}$ which in general is infinite and contains in particular the derivatives of all the fields $\mathfrak{F}_i$.

(b) There is a subset of fields $\{\mathcal{O}_j\} \in \{\mathfrak{F}_i\}$ called "quasi primary", that under global transformation (1.33) $x \to x'$ transform accordingly to

$$\mathcal{O}_j(x) \longrightarrow \mathcal{O}_j(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_j}{d}} \mathcal{O}_j(x), \tag{1.49}$$

where $\Delta_j := \dim[\mathcal{O}_j(x)]$.

(c) The rest of the $\{\mathfrak{F}_i\}$'s can be expressed as linear combinations of $\{\mathcal{O}_j\}$'s and their derivatives.

(d) There is a vacuum $|\Omega\rangle$ invariant under the conformal group.

These restrictions are enough to explore the properties of the correlation function in CFT. The general, N-point function is defined such that[1]

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_N)\rangle \equiv \frac{1}{\mathcal{Z}}\int \mathcal{D}\mathcal{O}\ \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_N)\, e^{-S_{\mathrm{CFT}}[\mathcal{O}]}, \tag{1.50}$$

where $\mathcal{Z}$ is the normalisation factor.

In particular we will focus on two-point functions

$$\mathcal{A}(x_1,x_2) := \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\,, \tag{1.51}$$

which are used to measure some generic properties of the local operators we want to examine. In particular, the symmetries of the theory will be reflected by the correlation function $\mathcal{A}$. Due to translational invariance of (1.50) (the value of the integral cannot change if we vary the integration variable), the two-point function can depend solely on the distance of the two points

$$\mathcal{A}(x_1,x_2) = \mathcal{A}(|x_1-x_2|) \equiv \mathcal{A}(|x_{12}|). \tag{1.52}$$

Then, according to (1.49) and (1.50) we have

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \left\langle \mathcal{O}(x_1')\mathcal{O}(x_2')\right\rangle\,, \tag{1.53}$$

where $\Delta_j = \dim[\mathcal{O}(x_j)]$. Now, (1.47) implies that under the dilatations $x \to \lambda x$ (1.53) reads

$$\mathcal{A}(|x_{12}|) = \lambda^{\Delta_1+\Delta_2}\mathcal{A}(\lambda|x_{12}|), \tag{1.54}$$

and this holds provided that

$$\mathcal{A}(|x_{12}|) = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}}, \tag{1.55}$$

where C_{12} is a constant determined by the normalisation of the fields. Finally, (1.48) tells us that

$$\mathcal{A}(|x_{12}|) = \gamma^{-\Delta_1}(x_1)\,\gamma^{-\Delta_2}(x_2)\,\mathcal{A}(|x'_{12}|), \tag{1.56}$$

and since under the special conformal rescaling (1.37) the distance $|x_{12}|$ behaves as

$$|x'_{12}| = \frac{|x_{12}|}{\gamma(x_1)\gamma(x_2)}, \tag{1.57}$$

then obviously we must have

$$\frac{C_{12}}{|x'{}_{12}|^{\Delta_1+\Delta_2}} = [\gamma(x_1)\gamma(x_2)]^{\frac{\Delta_1+\Delta_2}{2}}\frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}}. \tag{1.58}$$

[1] We work here in Euclidean space; the Minkowski space can be retrieved by letting $S_{\mathrm{CFT}} \to -iS_{\mathrm{CFT}}$.

Equations (1.56) and (1.58) combined imply that

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \left[\frac{\gamma(x_2)}{\gamma(x_1)}\right]^{\frac{\Delta_1-\Delta_2}{2}} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\,, \tag{1.59}$$

hence, the non-evanescence condition

$$\Delta_1 = \Delta_2, \qquad \text{or} \qquad \dim[\mathcal{O}(x_1)] = \dim[\mathcal{O}(x_2)]. \tag{1.60}$$

Summarising, the two-point function of the conformal field theory will obey

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \sim \frac{\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}. \tag{1.61}$$

In full analogy, one can determine the form of the 3 point function. Invariance under translations, rotations, and dilatations requires

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \sum_{\alpha,\beta,\gamma} \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\alpha}|x_{23}|^{\beta}|x_{13}|^{\gamma}}, \tag{1.62}$$

where the summation is restricted such that $\alpha+\beta+\gamma = \Delta_1+\Delta_2+\Delta_3$. Then, invariance under SCT sets

$$\alpha = \Delta_1+\Delta_2-\Delta_3, \quad \beta = \Delta_2+\Delta_3-\Delta_1, \quad \gamma = \Delta_3+\Delta_1-\Delta_2, \quad C_{\alpha\beta\gamma} = C_{123} \tag{1.63}$$

which gives the three point function for the conformal field theory.

1.2.3 Superconformal invariance of $\mathcal{N}=4$ SYM

It turns out that a maximally supersymmetric conformal field theories in four dimensions is the $\mathcal{N}=4$ supersymmetric Yang-Mills theory [18, 19]. It consists of one vector field $\mathcal{A}_\mu$, six scalars ϕ_n, $n=1,...,6$, and four Weyl fermions ψ^a_α, $\bar\psi^a_{\dot\alpha}$, where $\alpha=1,2$ and $\dot\alpha=1,2$ are $SL(2,\mathbb{C}) \sim SU(2)\times SU(2)$ spinor indices, and $a=1,...,4$ are indices for the fundamental and anti-fundamental representations of an internal $SO(6)\simeq SU(4)$ symmetry, known as an R-symmetry.

We assume all the fields to be in the adjoint representation of the gauge group $U(N)$, which means that they transform canonically under a local gauge transformation $\mathscr{U}(x)\in U(N)$ as follows

$$\mathcal{V} \to \mathscr{U}\mathcal{V}\mathscr{U}^{-1}, \qquad \mathcal{A}_\mu \to \mathscr{U}\mathcal{A}_\mu\mathscr{U}^{-1} - ig\,\partial_\mu\mathscr{U}\,\mathscr{U}^{-1}, \tag{1.64}$$

where g is a dimensionless coupling constant, and

$$\mathcal{V} = \{\psi^a_\alpha, \bar\psi^a_{\dot\alpha}, \phi_m\}. \tag{1.65}$$

However, it is more convenient to define the so-called covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu, \qquad \mathcal{D}_\mu\mathcal{V} := [\mathcal{D}_\mu, \mathcal{V}] = \partial_\mu\mathcal{V} - ig[\mathcal{A}_\mu, \mathcal{V}], \tag{1.66}$$

and then one can see that now for

$$\mathcal{W} = \{\mathcal{D}_\mu, \psi^a_\alpha, \bar\psi^a_{\dot\alpha}, \phi_m\}, \tag{1.67}$$

we have a uniform gauge transformation under $U(N)$

$$\mathcal{W} \to \mathscr{U}\mathcal{W}\mathscr{U}^{-1}. \tag{1.68}$$

On top of that we have a field strength

$$\mathcal{F}_{\mu\nu} := ig^{-1}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{1.69}$$

together with the associated Bianchi identity

$$\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]} = 0, \tag{1.70}$$

where $[\cdots]$ stands for antisymmetrisation.

The Lagrangian density for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory reads[2]

$$\begin{aligned} \mathscr{L}_{\text{SYM}}^{\mathcal{N}=4}[\mathcal{W}] &= \tfrac{1}{8}\text{tr}\, \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \tfrac{1}{2}\text{tr}\, \mathcal{D}_\mu\phi_n\mathcal{D}^\mu\phi_n - \tfrac{1}{4}g^2\text{tr}[\phi_m, \phi_n]^2 \\ &\quad + \text{tr}\, \bar{\psi}^a_{\dot{\alpha}}\, \sigma^{\dot{\alpha}\beta}_\mu \mathcal{D}^\mu \psi^a_\beta - \tfrac{i}{2}g\, \text{tr}\, \psi^a_\alpha \sigma^n_{ab}\epsilon^{\alpha\beta}[\phi_n, \psi^b_\beta] - \tfrac{i}{2}g\, \text{tr}\, \bar{\psi}^a_{\dot{\alpha}}\sigma^n_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}[\phi_n, \bar{\psi}^b_{\dot{\beta}}], \end{aligned} \tag{1.71}$$

where the spacetime location dependence is suppressed, and the trace runs not only over $U(N)$ indices, but also over spacetime μ, ν and spinor $\alpha, \dot{\alpha}, \beta, \dot{\beta}$ indices, over internal (R-symmetry) vector indices m, n, and over internal spinor indices a, b. Also, the matrices

$$\sigma^\mu_{\dot{\alpha}\beta} = \sigma^\mu_{\beta\dot{\alpha}} \tag{1.72}$$

are the chiral projections of the ten dimensional gamma matrices in four dimensions, and the matrices

$$\sigma^n_{ab} = -\sigma^n_{ba} \tag{1.73}$$

are the chiral projections of the ten dimensional gamma matrices in six dimensions. This stems from the fact that the $\mathcal{N} = 4$ SYM in 4D can be derived by dimensional reduction from $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in 10D.

$\mathcal{N} = 4$ SYM is a theory invariant under the $\mathcal{N} = 4$ super Poincaré algebra. This consists of the bosonic translations P_μ, eight supertranslations $\mathcal{Q}^a_\alpha$ together with their eight conjugates $\bar{\mathcal{Q}}^a_{\dot{\alpha}}$, plus Lorentz $SO(4)$ rotation generators $M_{\mu\nu}$, and internal $SO(6)$ rotation generators $\mathcal{R}_{mn}$. And so the infinitesimal supertranslations can be written as

$$\delta_{\text{sutra}} = a^\mu P_\mu + \varepsilon^{\alpha a}\mathcal{Q}^a_\alpha + \bar{\varepsilon}^{\dot{\alpha}a}\bar{\mathcal{Q}}^a_{\dot{\alpha}}. \tag{1.74}$$

The action of supertranslations on $\mathcal{N} = 4$ fields (1.67) $\delta_{\text{sutra}}\mathcal{W} = [\delta_{\text{sutra}}, \mathcal{W}]$ gives a set of transformations under which the action[3] of $\mathcal{N} = 4$ SYM remains unchanged

$$\begin{aligned} \delta_{\text{sutra}}\mathcal{D}_\mu &= ig\, \varepsilon^{\alpha a}\epsilon_{\alpha\beta}\, \sigma^{\beta\dot{\gamma}}_\mu \bar{\psi}^a_{\dot{\gamma}} + ig\, \bar{\varepsilon}^{\dot{\alpha}a}\epsilon_{\dot{\alpha}\dot{\beta}}\, \sigma^{\dot{\beta}\gamma}_\mu \psi^a_\gamma + \tfrac{i}{\sqrt{2}}g\, a^\nu \mathcal{F}_{\mu\nu}, \\ \delta_{\text{sutra}}\phi_m &= \varepsilon^{\alpha a}\sigma^{ab}_m \psi^b_\alpha + \bar{\varepsilon}^{\dot{\alpha}a}\sigma^{ab}_m \bar{\psi}^b_{\dot{\alpha}} + a^\mu \mathcal{D}_\mu \phi_m, \\ \delta_{\text{sutra}}\psi^a_\alpha &= -\tfrac{1}{2\sqrt{2}}\sigma^\mu_{\alpha\dot{\beta}}\, \epsilon^{\dot{\beta}\dot{\gamma}}\sigma^\nu_{\dot{\gamma}\delta}\, \varepsilon^{\delta a}\mathcal{F}_{\mu\nu} + \tfrac{i}{2}g\, \sigma^{ab}_m \sigma^{bc}_n \epsilon_{\alpha\beta}\, \varepsilon^{\beta c}[\phi_m, \phi_n] \\ &\quad + \sigma^{ab}_n \sigma^\mu_{\alpha\dot{\beta}}\, \bar{\varepsilon}^{\dot{\beta}b}\, \mathcal{D}_\mu\phi_n + a^\mu \mathcal{D}_\mu \psi^a_\alpha, \\ \delta_{\text{sutra}}\bar{\psi}^a_{\dot{\alpha}} &= -\tfrac{1}{2\sqrt{2}}\sigma^\mu_{\alpha\dot{\beta}}\, \epsilon^{\beta\gamma}\sigma^\nu_{\gamma\dot{\delta}}\, \bar{\varepsilon}^{\dot{\delta}a}\mathcal{F}_{\mu\nu} + \tfrac{i}{2}g\, \sigma^{ab}_m \sigma^{bc}_n \epsilon_{\dot{\alpha}\dot{\beta}}\, \bar{\varepsilon}^{\dot{\beta}c}[\phi_m, \phi_n] \\ &\quad + \sigma^{ab}_n \sigma^\mu_{\dot{\alpha}\beta}\, \bar{\varepsilon}^{\beta b}\, \mathcal{D}_\mu\phi_n + a^\mu \mathcal{D}_\mu \bar{\psi}^a_{\dot{\alpha}}. \end{aligned} \tag{1.75}$$

[2]We follow [20] with supersymmetry conventions.

[3]I emphasise that it is the action (2.3) of $\mathcal{N} = 4$ super Yang-Mills that is invariant, since we have to integrate over the spacetime in order to cancel the surface terms; also the algebra closes on-shell, and that means that the invariance will remain only up to the equations of motion.

The superconformal algebra can be summarised by the following commutation and anti-commutation relations (in addition to (1.40 - 1.45))

$$\{\mathcal{Q}^a_\alpha, \bar{\mathcal{Q}}^b_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}\,\delta^{ab} P_\mu \qquad \{\mathcal{Q}^a_\alpha, \mathcal{Q}^b_\alpha\} = \{\bar{\mathcal{Q}}^a_{\dot\alpha}\, \bar{\mathcal{Q}}^b_{\dot\alpha}\} = 0 \tag{1.76}$$

$$[P_\mu, \mathcal{Q}^a_\alpha] = [P_\mu, \bar{\mathcal{Q}}^a_{\dot\alpha}] = 0 \tag{1.77}$$

$$[\mathfrak{D}, \mathcal{Q}^a_\alpha] = -\tfrac{i}{2}\mathcal{Q}^a_\alpha \qquad [\mathfrak{D}, \bar{\mathcal{Q}}^a_{\dot\alpha}] = -\tfrac{i}{2}\bar{\mathcal{Q}}^a_{\dot\alpha} \tag{1.78}$$

$$[M^{\mu\nu}, \mathcal{Q}^a_\alpha] = i\sigma^{\mu\nu}_{\alpha\beta}\,\epsilon^{\beta\gamma}\mathcal{Q}^a_\gamma \qquad [M^{\mu\nu}, \bar{\mathcal{Q}}^a_{\dot\alpha}] = i\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma}\bar{\mathcal{Q}}^a_{\dot\gamma}, \tag{1.79}$$

where $\sigma^{\mu\nu}_{\alpha\beta}$ is an anti-symmetric combination of $\sigma^\mu_{\alpha\dot\alpha}$ and $\sigma^\nu_{\beta\dot\beta}$ (see for example [21, 22]). On top of that we have two sets of eight fermionic boosts $\mathcal{S}^a_{\dot\alpha}$, called superboosts, or special conformal supercharges, which toghether with conformal boosts K^μ and dilatations $\mathfrak{D}$ give the following superconformal algebra

$$[K^\mu, \mathcal{Q}^a_\alpha] = \sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{S}}^a_{\dot\beta} \qquad [K^\mu, \bar{\mathcal{Q}}^a_{\dot\alpha}] = \sigma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}\mathcal{S}^a_\beta \tag{1.80}$$

$$\{\mathcal{S}^a_\alpha, \bar{\mathcal{S}}^b_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}\,\delta^{ab} K_\mu \qquad \{\mathcal{S}^a_\alpha, \mathcal{S}^b_\alpha\} = \{\bar{\mathcal{S}}^a_{\dot\alpha}\, \bar{\mathcal{S}}^b_{\dot\alpha}\} = 0 \tag{1.81}$$

$$[\mathfrak{D}, \mathcal{S}^a_\alpha] = +\tfrac{i}{2}\mathcal{S}^a_\alpha \qquad [\mathfrak{D}, \bar{\mathcal{S}}^a_{\dot\alpha}] = +\tfrac{i}{2}\bar{\mathcal{S}}^a_{\dot\alpha}, \tag{1.82}$$

together with

$$\{\mathcal{Q}^a_\alpha, \mathcal{S}^b_\beta\} = -\epsilon_{\alpha\beta}\,\sigma^{mn}_{ab}\mathcal{R}_{mn} + \sigma^{\mu\nu}_{\alpha\beta}\,\delta^{ab}M_{\mu\nu} - \epsilon_{\alpha\beta}\,\delta^{ab}(\mathfrak{D} - \mathcal{C}) \tag{1.83}$$

$$\{\bar{\mathcal{Q}}^a_\alpha, \bar{\mathcal{S}}^b_\beta\} = +\epsilon_{\dot\alpha\dot\beta}\,\sigma^{mn}_{ab}\mathcal{R}_{mn} + \sigma^{\mu\nu}_{\dot\alpha\dot\beta}\,\delta^{ab}M_{\mu\nu} - \epsilon_{\dot\alpha\dot\beta}\,\delta^{ab}(\mathfrak{D} + \mathcal{C}), \tag{1.84}$$

where σ^{mn}_{ab} are constructed in analogy to $\sigma^{\mu\nu}_{\alpha\beta}$.

Summarising, the superconformal algebra, which is an invariance of $\mathcal{N}=4$ supersymmetric Yang-Mills theory, consists of (Lorentz and internal) rotations $M_{\mu\nu}$, $\mathcal{R}_{mn}$ respectively, of translations and supertranslations P_μ, $\mathcal{Q}^a_\alpha$, $\bar{\mathcal{Q}}^a_{\dot\alpha}$, boosts and superboosts K_μ, $\mathcal{S}^a_\alpha$, $\bar{\mathcal{S}}^a_{\dot\alpha}$, dilatation generator $\mathfrak{D}$, and finally chiral scalar charge (or hypercharge) $\mathcal{B}$, together with central charge $\mathcal{C}$. They all together can be expressed in terms of a supermatrix (supergroup) $U(2,2|4)$. This group, however, is reducible and can be reduced to an irreducible superconformal group $PU(2,2|4)$, which can be achieved by dropping hypercharge $\mathcal{B}$,[4] and setting central charge to zero in (1.83) and (1.84).

1.2.4 Chiral primaries

Let us consider the action of the dilatation operator on some local gauge invariant field $\mathcal{O}(x)$

$$[\mathfrak{D}, \mathcal{O}(x)] = \left(-i\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x); \tag{1.85}$$

it gives the conformal dimension of $\mathcal{O}(x)$, $\dim[\mathcal{O}(x)] = \Delta$. Let us for the sake of simplicity consider $\mathcal{O}(x=0)$ and see what happens to it if we act on it with the bosonic conformal boost K_μ

$$[\mathfrak{D}, [K_\mu, \mathcal{O}(0)]] = [[\mathfrak{D}, K_\mu], \mathcal{O}(0)] + [K_\mu, [\mathfrak{D}, \mathcal{O}(0)]] = -i(\Delta - 1)[K_\mu, \mathcal{O}(0)], \tag{1.86}$$

[4]The reason for that is that $\mathcal{B}$ is as external automorphism of this algebra and thus does not appear on the right-hand side of (1.83) and (1.84).

which tells us that K_μ lowers the dimension by 1. On top of that we know that unitarity of quantum field theories demands that local operators have positive dimensions, thus by acting on it with K_μ we should eventually reach some operator $\mathcal{O}^*(0)$ such that

$$[K_\mu, \mathcal{O}^*(0)] = 0. \tag{1.87}$$

$\mathcal{O}^*(0)$ is known as a conformal primary operator. One can obtain new operators by acting with the conformal algebra on the primaries.

Awareness of (1.85) allows us to realise quickly, by looking at (1.78) and (1.82), that $\mathcal{Q}$'s and $\bar{\mathcal{Q}}$'s have dimension $1/2$, whereas $\mathcal{S}$'s an $\bar{\mathcal{S}}$'s have dimension $-1/2$ (which in fact is consistent with (1.80) by the use of (1.86)). This gives us a hint that supercharges ($\mathcal{Q}$, $\bar{\mathcal{Q}}$, $\mathcal{S}$, $\bar{\mathcal{S}}$) can be used to create new operators by acting on a primary operator. In fact, the operators generated in such a way, along with the original operator, create an $SO(2\mathcal{N}, 2\mathcal{N})|_{\mathcal{N}=4}$ Clifford algebra, which is $2^{2\times 4} = 256$ dimensional. Obviously, if we can find operators annihilated by some of these supercharges, then the number of operators in the supermultiplet will be reduced.

Now let us take some other operator, say, $\mathcal{O}^{**}(0)$ that can be annihilated by all of the 16 special charges $\mathcal{S}$ and $\bar{\mathcal{S}}$, namely the one that fulfils

$$[\mathcal{S}^a_\alpha, \mathcal{O}^{**}(0)] = [\bar{\mathcal{S}}^a_{\dot\alpha}, \mathcal{O}^{**}(0)] = 0. \qquad \forall_{\alpha,\dot\alpha,a} \tag{1.88}$$

But that also means that the combination

$$-\{\mathcal{S}^b_\alpha, [\bar{\mathcal{S}}^a_{\dot\alpha}, \mathcal{O}^{**}(0)]\} - \{\bar{\mathcal{S}}^a_{\dot\alpha}, [\mathcal{S}^b_\alpha, \mathcal{O}^{**}(0)]\} \tag{1.89}$$

should vanish as well. Obviously then

$$-\{\mathcal{S}^b_\alpha, [\bar{\mathcal{S}}^a_{\dot\alpha}, \mathcal{O}^{**}(0)]\} - \{\bar{\mathcal{S}}^a_{\dot\alpha}, [\mathcal{S}^b_\alpha, \mathcal{O}^{**}(0)]\} = 0 = [\{\mathcal{S}^b_\alpha, \bar{\mathcal{S}}^a_{\dot\alpha}\}, \mathcal{O}^{**}(0)] \propto [K_\mu, \mathcal{O}^{**}(0)], \tag{1.90}$$

and a quick look at (1.87) tells us that while $\mathcal{O}^{**}(0)$ always satisfies (1.87), $\mathcal{O}^*(0)$ does not have to satisfy (1.90). This merely means that $\mathcal{O}^{**}(0)$ is a more specific operator than $\mathcal{O}^*(0)$; in fact it is its supersymmetric extension. For that reason $\mathcal{O}^{**}(0)$ is usually being referred to as a superconformal primary operator.

Suppose now that furthermore we have some operator $\mathcal{O}^{\mathrm{CPO}}(0)$ which gets annihilated by some of the supercharges $\mathcal{Q}$, namely

$$[\mathcal{Q}^a_\alpha, \mathcal{O}^{\mathrm{CPO}}(0)] = 0 \qquad \text{for some} \quad \alpha, a. \tag{1.91}$$

Such operators are called (superconformal) chiral primaries or BPS operators. Use of (1.89) and (1.91) will now yield that

$$\begin{aligned} 0 &= \{\mathcal{Q}^a_\alpha, [\mathcal{S}^b_\beta, \mathcal{O}^{\mathrm{CPO}}(0)]\} + \{\mathcal{S}^b_\beta, [\mathcal{Q}^a_\alpha, \mathcal{O}^{\mathrm{CPO}}(0)]\} = [\{\mathcal{Q}^a_\alpha, \mathcal{S}^b_\beta\}, \mathcal{O}^{\mathrm{CPO}}(0)] \\ &= [-\epsilon_{\alpha\beta}\, \sigma^{mn}_{ab} \mathcal{R}_{mn} + \sigma^{\mu\nu}_{\alpha\beta}\, \delta^{ab} M_{\mu\nu} - \epsilon_{\alpha\beta}\, \delta^{ab}\, \mathfrak{D},\, \mathcal{O}^{\mathrm{CPO}}(0)]. \end{aligned} \tag{1.92}$$

Let us then examine properties of chiral primaries for the case of $\mathcal{O}^{\mathrm{CPO}}(0)$ being a scalar, that is

$$[M_{\mu\nu}, \mathcal{O}^{\mathrm{CPO}}(0)] = 0. \tag{1.93}$$

Then, (1.92) yields that

$$\delta^{ab}[\mathfrak{D}, \mathcal{O}^{\mathrm{CPO}}(0)] = -\sigma_{ab}^{mn}[\mathcal{R}_{mn}, \mathcal{O}^{\mathrm{CPO}}(0)]. \tag{1.94}$$

Since $\mathcal{R}_{mn}$ $(m, n = 1, ..., 6)$ generates internal $SO(6)$ rotations, and $SO(6)$ is a rank 3 group, it has three commuting generators, say[5], $\mathcal{R}_{12}$, $\mathcal{R}_{34}$, $\mathcal{R}_{56}$, which generate the corresponding charges $(J_1,\ J_2,\ J_3)$. The σ_{ab}^{mn} will be the $SO(6)$ generators in the fundamental representation, and in particular, the commuting generators can be expressed as

$$\sigma_{ab}^{12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \sigma_{ab}^{34} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \quad \sigma_{ab}^{56} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \tag{1.95}$$

where off diagonal elements are zero. Then, (1.94) can be written for a chiral primary operator with R-charges e.g. $(J_1, 0, 0)$ as follows

$$\begin{pmatrix} \Delta - J_1 & & & \\ & \Delta - J_1 & & \\ & & \Delta + J_1 & \\ & & & \Delta + J_1 \end{pmatrix} \mathcal{O}^{\mathrm{CPO}}(0) = [\{\mathcal{Q}_\alpha^a, \mathcal{S}_\beta^b\}, \mathcal{O}^{\mathrm{CPO}}(0)], \tag{1.96}$$

where $\Delta = \dim[\mathcal{O}^{\mathrm{CPO}}(0)]$, and since (1.96) vanishes for some values of α, a, as defined in (1.91), it tells us that a scalar chiral primary operator with R-charges $(J_1, 0, 0)$ can be consistently annihilated by $\mathcal{Q}_\alpha^1$ and $\mathcal{Q}_\alpha^2$ if $\Delta = J_1$. The same argument involving the $\{\bar{\mathcal{Q}}, \bar{\mathcal{S}}\}$ anticommutator will tell us that they can also be annihilated by $\bar{\mathcal{Q}}_{\dot\alpha}^3$ and $\bar{\mathcal{Q}}_{\dot\alpha}^4$; this can be done for chiral primaries with R-charges $(0, J_2, 0)$ and $(0, 0, J_3)$. Furthermore, if we act on any of these operators with $\mathcal{R}_{mn}$ we will also get an operator that is a chiral primary.

Chiral primaries are protected operators. What does it mean? It is a general feature of operators in CFT that their dimensions depend on the coupling constant of the Yang-Mills theory. However, as shown above, chiral primaries are annihilated by half of the supercharges, and so the number of operators in the supermultiplet is now $2^4 = 16$. This number cannot change with the coupling, nor can the R-charge, since it too is an integer. For that reason Δ will always be equal to the R-charge, regardless of the strength of the coupling.

1.2.5 Conformal symmetry of $\mathcal{N} = 4$ SYM is maintained after quantisation

It is a rather common problem of a field theory that symmetries which are present at the classical level might get broken after quantisation. If the quantisation through a formalism of path integrals spoils these symmetries, then we speak of anomalies of this theory; in fact, conformal symmetry is usually anomalous. The way to deal with that is to regularise the theory to remove infinities, and by doing such one has to introduce a mass scale μ, in order to control the dimensionfullness of the series expansion properly. However that means that

[5] We denote these generators according to the planes of $\mathbf{S}^5$ sphere embedded in $\mathbb{R}^6$ on which they do act; for example $\mathcal{R}_{34}$ will generate rotations on the 34-plane, etc.

the correlation function itself will depend on some scale μ, which cannot happen if we want our two-point function to have a physical meaning. In order to deal with this problem, one assumes that the parameters of the quantum field theory also depend on the mass scale μ, such that the overall dependence will disappear.

In the case of $\mathcal{N}=4$ SYM, the only parameter is the coupling constant g, and dependence of μ is described by the beta function

$$\beta = \mu \frac{\partial g}{\partial \mu}. \tag{1.97}$$

However, the appearance of the beta function means that the scale invariance is broken. Fortunately, for $\mathcal{N}=4$ SYM it is believed that (see for example [23])

$$\beta = 0 \tag{1.98}$$

to all loops, and thus the conformal symmetry of $\mathcal{N}=4$ SYM survives quantisation.

1.3 The AdS/CFT correspondence

Having presented both, the AdS space and conformal field theory, it is at first glance certainly not obvious that a ten-dimensional string theory living in a hyperbolic-like space should have anything to do with the four dimensional maximally supersymmetric, conformal Yang-Mills theory, let alone being identical. Suprisingly, however, they are actually believed to be equivalent. We will now sketch a few arguments that led Maldacena [1] to formulate this conjecture, and then try to perform a relatively simple computation based on its postulates.

1.3.1 Type IIB string theory and gauge theories

The first sign that one could somehow relate type IIB string theory to some gauge theory is that a system of N coincident Dp-branes[6] is a classical solution of the low energy string effective action (in the Einstein frame)

$$S_{\mathrm{IIB}} \sim \int d^{10}x\sqrt{g}\left(R - \frac{1}{2}\, g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2}\sum_n \frac{1}{n!}\, e^{\frac{5-n}{2}\Phi} F_n^2 + \dots\right), \tag{1.99}$$

where Φ is a dilaton, $g_{\mu\nu}$ is the metric ($g \equiv |\det g_{\mu\nu}|$), R is the Ricci scalar curvature, and the n-form field strength F_n belongs to the Ramond-Ramond (R-R) sector; other terms in this action, like axion charge or the Neveu-Schwarz-Neveu-Schwarz (NS-NS) field, are irrelevant here, and hence dropped. On the other hand we know that in the low energy limit parallel Dp-branes realise $(p+1)$-dimensional $U(N)$ supersymmetric Yang-Mills theories. The solution of (1.99) has the following form

$$ds^2 = H^{-\frac{1}{2}}(r)\left[-f(r)\,dt^2 + \sum_{i=1}^{p}(dx^i)^2\right] + H^{\frac{1}{2}}(r)\left[f^{-1}(r)\,dr^2 + r^2 + r^2 d\Omega_{8-p}^2\right], \tag{1.100}$$

[6] To be more precise these are actually p-branes which are the solution of the low energy string action, however one can show that a p-brane and a Dp-brane both describe the same underlying object [15]; we are interested in the latter, since they are more physical. For more on Dp-branes see [25, 26, 27].

where

$$e^{\Phi} = H^{\frac{3-p}{4}}(r), \qquad H(r) = 1 + \frac{\kappa_p N}{r^{7-p}}, \qquad \kappa_p = \frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}} g_s, \tag{1.101}$$

$g_s \equiv e^{<\Phi>}$ being the string coupling, $\sqrt{\alpha'}$ is the fundamental string length, and since we consider non-extremal Dp-brane solution, $f(r) \approx 1$ and can be dropped. Notice that the metric (1.100) divides into the part which belongs to the world-volume of the brane, and to the part along the directions transverse to the brane. Furthermore, it is believed that a system of N coincident Dp-branes is described by the non-Abelian version of the Born-Infeld action. It can be argued that it takes the form [28]

$$S_{\mathrm{BI}} = -g_s \tau_p \int d^{p+1}\xi\, e^{-\Phi} \mathrm{tr} \sqrt{-\det\left(g_{\alpha\beta} + 2\pi\alpha' \mathcal{F}_{\alpha\beta}\right)}, \tag{1.102}$$

$g_{\alpha\beta}$ being the pullback to the Dp-brane world volume of the spacetime metric $g_{\mu\nu}$, and $2\pi\alpha'\mathcal{F}_{\alpha\beta} + \mathcal{B}_{\alpha\beta}$ is a gauge field that lives on the brane (but the $\mathcal{B}_{\alpha\beta}$ is dropped); the trace is symmetrised[7], and the determinant refers only to the Dp-brane $(p+1) \times (p+1)$-index structure of indices α, β. Above, the tension of the brane is given by

$$\tau_p \equiv \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_s}. \tag{1.103}$$

By expanding the action (1.102) in powers of $\mathcal{F}_{\alpha\beta}$ and demanding that the factor in front of the $\mathcal{F}^2$ term is

$$S_{\mathrm{BI}}^{\mathcal{F}^2} \sim -\frac{1}{4g_{\mathrm{YM}}^2} \int d^{p+1}\xi\, \mathcal{F}^{\mathsf{a}}_{\mu\nu} \mathcal{F}^{\mathsf{a}\,\mu\nu}, \tag{1.104}$$

as we expect to have in gauge theory, we obtain a relation between the string coupling and the gauge coupling

$$g_{\mathrm{YM}}^2 = \frac{2g_s}{\sqrt{\alpha'}} (2\pi\sqrt{\alpha'})^{p-2}. \tag{1.105}$$

It is worth stressing that even though we have established a connection between type IIB string theory and a gauge theory it does not mean that the duality has already been formulated. That is because we need to specify exactly (a) a very specific background for the string theory to propagate in, and (b) the exact type of gauge theory that is supposed to be compared with it. In fact, the connection between a string theory and a gauge theory has been noticed before Maldacena's discovery [30, 31].

1.3.2 The near-horizon approximation

The key point of this approach is now to look at the low energy limit of the set-up discussed above. Let us specifically consider $p = 3$, and a limit in which we approach the horizon at $r \sim 0$ but we keep the ratio of string length squared and r finite, namely

$$r \to 0, \qquad \alpha' \to 0, \qquad \mathsf{U} := \frac{r}{\alpha'} = \text{fixed}. \tag{1.106}$$

[7] In matter of fact, symmetrisation of the trace is correct only up to a certain order in the expansion of Born-Infeld action (1.102); at some point it breaks down and stops being a good approximation to the dynamics of Dp-branes (see for example [29]).

Then, the metric (1.100) will become

$$
\begin{aligned}
ds^2 &= \alpha' \left(\frac{\mathsf{U}^2}{\sqrt{4\pi g_s N}} dx_4^2 + \sqrt{4\pi g_s N} \left(\frac{d\mathsf{U}^2}{\mathsf{U}^2} + d\Omega_5^2 \right) \right) \\
&= \frac{\mathsf{U}^2}{R^2} d\tilde{x}_4^2 + R^2 \frac{d\mathsf{U}^2}{\mathsf{U}^2} + R^2 d\Omega_5^2,
\end{aligned} \tag{1.107}
$$

where $\tilde{x}$ is a suitably rescaled x, and

$$
R^4 := 4\pi g_s N \alpha'^2 \equiv g_{\mathrm{YM}}^2 N \alpha'^2, \tag{1.108}
$$

where we used (1.105). The form of the metric (1.107) is the manifold[8] $AdS_5 \times \mathbf{S}^5$, where two radii of AdS_5 and $\mathbf{S}^5$ are equal and given by

$$
R_{AdS_5} = R_{\mathbf{S}^5} = R. \tag{1.109}
$$

This clearly shows that the near horizon geometry of D3-branes in $AdS_5 \times \mathbf{S}^5$.

On the other hand, the field theory on N D3-branes is $\mathcal{N} = 4$ $U(N)$ super Yang-Mills at low energies[9]. Since excitations of the fields that live near the horizon have very small energies from the point of view of an outside observer we conclude that at low energies only these excitations will survive. So in the low energy limit we have two alternative descriptions which should be equivalent [1]. From the point of view of pure string theory, in the low energy limit $E \ll \frac{1}{\sqrt{\alpha'}}$, only massless states can be excited. In this limit, the open string states are massless excitations on the branes and are described by the theory that lives in the world-volume of the brane (in our case the $\mathcal{N} = 4$ SYM), whereas the closed string states are described by type IIB supergravity in the ten-dimensional bulk $AdS_5 \times \mathbf{S}^5$. On top of that, in the limit we will take ($g_s \to 0$) the massless open and closed string excitations do not interact, and thus gravity decouples from the brane being a "separate" theory.

1.3.3 Comparison of the string theory and the gauge theory

Statement that a ten dimensional string theory is equivalent to a four dimensional gauge theory is rather remarkable and it ought to be substantiated with a more convincing reasoning. Let us therefore try to compare the properties of

- type IIB string theory on $AdS_5 \times \mathbf{S}^5$ with $N = \int_{\mathbf{S}^5} F_5$ units of 5-form integer flux, and string coupling g_s, and
- $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$ in four dimensions.

First of all the type IIB string theory has 32 supercharges [25, 26] but even though $\mathcal{N} = 4$ SYM has only 16 of them, the superconformal extension gives us additional 16 fermionic generators, as we remember from section 1.2.3, confirming yet another agreement between these theories.

[8] One can see this explicitly by identifying $\mathsf{U} \equiv R^2 u$ in (1.107), dropping overall R^2, and comparing with (1.25).

[9] After all, as I mentioned before, the $\mathcal{N} = 4$ SYM in 4D can be derived by dimensional reduction of $\mathcal{N} = 1$ SYM in 10D.

Then, one should definitely compare global symmetries of both theories. The IIB string string on $AdS_5 \times \mathbf{S}^5$ theory has an isometry group $SO(2,4) \times SO(6)$, where the first part is the metric of pseudo-Euclidean hyperbolic space AdS_5, and $SO(6)$ are rotations of the $\mathbf{S}^5$ sphere. But the 32 Majorana spinor supercharges of IIB string theory (which are all preserved in this background) transform under this symmetry in such a way that the full invariance of the theory is given by the supergroup $PSU(2,2|4)$. This agrees with the symmetry group of superconformal $\mathcal{N}=4$ Yang-Mills, which is also given by $PSU(2,2|4)$, as mentioned in section 1.2.3. On top of that there is also the Montonen-Olive duality based on the group $SL(2,\mathbb{Z})$, which is an additional, discrete, global symmetry of both theories [32]. Thus we see that equivalence of these two theories seems very reasonable.

1.3.4 Implications of the correspondence

So how one could prove, or disprove, the AdS/CFT correspondence? It is certainly very difficult and here we present a heuristic argument. Let us first rewrite the most important equations of the correspondence

$$g^2_{\mathrm{YM}} = \frac{\lambda}{N} = 4\pi g_s; \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda}, \tag{1.110}$$

where we defined the 't Hooft coupling λ.

Now look at the near-horizon approximation (1.107). This approximation is adequate, provided that the radius of AdS_5 ($\mathbf{S}^5$) is very large in string units (that is the curvature of the background is small compared to the string scale). This can be achieved only if

$$R \gg \sqrt{\alpha'} \quad \Longleftrightarrow \quad \lambda \gg 1. \tag{1.111}$$

In this way we arrive at the ten dimensional classical supergravity compactified on $AdS_5 \times \mathbf{S}^5$, but on the other side we have to deal with a strongly coupled regime of $\mathcal{N}=4$ SYM, inaccessible by perturbative calculations. Also, the string excitations become infinitely heavy and decouple in that limit but, since we keep the radius of $\mathbf{S}^5$ fixed, we cannot at all neglect the Kaluza-Klein states associated with the compactification on that sphere. On the one hand, it is a good sign because having two completely different dual theories, with different properties, the potential overlap of a weakly coupled regime of both could yield some obscure contradictions; in this case we do not have to worry about that. However, the bad news is that the gauge/string duality is a kind of a weak/strong duality, making it hard to verify.

But then we can go even further than that and say that since supergravity is not a consistent quantum theory one should be able to extend the conjecture to any value of λ, yet for that one has to find a suitable substitute of classical supergravity. This was in fact the argument of Maldacena who argued that [1] the type IIB string theory on $AdS_5 \times \mathbf{S}^5$ and the $\mathcal{N}=4$ SYM theory are in fact the same, with parameters related through (1.110).

Then one can for example take the string coupling to be small $g_s \to 0$ such that $\lambda \propto g_s N =$ fixed, which corresponds to taking $N \to \infty$ ('t Hooft limit)[10], and then take λ to be large yet finite. In this way one can see that the classical supergravity is "replaced" by a free, non-interacting string theory. Unfortunately though, a quantisation of even a zero-genus type IIB string theory on $AdS_5 \times \mathbf{S}^5$ is a problem that remains unsolved.

[10] The 't Hooft limit is merely a requirement that the radii of AdS_5 and $\mathbf{S}^5$ be much larger than the Planck length, namely that $R^2 \gg \ell_p^2 = \sqrt{g_s}\alpha'$.

There is however a very good news in all this, namely that, as far as the low order in perturbation theory is concerned, there is a way to test the AdS/CFT duality by comparing spectrum of dual objects on both sides after all. This is possible by taking yet another limit: an infinite R-charge of some operators in a specific way, as suggested by Berenstein, Maldacena, and Nastase [5]. We will briefly discuss that in section 1.4.

1.3.5 The dictionary of Maldacena's conjecture

It is very important to emphasise that even though type IIB string theory is equivalent to $\mathcal{N}=4$ SYM theory (as of today all the tests of AdS/CFT correspondence seem to confirm so), these two theories live on two different spaces: IIB string theory lives on the $AdS_5 \times \mathbf{S}^5$ manifold, whereas $\mathcal{N}=4$ supersymmetric gauge theory lives on the boundary of AdS_5 which, from the point of view of the metric (1.107), is the four-dimensional Minkowski space. For that reason we need some kind of tool with the help of which we could relate the actions of these theories to each other in the boundary limit. This kind of proposal came about thanks to Gubser, Klebanov, Polyakov [2], and Witten [3] (GKPW), who suggested to relate the partitions functions of these two theories (we use the Euclidean metric (1.28) here)

$$\mathcal{Z}_{AdS}\left[\Phi_0(\vec{x})\right] \equiv \mathcal{Z}_{CFT}\left[\Phi_0(\vec{x})\right], \tag{1.112}$$

where $\Phi_0(\vec{x}) := \Phi(x_0 = 0, \vec{x})$ has two interpretations: On the gravity side these fields corresponds to the boundary values for the bulk fields $\{\Phi_i\}$ which propagate in the AdS space, and this includes not just the scalar fields but all the fields including the graviton and the gauge fields. On the conformal field theory side it corresponds to external sources (currents) coupled to operators in CFT. The idea therefore that we can obtain insertions of these operators by differentiation of the partition function (1.112) with respect to the sources. As suggested by GKPW, the ansatz for the precise relation of CFT on the boundary to AdS space[11] is

$$\begin{aligned}\mathcal{Z}_{CFT}\left[\Phi_0(\vec{x})\right] &= \int \mathcal{D}\mathcal{O}\,\exp\!\left[-S_{\text{CFT}}[\mathcal{O}(x)] + \int_{\partial(AdS_{d+1})} d^d x\,\Phi_0(\vec{x})\mathcal{O}_\Delta(\vec{x})\right] \\ &= \left\langle e^{\int d^d x\,\Phi_0(\vec{x})\mathcal{O}_\Delta(\vec{x})}\right\rangle .\end{aligned} \tag{1.113}$$

Note that we work in Euclidean space, and therefore the boundary of AdS_{d+1} space is a d-sphere $\mathbf{S}^d$. Then, as argued by GKPW, the partition function for AdS space should be

$$\mathcal{Z}_{AdS}\left[\Phi_0(\vec{x})\right] \sim e^{-N^2 S_{\text{sugra}}[\Phi] + \mathscr{O}(\alpha')} \times (\text{Quantum Corrections}). \tag{1.114}$$

We will now try to make a very specific and relatively simple check of AdS/CFT duality and calculate the two-point correlation function for the massive scalars in the bulk by the use of (1.112), and then compare it with the result of CFT (1.61). Since we consider $\lambda \gg 1$ we do not have to worry about corrections to the classical action in (1.114), and thus solely focus on the classical, massive, scalar theory in the bulk, described by

$$S[\Phi] = \frac{1}{2}\int_{AdS_{d+1}} d^{d+1}x\sqrt{g}\left(\partial_\mu\Phi(x)\partial^\mu\Phi(x) + m^2\Phi^2(x)\right), \tag{1.115}$$

[11] Note that we work again in d dimensions.

where m^2 is the Kaluza-Klein mass coming from the dimensional compactification. We will use the metric (1.23) in which coordinates it is the easiest to solve the equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,\partial^\mu\Phi(x)\right) = m^2\Phi^2(x) \tag{1.116}$$

at the boundary $x_0 = 0$ ($\tilde{y} = \infty$).

If, according to (1.23), we denote the metric on $\mathbf{S}^d$ as $\gamma_{\alpha\beta}$ then certainly

$$\sqrt{g} = \sinh^d\tilde{y}\,\sqrt{\gamma}, \tag{1.117}$$

and in these coordinates we would like to understand the behaviour of $\Phi(\tilde{y},\theta^\alpha)$ at the boundary $\tilde{y} = \infty$. Our choice of metric allows us to split the Laplacian such that one part depends solely on $\tilde{y}$, whereas the other one on all of the coordinates

$$\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\,\partial^\mu = \frac{1}{\sinh^d\tilde{y}}\,\frac{d}{d\tilde{y}}\,\sinh^d\tilde{y}\,\frac{d}{d\tilde{y}} - \frac{\mathsf{L}^2}{\sinh^2\tilde{y}}, \tag{1.118}$$

where

$$-\mathsf{L}^2 \equiv \frac{1}{\sqrt{\gamma}}\partial_\alpha\sqrt{\gamma}\partial^\alpha \tag{1.119}$$

is the Laplacian on the unit $\mathbf{S}^d$ sphere. We therefore see that for the large $\tilde{y}$ the dependence of the Laplacian on θ^α drops out, and the Klein-Gordon equation (1.116) becomes

$$\frac{1}{\sinh^d\tilde{y}}\,\frac{d}{d\tilde{y}}\left(\sinh^d\tilde{y}\,\frac{d}{d\tilde{y}}\,\Phi(\tilde{y},\theta^\alpha)\right) = m^2\,\Phi(\tilde{y},\theta^\alpha). \tag{1.120}$$

Due to spherical symmetry of the metric, one can expand Φ in spherical harmonics $\Phi(\tilde{y},\theta^\alpha) = \sum_n \Phi_n(\tilde{y})\mathcal{Y}_n(\theta^\alpha)$, and then (1.120) becomes

$$e^{-d\tilde{y}}\,\partial_{\tilde{y}}\left(e^{d\tilde{y}}\,\partial_{\tilde{y}}\Phi_n(\tilde{y})\right) = m^2\Phi_n(\tilde{y}) \tag{1.121}$$

will have two solutions $\Phi_n \sim 1$ and $\Phi_n \sim e^{\lambda\tilde{y}}$. Clearly, at the boundary the second solution dominates over the first one, and then (1.121) determines λ to be

$$\lambda(d+\lambda) = m^2 \qquad \Longrightarrow \qquad \lambda_\pm = \frac{1}{2}\left(-d \pm \sqrt{d^2+4m^2}\right). \tag{1.122}$$

Suppose now that $m^2 > 0$. Then the solution which will dominate at the boundary will be

$$\Phi(\tilde{y},\vec{x}) \sim \left(e^{\tilde{y}}\right)^{\lambda_+}\,\Phi_0(\vec{x}), \tag{1.123}$$

which will "blow up" at the boundary if $\lambda_+ > 0$. But suppose that we define

$$\mathsf{f}(\tilde{y},\vec{x}) := e^{-\tilde{y}}, \tag{1.124}$$

and then (1.123) will read

$$\Phi(\tilde{y},\vec{x}) \sim \mathsf{f}^{-\lambda_+}(\tilde{y},\vec{x})\,\Phi_0(\vec{x}). \tag{1.125}$$

The function f is just of the form needed to build a finite metric from the divergent AdS one, because the definition of Φ_0 as a function depends on the choice of particular f, which was the same choice used in (1.18). Following thus (1.20) we can rescale f arbitrarily, which on the other hand implies a freedom in Φ_0

$$\mathsf{f} \to e^{w}\mathsf{f} \qquad \Longrightarrow \qquad \Phi_0 \to e^{w\lambda_+}\Phi_0. \tag{1.126}$$

This transformation under conformal rescaling shows that Φ_0 must be understood as a conformal density of mass dimension $-\lambda_+$. Since the partition function (1.113) ought to be conformally invariant then we furthermore deduce that the field $\mathcal{O}_\Delta(\vec{x})$ has conformal dimension $\Delta \equiv d + \lambda_+$.

1.3.6 Two-point correlation function in AdS/CFT

Let us now calculate the two-point function for the field $\mathcal{O}_\Delta(\vec{x})$. For that purpose we use the metric of the form (1.28). As we remember, it has a boundary at $x^0 = 0$, which in Euclidean case corresponds to $\mathbb{R}^d$ together with a single point P at $x^0 = \infty$, giving a sphere $\mathbf{S}^d$. This can be achieved [3] by looking for a Green's function which is a solution of the Laplace equation on the Euclidean version of AdS_{d+1}, namely the (unit) ball $\mathcal{B}_{d+1}$ whose boundary value is a delta function at a point P on the boundary

$$\mathcal{K}(x;P) = \mathcal{K}(x^0,\vec{x};P), \tag{1.127}$$

where $x \in \mathcal{B}_{d+1}$, $\vec{x} \in \mathbb{R}^d$. Due to translational invariance at the boundary of (1.127) under $\vec{x} \to \vec{x}'$, $\mathcal{K}(x)$ will only be a function of x^0, and thus the Laplace equation will read

$$\frac{1}{\sqrt{g}}\partial_0\left(\sqrt{g}\,\partial^0\mathcal{K}(x^0)\right) = m^2\,\mathcal{K}(x^0). \tag{1.128}$$

Metric (1.28) yields that

$$g_{\mu\nu} = (x^0)^{-2}\delta_{\mu\nu}, \qquad \sqrt{g} = (x^0)^{-(d+1)}, \qquad \partial^0 = (x^0)^2\partial_0 \tag{1.129}$$

hence (1.128) becomes

$$\left[(x^0)^{d+1}\frac{d}{dx^0}\,(x^0)^{-d+1}\frac{d}{dx^0} - m^2\right]\mathcal{K}(x^0) = 0. \tag{1.130}$$

We seek for the solutions of the type

$$\mathcal{K}(x^0) \propto (x^0)^{d+\lambda} \tag{1.131}$$

and we see that the one which vanishes for $x^0 \to 0$ is the one which has

$$\lambda \equiv \lambda_+ = \frac{1}{2}\left(-d + \sqrt{d^2+4m^2}\right). \tag{1.132}$$

The full version of $\mathcal{K}(x)$ is found by an $SO(1,d+1)$ transformation that maps P to a finite point

$$x^\mu \longrightarrow \frac{x^\mu}{(x^0)^2+\vec{x}^2}, \qquad \mu = 0,1,...,d \tag{1.133}$$

yielding

$$\mathcal{K}(x) = c\frac{(x^0)^{d+\lambda_+}}{[(x^0)^2 + (\vec{x} - \vec{x}')^2]^{d+\lambda_+}}, \tag{1.134}$$

where we used translational invariance of $\mathcal{K}(x)$

$$\mathcal{K}(x) = \mathcal{K}(x^0, \vec{x}; \vec{0}) \equiv \mathcal{K}(x^0, \vec{x}; \vec{x}'). \tag{1.135}$$

Notice, however, that (1.134) does not become a delta function at $x^0 \to 0$, it is rather

$$\frac{(x^0)^{d+2\lambda_+}}{[(x^0)^2 + (\vec{x} - \vec{x}')^2]^{d+\lambda_+}} \stackrel{x^0 \to 0}{\longrightarrow} \delta^{(d)}(\vec{x} - \vec{x}'). \tag{1.136}$$

This can be intuitively understood when looking at the scalar field in the bulk, which can be expressed as

$$\begin{aligned} \Phi(x^0, \vec{x}) &= c \int d\vec{x}' \frac{(x^0)^{d+\lambda_+}}{[(x^0)^2 + (\vec{x} - \vec{x}')^2]^{d+\lambda_+}} \Phi_0(\vec{x}') \\ &= c\,(x^0)^{-\lambda_+} \int d\vec{x}' \frac{(x^0)^{d+2\lambda_+}}{[(x^0)^2 + (\vec{x} - \vec{x}')^2]^{d+\lambda_+}} \Phi_0(\vec{x}'), \end{aligned} \tag{1.137}$$

which merely confirms (1.125).

Let us now come back to the free scalar field action (1.115). One can clearly see that its on-shell version (that is with (1.116) included) would be

$$\begin{aligned} S[\Phi] &= \frac{1}{2} \int d^{d+1}x \sqrt{g} \left[\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, \Phi(x^0, \vec{x})\, \partial^\mu \Phi(x^0, \vec{x}) \right) \right] \\ &= \frac{1}{2} \int_{x^0=\epsilon}^{\infty} d\vec{x}\; \partial_0 \left[(x^0)^{-d+1}\, \Phi(x^0, \vec{x})\, \partial_0 \Phi(x^0, \vec{x}) \right] \\ &= -\frac{1}{2} \int d\vec{x} \left[(x^0)^{-d+1}\, \Phi(x^0, \vec{x})\, \partial_0 \Phi(x^0, \vec{x}) \right]_{x^0=\epsilon}. \end{aligned} \tag{1.138}$$

Then, using (1.137), we can easily show that

$$\partial_0 \Phi(x^0, \vec{x}) = c\,(d + \lambda_+)(x^0)^{d+\lambda_+-1} \int d\vec{x}'\, \frac{\Phi_0(\vec{x}')}{[(x^0)^2 + (\vec{x} - \vec{x}')^2]^{d+\lambda_+}} + \mathscr{O}[(x^0)^{d+1}], \tag{1.139}$$

hence

$$S[\Phi_0] = -\frac{c\,(d + \lambda_+)}{2} \iint d\vec{x}' d\vec{x}\, \frac{\Phi_0(\vec{x})\, \Phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2(d+\lambda_+)}}. \tag{1.140}$$

This yields the two-point function for an operator $\mathcal{O}_\Delta(\vec{x})$ with conformal dimension $\Delta = d + \lambda_+$

$$\left\langle \mathcal{O}_\Delta(\vec{x}) \mathcal{O}_\Delta(\vec{x}') \right\rangle = \frac{\delta}{\delta \Phi_0(\vec{x})} \frac{\delta}{\delta \Phi_0(\vec{x}')} \mathscr{Z}_{\mathrm{AdS}}[\Phi_0(\vec{y})] \bigg|_{\Phi_0 \equiv 0} \propto \frac{\text{const}}{|\vec{x} - \vec{x}'|^{2\Delta}}, \tag{1.141}$$

which perfectly agrees with (1.61).

In this way we have explicitly checked that two-point functions are well defined objects in AdS/CFT duality, and for that reason they can help us to compare many properties of dual objects in the gauge/string correspondence. For example, by comparing spectra of certain string states and their dual gauge states we will be able to test the conjecture, to some extent at least. Of course, this is precisely when the weak/strong duality problem comes about, and we will try to get around it by applying the BMN's double scaling limit; we will get back to this in the next section.

As a last remark, let us discuss the m^2 parameter. The assumption that $m^2 > 0$ led us to (1.125), and then to (1.137). That is because in the case of real, positive mass the linear combination of these solutions had one term which was dominant over the other one. However, it turns out [33] that for

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1 \tag{1.142}$$

both solutions (that is roots in (1.122)) may be chosen, hence two possible conformal field theories: one in which the operator has dimension Δ_+, and one in which it has dimension Δ_-, where

$$\Delta_\pm = d + \lambda_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2}\right). \tag{1.143}$$

This does not contradict itself, since Δ_+ is bounded from below by $\frac{d}{2}$ but there is no corresponding bound in d-dimensional CFT, whereas in the range (1.142) Δ_- is bounded from below by $\frac{d-2}{2}$, which is the unitarity bound on dimensions of scalars in d dimensional conformal field theories. For that reason Δ_- is crucial for consistency of the AdS/CFT duality. Furthermore, $m^2 < 0$ does not yield to tachyonic instabilities as one might suspect, since these appear when $m^2 < -\frac{d^2}{4}$, and not when $m^2 < 0$ [34, 35].

It turns out that in order to build a correlation function in the theory with dimension Δ_- one has to consider the most general solution of the scalar field $\Phi(x)$ at the boundary

$$\Phi(x^0, \vec{x}) \stackrel{x^0\to 0}{\longrightarrow} (x^0)^{d-\Delta_\pm}\left(\Phi_0(\vec{x}) + \mathscr{O}(x^0)\right) + (x^0)^{\Delta_\pm}\left(\mathcal{A}(\vec{x}) + \mathscr{O}(x^0)\right), \tag{1.144}$$

where (as we already know) $\Phi_0(\vec{x})$ represents a current in the AdS/CFT partition function (1.112), and (as we do not know yet) $\mathcal{A}(\vec{x})$ describes physical fluctuations that will be determined from the source by solving the classical equations

$$\mathcal{A}(\vec{x}) = \frac{1}{2\Delta_\pm - d}\left\langle \mathcal{O}(\vec{x})\right\rangle. \tag{1.145}$$

Since $\mathcal{A}(\vec{x})(2\Delta_\pm - d)$ is the variable conjugate to $\Phi_0(\vec{x})$, in order to interchange $\Delta_\pm$ and $d - \Delta_\pm$ one has to interchange $\Phi_0(\vec{x})$ and $(2\Delta_\pm - d)\mathcal{A}(\vec{x})$. This is a canonical transformation which for tree-level correlators reduces to a Legendre transform. Thus, the generating functional of correlators in the Δ_- theory must be obtained by Legendre-transforming the partition function of correlators in the Δ_+ theory. For details, please refer to [36].

Finally, let us do an explicit check for $m^2 = 0$. The dominant solution in $x^0 \to 0$ will be $\Delta_+ = d$. For the case of $AdS_5 \times \mathbf{S}^5$, $\Delta_+ = 4 \equiv \Delta$, and thus the correlation function will be of the form

$$\left\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{x}')\right\rangle \propto \frac{\text{const}}{|\vec{x} - \vec{x}'|^8}. \tag{1.146}$$

We know also that since (a) $\Phi_0(\vec{x})$ for $m = 0$ becomes a dilaton which is related to the string coupling as

$$g_s = e^{\Phi_0}, \tag{1.147}$$

(b) g_s is related to the Yang-Mills coupling through (1.105), and furthermore (c) the quick look at the $\mathcal{N} = 4$ SYM action (2.3) tells us that g^2_{YM} couples to $\mathrm{tr}\,\mathcal{F}^2$, then we should suspect that

$$\left\langle \mathrm{tr}\,\mathcal{F}^2(\vec{x})\,\mathrm{tr}\,\mathcal{F}^2(\vec{x}')\right\rangle \propto \frac{\mathrm{const}}{|\vec{x}-\vec{x}'|^8}, \tag{1.148}$$

or

$$\dim\,[\mathrm{tr}\,\mathcal{F}^2] = 4, \tag{1.149}$$

which is protected by R-symmetry [3]. This indeed does agree, since $\mathcal{A}_\mu(\vec{x})$ has conformal dimension 1, providing us with a nice consistency check for the AdS/CFT duality.

1.4 BMN operators and the double scaling limit

While the supergravity limit of type IIB string theory on $AdS_5 \times \mathbf{S}^5$, which corresponds to $\alpha' \to 0$, and $\lambda \to \infty$, began to be tested soon after the conjecture had been established (see for example [37]), the regime of non-interacting IIB string theory (with λ fixed yet large, $g_s \to 0$, $N \to \infty$) remained inaccessible due to enormous difficulty in quantisation of string theory in the $AdS_5 \times \mathbf{S}^5$ background. Then, Berenstein, Maldacena, and Nastase suggested [5] how to find the overlapping spectrum of dual operators through AdS/CFT correspondence on the pp-wave, which can be viewed as a first correction to the string theory in flat space.

The idea of pp-waves goes all the way to 1925 and comes from Brinkmann [38]. The "pp" term stands for plane-fronted waves with parallel propagation, and was introduced in 1962 by Ehlers and Kundt [39]. Then, in 1976 Penrose showed that plane waves can be obtained as limits of various backgrounds [40]. In the case of Berenstein, Maldacena, and Nastase the idea was to consider the trajectory of a particle that is moving very fast along the $\mathbf{S}^5$, and to focus on the geometry that this particle sees. Suppose it sits at the center of AdS, and rotates in $\mathbf{S}^3 \subset \mathbf{S}^5$ along one of the directions, say some θ (in global coordinates, same as (1.5) but this time for $\mathbf{S}^5$; see (3.47)), and then introduce the light-cone coordinates

$$x^+ = \frac{t+\theta}{2\mu}, \qquad x^- = \mu R^2(t-\theta), \tag{1.150}$$

where t is the time direction, and μ is some mass scale. By taking the the limit $R \to \infty$ one receives the parallel-plane (pp-wave) metric

$$ds^2 = -4dx^+dx^- - \mu^2 \sum_{I=1}^{8}(x^I)^2(dx^+)^2 + \sum_{I=1}^{8}(dx^I)^2, \tag{1.151}$$

which for $\mu \to 0$ reproduces the flat space string metric. Above, x^I are eight transverse coordinates, and come from both, AdS_5 and $\mathbf{S}^5$.

The energy in global coordinates in AdS is given by $E = i\partial_t$, and the angular momentum by $J = -i\partial_\theta$. The latter can be thought of as the rotation generator that rotates the 56-plane of $\mathbb{R}^6$. In terms of the dual CFT these are the energy and the R-charge of a state of the field theory on $\mathbb{R} \times \mathbf{S}^3$, or we can also say that $E = \Delta$ is the conformal weight of an operator on $\mathbb{R}^4$. The Hamiltonian will be

$$H = 2p^- = i\mu(\partial_t + \partial_\theta) = \mu(\Delta - J), \tag{1.152}$$

$$2p^+ = \frac{\Delta + J}{\mu R^2}, \tag{1.153}$$

and since $2p^+$ and $2p^-$ have to stay finite in $R \to \infty$, we can take

$$\Delta \sim J \to \infty \quad \text{such that} \quad J \sim R^2 \sim \sqrt{g_s N}. \tag{1.154}$$

And so we do the following. We take the 't Hooft limit $g_s \to 0$, $g_s N =$ fixed, and then take λ to be large, keeping $\frac{g_s N}{J^2}$ fixed and $\Delta - J$ as well, giving us the plane wave background with zero coupling.

The metric (1.151) leads us to the σ-model action which can be quantised rather easily[12], yielding the following Hamiltonian

$$2p^- = \sum_n N_n \sqrt{\mu^2 + \frac{4\pi g_s N}{J^2} n^2}, \tag{1.155}$$

where n labels the Fourier modes, N_n denotes the total occupation number of oscillatory mode, and the condition

$$P = \sum_n n N_n = 0 \tag{1.156}$$

is imposed.

Very generally speaking, (1.152) is used to construct the operators on the gauge side, plus the corresponding excitations, and then (1.155) is used to compare their spectra. This indeed can be done since the quantity $\frac{g_s N}{J^2}$ is being kept fixed. The spectrum is constructed such that, if we take $Z := \phi_5 + i\phi_6$, and then consider a chiral primary operator with R charge $(0, 0, J)$, then we can associate it with the vacuum state in light-cone gauge

$$|0, p^+\rangle_{\text{l.c.}} \mathrel{\widehat{=}} \mathrm{tr}[Z^J], \qquad (\Delta - J = 0) \tag{1.157}$$

up to a normalisation constant. We have 8 bosonic $(\alpha_0^{\mathcal{A},\dagger})$ + 8 fermionic $(\theta_0^{\mathcal{B},\dagger})$ modes $(\mathcal{A}, \mathcal{B} = 1, \ldots, 8)$ with $\Delta - J = 1$, for example applying the zero momentum bosonic oscillator $\alpha_0^{\dagger, i}$ (splitting $\mathcal{A} = \{i, \mu\}$)

$$\alpha_0^{i,\dagger} |0, p^+\rangle_{\text{l.c.}} \mathrel{\widehat{=}} \mathrm{tr}[\phi_i \, Z^J], \qquad i = 1, 2, 3, 4 \tag{1.158}$$

[12] The reason for this is that the pp-wave metric of type IIB string theory on $AdS_5 \times \mathbf{S}^5$ (1.151) acquires one additional term with respect to the flat space string theory, and this term has the form of a mass term. Since the quantisation of string theory with a harmonic oscillator-like potential is a problem which is already known and solved, quantisation of our model is rather straightforward.

and so on. If we consider a non-zero momentum creation oscillator of the type α^i_{-n} or $\tilde{\alpha}^i_{-n}$, where the operators without "$\sim$" denote the left-moving modes, and the ones with "$\sim$" denote the right-moving modes, then we need to include a phase, for example

$$\alpha^i_{-n}|0,p^+\rangle_{\text{l.c.}} \mathrel{\widehat{=}} \sum_{\ell=1}^{J} \text{tr}[Z^\ell \phi_i\, Z^{J-\ell}]\, e^{\frac{2\pi i n \ell}{J}}, \tag{1.159}$$

which will vanish due to the cyclicity of the trace; that is merelyy a restatement of (1.156). In this way we can build up a spectrum, for example

$$\alpha^i_{-n}\tilde{\alpha}^j_{-n}|0,p^+\rangle_{\text{l.c.}} \mathrel{\widehat{=}} \sum_{\ell=1}^{J} \text{tr}[\phi_i\, Z^\ell \phi_j\, Z^{J-\ell}]\, e^{\frac{2\pi i n \ell}{J}}, \tag{1.160}$$

and so forth. We just need to remember that the states whose total momentum is not zero along the string (and thus the "string" of Z's) lead to operators that are automatically zero by cyclicity of the trace; in this way we enforce (1.156) which is nothing but the Virasoro constraint on the string side (the number of left moving modes equals the number of right moving modes).

So what about the spectrum of the gauge side? Berenstein, Maldacena, and Nastase in [5] did a one-loop calculation of the correlation function and received that a contribution of a field ϕ_i inserted in the "string" of Z's gives a one-loop correction to the bare dimension of the operator

$$(\Delta - J)_n = \mu + \frac{g^2_{\text{YM}}N}{2\mu J^2}n^2, \tag{1.161}$$

which agrees with (1.155) expanded in small $\frac{g_s N}{J^2}$, due to (1.105). We will fill in many details into this derivations when considering a type IIB string on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the plane wave background.

Chapter 2

The Dilatation Operator

I have already mentioned that the two-point correlation function is a very well defined object from the point of view of AdS/CFT duality-conjecture. It is of particular importance as the correct establishment of a two-point function of a locally gauge invariant operator allows to read off its scaling dimension. Unfortunately, in practice it is not easy and there are two main reasons for that. First of all, this dimension depends on the coupling constant of super Yang-Mills theory, and therefore it has to be calculated loop by loop by means of field theory, which already makes the computation rather involved. Secondly, it is very rare that we can perform such calculation in a basis that yields anomalous dimensions (as corrections to the bare dimension) in diagonal form. Instead, one usually receives a matrix of anomalous dimensions which ought to be diagonalised. In the following chapter we present an alternative method of calculating this matrix, namely by applying the dilatation operator. Then, by the use of field theory, we come to its explicit one-loop form for the bosonic $SO(6)$ subsector of $\mathcal{N}=4$ SYM and show that it also yields the one-loop anomalous dimension matrix. Finally, we recalculate the precise form of both, the operator and the matrix for the $SU(3)$ and $SU(2)$ subsectors.

2.1 Correlation function of $\mathcal{N}=4$ SYM

The N-point correlation function of $\mathcal{N}=4$ SYM reads exactly like in (1.50), i.e. the two-point function will be

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \int \mathcal{D}\mathcal{W}\; \mathcal{O}[\mathcal{W}(x_1)]\mathcal{O}[\mathcal{W}(x_2)]\, e^{-S^{\mathcal{N}=4}_{\mathrm{SYM}}[\mathcal{W}]}, \tag{2.1}$$

where $\mathcal{W}$ is one of the $\mathcal{N}=4$ fields in some spacetime point x

$$\mathcal{W} \in \left\{\mathcal{D}^k\phi, \mathcal{D}^k\psi, \mathcal{D}^k\bar{\psi}, \mathcal{D}^k\mathcal{F}\right\}, \tag{2.2}$$

with appropriate indices, and where (2.1) ought to be properly normalised of course. The action is given by

$$S^{\mathcal{N}=4}_{\mathrm{SYM}}[\mathcal{W}] = \frac{2}{g^2_{\mathrm{YM}}}\int d^4x\, \mathscr{L}^{\mathcal{N}=4}_{\mathrm{SYM}}[\mathcal{W}, g=1], \tag{2.3}$$

where $g^2_{\rm YM}$ is a Yang-Mills coupling constant, and $\mathscr{L}^{\mathcal{N}=4}_{\rm SYM}$ is given by (1.71). The local, gauge invariant states in (2.1) are constructed such that we take

$$\mathcal{O}(x) = \operatorname{tr}\mathcal{W}(x)\cdots\mathcal{W}(x)\operatorname{tr}\mathcal{W}(x)\cdots\mathcal{W}(x)\cdots\,. \tag{2.4}$$

Due to the fact, however, that we will be mostly interested in the correlation functions in the planar (large N) limit, in which the field theory diagrams may connect only two single trace operators with genus zero, we will be considering a linear combination of single trace operators

$$\mathcal{O}(x) = \sum_{\substack{\text{appropriate}\\ \text{indices}}} \operatorname{tr}\mathcal{W}(x)\cdots\mathcal{W}(x). \tag{2.5}$$

The fields $\mathcal{W}(x)$ are in representation of the gauge group $U(N)$, and therefore we have

$$\mathcal{W}^{\mathfrak{ab}}(x) = \mathcal{W}^{(\mathsf{a})}(x)(T^{\mathsf{a}})^{\mathfrak{ab}}, \quad \mathfrak{a},\mathfrak{b} = 1,\ldots,N, \quad \mathsf{a} = 1,\ldots,N^2 \tag{2.6}$$

where T^{a}'s are the generators of $U(N)$, that is

$$[T^{\mathsf{a}},T^{\mathsf{b}}] = i\mathfrak{f}^{\mathsf{abc}}T^{\mathsf{c}}, \qquad \operatorname{tr}(T^{\mathsf{a}}T^{\mathsf{b}}) = \delta^{\mathsf{ab}}. \tag{2.7}$$

Therefore, the two-point correlator for $U(N)$ fields will read

$$\Big\langle \mathcal{W}^{\mathfrak{ab}}(x)\mathcal{W}^{\mathfrak{cd}}(y)\Big\rangle = \left(\begin{smallmatrix}\text{spacetime}\\ \text{propagator}\end{smallmatrix}\right) \times (\text{additional}) \times \delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}, \tag{2.8}$$

where "additional" denotes possible extra deltas coming from additional indices of a field $\mathcal{W}$.

In the perturbative theory it will be convenient to work with the two coupling constants g and $\tilde{g}$

$$\tilde{g}^2 := \frac{g^2_{\rm YM}}{16\pi^2}, \qquad g^2 = \tilde{g}^2 N = \frac{\lambda}{16\pi^2}, \tag{2.9}$$

where N is the number of colours in the $U(N)$ gauge group, and λ is the 't Hooft coupling, and for that reason we will perform all the field theory calculations with the following form of the action

$$S^{\mathcal{N}=4}_{\rm SYM}[\mathcal{W}] = \frac{1}{8\pi^2}\int d^4x\, \mathscr{L}^{\mathcal{N}=4}_{\rm SYM}[\mathcal{W}/\tilde{g}, \tilde{g} = \sqrt{g^2_{\rm YM}/16\pi^2}]. \tag{2.10}$$

Having said that, the two-point function will have the form (in the orthogonal basis)

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{C_{12}}{|x_{12}|^{2\Delta(g)}}, \tag{2.11}$$

where

$$\Delta(g) = \dim[\mathcal{O}(x_1)] = \dim[\mathcal{O}(x_2)] = \sum_{n=0}^{\infty} g^{2n}\Delta_{2n} \tag{2.12}$$

is a scaling dimension of a state $\mathcal{O}$, which depends on the coupling constant g, unless the state is a chiral primary of course. However, while the two- and the three-point functions of chiral primary operators are protected from radiative corrections, the higher-point functions do receive these corrections, and thus in the limit of infinite R-charge they simply diverge [24]. Above, the n index refers to $n^{\rm th}$ loop, and we note that in this expansion the terms with odd powers of g will not appear [20].

2.2 Anomalous dimensions from the two-point function

Soon after the BMN limit (section 1.4) was discovered people realised that an effective way to compute the scaling dimension $\Delta(g)$ of gauge operators is needed, because it can be immediately compared to the string side; the AdS/CFT conjecture can thus be tested explicitly, although only to some order in perturbation theory and in a specific limit. Unfortunately, the calculation of coupling-dependent Δ is a challenge in itself and requires quite a bit of field theory.

Let us now describe very briefly how the calculation should be performed up to one loop (generalisation to higher loops in this section will be rather straightforward here). In general, the constant C_{12} in (2.11) will depend on the coupling g and the divergent UV cutoff constant Λ (that sets the scale), and it is chosen such that

$$C_{12} = C_{12}(g,\Lambda) \to M(g)(1 - g^2\Delta_2 \ln\Lambda^2) + \mathscr{O}(g^3), \tag{2.13}$$

where we denoted

$$\Delta(g) \cong \Delta_0 + \gamma, \tag{2.14}$$

$\gamma \equiv g^2\Delta_2$ being the one-loop anomalous dimension. Then, expansion of (2.11), with (2.13) included, yields

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{M_0}{|x_{12}|^{2\Delta_0}} + \frac{g^2}{|x_{12}|^{2\Delta_0}}(M_2 - M_0\,\Delta_2 \ln|x_{12}\Lambda|^2), \tag{2.15}$$

where we expanded $M(g) \cong M_0 + g^2 M_2$. Obviously, in order to remove infinities from this expansion one has to renormalise the operator, and this is usually done by introducing the renormalisation factor by defining the renormalised basis

$$\widetilde{\mathcal{O}} \equiv \mathcal{Z}\cdot\mathcal{O}, \tag{2.16}$$

and then expect the desired correlator to be of the form

$$\langle \widetilde{\mathcal{O}}(x_1)\widetilde{\mathcal{O}}(x_2)\rangle = \frac{M_0}{|x_{12}|^{2\Delta_0}} + \frac{g^2}{|x_{12}|^{2\Delta_0}}(M_2 - M_0\,\Delta_2 \ln|x_{12}|^2), \tag{2.17}$$

and since

$$\langle \widetilde{\mathcal{O}}(x_1)\widetilde{\mathcal{O}}(x_2)\rangle = \langle \mathcal{Z}\mathcal{O}(x_1)\mathcal{Z}\mathcal{O}(x_2)\rangle = \mathcal{Z}^2\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle, \tag{2.18}$$

then comparison of (2.15) and (2.17) tells us that

$$\mathcal{Z}^2(g,\Lambda)\,C(g,\Lambda) = 1, \tag{2.19}$$

hence

$$\mathcal{Z}(g,\Lambda) = (1 - \gamma\ln\Lambda^2)^{-1/2} \cong 1 + \gamma\ln\Lambda. \tag{2.20}$$

This calculation, however, is done in diagonal basis, therefore we directly obtain the one-loop anomalous dimension. In practice though, it is almost never that easy because the states $\mathcal{O}(x)$ have appropriate matrix indices, and thus the coefficients in (2.15), (2.17), together with $\mathcal{Z}$ in (2.20), will also have them. Also, instead of the number $\Delta(g)$ we will rather have a (one-loop) matrix of anomalous dimensions, related to $\mathcal{Z}$ as

$$\Gamma^{\alpha}{}_{\beta} \equiv (\mathcal{Z}^{-1})^{\alpha}{}_{\gamma} \frac{d}{d\ln\Lambda} \mathcal{Z}^{\gamma}{}_{\beta}. \tag{2.21}$$

When we expand the operator $\mathcal{O}$ in some basis

$$\mathcal{O} = \xi^{\alpha}\mathcal{O}_{\alpha}, \tag{2.22}$$

such that γ can be obtained by the following relation

$$\Gamma(g)^{\alpha}{}_{\beta}\,\xi^{\beta} = \Delta(g)\,\xi^{\alpha}; \tag{2.23}$$

keeping that in mind, we see that (2.20) fulfils (2.21) for $\mathcal{Z}^{-1} \approx 1$.

Another important thing is that, as noticed in section 1.2.5, we cannot expand in the dimensionful parameter $\ln|x_{12}|$, and therefore we need to rescale the conformally invariant two-point correlation function such that we introduce some (mass) scale which will make the expansion dimensionless. Thus, we rescale $\mathcal{O}$ by exponential scale $\mu^{-\delta\Delta(g)}$, which gives

$$\mu^{-2g^2\Delta_2}\langle\bar{\mathcal{O}}(x_1)\mathcal{O}(x_2)\rangle_{\text{ren}} = \frac{M_0}{|x_{12}|^{2\Delta_0}} + \frac{g^2}{|x_{12}|^{2\Delta_0}}(M_2 - M_0\,\Delta_2 \ln|\mu x_{12}|^2). \tag{2.24}$$

Of course during this whole procedure we have to assume that $\Delta_2 \ll \ln|\mu x_{12}|^{-2}$ but arbitrariness of μ also makes (2.24) valid for any value of g. This arbitrary scale will, however, drop out when it is resummed to all orders in perturbation theory, and therefore it is nothing but an artefact of perturbation theory.

2.3 Anomalous dimensions from the dilatation operator

There are basically two equivalent approaches to calculate the dimension $\Delta(g)$ of an operator in (2.11). Either one expands the correlation function (2.11) in loops, by calculation of the appropriate integrals, as explained in section 2.2, or one derives an operator which somehow has all the field theory "inside" it, and then one uses it to obtain dimensions of various operators by acting on them with such an operator; this is precisely the dilatation operator $\mathfrak{D}$ from section 1.2!

We have seen that $\mathfrak{D} \in PSU(2,2|4)$, an invariance group of $\mathcal{N}=4$ SYM. An alternative way of defining it is to say that

$$\mathfrak{D} = \int d^3x\ \mathfrak{T}^{\mathcal{N}=4}_{t\mu} x^{\mu}, \tag{2.25}$$

where $\mathfrak{T}^{\mathcal{N}=4}_{\mu\nu}$ is the energy-momentum tensor of $\mathcal{N}=4$ SYM theory (this gives us the hint that in the string dual side it will be related to the energy of dual string states). In general, action of $\mathfrak{D}$ on $\mathcal{O}_{\alpha}$ from (2.22) would give us the anomalous dimension matrix (2.21), namely

$$\mathfrak{D} \circ \mathcal{O}_{\alpha} = \mathcal{O}_{\beta}\,\Gamma^{\beta}{}_{\alpha}, \tag{2.26}$$

and $\mathfrak{D} = \mathfrak{D}(g)$ can be also expanded in loops as

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \ldots \tag{2.27}$$

which, unlike (2.12), contains also odd powers of g, since it was shown in [44] that the double expansion turns out to be inconsistent. In this thesis, however, we will be interested only on terms up to order of $\mathscr{O}(g^2)$.

We will now follow [44] in derivation of the one-loop dilatation operator for the scalar subsector $SO(6) \subset PSU(2,2|4)$ of $\mathcal{N}=4$ super Yang-Mills, and assume that local operators (2.5) consist solely of scalars fields $\phi_i(x) = \phi_i^{(\mathsf{a})}(x)T^{\mathsf{a}}$ (with the flavour index $i = 1, ..., 6$, and the adjoint index $\mathsf{a} = 1, ..., N^2$) of the $\mathcal{N}=4$ $U(N)$ gauge theory, that is

$$\mathcal{O}(x) = \mathcal{C}_{i_1 i_2 \cdots i_L} \operatorname{tr}\left(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_L}\right)(x), \tag{2.28}$$

where $\mathcal{C}_{i_1 i_2 \cdots i_L}$ is a rank L tensor. It is instructive to mention that if this tensor is symmetric and traceless, operator $\mathcal{O}(x)$ becomes a chiral primary operator. Obviously, the bare dimension of (2.28) will be

$$\Delta_0 = L, \tag{2.29}$$

however we would very much like to derive the one-loop anomalous dimension matrix by deriving the one-loop dilatation operator and acting with it on the appropriate basis $\mathcal{O}_\alpha$. But let us perhaps first think about the tree-level dilatation operator, which clearly should reproduce (2.29). Since scalar fields have conformal dimension 1, then such operator should merely count their number, and this is exactly how we construct $\mathfrak{D}_0$, based on (1.85)

$$\mathfrak{D}_0 = \operatorname{tr}\phi_m\check{\phi}_m, \qquad m = 1, \ldots, 6 \tag{2.30}$$

where

$$\check{\phi}_m \equiv \frac{\delta}{\delta\phi_m} = T^{\mathsf{a}}\frac{\delta}{\delta\phi_m^{(\mathsf{a})}} \tag{2.31}$$

removes the field ϕ, and the other ϕ puts it back "by hand"[1]. Also, as shown in [20], the anomalous dilatation operator $\delta\mathfrak{D}(g) = \mathfrak{D}(g) - \mathfrak{D}_0$ must commute with the classical algebra of $PSU(2,2|4)$, hence

$$[\mathfrak{D}_0, \mathfrak{D}_k] = 0, \qquad k \geqslant 0. \tag{2.32}$$

The interesting question is what would be the one-loop dilatation operator and we will now try to answer this. Since we will be considering the two-point functions, we would like to distinguish the fields of operator (2.5) placed at some point $x_1 = x$ in spacetime from other fields placed in some other point, say, $x_2 = 0$. This we will achieve by abbreviating them (and hence the operators $\mathcal{O}$) with superscripts $\pm$, i.e.

$$\phi_m^+ = \phi_m(x), \qquad \phi_m^- = \phi_m(0). \tag{2.33}$$

Now we can evaluate the expectation value for the two point function, which we know to be

$$\langle\mathcal{O}^+\mathcal{O}^-\rangle = \exp\left(W_0[\partial/\partial\phi^\pm]\right)\exp\left(-S_{\text{int}}[\tilde{g},\phi^\pm]\right)\mathcal{O}^+\mathcal{O}^-\big|_{\phi=0}, \tag{2.34}$$

[1] In fact, it works like the number operator in quantum mechanics $\hat{N} = \sum_i a_i^\dagger a_i$.

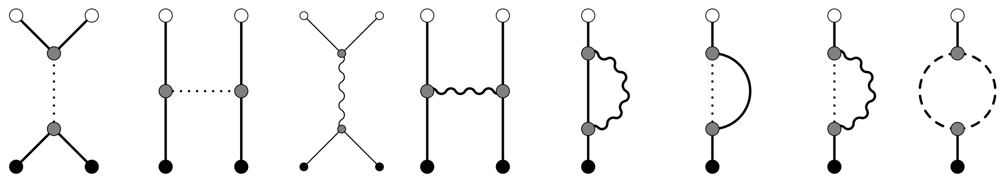

Figure 2.1: The one loop contribution to the two point function. The solid, wiggly, dashed and dotted lines represent scalars, gluons, fermions and ghost fields (that is non propagating auxiliary fields have quartic interactions) respectively. Diagrams 'SI' represent the scalar interaction, diagrams 'GE' - the gluon exchange, and diagrams 'SE' - the self energy.

where S_{int} are all non-kinetic terms of (2.3) that include scalar fields ϕ_m, and W_0 is the tree level connected Green's function

$$W_0 = \frac{1}{2} \iint dx\, dy\, \operatorname{tr} \check{\phi}(x) I_{xy} \check{\phi}(y), \tag{2.35}$$

I_{xy} being the propagator defined in (A.1). Obviously for $\tilde{g} = 0$ we will obtain the tree level expectation value

$$\langle \mathcal{O}^+ \mathcal{O}^- \rangle_{\substack{\text{tree} \\ \text{level}}} = \exp\left(W_0(x, \partial/\partial\phi^+, \partial/\partial\phi^-) \right) \mathcal{O}^+ \mathcal{O}^- \big|_{\phi=0}, \tag{2.36}$$

where W_0 now becomes

$$W_0(x, \check{\phi}^+, \check{\phi}^-) = I_{0x} \operatorname{tr} \check{\phi}^+_m \check{\phi}^-_m. \tag{2.37}$$

Since we put $\phi = 0$ in the end we need to make sure that all the fields ϕ^- in $\mathcal{O}^-$ are connected with ϕ^+ in $\mathcal{O}^+$, by the use of W_0. In particular, the classical (engineering) dimensions of $\mathcal{O}^+$ and $\mathcal{O}^-$ have to be equal, as predicted by conformal field theory.

Of course for $\tilde{g} \neq 0$ we have to deal with interactions, and evaluate the Feynman diagrams (up to one loop). The connected Green's functions can be read off the Lagrangian of the (regularised) $\mathcal{N} = 4$ SYM theory and they are depicted on figure 2.1.

The one loop correlator will take the form

$$\langle \mathcal{O}^+ \mathcal{O}^- \rangle_{\substack{\text{one} \\ \text{loop}}} = \exp\left(W_0(x, \check{\phi}^+, \check{\phi}^-) \right) (1 + \tilde{g}^2 W_2(x, \check{\phi}^+, \check{\phi}^-)) \mathcal{O}^+ \mathcal{O}^- \big|_{\phi=0}, \tag{2.38}$$

where $W_2(x, \check{\phi}^+, \check{\phi}^-)$ is the connected one loop Green's function, coming from the $\sim \tilde{g}^2$ expansion of $e^{-S_{\text{int}}}$ in (2.34). The diagrams can be evaluated to be (see [24, 44], but also [45, 46, 47])

$$\begin{aligned}
W_2^{\mathrm{SI}_1} &= \frac{1}{4} X_{00xx} \operatorname{tr} [\check{\phi}^+_m, \check{\phi}^+_n][\check{\phi}^-_m, \check{\phi}^-_n] \\
W_2^{\mathrm{SI}_2} &= \frac{1}{4} X_{00xx} \left(\operatorname{tr} [\check{\phi}^+_m, \check{\phi}^-_n][\check{\phi}^+_m, \check{\phi}^-_n] + \operatorname{tr} [\check{\phi}^+_m, \check{\phi}^-_n][\check{\phi}^-_m, \check{\phi}^+_n] \right) \\
W_2^{\mathrm{GE}_1} &\sim \operatorname{tr} [\check{\phi}^+_m, \check{\phi}^+_m][\check{\phi}^-_n, \check{\phi}^-_n] = 0 \\
W_2^{\mathrm{GE}_2} &= \left(-\frac{1}{2} \widetilde{H}_{0x0x} - Y_{00x} I_{0x} + \frac{1}{4} X_{00xx} \right) \operatorname{tr} [\check{\phi}^+_m, \check{\phi}^-_n][\check{\phi}^+_m, \check{\phi}^-_n] \\
W_2^{\mathrm{SE}} &= -Y_{00x} \operatorname{tr} [\check{\phi}^+_m, T^a][T^a, \check{\phi}^-_m],
\end{aligned} \tag{2.39}$$

where the integrals Y_{00x}, X_{00xx}, and H_{0x0x} are evaluated in appendix A. One can clearly see that the trace part of these functions represents the symmetry structure of the diagrams on figure 2.1: their legs want to attach to the external trace operators in the correlator just like the differential operators $\check{\phi}_m^{\pm}$ want to act on the fields in $\mathcal{O}^{\pm}$. Then, the spacetime factor and the statistical weight factor are represented by the integrals and coefficients in front of the trace.

We now use the Jacobi identity to show that

$$\mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_n^-][\check{\phi}_m^-, \check{\phi}_n^+] = \mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_n^+][\check{\phi}_m^-, \check{\phi}_n^-] - \mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_m^-][\check{\phi}_n^+, \check{\phi}_n^-],$$

and reexpress the scalar interaction as

$$W_2^{\mathrm{SI}} = \frac{1}{4} X_{00xx} \left[V_D(\check{\phi}^+, \check{\phi}^-) + V_F(\check{\phi}^+, \check{\phi}^-) + V_K(\check{\phi}^+, \check{\phi}^-) \right], \tag{2.40}$$

where

$$\begin{aligned} V_D(\check{\phi}^+, \check{\phi}^-) &= -\mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_m^-][\check{\phi}_n^+, \check{\phi}_n^-], \\ V_F(\check{\phi}^+, \check{\phi}^-) &= 2\mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_n^+][\check{\phi}_m^-, \check{\phi}_n^-], \\ V_K(\check{\phi}^+, \check{\phi}^-) &= \mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_n^-][\check{\phi}_m^+, \check{\phi}_n^-], \end{aligned} \tag{2.41}$$

and then classify all the W_2's according to their spacetime structure, obtaining

$$\begin{aligned} W_2^{\mathrm{X}} &= \frac{1}{4} X_{00xx} \left[V_F(\check{\phi}^+, \check{\phi}^-) + V_K(\check{\phi}^+, \check{\phi}^-) \right], \\ W_2^{\mathrm{H}} &= \frac{1}{2} \widetilde{H}_{0x0x} V_D(\check{\phi}^+, \check{\phi}^-), \\ W_2^{\mathrm{IY}} &= -Y_{00x} \left(I_{0x} \mathrm{tr}\,[\check{\phi}_m^+, \check{\phi}_m^-][\check{\phi}_n^+, \check{\phi}_n^-] + \mathrm{tr}\,[\check{\phi}_m^+, T^a][T^a, \check{\phi}_m^-] \right). \end{aligned} \tag{2.42}$$

Now, we make a very useful observation in (2.38). Since each $\check{\phi}^+$ is acting on ϕ^+ in $\mathcal{O}^+$, and then is being acted on by W_0, we can equally well substitute $\check{\phi}^+$ in W_2 by $I_{0x}^{-1}\phi^-$, since it has to be contracted with W_0 anyway, and hence come to the same result; the inversed propagator is nothing but a "recompense" of the spacetime structure. The only thing we have to remember about is to not to contract $\check{\phi}^-$ with ϕ^- inside the vertex and this is achieved by "sandwiching" W_2 with double colons which denote the normal ordering[2]. For this purpose we define the one loop effective vertex

$$V_2(x) = : W_2(x, I_{0x}^{-1}\phi, \check{\phi}) :, \tag{2.43}$$

and then (2.38) can be rewritten as

$$\langle \mathcal{O}^+ \mathcal{O}^- \rangle_{\substack{\text{one} \\ \text{loop}}} = \exp\left(W_0(x, \check{\phi}^+, \check{\phi}^-) \right) (1 + g^2 V_2^-(x)) \mathcal{O}^+ \mathcal{O}^- \big|_{\phi=0}. \tag{2.44}$$

Obviously, there is no difference between substituting

$$\check{\phi}^+ \to I_{0x}^{-1}\phi^- \quad \text{and} \quad \check{\phi}^- \to I_{0x}^{-1}\phi^+,$$

[2] Normal ordering is defined such that it moves all the "checks" to the right and in this way prevents self contractions inside the same vertex; we define it along with [20], namely $:\mathrm{tr}\,\phi_i\check{\phi}_j\phi_k\check{\phi}_l: = \mathrm{tr}\,\phi_i\check{\phi}_j\phi_k\check{\phi}_l - \delta_{jk} N \mathrm{tr}\,\phi_i\check{\phi}_l$.

therefore

$$V_2^- \mathrel{\widehat{=}} V_2^+, \tag{2.45}$$

where $\pm$ symbols naturally refer to the spacetime point in which (2.43) is defined, and the symbol $\widehat{=}$ denotes equivalence in an operator sense. Then the connected Green's functions can be written as

$$\begin{aligned} V_2^{\mathrm{X}} &= \frac{1}{4} X_{00xx} I_{0x}^{-2} \left[:V_F(\phi, \check{\phi}): + :V_K(\phi, \check{\phi}): \right], \\ V_2^{\mathrm{H}} &= \frac{1}{2} \widetilde{H}_{0x0x} I_{0x}^{-2} \ :V_D(\phi, \check{\phi}):, \\ V_2^{\mathrm{IY}} &= -Y_{00x} I_{0x}^{-1} \left(:\mathrm{tr}\, [\phi_m, \check{\phi}_m][\phi_n, \check{\phi}_n]: + :\mathrm{tr}\, [\phi_m, T^a][T^a, \check{\phi}_m]: \right). \end{aligned} \tag{2.46}$$

In the last term we can replace the normal ordering and hence absorb the second one into the first one

$$:\mathrm{tr}\, [\phi_m, \check{\phi}_m][\phi_n, \check{\phi}_n]: + 2N :\mathrm{tr}\, (\phi_m \check{\phi}_m): -2 :\mathrm{tr}\, (\phi_m)\, \mathrm{tr}\, (\check{\phi}_m): \tag{2.47}$$

$$= \mathrm{tr} :[\phi_m, \check{\phi}_m]::[\phi_n, \check{\phi}_n]: \equiv -\mathrm{tr}\, \boldsymbol{\eta}\, \boldsymbol{\eta}, \tag{2.48}$$

where

$$\boldsymbol{\eta} = i :[\phi_m, \check{\phi}_m]: \tag{2.49}$$

is the generator of gauge transformations. For that very reasons V_2^{IY} does not act on gauge invariant states $\mathcal{O}$, hence

$$V_2^{\mathrm{IY}} \mathrel{\widehat{=}} 0. \tag{2.50}$$

As for two other terms, we use (A.5) and (A.6) to find out that

$$\widetilde{H}_{0x0x} I_{0x}^{-2} = \left(-48\zeta(3)\epsilon + \mathscr{O}(\epsilon^2) \right) f(x), \tag{2.51}$$

where

$$f(x) = \frac{\Gamma(1-\epsilon)}{|\frac{1}{2}\mu^2 x^2|^{-\epsilon}}, \qquad \text{and} \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.52}$$

(the latter being the Riemann zeta function) and thus V_2^{H} doesn't contribute either when taking $\epsilon \to 0$. On the other hand (A.4) and (A.6) tell us that

$$X_{00xx} I_{0x}^{-2} = \left(\frac{2}{\epsilon} + 2 + \mathscr{O}(\epsilon^2) \right) f(x), \tag{2.53}$$

and we clearly see that for

$$V_2(x) = V_2^X(x). \tag{2.54}$$

The only thing left, basically, would be to renormalise the correlation function. This is done by introducing the renormalisation factor $\mathcal{Z}$, which we choose to be[3]

$$\mathcal{Z} = 1 - \frac{1}{2} \tilde{g}^2 V_2(1/\mu) + \mathscr{O}(\tilde{g}^3), \tag{2.55}$$

[3]Notice that the resemblance of (2.55) and (2.20) is not accidental.

then using (2.16) and (2.45) we can write

$$\langle \widetilde{\mathcal{O}}^{+} \widetilde{\mathcal{O}}^{-} \rangle_{\substack{\text{one}\\ \text{loop}}} = \exp\left(W_0(x, \check{\phi}^{+}, \check{\phi}^{-})\right)(1 + \tilde{g}^2 V_2^{-}(x) - \tilde{g}^2 V_2^{-}(1/\mu))\mathcal{O}^{+}\mathcal{O}^{-}\big|_{\phi=0}. \qquad (2.56)$$

On the other hand, a closer look at $V_2(x)$ tells us that the spacetime dependence appears solely through $f(x)$

$$V_2(x) = f(x)V_2 = \frac{\Gamma(1-\epsilon)}{|\frac{1}{2}\mu^2 x^2|^{-\epsilon}} V_2, \qquad (2.57)$$

a desired property of a renormalisable theory in dimensional regularisation, where

$$V_2 = \frac{1}{4}\left(\frac{2}{\epsilon} + 2 + \mathscr{O}(\epsilon^2)\right)\left[:V_F(\phi, \check{\phi}): \,+\, :V_K(\phi, \check{\phi}):\right]. \qquad (2.58)$$

We can now expand the following argument for small ϵ

$$V_2(x) - V_2(1/\mu) = [f(x) - f(1/\mu)]V_2 = \Gamma(1-\epsilon)\ln|\mu^2 x^2|\epsilon V_2, \qquad (2.59)$$

and then we let

$$\lim_{\epsilon \to 0}[V_2(x) - V_2(1/\mu)] = \ln|\mu x|^{-2}\mathfrak{D}_2, \qquad (2.60)$$

for

$$\mathfrak{D}_2 \equiv -\lim_{\epsilon \to 0} \epsilon V_2 \qquad (2.61)$$

which gives us the one loop correction to the dilatation operator

$$\mathfrak{D}_2 = -:\mathrm{tr}\,[\phi_m, \phi_n][\check{\phi}_m, \check{\phi}_n]: -\frac{1}{2}:\mathrm{tr}\,[\phi_m, \check{\phi}_n][\phi_m, \check{\phi}_n]:, \qquad (2.62)$$

and thus

$$\langle \widetilde{\mathcal{O}}^{+} \widetilde{\mathcal{O}}^{-} \rangle_{\substack{\text{one}\\ \text{loop}}} = \exp\left(W_0(x, \check{\phi}^{+}, \check{\phi}^{-})\right)(\ln|\mu x|^{-2}\tilde{g}^2 \mathfrak{D}_2^{-})\mathcal{O}^{+}\mathcal{O}^{-}\big|_{\phi=0}. \qquad (2.63)$$

However, in order to reobtain the dilatation operator in accordance with the definition (2.27), and thereby work with the coupling constant g, we redefine the dilatation operator (2.62) such that

$$\mathfrak{D}_2 \to \mathfrak{D}_2 = -N^{-1}\left(:\mathrm{tr}\,[\phi_m, \phi_n][\check{\phi}_m, \check{\phi}_n]: +\frac{1}{2}:\mathrm{tr}\,[\phi_m, \check{\phi}_n][\phi_m, \check{\phi}_n]:\right), \qquad (2.64)$$

and then we can speak of the dilatation operator (like in the beginning of this chapter) $\mathfrak{D}(g)$, instead of $\mathfrak{D}(\tilde{g})$.

The fact that although we consider the correlator of renormalised operators $\widetilde{\mathcal{O}}$, in practice we work with bare operators $\mathcal{O}$. This is because we choose to renormalise the dilatation operator instead of the states, according to

$$\mathfrak{D}_{\text{ren}} = \mathcal{Z}^{-1}\,\mathfrak{D}_{\text{bare}}\,\mathcal{Z}, \qquad (2.65)$$

which makes perfect sense, since $\mathfrak{D}_{\text{bare}}$, unlike (2.64), is expected to diverge, thus the operator $\mathfrak{D}_2$ in (2.64) is already renormalised.

2.4 A short manual for the dilatation operator

Let us calculate the $\mathcal{Z}$ operator explicitly. According to (2.55), and using (2.57), we can rewrite it as

$$\mathcal{Z} = \operatorname{tr}\phi_m\check{\phi}_m - \left(\frac{g_{YM}^2}{16\pi^2}\right)\frac{1}{2\epsilon}\left[:\operatorname{tr}[\phi_m,\phi_n][\check{\phi}_m,\check{\phi}_n]: \; + \frac{1}{2}:\operatorname{tr}[\phi_m,\check{\phi}_n][\phi_m,\check{\phi}_n]:\right], \tag{2.66}$$

where the tree level part we will simply denote hereafter as the identity operator. As an operator, it acts of course on the correlator $\langle\mathcal{O}^+\mathcal{O}^-\rangle$. The bad news, however, is that each operator in the correlator contains L different flavour indices and thus, when acting on it with $\mathcal{Z}$ (and thus $\mathfrak{D}(g)$), we will have to perform a certain number of mathematical operations which will be proportional to some permutation of these indices, which becomes quite tediuos. The good news though is that we are interested in the planar limit, and therefore our computation simplifies a lot since we just have to contract the nearest-neighbour fields, allowing us to use the following trick. Of all the set of scalar fields in an operator

$$\mathcal{O}^{\pm} = \operatorname{tr}\phi^{\pm}_{i_1}\phi^{\pm}_{i_2}\cdots\phi^{\pm}_{i_\ell}\phi^{\pm}_{i_{\ell+1}}\cdots\phi^{\pm}_{i_L} \tag{2.67}$$

we will distinguish two neighbouring ones, at site, say, (ℓ) and $(\ell+1)$, that is a "partial" operator

$$\mathcal{O}^{-\,\mathfrak{ab}}_{\ell,\ell+1} = \left(\phi^{-\,j_\ell}\phi^{-\,j_{\ell+1}}\right)^{\mathfrak{ab}}, \tag{2.68}$$

where $\mathfrak{a},\mathfrak{b}$ are colour indices, $j_\ell, j_{\ell+1}$ are flavour indices, and '$-$' denotes that the operator $\mathcal{O}^-$ is situated in point $x_2 = 0$. In analogy, we have its hermitian conjugate[4] in $x_1 = x$

$$\mathcal{O}^{+\,\mathfrak{cd}}_{\ell,\ell+1} = \left(\phi^+_{i_{\ell+1}}\phi^+_{i_\ell}\right)^{\mathfrak{cd}}. \tag{2.69}$$

Now, when evaluating the correlator, we contract the fields according to (2.36), namely

$$\langle\phi_i^{+\,\mathfrak{ab}}\phi_j^{-\,\mathfrak{cd}}\rangle = I_{0x}\,\delta_{ij}\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}, \tag{2.70}$$

therefore

$$\langle\mathcal{O}^{+\,\mathfrak{ab}}_{\ell,\ell+1}\mathcal{O}^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle = I_{0x}^2\,N\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}, \tag{2.71}$$

Actually, there could have also been other contribution apart from $\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}$, namely $\delta^{\mathfrak{ab}}\delta^{\mathfrak{cd}}$, but that would give negligible contributions in the planar limit (when contracting other fields in (2.67) and summing over $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ and $\mathfrak{d}$ afterwards), which we do not consider.

Evaluation of the one loop part of $\mathcal{Z}$ is just matter of a little algebraic exercise, which leads to

$$\langle\mathcal{O}^{+\,\mathfrak{ab}}_{\ell,\ell+1} :\operatorname{tr}[\phi^-_m,\phi^-_n][\check{\phi}^-_m,\check{\phi}^-_n]: \mathcal{O}^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle = 2I_{0x}^2\,N^2\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}(\delta^{j_{\ell+1}}_{i_\ell}\delta^{j_\ell}_{i_{\ell+1}} - \delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}), \tag{2.72}$$

and

$$\langle\mathcal{O}^{+\,\mathfrak{ab}}_{\ell,\ell+1} :\operatorname{tr}[\phi^-_m,\check{\phi}^-_n][\phi^-_m,\check{\phi}^-_n]: \mathcal{O}^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle = -I_{0x}^2\,N^2\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}\delta_{i_\ell i_{\ell+1}}\delta^{j_\ell j_{\ell+1}}, \tag{2.73}$$

[4]Although, since the fields are real it is just a transpose in this case.

which yields the $\mathcal{Z}$ factor in the flavour representation

$$\mathcal{Z}^{\cdots j_\ell j_{\ell+1}\cdots}_{\cdots i_\ell i_{\ell+1}\cdots} = \delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}} - \frac{\lambda}{16\pi^2}\ln\Lambda\left[2\delta^{j_{\ell+1}}_{i_\ell}\delta^{j_\ell}_{i_{\ell+1}} - 2\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}} - \delta_{i_\ell i_{\ell+1}}\delta^{j_\ell j_{\ell+1}}\right], \tag{2.74}$$

where

$$\ln\Lambda = \frac{1}{2\epsilon}, \tag{2.75}$$

and $\lambda = g^2_{YM}N$ is the 't Hooft coupling in the large N limit. This on the other hand, with (2.21), immediately yields the one-loop anomalous dimension matrix

$$(\Gamma_{\substack{\text{one}\\\text{loop}}})^{\cdots j_\ell j_{\ell+1}\cdots}_{\cdots i_\ell i_{\ell+1}\cdots} = \frac{\lambda}{16\pi^2}\left[2\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}} - 2\delta^{j_{\ell+1}}_{i_\ell}\delta^{j_\ell}_{i_{\ell+1}} + \delta_{i_\ell i_{\ell+1}}\delta^{j_\ell j_{\ell+1}}\right]. \tag{2.76}$$

In order, however, to consider the full sector of scalar fields

$$\mathcal{O}^+ = N^{-L/2}\mathrm{tr}\,\phi^+_{i_L}\cdots\phi^+_{i_1}, \qquad \mathcal{O}^- = N^{-L/2}\mathrm{tr}\,\phi^{-\,j_1}\cdots\phi^{-\,j_L}, \tag{2.77}$$

it is convenient to reexpress (2.76) as an operator acting on the flavour basis, namely

$$\langle\mathcal{O}^+\mathcal{O}^-\rangle = I^L_{0x}\left(\mathbb{1} - \frac{\lambda}{16\pi^2}\ln|2^{-1}\mu^2\Lambda^2x^2|\sum_{\ell=1}^{L}D_2^{\ell,\ell+1}\right)\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\cdots\delta^{j_L}_{i_L} + \text{cycles}. \tag{2.78}$$

Here, "cycles" mean that due to the trace cyclicity invariance we have L non trivial planar leading ways to contract these indices to form the flavour basis. Above

$$D_2^{\ell,\ell+1} \equiv 2\mathcal{I}_{\ell,\ell+1} - 2\mathcal{P}_{\ell,\ell+1} + \mathcal{K}_{\ell,\ell+1} \tag{2.79}$$

is the one-loop element of the anomalous dimension matrix (2.21) in an operator form, where its constituent operators act on the flavour basis in the following way

$$\begin{aligned}
\mathcal{I}_{\ell,\ell+1}\,\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}\cdots\delta^{j_L}_{i_L} &= \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}\cdots\delta^{j_L}_{i_L},\\
\mathcal{P}_{\ell,\ell+1}\,\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}\cdots\delta^{j_L}_{i_L} &= \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_{\ell+1}}_{i_\ell}\delta^{j_\ell}_{i_{\ell+1}}\cdots\delta^{j_L}_{i_L},\\
\mathcal{K}_{\ell,\ell+1}\,\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_\ell}_{i_\ell}\delta^{j_{\ell+1}}_{i_{\ell+1}}\cdots\delta^{j_L}_{i_L} &= \delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta_{i_\ell i_{\ell+1}}\delta^{j_\ell j_{\ell+1}}\cdots\delta^{j_L}_{i_L}
\end{aligned} \tag{2.80}$$

and are the identity, permutation and trace operator, respectively. In other words, these operators act in the tensor product $\mathcal{V}_\ell\otimes\mathcal{V}_{\ell+1} = \mathbb{R}^6\otimes\mathbb{R}^6$, of the 6^L-dimensional linear space $\mathcal{H} = \mathcal{V}_1\otimes\cdots\otimes\mathcal{V}_L$, as

$$\begin{aligned}
\mathcal{I}(u\otimes v) &= (u\otimes v),\\
\mathcal{P}(u\otimes v) &= (v\otimes u),\\
\mathcal{K}(u\otimes v) &= (u\cdot v)\sum_i \hat{\boldsymbol{e}}_i\otimes\hat{\boldsymbol{e}}_i,
\end{aligned} \tag{2.81}$$

where $\hat{\boldsymbol{e}}_i$ are a set of orthonormal unit vectors in $\mathbb{R}^6$. After having acted on the flavour basis, they give the $6^L\times 6^L$ anomalous dimension matrix for the full $SO(6)$ bosonic sector (with $\mathcal{Z}^{-1}\approx 1$)

$$\Gamma_{\substack{\text{one}\\\text{loop}}} = \frac{\lambda}{16\pi^2}\sum_{\ell=1}^{L}(2\mathcal{I}_{\ell,\ell+1} - 2\mathcal{P}_{\ell,\ell+1} + \mathcal{K}_{\ell,\ell+1})\,\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_L}_{i_L} + \text{cycles} \equiv g^2D_2, \tag{2.82}$$

where we supressed all of the indices. It is very important to note, however, that (2.64) and (2.66) are operators, whereas (2.74) and (2.82) are matrices.

This result is more important than one might think. As shown by Minahan and Zarembo [9] the one-loop anomalous dimension matrix (2.82) is the Hamiltonian of an integrable spin chain, which has $SO(6)$ symmetry and the spins transform in the vector representation. A basis of the spin chain Hilbert space is thus given by the states

$$|i_1, i_2, \ldots, i_L\rangle, \tag{2.83}$$

which corresponds, up to cyclic permutations, to single trace local operators (2.28). More suprisingly, it turns out that the integrability not only extends to the full sector of $\mathcal{N}=4$ SYM theory which is the $PSU(2,2|4)$ supergroup [41, 42], but also to higher loops [43, 44].

2.5 Splitting up $SO(6)$

Although the formalism of the dilatation operator of $\mathcal{N}=4$ SYM theory is particularly neat for the $SO(6)$ bosonic subsector, our aim is to express it in the language of $\mathcal{N}=1$ supersymmetric theories, from the point of view of which it consists of three chiral superfields $\mathfrak{W}_I$ in the adjoint representation. The reason is that the orbifolding of $\mathcal{N}=4$ theory to $\mathcal{N}=2$ theory will be performed by applying the projection conditions to these three complex scalars, defined as

$$\mathfrak{W}_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \equiv X, \quad \mathfrak{W}_2 = \tfrac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \equiv Y, \quad \mathfrak{W}_3 = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6) \equiv Z, \tag{2.84}$$

and their complex conjugates. In terms of representations, we split up $\mathbf{6} = \mathbf{3}+\mathbf{3}$ and employ an $SU(3)$ notation for the fields. In component form, we project a vector in $\mathcal{V}_\ell$

$$v^i \equiv (v^I, \bar{v}^I), \qquad i = 1, \ldots, 6, \quad I = 1, 2, 3, \tag{2.85}$$

to its components v^I and $\bar{v}^I = v^{\bar{I}}$. In this representation the permutation operator $\mathcal{P}$ and the trace operator $\mathcal{K}$ which act on $\mathcal{V}_\ell \otimes \mathcal{V}_{\ell+1}$ as

$$\begin{aligned}
\mathcal{I}(u \otimes v) &= (u \otimes v), \\
\mathcal{P}(u \otimes v) &= (v \otimes u), \\
\mathcal{K}(u \otimes v) &= (u \cdot v) \sum_I (\hat{\boldsymbol{e}}_I \otimes \hat{\boldsymbol{e}}_{\bar{I}} + \hat{\boldsymbol{e}}_{\bar{I}} \otimes \hat{\boldsymbol{e}}_I),
\end{aligned} \tag{2.86}$$

where

$$\begin{aligned}
u \otimes v &= u^I \otimes v^I + \bar{u}^I \otimes v^I + u^I \otimes \bar{v}^I + \bar{u}^I \otimes \bar{v}^I, \\
u \cdot v &= \sum_I (u^I \bar{v}^I + \bar{u}^I v^I),
\end{aligned}$$

and $\hat{\boldsymbol{e}}_I$ and $\hat{\boldsymbol{e}}_{\bar{J}}$ are vectors of an orthonormal basis in $\mathcal{V}$. This suggests that every delta function in (2.74), (2.76), or in the flavour basis should change accordingly to (2.85), that is

$$\delta_i^{\ j} \to \delta_I^{\ \bar{J}} + \delta_{\bar{I}}^{\ J}. \tag{2.87}$$

We would however like to reproduce this result by the field theory calculations. For this, there are two steps that ought to be performed. First, we need to reexpress the dilatation operator (2.64) in terms of the six complex scalars from (2.84), that is $\mathfrak{W}_I$ and $\overline{\mathfrak{W}}_I = \mathfrak{W}_{\bar{I}}$, $(I = 1, 2, 3)$, and then and redo all the field theory calculations. As a consistency check we expect that the one loop anomalous dimension matrix (2.82) will be reproduced, with the basis changed accordingly to (2.87).

2.5.1 Splitting up the dilatation operator

The aim is to express the dilatation operator $\mathfrak{D}_2$ in terms of $\mathfrak{W}$ and $\overline{\mathfrak{W}}$. First we notice that since index i at ϕ_i takes on six possible values, one could always rewrite an arbitrary function of these fields in the following way

$$\sum_{i=1}^{6} F(\phi_i) = \sum_{I=1}^{3} \left[F(\phi_{2I-1}) + F(\phi_{2I})\right]. \tag{2.88}$$

On the other hand, the complex scalars are defined such that

$$\mathfrak{W}_I = \frac{1}{\sqrt{2}}(\phi_{2I-1} + i\phi_{2I}), \qquad \overline{\mathfrak{W}}_I = \frac{1}{\sqrt{2}}(\phi_{2I-1} - i\phi_{2I}), \quad I = 1, 2, 3, \tag{2.89}$$

and thus

$$\phi_{2I-1} = \frac{1}{\sqrt{2}}(\mathfrak{W}_I + \overline{\mathfrak{W}}_I), \qquad \phi_{2I} = \frac{1}{i\sqrt{2}}(\mathfrak{W}_I - \overline{\mathfrak{W}}_I), \tag{2.90}$$

Hence we have

$$\check{\mathfrak{W}}_I = \frac{\delta}{\delta \mathfrak{W}_I} = \frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\phi_{2I-1}} - i\frac{\delta}{\delta\phi_{2I}}\right), \tag{2.91}$$

and

$$\check{\overline{\mathfrak{W}}}_I = \frac{\delta}{\delta \overline{\mathfrak{W}}_I} = \frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\phi_{2I-1}} + i\frac{\delta}{\delta\phi_{2I}}\right). \tag{2.92}$$

Now, we can take advantage of (2.88) and write that

$$\sum_{m,n=1}^{6} \mathrm{tr}\,[\phi_m, \check{\phi}_n][\phi_m, \check{\phi}_n] = \sum_{I,J=1}^{3} \left(2\mathrm{tr}[\check{\overline{\mathfrak{W}}}_I, \mathfrak{W}_J][\check{\mathfrak{W}}_I, \overline{\mathfrak{W}}_J] + 2\mathrm{tr}[\check{\overline{\mathfrak{W}}}_I, \overline{\mathfrak{W}}_J][\check{\mathfrak{W}}_I, \mathfrak{W}_J]\right),$$

and

$$\begin{aligned}&\sum_{m,n=1}^{6} \mathrm{tr}\,[\phi_m, \phi_n][\check{\phi}_m, \check{\phi}_n] \\ &\quad = \sum_{I,J=1}^{3} \left(\mathrm{tr}[\overline{\mathfrak{W}}_I, \overline{\mathfrak{W}}_J][\check{\overline{\mathfrak{W}}}_I, \check{\overline{\mathfrak{W}}}_J] + \mathrm{tr}[\mathfrak{W}_I, \mathfrak{W}_J][\check{\mathfrak{W}}_I, \check{\mathfrak{W}}_J] + 2\mathrm{tr}[\overline{\mathfrak{W}}_I, \mathfrak{W}_J][\check{\overline{\mathfrak{W}}}_I, \check{\mathfrak{W}}_J]\right).\end{aligned}$$

This suggests that one could express the dilatation operator $\mathfrak{D}_2$ in our new basis (call it $\mathfrak{D}_2'$) in terms of its holomorphic, anti-holomorphic and non-holomorphic parts

$$\mathfrak{D}_2' = N^{-1}\left(\mathfrak{D}_2^{h} + \mathfrak{D}_2^{\bar{h}} + \mathfrak{D}_2^{h\bar{h}}\right), \tag{2.93}$$

where

$$
\begin{aligned}
\mathfrak{D}_2^{h} &= -\mathrm{tr}[\mathfrak{W}_I,\mathfrak{W}_J][\check{\mathfrak{W}}_I,\check{\mathfrak{W}}_J]\\
\mathfrak{D}_2^{\bar{h}} &= -\mathrm{tr}[\overline{\mathfrak{W}}_I,\overline{\mathfrak{W}}_J][\check{\overline{\mathfrak{W}}}_I,\check{\overline{\mathfrak{W}}}_J]\\
\mathfrak{D}_2^{h\bar{h}} &= -2\mathrm{tr}[\mathfrak{W}_I,\overline{\mathfrak{W}}_J][\check{\mathfrak{W}}_I,\check{\overline{\mathfrak{W}}}_J]-\mathrm{tr}[\mathfrak{W}_I,\check{\mathfrak{W}}_J][\overline{\mathfrak{W}}_I,\check{\overline{\mathfrak{W}}}_J]-\mathrm{tr}[\mathfrak{W}_I,\check{\overline{\mathfrak{W}}}_J][\overline{\mathfrak{W}}_I,\check{\mathfrak{W}}_J]
\end{aligned}
\tag{2.94}
$$

2.5.2 Splitting up the basis

Unlike in the $SO(6)$ case, instead of one basis (2.67) we now have 2^L different, gauge invariant fields which get contracted with each other, for example

$$
\mathcal{O}=\mathrm{tr}\left(\mathfrak{W}_{I_1}\mathfrak{W}_{I_2}\overline{\mathfrak{W}}_{I_3}\mathfrak{W}_{I_4}\overline{\mathfrak{W}}_{I_5}\cdots\overline{\mathfrak{W}}_{I_{k-1}}\mathfrak{W}_{I_k}\mathfrak{W}_{I_{k+1}}\cdots\mathfrak{W}_{I_{L-1}}\overline{\mathfrak{W}}_{I_L}\right), \tag{2.95}
$$

and hence different contraction rules, namely

$$
\langle\mathfrak{W}_I^{+\,\mathfrak{ab}}\,\overline{\mathfrak{W}}_J^{-\,\mathfrak{cd}}\rangle = I_{0x}\delta_{I\bar{J}}\,\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}, \tag{2.96}
$$

$$
\langle\overline{\mathfrak{W}}_I^{+\,\mathfrak{ab}}\,\mathfrak{W}_J^{-\,\mathfrak{cd}}\rangle = I_{0x}\delta_{\bar{I}J}\,\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}, \tag{2.97}
$$

and all others equal to zero. Having said that, in analogy to (2.68) we can define a "partial" operator $\mathcal{O}'$ in this basis as a transformation from $\mathcal{O}$ to the new representation, that is

$$
\mathcal{O}^{\mathfrak{ab}}_{\ell,\ell+1}=(\phi^{j_\ell}\phi^{j_{\ell+1}})^{\mathfrak{ab}}\ \longrightarrow\ \mathcal{O}'^{\mathfrak{ab}}_{\ell,\ell+1}=\frac{1}{2}\left[(\phi^{2J_\ell-1}+\phi^{2J_\ell})(\phi^{2J_{\ell+1}-1}+\phi^{2J_{\ell+1}})\right]^{\mathfrak{ab}},
$$

where we used the fact that

$$
\phi^{j}\ \longrightarrow\ \frac{1}{\sqrt{2}}(\phi^{2J-1}+\phi^{2J}).
$$

It is now straightforward to apply (2.90) and write out our new "partial" basis in the following way

$$
\mathcal{O}'^{\mathfrak{ab}}_{\ell,\ell+1}=\frac{1}{2}\left(\mathfrak{W}^{J_\ell}\overline{\mathfrak{W}}^{J_{\ell+1}}+\overline{\mathfrak{W}}^{J_\ell}\mathfrak{W}^{J_{\ell+1}}-i\mathfrak{W}^{J_\ell}\mathfrak{W}^{J_{\ell+1}}+i\overline{\mathfrak{W}}^{J_\ell}\overline{\mathfrak{W}}^{J_{\ell+1}}\right)^{\mathfrak{ab}}. \tag{2.98}
$$

The tree-level basis can be achieved by computing

$$
\begin{aligned}
\langle\mathcal{O}'^{+\,\mathfrak{ab}}_{\ell,\ell+1}\mathcal{O}'^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle &= I_{0x}^2\,N\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}\left(\delta_{I_\ell}^{\ \bar{J}_\ell}\delta_{I_{\ell+1}}^{\ \bar{J}_{\ell+1}}+\delta_{I_\ell}^{\ \bar{J}_\ell}\delta_{\bar{I}_{\ell+1}}^{\ J_{\ell+1}}+\delta_{\bar{I}_\ell}^{\ J_\ell}\delta_{I_{\ell+1}}^{\ \bar{J}_{\ell+1}}+\delta_{\bar{I}_\ell}^{\ J_\ell}\delta_{\bar{I}_{\ell+1}}^{\ J_{\ell+1}}\right)\\
&= I_{0x}^2\,N\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}\left(\delta_{I_\ell}^{\ \bar{J}_\ell}+\delta_{\bar{I}_\ell}^{\ J_\ell}\right)\left(\delta_{I_{\ell+1}}^{\ \bar{J}_{\ell+1}}+\delta_{\bar{I}_{\ell+1}}^{\ J_{\ell+1}}\right),
\end{aligned}
\tag{2.99}
$$

which not only fulfils our expectations but also gives us the idea that

$$
\langle\mathcal{O}'^{+}\mathcal{O}'^{-}\rangle = I_{0x}^L\left(\delta_{I_1}^{\ \bar{J}_1}+\delta_{\bar{I}_1}^{\ J_1}\right)\left(\delta_{I_2}^{\ \bar{J}_2}+\delta_{\bar{I}_2}^{\ J_2}\right)\cdots\left(\delta_{I_L}^{\ \bar{J}_L}+\delta_{\bar{I}_L}^{\ J_L}\right)+\text{cycles}, \tag{2.100}
$$

where we followed exactly as in the case of tree-level part of (2.78).

The one-loop correlation function can be obtained by calculating

$$
\langle\mathcal{O}'^{+\,\mathfrak{ab}}_{\ell,\ell+1}\mathfrak{D}_2\mathcal{O}'^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle \equiv -\sum_{p=1}^{6}\langle\mathfrak{D}_{2,p}\rangle, \tag{2.101}
$$

where

$$\begin{aligned}
\langle\mathfrak{D}_{2,1}\rangle &= \langle(\overline{\mathfrak{W}}^{+}_{I_{\ell+1}}\overline{\mathfrak{W}}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\mathfrak{W}^{-\,J_\ell}\mathfrak{W}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,\\
\langle\mathfrak{D}_{2,2}\rangle &= \langle(\mathfrak{W}^{+}_{I_{\ell+1}}\overline{\mathfrak{W}}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\overline{\mathfrak{W}}^{-\,J_\ell}\mathfrak{W}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,\\
\langle\mathfrak{D}_{2,3}\rangle &= \langle(\mathfrak{W}^{+}_{I_{\ell+1}}\overline{\mathfrak{W}}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\mathfrak{W}^{-\,J_\ell}\overline{\mathfrak{W}}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,\\
\langle\mathfrak{D}_{2,4}\rangle &= \langle(\overline{\mathfrak{W}}^{+}_{I_{\ell+1}}\mathfrak{W}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\overline{\mathfrak{W}}^{-\,J_\ell}\mathfrak{W}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,\\
\langle\mathfrak{D}_{2,5}\rangle &= \langle(\overline{\mathfrak{W}}^{+}_{I_{\ell+1}}\mathfrak{W}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\mathfrak{W}^{-\,J_\ell}\overline{\mathfrak{W}}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,\\
\langle\mathfrak{D}_{2,6}\rangle &= \langle(\mathfrak{W}^{+}_{I_{\ell+1}}\mathfrak{W}^{+}_{I_{\ell}})^{\mathfrak{ab}}\,\mathfrak{D}_2\,(\overline{\mathfrak{W}}^{-\,J_\ell}\overline{\mathfrak{W}}^{-\,J_{\ell+1}})^{\mathfrak{cd}}\rangle,
\end{aligned}$$

where all the other terms were dropped due to the fact they vanish during contractions. Furthermore, we notice that in $\langle\mathfrak{D}_{2,1}\rangle$ only $\mathfrak{D}_2^{h}$ does not give vanishing contribution, as well as $\mathfrak{D}_2^{\bar h}$ in $\langle\mathfrak{D}_{2,6}\rangle$, and $\mathfrak{D}_2^{h\bar h}$ in four other terms. Let us now write that

$$\langle\mathfrak{D}_{2,p}\rangle = I_{0x}^2 N\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}} D_{2,p}, \tag{2.102}$$

where $D_{2,p}$ for $p = 1,...,6$ expresses only the flavour symmetry structure. Thus one can easily verify that

$$\begin{aligned}
D_{2,1} &= 2\left(\delta_{\bar I_\ell}^{\;J_{\ell+1}}\delta_{\bar I_{\ell+1}}^{\;J_\ell} - \delta_{\bar I_\ell}^{\;J_\ell}\delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}}\right),\\
D_{2,2} &= 2\delta_{\bar I_\ell}^{\;J_{\ell+1}}\delta_{I_{\ell+1}}^{\;\bar J_\ell} - \delta_{\bar I_\ell I_{\ell+1}}\delta^{\bar J_\ell J_{\ell+1}},\\
D_{2,3} &= -2\delta_{\bar I_\ell}^{\;J_\ell}\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}} - \delta_{\bar I_\ell I_{\ell+1}}\delta^{J_\ell \bar J_{\ell+1}},\\
D_{2,4} &= -2\delta_{I_\ell}^{\;\bar J_\ell}\delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}} - \delta_{I_\ell \bar I_{\ell+1}}\delta^{J_\ell \bar J_{\ell+1}},\\
D_{2,5} &= 2\delta_{I_\ell}^{\;\bar J_{\ell+1}}\delta_{\bar I_{\ell+1}}^{\;J_\ell} - \delta_{I_\ell \bar I_{\ell+1}}\delta^{\bar J_\ell J_{\ell+1}},\\
D_{2,6} &= 2\left(\delta_{I_\ell}^{\;\bar J_{\ell+1}}\delta_{I_{\ell+1}}^{\;\bar J_\ell} - \delta_{I_\ell}^{\;\bar J_\ell}\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}}\right),
\end{aligned} \tag{2.103}$$

hence the one-loop dilatation matrix

$$\begin{aligned}
D_2 = \sum_{p=1}^{6} D_{2,p} &= 2\delta_{I_\ell}^{\;\bar J_{\ell+1}}\delta_{I_{\ell+1}}^{\;\bar J_\ell} + 2\delta_{I_\ell}^{\;\bar J_{\ell+1}}\delta_{\bar I_{\ell+1}}^{\;J_\ell} + 2\delta_{\bar I_\ell}^{\;J_{\ell+1}}\delta_{I_{\ell+1}}^{\;\bar J_\ell} + 2\delta_{\bar I_\ell}^{\;J_{\ell+1}}\delta_{\bar I_{\ell+1}}^{\;J_\ell}\\
&- 2\delta_{I_\ell}^{\;\bar J_\ell}\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}} - 2\delta_{I_\ell}^{\;\bar J_\ell}\delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}} - 2\delta_{\bar I_\ell}^{\;J_\ell}\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}} - 2\delta_{\bar I_\ell}^{\;J_\ell}\delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}}\\
&- \delta_{\bar I_\ell I_{\ell+1}}\delta^{\bar J_\ell J_{\ell+1}} - \delta_{\bar I_\ell I_{\ell+1}}\delta^{J_\ell \bar J_{\ell+1}} - \delta_{I_\ell \bar I_{\ell+1}}\delta^{\bar J_\ell J_{\ell+1}} - \delta_{I_\ell \bar I_{\ell+1}}\delta^{J_\ell \bar J_{\ell+1}}
\end{aligned}$$

and therefore

$$\begin{aligned}
\langle\mathcal{O}'^{+\,\mathfrak{ab}}_{\ell,\ell+1}\mathfrak{D}_2\mathcal{O}'^{-\,\mathfrak{cd}}_{\ell,\ell+1}\rangle = -I_{0x}^2 N\delta^{\mathfrak{ad}}\delta^{\mathfrak{bc}}\Big[&2\left(\delta_{I_\ell}^{\;\bar J_{\ell+1}} + \delta_{\bar I_\ell}^{\;J_{\ell+1}}\right)\left(\delta_{I_{\ell+1}}^{\;\bar J_\ell} + \delta_{\bar I_{\ell+1}}^{\;J_\ell}\right)\\
&- 2\left(\delta_{I_\ell}^{\;\bar J_\ell} + \delta_{\bar I_\ell}^{\;J_\ell}\right)\left(\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}} + \delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}}\right) - \left(\delta_{\bar I_\ell I_{\ell+1}} + \delta_{I_\ell \bar I_{\ell+1}}\right)\left(\delta_{\bar J_\ell J_{\ell+1}} + \delta_{J_\ell \bar J_{\ell+1}}\right)\Big],
\end{aligned}$$

giving the one-loop anomalous dimension matrix

$$\begin{aligned}
\Gamma_{\substack{\text{one}\\ \text{loop}}} = g^2 D_2 = \frac{\lambda}{16\pi^2}\Big[&2\left(\delta_{I_\ell}^{\;\bar J_\ell} + \delta_{\bar I_\ell}^{\;J_\ell}\right)\left(\delta_{I_{\ell+1}}^{\;\bar J_{\ell+1}} + \delta_{\bar I_{\ell+1}}^{\;J_{\ell+1}}\right)\\
&- 2\left(\delta_{I_\ell}^{\;\bar J_{\ell+1}} + \delta_{\bar I_\ell}^{\;J_{\ell+1}}\right)\left(\delta_{I_{\ell+1}}^{\;\bar J_\ell} + \delta_{\bar I_{\ell+1}}^{\;J_\ell}\right) + \left(\delta_{\bar I_\ell I_{\ell+1}} + \delta_{I_\ell \bar I_{\ell+1}}\right)\left(\delta^{\bar J_\ell J_{\ell+1}} + \delta^{J_\ell \bar J_{\ell+1}}\right)\Big],
\end{aligned} \tag{2.104}$$

where we supressed flavour indices on the left-hand side. This fully reproduces the result (2.76), only with indices (i, j) replaced by $(I, \bar{J})$ and $(\bar{I}, J)$ according to (2.87), as predicted in (2.85). Generalisation of (2.104) to an equivalent of (2.82) is rather straightforward.

2.6 $SU(3)$ and $SU(2)$ bosonic subsectors

By following the same logic as we have done so far we could break **6** even more, by letting $\mathbf{3} = \mathbf{2} + \mathbf{1}$. This construction implies that

$$v^I \equiv (v^\alpha, \xi), \qquad \alpha = 1, 2, \tag{2.105}$$

where its components are an $SU(2)$ doublet (v^α) and a singlet (ξ). There is a freedom in forming these doublets, the fact that will have important physical implications when orbifolding to $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. For our convenience (which will become subsequently justified) we choose to consider the following doublets:

$$\chi_\alpha^{\mathrm{SU(2)_R}} := \begin{pmatrix} X \\ \bar{Y} \end{pmatrix}, \qquad \psi_\alpha^{\mathrm{SU(2)_L}} := \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \phi_\alpha^{SU(2)_Z} := \begin{pmatrix} X \\ Z \end{pmatrix}, \qquad \alpha = 1, 2. \tag{2.106}$$

For these, the one-loop dilatation operator becomes again

$$\mathfrak{D}_2' = N^{-1} \left(\mathfrak{D}_2^h + \mathfrak{D}_2^{\bar{h}} + \mathfrak{D}_2^{h\bar{h}} \right), \tag{2.107}$$

but where now

$$\begin{aligned}
\mathfrak{D}_2^h =& - \mathrm{tr}[\Psi_\alpha, \Psi_\beta][\check{\Psi}_\alpha, \check{\Psi}_\beta] - 2\mathrm{tr}[\Psi_\alpha, \varphi][\check{\Psi}_\alpha, \check{\varphi}], \\
\mathfrak{D}_2^{\bar{h}} =& - \mathrm{tr}[\bar{\Psi}_\alpha, \bar{\Psi}_\beta][\check{\bar{\Psi}}_\alpha, \check{\bar{\Psi}}_\beta] - 2\mathrm{tr}[\bar{\Psi}_\alpha, \bar{\varphi}][\check{\bar{\Psi}}_\alpha, \check{\bar{\varphi}}], \\
\mathfrak{D}_2^{h\bar{h}} =& - 2\mathrm{tr}[\Psi_\alpha, \bar{\Psi}_\beta][\check{\Psi}_\alpha, \check{\bar{\Psi}}_\beta] - 2\mathrm{tr}[\Psi_\alpha, \bar{\varphi}][\check{\Psi}_\alpha, \check{\bar{\varphi}}] - 2\mathrm{tr}[\bar{\Psi}_\alpha, \varphi][\check{\bar{\Psi}}_\alpha, \check{\varphi}] \\
& - 2\mathrm{tr}[\varphi, \bar{\varphi}][\check{\varphi}, \check{\bar{\varphi}}] - \mathrm{tr}[\Psi_\alpha, \check{\Psi}_\beta][\bar{\Psi}_\alpha, \check{\bar{\Psi}}_\beta] - \mathrm{tr}[\Psi_\alpha, \check{\bar{\Psi}}_\beta][\bar{\Psi}_\alpha, \check{\Psi}_\beta] \\
& - \mathrm{tr}[\Psi_\alpha, \check{\varphi}][\bar{\Psi}_\alpha, \check{\bar{\varphi}}] - \mathrm{tr}[\Psi_\alpha, \check{\bar{\varphi}}][\bar{\Psi}_\alpha, \check{\varphi}] - \mathrm{tr}[\varphi, \check{\Psi}_\alpha][\bar{\varphi}, \check{\bar{\Psi}}_\alpha] \\
& - \mathrm{tr}[\varphi, \check{\bar{\Psi}}_\alpha][\bar{\varphi}, \check{\Psi}_\alpha] - \mathrm{tr}[\varphi, \check{\varphi}][\bar{\varphi}, \check{\bar{\varphi}}] - \mathrm{tr}[\varphi, \check{\bar{\varphi}}][\bar{\varphi}, \check{\varphi}],
\end{aligned} \tag{2.108}$$

where $\Psi = \{\chi, \psi, \phi\}$, and φ the remaining $U(1)$ field (which will be $\varphi = Z$ i the case of $SU(2)_L$ and $SU(2)_R$, and $\varphi = Y$ in the case of $SU(2)_Z$). This breaks our dilatation operator into $\mathbf{2}+\mathbf{1}+\mathbf{2}+\mathbf{1}$. The basis can be dealt with in a similar fashion, keeping in mind (2.105), and the correlation function can be derived again in this representation, however we will not go through the whole derivation here. The reason is that the way of proceeding will be fully analogous to the previous case and we expect to get an identical result from the physical point of view. Therefore, we will stop here and leave it to a curious reader as an exercise.

Instead, I would like to show that the trace operator $\mathcal{K}$ in (2.82) vanishes for $SU(3)$ and $SU(2)$ bosonic subsectors and this can be easily done with the treatment presented above. In these sectors the gauge invariant operators can be built only out of $\mathfrak{W}_I$'s for $SU(3)$, and Ψ_α's for $SU(2)$. Thus, the $SO(6)$ operators (2.28) will become for the $SU(3)$ case

$$\mathcal{O}(x) = \mathrm{tr}[\mathfrak{W}_{I_1} \mathfrak{W}_{I_2} \cdots \mathfrak{W}_{I_L}](x), \tag{2.109}$$

and for the $SU(2)$ case

$$\mathcal{O}(x) = \mathrm{tr}[\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\Psi_{\alpha_L}](x). \tag{2.110}$$

For that very reason the dilatation operator (2.94) for the $SU(3)$ case becomes

$$\mathfrak{D}_2 = -N^{-1}\mathrm{tr}[\mathfrak{W}_I, \mathfrak{W}_J][\check{\mathfrak{W}}_I, \check{\mathfrak{W}}_J] \tag{2.111}$$

and the dilatation operator (2.108) for the $SU(2)$ case becomes

$$\mathfrak{D}_2 = -\,N^{-1}\mathrm{tr}[\Psi_\alpha, \Psi_\beta][\check{\Psi}_\alpha, \check{\Psi}_\beta]. \tag{2.112}$$

Both, in $SU(2)$ and $SU(3)$ it will be of the form

$$\Gamma^{SU(2),SU(3)}_{\substack{\text{one}\\\text{loop}}} = \frac{\lambda}{8\pi^2}\sum_{\ell=1}^{L}\left(\mathcal{I}_{\ell,\ell+1} - \mathcal{P}_{\ell,\ell+1}\right), \tag{2.113}$$

which, recognised as the Hamiltonian of the XXX Heisenberg spin chain with L lattice sites, consists solely of the identity operator and the permutation operator. Also, we can see that in terms of our new basis, the trace operator $\mathcal{K}$ is merely the consequence of hitting the non-holomorphic structure in the basis (e.g. $\cdots\mathfrak{W}_{I_\ell}\overline{\mathfrak{W}}_{I_{\ell+1}}\cdots$) with the dilatation operator, which can actually be understood just by looking at $\mathcal{K}$'s definition in (2.86).

Summary

In this chapter we have presented an alternative method of calculating anomalous dimension matrices - the dilatation operator, and used it explicitly to obtain the form of the one-loop anomalous dimension matrix for the $SO(6)$ bosonic subsector of $\mathcal{N}=4$ SYM theory; we have mentioned that it is equivalent to the Hamiltonian of an integrable spin chain. (Since our notation might be slightly confusing for the reader, we recap the symbols we will be using throughout the next two chapters in table 2.1.) Afterwards, we showed that one can reproduce the same matrix in different basis, namely in (2.85) and (2.105). This also showed us what will be its form in the $SU(3)$ and $SU(2)$ bosonic subsectors.

	operator	matrix	eigenvalues
full expansion, up to one loop	$\mathfrak{D}(g)$	$\Gamma(g)$	$\Delta(g)$
terms next to g^2	$\mathfrak{D}_2$	D_2	Δ_2
one-loop anomalous expressions	$\hat{\Gamma}_{\substack{\text{one}\\\text{loop}}}$	$\Gamma_{\substack{\text{one}\\\text{loop}}}$	γ

Table 2.1: Summary of different symbols for the dilatation operator, the anomalous dimension matrix, and its eigenvalues, the anomalous dimensions; we do not write any terms of the order of $\mathcal{O}(g^3)$.

Chapter 3

Quantisation of string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the pp-wave limit

Orbifolds arise in a purely geometric context by generalising the notion of manifolds to allow a discrete set of singular points. Thus, having some manifold $\mathcal{M}$ we could act on it with a discrete group action $\mathcal{G} : \mathcal{M} \to \mathcal{M}$. This action is said to possess a fixed point $x \in \mathcal{M}$ if for some $\mathfrak{g} \in \mathcal{G}$ ($\mathfrak{g} \neq$ identity) we have $\mathfrak{g}x = x$. Then, by identifying points under the relation $x \sim \mathfrak{g}x$ for all $\mathfrak{g} \in \mathcal{G}$, we construct the quotient space $\mathcal{M}/\mathcal{G}$. Each point is identified with its orbit (in the mathematical sense) under $\mathcal{G}$, hence the name orbifold. If the group $\mathcal{G}$ acts freely (no fixed points) then we have the special case of orbifold which is an ordinary manifold. Otherwise, the points of the orbifold corresponding to the fixed point set have discrete identifications of their tangent spaces, and are not manifold points.

This set-up is crucial for properties of a (string) theory compactified on an orbifold. Since fields of this theory can propagate on it freely, and on the other hand some points of orbifold are identified through $x \sim \mathfrak{g}x$, as a consequence one relaxes the boundary conditions for the fields, allowing by the same time to add new states to it. These new states are called the twisted sectors of the theory.

Although orbifolds in context of string theory have come about sometime ago already (see for example [49, 50, 52]), their application to AdS/CFT conjectrure is particularly exciting. The reason is that the singular points of the orbifold can break supersymmetry of strings propagating on them, and therefore it is expected that a similar occurance might take place on the gauge dual side. Then, it would be very interesting to examine if the corresponding gauge theory with less supersymmetry is integrable. This breaking might hopefully lead us to theories with zero supersymmetry, like QCD. In this chapter we will try to consider a very specific orbifold, and formulate the AdS/CFT duality based on the new set-up, which we develop in the next few sections, and finally test it by comparing the spectrum up to the lowest order, by the use of the Bethe ansatz procedure [51].

3.1 The orbifold/quiver gauge correspondence

I have already mentioned in section 1.3.1 that a connection between string theory and gauge theories appeared before the discovery of Maldacena's conjecture. Orbifolds entered in that context also before 1997; it has been proved [53] that placing D3-branes at orbifolds results in a certain class of four dimensional field theories, usually referred to as quiver gauge theories. Then, it was shown [54, 55] that these theories admit an AdS dual where the compact 5-manifold is an orbifold on $\mathbf{S}^5$.

In particular, we consider a discrete symmetry group $\mathcal{G} = \mathbb{Z}_M$ and act with it on the five sphere $\mathbf{S}^5$, embedded in $\mathbb{R}^6 \sim \mathbb{C}^3$ with

$$\sum_{i=1}^{3} |z_i|^2 = R^2, \tag{3.1}$$

such that

$$(z_1, z_2, z_3) \sim (\omega z_1, \omega^{-1} z_2, z_3), \qquad \omega := e^{2\pi i/M}. \tag{3.2}$$

As a result, we will obtain the orbifold $\mathbb{C}^2/\mathbb{Z}_M \times \mathbb{C}$. From a point of view of Dp-brane dynamics, we consider N D3-branes transverse to the 6-dimensional complex space $\mathbb{C}^3/\mathbb{Z}_M$, and then the covering space is the "parent" space with NM D3-branes, with the $\mathcal{N} = 4$ SYM theory of gauge group $SU(NM)$. Its projection to the "daughter" space gives [53] an $\mathcal{N} = 2$ superconformal field theory living on each of the four-dimensional world volumes of M stacks of N D3-branes. It has the R-symmetry group $SU(2)_R \times U(1)$, and the gauge group

$$U(N)^{(1)} \times U(N)^{(2)} \times \cdots \times U(N)^{(M)}, \tag{3.3}$$

where we impose

$$U(N)^{(M+a)} \leftrightarrow U(N)^{(a)}. \tag{3.4}$$

The new $\mathcal{N} = 2$ supersymmetric quiver gauge theory (QGT) is a "daughter" of the "parent" $\mathcal{N} = 4$ SYM theory and from the mathematical sense it is its projection from the covering space to the "daughter" space by the group $\mathcal{G}$.

The fact that the holographic dual of the $\mathcal{N} = 2$ quiver gauge theory is type IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ is very non-trivial and has been first argued in [54] and shortly afterwards in [55]. It has been also shown [56] that the correlation functions of the "daughter" theory are identical to those of the "parent" theory, up to the following rescaling of the coupling constant

$$g_{\mathrm{YM}}^2 \to \frac{1}{M} (g_{\mathrm{YM}}^{(a)})^2. \qquad \forall_{a=1,\dots,M} \tag{3.5}$$

This gives us some ideas about the form of the basic quantities of this kind of AdS/CFT duality. First of all, we expect that the analogue of (1.105) for $p = 3$ would be

$$(g_{\mathrm{YM}}^{(a)})^2 \equiv g_{\mathrm{QGT}}^2 = 4\pi g_s M = g_{\mathrm{YM}}^2 M, \tag{3.6}$$

and thus the radius of AdS_5 and $\mathbf{S}^5$ would be

$$R^2 = \sqrt{4\pi g_s \alpha'^2 NM}. \tag{3.7}$$

The volume of $\mathbf{S}^5/\mathbb{Z}_M$ will thus be reduced by a factor of M compared to that of a covering space $\mathbf{S}^5$. Furthermore, there will now be MN units of Ramond-Ramond 5-form flux through the five sphere $\mathbf{S}^5$ (and thus AdS_5), and since the five sphere contains M copies of a fundamental domain that are identified by the orbifold group, there will be N units of

$$\begin{tikzcd}
AdS_5 \times S^5 \arrow[r, leftrightarrow] \arrow[d, "\mathbb{Z}_M"] & \mathcal{N}=4 \;\; SYM \arrow[d, "\mathbb{Z}_M"] \\
AdS_5 \times S^5/\mathbb{Z}_M \arrow[r, leftrightarrow] & \mathcal{N}=2 \;\; QGT
\end{tikzcd}$$

Figure 3.1: The scheme of relation between $\mathcal{N}=4$ SYM, $\mathcal{N}=2$ quiver gauge theory, and their corresponding type IIB string theories.

flux per fundamental domain, and hence N units through the orbifold $\mathbf{S}^5/\mathbb{Z}_M$. Finally, the 't Hooft coupling in each of the domain copies is

$$\lambda' := g^2_{\mathrm{QGT}} N = 4\pi g_s NM, \tag{3.8}$$

which is the same as the 't Hooft coupling on the original NM D3-branes before orbifolding, for which the Yang-Mills coupling was equal to $4\pi g_s$.

The plan for the rest of the chapter is the following. We start with the gauge side and present the formalism of orbifolding the "parent" $\mathcal{N}=4$ SYM theory to its "daughter" $\mathcal{N}=2$ QGT, present content fields of the latter, derive the $\mathcal{N}=2$ bosonic action, and show that it is $SU(2)_R \times U(1)$ invariant. Then we move to the string side and quantise the type IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the pp-wave background, emphasising differences between this theory and the one reviewed in section 1.4. Afterwards, the global symmetry analysis based on [57] allows us to construct the $\mathcal{N}=2$ MRV operators on the gauge side, and then match them to the string states. Finally, we derive the dilatation operator for $\mathcal{N}=2$ QGT, use it to calculate the one-loop anomalous dimensions of the MRV states. We compare the results to the string spectrum, derived during the quantisation of string theory in the triple scaling limit, thereby testing to lowest order the type IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M/\mathcal{N}=2$ QGT duality. For a scheme of all this, please refer to figure 3.1.

3.2 $\mathcal{N}=2$ quiver gauge theory from $\mathcal{N}=4$ SYM

The $\mathcal{N}=2$ superconformal quiver gauge theory has the following structure:

- M vector multiplets

$$(\mathcal{A}^a_\mu, \Phi^a, \psi^a, \psi^a_\Phi), \qquad a = 1, ..., M \tag{3.9}$$

  where Φ^a is a complex scalar, together with its superpartner - the Weyl fermion ψ^a_Φ. $\mathcal{A}^a_\mu$ is the gauge field, and ψ^a is the gaugino. All these fields transform in the adjoint of $U(N)^{(a)}$.

- M bi-fundamental hypermultiplets

$$(A^a, B^a, \chi^a_A, \chi^a_B), \qquad a = 1, ..., M. \tag{3.10}$$

  The complex scalar field A^a and its superpartner χ^a_A transform in the $(N^a, \overline{N}^{a+1})$ of $U(N)^{(a)} \times U(N)^{(a+1)}$. The pair B^a and χ^a_B transform in the $(\overline{N}^a, N^{a+1})$.

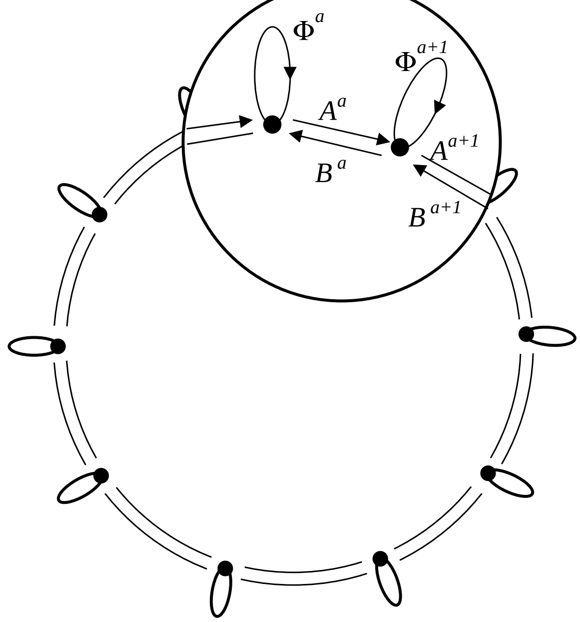

Figure 3.2: The quiver (moose) diagram. The $\mathcal{N}=2$ fields come about as "translations" in quiver space; the quiver space is periodic under M and has a fixed set of M quiver-dots, representing gauge groups $U(N)^{(a)}$.

This matter content of the gauge theory can be succinctly summarised in the form of a quiver (moose) diagram (see figure 3.2). It shows M quiver-dots in quiver space where each dot (fundamental domain) represents the $U(N)$ gauge group, on the string side described by a stack of D3-branes, and all the $U(N)$ groups are linked together by matter in bi-fundamental representation. In this way one can view the vector multiplet fields $(\mathcal{A}^a_\mu, \Phi^a, \psi^a, \psi^a_\Phi)$ as "translations" from quiver-dot (a) to the same one (a), whereas the bi-fundamental hypermultiplet fields $(A^a, B^a, \chi^a_A, \chi^a_B)$ come about as translations from quiver-dot (a) to quiver-dot $(a \pm 1)$.

3.2.1 The set-up

The orbifold projection of $\mathcal{N}=4$ SYM theory to $\mathcal{N}=2$ supersymmetric QGT is, as discussed in [53] and [56], performed by embedding the orbifold group $\mathcal{G}=\mathbb{Z}_M$, which is a subgroup of the bosonic $SO(6)$ R-symmetry, into the gauge group. In the following section, however, we will refrain to the bosonic subsector of $\mathcal{N}=4$ theory SYM and orbifold it, receiving the bosonic $\mathcal{N}=2$ quiver gauge theory. Let us denote an $\mathcal{N}=4$ bosonic field in $\mathcal{N}=1$ language as $\mathcal{Y}$. This field transforms under the action of an element $\gamma \in \mathbb{Z}_M$ as follows

$$\mathcal{Y} \longrightarrow \mathfrak{U}(\gamma)\,(\mathcal{R}_\gamma \circ \mathcal{Y})\,\mathfrak{U}^\dagger(\gamma) = \mathcal{Y}, \tag{3.11}$$

where $\mathfrak{U}(\gamma)$ is the representation of the element as a matrix of the gauge group, and $\mathcal{R}_\gamma$ is the corresponding element of the R-symmetry group. Therefore, (3.11) should be viewed merely as a constraint on the field $\mathcal{Y}$.

As explained from the point of view of Dp-brane dynamics, if the gauge group of $\mathcal{N}=4$ SYM is chosen to be $U(MN)$, then the projection by $\mathbb{Z}_M$ creates a quiver theory that has a residual $U(N)^M$ local symmetry. $\mathfrak{U}(\gamma)$ can be chosen as a $MN \times MN$ diagonal matrix

$$\mathfrak{U}(\gamma) = \mathrm{diag}\left(\mathbb{1}, \omega, \omega^2, \dots, \omega^{M-1}\right), \tag{3.12}$$

where each entry is an $N \times N$ unit matrix. The field $\mathcal{Y}$ of the orbifolded theory is then defined by the constraint

$$\mathcal{Y} = \omega^{s_{\mathcal{Y}}}\ \mathfrak{U}(\gamma)\, \mathcal{Y}\, \mathfrak{U}^\dagger(\gamma). \tag{3.13}$$

Since $\mathbb{Z}_M \in$ R-symmetry, $\mathcal{R}_\gamma$ will just be a representation of orbifold action on fields with internal indices, and since we know from section that 1.2.4 that $\mathcal{R}_{mn}$ operator generates three $U(1)$ Cartan charges of $SO(6)$, J_1, J_2, and J_3, then we realise that $\mathcal{R}_\gamma$ has to act on the three planes (12-plane, 34-plane, and 56-plane) of $\mathbf{S}^5$ embedded in $\mathbb{R}^6 \sim \mathbb{C}^3$ as well. Finally, a quick look at the identification (3.2) allows us to write that

$$\mathcal{R}_\gamma \circ X = \omega X, \qquad \mathcal{R}_\gamma \circ Y = \omega^{-1} Y, \qquad \mathcal{R}_\gamma \circ Z = Z, \tag{3.14}$$

and from (3.13) we deduce that

$$s_X = s_{\bar{Y}} = +1, \qquad s_{\bar{X}} = s_Y = -1, \qquad s_Z = s_{\bar{Z}} = 0. \tag{3.15}$$

As a result, we have the following commutation relations

$$\mathfrak{U}(\gamma)\begin{pmatrix} X \\ \bar{Y} \end{pmatrix} = \omega^{-1}\cdot\begin{pmatrix} X \\ \bar{Y} \end{pmatrix}\mathfrak{U}(\gamma), \qquad \mathfrak{U}(\gamma)\begin{pmatrix} \bar{X} \\ Y \end{pmatrix} = \omega^{+1}\cdot\begin{pmatrix} \bar{X} \\ Y \end{pmatrix}\mathfrak{U}(\gamma),$$
$$\mathfrak{U}(\gamma)\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix} = \begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}\mathfrak{U}(\gamma). \tag{3.16}$$

Thus, the components surviving the projection are

$$X = \begin{pmatrix} 0 & A^1 & & & \\ & 0 & A^2 & & \\ & & 0 & & \\ & & & \ddots & A^{M-1} \\ A^M & & & & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & & & & B^M \\ B^1 & 0 & & & \\ & B^2 & 0 & & \\ & & & \ddots & \\ & & & B^{M-1} & 0 \end{pmatrix},$$
$$Z = \begin{pmatrix} \Phi^1 & & & & \\ & \Phi^2 & & & \\ & & \Phi^3 & & \\ & & & \ddots & \\ & & & & \Phi^M \end{pmatrix}, \qquad \mathcal{A}_\mu = \begin{pmatrix} \mathcal{A}_\mu^1 & & & & \\ & \mathcal{A}_\mu^2 & & & \\ & & \mathcal{A}_\mu^3 & & \\ & & & \ddots & \\ & & & & \mathcal{A}_\mu^M \end{pmatrix}, \tag{3.17}$$

where we pointed out that $\mathcal{A}_\mu^a$ transforms in the adjoint representation of $U(N)^{(a)}$. These are the projected fields of $\mathcal{N}=4$ SYM corresponding to the orbifold projections acting on the Chan-Paton factors of the open string ending on the NM D3-branes in the covering space of the orbifold space. Each non-vanishing entry of the above matrices is an $N \times N$ matrix and corresponds to the bosonic field of the $\mathcal{N}=2$ quiver gauge theory.

3.2.2 $\mathcal{N}=2$ QGT bosonic action

In order to obtain the (bosonic) action of $\mathcal{N}=2$ quiver gauge theory we will orbifold the bosonic part of the Euclidean $\mathcal{N}=4$ SYM action (2.3)

$$S^{\mathcal{N}=4}_{\text{bosonic}} = \frac{1}{g^2_{YM}} \int d^4x \, \mathrm{tr} \left(\frac{1}{4} \mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} + \mathcal{D}_\mu \phi_m \mathcal{D}_\mu \phi_m - \frac{1}{2} [\phi_m, \phi_n][\phi_m, \phi_n] \right). \tag{3.18}$$

It can be rewritten in terms of the complex scalars (2.84) by the use of a little bit of algebra as:

$$\mathrm{tr}\, \mathcal{D}_\mu \phi_m \mathcal{D}_\mu \phi_m = 2\mathrm{tr}\, \mathcal{D}_\mu \mathfrak{W}_I \mathcal{D}_\mu \overline{\mathfrak{W}}_I, \tag{3.19}$$

where the appropriate sum is assumed, and

$$V[\phi] \equiv \frac{1}{2} \sum_{m,n=1}^{6} \mathrm{tr}\, [\phi_m, \phi_n]^2 = \sum_{I,J=1}^{3} \left(\mathrm{tr}\, |[\mathfrak{W}_I, \mathfrak{W}_J]|^2 + \mathrm{tr}\, |[\mathfrak{W}_I, \overline{\mathfrak{W}}_J]|^2 \right), \tag{3.20}$$

where we defined

$$|[A, B]|^2 := [A, B][\bar{B}, \bar{A}], \tag{3.21}$$

yielding

$$S^{\mathcal{N}=4}_{\text{scalar}} = \frac{1}{g^2_{YM}} \int d^4x \left[2\mathrm{tr}\, \mathcal{D}_\mu \mathfrak{W}_I \mathcal{D}_\mu \overline{\mathfrak{W}}_I - \mathrm{tr}\, |[\mathfrak{W}_I, \mathfrak{W}_J]|^2 - \mathrm{tr}\, |[\mathfrak{W}_I, \overline{\mathfrak{W}}_J]|^2 \right]. \tag{3.22}$$

We also note that the vertex could be also expressed in terms of the F-terms and D-terms by simply applying the Jacobi identity to (3.20), getting

$$V[\phi] \equiv \mathcal{L}_F + \mathcal{L}_D, \tag{3.23}$$

where

$$\mathcal{L}_F = -2 \sum_{I,J=1}^{3} \mathrm{tr}\, [\mathfrak{W}_I, \mathfrak{W}_J][\overline{\mathfrak{W}}_I, \overline{\mathfrak{W}}_J], \tag{3.24}$$

$$\mathcal{L}_D = \sum_{I,J=1}^{3} \mathrm{tr}\, [\mathfrak{W}_I, \overline{\mathfrak{W}}_I][\mathfrak{W}_J, \overline{\mathfrak{W}}_J]. \tag{3.25}$$

Our aim is to express the $\mathcal{N}=4$ action in terms of the multiplets of the R-symmetry of $\mathcal{N}=2$ theory which, as we know already, is $SU(2)_R \times U(1)$. The reason is that this is the symmetry group which does not get broken by the action of $\mathcal{G} = \mathbb{Z}_M$. For that reason let us define an $SU(2)_R$ scalar doublet $(\alpha, \beta = 1, 2)$

$$\chi_\alpha = \begin{pmatrix} \mathfrak{W}_1 \\ \overline{\mathfrak{W}}_2 \end{pmatrix} = \begin{pmatrix} X \\ \bar{Y} \end{pmatrix}, \quad \text{and its hermitian conjugate} \quad \bar{\chi}^\alpha = \begin{pmatrix} \overline{\mathfrak{W}}_1 \\ \mathfrak{W}_2 \end{pmatrix} = \begin{pmatrix} \bar{X} \\ Y \end{pmatrix}, \tag{3.26}$$

which are nothing but spinors of $SU(2) \simeq Spin(3)$ group, and thus $\mathcal{M} \in SU(2)$ will act on χ_α such that

$$\chi_\alpha \rightarrow (\mathcal{M}\chi)_\alpha = (\mathcal{M})_\alpha{}^\beta \chi_\beta, \tag{3.27}$$

where

$$(\mathcal{M})_\alpha{}^\beta = \exp\left[\frac{i}{2}\varphi^j(\sigma^j)_\alpha{}^\beta\right], \tag{3.28}$$

with σ^j $(j=1,2,3)$ being generators of $SU(2)$ - the Pauli matrices. Then, since

$$\bar{\chi}^\alpha \to \bar{\chi}^\beta(\mathcal{M}^\dagger)_\beta{}^\alpha, \tag{3.29}$$

the combination

$$\bar{\chi}^\alpha\chi_\alpha \to \bar{\chi}^\beta(\mathcal{M}^\dagger)_\beta{}^\alpha(\mathcal{M})_\alpha{}^\gamma\chi_\gamma = \bar{\chi}^\beta\chi_\beta \tag{3.30}$$

will always be $SU(2)$ invariant. Therefore keeping in mind that

$$\overline{(\chi_\alpha)} = \bar{\chi}^\alpha, \tag{3.31}$$

one can prove that

$$\begin{aligned}
&\mathrm{tr}[\chi_\alpha,\bar{\chi}^\beta][\chi_\beta,\bar{\chi}^\alpha] = \mathrm{tr}|[\mathfrak{W}_1,\overline{\mathfrak{W}}_1]|^2 + \mathrm{tr}|[\mathfrak{W}_2,\overline{\mathfrak{W}}_2]|^2 + \mathrm{tr}|[\mathfrak{W}_1,\mathfrak{W}_2]|^2 && (3.32)\\
&\mathrm{tr}[\chi_\alpha,Z][\bar{Z},\bar{\chi}^\alpha] + \mathrm{tr}[\bar{\chi}^\alpha,Z][\bar{Z},\chi_\alpha] \\
&\qquad = \mathrm{tr}|[\mathfrak{W}_1,\mathfrak{W}_3]|^2 + \mathrm{tr}|[\mathfrak{W}_2,\mathfrak{W}_3]|^2 + \mathrm{tr}|[\mathfrak{W}_1,\overline{\mathfrak{W}}_3]|^2 + \mathrm{tr}|[\mathfrak{W}_2,\overline{\mathfrak{W}}_3]|^2 && (3.33)\\
&\mathrm{tr}[\chi_\alpha,\chi_\beta][\bar{\chi}^\beta,\bar{\chi}^\alpha] = 2\mathrm{tr}|[\mathfrak{W}_1,\overline{\mathfrak{W}}_2]|^2 && (3.34)
\end{aligned}$$

and thus we can rewrite the full vertex (3.20) as

$$V[\chi,\bar{\chi},Z] = \mathrm{tr}|[\chi_\alpha,\bar{\chi}^\beta]|^2 + \mathrm{tr}|[\chi_\alpha,\chi_\beta]|^2 + 2\mathrm{tr}|[\chi_\alpha,Z]|^2 + 2\mathrm{tr}|[\bar{\chi}^\alpha,Z]|^2 + \mathrm{tr}|[Z,\bar{Z}]|^2, \tag{3.35}$$

giving an explicitly $SU(2)_R \times U(1)$ invariant $\mathcal{N}=4$ scalar vertex. The kinetic terms are trivial to work out, and thus we end up with

$$S^{\mathcal{N}=4}_{\text{scalar}} = \frac{1}{g^2_{YM}}\int d^4x\left[2\mathrm{tr}\,\mathcal{D}_\mu\chi_\alpha\mathcal{D}_\mu\bar{\chi}^\alpha + 2\mathrm{tr}\,\mathcal{D}_\mu Z\mathcal{D}_\mu\bar{Z} - V[\chi,\bar{\chi},Z]\right]. \tag{3.36}$$

The last thing to do is to take advantage of the fact that all terms are $SU(2)_R \times U(1)$ invariant, and also traced. For that reason orbifolding can be performed in terms of $SU(2)_R$ spinors. We will thus use

$$\begin{aligned}
\chi_\alpha &:= \chi_\alpha^{kl} = \delta^{a,b-1}\otimes(\chi^a_\alpha)^{IJ},\\
\bar{\chi}^\alpha &:= \bar{\chi}^{\alpha\,kl} = \delta^{a,b+1}\otimes(\bar{\chi}^{\alpha,\,a-1})^{IJ},\\
Z &:= Z^{kl} = \delta^{ab}\otimes(\Phi^a)^{IJ},
\end{aligned} \tag{3.37}$$

(a not summed) for $k,l=1,\ldots,MN$; $a,b=1,\ldots,M$; and $I,J=1,\ldots,N$, and where

$$\chi^a_\alpha = \begin{pmatrix} A^a \\ \bar{B}^a\end{pmatrix}, \qquad \bar{\chi}^{\alpha,\,a} = \begin{pmatrix} \bar{A}^a \\ B^a\end{pmatrix}. \tag{3.38}$$

For example,

$$\begin{aligned}
\chi^{kl}_\alpha\bar{\chi}^{\alpha\,lm} &= \left(\delta^{a,b-1}\otimes\chi^a_\alpha\right)^{kl}\left(\delta^{a,b+1}\otimes\bar{\chi}^{\alpha,\,a-1}\right)^{lm}\\
&= \delta^{a,b-1}\delta^{b,c+1}\otimes\chi^a_\alpha\bar{\chi}^{\alpha,\,b-1} = \delta^{ab}\otimes\chi^a_\alpha\bar{\chi}^{\alpha,\,a},
\end{aligned} \tag{3.39}$$

where the $b-1$ index on the second field ($\bar{\chi}$) is fixed by the first delta $\delta^{a,b-1}$; all the other multiplications are performed in complete analogy. Obviously

$$\chi^a \leftrightarrow \chi^{a,a+1}, \qquad \bar{\chi}^a \leftrightarrow \bar{\chi}^{a+1,a}. \tag{3.40}$$

And thanks to that, the kinetic term can be rewritten simply as

$$\mathrm{tr}\, \mathcal{D}_\mu \chi_\alpha \mathcal{D}_\mu \bar{\chi}^\alpha = \mathrm{tr}\, \mathcal{D}_\mu \chi^a_\alpha \mathcal{D}_\mu \bar{\chi}^{\alpha,\, a}, \tag{3.41}$$

where the trace on the left-hand side is over the NM-dimensional covering space, and the one on the right-hand side is over the N-dimensional space, but it assumes the sum $\sum_{a=1}^{M}$. Above

$$\mathcal{D}_\mu \chi^a_\alpha = \partial_\mu \chi^a_\alpha - i(\mathcal{A}^a_\mu \chi^a_\alpha - \chi^a_\alpha \mathcal{A}^{a+1}_\mu) \equiv \partial_\mu \chi^a_\alpha - i[\mathcal{A}_\mu, \chi_\alpha]^{\,a,a+1}, \tag{3.42}$$

$$\mathcal{D}_\mu \bar{\chi}^{\alpha,\, a} = \partial_\mu \bar{\chi}^{\alpha,\, a} - i(\mathcal{A}^{a+1}_\mu \bar{\chi}^{\alpha,\, a} - \bar{\chi}^{\alpha,\, a} \mathcal{A}^a_\mu) \equiv \partial_\mu \bar{\chi}^{\alpha,\, a} - i[\mathcal{A}_\mu, \bar{\chi}^\alpha]^{\,a+1,a}. \tag{3.43}$$

Periodicity condition $\mathcal{Y}^a = \mathcal{Y}^{a+M}$ should be kept in mind.

With this knowledge the vertex ingredients (3.35) can be worked out rather easily to be

$$\begin{aligned}
\mathrm{tr}|[\,\chi_\alpha, \bar{\chi}^\beta\,]|^2 &= \mathrm{tr}|\chi^{a+1}_\alpha \bar{\chi}^{\beta,\, a+1} - \bar{\chi}^{\beta,\, a} \chi^a_\alpha|^2, \\
\mathrm{tr}|[\,\chi_\alpha, \chi_\beta\,]|^2 &= \mathrm{tr}|\chi^a_\alpha \chi^{a+1}_\beta - \chi^a_\beta \chi^{a+1}_\alpha|^2, \\
\mathrm{tr}|[\,\chi_\alpha, Z\,]|^2 &= \mathrm{tr}|\chi^a_\alpha \Phi^{a+1} - \Phi^a \chi^a_\alpha|^2, \\
\mathrm{tr}|[\,\bar{\chi}^\alpha, Z\,]|^2 &= \mathrm{tr}|\bar{\chi}^{\alpha,\, a} \Phi^a - \Phi^{a+1} \bar{\chi}^{\alpha,\, a}|^2, \\
\mathrm{tr}|[\,Z, \bar{Z}\,]|^2 &= \mathrm{tr}|[\Phi^a, \bar{\Phi}^a]|^2.
\end{aligned} \tag{3.44}$$

This all yields the bosonic action for the $\mathcal{N}=2$ quiver gauge theory

$$\begin{aligned}
S^{\mathcal{N}=2}_{\text{bosonic}} = \frac{1}{g^2_{YM}} \int d^4x \Big(&\frac{1}{4} \mathrm{tr}\, \mathcal{F}^a_{\mu\nu} \mathcal{F}^a_{\mu\nu} + 2\mathrm{tr}\, \mathcal{D}_\mu \chi^a_\alpha \mathcal{D}_\mu \bar{\chi}^{\alpha,\, a} + 2\mathrm{tr}\, \mathcal{D}_\mu \Phi^a \mathcal{D}_\mu \bar{\Phi}^a \\
&- \mathrm{tr}|\chi^{a+1}_\alpha \bar{\chi}^{\beta,\, a+1} - \bar{\chi}^{\beta,\, a} \chi^a_\alpha|^2 - \mathrm{tr}|\chi^a_\alpha \chi^{a+1}_\beta - \chi^a_\beta \chi^{a+1}_\alpha|^2 \\
- 2\,\mathrm{tr}|\chi^a_\alpha \Phi^{a+1} - \Phi^a \chi^a_\alpha|^2 &- 2\,\mathrm{tr}|\bar{\chi}^{\alpha,\, a} \Phi^a - \Phi^{a+1} \bar{\chi}^{\alpha,\, a}|^2 - \mathrm{tr}|[\Phi^a, \bar{\Phi}^a]|^2 \Big),
\end{aligned} \tag{3.45}$$

where the $\mathcal{N}=2$ QGT field strength is given by

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu - i[\mathcal{A}^a_\mu, \mathcal{A}^a_\nu]. \tag{3.46}$$

3.3 Double scaling limit and the DLCQ pp-wave

Now we proceed to presenting the procedure of quantising type IIB string theory on the $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ orbifold, following Mukhi, Rangamani, and E. Verlinde (MRV) [57][1]. The crucial thing here is to keep the number of elements in the discrete group $|\mathcal{G}| = M$ as a free parameter, and then take it to be infinite (together with N) but in a very specific way, namely keeping the ratio N/M constant. We can immediately see that the radius of

[1] In fact, some work on reproducing the pp-wave limit for orbifolded string theories had been done before MRV (e.g. in [58, 59, 60, 61]), however it is the MRV paper where the limit in which we obtain the DLCQ string spectrum was first noticed.

AdS_5 (and thus $\mathbf{S}^5$) will become infinite in this limit, however it turns out that orbifolding procedure compactifies one of the directions (we will shortly see which one), and the radius of this compactification will be held finite.

Why do we take such a limit in the first place? The idea is to look at the geometry seen by a particle moving on a light-like (null) geodesic along a great circle of the $\mathbf{S}^5$ sphere. This has been successfully done for the smooth manifold $\mathbf{S}^5$ in [5] in the context of $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills theory (see section 1.4), resulting in a pp-wave in their limit, which is to take N infinitely large. The crucial part there was to keep $\frac{g_s N}{J^2}$ finite, which on the one hand gave the possibility to take a limit crucial for validity of Maldacena's conjecture, and on the other hand gave access to an overlapping spectrum of dual gauge operators and string states; one is then able to compare these spectra and thus test the AdS/CFT duality.

In the case of orbifold $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ the situation complicates a little bit since now a difference in choice of a null geodesic arises: depending whether we will take the trajectory to lie along the singularities of the orbifold or not (as explained in [58]). If we choose it to lie along the singular locus we will obtain a pp-wave background that has the $\mathbb{Z}_M$ ALE singularity as part of its transverse space, and the result will be the orbifolded version of maximally supersymmetric pp-wave with the original 16 supercharges of $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$. Avoidance of the singularities, will result in the maximally supesymmetric background with the supersymmetry enhanced to 32 supercharges, and this is the case we are interested in. For that reason we expect that our gauge states, even though they will be the $\mathcal{N}=2$ gauge states, will form a multiplet of $\mathcal{N}=4$ states. Furthermore, as mentioned above, we shall take N and M large, though keeping N/M constant, and therefore also $\frac{g_s N}{M}$, in analogy to the BMN case[2]

The metric of $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ can be written in global coordinates (c.f. section 1.1.1) as

$$\begin{aligned} ds^2 = R^2 \big[& -\cosh^2\!\rho\, dt^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega_3^2 \\ & + d\alpha^2 + \sin^2\!\alpha\, d\theta^2 + \cos^2\!\alpha \left(d\gamma^2 + \cos^2\!\gamma\, d\chi^2 + \sin^2\!\gamma\, d\phi^2 \right) \big] , \end{aligned} \tag{3.47}$$

where $d\Omega_3^2$ in the first line is the unit 3-sphere. The second one describes the metric for an $\mathbf{S}^5$ embedded in a six dimensional space containing a $\mathbb{Z}_M$ (ALE) singularity. The relationship with the complex z_i coordinates and the angles of the sphere is

$$z_1 = R\cos\alpha\cos\gamma\, e^{i\chi}, \quad z_2 = R\cos\alpha\sin\gamma\, e^{i\phi}, \quad z_3 = R\sin\alpha\, e^{i\theta}. \tag{3.48}$$

This implies that with embedding (3.2) χ and ϕ can be written in terms of usual 2π-periodic angles as

$$\begin{aligned} & \chi = \frac{\eta}{M}, \qquad \phi = -\frac{\eta}{M} + \varrho, \\ & \eta \sim \eta + 2\pi, \quad \varrho \sim \varrho + 2\pi, \end{aligned} \tag{3.49}$$

[2]The appealing power-difference between $\frac{g_s N}{M}$ and $\frac{g_s N}{J^2}$ in the denominator stems from the fact that either we rescale the coupling constant in the former by $1/M$ during orbifolding, or we rescale N by a factor of M in the latter (where $J \sim M$ for long operators), consistently with the properties of orbifold/quiver gauge correspondence, presented in section 3.1.

which on the other hand leads us to the conclusion that the $\mathbb{C}^2/\mathbb{Z}_M \times \mathbb{C}$ orbifold induces an $\mathbf{S}^5/\mathbb{Z}_M$ orbifold given by the identification

$$\chi \sim \chi + \frac{2\pi\mathfrak{m}}{M}, \qquad \phi \sim \phi - \frac{2\pi\mathfrak{m}}{M} + 2\pi\mathfrak{m} \tag{3.50}$$

where $\mathfrak{m}$ is some integer.

In order to take the pp-wave limit, we will consider a particle moving along the θ direction and sitting at $\rho = 0$ and $\alpha = 0$. We define

$$r \equiv \rho R, \quad w \equiv \alpha R, \quad y \equiv \gamma R, \tag{3.51}$$

and introduce the light cone coordinates

$$x^+ \equiv \frac{1}{2}(t+\chi), \qquad x^- \equiv \frac{R^2}{2}(t-\chi). \tag{3.52}$$

Making the proper substitutions in the metric, and also introducing the mass parameter μ by rescaling $x^+ \to \mu x^+$, and $x^- \to \frac{1}{\mu}x^-$, we arrive at

$$\begin{aligned} ds^2 = R^2 \Bigg[&- \cosh^2 \frac{r}{R} \left(\mu dx^+ + \frac{1}{\mu R^2} dx^-\right)^2 + \frac{dr^2}{R^2} + \sinh^2 \frac{r}{R}\, d\Omega_3^2 \\ &+ \frac{dw^2}{R^2} + \sin^2 \frac{w}{R}\, d\theta^2 + \cos^2 \frac{w}{R} \left(\frac{dy^2}{R^2} + \cos^2 \frac{y}{R} \left(\mu dx^+ - \frac{1}{\mu R^2} dx^-\right)^2 + \sin^2 \frac{y}{R}\, d\phi^2 \right) \Bigg], \end{aligned}$$

which in the $R \to \infty$ limit becomes

$$\begin{aligned} ds^2 =& -4dx^+dx^- - \mu^2(r^2+w^2+y^2)(dx^+)^2 + dr^2 + r^2 d\Omega_3^2 + dw^2 + w^2 d\theta^2 + dy^2 + y^2 d\phi^2 \\ \equiv& -4dx^+dx^- - \mu^2 \sum_{I=1}^{8} (x^I)^2 (dx^+)^2 + \sum_{I=1}^{8} (dx^I)^2, \end{aligned} \tag{3.53}$$

where the transverse coordinates x^I were introduced. There is also a Ramond-Ramond flux in the geometry (3.53)

$$F_{+1234} = F_{+5678} \propto \mu, \tag{3.54}$$

in the Penrose limit, thus explicitly showing that the transverse $SO(8)$ invariance of the metric (3.53) is broken into $SO(4) \times SO(4)$ subgroup. One should also notice that putting $\mu \to 0$ reproduces the string theory in the flat background, as intuitively expected.

The fact that the metric (3.53) is exactly the same as (1.151) is merely a consequence of the maximally supersymmetric background, however there is a very important difference between this limit and the standard plane-wave limit [5], namely that here the light-like direction x^- is now compact. From (3.50) we see that the light cone coordinates obey

$$x^+ \sim x^+ + \frac{\mu\pi}{M}\mathfrak{m}, \qquad x^- \sim x^- + \frac{\pi R^2}{\mu M}\mathfrak{m}. \tag{3.55}$$

This combined shift in x^+ and x^- has to be accompanied by a simultaneous shift in $\phi \sim \phi - 2\pi/M$. For $M \to \infty$ we see that x^+ and ϕ stay unchanged, whereas x^- becomes periodic in this limit, with the period

$$\frac{\pi R^2}{\mu M} \equiv 2\pi R_- \quad \Longrightarrow \quad R_- = \frac{\alpha'}{\mu}\sqrt{\pi g_s \frac{N}{M}} = \text{finite}. \tag{3.56}$$

As a consequence, the corresponding light cone momentum $2p^+$ is quantised in units of $1/R_-$, namely

$$2p^+ = \frac{\boldsymbol{k}}{R_-}, \qquad \boldsymbol{k} = 1, 2, 3, \ldots \tag{3.57}$$

The conclusion is that the Penrose limit of $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ with $M \to \infty$ in this particular way leads to a Discrete Light Cone Quantisation (DLCQ) of the string on a plane wave background, in which the null direction x^- is periodic, around which the strings can wrap, as we will see very soon.

3.4 DLCQ quantisation of the type IIB plane-wave string

The string (bosonic) σ-model action[3] of the metric (3.53) is given in conformal gauge by

$$S = -\frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\sigma \left(-4\partial_\alpha x^+ \partial^\alpha x^- + \partial_\alpha x^I \partial^\alpha x^I - \mu^2 (x^I)^2 \partial_\alpha x^+ \partial^\alpha x^+\right). \tag{3.58}$$

The equations of motion are derived to be

$$\partial_\alpha \partial^\alpha x^+ = 0, \tag{3.59}$$

$$\partial_\alpha \partial^\alpha x^I - \mu^2 x^I (\partial_\alpha x^+ \partial^\alpha x^+) = 0. \tag{3.60}$$

The first equation (3.59) allows us to choose the light-cone gauge

$$x^+ = \alpha' p^+ \tau, \tag{3.61}$$

which implies that we can rewrite the action (3.58) in this gauge (assuming that the fields vanish at infinite time)

$$S^{\text{l.c.}} = -\frac{1}{4\pi\alpha'}\iint d\tau d\sigma \left(\partial_\alpha x^I \partial^\alpha x^I - \eta^2 (x^I)^2\right), \tag{3.62}$$

where $\eta := \alpha' p^+ \mu$. Then the second equation of motion (3.60) becomes

$$(\partial_\alpha \partial^\alpha - \eta^2) x^I = 0, \tag{3.63}$$

which can be solved to (when quantising this string sigma model, we start by using the oscillator notation consistent with section 1.4)

$$x^I(\tau, \sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{i}{\sqrt{\omega_n}} \left(\alpha_n^I e^{-i(\omega_n \tau + k_n \sigma)} - \tilde{\alpha}_n^I e^{i(\omega_n \tau + k_n \sigma)}\right), \tag{3.64}$$

[3]We use the conformally invariant metric $\eta_{\alpha\beta} = \text{diag}(-1, +1)$, c.f. (4.1).

where

$$\omega_n^2 = k_n^2 + \eta^2, \tag{3.65}$$

and furthermore the boundary condition for the closed string $x^I(\tau,\sigma) = x^I(\tau,\sigma+2\pi)$ gives that $k_n = n$, hence

$$\omega_n = \mathrm{sgn}(n)\sqrt{n^2+\eta^2}. \tag{3.66}$$

Also, introducing the zero modes

$$\alpha_0^I = \sqrt{\frac{\alpha'}{2\eta}}\left(p_0^I - i\frac{\eta}{\alpha'}x_0^I\right), \tag{3.67}$$

we can write down the solution of the transverse coordinates

$$\begin{aligned} x^I(\tau,\sigma) =& \cos(\eta\tau)x_0^I + \frac{1}{\eta}\sin(\eta\tau)\alpha' p_0^I \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{\sqrt{\omega_n}}\left(\alpha_n^I e^{-i(\omega_n\tau+n\sigma)} - \tilde{\alpha}_n^I e^{i(\omega_n\tau+n\sigma)}\right), \end{aligned} \tag{3.68}$$

and hence the corresponding momenta

$$\begin{aligned} p^I(\tau,\sigma) = \partial_\tau x^I(\tau,\sigma) =& \cos(\eta\tau)\alpha' p_0^I - \sin(\eta\tau)\eta x_0^I \\ &+ \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\sqrt{\omega_n}\left(\alpha_n^I e^{-i(\omega_n\tau+n\sigma)} - \tilde{\alpha}_n^I e^{i(\omega_n\tau+n\sigma)}\right). \end{aligned} \tag{3.69}$$

The null direction can be expanded in a very similar fashion, although (3.55) tells us that the boundary condition will now read

$$x^-(\tau,\sigma+2\pi) - x^-(\tau,\sigma) = (2\pi R_-)\mathfrak{m}, \tag{3.70}$$

meaning that strings can wind around this direction $\mathfrak{m}$-times.

The light-cone Hamiltonian and the momentum constraints can be derived from the condition that the energy-momentum tensor vanishes, namely

$$2\alpha' p^+\partial_\tau x^- = \frac{1}{2}\left[(\partial_\tau x^I)^2 + (\partial_\sigma x^I)^2 + \mu^2(x^I)^2\right], \tag{3.71}$$

$$2\alpha' p^+\partial_\sigma x^- = \partial_\tau x^I\partial_\sigma x^I, \tag{3.72}$$

where (3.71) yields

$$H^{\,\mathrm{l.c.}} = \frac{1}{\alpha' p^+}\int_0^{2\pi} d\sigma\left[\frac{1}{2}(p^I)^2 + (\partial_\sigma x^I)^2 + \mu^2(x^I)^2\right]. \tag{3.73}$$

The canonical quantisation of the theory is performed by imposition of the commutation relations (in the classical picture we have Poisson brackets instead)

$$\left[x^I(\tau,\sigma), \Pi^J(\tau,\sigma')\right] = i\delta^{IJ}\delta(\sigma-\sigma'), \tag{3.74}$$

where

$$\Pi^J(\tau,\sigma) \equiv \frac{p^J(\tau,\sigma)}{2\pi\alpha'}, \tag{3.75}$$

and which implies that

$$[\alpha_n^I, \alpha_m^J] = [\tilde{\alpha}_n^I, \tilde{\alpha}_m^J] = \delta_{n+m,0}\,\delta^{IJ}, \qquad [\alpha_n^I, \tilde{\alpha}_m^J] = 0. \tag{3.76}$$

Clearly, now the Hamiltonian can be rewritten in terms of oscillator modes

$$H^{\,\mathrm{l.c.}} = \mu(\alpha_0^{\dagger\,I}\alpha_0^I) + \frac{1}{\alpha' p^+}\sum_{n=1}^{\infty}\sqrt{n^2 + (\alpha' p^+ \mu)^2}\left[\alpha_{-n}^I\alpha_n^I + \tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I\right]. \tag{3.77}$$

Since the states of the string will be in our model characterised by their discrete light-cone momentum $\boldsymbol{k}$, and the wrapping number $\mathfrak{m}$, the Fock vacuum can be defined such that it is annihilated by the positive modes

$$\alpha_n^I|\boldsymbol{k},\mathfrak{m}\rangle = \tilde{\alpha}_n^I|\boldsymbol{k},\mathfrak{m}\rangle = 0, \qquad n \geqslant 1. \tag{3.78}$$

This could be substituted with just one equation when defining

$$a_n^\dagger := \begin{cases} \alpha_{-n} & n>0 \\ \tilde{\alpha}_{-|n|} & n<0 \\ \alpha_0^\dagger & n=0 \end{cases}, \qquad \text{and} \qquad a_n := \begin{cases} \alpha_n & n>0 \\ \tilde{\alpha}_{|n|} & n<0 \\ \alpha_0 & n=0 \end{cases}, \tag{3.79}$$

which ought to be properly normalised of course, and then the condition will read

$$a_n^I|\boldsymbol{k},\mathfrak{m}\rangle = 0, \qquad n \geqslant 1. \tag{3.80}$$

On top of that we have a level matching condition, as a consequence of the reparametrisation invariance: equation (3.72), together with (3.70) yield that

$$\boldsymbol{k}\mathfrak{m} = \sum_{n=1}^{\infty} n\,N_n, \tag{3.81}$$

where

$$N_n = \sum_{I=1}^{8} a_n^{I\,\dagger} a_n^I. \tag{3.82}$$

Other string states (than the ground state) are built by acting with transverse oscillators

$$\prod_{j=1}^{\mathcal{P}} a_{n_j}^{I_j}|\boldsymbol{k},\mathfrak{m}\rangle, \tag{3.83}$$

and the full level matching condition thus reads

$$\sum_{j=1}^{\mathcal{P}} n_j = \boldsymbol{k}\mathfrak{m}, \tag{3.84}$$

and the Hamiltonian

$$2p^- = H = \sum_{n=0}^{\infty} N_n \sqrt{\mu^2 + g_{\mathrm{QGT}}^2 \frac{N}{M^2}\left(\frac{n}{\boldsymbol{k}}\right)^2}, \tag{3.85}$$

hence the spectrum

$$E = \sum_{j=1}^{\mathcal{P}} \sqrt{\mu^2 + g_{\mathrm{QGT}}^2 \frac{N}{M^2}\left(\frac{n_j}{\boldsymbol{k}}\right)^2}. \tag{3.86}$$

When comparing this to the gauge spectrum we will usually put $\mu = 1$, merely for convenience.

3.5 Identification of charges

The R-symmetry group of the parent gauge theory is $SO(6) \sim SU(4)$, which has a subgroup $SO(4) \times U(1)$, where $SO(4) \simeq SU(2)_L \times SU(2)_R$. Accordingly to section 3.2.1, we identify A^a, B^a, and Φ, with z_1, z_2, and z_3 respectively, and note that (A^a, B^a) form a doublet under $SU(2)_L$, and $(A^a, \bar{B}^a)$ - a doublet under $SU(2)_R$. Then we note that orbifolding breaks the $SU(2)_L$ symmetry, so that the R-symmetry of the original $\mathcal{N} = 4$ SYM, that is $SO(6)$, becomes the $SU(2)_R \times U(1)$ R-symmetry of $\mathcal{N} = 2$ SYM. These are the facts that will help us to identify the relevant charges for these symmetries.

There are three important quantum numbers that possess their duals on both sides of the conjecture. The first one is the string energy $i\partial_t$, on the string side, and the conformal dimension operator Δ on the gauge side. Two others are $U(1)$ charges, and by convention they are generated by $e^{4\pi i J}$.

The first one is the $U(1)_R \subset SU(2)_R$ charge, and it acts on the A^a and B^a fields as follows

$$A^a \to e^{i\xi} A^a, \qquad B^a \to e^{i\xi} B^a, \qquad 0 \leqslant \xi < 2\pi. \tag{3.87}$$

This $U(1)_R$ charge is generated by $e^{4\pi i J_R}$, where J_R corresponds to the Killing vector

$$J_R = \frac{1}{2}(J_1 + J_2) = -\frac{i}{2}(\partial_\chi + \partial_\phi), \tag{3.88}$$

where $J_1 \equiv i\frac{\partial}{\partial\chi}$ and $J_2 \equiv i\frac{\partial}{\partial\phi}$. Obviously, the fields A^a and B^a of the hypermultiplet both have fractional charges $\frac{1}{2}$ under J_R.

Another $U(1)$ charge is the $U(1)_L \subset SU(2)_L$ is not that obvious since $SU(2)_L$ is broken by orbifolding; it rotates the A^a and B^a fields in the opposite directions in the following way

$$A^a \to e^{i\zeta} A^a, \qquad B^a \to e^{-i\zeta} B^a, \qquad 0 \leqslant \zeta < \frac{2\pi}{M}, \tag{3.89}$$

and is generated by $e^{4\pi i \tilde{J}_L}$, of which $\tilde{J}_L$ corresponds to the Killing vector

$$\tilde{J}_L = \frac{1}{2}(J_1 - J_2) = -\frac{i}{2}(\partial_\chi - \partial_\phi), \tag{3.90}$$

under which the fields A^a and B^a have fractional charge $\frac{1}{2M}$ and $-\frac{1}{2M}$, respectively. For the sake of convenience we redefine $J_L \equiv M\tilde{J}_L$, and summarise all charges and Killing vectors as

$$\Delta = i\partial_t, \qquad J_L = -\frac{i}{2M}(\partial_\chi - \partial_\phi), \qquad J_R = -\frac{i}{2}(\partial_\chi + \partial_\phi), \tag{3.91}$$

which immediately implies that the light-cone momenta are

$$H = 2p^- = i(\partial_t + \partial_\chi) = \Delta - MJ_L - J_R, \tag{3.92}$$
$$2p^+ = i\frac{(\partial_t - \partial_\chi)}{R^2} = \frac{\Delta + MJ_L + J_R}{2MR_-}. \tag{3.93}$$

As explained in [5], in order to relate gauge theory to the string states we must look for operators that have both p^- and p^+ finite. Since $R \to \infty$, this means that Δ and $MJ_L + J_R$ must be both large, while their difference remains finite. Physical gauge invariant operators should have half-integral values, for J_L and J_R, which implies that (a) MJ_L automatically becomes large when $M \to \infty$, even when J_L is kept fixed, and this on the other hand that (b) J_R scales like like M. The scaling dimension we do not have to worry about since BPS bound implies that $\Delta \geqslant MJ_L + J_R$. This assures that both p^- and p^+ stay finite in the plane wave background.

	A^a	B^a	$\bar{A}^a$	$\bar{B}^a$	Φ^a	$\bar{\Phi}^a$	χ^a_A	$\bar{\chi}^a_B$	χ^a_B	$\bar{\chi}^a_A$	$\bar{\psi}^a_\Phi$	$\bar{\psi}^a$	ψ^a_Φ	ψ^a
Δ	1	1	1	1	1	1	3/2	3/2	3/2	3/2	3/2	3/2	3/2	3/2
MJ_L	1/2	-1/2	-1/2	1/2	0	0	1/2	1/2	-1/2	-1/2	0	0	0	0
J_R	1/2	1/2	-1/2	-1/2	0	0	0	0	0	0	1/2	1/2	-1/2	-1/2
H	0	1	2	1	1	1	1	1	2	2	1	1	2	2

Table 3.1: The Δ, J_L and J_R eigenvalues for bosonic and fermionic operators.

The full list of eigenvalues of all the $\mathcal{N} = 2$ fields is summarised in table 3.1. To what has been said above we should add that the fields Φ^a should be neutral under the action of $e^{4\pi i J}$, for they correspond to translation of the original NM D3-branes along the transverse $\mathbb{R}^2$ that is unaffected by the orbifold group. Complex conjugation and supersymmetry give us the remaining charge assignments for the fermions and all the conjugate fields.

3.6 Matching the string states with the gauge operators

By the use of our dictionary in table 3.1 we can now try to match the appropriate string states to the gauge theory states and compare their spectra. We continue to follow [57], where they based to some extent on [5], however important subtleties had to be taken into account, namely the discrete light-cone momenta $\boldsymbol{k}$, and the discrete winding $\mathfrak{m}$.

Let us first construct the ground state with no winding $\mathfrak{m}$. Obviously, this will be a state with $H = 0$, thus according to (3.92) with

$$\Delta = MJ_L + J_R. \tag{3.94}$$

Inserting this into (3.93), and taking advantage of (3.57) gives

$$2p^{+} = \frac{\Delta}{\boldsymbol{k}M} \cdot 2p^{+}, \tag{3.95}$$

hence

$$\Delta = \boldsymbol{k}M. \tag{3.96}$$

We can now build the ground state gauge operator by looking at table 3.1 and realising that the only field that can be used in constructing the ground state is the one that has $H = 0$ itself, namely the field A^a, which transforms under a local element of the gauge group $\mathscr{U}(x)^{(a)} \in U(N)^{(a)}$ as

$$A^a \to \mathscr{U}(x)^{(a)} A^a \, (\mathscr{U}(x)^{(a)})^{-1}, \tag{3.97}$$

and hence

$$|\boldsymbol{k}, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \frac{1}{\sqrt{\mathcal{C}}} \operatorname{tr}\Big(A^1(x)A^2(x) \cdots A^{\boldsymbol{k}M}(x)\Big), \qquad \mathcal{C} = N^{\boldsymbol{k}M}, \tag{3.98}$$

which, due to (3.4), can be rewritten as

$$|\boldsymbol{k}, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \frac{1}{\sqrt{\mathcal{C}}} \operatorname{tr}\big(A^1(x)A^2(x) \cdots A^{M}(x)\big)^{\boldsymbol{k}}, \qquad \mathcal{C} = (N^M)^{\boldsymbol{k}}. \tag{3.99}$$

This gives us an idea that $\boldsymbol{k}$ tells us how many times a "string" of $\mathcal{N} = 2$ fields A^a wraps around a quiver diagram[4], as illustrated in figure 3.3. The normalisation constants are obtained from calculation of the two-point correlation function in the non-interacting case, where each A^a is contracted with the corresponding object, according to the usual contraction rules (they will be discussed more extensively in section 3.8.1).

The first excited state has $H = 1$, and on the string side it is obtained by acting with the proper excitation $a_0^{I\,\dagger}$. There are eight bosonic zero mode oscillators, corresponding to the transverse coordinates $x^I(\tau, \sigma)$, thus eight states with $H = 1$ are expected on the gauge side. Indeed, four of these are obtained by appropriate combinations of the Φ^a, $\bar{\Phi}^a$, B^a, and $\bar{B}^a$ fields, and four others - of combinations of covariant derivatives $\mathcal{D}_I^{(a)}$. The only thing we have to be careful about is the insertion of these fields in the ground state (3.98), so that they remain gauge invariant. Also, these fields can be inserted in any of the $\boldsymbol{k}$ strings of A^a's, thus we write (dropping the spacetime dependence)

$$a_0^{\Phi,\dagger}|\boldsymbol{k}, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \frac{1}{\sqrt{\mathcal{C}_\Phi}} \sum_{\ell=1}^{\boldsymbol{k}M} \operatorname{tr}\Big(A^1A^2 \cdots A^{\ell-1}\Phi^{\ell}A^{\ell} \cdots A^M(A^1 \cdots A^M)^{\boldsymbol{k}-1}\Big), \tag{3.100}$$

and in the same manner for $\bar{\Phi}^a$, together with

$$a_0^{B,\dagger}|\boldsymbol{k}, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \frac{1}{\sqrt{\mathcal{C}_B}} \sum_{\ell=1}^{\boldsymbol{k}M} \operatorname{tr}\Big(A^1A^2 \cdots A^{\ell}B^{\ell}A^{\ell} \cdots A^M(A^1 \cdots A^M)^{\boldsymbol{k}-1}\Big), \tag{3.101}$$

$$a_0^{\bar{B},\dagger}|\boldsymbol{k}, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \frac{1}{\sqrt{\mathcal{C}_{\bar{B}}}} \sum_{\ell=1}^{\boldsymbol{k}M} \operatorname{tr}\Big(A^1A^2 \cdots A^{\ell-1}\bar{B}^{\ell}A^{\ell+1} \cdots A^M(A^1 \cdots A^M)^{\boldsymbol{k}-1}\Big), \tag{3.102}$$

[4]Or, in other words, how many times these fields wind around in quiver space.

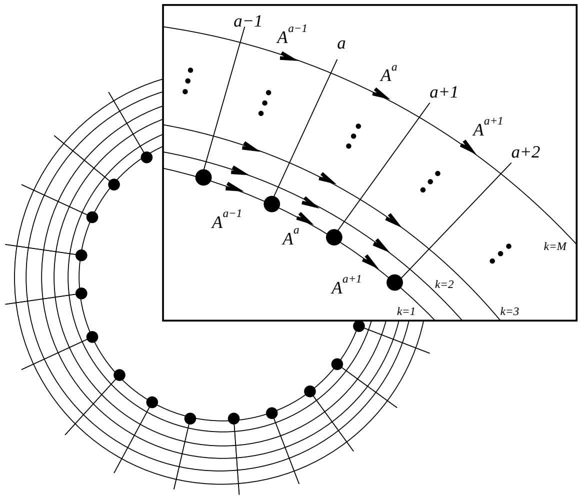

Figure 3.3: The ground state of $\mathcal{N} = 2$ theory, seen as $\boldsymbol{k}M$ of A^a fields, wrapping around a quiver (moose) diagram.

where above

$$\mathcal{C}_\Phi = M \cdot N^{\boldsymbol{k}M+1}, \qquad \mathcal{C}_B = M \cdot N^{\boldsymbol{k}M+2}, \qquad \mathcal{C}_{\bar{B}} = M \cdot N^{\boldsymbol{k}M}. \tag{3.103}$$

The above sum is rather intuitive, since we need to make sure to integrate the operator over all possible fluctuations of the bosonic particle in the zero mode. In [5] these insertions were all equivalent to each other, but here the situation is somewhat different, hence the sum. Also, we will drop the normalisation of operators hereafter. There are also eight fermionic modes but we will treat fermions as being outside the scope of this thesis, and focus solely on the bosonic sector.

An important difference between zero winding $\mathfrak{m}$ and non-zero winding is that in the latter case the insertion of impurities[5] in various position in the A^a-chain produces phases. As suggested by [57], we can construct

$$a_n^{\Phi,\dagger}|\boldsymbol{k}, \mathfrak{m}\rangle = \sum_{\ell=1}^{\boldsymbol{k}M} \mathrm{tr}\Big(A^1 A^2 \cdots A^{\ell-1}\Phi^\ell A^\ell \cdots A^{\boldsymbol{k}M}\Big)\,\omega^{\frac{\mathfrak{m}\ell}{\boldsymbol{k}}}, \tag{3.104}$$

where[6] $\omega = e^{\frac{2\pi i}{M}}$. Other fields can be inserted with complete analogy to the $\mathfrak{m} = 0$ case. With this phase, one can see that due to the level matching condition (3.84) the winding in (3.98) has to be put to zero, because inserting the phase in the state and integrating with the sum according to $\sum_{\ell=1}^{M} \omega^{\frac{\mathfrak{m}\ell}{\boldsymbol{k}}}$ will kill the whole state, exactly like in the case of (1.159). The fact, however, that we sum over M non-equivalent insertions of Φ^a in the A^a-vacuum in (3.104) implies that it does not vanish identically, unlike in the BMN-case. That

[5] I will refer to these fields as impurities in the ground state, and the excited states will be impured with these fields. Also, impurities will be sometimes called magnons.

[6] The overall phase above $\omega^{\frac{\mathfrak{m}\ell}{\boldsymbol{k}}} = e^{\frac{2\pi i \mathfrak{m}\ell}{\boldsymbol{k}M}}$ is a "naive" Fourier transform, taken in analogy to the BMN paper [5], however we will see soon that this choice (as well as the one in (3.105)) is reproduces the correct result when comparing to the plane wave ansatz and solving it (see section 3.8).

is because the insertions are non-equivalent here, (c.f. with the level matching condition (3.84)). However, this operator is usually referred to as quasi-protected, because it can be shown [62] that although it is not fully protected (like the one impurity operator in the BMN limit), all its non-planar corrections vanish, and therefore the spectrum of this state in string theory does not receive string loop corrections.

The next thing to do would be to include more impurities like for example the two-insertion case

$$a_{n_1}^{\Phi,\dagger}a_{n_2}^{\Phi,\dagger}|\boldsymbol{k},\mathfrak{m}\rangle = \sum_{\substack{\ell_1=1\\ \ell_2\geqslant\ell_1}}^{\boldsymbol{k}M} \mathrm{tr}\Big(A^1A^2\cdots A^{\ell_1-1}\Phi^{\ell_1}A^{\ell_1}\cdots A^{\ell_2-1}\Phi^{\ell_2}A^{\ell_2}\cdots A^{\boldsymbol{k}M}\Big)\,\omega^{\frac{1}{\boldsymbol{k}}\sum_i n_i\ell_i}, \tag{3.105}$$

where

$$\sum_{i=1}^{\mathcal{P}=2} n_i = \boldsymbol{k}\mathfrak{m}, \tag{3.106}$$

and in analogy for $\mathcal{P} > 2$. Notice that for $\mathfrak{m} = 0$ the overall phase in (3.105) will still appear, only that then the level matching condition will read $\sum_i n_i = 0$. In the rest of this thesis I will refer to the $\mathcal{N} = 2$ QGT states constructed in this section as the "$\mathcal{N} = 2$ MRV states".

3.7 Rewriting $\mathcal{N} = 2$ MRV states in $\mathcal{N} = 4$ notation

It would certainly be a good news if we should be able to reproduce our $\mathcal{N} = 2$ MRV states, from section (3.6), from well known $\mathcal{N} = 4$ states, since then one could apply all the developed tools for the $\mathcal{N} = 4$ theory and make the analysis more advanced, e.g. the diagonalisation analysis could be perhaps performed more efficiently.

If we consider the orbifolded matrices (3.17) we will learn that the ground state (3.98) can be expressed as (we will focus on the $\boldsymbol{k} = 1$ case first, and then give arguments on how to generalise it to an arbitrary $\boldsymbol{k}$)

$$|\boldsymbol{k} = 1, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \mathrm{tr}(X^M), \tag{3.107}$$

and will reproduce the state (3.98) M times, merely reflecting the fact that the gauge group has changed from $SU(N)$ to $SU(MN)$. The one impurity state[7] can be written as

$$a_n^{Z,\dagger}|\boldsymbol{k} = 1, \mathfrak{m} = 0\rangle \mathrel{\widehat{=}} \sum_{\ell=1}^{M} \mathrm{tr}(X^\ell Z X^{M-\ell}). \tag{3.108}$$

We could keep inserting these fields in full analogy to what we did in section 1.4, however based on considerations in section 3.1 we can suspect what kind of a problem will occur here. Since this notation is inherited from the parent $\mathcal{N} = 4$ theory, we cannot expect that the $\mathcal{N} = 2$ MRV states with $\mathfrak{m} \neq 0$ can be simply reproduced in $\mathcal{N} = 4$ notation. The

[7] Here, in analogy to the $\mathcal{N} = 2$ states we will focus on the case of $\Phi^a \leftrightarrow Z$ impurities, although everything here can be generalised to any other particle with ease.

reason for that is that the winding states are not present in the parent theory, but originate as twisted sectors in the orbifold. Therefore, in order to write down the states with $\mathfrak{m} \neq 0$ in $\mathcal{N}=4$ notation one has to come up with something a little bit more inventive.

The way out of this problem, as suggested in [57], is use the representation of the element $\gamma \in \mathbb{Z}_M$ as a matrix of the gauge group (3.12) to the power of $\mathfrak{m}$, namely

$$\mathfrak{U}^{\mathfrak{m}} \tag{3.109}$$

and insert it in the trace of gauge fields[8]. For example, the one-impurity state can be written as

$$a_n^{Z,\dagger}|\boldsymbol{k}=1,\mathfrak{m}\rangle \mathrel{\widehat{=}} \operatorname{tr}(\mathfrak{U}^{\mathfrak{m}} Z X^M), \tag{3.110}$$

and also, equivalence of these states (up to the factor of M)[9] is rather straightforward to prove. A slightly more difficult case to show is the $\mathcal{P}$-impurity ($\mathcal{P}>1$) case, since then the proof requires symmetrisation over the inserted states (see appendix B) but nevertheless these states do match, thus

$$\begin{aligned} a_{n_1}^{Z,\dagger} a_{n_2}^{Z,\dagger}|\boldsymbol{k}=1,\mathfrak{m}\rangle &\mathrel{\widehat{=}} \sum_{\ell_2 \geqslant \ell_1}^{M} \sum_{\ell_1=1}^{M} \operatorname{tr}(X^{\ell_1} Z_{n_1} X^{\ell_2-\ell_1} Z_{n_2} X^{M-\ell_2}) \\ &= \sum_{\ell_1,\ell_2} \omega^{\sum_i n_i \ell_i} \operatorname{tr}(\mathfrak{U}^{\mathfrak{m}} X^{\ell_1} Z X^{\ell_2-\ell_1} Z X^{M-\ell_2}), \end{aligned} \tag{3.111}$$

where we defined

$$Z_{n_i} := Z \cdot \mathfrak{U}^{n_i}, \tag{3.112}$$

took advantage of (3.16), and kept in mind that

$$\sum_i n_i = \mathfrak{m}, \tag{3.113}$$

which can be easily checked by letting $\ell_i \to \ell_i + 1$. Obviously the number of impurities (magnons) can be increased in a straightforward manner.

A generalisation to an arbitrary $\boldsymbol{k}$ can be performed by defining the $\boldsymbol{k}MN \times \boldsymbol{k}MN$ matrix

$$\widetilde{\mathfrak{U}}(\gamma) := \operatorname{diag}\left(\mathbb{1}, \omega^{\frac{1}{\boldsymbol{k}}}, \omega^{\frac{2}{\boldsymbol{k}}}, \ldots, \omega^{\frac{\boldsymbol{k}M-1}{\boldsymbol{k}}}\right), \tag{3.114}$$

and consistently rescaling the dimension of matrices considered in (3.17) from $MN \times MN$ to $\boldsymbol{k}MN \times \boldsymbol{k}MN$, so that for example

$$(X)^{kl} = \delta^{a,b-1} \otimes (A^a)^{IJ}, \qquad \begin{array}{l} k,l = 1,\ldots,\boldsymbol{k}MN \\ a,b = 1,\ldots,\boldsymbol{k}M \\ I,J = 1,\ldots,N \end{array} \tag{3.115}$$

[8] I will hereafter refer to it as a "twist-matrix", as it represents the twisted sectors of orbifolded theory.

[9] The fact that $\mathcal{N}=2$ MRV states and $\mathcal{N}=4$ MRV states are equivalent up to the factor of M is merely a consequence of the correlation function being equal up to the rescaling of the coupling constant (3.5).

thus tensoring now the $N \times N$ $\mathcal{N}=2$ fields with a $\boldsymbol{k}M \times \boldsymbol{k}M$ unity matrix; all the other fields are tensored in analogy to (3.37), based on (3.17). Then, the most general state can be written as

$$\begin{aligned}\prod_{i=1}^{\mathcal{P}} a_{n_i}^{Z,\dagger}|\boldsymbol{k},\mathfrak{m}\rangle &\mathrel{\widehat{=}} \sum_{\ell_{\mathcal{P}}\geqslant\ldots\geqslant\ell_2\geqslant\ell_1}^{\boldsymbol{k}M} \operatorname{tr}(X^{\ell_1} Z_{n_1} X^{\ell_2-\ell_1} Z_{n_2}\cdots Z_{n_{\mathcal{P}}} X^{\boldsymbol{k}M-\ell_{\mathcal{P}}}) \\ &= \sum_{\ell_{\mathcal{P}}\geqslant\ldots\geqslant\ell_2\geqslant\ell_1}^{\boldsymbol{k}M} \omega^{\frac{1}{k}\sum_i n_i\ell_i} \operatorname{tr}(\widetilde{\mathfrak{U}}^{\boldsymbol{k}\mathfrak{m}} X^{\ell_1} Z X^{\ell_2-\ell_1} Z\cdots Z X^{\boldsymbol{k}M-\ell_{\mathcal{P}}}) \\ &= \sum_{\ell_{\mathcal{P}}\geqslant\ldots\geqslant\ell_2\geqslant\ell_1}^{\boldsymbol{k}M} \omega^{\frac{1}{k}\sum_i n_i\ell_i} \operatorname{tr}(\mathfrak{U}^{\mathfrak{m}} X^{\ell_1} Z X^{\ell_2-\ell_1} Z\cdots Z X^{\boldsymbol{k}M-\ell_{\mathcal{P}}}),\end{aligned} \tag{3.116}$$

and as a consistency check we can again rescale $\ell_i \to \ell_i + 1$, obtaining the same state multiplied by an overall phase

$$\omega^{\frac{1}{k}\sum_i n_i}\omega^{-\mathfrak{m}}, \tag{3.117}$$

and by requiring that it is equal to one (so that the whole state does not vanish) we reproduce the most general level matching condition (3.84). From now on, I will refer to the $\mathcal{N}=2$ MRV states in $\mathcal{N}=4$ notation as the "$\mathcal{N}=4$ MRV states".

3.8 Spectrum comparison

In this section we would like to calculate the one-loop anomalous dimension of the MRV states. The $\mathcal{N}=2$ MRV states and the $\mathcal{N}=4$ MRV states ought to be approached separately, although we expect the results to be the same. In the case of dealing with the former we need the $\mathcal{N}=2$ dilatation operator, which we need to derive, and in the case of $\mathcal{N}=4$ MRV states we need to deal with the twist matrix $\mathfrak{U}^{\mathfrak{m}}$ inside the state, and therefore we will treat both cases independently below.

3.8.1 $\mathcal{N}=2$ MRV states

The $\mathcal{N}=2$ one-loop anomalous dimension matrix

Since the $\mathcal{N}=2$ quiver gauge theory can be obtained by the action of the discrete group $\mathbb{Z}_M$, which gives us the constraint (3.11), we can use it to obtain the dilatation operator for the $\mathcal{N}=2$ QGT. The fact that orbifolding does not destroy the structure of the dilatation operator can be explained by noting that the $\mathcal{N}=4$ dilatation operator commutes with the R-symmetry generators of $SO(6)$, and hence with $\mathbb{Z}_M \subset SO(6)$ with respect to which we perform orbifolding. As the $\mathcal{N}=2$ dilatation operator thus ought to be invariant under $SU(2)_R \times U(1)$, we will aim to express it in terms of χ^a_α and Φ^a fields. This can be achieved by applying (3.13) to (2.108), i.e. using the same abbreviation as in (3.37)

$$\begin{aligned}(\chi_\alpha)^{kl} &= \delta^{a,b-1}\otimes(\chi^a_\alpha)^{IJ} \\ (\check{\chi}_\alpha)^{kl} &= \delta^{a,b+1}\otimes(\check{\chi}^{a-1}_\alpha)^{IJ} \\ Z^{kl} &= \delta^{ab}\otimes(\Phi^a)^{IJ} \\ \check{Z}^{kl} &= \delta^{ab}\otimes(\check{\Phi}^a)^{IJ}\end{aligned} \tag{3.118}$$

where we chose $\Psi \equiv \chi$, and $\varphi \equiv Z$, and then the dilatation operator for $\mathcal{N} = 2$ QGT after a somewhat tedious computation becomes

$$\mathfrak{D}_2^{\mathcal{N}=2} = N^{-1}\left(\mathfrak{D}_2^{h} + \mathfrak{D}_2^{\bar{h}} + \mathfrak{D}_2^{h\bar{h}}\right), \tag{3.119}$$

where

$$\begin{aligned}
\mathfrak{D}_2^{h} = {} & 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\chi_\beta^{a+1}\check{\chi}_\beta^{a+1}\check{\chi}_\alpha^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\chi_\beta^{a+1}\check{\chi}_\alpha^{a+1}\check{\chi}_\beta^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\Phi^{a+1}\check{\chi}_\alpha^a\check{\Phi}^a) \\
& + 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\Phi^{a+1}\check{\Phi}^{a+1}\check{\chi}_\alpha^a) + 2\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\chi_\alpha^a\,\check{\chi}_\alpha^a\check{\Phi}^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\chi_\alpha^a\,\check{\Phi}^{a+1}\check{\chi}_\alpha^a),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}_2^{\bar{h}} = {} & 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\bar{\chi}^{\beta,\,a-2}\check{\bar{\chi}}^{\beta,\,a-2}\check{\bar{\chi}}^{\alpha,\,a-1}) - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\bar{\chi}^{\beta,\,a-2}\check{\bar{\chi}}^{\alpha,\,a-2}\check{\bar{\chi}}^{\beta,\,a-1}) \\
& - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\bar{\Phi}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\check{\bar{\Phi}}^{a}) + 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\bar{\Phi}^{a-1}\check{\bar{\Phi}}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}) \\
& + 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\Phi}^a\bar{\chi}^{\alpha,\,a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\check{\bar{\Phi}}^{a}) - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\Phi}^a\bar{\chi}^{\alpha,\,a-1}\check{\bar{\Phi}}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}_2^{h\bar{h}} = {} & 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\bar{\chi}^{\beta,\,a}\check{\bar{\chi}}^{\beta,\,a}\check{\chi}_\alpha^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\bar{\chi}^{\beta,\,a}\check{\chi}_\alpha^{a-1}\check{\bar{\chi}}^{\beta,\,a-1}) + 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\beta,\,a-1}\chi_\alpha^{a-1}\check{\chi}_\alpha^{a-1}\check{\bar{\chi}}^{\beta,\,a-1}) \\
& - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\beta,\,a-1}\chi_\alpha^{a-1}\check{\bar{\chi}}^{\beta,\,a}\check{\chi}_\alpha^a) + 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\bar{\Phi}^{a+1}\check{\bar{\Phi}}^{a+1}\check{\chi}_\alpha^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\bar{\Phi}^{a+1}\check{\chi}_\alpha^a\check{\bar{\Phi}}^a) \\
& + 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\Phi}^a\chi_\alpha^a\,\check{\chi}_\alpha^a\check{\bar{\Phi}}^a) - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\Phi}^a\chi_\alpha^a\,\check{\bar{\Phi}}^{a+1}\check{\chi}_\alpha^a) + 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\Phi^{a-1}\check{\Phi}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}) \\
& - 2\sum_{a=1}^{M}\mathrm{tr}(\bar{\chi}^{\alpha,\,a-1}\Phi^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\check{\Phi}^a) + 2\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\bar{\chi}^{\alpha,\,a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\check{\Phi}^a) \\
& - 2\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\bar{\chi}^{\alpha,\,a-1}\check{\Phi}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}) - 2\sum_{a=1}^{M}\mathrm{tr}[\Phi^a,\bar{\Phi}^a][\check{\Phi}^a,\check{\bar{\Phi}}^a] - \sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\chi}_\beta^a\,\bar{\chi}^{\alpha,\,a-1}\check{\bar{\chi}}^{\beta,\,a-1})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\chi}_\beta^a\,\check{\bar{\chi}}^{\beta,\,a}\,\bar{\chi}^{\alpha,\,a})+\sum_{a=1}^{M}\mathrm{tr}(\check{\chi}_\beta^{a-1}\chi_\alpha^{a-1}\bar{\chi}^{\alpha,\,a-1}\check{\bar{\chi}}^{\beta,\,a-1})-\sum_{a=1}^{M}\mathrm{tr}(\check{\chi}_\beta^{a-1}\chi_\alpha^{a-1}\check{\bar{\chi}}^{\beta,\,a}\,\bar{\chi}^{\alpha,\,a})\\
&-\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\bar{\chi}}^{\beta,\,a+1}\bar{\chi}^{\alpha,\,a+1}\check{\chi}_\beta^a)-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\chi}}^{\beta,\,a}\,\chi_\alpha^{a+1}\bar{\chi}^{\alpha,\,a+1}\check{\chi}_\beta^a)-\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\bar{\chi}}^{\beta,\,a+1}\check{\chi}_\beta^{a+1}\bar{\chi}^{\alpha,\,a})\\
&-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\chi}}^{\beta,\,a}\,\chi_\alpha^{a+1}\check{\chi}_\beta^{a+1}\bar{\chi}^{\alpha,\,a})-\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\Phi}^{a+1}\bar{\chi}^{\alpha,\,a}\check{\bar{\Phi}}^a)+\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\Phi}^{a+1}\check{\bar{\Phi}}^{a+1}\bar{\chi}^{\alpha,\,a})\\
&+\sum_{a=1}^{M}\mathrm{tr}(\check{\Phi}^a\chi_\alpha^a\,\bar{\chi}^{\alpha,\,a}\check{\bar{\Phi}}^a)-\sum_{a=1}^{M}\mathrm{tr}(\check{\Phi}^a\chi_\alpha^a\,\check{\bar{\Phi}}^{a+1}\bar{\chi}^{\alpha,\,a})-\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\bar{\Phi}}^{a+1}\bar{\chi}^{\alpha,\,a}\check{\Phi}^a)\\
&-\sum_{a=1}^{M}\mathrm{tr}(\chi_\alpha^a\,\check{\bar{\Phi}}^{a+1}\check{\Phi}^{a+1}\bar{\chi}^{\alpha,\,a})-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\Phi}}^a\chi_\alpha^a\,\bar{\chi}^{\alpha,\,a}\check{\Phi}^a)-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\Phi}}^a\chi_\alpha^a\,\check{\Phi}^{a+1}\bar{\chi}^{\alpha,\,a})\\
&-\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\check{\chi}_\alpha^{a-1}\bar{\Phi}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1})-\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\check{\chi}_\alpha^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\bar{\Phi}^a)-\sum_{a=1}^{M}\mathrm{tr}(\check{\chi}_\alpha^{a-1}\Phi^{a-1}\bar{\Phi}^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1})\\
&-\sum_{a=1}^{M}\mathrm{tr}(\check{\chi}_\alpha^{a-1}\Phi^{a-1}\check{\bar{\chi}}^{\alpha,\,a-1}\bar{\Phi}^a)-\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\check{\bar{\chi}}^{\alpha,\,a}\bar{\Phi}^{a+1}\check{\chi}_\alpha^a)-\sum_{a=1}^{M}\mathrm{tr}(\Phi^a\check{\bar{\chi}}^{\alpha,\,a}\check{\chi}_\alpha^a\,\bar{\Phi}^a)\\
&-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\chi}}^{\alpha,\,a}\,\Phi^{a+1}\bar{\Phi}^{a+1}\check{\chi}_\alpha^a)-\sum_{a=1}^{M}\mathrm{tr}(\check{\bar{\chi}}^{\alpha,\,a}\,\Phi^{a+1}\check{\chi}_\alpha^a\,\bar{\Phi}^a)-\sum_{a=1}^{M}\mathrm{tr}[\Phi^a,\check{\Phi}^a][\bar{\Phi}^a,\check{\bar{\Phi}}^a]\\
&-\sum_{a=1}^{M}\mathrm{tr}[\Phi^a,\check{\bar{\Phi}}^a][\bar{\Phi}^a,\check{\Phi}^a].
\end{aligned}
$$

One impurity

So what are the one-loop corrections to the conformal dimension of the $\mathcal{N}=2$ states? Obviously, the ground states with $H=0$ cannot receive any corrections but then what about the $H=1$ states? We have already mentioned that they are quasi-protected, that is do not receive non-planar corrections, so what are the planar ones? Let us consider the one impurity state

$$
\mathcal{O}_n^{\mathcal{N}=2}=\sum_{a=1}^{M}\Psi_a\,\mathcal{O}_n^a, \tag{3.120}
$$

where

$$
\mathcal{O}_n^a=\mathrm{tr}(A^1A^2\cdots A^{a-1}\Phi^aA^a\cdots A^M), \tag{3.121}
$$

and we impose the one-particle plane-wave ansatz

$$
\Psi_a=e^{ipa}, \tag{3.122}
$$

which represents the propagation of the particle in the chain of "vacuum" particles A^a.

In order to calculate the anomalous dimension of this operator one needs to find out how does the $\mathcal{N}=2$ anomalous dimension matrix operator

$$
\hat{\Gamma}^{\mathcal{N}=2}_{\substack{\text{one}\\ \text{loop}}}:=\frac{g_{\mathrm{QGT}}^2N}{16\pi^2}\,\mathfrak{D}_2^{\mathcal{N}=2} \tag{3.123}
$$

act on the state (3.120). Let us notice that the coupling constant in (3.123) is now g^2_{QGT} instead of g^2_{YM}. That is because the coupling constant has changed accordingly to the theory we are dealing with; the rest is just the same as in the case of $\mathcal{N}=4$ SYM. The contraction rules are now the following (in analogy to 2.70)

$$\begin{aligned} \langle (\bar{\chi}^{+\,a}_{\alpha})^{mn} (\chi^{-\,b}_{\beta})^{kl} \rangle = \langle (\chi^{+\,a}_{\alpha})^{mn} (\bar{\chi}^{-\,b}_{\beta})^{kl} \rangle = I_{0x}\, \delta_{\alpha\beta}\, \delta^{ab}\, \delta^{ml} \delta^{nk}, \\ \langle (\bar{\Phi}^{+\,a})^{mn} (\Phi^{-\,b})^{kl} \rangle = \langle (\Phi^{+\,a})^{mn} (\bar{\Phi}^{-\,b})^{kl} \rangle = I_{0x}\, \delta^{ab}\, \delta^{ml} \delta^{nk}, \\ (\alpha,\beta = 1,2, \quad a,b = 1,\ldots,M, \quad m,n,k,l = 1,\ldots,N), \end{aligned} \tag{3.124}$$

and all the other ones zero. In fact, the dilatation operator of $\mathcal{N}=2$ QGT will now be the same as in the case of $\mathcal{N}=4$ SYM, only with g^2_{YM} replaced by g^2_{QGT} (c.f. (2.82)), and that can be understood, for example, by looking at the way (3.119) acts on our one impurity state (3.120). Obviously, the action on the "vacuum" chain

$$\cdots A^a A^{a+1} \cdots \tag{3.125}$$

will be zero, however encountering the impured fields we will have e.g.

$$\mathfrak{D}_2^{\mathcal{N}=2} \circ (\Phi A)^{a,a+1} := \mathfrak{D}_2^{\mathcal{N}=2} \circ \Phi^a A^a = \Phi^a A^a - A^a \Phi^{a+1} = [\Phi, A]^{a,a+1}, \tag{3.126}$$

thus we see that the dilatation operator does not "touch" the overall orbifold structure, simply either permuting two fields (permutation operator) or leaving them invariant (identity operator). Thus we can write that

$$\begin{aligned} \hat{\Gamma}^{\mathcal{N}=2}_{\substack{\text{one}\\ \text{loop}}} \circ \mathcal{O}_n^{\mathcal{N}=2} &= \frac{\lambda'}{8\pi^2} \sum_{a=1}^{M} \Psi_a \left[2\mathcal{O}_n^a - \mathcal{O}_n^{a-1} - \mathcal{O}_n^{a+1} \right] \\ &= \frac{\lambda'}{8\pi^2} \sum_{a=1}^{M} (2\Psi_a - \Psi_{a-1} - \Psi_{a+1}) \mathcal{O}_n^a \\ &= \frac{\lambda'}{8\pi^2} (2 - e^{-ip} - e^{ip}) \mathcal{O}_n^{\mathcal{N}=2}, \end{aligned} \tag{3.127}$$

where $\lambda' = g^2_{\mathrm{QGT}} N$, and hence the one-loop anomalous dimension of (3.120)

$$\gamma = \frac{\lambda'}{2\pi^2} \sin^2\left(\frac{p}{2}\right). \tag{3.128}$$

Now we are almost done, we just have to notice that

$$\mathcal{O}_n^a = \mathcal{O}_n^{a+M}, \tag{3.129}$$

which simply follows from (3.4), and impose that on (3.122), getting

$$p = \frac{2\pi}{M} \times \mathsf{q}, \tag{3.130}$$

where q is some integer. In order to specify this integer we have to realise that the translation operator

$$\mathcal{U} \equiv e^{i\mathsf{P}}, \tag{3.131}$$

P being the overall momentum, which translates each site to the right by one site, acting on (3.104), with $\boldsymbol{k}=1$, gives us the momentum constraint

$$e^{i\mathsf{P}}|\text{phys}\rangle = \omega^{-\mathfrak{m}}|\text{phys}\rangle = e^{ip}|\text{phys}\rangle, \tag{3.132}$$

where $|\text{phys}\rangle$ represents a single trace operator, hence

$$p = -\frac{2\pi\mathfrak{m}}{M}, \tag{3.133}$$

and since the level matching condition reads

$$n = \mathfrak{m}, \tag{3.134}$$

therefore $\mathfrak{q}$ is

$$\mathfrak{q} = -n = -\mathfrak{m}. \tag{3.135}$$

For that reason our one-loop anomalous dimension in the large M limit will be

$$\gamma = \frac{\lambda'}{2M^2}n^2 = \frac{1}{2}\frac{g^2_{\text{QGT}}N}{M^2}n^2, \tag{3.136}$$

and this agrees with the spectrum (3.86) when the square root is expanded in small g^2_{QGT}.

Two impurities

We now proceed to the states (3.105). Let us denote the full state as

$$\mathcal{O}^{\mathcal{N}=2}_{n_1,n_2} = \sum_{\ell_2\geqslant\ell_1} \Psi_{\ell_1,\ell_2}\, \mathcal{O}^{\ell_1,\ell_2}_{n_1,n_2}, \tag{3.137}$$

where

$$\mathcal{O}^{\ell_1,\ell_2}_{n_1,n_2} = \operatorname{tr}(A^1\cdots A^{\ell_1-1}\Phi^{\ell_1}A^{\ell_1}\cdots A^{\ell_2-1}\Phi^{\ell_2}A^{\ell_2}\cdots A^M), \tag{3.138}$$

but this time the plane-wave ansatz will be as follows

$$\Psi_{\ell_1,\ell_2} \equiv \mu_1^{\ell_1}\mu_2^{\ell_2} + \mathcal{S}(p_2,p_1)\mu_2^{\ell_1}\mu_1^{\ell_2}, \qquad \mu_j := e^{ip_j} \tag{3.139}$$

construction of which reflects the fact that, if the system is integrable, then the two excited particles either pass through each other, or exchange momenta with an amplitude given by the $\mathcal{S}$-matrix $\mathcal{S}(p_2,p_1)$. The scattering process in non-diffractive if the individual momenta p_j are individually conserved, which for the two-magnon case will always be true, due to the momentum conservation.

The integrability is assumed since the ground state (3.98) is mapped to the spin-chain ferromagnetic state $|\text{phys}\rangle = |\downarrow\downarrow\cdots\downarrow\rangle$, which is merely an $SU(2)$ analogue of (2.83). This is because for the $SU(2)$ subsector the indices take on two different values, and thus we identify them with two positions of the spin: up and down. Likewise, for the $SO(6)$ case we can identify each site of the chain with the spin which can flip in all of the 6 directions of the $\mathbf{S}^5$ sphere embedded in $\mathbb{R}^6$, however as far as we are concerned, this model does not have an analogue in the real world. The situation with the $SU(2)$ case is completely different, as the model of spins flipping up and down is very well known from statistical physics. One more thing that needs to be taken into account is the winding number $\mathfrak{m}$, which on the gauge dual side becomes a quantum number, and therefore the ground state ought to be correctly rewritten as $|\text{phys}\rangle = |\mathfrak{m} = 0; \downarrow\downarrow \cdots \downarrow\rangle$. Its excitations are viewed as the ground state impured with excited spins-up (magnons), for example (3.104) is mapped to $|\text{phys}\rangle = |\mathfrak{m}; \downarrow\downarrow \cdots \downarrow\uparrow\downarrow \cdots \downarrow\rangle$, and of course integrated over the insertions of the magnon.

This model is known to be integrable due to existence of $L-1$ higher charges $\mathfrak{Q}_k$ which commute with Hamiltonian (alias anomalous dimension matrix (2.113)), and among themselves, i.e. $[\mathfrak{Q}_k, \mathfrak{Q}_\ell] = 0$. And so, in the $SU(2)$ sector

$$\mathfrak{Q}_2 := \mathcal{H}_{XXX_{1/2}} \equiv \sum_{\ell=1}^{L} \left(\mathcal{I}_{\ell,\ell+1} - \mathcal{P}_{\ell,\ell+1}\right) = \frac{1}{2}\sum_{\ell=1}^{L} \left(\mathcal{I}_{\ell,\ell+1} - \vec{\boldsymbol{\sigma}}_\ell \cdot \vec{\boldsymbol{\sigma}}_{\ell+1}\right) \tag{3.140}$$

$$\mathfrak{Q}_3 := \sum_{\ell=1}^{L} \left(\vec{\boldsymbol{\sigma}}_\ell \times \vec{\boldsymbol{\sigma}}_{\ell+1}\right) \cdot \vec{\boldsymbol{\sigma}}_{\ell+2}, \tag{3.141}$$

and so on (see [63]).

The procedure of solving the Schrödinger equation goes in analogy to the previous case, however this time we need to consider separately the case when two excitation particles meet at $\ell_1 = \ell_2$. The symmetrisation requirement allows us to recast the problem to the action of D_2 on the wave functions Ψ_{ℓ_1,ℓ_2}. Thus the two Schrödinger equations read

$$\gamma\Psi_{\ell_1,\ell_2} = \frac{\lambda'}{8\pi^2}\left(4\Psi_{\ell_1,\ell_2} - \Psi_{\ell_1-1,\ell_2} - \Psi_{\ell_1+1,\ell_2} - \Psi_{\ell_1,\ell_2-1} - \Psi_{\ell_1,\ell_2+1}\right), \qquad \ell_1 < \ell_2, \tag{3.142}$$

and

$$\gamma\Psi_{\ell_1,\ell_1} = \frac{\lambda'}{8\pi^2}\left(2\Psi_{\ell_1,\ell_1} - \Psi_{\ell_1-1,\ell_1} - \Psi_{\ell_1,\ell_1+1}\right), \qquad \ell_1 = \ell_2. \tag{3.143}$$

The first equation (3.142), when plugging in (3.139), yields that

$$\gamma = \frac{\lambda'}{2\pi^2}\sum_{i=1}^{2} \sin^2\left(\frac{p_i}{2}\right), \tag{3.144}$$

which can be solved by determining the corresponding momenta p_i.

The second equation (3.143) should not be solved separately, unlike (3.142), as it will not yield anything new, but be rather viewed as an "indication" in determining a proper constraint from (3.142) for $\ell_1 = \ell_2$; it can be simply obtained by subtracting (3.143) from (3.142), getting

$$2\Psi_{\ell_1,\ell_1} = \Psi_{\ell_1+1,\ell_1} + \Psi_{\ell_1,\ell_1-1}, \tag{3.145}$$

which, by the use of (3.139), determines the $\mathcal{S}$-matrix to be

$$\mathcal{S}(p_2,p_1) = -\frac{\mu_1}{\mu_2} \cdot \frac{\mu_1\mu_2 - 2\mu_2 + 1}{\mu_1\mu_2 - 2\mu_1 + 1} = \mathcal{S}^{-1}(p_1,p_2). \tag{3.146}$$

The boundary condition

$$\mathcal{O}^{\ell_1,M+1}_{n_1,n_2} = \mathcal{O}^{1,\ell_1}_{n_1,n_2} \quad \Longrightarrow \quad \Psi_{\ell_1,M+1} = \Psi_{1,\ell_1}, \tag{3.147}$$

yields the equation

$$\mu_2^M \left(\frac{\mu_1}{\mu_2}\right)^{\ell_1-1} + \ \mathcal{S}(p_2,p_1)\ \mu_1^M = 1 \ + \ \mathcal{S}(p_2,p_1) \left(\frac{\mu_1}{\mu_2}\right)^{\ell_1-1}, \tag{3.148}$$

and since it should hold for any value for any value of ℓ_1 we conclude that

$$\mathcal{S}(p_2,p_1) = \mu_2^M = \mu_1^{-M}, \tag{3.149}$$

which together with (3.146) gives us

$$\mathcal{S}(p_2,p_1) = \mu_2^M, \qquad \mathcal{S}(p_1,p_2) = \mu_1^M, \tag{3.150}$$

and

$$(\mu_1\mu_2)^M = 1 \quad \Longrightarrow \quad p_1 + p_2 = \frac{2\pi}{M} \times \mathfrak{q}, \tag{3.151}$$

where again $\mathfrak{q}$ is some integer, and again it can be determined by considering the translation condition

$$\mathcal{U}|\text{phys}\rangle = e^{i\mathsf{P}}|\text{phys}\rangle = \omega^{-\mathfrak{m}}|\text{phys}\rangle = e^{i(p_1+p_2)}|\text{phys}\rangle, \tag{3.152}$$

hence

$$\mathfrak{q} = -\mathfrak{m}. \tag{3.153}$$

The equations (3.146), (3.150), and (3.152) determine the Bethe equations

$$\omega^{\mathfrak{m}}\mu_1^L = -\frac{\mu_1\mu_2 - 2\mu_1 + 1}{\mu_1\mu_2 - 2\mu_2 + 1}, \qquad \omega^{\mathfrak{m}}\mu_2^L = -\frac{\mu_1\mu_2 - 2\mu_2 + 1}{\mu_1\mu_2 - 2\mu_1 + 1}, \tag{3.154}$$

where obviously $L = M + 2$. They can be solved in the large M limit as

$$e^{ip_1(M+2)} = -\,\omega^{-\mathfrak{m}}\,\frac{\omega^{-\mathfrak{m}} - 2e^{ip_1} + 1}{\omega^{-\mathfrak{m}} - 2e^{ip_2} + 1} = \frac{-2e^{ip_1} + 1 + \omega^{-\mathfrak{m}}}{2e^{-ip_1} - 1 - \omega^{\mathfrak{m}}} \overset{M\to\infty}{\approx} \frac{-2e^{ip_1} + 2}{2e^{-ip_1} - 2}, \tag{3.155}$$

hence

$$e^{ip_1(M+1)} = 1 \quad \Longrightarrow \quad p_1 \overset{M\to\infty}{\approx} \frac{2\pi}{M} \times \mathfrak{q}, \tag{3.156}$$

which can be reasonably identified with

$$\mathfrak{q} = -n_1, \tag{3.157}$$

because if

$$p_1 = \frac{2\pi}{M}(-n_1), \qquad p_2 = \frac{2\pi}{M}(-n_2), \tag{3.158}$$

then (3.158) together with (3.152) reproduce the level matching condition

$$n_1 + n_2 = \mathfrak{m}. \tag{3.159}$$

The only thing left now is to write down the expression for the anomalous dimension, which is in $M \to \infty$

$$\gamma \cong \frac{1}{2}\frac{g_{\text{QGT}}^2 N}{M^2}(n_1^2 + n_2^2), \tag{3.160}$$

and again agrees with the string spectrum (3.86).

3.8.2 $\mathcal{N}=4$ MRV states

Let us leave out one-impurity states (as they are not that interesting on this side, and not that difficult to calculate either), and let us focus straight away on the $\mathcal{N}=4$ MRV states with two impurities

$$\mathcal{O}^{\mathcal{N}=4}_{n_1,n_2} = \sum_{\ell_2\geqslant\ell_1} \Psi_{\ell_1,\ell_2}\, \mathcal{O}^{\ell_1,\ell_2}_{n_1,n_2}, \tag{3.161}$$

where

$$\mathcal{O}^{\ell_1,\ell_2}_{n_1,n_2} = \mathrm{tr}\,[\mathfrak{U}^{\mathtt{m}} X^{\ell_1-1} \underset{\ell_1^{\mathrm{th}}}{Z}\, X^{\ell_2-\ell_1-1} \underset{\ell_2^{\mathrm{th}}}{Z}\, X^{L-\ell_2}] \tag{3.162}$$

and the plane wave ansatz has the same form as previously, that is (3.139). Also, using (3.16) we can circle around with the twist matrix $\mathfrak{U}^{\mathtt{m}}$, passing all the particles in the trace, yielding the same state with an overall phase

$$\omega^{\mathtt{m}(2-L)}, \tag{3.163}$$

and by requiring that it is equal to one so that the whole state does not vanish we arrive at

$$L = 0 \bmod M+2. \tag{3.164}$$

Since the fields are the ones of $\mathcal{N}=4$ SYM (only in the $SU(MN)$ representation) we can just use the very well known anomalous dimension matrix for $\mathcal{N}=4$ theory and write down the Schrödinger equations for (3.162)

$$\gamma\Psi_{\ell_1,\ell_2} = \frac{\lambda''}{8\pi^2}\left(4\Psi_{\ell_1,\ell_2} - \Psi_{\ell_1-1,\ell_2} - \Psi_{\ell_1+1,\ell_2} - \Psi_{\ell_1,\ell_2-1} - \Psi_{\ell_1,\ell_2+2}\right), \quad \ell_2 > \ell_1+1 \tag{3.165}$$

where[10] $\lambda'' := g^2_{\mathrm{YM}}NM$, and

$$\gamma\Psi_{\ell_1,\ell_1+1} = \frac{\lambda''}{8\pi^2}\left(2\Psi_{\ell_1,\ell_1+1} - \Psi_{\ell_1-1,\ell_1+1} - \Psi_{\ell_1,\ell_1+2}\right), \qquad \ell_2 = \ell_1+1. \tag{3.166}$$

Then, like previously, (3.165) yields

$$\gamma = \frac{\lambda''}{2\pi^2}\sum_{i=1}^{2}\sin^2\left(\frac{p_i}{2}\right) = \frac{\lambda''}{8\pi^2}\sum_{i=1}^{2}\frac{1}{u_i^2+1/4}, \tag{3.167}$$

where for future reference we introduced the so-called Bethe roots

$$u_j := \frac{1}{2}\cot\left(\frac{p_j}{2}\right), \tag{3.168}$$

and then the analogy of the equation (3.145) will read

$$2\Psi_{\ell_1,\ell_1+1} = \Psi_{\ell_1,\ell_1} + \Psi_{\ell_1+1,\ell_1+1}, \tag{3.169}$$

[10] I want to emphasise again that we are dealing with fields in the $SU(MN)$ representation. For that reason the action of the dilatation operator will now produce a factor of MN instead of N when contracting two, nearest-neighbor colour deltas $\delta^{\mathfrak{ab}}$.

implying that

$$\mathcal{S}(p_2,p_1) = -\frac{\mu_1\mu_2 - 2\mu_2 + 1}{\mu_1\mu_2 - 2\mu_1 + 1} = \mathcal{S}^{-1}(p_1,p_2), \tag{3.170}$$

and since

$$e^{ip_j} = \frac{\cot(p_j/2)+i}{\cot(p_j/2)-i} = \frac{u_j + i/2}{u_j - i/2}, \tag{3.171}$$

then

$$\mathcal{S}(p_2,p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}. \tag{3.172}$$

So far we have proceeded with close similarity to the $\mathcal{N}=2$ notation case, however now let us notice an important subtlety in this case, namely when a dilatation operator encounters the twist matrix between the fields it acts on. Obviously in the case of

$$\cdots X \mathfrak{U}^{\mathsf{m}} X \cdots \tag{3.173}$$

the permutation and the identity operators will give the same result when acting on the sequence above, making the anomalous dimension matrix vanish at this site, however the situation becomes more interesting when we have for example

$$\cdots Z \,\mathfrak{U}^{\mathsf{m}} X \cdots \tag{3.174}$$

since then

$$\mathcal{I} \circ (Z\,\mathfrak{U}^{\mathsf{m}} X) = Z\,\mathfrak{U}^{\mathsf{m}} X \tag{3.175}$$

$$\mathcal{P} \circ (Z\,\mathfrak{U}^{\mathsf{m}} X) = \omega^{-\mathsf{m}}\,(X \mathfrak{U}^{\mathsf{m}} Z), \tag{3.176}$$

therefore the dilatation matrix will be different than in the case of $\mathsf{m}=0$. However, the way to deal with this is to impose the proper boundary conditions on the basis in the following way. If we associate a vacuum particle X at site i with a down-spin site $|\downarrow\rangle_{i^{\rm th}}$, and the excitation particle Z at site i with an up-spin $|\uparrow\rangle_{i^{\rm th}}$, then the proper boundary conditions are defined in the following way

$$\begin{aligned} |\downarrow\rangle_{(L+1)^{\rm th}} &= |\downarrow\rangle_{1^{\rm st}}, \\ |\uparrow\rangle_{(L+1)^{\rm th}} &= \omega^{-\mathsf{m}}\,|\uparrow\rangle_{1^{\rm st}}. \end{aligned} \tag{3.177}$$

Then the dilatation operator becomes exactly the same as in the $\mathcal{N}=4$ case.

A completely equivalent way of looking at this is to reproduce the Schrödinger equation for $\mathcal{O}^{\ell_1,L}_{n_1,n_2}$, namely

$$\gamma\,\mathcal{O}^{\ell_1,L}_{n_1,n_2} = \frac{\lambda''}{8\pi^2}\left[4\mathcal{O}^{\ell_1,L}_{n_1,n_2} - \mathcal{O}^{\ell_1-1,L}_{n_1,n_2} - \mathcal{O}^{\ell_1+1,L}_{n_1,n_2} - \mathcal{O}^{\ell_1,L-1}_{n_1,n_2} - \mathcal{O}^{\ell_1,L+1}_{n_1,n_2}\right], \tag{3.178}$$

but then

$$\mathcal{O}^{\ell_1,L+1}_{n_1,n_2} = \omega^{-\mathsf{m}}\,\mathrm{tr}\left[\mathfrak{U}^{\mathsf{m}} Z X^{\ell_1-2} Z X^{L-\ell_1}\right] = \omega^{-\mathsf{m}}\,\mathcal{O}^{1,\ell_1}_{n_1,n_2}. \tag{3.179}$$

Imposing this boundary condition on (3.139) gives

$$\mu_2^L \left(\frac{\mu_1}{\mu_2}\right)^{\ell_1 - 1} + \; \mathcal{S}(p_2, p_1)\, \mu_1^L = \omega^{-\mathfrak{m}} \left[1 \;+\; \mathcal{S}(p_2, p_1) \left(\frac{\mu_1}{\mu_2}\right)^{\ell_1 - 1}\right], \tag{3.180}$$

hence the quantisation of momenta

$$(\mu_1 \mu_2)^L = \omega^{-2\mathfrak{m}}, \tag{3.181}$$

and

$$\mathcal{S}(p_1, p_2) = \mu_1^L \omega^{\mathfrak{m}} \quad \Longrightarrow \quad \mathcal{S}(p_1, p_2)\mathcal{S}(p_2, p_1) = (\mu_1 \mu_2)^L \omega^{2\mathfrak{m}} = 1. \tag{3.182}$$

Equations (3.172) and (3.182) yield

$$\omega^{\mathfrak{m}} \left(\frac{u_i + i/2}{u_i - i/2}\right)^L = \prod_{\substack{j=1 \\ j \neq i}}^{\mathcal{P}=2} \frac{u_i - u_j + i}{u_i - u_j - i}, \tag{3.183}$$

together with the translation condition

$$\omega^{\mathfrak{m}} \prod_{i=1}^{\mathcal{P}=2} \left(\frac{u_i + i/2}{u_i - i/2}\right) = 1, \tag{3.184}$$

giving the full set of Bethe equations. Above, $\mathcal{P}$ is the number of magnons (and here it is specifically equal to 2).

Finally, a few remarks. First of all, let us notice that in spite of difference between two approaches in calculating the Bethe equations for the MRV states - in the $\mathcal{N} = 2$ notation case, and in the $\mathcal{N} = 4$ notation case - the result (3.183) is exactly the same as the result (3.154), only keeping in mind that

$$\mu_j \equiv \frac{u_j + i/2}{u_j - i/2}, \tag{3.185}$$

and it leads to the same solution

$$\gamma \cong \frac{1}{2} \frac{g_{\mathrm{YM}}^2 N M}{M^2} (n_1^2 + n_2^2), \tag{3.186}$$

which is equal to (3.160), as (3.6) implies that $\lambda' = \lambda''$.

Secondly, although the comparison above has been done for $\boldsymbol{k} = 1$, it can be easily generalised for $\boldsymbol{k} > 1$, by simply rescaling $M \to \boldsymbol{k} M$ in both, the $\mathcal{N} = 2$ notation case and the $\mathcal{N} = 4$ notation case. One can then convince oneself that the most general free spectrum (3.86) can be reproduced.

Another thing is that one can notice that the same result as (3.160), and (3.186) could have been obtained by considering the phases $\omega^{\frac{1}{k}\sum_i n_i \ell_i}$ as in (3.105) instead of the plane-wave ansatz. The reason for that is that these phases are a naive Fourier tansform of the trace operators and merely reproduce the non-interactive spectrum of magnons. As long as we consider low order in perturbation theory, or a very small number of magnons, the assumption that they do not interact is reasonable. However, as soon as we add many magnons to the trace, or simply calculate higher loops, the phases in (3.105) will become useless for the comparison of the spectrum. For that very reason one uses the Bethe ansatz (3.139), as it allows for a relatively straightforward generalisation to a perturbative asymptotic Bethe ansatz (PABA), making the more precise calculation of the spectrum possible [64]. An explicit application of PABA to the case considered in this thesis, meaning type IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ and the dual $\mathcal{N}=2$ quiver gauge side, has been studied in [65]. There, the explicit three-loop spectrum comparison has been performed and the disagreement at three lopps has been found, exactly like in the case of parent $\mathcal{N}=4$ SYM theory.

At the end it is also worth pointing out that another, interesting limit, alternative to the one in [57], was found by Bertolini, de Boer, Harmark, Imeroni, and Obers (HIOBB) in [66]. It was shown there that the IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the new pp-wave limit with space-like circle can be quantised, and its states can be matched to the $\mathcal{N}=2$ QGT states. The most important difference between the HIOBB limit and the MRV limit is that in the former case the string spectrum acquires an extra, winding zero-mode, proportional to the string winding number $\mathfrak{m}$, and coming from the space-like circles of the pp-wave. However, while the string theory in this limit does not become much more complicated than in the case of MRV limit, the calculations on the gauge side are very difficult since now there are two fields building a ground state (say, A^a and B^a), making the field theory computations rather involved. This is also the reason why the gauge spectrum calculations in [66] have been perfomed for a modest (when compared to the MRV case) set of gauge states (c.f. [66] and [62, 65]).

Summary

In this chapter we have introduced the procedure of orbifolding the AdS/CFT duality on both sides. The type IIB string theory on $AdS_5 \times \mathbf{S}^5$ after orbifolding by the discrete group $\mathcal{G}=\mathbb{Z}_M$ now lives on the $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ orbifold, where $\mathcal{G}$ acts only on the $\mathbf{S}^5$ through embedding (3.2), thereby breaking its isometry group to $SU(2)_R \times U(1)$. Its gauge dual side is the $\mathcal{N}=2$ quiver gauge theory with the corresponding R-symmetry.

Basing on this knowledge, we have quantised the string theory $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the pp-wave limit, proceeding in full analogy to the $\mathcal{N}=4$ case, considered in section 1.4. The difference in our case, however, is that now the null direction becomes compactified, causing the strings to wrap $\mathfrak{m}$ times around it. The global symmetry arguments have led to establishment of the dictionary between the string states and the quiver gauge invariant operators; both of them possess an additional quantum number, being the string winding $\mathfrak{m}$. On top of that, we have shown that the quiver $\mathcal{N}=2$ MRV gauge states can be rewritten in an $\mathcal{N}=4$ notation without any loss of information about the original $\mathcal{N}=2$ state.

Having established the dictionary of orbifolded Maldacena's conjecture, we have also orbifolded the $\mathcal{N}=4$ one-loop dilatation operator to the $\mathcal{N}=2$ one-loop dilatation operator in the explicit $SU(2)_R \times U(1)$ invariant form, and acted with it on the $\mathcal{N}=2$ MRV states. Furthermore, we have shown that action of the original $\mathcal{N}=4$ one-loop dilatation operator on the $\mathcal{N}=4$ MRV states gives the same Bethe equations as in the $\mathcal{N}=2$ case, at least up to two impurities in the ground state, yielding the same one-loop anomalous dimension as the one predicted on the dual string side.

It is rather intuitive that the situation gets more interesting when increasing the level of magnonisation, as then we will not be able to neglect the interactions between the magnons. Fortunately, it turns out that integrability ("obtained" by mapping these states to the Hamiltonian of an integrable spin chain) implies that, for larger number of magnons than two, Bethe's equations will have exactly the same form as (3.183) and (3.184) only for $\mathcal{P} > 2$. Unfortunalety, solving them is another matter and a fully different approach has to be undertaken. We will deal with this problem in the next chapter.

It is also worth pointing out that the full set of one-loop Bethe equations for the $\mathbb{Z}_M$ orbifold of $\mathcal{N} = 4$ SYM has been worked out by symmetry arguments by Beisert and Robain in [67].

Chapter 4

Semiclassical spinning strings in $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$

Although quantisation of string theory in the pp-wave limit is a very nice way to test the AdS/CFT correspondence, it is most certainly a very specific one as well. As of today, it is still unknown how to quantise the string theory on $AdS_5 \times \mathbf{S}^5$ in the fully curved background, and it is a very bad news since this would provide us with an extremely serious test of the AdS/CFT duality. It has been noticed, however, that one could find classical solutions of the full string theory on $AdS_5 \times \mathbf{S}^5$ in curved space, and by taking the limit of large spins one can show that quantum effects are suppressed, and already the classical solution yields a good approximation for the full energy [68, 69, 70]. Furthermore, Frolov and Tseytlin argued in [6] that many of these spinning string solutions have an expansion which is in qualitative agreement with the loop expansion of gauge theory, and their conjecture has been confirmed in many cases since, for example in [73, 74, 76]. In this chapter, we will follow Ideguchi [75] in examining how do the semclassical solutions look for the orbifolded string theory on $AdS_5 \times \mathbf{S}^5$, namely on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$.

4.1 Strings rotating in $\mathbb{R}_t \times \mathbf{S}^5/\mathbb{Z}_M$

4.1.1 Geometry and symmetries

We will start with the Polyakov action for bosonic string theory

$$S = -\frac{1}{4\pi\alpha'} \iint d\tau d\sigma\, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}, \qquad \begin{array}{l} \alpha, \beta = 1, 2 \\ \mu, \nu = 1, ..., 10 \end{array} \tag{4.1}$$

where we specify $\eta^{\alpha\beta} = \mathrm{diag}(-1, 1)$ to be a conformally flat world-sheet metric, bosonic fields X^μ are functions of world sheet time and space $X^\mu = X^\mu(\tau, \sigma)$, and $G_{\mu\nu}$ is the spacetime metric. It is worth noticing that this action yields the following equations of motion

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta X^\mu = 0, \tag{4.2}$$

and using the "dot-prime" notation, namely

$$\dot{X} := \partial_\tau X, \tag{4.3}$$

$$X' := \partial_\sigma X, \tag{4.4}$$

one can calculate the variation with respect to the world-sheet metric, demanding that they hold regardless of the way we rescale it, obtaining (c.f. (1.30))

$$\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\,(-\dot{X}^2 + X'^2) = 0, \tag{4.5}$$

which gives to independent equations

$$\dot{X} \cdot X' = 0, \tag{4.6}$$
$$\dot{X}^2 + X'^2 = 0. \tag{4.7}$$

These two equations reflect the fact that the conformal invariance of the world-sheet requires that the total 2-dimensional energy momentum tensor vanishes.

The bosonic action (4.1) for $AdS_5 \times \mathbf{S}^5$ ought to manifest its bosonic isometry group[1] $SO(2,4) \times SO(6)$, hence

$$S_{AdS_5\times\mathbf{S}^5} = -\frac{R^2}{4\pi\alpha'}\iint d\tau d\sigma\left[G^{(AdS_5)}_{mn}\partial_\alpha X^m\partial^\alpha X^n + G^{(\mathbf{S}^5)}_{mn}\partial_\alpha Y^m\partial^\alpha Y^n\right], \tag{4.8}$$

where we split $X^\mu = \{X^m, Y^n\}$ for $m,n = 1,2,...,5$, and where $R^2 = \sqrt{4\pi g_s\alpha'^2 N}$. The metric we rewrite in global coordinates, exactly like in (3.47)

$$\begin{aligned} ds^2_{AdS_5} &= d\rho^2 - \cosh^2\!\rho\, dt^2 + \sinh^2\!\rho\left(d\psi^2 + \cos^2\!\psi\, d\varphi^2 + \sin^2\!\psi\, d\xi^2\right)\\ ds^2_{\mathbf{S}^5} &= d\alpha^2 + \sin^2\!\alpha\, d\theta^2 + \cos^2\!\alpha\left(d\gamma^2 + \cos^2\!\gamma\, d\chi^2 + \sin^2\!\gamma\, d\phi^2\right). \end{aligned} \tag{4.9}$$

Obviously, this metric arises from a parametrisation of the five sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1, \tag{4.10}$$

and the anti-de Sitter space

$$-y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_5^2 = -1, \tag{4.11}$$

by defining[2]

$$\begin{aligned} y_0 &= \cosh\rho\sin t & x_1 &= \cos\alpha\cos\gamma\cos\chi\\ y_1 &= \sinh\rho\cos\psi\cos\varphi & x_2 &= \cos\alpha\cos\gamma\sin\chi\\ y_2 &= \sinh\rho\cos\psi\sin\varphi & x_3 &= \cos\alpha\sin\gamma\cos\phi\\ y_3 &= \sinh\rho\sin\psi\cos\xi & x_4 &= \cos\alpha\sin\gamma\sin\phi\\ y_4 &= \sinh\rho\sin\psi\sin\xi & x_5 &= \sin\alpha\cos\theta\\ y_5 &= \cosh\rho\cos t & x_6 &= \sin\alpha\sin\theta, \end{aligned} \tag{4.12}$$

and thus the embedding coordinates are given by

$$X^m = (\rho, \psi, \varphi, \xi, t), \qquad Y^m = (\alpha, \gamma, \chi, \phi, \theta).$$

where $X^m = X^m(\tau,\sigma)$ and $Y^m = Y^m(\tau,\sigma)$.

[1] Due to the fact that we have already discussed orbifolding extensively in chapter (3) we will first describe the setup for spinning strings in $AdS_5 \times \mathbf{S}^5$ and then act with $\mathbb{Z}_M$. We are allowed to do this provided that we keep in mind all the consequences of orbifolding.

[2] Note that we rescaled the metric by $\frac{1}{R^2}$, which is exactly like we did a few times in section 1.1.

The on-shell generators of conserved charges are

$$S_{PQ} = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\, (y_P\, \dot{y}_Q - y_Q\, \dot{y}_P) = -S_{QP} \equiv \sqrt{\lambda}\, \mathcal{S}_{PQ}, \tag{4.13}$$

$$J_{MN} = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\, (x_M\, \dot{x}_N - x_N\, \dot{x}_M) = -J_{NM} \equiv \sqrt{\lambda}\, \mathcal{J}_{MN}, \tag{4.14}$$

where $P, Q = \{0,1,2,3,4,5\}$, and $M, N = \{1,2,3,4,5,6\}$, and where $\sqrt{\lambda} = R^2/\alpha'$. The fact that they are antisymmetric tells us that there are $15+15$ of them, and their Cartan subalgebra consists of $3+3$ generators of $SO(2,4)\times SO(6)$, corresponding to the $3+3$ linear isometries of the $AdS_5 \times \mathbf{S}^5$ metric (4.9), that is: the translations in AdS time t

$$S_{50} \equiv E = \sqrt{\lambda}\, \mathcal{E}, \tag{4.15}$$

translations in the AdS two angles φ and ξ

$$S_1 \equiv S_{12} = \sqrt{\lambda}\, \mathcal{S}_1, \qquad S_2 \equiv S_{34} = \sqrt{\lambda}\, \mathcal{S}_2, \tag{4.16}$$

and the rotations in three angles of S^5, namely χ, ϕ, and θ

$$J_1 \equiv J_{12} = \sqrt{\lambda}\, \mathcal{J}_1, \qquad J_2 \equiv J_{34} = \sqrt{\lambda}\, \mathcal{J}_2, \qquad J_3 \equiv J_{56} = \sqrt{\lambda}\, \mathcal{J}_3. \tag{4.17}$$

These conserved charges are (target-space) energy E, two spins S_1, and S_2, as well as three angular momenta J_1, J_2, J_3, and in the full quantum theory these should take quantised values. Therefore, for a solution to have a consistent, semiclassical approximation, that is to correspond to an eigenstate of the Hamiltonian which carries these quantum numbers, all other non-Cartan (non-commuting) components of the symmetry generators (4.13), (4.14) should vanish [69].

4.1.2 Semiclassical solutions

We are interested in closed strings located in the centre of AdS_5, with the time coordinate proportional to the world-sheet time τ, and rotating in $\mathbf{S}^5$ with three different spins, and for that we make the following ansatz for the embedding field components

$$\begin{array}{llll} t = \kappa\tau, & \rho = 0 = \psi = \varphi = \xi, & \alpha = \alpha_0, & \gamma = \gamma_0, \\ \chi = w_1\tau + m_1\sigma, & \phi = w_2\tau + m_2\sigma, & \theta = w_3\tau + m_3\sigma, & \end{array} \tag{4.18}$$

where α_0 and γ_0 are constants. Also, the closed string periodicity condition

$$x_M(\tau,\sigma) = x_M(\tau, \sigma + 2\pi) \tag{4.19}$$

requires that m_i, which count the number of times the string wraps around $\mathbf{S}^5$ in one of the three directions χ, ϕ, or θ, are integer.

Here it is a good point to introduce the orbifolding procedure; its consequences are practically the same as in chapter 3. First of all we need to go into the covering space and thus rescale $N \to NM$. Then, the global metric will still be (4.9), but the (dropped) radius-squared will read $R^2 = \sqrt{4\pi g_s \alpha'^2 NM}$.

Secondly, we remember that the embedding (3.2) implies the identification (3.50). This, simultaneously imposed with the periodicity condition (4.19), influences our ansatz in (4.18) such that (see [71])

$$m_1 = \tilde{m}_1 + \frac{\mathfrak{m}}{M}, \qquad m_2 = \tilde{m}_2 - \frac{\mathfrak{m}}{M}, \qquad m_3 = \tilde{m}_3, \tag{4.20}$$

($\mathfrak{m} = 0, 1, 2, ..., M-1$), where we first changed $m_i \to \tilde{m}_i$ in (4.18), and then defined these angles again

$$\chi = w_1\tau + m_1\sigma, \qquad \phi = w_2\tau + m_2\sigma, \qquad \theta = w_3\tau + m_3\sigma,$$

where the relation between m_i and $\tilde{m}_i$ is given by (4.20).

Another consequence of orbifolding is that in orbifolded theory we have to restrict string states to states that are invariant under the orbifolding action $e^{4\pi \tilde{J}_L} = e^{2\pi i(J_1 - J_2)}$ (c.f. section 3.5), hence the relation between two angular momenta

$$J_1 - J_2 = 0 \mod M. \tag{4.21}$$

It is very important to notice the following. String theories living on $\mathbf{S}^5$ sphere obey (bosonic) symmetry $SO(6)$, and hence three commuting Cartan currents J_1, J_2, and J_3. The action of $\mathbb{Z}_M$ breaks this symmetry to $SU(2) \times U(1)$, which we know has two Cartan generators. On the other hand, the movement of semiclassical strings spinning in $\mathbf{S}^5/\mathbb{Z}_M$ will still be described by three angular momenta J_i, therefore the equation (4.21) relates two of these to each other. The fact that this relation is ambiguous merely reflects the boundary condition (3.4) of $U(N)^M$.

With configuration (4.18) the embedding coordinates become

$$\begin{array}{c} y_0 = \sin(\kappa\tau) \\ y_1 = y_2 = y_3 = y_4 = 0 \\ y_5 = \cos(\kappa\tau) \end{array} \qquad \begin{array}{l} x_1 = \cos\alpha_0 \cos\gamma_0 \cos(w_1\tau + m_1\sigma) \\ x_2 = \cos\alpha_0 \cos\gamma_0 \sin(w_1\tau + m_1\sigma) \\ x_3 = \cos\alpha_0 \sin\gamma_0 \cos(w_2\tau + m_2\sigma) \\ x_4 = \cos\alpha_0 \sin\gamma_0 \sin(w_2\tau + m_2\sigma) \\ x_5 = \sin\alpha_0 \cos(w_3\tau + m_3\sigma) \\ x_6 = \sin\alpha_0 \sin(w_3\tau + m_3\sigma), \end{array} \tag{4.22}$$

and the σ-model action for semiclassical strings on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ will turn into

$$S = -\frac{R^2}{4\pi\alpha'} \iint d\tau d\sigma \left[L_{AdS_5} + L_{\mathbf{S}^5/\mathbb{Z}_M}\right], \tag{4.23}$$

where $R^2 = \sqrt{4\pi g_s \alpha'^2 NM}$, and

$$\begin{aligned} L_{AdS_5} &= \eta^{PQ}\partial_\alpha y_P\, \partial^\alpha y_Q + \widetilde{\Lambda}\left(\eta^{PQ} y_P\, y_Q + 1\right), \qquad && P, Q = \{0, 1, 2, 3, 4, 5\} \\ L_{\mathbf{S}^5/\mathbb{Z}_M} &= \partial_\alpha x_M \partial^\alpha x_M + \Lambda\left(x_M x_M - 1\right). && M = 1, 2, 3, 4, 5, 6 \end{aligned}$$

Above, $\eta^{PQ} = \eta_{PQ} = \mathrm{diag}(-1, +1, +1, +1, +1, -1)$, and Λ and $\widetilde{\Lambda}$ are Lagrange multipliers which are inserted in (4.23) by hand in order to reproduce (4.10) and (4.11) through the equations of motion. The equations of motion for the AdS_5 part are

$$(\partial^2 - \widetilde{\Lambda}) y_P = 0, \tag{4.24}$$

$$\eta^{PQ}\, y_P\, y_Q = -1, \tag{4.25}$$

among which (4.24) yields that

$$\widetilde{\Lambda} = \eta^{PQ}\partial_\alpha y_P\, \partial^\alpha y_Q = \kappa^2, \tag{4.26}$$

and (4.25) reproduces (4.11). The $\mathbf{S}^5/\mathbb{Z}_M$ part gives the following equations of motion

$$(\partial^2 - \Lambda)x_M = 0, \tag{4.27}$$
$$x_M x_M = 1. \tag{4.28}$$

Again, (4.28) trivially satisfies (4.10), and (4.27) can be most easily solved by considering complex coordinates

$$(\partial^2 - \Lambda)z_I = 0, \qquad I = 1,2,3, \tag{4.29}$$

where

$$z_1 \equiv x_1 + ix_2, \qquad z_2 \equiv x_3 + ix_4, \qquad z_3 \equiv x_5 + ix_6,$$

which are nothing but (3.48), rescaled by R in accordance with (4.9). Then, using (4.18) we obtain

$$\Lambda = \partial_\alpha x_M\, \partial^\alpha x_M = w_1^2 - m_1^2 = w_2^2 - m_2^2 = w_3^2 - m_3^2 \equiv -\nu^2. \tag{4.30}$$

On top of that the conformal invariance of the world-sheet requires that the total 2-dimensional energy-momentum tensor vanishes (that is in full analogy to (4.6) and (4.7)), namely

$$\eta^{PQ}(\dot{y}_P\, \dot{y}_Q + y'_P\, y'_Q) + \dot{x}_M \dot{x}_M + x'_M x'_M = 0, \tag{4.31}$$

and

$$\eta^{PQ}\dot{y}_P\, y'_Q + \dot{x}_M x'_M = 0. \tag{4.32}$$

Of these conditions, (4.31), together with (4.18), gives that

$$-\kappa^2 + (w_1^2 + m_1^2)\sin^2\!\alpha_0 \cos^2\!\gamma_0 + (w_2^2 + m_2^2)\sin^2\!\alpha_0 \sin^2\!\gamma_0 + (w_3^2 + m_3^2)\cos^2\!\alpha_0 = 0, \tag{4.33}$$

and (4.32) that

$$w_1\, m_1 \sin^2\!\alpha_0 \cos^2\!\gamma_0 + w_2\, m_2 \sin^2\!\alpha_0 \sin^2\!\gamma_0 + w_3\, m_3 \cos^2\!\alpha_0 = 0. \tag{4.34}$$

This yields all the (semclassical) dynamics for a string moving in $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ we need in order to solve the relevant charges as functions of the quantities introduced in (4.18). Before we proceed, however, let us now stop for a moment and say something about the integrability of our model.

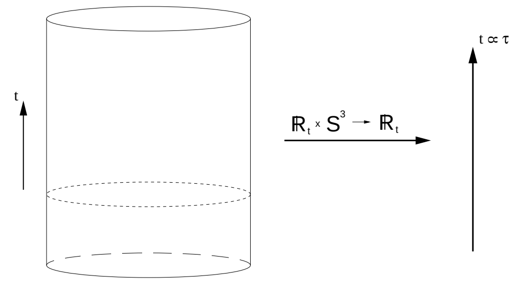

Figure 4.1: By considering AdS_5 space (which can be viewed as a bulk cylinder with boundary $\mathbb{R}_t \times S^3$) with $\rho = \psi = \varphi = \xi = 0$, and taking conformal symmetry under consideration, we come to its very simple representation as a one-dimensional $\mathbb{R}_t$, with time coordinate proportional to the world-sheet time $t \propto \tau$.

4.1.3 Remark on integrability

The equations of motion for the σ-model (4.23) and ansatz (4.18), together with conformal constraints (4.31) and (4.32), can be rewritten with the help of new coordinates

$$\sigma^{\pm} := \frac{1}{2}(\tau \pm \sigma), \qquad \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}, \tag{4.35}$$

as

$$\partial_+\partial_- x_M - (\partial_+ x_N \partial_- x_N) x_M = 0, \qquad \partial_+\partial_- t = 0, \tag{4.36}$$
$$(\partial_+ x_M)^2 = (\partial_+ t)^2, \qquad (\partial_- x_M)^2 = (\partial_- t)^2. \tag{4.37}$$

The equation of motion $\partial_+\partial_- t$ can be solved most generally as

$$t = f_+(\tau + \sigma) + f_-(\tau - \sigma), \tag{4.38}$$

Also, the equations (4.36) and (4.37) are invariant under conformal transformation $\sigma^{\pm} \to \xi_{\pm}(\sigma^{\pm})$, where $\xi_{\pm}$ is an arbitrary function, so given a solution $x_M(\sigma^+, \sigma^-)$ one can always find a new solution $\tilde{x}_M(\sigma^+, \sigma^-) = x_M(\xi_+(\sigma^+), \xi_-(\sigma^-))$. For that reason we can rescale $(\partial_+ x_M)(\partial_+ x_M)$ and $(\partial_- x_M)(\partial_- x_M)$ such that they are equal to some constant, which is equivalent to setting

$$f_+(\tau + \sigma) = \frac{1}{2}\kappa(\tau - \sigma), \qquad f_-(\tau - \sigma) = \frac{1}{2}\kappa(\tau - \sigma), \qquad \kappa = \text{const.} \tag{4.39}$$

Hence

$$t = \kappa\tau. \tag{4.40}$$

We therefore see that this time-ansatz in (4.18) is a straightforward consequence of restricting the motion to the $\mathbb{R}_t$ part of AdS, together with conformal residual symmetry of the string world-sheet (see figure 4.1).

To describe the $\mathbf{S}^5$ part better, let us introduce an $O(6)$ vector

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \tag{4.41}$$

and define a matrix

$$\mathfrak{g}_{MN} := e^{i\pi \mathfrak{P}_{MN}}, \tag{4.42}$$

where

$$\mathfrak{P} = \vec{X} \cdot \vec{X}^t, \qquad \text{or} \qquad \mathfrak{P}_{MN} = x_M x_N \tag{4.43}$$

is a projector, due to $x_M x_M = 1$, and hence

$$\mathfrak{g}_{MN} = 1 - 2\mathfrak{P}_{MN}. \tag{4.44}$$

Since $\mathfrak{g} \in O(6)$, it is orthogonal

$$(\mathfrak{g})_{MN}\, (\mathfrak{g}^t)_{NP} = \delta_{MP},$$

but also symmetric

$$(\mathfrak{g})_{MN} = (\mathfrak{g}^t)_{MN},$$

hence

$$(\mathfrak{g}^t)_{MN} = (\mathfrak{g}^{-1})_{MN}. \tag{4.45}$$

Then we define conserved currents in $O(6)$

$$\mathfrak{j}^{(1)}_\alpha \equiv \mathfrak{g}^{-1}\partial_\alpha \mathfrak{g}, \qquad \partial^\alpha \mathfrak{j}^{(1)}_\alpha = 0, \tag{4.46}$$

which in terms of their components are

$$(\,\mathfrak{j}^{(1)}_\alpha)_{MN} = 2(x_M \partial_\alpha x_N - x_N \partial_\alpha x_M). \tag{4.47}$$

The conserved current also satisfies the flatness condition, which in terms of $\sigma^\pm$ coordinates can be written as

$$\partial_+\, \mathfrak{j}^{(1)}_- - \partial_-\, \mathfrak{j}^{(1)}_+ + [\,\mathfrak{j}^{(1)}_+,\, \mathfrak{j}^{(1)}_-] = 0. \tag{4.48}$$

Then, defining

$$D_\alpha \equiv \partial_\alpha + \mathfrak{j}^{(1)}_\alpha$$

and having

$$[D_\alpha,\, \mathfrak{j}^{(1)}_\alpha] = 0, \tag{4.49}$$

one can construct an infinite "tower" of conserved non-local currents $\mathfrak{j}_\alpha^{(n)}$ by the following iterative procedure

$$\mathfrak{j}_\alpha^{(n)} = \epsilon_{\alpha\beta}\,\partial^\beta \chi^{(n)}, \qquad \mathfrak{j}_\alpha^{(n+1)} = D_\alpha \chi^{(n)}, \qquad \chi^{(0)} = 1. \tag{4.50}$$

And then we get an infinite set of conserved charges

$$\mathfrak{Q}^{(n)} \equiv \int_0^{2\pi} d\sigma\, \mathfrak{j}_\tau^{(n)}(\tau,\sigma), \tag{4.51}$$

where we can see that for example that (c.f. (4.14))

$$\mathfrak{Q}^{(1)}_{MN} = 2\int_0^{2\pi} d\sigma (x_M \partial_\tau x_N - x_N \partial_\tau x_M) = 4\pi \mathcal{J}_{MN}. \tag{4.52}$$

The infinite set of non-local, conserved Nöther charges in (4.51) is a sign of integrability of semiclassical string theory on $\mathbb{R}_t \times \mathbf{S}^5$ [72]. We also see that since the main implication of orbifolding in this model is the identification (4.20), the orbifolding procedure is "absorbed" in coordinates x_M. The infinte set of charges (4.51) can thus be always reproduced with the new coordinates x_M, and therefore we expect that the integrability of this model cannot be spoiled by $\mathbb{Z}_M$.

4.1.4 Semiclassical solutions continued

Now we would like to "compress" all the results into the expressions for conserved charges. Since we specify the motion of strings to $\mathbb{R}_t \times \mathbf{S}^5/\mathbb{Z}_M$ we expect that the only non-vanishing conserved charges will be the energy E, and three angular momenta J_I, $I = 1, 2, 3$, and indeed using (4.13 - 4.17) one can convince oneself that for our ansatz (4.18) two spins of AdS_5 will vanish, leaving all the other charges to be

$$\begin{aligned} \mathcal{E} &= \kappa, \\ \mathcal{J}_1 &= \sin^2\!\alpha_0 \cos^2\!\gamma_0\, w_1, \\ \mathcal{J}_2 &= \sin^2\!\alpha_0 \sin^2\!\gamma_0\, w_2, \\ \mathcal{J}_3 &= \cos^2\!\alpha_0\, w_3, \end{aligned} \tag{4.53}$$

with the help of which (and (4.30)) we can rewrite (4.33) and (4.34) as

$$\begin{aligned} \mathcal{E}^2 &= 2w_1^2 \sin^2\!\alpha_0 \cos^2\!\gamma_0 + 2w_2^2 \sin^2\!\alpha_0 \sin^2\!\gamma_0 + 2w_3^2 \cos^2\!\alpha_0 - \nu^2 \\ &= 2\sum_{I=1}^3 w_I \mathcal{J}_I - \nu^2 = 2\sum_{I=1}^3 \sqrt{m_I^2 + \nu^2}\, J_I - \nu^2, \end{aligned} \tag{4.54}$$

and

$$m_1 \mathcal{J}_1 + m_2 \mathcal{J}_2 + m_3 \mathcal{J}_3 = 0, \tag{4.55}$$

respectively. On top of that, (4.30) itself can be rewritten as

$$\sum_{I=1}^3 \frac{\mathcal{J}_I}{\sqrt{m_I^2 + \nu^2}} = 1. \tag{4.56}$$

Now the aim is to express the energy $\mathcal{E}$ in terms of angular momenta $\mathcal{J}_I$ and winding numbers m_I. In order to do this we first need to solve (4.56) in terms of ν, determining ν as a function of $\mathcal{J}_I$ and m_I, and then substitute the result into (4.54). The conditions (4.55) will be imposed at the very end, implying that for a given angular momenta $\mathcal{J}_I$ the solution exists only for a special choice of the integers m_I.

Thus we first solve (4.56). Assuming that $\nu^2 > m_I^2$, one can expand the series (4.56) in $m_I^2/\nu^2 < 1$, with the total angular momentum

$$\mathcal{J} \equiv \sum_{I=1}^{3} \mathcal{J}_I, \tag{4.57}$$

as

$$|\nu| = \sum_{I=1}^{3} \frac{\mathcal{J}_I}{\sqrt{1+\frac{m_I^2}{\nu^2}}} = \mathcal{J}\left(1 - \frac{1}{2\nu^2}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} + \frac{3}{8\nu^4}\sum_{I=1}^{3} m_I^4 \frac{\mathcal{J}_I}{\mathcal{J}} - \dots\right),$$

and then eliminate ν from the right-hand side of the expansion, replacing it with $\mathcal{J}$, obtaining

$$|\nu| = \mathcal{J}\left[1 - \frac{1}{2\mathcal{J}^2}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} + \frac{3}{8\mathcal{J}^4}\sum_{I=1}^{3} m_I^4 \frac{\mathcal{J}_I}{\mathcal{J}} - \frac{1}{2\mathcal{J}^4}\left(\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}}\right)^2 + \dots\right]. \tag{4.58}$$

Furthermore, since the energy (4.54) can also be expanded in $m_I^2/\nu^2 < 1$ as

$$\begin{aligned}\mathcal{E}^2 &= 2\sum_{I=1}^{3}\sqrt{m_I^2+\nu^2}\, J_I - \nu^2 \\ &= 2|\nu|\mathcal{J}\left(1 + \frac{1}{2\nu^2}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} - \frac{1}{8\nu^4}\sum_{I=1}^{3} m_I^4 \frac{\mathcal{J}_I}{\mathcal{J}} + \dots\right) - \nu^2,\end{aligned} \tag{4.59}$$

we can again get rid of $|\nu|$ by expressing it in terms of $\mathcal{J}$ using (4.58), getting

$$\mathcal{E}^2 = \mathcal{J}^2\left[1 + \frac{1}{\mathcal{J}^2}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} - \frac{1}{4\mathcal{J}^4}\sum_{I=1}^{3} m_I^4 \frac{\mathcal{J}_I}{\mathcal{J}} + \frac{1}{4\mathcal{J}^4}\left(\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}}\right)^2 + \dots\right],$$

or

$$\mathcal{E} = \mathcal{J}\left[1 + \frac{1}{2\mathcal{J}^2}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} - \frac{1}{8\mathcal{J}^4}\sum_{I=1}^{3} m_I^4 \frac{\mathcal{J}_I}{\mathcal{J}} + \dots\right], \tag{4.60}$$

where the term $\sim \left(\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}}\right)^2$ has vanished. As we are interested, however, in the first order expansion, the expression we will try to solve will be

$$\mathcal{E} = \mathcal{J} + \frac{1}{2\mathcal{J}}\sum_{I=1}^{3} m_I^2 \frac{\mathcal{J}_I}{\mathcal{J}} + \dots, \tag{4.61}$$

with the help of the constraint

$$\sum_{I=1}^{3} m_I \mathcal{J}_I = 0, \tag{4.62}$$

where we take both, $\mathcal{E}$ and $\mathcal{J}$ to be large such that

$$\mathcal{E} - \mathcal{J} = \text{fixed}, \quad \text{and} \quad \frac{1}{\mathcal{J}^2} = \frac{g_{\mathrm{QGT}}^2 N}{L^2} \sim \frac{g_{\mathrm{QGT}}^2 N}{M^2} = \text{fixed}, \tag{4.63}$$

and then expand in small $\mathcal{J}^{-2}$, thus exactly like in the case of (3.86). Alternatively, using (4.15) and (4.17), together with $L \equiv \sqrt{\lambda}\,\mathcal{J}$,

$$\begin{aligned} E &= L + \frac{\lambda}{2L^2} \sum_{I=1}^{3} m_I^2 J_I + \dots, \\ \sum_{I=1}^{3} m_I J_I &= 0. \end{aligned} \tag{4.64}$$

4.1.5 Specification to $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$

Now we will specify the energy solutions to $SU(2)$ subsector, namely consider strings spinning in $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$, by returning to the doublets defined in (2.106). Let us first discuss the doublet consisting of the particles considered in section 3.8

$$\phi_\alpha^{SU(2)_Z} = \begin{pmatrix} X \\ Z \end{pmatrix}.$$

On the field theory side, by an argument of classical scaling dimensions and charges, the conformal operators composed of these doublets should have J_1 X's, and J_3 Z's. On the string dual side we thus expect $\mathcal{J}_1, \mathcal{J}_3 \neq 0$, and $\mathcal{J}_2 = 0$. Since we will be viewing these states as built of X particles, and "magnonised" with Z's, then obviously $\mathcal{J}_1 > \mathcal{J}_3$, but also $\mathcal{J}_3 > 0$. The anomalous dimension, together with the constraint (4.55), will now read

$$\widetilde{\gamma} \equiv \mathcal{E} - \mathcal{J} = \frac{1}{2\mathcal{J}^2}(m_1^2 \mathcal{J}_1 + m_3^2 \mathcal{J}_3), \tag{4.65}$$

$$m_1 \mathcal{J}_1 + m_3 \mathcal{J}_3 = 0. \tag{4.66}$$

Equation (4.66) tell us that since $\mathcal{J}_1, \mathcal{J}_3 > 0$, then $\operatorname{sgn}(m_1 m_3) \leqslant 0$. Then, knowing that $\mathcal{J}_1 + \mathcal{J}_3 = \mathcal{J}$, we obtain from (4.66)

$$\mathcal{J}_1 = \frac{m_3}{m_3 - m_1}\mathcal{J}, \qquad \mathcal{J}_3 = \frac{m_1}{m_1 - m_3}\mathcal{J},$$

and insert in (4.65) to get

$$\widetilde{\gamma}_{SU(2)_Z} = -\frac{1}{2\mathcal{J}} m_1 m_3,$$

which has to be non-negative. Thus if $\gamma = \sqrt{\lambda}\,\widetilde{\gamma}$, then

$$\gamma_{SU(2)_Z} = E - L = -\frac{\lambda}{2L} m_1 m_3. \tag{4.67}$$

Finally, (4.20) yields that

$$\gamma_{SU(2)_Z} = \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{\mathfrak{m}}{M}\right)\tilde{m}_3, \tag{4.68}$$

where we let $\tilde{m}_3 \to -\tilde{m}_3$, so that $\gamma \geqslant 0$.

In order to calculate now the energy correction for the $SU(2)_R$ doublet-case

$$\chi_\alpha^{SU(2)_R} = \begin{pmatrix} X \\ \bar{Y} \end{pmatrix}$$

we proceed in full analogy to the previous case. Gauge dual states have J_1 X's, and $|J_3|$ $\bar{Y}$'s, hence on the string side we expect $\mathcal{J}_1, \mathcal{J}_2 \neq 0$, and $\mathcal{J}_3 = 0$. Furthermore $\mathcal{J}_1 > 0$, $\mathcal{J}_2 < 0$, hence $\mathrm{sgn}(m_1 m_2) \geqslant 0$, and for that reason

$$\gamma = \frac{\lambda}{2L} m_1 m_2.$$

Then, using (4.20) we get

$$\gamma_{SU(2)_R} = \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{\mathfrak{m}}{M}\right)\left(\tilde{m}_2 - \frac{\mathfrak{m}}{M}\right). \tag{4.69}$$

Finally, the $SU(2)_L$-case, that is when gauge states are composed of the $SU(2)_L$ doublets

$$\psi_\alpha^{SU(2)_L} = \begin{pmatrix} X \\ Y \end{pmatrix},$$

and having thus J_1 X's, and J_2 Y's. Since $\mathcal{J}_1 > \mathcal{J}_2 > 0$, and $\mathcal{J}_3 = 0$, the winding momenta fulfil $\mathrm{sgn}(m_1 m_2) \leqslant 0$, and

$$\gamma_{SU(2)_L} = \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{\mathfrak{m}}{M}\right)\left(\tilde{m}_2 + \frac{\mathfrak{m}}{M}\right), \tag{4.70}$$

where we let $\tilde{m}_2 \to -\tilde{m}_2$. This concludes the lowest order ($\sim g^2$) calculation of circular 2-spin semiclassical energy solutions of our strings in $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$. We now proceed to the dual gauge theory side.

4.2 The dual $\mathcal{N}=2$ gauge theory side

In order to compare the results from the previous section, we need to calculate anomalous dimensions as a function of angular momenta $\gamma = \gamma(J_I)$ of the gauge operators dual to the strings spinning in $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$. We will therefore generalise the procedure of diagonalising the $\mathcal{N}=2$ dilatation matrix from section 3.8 by examining the case of $\mathcal{P} > 2$. For that reason, interactions between magnons (even though we still take $M \to \infty$) cannot be neglected, and the Bethe ansatz procedure will not produce the same eigenvalues as the free-magnon phases as in (3.104), or (3.105). Instead, we will use so-called rational solutions and take the thermodynamic limit by letting $L \to \infty$, but since Bethe roots scale like $u_i \sim L$, the Bethe equations will become ambiguous; ambiguity for each of $\mathcal{P}$ equations will be reflected by existence of (by assumptions) finite numbers of so called lattice momenta $\tilde{n}_j$ ($j = 1, ..., \mathcal{P}$) of each of the j^{th} magnon. We will shortly examine this more thoroughly.

Let us first try to derive Bethe's equations for an arbitrary number of magnons $\mathcal{P}$. In fact, it will not be a derivation as much as it will be a proof for I have already mentioned in the end of chapter 3 that they will be exactly the same as in the $\mathcal{P}=2$ case. Although it is rather intuitive that this should be the case (a j^{th} magnon travelling around a chain will pass the twist matrix only once, leaving the same phase regardless of which magnon we will push around), it is certainly not that obvious to prove it. Let us do it then, for the case of Z-impurity, like in section 3.8.

4.2.1 Bethe equations for the full $SU(2)$ subsector

Before presenting the proof, let me just say that when deriving Bethe's equations for the $\mathcal{N}=4$ MRV states in section 3.8 we have used the boundary condition (3.179). Instead, we could have defined these states such that they obey the boundary condition for the $\mathcal{N}=2$ MRV states (3.147), getting exactly the same result (3.183); this means that these approaches are equivalent. In this section we will undertake the second approach and try to solve the $\mathcal{P}$-magnon case for $\mathcal{N}=4$ MRV states by imposing the boundary condition (3.147).

Let us define the $\mathcal{P}$-magnon $\mathcal{N}=2$ MRV and $\mathcal{N}=4$ MRV states

$$\mathcal{O}^{\mathcal{N}=2}_{n_1,\dots,n_\mathcal{P}} = \sum_{\ell_\mathcal{P}\geqslant\dots\geqslant\ell_1} \Psi_{\ell_1,\ell_2,\dots,\ell_\mathcal{P}}\, \mathcal{O}^{\ell_1,\dots,\ell_\mathcal{P}}_{n_1,\dots,n_\mathcal{P}}, \tag{4.71}$$

$$\mathcal{O}^{\mathcal{N}=4}_{n_1,\dots,n_\mathcal{P}} = \sum_{\ell_\mathcal{P}\geqslant\dots\geqslant\ell_1} \Psi_{\ell_1,\ell_2,\dots,\ell_\mathcal{P}}\, \widetilde{\mathcal{O}}^{\ell_1,\dots,\ell_\mathcal{P}}_{n_1,\dots,n_\mathcal{P}}, \tag{4.72}$$

where[3]

$$\mathcal{O}^{\ell_1,\dots,\ell_\mathcal{P}}_{n_1,\dots,n_\mathcal{P}} = \operatorname{tr}\left(A^1\cdots A^{\ell_1-1}\Phi^{\ell_1}A^{\ell_1}\cdots A^{\ell_2-1}\Phi^{\ell_2}A^{\ell_2}\cdots A^{\ell_\mathcal{P}-1}\Phi^{\ell_\mathcal{P}}A^{\ell_\mathcal{P}}\cdots A^M\right), \tag{4.73}$$

and

$$\widetilde{\mathcal{O}}^{\ell_1,\dots,\ell_\mathcal{P}}_{n_1,\dots,n_\mathcal{P}} = \operatorname{tr}\left[\mathfrak{U}^{\mathfrak{m}}X^{\ell_1-1}ZX^{\ell_2-\ell_1}Z\cdots ZX^{M-\ell_\mathcal{P}+1}\right], \tag{4.74}$$

and in this way the both states obey the same boundary conditions (here for the 1^{st} magnon)

$$\begin{pmatrix}\mathcal{O}\\ \widetilde{\mathcal{O}}\end{pmatrix}^{\ell_2,\ell_3,\dots,\ell_\mathcal{P},\ell_1+M}_{n_1,\dots,n_\mathcal{P}} = \begin{pmatrix}\mathcal{O}\\ \widetilde{\mathcal{O}}\end{pmatrix}^{\ell_1,\dots,\ell_\mathcal{P}}_{n_1,\dots,n_\mathcal{P}} \tag{4.75}$$

which will be imposed on the plane-wave ansatz $\Psi_{\ell_1,\ell_2,\dots,\ell_\mathcal{P}}$. Now, if this system is integrable (and we assume it is since we map it to the XXX Heisenberg model as mentioned in section 3.8), we can with clear conscience presuppose it is non-diffractive, and impose the following ansatz

$$\Psi_{\ell_1,\ell_2,\dots,\ell_\mathcal{P}} = \sum_{\sigma\in\mathrm{Perm}(\mathcal{P})} \exp\left[i\sum_{i=1}^{\mathcal{P}} p_{\sigma(i)}\ell_i + \frac{i}{2}\sum_{i<j}\theta_{\sigma(i),\sigma(j)}\right] \equiv \sum_{\sigma\in\mathrm{Perm}(\mathcal{P})} \Psi_{\ell_1,\ell_2,\dots,\ell_\mathcal{P}}(\sigma), \tag{4.76}$$

as a non-diffractive generalisation of (3.139). Above,

$$e^{i\theta_{ij}} \equiv \mathcal{S}(p_i,p_j). \tag{4.77}$$

[3] Again, although we put $\boldsymbol{k}=1$ here, the case of an arbitrary $\boldsymbol{k}$ can be obtained by rescaling $M\to\boldsymbol{k}M$.

Now, the boundary condition[4] reads

$$\Psi_{\ell_1,\ell_2,\ldots,\ell_{\mathcal{P}}} = \Psi_{\ell_2,\ldots,\ell_{\mathcal{P}},\ell_1+M} \equiv \sum_{\sigma\in\mathrm{Perm}(\mathcal{P})} \Psi_{\ell_2,\ldots,\ell_{\mathcal{P}},\ell_1+M}(\sigma), \tag{4.78}$$

where

$$\Psi_{\ell_2,\ldots,\ell_{\mathcal{P}},\ell_1+M}(\sigma') = \exp\left[i\sum_{i=2}^{\mathcal{P}} p_{\sigma'(i-1)}\ell_i + p_{\sigma'(\mathcal{P})}(\ell_1+M) + \frac{i}{2}\sum_{i<j}\theta_{\sigma'(i),\sigma'(j)}\right]. \tag{4.79}$$

Having thus two different expressions we need to fix the terms on both sides of (4.78), such that $p_{\sigma'(i-1)} = p_{\sigma(i)}$, and thus we demand that

$$\begin{array}{ll} \sigma'(i-1) = \sigma(i), & \qquad i = 2,3,\ldots,\mathcal{P} \\ \sigma'(\mathcal{P}) = \sigma(1). & \end{array} \tag{4.80}$$

Then we can use the fact that

$$\begin{aligned} \sum_{i=1}^{\mathcal{P}-1}\sum_{j>i}^{\mathcal{P}}\theta_{\sigma(i),\sigma(j)} &= \sum_{j=2}^{\mathcal{P}}\theta_{\sigma(1),\sigma(j)} + \sum_{i=2}^{\mathcal{P}-1}\sum_{j>i}^{\mathcal{P}}\theta_{\sigma(i),\sigma(j)} \\ &= \sum_{j=2}^{\mathcal{P}}\theta_{\sigma(1),\sigma(j)} + \sum_{i=1}^{\mathcal{P}-2}\sum_{j>i}^{\mathcal{P}-1}\theta_{\sigma'(i),\sigma'(j)} \\ &= \sum_{j=2}^{\mathcal{P}}\theta_{\sigma(1),\sigma(j)} + \sum_{i=1}^{\mathcal{P}-1}\sum_{j>i}^{\mathcal{P}}\theta_{\sigma'(i),\sigma'(j)} - \sum_{i=1}^{\mathcal{P}-1}\theta_{\sigma'(i),\sigma'(\mathcal{P})} \\ &= \sum_{i<j}\theta_{\sigma'(i),\sigma'(j)} + 2\sum_{i=2}^{\mathcal{P}}\theta_{\sigma(1),\sigma(i)}, \end{aligned}$$

hence

$$\begin{aligned} \Psi_{\ell_2,\ldots,\ell_{\mathcal{P}},\ell_1+M}(\sigma') &= \exp\left[i\sum_{i=2}^{\mathcal{P}} p_{\sigma'(i-1)}\ell_i + p_{\sigma'(\mathcal{P})}(\ell_1+M) + \frac{i}{2}\sum_{i<j}\theta_{\sigma'(i),\sigma'(j)}\right] \\ &= \exp\left[i\sum_{i=1}^{\mathcal{P}} p_{\sigma(i)}\ell_i + p_{\sigma(1)}M + \frac{i}{2}\left(\sum_{i<j}\theta_{\sigma(i),\sigma(j)} - 2\sum_{i=2}^{\mathcal{P}}\theta_{\sigma(1),\sigma(i)}\right)\right], \end{aligned}$$

and by the use of (4.78) we obtain an $SU(2)$ eigenvalue equation

$$e^{ip_1M} = \exp\left(i\sum_{i=2}^{\mathcal{P}}\theta_{1,i}\right) = \prod_{i=2}^{\mathcal{P}}\mathcal{S}(p_1,p_i), \tag{4.81}$$

which can be easily generalised to an arbitrary j^{th} magnon

$$e^{ip_jM} = \exp\left(i\sum_{\substack{i=1\\ i\neq j}}^{\mathcal{P}}\theta_{ji}\right) = \prod_{\substack{i=1\\ i\neq j}}^{\mathcal{P}}\mathcal{S}(p_j,p_i). \tag{4.82}$$

[4] In fact it is one of $\mathcal{P}$ boundary conditions, however in practice they are all identical.

On top of that we have a translation condition (which is a $\mathcal{P}$-magnon version of (3.132))

$$\mathcal{U}|\mathrm{phys}\rangle = \omega^{-\mathfrak{m}}|\mathrm{phys}\rangle, \tag{4.83}$$

or

$$\Psi_{\ell_1+1,\ell_2+1,\ldots,\ell_{\mathcal{P}}+1} = \omega^{-\mathfrak{m}}\,\Psi_{\ell_1,\ell_2,\ldots,\ell_{\mathcal{P}}}, \tag{4.84}$$

which using (4.76) gives

$$\prod_{i=1}^{\mathcal{P}} \mu_i = \prod_{i=1}^{\mathcal{P}} e^{ip_i} = \omega^{-\mathfrak{m}}. \tag{4.85}$$

Now we derive the Schrödinger equations in complete analogy to (3.142) and (3.143), only that now we have $(\mathcal{P}-1)$ of them. They yield one-loop correction of the conformal dimension

$$\gamma = \frac{\lambda'}{8\pi^2}\sum_{i=1}^{\mathcal{P}}\frac{1}{u_i^2+1/4}, \tag{4.86}$$

where $\lambda' = g^2_{\mathrm{QGT}}N = g^2_{\mathrm{YM}}MN$, and the $\mathcal{P}$ magnon $\mathcal{S}$-matrix

$$\mathcal{S}_{ji}(u) = \frac{\mu_i}{\mu_j}\cdot\frac{u_i-u_j+i}{u_i-u_j-i}, \tag{4.87}$$

where we redefined

$$\mathcal{S}_{ji}(p) \equiv \mathcal{S}(p_j,p_i). \tag{4.88}$$

Then (4.82) implies that

$$\begin{aligned}\mu_j^M &= \prod_{\substack{i=1\\ i\neq j}}^{\mathcal{P}}\frac{\mu_i}{\mu_j}\cdot\frac{u_i-u_j+i}{u_i-u_j-i} = \left(\prod_{i\neq j}^{\mathcal{P}}\frac{\mu_i}{\mu_j}\right)\prod_{i\neq j}^{\mathcal{P}}\frac{u_i-u_j+i}{u_i-u_j-i}\\ &= (\mu_j)^{1-\mathcal{P}}\left(\prod_{i\neq j}^{\mathcal{P}}\mu_i\right)\prod_{i\neq j}^{\mathcal{P}}\frac{u_i-u_j+i}{u_i-u_j-i} = (\mu_j)^{-\mathcal{P}}\omega^{-\mathfrak{m}}\prod_{i\neq j}^{\mathcal{P}}\frac{u_i-u_j+i}{u_i-u_j-i},\end{aligned} \tag{4.89}$$

where we used (4.85). And thus, using (3.185) we rewrite Bethe equations (4.82) and (4.85) as

$$\omega^{\mathfrak{m}}\left(\frac{u_i+i/2}{u_i-i/2}\right)^L = \prod_{\substack{j=1\\ j\neq i}}^{\mathcal{P}}\frac{u_i-u_j+i}{u_i-u_j-i}, \qquad L = M+\mathcal{P} \tag{4.90}$$

$$\omega^{\mathfrak{m}}\prod_{i=1}^{\mathcal{P}}\left(\frac{u_i+i/2}{u_i-i/2}\right) = 1, \tag{4.91}$$

in agreement with (3.183) and (3.184), and the earlier statement that they hold for $\mathcal{P}>2$.

4.2.2 Solution

As noted before, although Bethe equations for $\mathcal{P} = 2$ and $\mathcal{P} > 2$ are the same it does not mean that their solutions are alike. Looking at (3.168), and remembering that $p_i \sim \frac{2\pi}{L} \times$ integer, we see that

$$u_i \sim L,$$

and thus it is natural to define rescaled Bethe roots

$$\mathsf{x}_i := \frac{u_i}{L},$$

which stay finite at $L \to \infty$. In the case of Z (Φ^a) impurities we have $\mathcal{P} = J_3$. Taking the logarithm of (4.90), and keeping in mind that

$$\ln\left(\frac{x+\epsilon}{x-\epsilon}\right) \xrightarrow{\epsilon\to 0} \frac{2\epsilon}{x}, \tag{4.92}$$

we receive

$$\frac{1}{\mathsf{x}_j} = \frac{2}{L}\sum_{i\neq j}^{J_3} \frac{1}{\mathsf{x}_j - \mathsf{x}_i} - 2\pi\left(\tilde{n}_j + \frac{\mathfrak{m}}{M}\right), \tag{4.93}$$

where $2\pi\left(\tilde{n}_j + \frac{\mathfrak{m}}{M}\right)$ comes as a lattice twisted momentum and stems from the logarithmic cut of the complex plane. Let us assume that we have an even number of magnons $\mathcal{P} = 2J$. Up to first approximation, the Bethe roots may be placed around the positions

$$u_i = \frac{-L}{2\pi\left(\tilde{n}_j + \frac{\mathfrak{m}}{M}\right)}, \tag{4.94}$$

however as we increase the number of magnons, the interactions between them cannot be neglected, and situation gets more complicated when more than one magnon occupies the same momentum state. Then, since they do not have the same rapidities, the interaction term pushes the roots apart and the rapidities split in the complex plane. As a result, the roots form a "string", roughly parallel to the imaginary axis, with the separation between the adjacent roots scaling as $\Delta\mathsf{x} \sim 1/L$. Also, since the distribution of roots is invariant under reflection about the real axis[5], and since for $0 < 2J/L \ll 1/2$ it is energetically favourable to evenly distribute the roots on the ± 1 branches, the distribution looks like that in figure 4.2; see [76, 77]. For that reason, since roots with the same mode number form a continuous contour in the complex plane in the scaling limit $L \to \infty$, it is reasonable to assume that $\tilde{n}_j \equiv \tilde{n} = 1, 2, 3, \ldots$, and for the sake of simplicity of our notation we change $\tilde{n} \to a$. With the same argument, (4.91) can be rewritten in the form

$$\mathsf{P} = \frac{1}{L}\sum_{j=1}^{J_3} \frac{1}{\mathsf{x}_j} = -2\pi\left(b + \frac{\mathfrak{m}}{M}\right). \tag{4.95}$$

[5]That means that for every root x_i there exist a root x_j such that $\mathsf{x}_j = \mathsf{x}_i^*$.

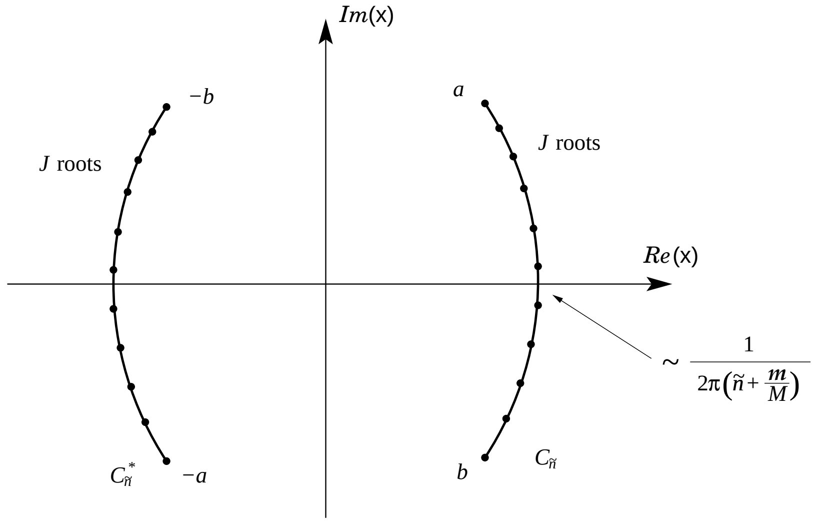

Figure 4.2: Distribution of Bethe roots for large L, which concentrate on $2\tilde{n}$ cuts; here we sketch only the $\tilde{n}^{\text{th}}$ pair, where $\tilde{n} = 1, 2, 3\ldots$, and $\mathfrak{m} = 0, 1, 2, \ldots, M-1$.

Also, (4.86) can be reexpressed as

$$\gamma = \frac{\lambda'}{8\pi^2 L^2} \sum_{i=1}^{J_3} \frac{1}{\mathsf{x}_i^2}. \tag{4.96}$$

In order to solve (4.93) for (4.96) it is very much helpful to define a resolvent

$$\mathsf{G}(\mathsf{x}) := \frac{1}{L} \sum_{j=1}^{J_3} \frac{1}{\mathsf{x} - \mathsf{x}_j}, \tag{4.97}$$

thanks to which we can rewrite (4.95) as

$$\mathsf{G}(0) = 2\pi \left(b + \frac{\mathfrak{m}}{M} \right) = -\mathsf{P}, \tag{4.98}$$

and

$$\gamma = -\frac{\lambda'}{8\pi^2 L} \mathsf{G}'(0). \tag{4.99}$$

Next, using the symmetricity of the sum

$$\sum_{j\neq i} \frac{1}{(\mathsf{x}_j - \mathsf{x}_i)(\mathsf{x} - \mathsf{x}_j)} = \sum_{i\neq j} \frac{1}{(\mathsf{x}_i - \mathsf{x}_j)(\mathsf{x} - \mathsf{x}_i)},$$

where the summation goes after both i and j, we can write that

$$\begin{aligned} \sum_{j\neq i} \frac{1}{(\mathsf{x}_j - \mathsf{x}_i)(\mathsf{x} - \mathsf{x}_j)} &= \frac{1}{2} \sum_{j\neq i} \frac{1}{\mathsf{x}_j - \mathsf{x}_i} \left(\frac{1}{\mathsf{x} - \mathsf{x}_j} - \frac{1}{\mathsf{x} - \mathsf{x}_i} \right) \\ &= \frac{1}{2} \sum_{j\neq i} \frac{1}{(\mathsf{x} - \mathsf{x}_j)(\mathsf{x} - \mathsf{x}_i)} = \frac{1}{2} \sum_{i,j} \frac{1}{(\mathsf{x} - \mathsf{x}_j)(\mathsf{x} - \mathsf{x}_i)} - \frac{1}{2} \sum_{j} \frac{1}{(\mathsf{x} - \mathsf{x}_j)^2}. \end{aligned} \tag{4.100}$$

On the other hand

$$\sum_j \frac{1}{\mathsf{x}_j(\mathsf{x}-\mathsf{x}_j)} = \frac{1}{\mathsf{x}}\sum_j\left(\frac{1}{\mathsf{x}_j}+\frac{1}{\mathsf{x}-\mathsf{x}_j}\right). \tag{4.101}$$

Then, multiplying (4.93) by $\sum_j \frac{1}{\mathsf{x}-\mathsf{x}_j}$, and using (4.100) together with (4.101) we obtain

$$\frac{L}{\mathsf{x}}\left[-2\pi\left(b+\frac{\mathfrak{m}}{M}\right)+\mathsf{G}(\mathsf{x})\right] = L\mathsf{G}^2(\mathsf{x})+\mathsf{G}'(\mathsf{x})-2\pi L\left(a+\frac{\mathfrak{m}}{M}\right)\mathsf{G}(\mathsf{x}),$$

or, dropping the terms of the order $\mathcal{O}(1/L)$,

$$\mathsf{x}\mathsf{G}^2(\mathsf{x}) = \mathsf{G}(\mathsf{x})\left[1+2\pi\mathsf{x}\left(a+\frac{\mathfrak{m}}{M}\right)\right]-2\pi\left(b+\frac{\mathfrak{m}}{M}\right). \tag{4.102}$$

It can be solved to be

$$\mathsf{G}(\mathsf{x}) = \frac{1}{2\mathsf{x}}\left[1+2\pi\mathsf{x}\left(a+\frac{\mathfrak{m}}{M}\right)-\sqrt{\left[1+2\pi\mathsf{x}\left(a+\frac{\mathfrak{m}}{M}\right)\right]^2-8\pi\mathsf{x}\left(b+\frac{\mathfrak{m}}{M}\right)}\,\right], \tag{4.103}$$

where the minus sign was fixed by the constraint (4.98). Then, $\mathsf{G}(\mathsf{x})$ can be expanded around zero to get that

$$\mathsf{G}'(0) = (2\pi)^2\left(b+\frac{\mathfrak{m}}{M}\right)\left[\left(b+\frac{\mathfrak{m}}{M}\right)-\left(a+\frac{\mathfrak{m}}{M}\right)\right], \tag{4.104}$$

hence

$$\gamma_{SU(2)_Z} = \frac{\lambda'}{2L}(a-b)\left(b+\frac{\mathfrak{m}}{M}\right), \tag{4.105}$$

and if we redefine $a-b\equiv\tilde{m}_3$, $b\equiv\tilde{m}_1$, we will reproduce the result (4.68).

If we now impure the MRV states (4.73) and (4.74) with the $\bar{B}^a(\bar{Y})$ fields instead of the $\Phi^a(Z)$ fields as above, we will deal with the case of $SU(2)_R$ doublets, and this fact itself is enough to not to perform any calculations. The reson is that one can express both, the dilatation operator and the gauge theory states, in terms of these doublets which, as we know from chapter 3, do survive orbifolding. For that reason, the $\mathcal{N}=2$ dilatation operator will be exactly the same as the $\mathcal{N}=4$ dilatation operator, which means that there is no need for the twisted boundary conditions (3.177), and therefore the Bethe equations (4.90) will be the same as for $\mathcal{N}=4$ SYM theory. Hence, since $\mathfrak{m}=0$ in (4.90), and the translation condition (4.84) does not change, the equations (4.93) and (4.95) will read

$$\frac{1}{\mathsf{x}_j} = \frac{2}{L}\sum_{i\neq j}^{|J_2|}\frac{1}{\mathsf{x}_j-\mathsf{x}_i}-2\pi a, \tag{4.106}$$

and

$$\frac{1}{L}\sum_{j=1}^{|J_2|}\frac{1}{\mathsf{x}_j} = -2\pi\left(b+\frac{\mathfrak{m}}{M}\right), \tag{4.107}$$

respectively. They can be solved in exactly the same way as the previous case to yield the following anomalous dimension

$$\gamma_{SU(2)_R} = \frac{\lambda'}{2L}\left(a - b - \frac{\mathfrak{m}}{M}\right)\left(b + \frac{\mathfrak{m}}{M}\right), \tag{4.108}$$

which after the identification $b = \tilde{m}_1$, and $a - b = \tilde{m}_2$ reproduces the result (4.69).

In the case of $SU(2)_L$-doublet we proceed in full analogy to the analysis for the case of Z (Φ^a) impurities. The difference is that since now we deal with the Y (B^a) impurities, the $\mathcal{P}$ magnon $\mathcal{N} = 2$ MRV and $\mathcal{N} = 4$ MRV states will be defined as in (4.71) and (4.72), but where now

$$\mathcal{O}^{\ell_1,\ldots,\ell_\mathcal{P}}_{n_1,\ldots,n_\mathcal{P}} = \mathrm{tr}\left(A^1 \cdots A^{\ell_1-1}B^{\ell_1-1}A^{\ell_1-1}A^{\ell_1}\cdots A^{\ell_\mathcal{P}-1}B^{\ell_\mathcal{P}-1}A^{\ell_\mathcal{P}-1}A^{\ell_\mathcal{P}}\cdots A^M\right), \tag{4.109}$$

and

$$\widetilde{\mathcal{O}}^{\ell_1,\ldots,\ell_\mathcal{P}}_{n_1,\ldots,n_\mathcal{P}} = \mathrm{tr}\left[\mathfrak{U}^{\mathfrak{m}}X^{\ell_1-1}YX^{\ell_2-\ell_1+1}YX^{\ell_3-\ell_2+1}Y\cdots YX^{M-\ell_\mathcal{P}+2}\right], \tag{4.110}$$

and the same boundary conditions as in (4.75). Obviously the Y (B^a) magnons will meet at $\ell_{i+1} = \ell_i - 1$ ($i = 1, ..., \mathcal{P}-1$), and thus the Schrödinger equations yield a different analogue of (3.145)

$$2\Psi_{\ell_i,\ell_i-1} = \Psi_{\ell_i,\ell_i-2} + \Psi_{\ell_i+1,\ell_i-1}, \tag{4.111}$$

giving a different (than in (4.87)) $\mathcal{S}$-matrix, namely

$$\mathcal{S}(p_j,p_i) = \left(\frac{\mu_i}{\mu_j}\right)^2 \frac{u_i - u_j + i}{u_i - u_j - i}, \tag{4.112}$$

due to which the Bethe equations (4.82) can be easily shown to be

$$\omega^{2\mathfrak{m}}\left(\frac{u_i + i/2}{u_i - i/2}\right)^L = \prod_{\substack{j=1\\ j\neq i}}^{\mathcal{P}} \frac{u_i - u_j + i}{u_i - u_j - i}, \qquad L = M + 2\mathcal{P} \tag{4.113}$$

$$\omega^{\mathfrak{m}}\prod_{i=1}^{\mathcal{P}}\left(\frac{u_i + i/2}{u_i - i/2}\right) = 1. \tag{4.114}$$

In the thermodynamic limit they become

$$\begin{aligned}\frac{1}{\mathsf{x}_j} &= \frac{2}{L}\sum_{i\neq j}^{J_2}\frac{1}{\mathsf{x}_j - \mathsf{x}_i} - 2\pi\left(a + \frac{2\mathfrak{m}}{M}\right),\\ \frac{1}{L}\sum_{j=1}^{J_2}\frac{1}{\mathsf{x}_j} &= -2\pi\left(b + \frac{\mathfrak{m}}{M}\right),\end{aligned} \tag{4.115}$$

hence the anomalous dimension

$$\gamma_{SU(2)_L} = \frac{\lambda'}{2L}\left(a - b + \frac{\mathfrak{m}}{M}\right)\left(b + \frac{\mathfrak{m}}{M}\right), \tag{4.116}$$

which agrees with (4.70) after the identification $b \equiv \tilde{m}_1$, and $a - b \equiv \tilde{m}_2$.

Remark on algebraic curve of $AdS_5 \times \mathbf{S}^5$

Although realisation of existance of semiclassical, spinning string solutions in $AdS_5 \times \mathbf{S}^5$ in the large spin limit has been rather successfully tested, as mentioned in the beginning of this chapter, derivation of the exact solutions is very difficult. The reason for this is that the complexity of the functions, from which we obtain these solutions, increases with the complexity of the solutions. The functions that occur are of algebraic, elliptic, or hyperelliptic type, and many other, often much more involved ones. Even if every solution can be found using a suitable choice of functions, it will be impossible to understand their generic structure having infinitely many of them. For that reason, people changed the approach and started to ask themselves how is the spectrum of string solutions organised? This kind of classification was performed for bosonic strings on $\mathbb{R} \times \mathbf{S}^3$ in [78] where it was shown that for each solution of the equations of motion there exists a corresponding, hyperelliptic curve, and that the quantities such as the energy and Noether charges, were identified in the algebraic curve. At this point, the logic can be reversed and one can investigate those curves which correspond to some classical solutions and this, on the other hand, leads to a solution of the spectral problem in terms of algebraic curves. From these curves one can read off a lot of information on how to classify string solutions, which is a step towards understanding how to quantise the classical string theory on $AdS_5 \times \mathbf{S}^5$. Alternatively, one can study the algebraic curve for the gauge theory, and thus test the AdS/CFT duality. For the work done in this area see for example [78, 79, 80, 81]. It would be also interesting to see how does the algebraic curve behave under orbifolding of AdS/CFT duality by the group $\mathcal{G} = \mathbb{Z}_M$.

Summary

In this chapter we have considered strings rotating in $\mathbb{R}_t \times \mathbf{S}^5 \subset AdS_5 \times \mathbf{S}^5$, with large, corresponding quantum numbers. We have examined how does the situation look when we orbifold the $\mathbf{S}^5$ sphere, and it has turned out that we merely have to redefine the winding modes of the strings spinning in three directions of the $\mathbf{S}^5/\mathbb{Z}_M$ sphere according to (4.20). Afterwards, we have solved the energy of the strings spinning in $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M \subset \mathbb{R}_t \times \mathbf{S}^5/\mathbb{Z}_M$ for the winding modes, and then for their total angular momentum, which is supposed to be taken large. The results (up to one loop) have been compared with the spectrum of the dual gauge operators in $SU(2)$ bosonic subsector, and for that to happen we had to derive Bethe's equations for an arbitrary number of magnons, and then apply the method of rational solutions to solve the Bethe roots in the limit of very long gauge operators. Finally, the one-loop agreement has been confirmed.

Outlook

In this thesis we have presented a review of the work that has been done concerning the AdS/CFT duality projected by the orbifold group $\mathcal{G} = \mathbb{Z}_M$. The resulting type IIB string theory lives on the ten dimensional orbifold $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$, the global isometry of which is $SU(2,2) \times SU(2)_R \times U(1)$ as the orbifolding breaks one of two $SU(2)$'s in the $SO(6)$ rotation group of five sphere. On the gauge dual side, $SU(2)_R \times U(1)$ becomes the R-symmetry of the corresponding conformal field theory, which in this case is the $\mathcal{N} = 2$ superconformal quiver gauge theory, with the local gauge group $U(N)^M$.

However, as in the case of original Maldacena's conjecture, the symmetry arguments are not enough, and therefore one would like to test this type of string theory on orbifold/quiver gauge theory duality. As noticed by Mukhi, Rangamani, and Verlinde [57], there is a very convenient limit in which one can rather easily quantise the IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ in the pp-wave limit. This orbifold-analogy of the BMN limit of $\mathcal{N} = 4$ SYM allows for an establishment of the dictionary between the string states and the quiver gauge states, and for the calculation of the free (non-interactive) energy spectrum on the string side. The main difference between the MRV limit and the BMN limit is that in the former case the null direction becomes compactified, and strings can wind around this direction $\mathfrak{m}$ times. In fact, after quantisation the winding $\mathfrak{m}$ becomes a quantum number of both, string and gauge states, since the states with different values of $\mathfrak{m}$ do not mix with each other [62, 65, 75]. It also turns out that the gauge MRV states can be written in both, the $\mathcal{N} = 2$ notation and in the $\mathcal{N} = 4$ notation, and we map them to the "twisted" spin chain ferromagnetic states. Afterwards, we calculate the spectrum of MRV operators in the $SU(2)$ bosonic subsector at one-loop, expecting that it is the same in both notations, up to the rescaling of the coupling constant of the "parent" and daughter "theories". For the $\mathcal{N} = 2$ MRV states we use earlier derived $\mathcal{N} = 2$ one-loop dilatation operator, and for the $\mathcal{N} = 4$ MRV states we use the "parent" $\mathcal{N} = 4$ one-loop dilatation operator, together with the twisted boundary conditions. Finally, we find that the one-loop anomalous dimension of the MRV operators in the $SU(2)$ subsector agrees with the result obtained on the string side.

Another idea to test our string on orbifold/quiver gauge correspondence was to consider the semiclassical solutions of spinning strings in $\mathbb{R}_t \times \mathbf{S}^5/\mathbb{Z}_M$, in the limit of large conserved charges, since then quantum effects are suppressed. After solving the energy of spinning strings for the angular momenta and winding modes, coming from wrapping of the strings around three different directions of five sphere, we specify the spinning sector to a submanifold $\mathbb{R}_t \times \mathbf{S}^3/\mathbb{Z}_M$, in order to compare it with an $SU(2)$ subsector on the dual gauge side. The Bethe equations for the MRV states in the $SU(2)$ representation are solved by the procedure of rational solutions, and the spectrum of spinning strings and their gauge duals up to one loop agrees perfectly. Having carried all of this out, one might ask oneself a question: is further examination of IIB string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M/\mathcal{N}=2$ quiver gauge theory duality worth an effort? After all it shows many of the same properties as its "parent" duality, like for example the same anomalous dimension matrix (although modified Bethe's equations), the same correction to the first and second loop (with the coupling constant rescaled properly of course), and the same discrepancy at three loops. However, one has to notice that these tests are performed in the large R-charge limit, and hence it might be that some effects become suppressed when the number of excitations is small compared to the length of the spin chain. For that reason, it would be interesting to study how does the orbifolding procedure influence the spectral (algebraic) curve of the sigma model on $AdS_5 \times \mathbf{S}^5$, and the distribution of Bethe roots on the gauge side.

Also, orbifolding the original duality by the group $\mathcal{G} = \mathbb{Z}_M$ might be viewed as a first step towards consideration of more complicated orbifolds, for example non-Abelian orbifolds. There are various ways of breaking one of $SU(2) \sim SO(3)$ groups of $SO(6)$ (for example by orbifolds of polyhedronic type) in order to obtain a theory with $\mathcal{N}=2$ supersymmetries, and it would be very interesting to examine if such $\mathcal{N}=2$ theories are integrable. Obviously, one does not have to restrict oneself to $\mathcal{N}=2$ theories but also consider a more general class of orbifolds that break supersymmetry to $\mathcal{N}=1$ or even $\mathcal{N}=0$, and look for integrability. In spite of some work done in this direction, this certainly is an area that remains rather unexplored (see [67, 82, 83], and also [84, 85]).

Appendix A

One-loop integrals

In this appendix we present the formulas necessary for the field theory calculations in chapter 2, according to the action (2.10). The scalar propagator in our notation will be of the form

$$I_{xy} = \frac{\Gamma(1-\epsilon)}{|\frac{1}{2}(x-y)^2|^{1-\epsilon}}. \tag{A.1}$$

The integrals appearing at the one-loop connected Green's functions (2.39) are the following

$$\begin{aligned}
Y_{x_1\,x_2\,x_3} &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1 z} I_{x_2 z} I_{x_3 z}, \\
X_{x_1\,x_2\,x_3\,x_4} &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}z}{(2\pi)^{2-\epsilon}} I_{x_1 z} I_{x_2 z} I_{x_3 z} I_{x_4 z}, \\
\widetilde{H}_{x_1\,x_2\,x_3\,x_4} &= \frac{1}{2}\mu^{2\epsilon} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 \int \frac{d^{4-2\epsilon}z_1 d^{4-2\epsilon}z_2}{(2\pi)^{4-2\epsilon}} I_{x_1 z_1} I_{x_2 z_1} I_{z_1 z_2} I_{z_2 x_3} I_{z_2 x_4}.
\end{aligned} \tag{A.2}$$

When evaluated in two-point functions, they yield [48] that

$$\frac{Y_{00x}}{I_{0x}} = \frac{1}{\epsilon(1-2\epsilon)} f(x), \tag{A.3}$$

$$\frac{X_{00xx}}{I_{0x}^2} = \frac{2(1-3\epsilon)\gamma}{\epsilon(1-2\epsilon)^2} f(x), \tag{A.4}$$

$$\frac{\widetilde{H}_{0x0x}}{I_{0x}^2} = -\frac{2(1-3\epsilon)(\gamma-1)}{\epsilon^2(1-2\epsilon)} f(x), \tag{A.5}$$

where

$$f(x) = \frac{\Gamma(1-\epsilon)}{|\frac{1}{2}\mu^2 x^2|^{-\epsilon}}, \quad \gamma = \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-3\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)} = 1 + 6\zeta(3)\epsilon^3 + \mathscr{O}(\epsilon^4). \tag{A.6}$$

This, together with the formalism from section 2.3, is enough to derive the one-loop dilatation operator for $\mathcal{N}=4$ supersymmetric Yang-Mills theory.

Appendix B

Proof of the equivalence of MRV states

In this appendix we want to prove that the two magnon $\mathcal{N}=4$ MRV state

$$\alpha^{Z_1}_{-n_1}\alpha^{Z_2}_{-n_2}|k=1,\mathfrak{m}\rangle = \sum_{\ell_2 \geqslant \ell_1}^{M} \mathrm{tr}\,(X^{\ell_1}Z_{n_1}X^{\ell_2-\ell_1}Z_{n_2}X^{M-\ell_2}) \tag{B.1}$$

is equivalent to the $\mathcal{N}=2$ MRV state

$$\alpha^{\Phi_1}_{-n_1}\alpha^{\Phi_2}_{-n_2}|k=1,\mathfrak{m}\rangle = \sum_{\ell_2 \geqslant \ell_1}^{M} \omega^{\sum_i n_i\ell_i}\,\mathrm{tr}\,(A^1A^2\cdots A^{\ell_1-1}\Phi_1^{\ell_1}A^{\ell_1}\cdots A^{\ell_2-1}\Phi_2^{\ell_2}A^{\ell_2}\cdots A^M). \tag{B.2}$$

In contrary to chapter 3, I enumerated particles with which we impure the ground state, since this allows to give a crucial argument proving the equivalence of (B.1) and (B.2), up to the overall factor of M. Since the string creation operators α_{-n_1} and α_{-n_2} commute with each other there is no given hint which particle in the ground state should appear on the right-hand side with respect to the other one, and thus it is essential to consider a symmetric contributions of these states. Let us denote the state (B.1) as

$$|k=1,\mathfrak{m};Z_1,Z_2\rangle^{\mathcal{N}=4}, \tag{B.3}$$

and the state (B.2) as

$$|k=1,\mathfrak{m};\Phi_1,\Phi_2\rangle^{\mathcal{N}=2}. \tag{B.4}$$

Then the two impurity state

$$|k=1,\mathfrak{m};Z_i,i=1,2\rangle^{\mathcal{N}=4} \equiv \frac{1}{\sqrt{2}}\left(|k=1,\mathfrak{m};Z_1,Z_2\rangle^{\mathcal{N}=4}+|k=1,\mathfrak{m};Z_2,Z_1\rangle^{\mathcal{N}=4}\right), \tag{B.5}$$

ought to be equal to (up to the factor of M)

$$|k=1,\mathfrak{m};\Phi_i,i=1,2\rangle^{\mathcal{N}=2} \equiv \frac{1}{\sqrt{2}}\left(|k=1,\mathfrak{m};\Phi_1,\Phi_2\rangle^{\mathcal{N}=2}+|k=1,\mathfrak{m};\Phi_2,\Phi_1\rangle^{\mathcal{N}=2}\right). \tag{B.6}$$

The proof goes as follows.

The state (B.1) can be orbifolded by the use of (3.17) as

$$\begin{aligned}
&|k=1,\mathfrak{m};Z_i,Z_j\rangle \\
&\quad=\sum_{a=1}^{M}\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\omega^{n_i(\ell_1+a)+n_j(\ell_2+a)}\mathrm{tr}\left(A^{1+a}A^{2+a}\cdots\Phi_i^{\ell_1+a}\cdots\Phi_j^{\ell_2+a}\cdots A^{M+a}\right)\\
&\quad\equiv\sum_{a=1}^{M}\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\mathfrak{f}_{ij}^{a}(\ell_1,\ell_2),
\end{aligned}\tag{B.7}$$

for $i,j=1,2$ and $i\neq j$. We can immediately see that the $\mathfrak{f}^{a=M}$ element of the sum reproduces (B.2) but what about $M-1$ other elements? Let us try to prove that the $\mathfrak{f}^{a=1}$ element is the same as the $\mathfrak{f}^{a=M}$ element. I denote the $\mathfrak{f}^{a=M}$ element as

$$\mathfrak{f}_{ij}^{M}(\ell_1,\ell_2)=\mathrm{tr}(A^1\cdots\Phi_i^{\ell_1}\cdots\Phi_j^{\ell_2}\cdots A^M)\omega^{n_i\ell_1+n_j\ell_2},\tag{B.8}$$

and the $\mathfrak{f}^{a=1}$ element as

$$\mathfrak{f}_{ij}^{1}(\ell_1,\ell_2)=\mathrm{tr}(A^1\cdots\Phi_i^{\ell_1+1}\cdots\Phi_j^{\ell_2+1}\cdots A^M)\omega^{n_i(\ell_1+1)+n_j(\ell_2+1)}.\tag{B.9}$$

We now write that

$$\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\mathfrak{f}_{ij}^{M}(\ell_1,\ell_2)=\sum_{\ell_2=1}^{M}\mathfrak{f}_{ij}^{M}(1,\ell_2)+\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=2}^{M}\mathfrak{f}_{ij}^{M}(\ell_1,\ell_2),\tag{B.10}$$

and

$$\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\mathfrak{f}_{ij}^{1}(\ell_1,\ell_2)=\sum_{\ell_2\geqslant\ell_1}^{M-1}\sum_{\ell_1=1}^{M-1}\mathfrak{f}_{ij}^{1}(\ell_1,\ell_2)+\sum_{\ell_1=1}^{M}\mathfrak{f}_{ij}^{1}(\ell_1,M).\tag{B.11}$$

Clearly,

$$\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=2}^{M}\mathfrak{f}_{ij}^{M}(\ell_1,\ell_2)=\sum_{\ell_2\geqslant\ell_1}^{M-1}\sum_{\ell_1=1}^{M-1}\mathfrak{f}_{ij}^{1}(\ell_1,\ell_2),\tag{B.12}$$

however

$$\sum_{\ell_1=1}^{M}\mathfrak{f}_{ij}^{1}(\ell_1,M)=\sum_{\ell_1=2}^{M}\mathfrak{f}_{ji}^{M}(1,\ell_1)+\mathfrak{f}_{ij}^{M}(1,1),\tag{B.13}$$

and therefore

$$\begin{aligned}
&\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\left\{\mathfrak{f}_{12}^{1}(\ell_1,\ell_2)+\mathfrak{f}_{21}^{1}(\ell_1,\ell_2)\right\}\\
&\qquad=\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=2}^{M}\left\{\mathfrak{f}_{12}^{M}(\ell_1,\ell_2)+\mathfrak{f}_{21}^{M}(\ell_1,\ell_2)\right\}+\sum_{\ell_1=1}^{M}\left\{\mathfrak{f}_{12}^{M}(1,\ell_1)+\mathfrak{f}_{21}^{M}(1,\ell_1)\right\}\\
&\qquad=\sum_{\ell_2\geqslant\ell_1}^{M}\sum_{\ell_1=1}^{M}\left\{\mathfrak{f}_{12}^{M}(\ell_1,\ell_2)+\mathfrak{f}_{21}^{M}(\ell_1,\ell_2)\right\}.
\end{aligned}\tag{B.14}$$

Hence,

$$|k=1,\mathfrak{m};Z_i,i=1,2\rangle^{\mathcal{N}=4} = M|k=1,\mathfrak{m};\Phi_i,i=1,2\rangle^{\mathcal{N}=2}. \tag{B.15}$$

This could be generalised to any $a = 1,\ldots,M$, and also to an arbitrary number of magnons, keeping in mind that the symmetrisation should be taken accordingly to the number of inserted particles.

This completes the proof.

Acknowledgements

First of all, I would like to thank my supervisor, Niels Obers for choosing such a fascinating topic for my Master's project, and a very helpful guidance throughout the whole year. I would also like to thank my family for their unfaltering support and belief. Then I want to express my great appreciation to Charlotte Kristjansen, Romuald Janik, Troels Harmark, Shinji Hirano, and Niklas Beisert for helpful comments and discussions, and to Mads T. Frandsen, Johannes Weirich, and Thomas Olsen for wholesome conversations on the subject. Furthermore, I would like to acknowledge most of the people above again for precious remarks concerning the text. Finally, great thanks to all my friends, but especially Zofia for keeping me down-to-earth, and Michal and Asia for being there when I needed it.
Thank you all!

Bibliography

[1] J. M. Maldacena, *"The Large N limit of superconformal field theories and supergravity"*, Adv. Theor. Math. Phys. 2 (1998) 231, `hep-th/9711200`

[2] S.S. Gubser, I. R. Klebanov, A. M. Polyakov, *"Gauge theory correlators from noncritical string theory"*, Phys. Lett. B428 (1998) 105, `hep-th/9802109`

[3] E. Witten, *"Anti-de Sitter space and holography"* Adv. Theor. Math. Phys. 2 (1998) 253, `hep-th/9802150`

[4] G. 't Hooft, *"A Planar Diagram Theory for Strong Interactions"*, Nucl. Phys. B72 (1974) 461

[5] D. Berenstein, J. M. Maldacena, H. Nastase, *"Strings in flat space and pp waves from $\mathcal{N}=4$ superYang-Mills"*, JHEP 0204 (2002) 013, `hep-th/0202021`

[6] S. Frolov, A.A. Tseytlin, *"Multispin string solutions in $AdS_5 \times \mathbf{S}^5$"*, Nucl. Phys. B668 (2003) 77, `hep-th/0304255`

[7] G. Arutyunov, S. Frolov, M. Staudacher, *"Bethe ansatz for quantum strings"*, JHEP 0410 (2004) 016, `hep-th/0406256`

[8] N. Gromov, P. Vieira, *"Constructing the AdS/CFT dressing factor"*, `hep-th/0703266`

[9] J.A. Minahan, K. Zarembo, *"The Bethe ansatz for $\mathcal{N}=4$ superYang-Mills"*, JHEP 0303 (2003) 013, `hep-th/0212208`

[10] S. J. Avis, C. J. Isham, and D. Storey, *"Quantum Field Theory in Anti-de Sitter Spacetime"*, Phys. Rev. D18 (1978) 3565

[11] I. Bengtsson, *"Anti-de Sitter space" - Lecture notes*, http://www.physto.se/∼ingemar

[12] G. W. Gibbons, *"Anti-de-Sitter Spacetime and Its Uses"*, Lecture Notes in Phys. 537 (2000) 102

[13] J. L. Petersen, *"Introduction to the Maldacena conjecture on AdS/CFT"*, Int. J. Mod. Phys. A14 (1999) 3597-3672, `hep-th/9902131`

[14] G. F. R. Ellis, S. W. Hawking, *"The large scale structure of space-time"*. Cambridge university press (1973), `(see pages 131-134)`

[15] E. D'Hoker, D. Z. Freedman, *"Supersymmetric gauge theories and the AdS/CFT correspondence"*, Boulder 2001, Strings, branes and extra dimensions: 3-158, `hep-th/0201253`

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, Y. Oz, *"Large N field theories, string theory and gravity"*, Phys. Rept. 323 (2000) 183-386, `hep-th/9905111`

[17] P. Ginsparg, *"Applied Conformal Field Theory"*, Proceedings of the Les Houches school on Fields, Strings and Critical Phenomena (1988), `hep-th/9108028`

[18] F. Gliozzi, J. Scherk, D. I. Olive, *"Supersymmetry, Supergravity theories and the Dual Spinor Model"*, Nucl. Phys. B122 (1977) 253

[19] L. Brink, J. H. Schwarz, J. Scherk, *"Supersymmetric Yang-Mills Theories"*, Nucl. Phys. B121 (1977) 77

[20] N. Beisert, *"The Dilatation Operator $\mathcal{N}=4$ Super Yang-Mills Theory and Integrability"*, Phys. Rept. 405 (2005) 1, `hep-th/0407277`

[21] M. Sohnius, *"Introducing Supersymmetry"*, Phys. Rep. 128 (1985) 39

[22] J. Wess and J. Bagger, *"Supersymmetry and Supergravity"*, 2nd edition, Princeton University Press (1992)

[23] M. F. Sohnius, P. C. West, *"Conformal invariance in $\mathcal{N}=4$ supersymmetric Yang-Mills Theory"*, Phys. Lett. B100 (1981) 245

[24] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff, M. Staudacher, *"BMN correlators and operator mixing in $\mathcal{N}=4$ superYang-Mills theory"*, Nucl. Phys. B650 (2003) 125, `hep-th/0208178`

[25] J. Polchinski, *"String Theory"*, Cambridge University Press, Cambridge (1998)

[26] C. V. Johnson, *"D-Branes"*, Cambridge University Press, Cambridge (2003)

[27] J. Polchinski, *"Tasi lectures on D-branes"*, Boulder 1996, Fields, strings and duality: 293-356, `hep-th/9611050`

[28] A. A. Tseytlin, *"On nonAbelian generalization of Born-Infeld action in string theory"*, Nucl. Phys. B501 (1997) 41, `hep-th/9701125`

[29] A. Sevrin, J. Troost, W. Troost, *"The nonAbelian Born-Infeld action at order F**6"*, Nucl. Phys. B603 (2001) 389, `hep-th/0101192`

[30] J. Polchinski, *"Combinatorics of boundaries in string theory"*, Phys. Rev. D50 (1994) 6041, `hep-th/9407031`

[31] E. Witten, *"Bound states of strings and p-branes"*, Nucl. Phys. B460 (1996) 335, `hep-th/9510135`

[32] C. Montonen, D. I. Olive, *"Magnetic Monopoles as Gauge Particles?"*, Phys. Lett. B72 (1977) 117

[33] P. Breitenlohner, D. Z. Freedman, *"Stability in Gauged Extended Supergravity"*, Annals Phys. 144 (1982) 249

[34] P. Breitenlohner, D. Z. Freedman, *"Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity"*, Phys. Lett. B115 (1982) 197

[35] L. Mezincescu, P.K. Townsend, *"Stability At A Local Maximum In Higher Dimensional Anti-De Sitter Space And Applications To Supergravity"*, Annals Phys. 160 (1985) 406

[36] I. R. Klebanov, E. Witten, *"AdS/CFT correspondence and symmetry breaking"*, Nucl. Phys. B556 (1999) 89, `hep-th/9905104`

[37] M. Bianchi, M. B. Green, S. Kovacs, G. Rossi, *"Instantons in supersymmetric Yang-Mills and D instantons in IIB superstring theory"*, JHEP 9808 (1998) 013, `hep-th/9807033`

[38] H. W. Brinkmann, *"Einstein spaces which are mapped conformally on each other"*, Math. Ann. 18 (1925) 119

[39] J. Ehlers, W. Kundt, *"Exact solutions of the gravitational field equations"*, Gravitation: an Introduction to Current Research: 49 (1962) 101, `(see sections 2-5)`

[40] R. Penrose, *"Any spacetime has a plane wave as a limit"*, Differential geometry and relativity, Reidel, Dordrecht (1976), `(see pages 271-275)`

[41] N. Beisert, *"The complete one loop dilatation operator of N=4 super Yang-Mills theory"*, Nucl. Phys. B676 (2004) 3, `hep-th/0307015`

[42] N. Beisert, M. Staudacher, *"The $\mathcal{N}=4$ SYM integrable super spin chain"*, Nucl. Phys. B670 (2003) 439, `hep-th/0307042`

[43] N. Beisert, *"The $\mathfrak{su}(2|3)$ dynamic spin chain"*, Nucl. Phys. B682 (2004) 487, `hep-th/0310252`

[44] N. Beisert, C. Kristjansen, M. Staudacher, *"The Dilatation operator of conformal $\mathcal{N}=4$ superYang-Mills theory"*, Nucl. Phys. B664 (2003) 131, `hep-th/0303060`

[45] C. Kristjansen, J. Plefka, G.W. Semenoff, M. Staudacher, *"A New double scaling limit of $\mathcal{N}=4$ superYang-Mills theory and PP wave strings"*, Nucl. Phys. B643 (2002) 3, `hep-th/0205033`

[46] D. J. Gross, A. Mikhailov, R. Roiban, *"Operators with large R charge in $\mathcal{N}=4$ Yang-Mills theory*, Annals Phys. 301 (2002) 31, `hep-th/0205066`

[47] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, W. Skiba, *"PPwave string interactions from perturbative Yang-Mills theory"*, JHEP 0207 (2002) 017, `hep-th/0205089`

[48] D.I. Kazakov, *"Calculation of Feynman integrals by the method of 'uniqueness' "*, Theor. Math. Phys. 58 (1984) 223

[49] L. J. Dixon, J. A. Harvey, C. Vafa, E. Witten, *"Strings on Orbifolds"*, Nucl. Phys. B261 (1985) 678

[50] I. Antoniadis, K. Benakli, *"Limits on extra dimensions in orbifold compactifications of superstrings"*, Phys. Lett. B326 (1994) 69, `hep-th/9310151`

[51] N. Beisert, M. Staudacher, *"Long-range* $\mathfrak{psu}(2,2|4)$ *Bethe Ansatze for gauge theory and strings*, Nucl. Phys. B727 (2005) 1, hep-th/0504190

[52] E. Witten, *"Five-branes and M theory on an orbifold"*, Nucl. Phys. B463 (1996) 383, hep-th/9512219

[53] M. R. Douglas, G. W. Moore, *"D-branes, quivers, and ALE instantons"*, hep-th/9603167

[54] S. Kachru, E. Silverstein, *"4D conformal theories and strings on orbifolds"*, Phys. Rev. Lett. 80 (1998) 4855, hep-th/9802183

[55] A. E. Lawrence, N. Nekrasov, C. Vafa, *"On conformal field theories in four-dimensions"*, Nucl. Phys. B533 (1998) 199, hep-th/9803015

[56] M. Bershadsky, A. Johansen, *"Large N limit of orbifold field theories"*, "Nucl. Phys. B536 (1998) 141, hep-th/9803249

[57] S. Mukhi, M. Rangamani, E. Verlinde, *"Strings from Quivers, Membranes from Moose"*, JHEP 0205 (2002) 023, hep-th/0204147

[58] M. Alishahiha, M. M. Sheikh-Jabbari, *"The pp wave limits of orbifolded* $AdS_5 \times \mathbf{S}^5$*"*, Phys. Lett. B535 (2002) 328, hep-th/0203018

[59] N.-W. Kim, A. Pankiewicz, S.-J. Rey, S. Theisen, *"Superstring on PP wave orbifold from large N quiver gauge theory"*, Eur. Phys. J. C25 (2002) 327, hep-th/0203080

[60] T. Takayanagi, S. Terashima, *"Strings on orbifolded PP waves"*, JHEP 0206 (2002) 036, hep-th/0203093

[61] E. Floratos, A. Kehagias, *"Penrose limits of orbifolds and orientifolds"*, JHEP 0207 (2002) 031, hep-th/0203134

[62] G. De Risi, G. Grignani, M. Orselli, G. W. Semenoff, *"DLCQ string spectrum from N=2 SYM theory"*, JHEP 0411 (2004) 053, hep-th/0409315

[63] M. P. Grabowski, P. Mathieu, *"Quantum integrals of motion for the Heisenberg spin chain"*, hep-th/9403149

[64] M. Staudacher, *"The Factorized S-matrix of CFT/AdS"*, JHEP 0505 (2005) 054, hep-th/0412188

[65] D. Astolfi, V. Forini, G. Grignani, G. W. Semenoff, *"Finite size corrections and integrability of mathcal* $\mathcal{N}=2$ *SYM and DLCQ strings on a pp-wave"*, JHEP 0609 (2006) 056, hep-th/0606193

[66] M. Bertolini, J. de Boer, T. Harmark, E. Imeroni, N.A. Obers, *"Gauge theory description of compactified pp waves"*, JHEP 0301 (2003) 016, hep-th/0209201

[67] N. Beisert, R. Roiban, *"The Bethe ansatz for* $\mathbb{Z}_S$ *orbifolds of* $\mathcal{N}=4$ *super Yang-Mills theory"*, JHEP 0511 (2005) 037, hep-th/0510209

[68] S.S. Gubser, I.R. Klebanov, A. M. Polyakov, *"A Semiclassical limit of the gauge/string correspondence"*, Nucl. Phys. B636 (2002) 99, `hep-th/0204051`

[69] S. Frolov, A.A. Tseytlin, *"Semiclassical quantization of rotating superstring in* $AdS_5 \times \mathbf{S}^5$*"*, JHEP 0206 (2002) 007, `hep-th/0204226`

[70] J.G. Russo, *"Anomalous dimensions in gauge theories from rotating strings in* $AdS_5 \times \mathbf{S}^5$*"*, JHEP 0206 (2002) 038, `hep-th/0205244`

[71] G. Mandal, N. V. Suryanarayana, S. R. Wadia, *"Aspects of semiclassical strings in* AdS_5*"*, Phys. Lett. B543 (2002) 81, `hep-th/0206103`

[72] A.A. Tseytlin, *"Spinning strings and AdS/CFT duality"*, M. Shifman, et al.: From fields to strings, vol. 2 1648-1707, (World Scientific, 2004), `hep-th/0311139`

[73] S. Frolov, A.A. Tseytlin, *"Quantizing three spin string solution in* $AdS_5 \times \mathbf{S}^5$*"*, JHEP 0307 (2003) 016, `hep-th/0306130`

[74] N. Beisert, S. Frolov, M. Staudacher, A.A. Tseytlin, *"Precision spectroscopy of AdS/CFT"*, JHEP 0310 (2003) 037, `hep-th/0308117`

[75] K. Ideguchi, *"Semiclassical strings on* $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_M$ *and operators in orbifold field theories"*, JHEP 0409 (2004) 008, `hep-th/0408014`

[76] N. Beisert, J.A. Minahan, M. Staudacher, K. Zarembo, *"Stringing spins and spinning strings"*, JHEP 0309 (2003) 010, `hep-th/0306139`

[77] A. Kundu, *"Refinement of Bethe ansatz string and its alternative"*, `cond-mat/0011467`

[78] V.A. Kazakov, A. Marshakov, J.A. Minahan, K. Zarembo, *"Classical/quantum integrability in AdS/CFT"*, JHEP 0405 (2004) 024, `hep-th/0402207`

[79] N. Beisert, V.A. Kazakov, K. Sakai, *"Algebraic curve for the* $SO(6)$ *sector of AdS/CFT"*, Commun. Math. Phys. 263 (2006) 611, `hep-th/0410253`

[80] S. Schafer-Nameki, *"The Algebraic curve of 1-loop planar* $\mathcal{N} = 4$ *SYM"*, Nucl. Phys. B714 (2005) 3, `hep-th/0412254`

[81] N. Beisert, V.A. Kazakov, K. Sakai, K. Zarembo, *"The Algebraic curve of classical superstrings on* $AdS_5 \times \mathbf{S}^5$*"*, Commun. Math. Phys. 263 (2006) 659, `hep-th/0502226`

[82] X.-J. Wang, Y.-S. Wu, *"Integrable spin chain and operator mixing in* $\mathcal{N} = 1, 2$ *supersymmetric theories"*, Nucl. Phys. B683 (2004) 363, `hep-th/0311073`

[83] D. Sadri, M. M. Sheikh-Jabbari, *"Integrable spin chains on the conformal moose"*, JHEP 0603 (2006) 024, `hep-th/0510189`

[84] P. H. Frampton, T. W. Kephart, *"Chiral fermions and AdS/CFT duality for a non-Abelian orbifold"*, Phys. Lett. B485 (2000) 403, `hep-th/9912028`

[85] B. A. Burrington, J. T. Liu, L. A. Pando Zayas, *"Finite Heisenberg groups from non-Abelian orbifold quiver gauge theories"*, `hep-th/0701028`