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Dipole Moments of Black Branes

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Abstract

In recent years it has been shown that higher-order corrections, namely pole-dipole corrections, to the blackfold approach reveal that higher dimensional black branes possess some attributes analogous to conventional elastic materials. Pole-dipole order corrections to the effective stress-energy tensor of charged fluid branes living on a submanifold of a background metric result in bending moments, the effect of which is captured by a relativistic generalization of the Young modulus and piezoelectric moduli. Applying the correction to electrically charged black branes one uncovers that their behaviour is captured by relativistic generalizations of classical electroelasticity theory. In the thesis the recent developments on the dynamics of charged blackfolds and their electroelastic behaviour to higher-order corrections is reviewed.

"Some people play bingo. Other people do something fun."
- Kim "Kanonarm" Köbke

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Introduction

The idea that a sufficiently large concentration of mass can produce a gravitational field so intense that not even light can escape its pull can be traced as far back as 1783 when John Michell calculated that a ray of light could be trapped on a massive body under the assumption that light is affected by the pull of gravity. At the time there was no sound reasoning as to why a massless phenomenon such as light should adhere to the force of gravity in the same way as massive objects do. A rigorous new theory of gravity saw the light of day as Albert Einstein published his general theory of relativity, explaining gravity as a curvature of spacetime. A localized matter or energy source (encoded by the stress-energy tensor $T_{\mu\nu}$) curves spacetime, captured by its metric $g_{\mu\nu}$, according a set of partial differential equations termed the *Einstein field equations*,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1}$$

Test particles in free fall will then move in straight lines in the curved spacetime (called geodesics), governed by the geodesic equation,

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \tag{2}$$

where the Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ depend on the first derivatives of the metric. By looking at some solutions of the field equations one finds that the existence of objects which curve spacetime sufficiently, so that not even light can escape, is unavoidable. As an example, the unique spherically symmetric static solution to the vacuum equations $R_{\mu\nu} = 0$ in four dimensions reads

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3}$$

This solution is known as the Schwarzschild solution. It contains a singularity at $r = 0$ where the Riemann curvature is infinite and an *event horizon* at $r = r_0$, beyond which even null geodesics are confined. Solutions containing singularities hidden behind an event horizon are known as *black holes*. Classically black holes only absorb, emitting no radiation, and so they must have a non-existent temperature.

Stephen Hawking however showed by methods of semiclassical gravity, in which matters fields are given a quantum treatment, that black holes radiate with a blackbody spectrum and, in consequence, have a non zero temperature (coined the Hawking temperature) and entropy. To fully understand the nature of black holes a quantum description of gravity is in order. In 4-dimensional gravity the uniqueness of solutions is limiting, the unique static solution in vacuum being the Schwarzschild metric and in the stationary case, the Kerr metric. For non-static black holes in $D > 4$ pure gravity (such as the higher-dimensional generalization of the Kerr black hole, introduced by Myers and Perry in [1]) there exist no uniqueness theorems, giving rise to a wide range of new solutions. By far the richest collection of black hole configurations have been found in five dimensions using special ansätze [2] [3] [4]. The special ansätze we refer to are based of symmetry properties and inverse scattering techniques [5] [6] [7] [8]. By examining black hole solutions where we look at the dimensionality as a tunable parameter, we discover that some attributes are universal while others are dimensionally dependent, such as uniqueness and horizon topologies. It has become clear that the analysis of higher-dimensional solutions provides a larger, more intricate field from which we can extract knowledge of black holes.

The study of higher-dimensional black holes is especially relevant for string theory. Type-II string theory is formulated $d = 10$, which can also be arrived at from the $d = 11$ M-theory through duality transformations. String theory provides a quantum description of gravity and yields a large variety of black hole solutions. Further understanding of black holes in string theory might provide answers to long overdue problems. An example is the information-loss paradox. If a black hole radiates purely thermally then all information about the original infalling matter would be destroyed once the black hole has dissipated, in violation with the law of conservation of information. A complete quantum description of black holes should resolve this problem. The objects most fitting for our discussion of black holes in string theory are *p-branes*.

In recent years an effective world-volume theory has been presented that describes the dynamics of black branes, living on a submanifold embedded in a dynamical background spacetime, that have two widely separated horizon length scales [9] [10] [11] [12]. Black p-branes carrying Ramond-Ramond charges of field strength F_{p+2} are of high significance to supergravity theories, the low-energy limits of supersymmetric string/M-theories. These black p-branes can also carry a q-brane charge $(0 < q \leq p)$, dissolved in its world-volume, with associated field strength F_{q+2}. In [13] corrections to the monopole approximation of the stress-energy tensor are employed to $1st$ order revealing fine structure of neutral bent black p-branes. The corrections result in a dipole of the world-volume stress-energy, controlled by response coefficients that are interpreted as a relativistic generalization of the Young modulus of elastic materials. In [14] $1st$-order corrections were extended onto blackfolds carrying q-brane charge, in that case the gauge field acquires a electric dipole moment analogous to the piezoelectric moduli.

The aim of the thesis is to study higher-order corrections to black p-branes and review the different concepts and tools needed to develop explicit expressions of those corrections.

The structure of the thesis will be as follows. Chapter 1 will be an introduction to charged p-brane solutions occurring in the low-energy regime of string/M-theory. We will present the general procedure of obtaining their conserved quantities and thermodynamics. BPS states will be discussed as well as duality transformations and dimensional reduction of solutions and, leading from that discussion there will be a simple example of a solution generating technique, which is necessary for the developments in chapter 4. In chapter 2 the general framework of the blackfold approach for charged black p-branes will be reviewed. We will demonstrate how the method is used for charged blackfolds in supergravity carrying a dissolved q-brane charge. Chapters 3 and 4 will be the main point of this thesis, concentrating on the pole-dipole corrections to the blackfold approach. In chapter 3 the previous work done in [13] and [14] will be reviewed, where the effective world-volume theory is corrected up to dipole order for, first, neutral blackfolds, then blackfolds carrying higher form charge. In chapter 4 the steps leading to quantitative expressions describing the bending effects of blackfolds carrying higher-form charge are reviewed and the results of [14] presented.

Chapter 1

Black Branes

What is a brane? A *brane* is a generalization of a point particle, for example, a point particle is a 0-brane, a string is a 1-brane, and a p-brane is a arbitrary p-dimensional generalization. The concept of p-branes arises in supergravity theories. In supergravity theory, the low energy limit of superstring theory, in specific cases, a p-brane is the low energy limit of D-branes. The p-brane solutions occurring in supergravity can be well analyzed by classical methods and describe the non-perturbative sector of string theory. In this chapter the necessary concepts and tools needed for our discussion in the coming chapters are introduced. We take a general look at p-branes in arbitrary dimensions and the ones arising from $d = 10$ and $d = 11$ supergravities. Different string theories are intimately related to each other as well as M-theory through duality symmetries. These symmetries are presented, along with an example of how they are used to produce new solutions. The chapter is concluded with the introduction of the concept of smeared brane solutions.

1.1 Introductory on branes

Let's examine a bit the black brane pure gravity solutions to the Einstein field equations. Neutral black branes can be modeled classically by embedding p dimensional extended objects into a background metric. A simple way of introducing the idea is to first take a look at the generalization of the Schwarzschild solution to arbitrary dimensions, known as the Schwarzschild-Tangherlini solution

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \quad f = 1 - \frac{r_0^{D-3}}{r^{D-3}}, \tag{1.1}$$

where r_0 is the horizon radius. This solution can be modified to represent a Minkowskian neutral static black brane with a $p+1$ dimensional world-volume $\mathcal{W}_{p+1}$ by adding p extra flat directions to the metric. It is straightforward to show that such a procedure is also a solution to the Einstein equations. We write it as

$$ds^2 = -f dt^2 + \sum_{i=1}^{p} dx_i^2 + f^{-1} dr^2 + r^2 d\Omega_{n+1}^2 \quad , \quad f = 1 - \tfrac{r_0^n}{r^n}, \tag{1.2}$$

where we have defined $n = D-p-3$. The position of the brane in directions transverse the world-volume is characterized by $D-p-1$ coordinates. To make the solution more general we should consider the possibility of velocity fields living on the world-volume. Let us introduce so called world-volume coordinates by $\sigma^a = (t, x^i)$. Applying a boost, u^a (normalized by $u^a u_a = -1$), in the world-volume directions gives us the form

$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + f^{-1} dr^2 + r^2 d\Omega_{n+1}. \tag{1.3}$$

This is the most general form of flat neutral p-brane solution when only considering pure gravity.

A more suited way of working out black brane solutions in relation to this thesis is to define an action, incorporating the fields we wish present, and work out its solutions. In string theory one generally looks for equations of motion from a specific action, so in context of the continuing discussion it is useful to look at a generic example. A general charged dilatonic p-brane solution can be arrived at by defining an appropriate action. The fields entering the action should then be a graviton, $g_{\mu\nu}$, a gauge field $C_{[p+1]}$, with field strength $(dC)_{[p+2]}$, and a scalar dilaton ϕ. The general form of the action is

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(d\phi)^2 - \frac{1}{2(p+2)!} e^{a\phi} (dC)_{p+2}^2 \right), \tag{1.4}$$

where $n = D-p-3$, G is the D-dimensional Newton's constant and the exponent, a, is the dilaton coupling constant. Assigning the coupling constant the value $a^2 \equiv \frac{4}{N} - \frac{2(p+1)n}{D-2}$ yields a special class of solutions which are relevant to the analysis in chapters 3 and 4. The solutions to each of the fields are as follows. The line element of the flat dilatonic charged p-brane reads

$$ds^2 = H^{-\frac{Nn}{D-2}} \Big(-f dt^2 + \sum_{i=1}^{p} dx_i^2 \Big) + H^{\frac{N(p+1)}{D-2}} (f^{-1} dr^2 + r^2 d\Omega_{n+1}^2). \tag{1.5}$$

While the dilaton and gauge field solutions are

$$e^{2\phi} = H^{aN}, \quad A_{p+1} = \sqrt{N} \coth\alpha (H^{-1} - 1) dt \wedge dx_1 \wedge \ldots \wedge dx_p. \tag{1.6}$$

With f being defined in the same way as earlier and the function H defined as

$$H = 1 + \frac{r_0^n \sinh^2 \alpha}{r^n}. \tag{1.7}$$

We need to restrict $a^2 \geq 0$ in order for the dilaton not to enter as a ghost. This restriction leads to a limit on N, as it is included in the definition of a, so

$$N \leq 2\left(\frac{1}{n} + \frac{1}{p+1}\right). \tag{1.8}$$

Notice that taking the boost parameter $\alpha \to 0$, the solution for the metric (1.5) becomes the same as for the neutral p-brane (1.2). The radial coordinate is defined in the directions transverse to the world-volume $|x_\perp| \equiv r$. We will discuss these solutions further in chapter 2 in the context of black branes carrying higher-form charge, then put this class of solutions to use in the results of chapter 4, so they play an integral part in context of this thesis.

We can obtain the different branes corresponding to either type-II string theory or M-theory by plugging in the relevant quantities. For example, we could plug in $D = 11$, $N = 1$, and $p = 2, 5$, resulting in the solutions for the M-branes. The H functions are harmonic functions as they are required to be solutions to the Laplace equation $\partial_\perp^2 H_p(r) = 0$. The partial differential operator $\partial_\perp$ acts only on the coordinates orthogonal to the brane.

The general method of extracting conserved quantities of a system from the far-region asymptotics of the fields characterizing a given theory is outlined in section 1.4.

1.2 Supergravity actions

As a concrete example of an action leading to charged dilatonic p-brane solutions, we take a look at type-II string theory. In chapter 4, the preceding discussion of chapters 2 and 3 are given an explicit relation to type-II string theory, so in this section, we say a few words about it.

Type-II string theories have two sectors of massless modes, NS-NS and R-R. The low-energy limit of a supersymmetric string theory is called a *supergravity* theory. Let us write the total action for type-IIA supergravity, split in terms of the different massless sectors and the Chern-Simons term (that incorporates the intersections of branes),

$$\begin{aligned} S_{\text{IIA}} &= S_{\text{NS-NS}} + S_{\text{R-R}} + S_{\text{C-S}}, \\ S_{\text{NS-NS}} &= \frac{1}{16\pi G}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big(R + 4(d\phi)^2 - \frac{1}{2}(dB_2)^2\Big), \\ S_{\text{R-R}} &= -\frac{1}{2}\frac{1}{16\pi G}\int d^{10}x\Big((dC_1)^2 + (dC_3 + dC_1\wedge dB_2)^2\Big), \\ S_{\text{C-S}} &= -\frac{1}{2}\frac{1}{16\pi G}\int B_2\wedge dC_3\wedge dC_3. \end{aligned} \tag{1.9}$$

In type-IIA we have the C_n gauge fields with odd numbered n, the two independent ones being C_1 and C_3. In addition to the two forementioned R-R forms from there exist equally fundamental objects, related through Hodge duality. The Hodge operator, $*$ acting on a p-form, maps it to a dual $(D-p)$-form. Acting twice with the Hodge operator maps a p-form onto itself. So, in addition to C_1 and C_3 we have their (magnetic) duals C_7 and C_5. In IIB the dimensionality of the R-R fields is even numbered, consisting of the axion C_0, C_2, and C_4 which has a self-dual field strength $*dC_4 = dC_4$. The object naturally coupling to the R-R four-form would be a three-brane, that object would then be a self-dual object, so equal to its magnetically dual counterpart. Another fundamental dual object in string theory is the NS5-brane. Consider that the $10d$ fundamental superstring couples to the NS-NS two-form B_2. The dual of B_2 is a six-form potential, then the object it would couple to is a five-brane. Thus the magnetic dual object to the superstring is the NS5-brane. One might also note that in four dimensions it is only a numerical coincidence that the dual to a electrically charged particle is again a particle.

An in depth review on black holes and p-branes in string theory that contains a wide variety of subjects relevant to this thesis is [15].

R-R p-brane states are known not be a part of the perturbative spectrum of string theory. The study of p-branes and their magnetic solitonic duals probe the structure of non-perturbative string theory.

We refrain from writing and examining the equations of motion of the supergravity actions since they are not particularly relevant for the purpose of the thesis. Continuing our discussion of supersymmetric string theory, we present some of the basics of supersymmetry and the concept of BPS-states.

1.3 Supersymmetry and BPS-states

A very effective way of probing the non-perturbative aspect of string theory is through the study of solitonic solutions. Solutions of non-linear field equations of a theory cannot be found as perturbations of linearized field equations [16]. The stability of these solutions is ensured by the attribute that they carry a topological charge. Solitons are localized, finite energy solutions of the classical equations of motion for a non-linear field theory. Their mass density is usually inversely proportional to a positive power of a dimensionless coupling constant so they become very massive at weak coupling and will be vanishing in the perturbative sector of a field theory. Reviews on supergravity solitons can be found in [15] [17]. Hence, the brane mass will be very large according to the classical mass formula of quantum theory. However, we have to address the question of whether the classical mass formula can be trusted.

Supersymmetric states saturate an inequality resulting from the supersymmetry algebra. For a four dimensional, $\mathcal{N} = 2$ generator theory a majorana spinor Q is usually written in terms of Weyl spinors Q_α, $Q_{\dot{\alpha}}$:

$$Q = \begin{pmatrix} Q_\alpha \\ Q_{\dot{\alpha}} \end{pmatrix}. \tag{1.10}$$

We can switch to a basis of creation and annihilation operators defined by

$$a_\alpha^\pm = \frac{1}{\sqrt{2}}(Q_\alpha^1 \pm \epsilon_{\alpha\beta} Q_\beta^{2*}). \tag{1.11}$$

In this new basis the usual extended supersymmetry algebra in the former basis becomes a fermionic oscillator algebra. Working in rest frame yields simple anticommutation rules,

$$\begin{aligned} \{a_\alpha^+, a_\beta^{+\dagger}\} &= \delta_{\alpha\beta}(M + Z), \\ \{a_\alpha^-, a_\beta^\dagger\} &= \delta_{\alpha\beta}(M - Z), \end{aligned} \tag{1.12}$$

with central charge Z. With all the rest of the anticommutators vanishing. It then naturally follows from unitarity of the operators that

$$M \geq a|Z|. \tag{1.13}$$

This inequality is called the Bogomolny'i bound, with "a" being the coupling constant of the theory in which we work. This is also generally valid for higher $\mathcal{N}$ and dimensions. Such states must be annihilated by at least one of the supersymmetry generators. These states are associated with both the perturbative and non-perturbative spectrum of string theory. No quantum corrections are received by BPS states due to the renormalization theorem of supersymmetry [18]. One can not trust the ADM mass of non-extremal branes (not saturating the BPS-bound) because quantum corrections are needed. In $10d$ the BPS supergravity solutions in string frame are

$$\begin{aligned} ds^2 &= H_p(r)^{-\frac{1}{2}}\big(-dt^2 + dx_{\|}^2\big) + H_p(r)^{\frac{1}{2}} dx_{\perp}^2, \\ e^{\Phi} &= H_p(r)^{\frac{1}{4}(3-p)}, \\ C_{01\ldots p} &= g_s^{-1}(1 - H_p(r)^{-1}). \end{aligned} \tag{1.14}$$

The function H_p is a harmonic function, as mentioned in section 1.1, a result of $\partial_{\perp}^2 H_p(r) = 0$, and g_s is the string coupling constant. The solution of the Laplace equation is

$$H_p = 1 + \frac{c_p g_s N_p l_s^{7-p}}{r^{7-p}}. \tag{1.15}$$

The number of p-branes is written as N_p.

1.4 Conserved quantities and thermodynamics

1.4.1 Conserved quantities

A important step is to quantify the physical quantities which are conserved under the equations of motions of our theory. The standard way of defining mass and angular momenta is by the method of extracting the information as if we were looking at a non-relativistic system. It should be noted that this procedure is only valid for p-branes of $p < 7$, the reason being that if the dimension of the brane is larger then the spacetime will not be asymptotically flat and a linearized gravity approximation will not make sense. We take a infinitesimal perturbation to the spacetime metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} << 1$. We should note that linearized gravity is a very useful tool and comes to use in the coming chapters. The time derivatives are neglected and we assume the tt-component of the stress-energy tensor to be much larger than the rest. The gravitational part of the action is the one relevant to these calculations

$$S = \int d^D x \bigg(\frac{\sqrt{-g} R[g]}{16\pi G_D} + \mathcal{L}_m \bigg). \tag{1.16}$$

Where G_D is the D dimensional Newton's constant and $\mathcal{L}_m$ is the Lagrangian of the matter fields. The equations of motion leading from the action are the D dimensional Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_D T^{(m)}_{\mu\nu}. \tag{1.17}$$

We can partially fix the diffeomorphism symmetry of the deviation to the metric by working in a harmonic gauge,

$$\partial_\nu (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^\lambda_\lambda) = 0. \tag{1.18}$$

This harmonic gauge condition makes it possible for us to identify $R_{\mu\nu} = (\partial^i \partial_i) h_{\mu\nu}$. Putting that identification into the Einstein equations results in a Laplace equation for $h_{\mu\nu}$, to which a general solution is well known. The solution reads

$$\begin{aligned} h_{\mu\nu}(x) &= \frac{16\pi G_D}{(D-3)\Omega_{D-2}} \int d^{D-1}\vec{y} \frac{\tilde{T}_{\mu\nu}(|\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|^{D-3}}, \\ \tilde{T}_{\mu\nu} &\equiv - \left[T^{(m)}_{\mu\nu} - \frac{1}{(D-2)} \eta_{\mu\nu} T^{(m)\lambda}_{\lambda} \right]. \end{aligned} \tag{1.19}$$

Where $\Omega_n = \mathrm{area}(S^n)$. Expanding this expression in moments and evaluating in rest frame gives us a form convenient for reading the mass and angular momenta off the metric. Then the ADM mass and angular momenta can then be read off from the metric

$$\begin{aligned} g_{tt} &\to -1 + \frac{16\pi G_D}{(D-2)\Omega_{D-2}} \frac{M}{r^{D-3}} + \ldots \\ g_{ij} &\to 1 + \frac{16\pi G_D}{(D-2)(D-3)\Omega_{D-2}} \frac{M}{r^{D-3}} + \ldots \\ g_{ti} &\to \frac{16\pi G_D}{\Omega_{D-2}} \frac{x^j J^{ij}}{r^{D-1}} + \ldots \end{aligned} \tag{1.20}$$

Now, lets move on to the definition of the conserved charge. For a p-brane charged under a R-R potential C_{p+1} the relevant field equation is

$$d * (dC_{p+1}) \propto *(J_{p+1}). \tag{1.21}$$

Where J_{p+1} is the conserved current of the system. The conserved Noether (electric) charge of the brane is the integral of the current, or equally, the integral of the dual field strength of $(dC)_{p+2}$ over the sphere S^{D-p-2},

$$Q_e = \int_{S^{D-p-2}} *(dC)_{p+2}. \tag{1.22}$$

We can calculate an associated magnetic charge in a dual frame by simply replacing $*(dC)$ with (dC) in the integral

$$Q_m = \int_{S^{p+2}} (dC)_{p+2} \tag{1.23}$$

Analogously to conventional four dimensional electromagnetism the potential obeys the Bianchi identity $d([dC]_{p+2}) = 0$. From that we can write a purely topological charge

$$P_{D-p-3} = \int_{S^{p+2}} (dC)_{p+2}. \tag{1.24}$$

For other branes than Dp-branes these derivations would have been slightly different but we are concentrating on the cases most relevant to black holes in string theory. In chapter 4 we will come back to some of these aspects in discussing a magnetic dual coupling of p-branes to q-form charge.

1.4.2 Black hole thermodynamics

The laws of black hole thermodynamic are in close analogy to the laws of thermodynamics, first put forth in [19]. They depend on the existence of classical no-hair theorems which state that a black hole is unique through the determination of its conserved quantities. The zeroth law states that the surface gravity, defined assuming the event horizon is a Killing horizon and denoted as $\hat{\kappa}$, is constant over the horizon of a stationary black hole. The surface gravity is determined by evaluating the gradient of the norm of the generating Killing field at the distance of the horizon from the origin point. Perturbed stationary black holes yield a relationship between the change in mass/energy to the change in angular momentum, horizon area, and charge. The first law states

$$dM = \hat{\kappa}\frac{dA}{8\pi} + \omega_H dJ + \Phi_e dQ, \tag{1.25}$$

where ω_H is the horizon angular velocity, $\omega_H = \frac{d\phi}{dt}|_{r=r_0}$ with ϕ as the angular coordinate of rotation, and Φ_e is the electrostatic potential of a charged black hole, $\Phi_e = -A_\mu k^\mu|_{r=r_0}$ where A_μ is the gauge field and k^μ is the Killing generator. The second law states, assuming the weak energy condition, that the horizon area cannot decrease with time through any physical processes,

$$\frac{dA}{dt} \geq 0. \tag{1.26}$$

The third law is that the surface gravity cannot vanish by means of a physical process such as emission of radiation.

The temperature of a thermally radiating black hole was calculated by Stephen Hawking by methods of semiclassical gravity and was found to be

$$T_H = \frac{\hbar\hat{\kappa}}{2\pi}. \tag{1.27}$$

The Bekenstein-Hawking entropy of a black hole in arbitrary dimension is

$$S_{\text{BH}} = \frac{A_d}{4\hbar G_d}, \tag{1.28}$$

where G_d is the d-dimensional Newton's constant and A_d is the area of the event horizon. It is noteworthy that the entropy of a black hole scales as area rather than, more intuitively, the volume. Susskind and 't Hooft proposed a principle in string theory called the holographic principle where the fundamental degrees of freedom of a system are characterized by a quantum field theory in one less dimension and with a Planck-scale ultra-violet cutoff. An explicit example of a conjecture in string theory that has a precise example of this principle is the AdS/CFT correspondence (anti-de Sitter/Conformal field theory). In fact, in context of the conclusion of chapter 4, where we review the derivation of a set of response coefficients characterizing the bending of charged black branes, there are signs of relevance and further development in relation to AdS/CFT. A review on holographic principle and an introduction to AdS/CFT can be found in the bibliography, [20] [21].

1.5 Dualities and solution-generating techniques

Duality transformations show the equivalence between theories. In superstring/M-theory we have three dualities, S- T- and U-duality. T-duality exchanges IIA and IIB by dimensional uplift and compactification on a torus. S-duality is a manifestation of the symmetry group of IIB and maps a theory on to the same IIB theory with an inverse string coupling constant. M-theory is a eleven dimensional theory, $d = 11$ being the maximal dimension for supersymmetry. The strong coupling limit of IIA is equivalent to M-theory, related by S-duality. Duality transformations can be utilized to obtain new solutions from already known solutions. The procedure of this solution generating technique is done by a uplift-boost-reduction. The known solution is lifted by adding an additional dimension that possesses a translational invariance. A boost is then performed along the direction of the new dimension and finally the extra dimension is compactified by either a Kaluza-Klein reduction or one of the duality transformations of the theory.

1.5.1 Kaluza-Klein reduction

Kaluza-Klein dimensional reduction is done by compactifying a dimension of the metric on a circle. This method is of high importance in chapter 4 where it will be used to generate charge from a bent neutral brane solution. We will provide an example of such a procedure in section 1.5.4. In Einstein frame a $d + 1$ dimensional metric decomposes as

$$ds^2_{d+1} = e^{2\alpha\phi} ds^2_d + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2, \tag{1.29}$$

$$\alpha^2 = 1/[2(D-1)(D-2)], \quad \beta = (2-D)\alpha. \tag{1.30}$$

With A_μ as a gauge field, ϕ the dilaton, and μ are the d-dimensional spacetime indices. The Lagrangian density will decompose as

$$\sqrt{-g_{d+1}} R_{d+1} = \sqrt{-g_d}\big(R_d - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(d-1)\alpha\phi} F^2_{(2)}\big), \tag{1.31}$$

with $F_{(2)}$ as the field strength of the gauge potential A_μ.

There is an intimate relation between the eleven dimensional M-theory and type-IIA string theory. They are simply related through dimensional reduction by compactification on a circle. Denote the 11th coordinate x_{11} and wrap the dimension on a circle of radius

$$R_{11} = g_s l_s. \tag{1.32}$$

Then the supergravity fields will decompose as

$$
\begin{aligned}
ds_{11}^2 &= e^{-2\Phi/3} ds_{10}^2 + e^{4\Phi/3}(dx^{11} + C_{1\mu} dx^{\mu})^2, \\
(\partial A_3) &= e^{4\Phi/3}(\partial C_3 - 2H_3 C_1) + \frac{1}{2} e^{\Phi/3}(\partial B_2) dx_{11}.
\end{aligned}
\tag{1.33}
$$

If the direction that the Kaluza-Klein reduction is used to compactify is boosted then from the lower-dimensional perspective, the off-diagonal components of the metric can be interpreted as a gauge-field. Thus by boosting a direction of a neutral solution and using Kaluza-Klein reduction to compactify that direction we can obtain a new solution charged under a gauge field.

1.5.2 T-duality

This symmetry is extremely relevant to our discussion in chapters three and four. It plays an integral role in generating the higher-form charge for elastically perturbed p-brane solutions in type-II string theory carrying a smeared q-brane charge (the concept of smearing is introduced at the end of this chapter).

T-duality is an inherent symmetry property in superstring theory. This can be demonstrated by noting that, although the massless spectrum of type-IIA- and B theories differ in ten dimensions, their spectrum matches when an isometry direction is compactified. T-duality can be seen as a map between type-IIA string theory compactified on a circle of radius R to type-IIB of radius l_s^2/R. The two theories are said to be T-dual to each other with IIA compactified on a large radius and IIB on a small radius. It can also be performed along a transverse direction, for example to generate a KK monopole. It generally changes the dimension of a brane depending on whether it is performed along a world-volume direction or transverse to the world-volume.

In addition to exchanging the two type-II theories, T-duality exchanges the winding and momentum modes of fundamental strings $F1$. The radius in string units is interchanged and the string coupling is left unchanged in one lower dimension. We illustrate this effect on the units by

$$
\frac{\tilde{R}}{\tilde{l}_s} = \frac{l_s}{R} \quad , \quad \frac{\tilde{g}_s}{\sqrt{\tilde{R}/\tilde{l}_s}} \quad , \quad \tilde{l}_s = l_s. \tag{1.34}
$$

A general set of rules, known as Buscher's rules, tells us how T-duality acts on the fields of a theory. For an isometry direction, z, Buscher's rules for the string metric $G_{\mu\nu}$, the two-form field $B_{\mu\nu}$, the scalar dilaton Φ, and the R-R fields C_n, acting on NS-NS fields are as follows [15] [22]:

$$\begin{aligned} e^{2\tilde{\Phi}} = e^{2\Phi}/G_{zz}, \quad \tilde{G}_{zz} = /G_{zz}, \quad \tilde{G}_{\mu z} = B_{\mu z}/G_{zz}, \\ \tilde{G}_{\mu\nu} = G_{\mu\nu} - (G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z})/G_{zz}, \\ \tilde{B}_{\mu\nu} = B_{\mu\nu} - (B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z})/G_{zz}. \end{aligned} \tag{1.35}$$

The gauge field transforms as

$$\tilde{C}^{(n)}_{\mu\ldots\nu\alpha z} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}G_{|\alpha]z}}{G_{zz}}. \tag{1.36}$$

The above equation is not the most general expression, although valid in special cases.

1.5.3 S-duality

This particular symmetry does not play a role in the developments reviewed in the next chapters, however it is worth mentioning a few words about it. Type-IIB string theory has a non-perturbative $SL(2, Z)$ symmetry. That symmetry is manifested in the S-duality. It maps the coupling constant to the inverse of another theory, $g_s \to 1/g_s$. In the low energy limit it is a $SL(2, \mathbb{R})$. If we look at the Z_2 subgroup of the symmetry, where $C_0 = 0$, we can observe some direct effects of S-duality. I we define $\tau = C^{(0)} + ie^{-\phi}$ we have

$$\begin{aligned} \tau &\to -1/\tau, \\ B_2 &\leftrightarrow C_2, \\ C_4^+ &\to C_4^+. \end{aligned} \tag{1.37}$$

In terms of the branes of the theory it means an interchange of $F1$ with $D1$ and $NS5$ with $D5$, while the $D3$ brane is left invariant.

1.5.4 Solution-generating techniques

Solution-generating techniques are important for our discussion in chapter 4, the examples of which we look at will be particularly relevant. In chapter 4 the steps leading to a bent neutral p-brane solution are presented and that solution is then used as a seed solution for the particular example shown in this section. We can obtain new solutions consisting of new desired attributes by employing different solution generating techniques. Kaluza-Klein reduction will be useful for obtaining solutions carrying a Maxwell charge (a $q = 0$ form) and T-duality will make it possible to fairly straightforwardly obtain black brane solutions with higher form gauge fields.

Let us look at an example of an uplift-boost-reduction procedure with the purpose of obtaining a dilatonic black p-brane with a Maxwell charge and KK dilaton coupling from a neutral solution. We can take a D-dimensional metric ds^2_D and add $m+1$ flat directions by taking a direct product with $\mathbb{R}^{m+1}$. It can be checked that such a procedure will also satisfy the Einstein equations. Let's write it as

$$ds^2_{d+1} = ds^2_D + \sum_{i=1}^{m}(dy_i)^2 + dx^2, \tag{1.38}$$

with $d = \tilde{p} + m + n + 3$. Adding extra dimensions to a solution is known as a *lift*. One of the extra dimensions added is separated from the other because it will be the isometry direction along which the boost and reduction will be performed. A boost will map a solution to another solution. The next step is to perform a uniform boost in the t- and x-directions with $[c_\kappa, s_\kappa]$ ($c_\kappa \equiv \cosh\kappa, s_\kappa \equiv \sinh\kappa$), with rapidity κ. The relevant metric components will then be:

$$\begin{aligned}
g^{(d+1)}_{tt} &= g_{tt}c^2_\kappa + s^2_\kappa, \\
g^{(d+1)}_{xx} &= g_{tt}s^2_\kappa + c^2_\kappa, \\
g^{(d+1)}_{tx} &= s_\kappa c_\kappa(g_{tt} + 1), \\
g^{(d+1)}_{tz_i} &= c_\kappa g_{tz_i}, \\
g^{(d+1)}_{xz_i} &= s_\kappa g_{tz_i}.
\end{aligned} \tag{1.39}$$

Now that we have the boosted metric we can perform the Kaluza-Klein reduction on the x-dimension. The dimensional reduction follows decomposition mentioned before, (1.25) and (1.27). Following that decomposition we find the expression for the d-dimensional metric, the newly acquired gauge field, and dilaton,

$$g^{(d)}_{\mu\nu} = e^{-2\alpha\phi}\left(g^{(d+1)}_{\mu\nu} - \frac{g^{(d+1)}_{\mu x}g^{(d+1)}_{\nu x}}{g^{(d+1)}_{xx}}\right), \qquad A_\mu = \frac{g^{(d+1)}_{x\mu}}{g^{(d+1)}_{xx}}, \qquad e^{2(2-d)\alpha\phi} = g^{(d+1)}_{xx}. \tag{1.40}$$

If the solution acquired after the uplift-boost-reduction procedure is identified as a solution in string theory it can be worked on further by using T-duality. The m remaining directions, which we are left with are produced in order to gain a higher-form charge. We leave the procedure of using T-duality until chapter 4, where it is used to compactify the m extra dimensions, given a particular seed solution identified with type-II string theory.

1.6 Smearing of branes and making black holes

The procedure of *smearing* branes results in a larger brane solution. It involves setting up an infinite array of branes, then approximating the sum by taking their periodicity to be infinitesimally small. Because $\partial_\perp^2$ is a linear operator we can construct a multi-center solution, $H_{\bar{p}}$. For a multi-center solution of only one kind of BPS branes the gauge field forces cancel the gravitational and dilatonic forces, so the system will be in static equilibrium. Thus from the metric (1.14) of the extremal BPS p-brane, we can write

$$H_{\bar{p}} = 1 + c_p g_s N_p l_s^{7-p} \sum_i \frac{1}{|x_\perp - x_{\perp i}|^{7-p}}. \tag{1.41}$$

Taking the sum from minus infinity to infinity with the periodicity of the branes being $2\pi R$ we can approximate the sum as an integral if we assume $x_\perp >> R$. Making appropriate changes of variables makes it possible to write that integral in a form that is straightforwardly solvable:

$$H_{\bar{p}} \approx 1 + c_p g_s N_p l_s^{7-p} \frac{1}{2\pi R} \frac{1}{\hat{r}^{7-[p+1]}} \int du \frac{1}{\sqrt{1+u^2}^{(7-p)}}. \tag{1.42}$$

Which with the integral solved and a little bit of rewriting becomes

$$H_{\bar{p}} \approx 1 + \left[\frac{N_p}{(R/l_s)}\right] g_s c_{p+1} \left(\frac{l_s}{\hat{r}}\right)^{7-[p+1]}. \tag{1.43}$$

In the limit we applied we can look at an array of branes as making up a linear density of branes. We call $H_{\bar{p}}$ a smeared harmonic function. Comparing this result with a harmonic function for a $(p+1)$ brane we can read off that the linear density of p-branes is the same as the number of $(p+1)$-branes:

$$N_{p+1} = \frac{N_p}{(R/l_s)}. \tag{1.44}$$

Through special intersection rules for BPS p-branes we can construct black hole solutions. The method used is known as the harmonic function rule. BPS black holes are extremal, so their temperature is zero. The harmonic function rule is a systematic ansatz for construction of supergravity solutions of pairwise intersections of BPS branes, developed in [23] [24] [25]. It involves both parallel and perpendicular intersections of branes. We restrict ourselves to harmonic functions that only depend on the transverse coordinates. The metric is factorized into a product structure with respect to intersection rules that we will not go into here. This procedure results in smeared intersecting brane solutions. As an example of the harmonic function rule we can construct a system with a $D5$ brane with a smeared $D1$. If we define a overall transverse coordinate, $r^2 = x_\perp^2 \equiv \sum_{i=1}^{4}(x^i)^2$, in string frame it gives the metric with the $D1$ brane smeared in directions $x^2 ... x^5$

$$\begin{aligned} ds_{10}^2 = H_1(r)^{-\frac{1}{2}} H_5(r)^{-\frac{1}{2}}(-dt^2 + dx_1^2) + H_1(r)^{\frac{1}{2}} H_5^{-\frac{1}{2}} dx_{2...5}^2 + \\ H_1(r)^{\frac{1}{2}} H_5^{\frac{1}{2}}(dr^2 + r^2 d\Omega_3^2). \end{aligned} \tag{1.45}$$

While the dilaton and gauge fields are

$$e^\Phi = H_1(r)^{\frac{1}{2}} H_5(r)^{-\frac{1}{2}}, \tag{1.46}$$

$$C_{01} = g_s^{-1}[1 - H_1(r)^{-1}], \quad C_{01...5} = g_s^{-1}[1 - H_5(r)^{-1}]. \tag{1.47}$$

Dimensional reduction can be performed for the solution to acquire different horizons, thus resulting in different entropies. Aside from this example, many other solutions can be constructed using the harmonic function rule.

The solutions constructed this way are generally extremal. The same kinds of BPS branes can be put together with the multi-center construction because they are in static equilibrium with each other. In many cases, non-extremal branes cannot satisfy this equilibrium, causing them to attract to each other and thus they will not satisfy the harmonic function rule.

Chapter 2

The Blackfold Approach

The blackfold approach is a framework in which we capture the long-range effective dynamics of fluid black branes living on a dynamical submanifold of a background metric. We call such objects *blackfolds*. One requirement for the physical system to which we want to apply the method is that the black object has two widely separated length scales of its horizon. In four dimensions all black hole horizons are approximately a sphere, the main reason for which is the Kerr bound $J \leq GM^2$. In higher dimensions we are not restricted by the Kerr bound and so we can have ultra-spinning black holes. In the case of a ultra-spinning Myers-Perry black hole, the result of taking the limit of the length scale associated with the angular momentum being much larger then the length scale of the mass is a pancaked horizon, equivalent to a black brane. If we identify the larger scale as R and the smaller, horizon thickness, as r_0, we are able to approximate the dynamics of the blackfold in expansions of the infinitesimal factor $r_0/R << 1$. This method gives us the opportunity to discover new higher dimensional black hole solutions by wrapping black branes along a submanifold of a desired topology [11] [10] [12]. The discussion of the blackfold approach presented here will mostly base on the work of [26] and [27] where the blackfold approach is applied to branes carrying brane currents.

2.1 Blackfolds

2.1.1 Relevant tools

A few mathematical tools are needed for the discussion of the blackfold approach. The tools relevant to the world-volume geometry have to be put forth. Embedding of a black brane with world-volume $\mathcal{W}_{p+1}$ in a background metric $g_{\mu\nu}$ is parameterized by two main quantities. First, we note that the position of the brane in the background metric is written in terms of coordinates $X^\mu(\sigma^a)$. The first embedding quantity we mention is the induced world-volume metric of the brane,

$$\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{2.1}$$

where $a, b = 0, \ldots, p$ and $\mu, \nu = 0, \ldots, D-1$. The second quantity is the first fundamental form of the submanifold

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab}, \tag{2.2}$$

which acts as a projection operator onto the world-volume coordinates. We can similarly define a projector onto the space transverse to the world-volume,

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu}. \tag{2.3}$$

Tensors living in the world-volume $\mathcal{W}_{p+1}$ can only have well defined covariant derivatives in the tangential directions, so we define a tangential covariant derivative as

$$\bar{\nabla}_\mu = h^\nu_\mu \nabla_\nu. \tag{2.4}$$

Lastly we define the *extrinsic curvature tensor* as

$$K_{\mu\nu}^{\ \ \rho} = h^\sigma_\mu \bar{\nabla}_\nu h^\rho_\sigma. \tag{2.5}$$

2.1.2 The blackfold equations

Let us consider a stress-energy tensor, $T^{\mu\nu}$, of a black brane. It has a support on the $p+1$ dimensional world-volume. Having no orthogonal components, it satisfies the tangentiality condition

$$\perp^\rho_\mu T^{\mu\nu} = 0. \tag{2.6}$$

We make two basic assumptions of the stress-energy tensor. The first is that it derives from the dynamics of general relativity, the second being that spacetime diffeomorphism invariance holds. Since the stress-energy tensor is vanishing in orthogonal components we write the conservation equations with the tangential covariant derivative,

$$\bar{\nabla}_\mu T^{\mu\rho} = 0. \tag{2.7}$$

Decomposing the expression into tangential and orthogonal components gives us specially intrinsic and extrinsic equations of the brane dynamics. The decomposition can be done in the following way,

$$\begin{aligned} \bar{\nabla}_\mu T^{\mu\rho} &= \bar{\nabla}_\mu (T^{\mu\nu} h^\rho_\nu) = T^{\mu\nu} \bar{\nabla}_\mu h^\rho_\nu + h^\rho_\nu \bar{\nabla}_\mu T^{\mu\nu} \\ &= T^{\mu\nu} h^\sigma_\nu \bar{\nabla}_\mu h^\rho_\sigma + h^\rho_\nu \bar{\nabla}_\mu T^{\mu\nu} \\ &= T^{\mu\nu} K^\rho_{\mu\nu} + \partial_b X^\rho D_a T^{ab}. \end{aligned} \tag{2.8}$$

The D equations can be split into, respectively, $D-p-1$ extrinsic equations and $p+1$ intrinsic equations,

$$T^{\mu\nu} K^\rho_{\mu\nu} = 0, \tag{2.9}$$

$$D_a T^{ab} = 0. \tag{2.10}$$

The extrinsic equation (2.9) is a special case of Carter's equation [26] [28]. Here we are ignoring any background fields that might couple to the p-brane. If we define $H_{[p+2]}$ to be the field strength of a given background potential, Carter's equation written out fully are [28]

$$T^{\mu\nu} K^\rho_{\mu\nu} = \frac{1}{(q+1)!} \perp^\rho_\sigma J_{\mu_0 \ldots \mu_q} H^{\mu_0 \ldots \mu_q \sigma}. \tag{2.11}$$

The extrinsic equations are generalizations of the geodesic equation for free point particles to p-branes. The similarity of the extrinsic equations to the geodesic equation can be seen by writing them explicitly in terms of the embedding $X^\mu(\sigma^a)$,

$$T^{ab}(D_a \partial_b X^\rho + \Gamma^\rho_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) = 0. \tag{2.12}$$

While the geodesic equation for free falling point particles reads,

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{2.13}$$

2.2 Dynamics of blackfolds carrying electric charge

The framework for blackfolds carrying a brane charge was first developed in [26] [27]. With the inclusion of q-brane charges carried by the p-brane it is necessary to introduce some new quantities. Aside from our p-brane being charged under a p-form R-R potential we can have a dissolved q-brane charge on the world-volume $\mathcal{W}_{p+1}$. Now we should have a local charge density $\mathcal{Q}_q$ of q-branes on the world-volume. We define Φ_q as the local potential conjugate to the charge density. The induced metric along the world-volume of the q-brane current we identify as $h_{ab}^{(q)}$. In addition we have the global counterparts to the local charge density and conjugate, Q_q (note that $\mathcal{Q}_q = Q_q$) and $\Phi_H^{(q)}$.

This section will start out by outlining the dynamics of branes carrying a 0-brane (particle) current, then move up to branes carrying string currents and finally higher form charges. There are a lot of properties that stay the same in the analysis throughout the different cases. It consists of first analyzing the form of the stress-energy tensor in presence of a brane charge, then examine how the extrinsic and intrinsic dynamics look for the relevant cases.

Let's start by saying a few words about the simple case of branes carrying Maxwell charge. When $q = 0$ the fluid brane retains its spatial isotropy and is characterized by a eigenvalue energy density ε. We can write the stress-energy tensor in a perfect fluid form,

$$T_{ab} = \varepsilon u_a u_b + P(\gamma_{ab} + u_a u_b), \tag{2.14}$$

with pressure P and a unit-normalized timelike eigenvector u. Defining a charge density $\mathcal{Q}$ on the fluid, we write the particle current proportional to the velocity vector,

$$J_a = \mathcal{Q} u_a. \tag{2.15}$$

Let us move on and examine the attributes of branes carrying higher-form ($q > 0$) charge.

2.2.1 Branes carrying a string current

Branes carrying a $q > 0$ current have a broken isotropy. Normally, with a spatially isotropic brane the stress-energy tensor would follow the form of (2.14). Since, in this case, we have a string current, there is a two-form world-volume current J_{ab} present associated with it. We can define a spacelike vector $v^b = u_a J^{ab}$, which is orthogonal to u. Demanding v to be normalized to one makes it possible for us to write the two-current as

$$J_{ab} = \mathcal{Q}(u_a v_b - v_a u_b). \tag{2.16}$$

In the absence of dissipative currents, v is an eigenvector of T_{ab}. The new velocity vector v characterizes the directions along which the dissolved string charges lie. Although, not totally isotropic, the stress-energy tensor of the fluid is still isotropic in transverse spatial directions. So we can rewrite the stress-energy tensor split in terms of parallel and orthogonal pressures,

$$T_{ab} = \varepsilon u_a u_b + P_{\|} v_a v_b + P_{\perp}(\gamma_{ab} + u_a u_b - v_a v_b). \tag{2.17}$$

The introduction of the vector v through the string charge breaks the isotropy. This pressure difference between $P_{\|}$ and $P_{\perp}$ is essentially the result of the energy density of the strings $\Phi\mathcal{Q}$. So generally for branes carrying a higher form charge we have a pressure difference, in this case,

$$P_{\perp} - P_{\|} = \Phi\mathcal{Q}. \tag{2.18}$$

Where Φ is the string chemical potential.

Branes carrying string currents obey some recognizable thermodynamic relations, on their world-volume thermodynamic equilibrium is satisfied locally which leads to the first law

$$d\varepsilon = \mathcal{T} ds + \Phi d\mathcal{Q}, \tag{2.19}$$

where $\mathcal{T}$ is the local temperature. The setting also satisfies the thermodynamic Gibbs-Duhem relations,

$$\varepsilon + P_{\perp} = \mathcal{T} s + \Phi\mathcal{Q}, \tag{2.20}$$

$$dP_{\perp} = s d\mathcal{T} + \mathcal{Q} d\Phi, \quad dP_{\|} = s d\mathcal{T} - \Phi d\mathcal{Q}. \tag{2.21}$$

The current and the stress-energy tensor must obey, respectively, the current continuity equations and the intrinsic fluid equations,

$$d * J = 0, \quad D_a T^{ab} = 0. \tag{2.22}$$

2.2.2 Branes carrying p-brane charge

A p-brane current carried by a fluid brane with world-volume $\mathcal{W}_{p+1}$ has the form

$$J = Q_p \hat{V}_{(p+1)}. \tag{2.23}$$

Here we have introduced $\hat{V}_{(p+1)}$ as the volume form on $\mathcal{W}_{p+1}$. The charge is assumed to be conserved, and so it must be constant along the world-volume, $\partial_a Q_p = 0$. The characterizing quantities in this case is the same as for a neutral fluid, the velocity vector u and the energy density ε. The perfect fluid is characterized by the stress-energy tensor and the equation of state, where Q_p manifests. The form of the stress-energy tensor is that of (2.14). As in the previous case of branes carrying string charge, locally the fluid satisfies the thermodynamic relations

$$d\varepsilon = \mathcal{T} ds \quad , \quad \varepsilon + P = \mathcal{T} s, \tag{2.24}$$

with local temperature $\mathcal{T}$ and entropy density s. The conservation of entropy can be seen from parallel components of the fluid equations $D_a T^{ab} = 0$ to the local velocity u^a and using the thermodynamic relations (2.24),

$$D_a(s u^a) = 0. \tag{2.25}$$

The transverse part of the equation results in the Euler force equations,

$$(\gamma^{ab} + u^a u^b)(\dot{u}_b + \partial_b \ln \mathcal{T}) = 0. \tag{2.26}$$

The extrinsic elastic dynamics, or the dynamics of the fluid within the background spacetime, are governed by Carter's equations (2.11). We will however neglect background fields as they are not considered in the chapters to come. The extrinsic equations can thus be written as

$$-PK^{\rho} = \perp^{\rho}_{\mu} s \mathcal{T} \dot{u}^{\mu}. \tag{2.27}$$

The interpretation of this equation is that it captures the acceleration in the orthogonal directions to the world-volume due to the force exerted by the effect of the extrinsic curvature.

- Stationary configurations

In continuation of the discussion of branes carrying a p-brane current we look at stationary configurations. The bent brane solutions we construct in chapter 4 are stationary so it is informative in context of the later chapters to look at stationary solutions. The velocity field is now aligned with a Killing vector k along the world-volume. We assume that $k = k^{\mu}\partial_{\mu}$ generates isometries in both the world-volume and the background spacetime. In that way we write,

$$u = \frac{k}{|k|} \quad , \quad \nabla_{(\mu}k_{\nu)} = 0, \tag{2.28}$$

and $\dot{u}_{\mu} = \partial_{\mu} \ln |k|$. In this setup the local temperature $\mathcal{T}$ is simply the redshifted global temperature T by the absolute value of the killing vector,

$$\mathcal{T}(\sigma^a) = \frac{T}{|k|}. \tag{2.29}$$

With the stationary configuration determined in this way the extrinsic equations (2.27) take the form

$$K^{\rho} = \perp^{\rho\mu}\partial_{\mu} \ln(-P). \tag{2.30}$$

It must be specified in which way k is related to the background Killing vector ξ, that signifies unit-time translations at asymptotic infinity, to determine the physical quantities of the system. We write

$$k = \xi + \Omega\chi. \tag{2.31}$$

Here χ is a spacelike Killing vector orthogonal to ξ, a generator of rotations, and Ω is a angular velocity constant. The Killing vector ξ is assumed to be orthogonal to spacelike hypersurfaces $\mathcal{B}_p$ on the world-volume, and we say that the unit-normal is

$$n^a = \frac{1}{R_0}\xi^a|_{\mathcal{W}_{p+1}}, \tag{2.32}$$

where R_0 measures the local gravitational shifts between points on the world-volume. Working out the properties a little bit and writing integrations on the world-volume as integrals over the spacelike hypersurfaces $\mathcal{B}_p$ with measure $dV_{(p)}$ [27] results in the mass, angular momentum, and entropy, obtained as integrals over spacelike hypersurfaces $\mathcal{B}_p$ with a measure $dV_{(p)}$,

$$\begin{aligned} M &= \int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \xi^b, \quad J = -\int_{\mathcal{B}_p} dV_{(p)} T_{ab} n^a \chi^b, \\ S &= -\int_{\mathcal{B}_p} dV_{(p)} s u_a n^a. \end{aligned} \tag{2.33}$$

Although a different way of extracting the conserved quantities, this is effectively equivalent to what we did before in the simple case of linearized gravity in chapter 1.

2.2.3 Thermodynamic properties of the blackfold fluid

Moving on to read off the characterizing quantities from the effective stress-energy tensor will be relevant to the discussion in chapters 3 and 4. Following the charged dilatonic p-brane solutions of the action presented in section 1.1 with (1.5), conserved quantities and thermodynamics can be calculated from the effective stress-energy tensor as in terms of the horizon thickness r_0 and charge parameter α. We have

$$\varepsilon = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n (n+1+nN\sinh^2\alpha), \quad P = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n (1+nN\sinh^2\alpha). \tag{2.34}$$

The local temperature $\mathcal{T}$, the local entropy density s, the charge density Q_p and the local chemical potential Φ are

$$\begin{aligned} \mathcal{T} &= \frac{n}{4\pi r_0 (\cosh\alpha)^N}, \quad s = \frac{\Omega_{(n+1)}}{4G} r_0^{n+1} (\cosh\alpha)^N, \\ Q_p &= \frac{\Omega_{(n+1)}}{16\pi G} n\sqrt{N} r_0^n \cosh\alpha \sinh\alpha, \quad \Phi = \sqrt{N}\tanh\alpha. \end{aligned} \tag{2.35}$$

The chemical potential Φ is the value of the difference of the gauge field estimated at the horizon and at spatial infinity. The Gibbs free energy, defined as

$$\mathcal{G} = \varepsilon - \mathcal{T}s - \Phi_p Q_p = -P - \Phi_p Q_p \tag{2.36}$$

takes the form,

$$\mathcal{G} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n. \tag{2.37}$$

The parameters r_0 and α encode the short scale structure of the geometry of the black brane. The inclusion of charge results in an additional tension on the brane world-volume, which can becomes apparent when certain relations are worked out between the physical quantities [27].

The Gibbs free energy density can be expressed as

$$\mathcal{G} = \frac{1}{n}\mathcal{T}s, \tag{2.38}$$

which can be interpreted as the Smarr relation in a Minkowski background.

2.2.4 Boundaries, extremality, and near-extremality

The fact that the charge Q_p has to be conserved restricts blackfolds with p-brane charges to no open boundaries. It is however possible that the blackfold meets the end of another brane that carries the charge.

When these kinds of black branes are taken to extremal limits ($r_0 \to 0$ and $\alpha \to \infty$) they reach a point where their horizon becomes degenerate. Most of those solutions have a singular horizon in the extremal limit but they can be approached by limits of solutions that have non-degenerate solutions [29]. Even though the intrinsic dynamics are not that of fluid dynamics, the extremal limit solutions can have good physical interpretations and we are able to apply the blackfold method on them.

It can be shown that the ground state of the charged p-brane corresponds to the extremal limit so we can examine near-extremal p-branes, with a given charge Q_p, by writing the excited states of the stress-energy tensor as

$$T_{ab}^{(exc)} = T_{ab} - T_{ab}^{(ground)}. \tag{2.39}$$

We can obtain the near-extremal limit by taking this difference to be as small as possible, which happens when the chemical potential $\Phi \simeq \sqrt{N}$ [27]. When $N/2 - 1/n = 1/(p+1)$ the stress tensor is traceless, resulting in a vanishing of the dilaton coupling. There are many known cases of these kinds of branes in string theory.

This point does not play a particular role in this thesis but it was worth mentioning a few words about it. Let us move on to the discussion of fluid branes carrying a higher-form ($q \leq p$) charge.

2.2.5 Black branes with $q \leq p$ currents

In the coming chapters solutions and dynamics of bent p-branes carrying higher-form charge is examined, so the discussion of $q > 1$ is particularly important. In chapter 4 a class of stationary bent p-brane solutions in type-II string theory is obtained that carry smeared $q \leq p$ charges. We will present the dynamics and thermodynamic properties for the general case of a p-brane carrying a arbitrary dissolved q-brane charge, only restricted by $0 \leq q \leq p$. For the purpose of this thesis it is enough that we constrain ourselves to the case of a perfect fluid description of the p-brane. This assumption is enough to construct stationary configurations.

Each of the q-branes living on the world-volume $\mathcal{W}_{p+1}$ constitutes a subworld-volume $\mathcal{C}_{q+1}$, to which we associate a unit volume-form $\hat{V}_{q+1}$. The conserved current is defined through the volume form and the charged density,

$$J_{q+1} = \mathcal{Q}_q \hat{V}_{q+1}. \tag{2.40}$$

We will assume that J_{p+1} satisfies the continuity equation for the higher-form current $d * J_{(q+1)} = 0$. As we saw before when there is a charge of higher form than Maxwell ($q = 0$) charge, the q-brane currents cause a pressure difference in directions orthogonal and parallel to the brane, making the fluid anisotropic.

The thermodynamic relations

$$d\varepsilon = \mathcal{T} ds + \sum_q{}' \Phi_q d\mathcal{Q}_q, \tag{2.41}$$

are satisfied locally in the fluid. The prime on the sum indicates that $q = p$ is not included. Introducing the Gibbs free energy density as

$$\mathcal{G} = \varepsilon - \mathcal{T} s - \sum_q \Phi_q \mathcal{Q}_q, \tag{2.42}$$

we can write the stress-energy tensor as

$$T_{ab} = \mathcal{T} s u_a u_b - \mathcal{G} \gamma_{ab} - \sum_q \Phi_q \mathcal{Q}_q h^{(q)}_{ab}. \tag{2.43}$$

Where $h^{(q)}_{ab}$ is the induced metric on the q-brane world-volume $\mathcal{C}_{q+1}$. The authors of [27] point out that this form of the stress-energy tensor is not completely proven to be the universal form of any charged brane, but it can safely be presumed to be true for a large class of branes and for black branes in type-II/M supergravity along with their toroidal compactifications. There are two noteworthy points that can be made at this point. The brane densities $\mathcal{Q}_q$ are constant along their world-volume and can only vary in $p-q$ directions transverse to the current so they are in a sense quasi-local. The other point is that the world-volume of two different brane currents with $q \leq q'$ do not necessarily intersect so they can in fact have currents along different directions.

The Smarr relation for p-branes in type-II/M supergravity, compactified on a torus reads

$$\varepsilon = \frac{n+1}{n}\mathcal{T}s + \sum_q \Phi_q \mathcal{Q}_q, \tag{2.44}$$

which can be rewritten using (2.42) as

$$\mathcal{G} = \frac{1}{n}\mathcal{T}s. \tag{2.45}$$

Thus we are able to express the general stress-energy tensor more conveniently as

$$T_{ab} = \mathcal{T}s\left(u_a u_b - \frac{1}{n}\gamma_{ab}\right) - \sum_q \Phi_q \mathcal{Q}_q h^{(q)}_{ab}. \tag{2.46}$$

The extrinsic equations for a fluid black p-brane carrying a q-brane charge now take the form

$$\mathcal{T}s\perp^{\rho}_{\mu}\dot{u}^{\mu} = \frac{1}{n}\mathcal{T}sK^{\rho} + \perp^{\rho}_{\mu}\sum_q \Phi_q \mathcal{Q}_q K^{\mu}_{(q)}, \qquad K^{\mu}_{(q)} = h^{ab}_{(q)}K^{\mu}_{ab}. \tag{2.47}$$

$K^{\mu}_{(q)}$ is the mean curvature vector of the embedding of the q-brane world-volume in the background spacetime.

2.3 Stationary charged blackfolds

We move on to the determination of the charges, potential, and thermodynamics of stationary perfect fluid branes carrying a q-brane charge. Again, as before, they have a velocity vector aligned with a Killing vector. The temperature redshifts as

$$\mathcal{T}(\sigma^a) = \frac{T}{|k|} \quad , \tag{2.48}$$

where T is the global temperature. The local temperature is obtained simply by redshifting the global temperature. According to the equipotential condition for electric equilibrium [26], the potential $\Phi_q(\sigma)$ in the stationary case does not depend on the Killing time nor directions transverse to the current. The global q-brane potential for the stationary configuration is thus obtained as an integration over spatial directions along the current,

$$\Phi_H^{(q)} = \int d^q\sigma |h^{(q)}(\sigma^a)|^{1/2} \Phi_q(\sigma), \tag{2.49}$$

which is a constant.

The relation of the Killing vector k to the killing vector of the background spacetime is assumed to be the same as the stationary configuration for the branes carrying a p-brane current. The conserved quantities would then be obtained in the same way.

The q-brane charge is obtained by considering the spatial sections of $\mathcal{C}_{q+1}$, the world-volume of the q-brane, that are orthogonal to the normal n^a. We arrive at it by integrating the charge density over transverse directions to the current, with the introduction of a unit q-form, $\omega_{(q)}$, orthogonal to n^a. So we have

$$Q_q = -\int_{\mathcal{B}_{p-q}} dV_{(p-q)} J_{(q+1)} \cdot (n \wedge \omega_{(q)}) = \int_{\mathcal{B}_{p-q}} dV_{p-q} \sqrt{-h_{ab}^{(q)} n^a n^b}\, \mathcal{Q}_q(\sigma) \quad . \tag{2.50}$$

And the global chemical potential becomes

$$\Phi_H^{(q)} = \int dV_{(q)} \frac{R_0}{\sqrt{-h_{ab}^{(q)} n^a n^b}} \Phi_q(\sigma) \quad . \tag{2.51}$$

These expressions for a general q-brane current simplify into the special cases by simply plugging in the appropriate values of q.

The extrinsic equations can be obtained for the stationary configurations from varying the action

$$I = -\int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\mathcal{G} = -\delta t \Big(M - TS - \Omega J - \sum_q \Phi_H^{(q)} Q_q \Big) \quad , \tag{2.52}$$

while keeping T, Ω, and $\Phi_H^{(q)}$ constant. This leads to us seeing that the analogous global first law of thermodynamics for variations of the brane embedding is equivalent to the extrinsic equations,

$$dM = TdS + \Omega dJ + \sum_q \Phi_H^{(q)} dQ_q \quad . \tag{2.53}$$

2.4 Black p-brane solutions with dissolved q-brane charges

Since our discussion will ultimately lead us obtaining bent brane solutions carrying smeared higher-form charge it is natural to review the leading order solutions. The appendix of [26] shows in detail how to obtain black p-brane solutions carrying dissolved q-brane charge starting from a class of solutions obtained by Gibbons and Maeda. It is a solution of an action of the same form as introduced in chapter 1 with (1.4). The details of the derivation of the solution will not be presented but the outline of the steps leading to it will be presented as a special class of the Gibbons-Maeda solutions will become important later.

The Gibbons-Maeda solution is an Einstein-Maxwell-dilaton (EMD) solution of a specific action for a $d = n + 3$ spherical charged black hole with event horizon r_0 and charge parameter α. The desired solution is arrived at by a sequence of uplifts of the solution. The solution is first uplifted to $n + q + 3$ dimensions to obtain q-brane charges in the same number of extended directions then, again uplifted by $p - q$ extra dimensions, leading to the final solution.

The starting solution can be put in the form,

$$d\tilde{s}^2 = -fh^{-\tilde{A}}dt^2 + h^{\tilde{B}}(f^{-1}dr^2 + r^2 d\Omega_{n+1}^2). \tag{2.54}$$

In this expression f is defined in the same way as is chapter 1, h is a radial function that incorporates the charge parameter α, and the coefficients $\tilde{A}$ and $\tilde{B}$ are related to the dilaton coupling $\tilde{a}$ of the theory. The tilde marks over the quantities is simply to identify them as the ones associated with the d-dimensional starting solution, in contrast to the uplifted ones. We write the first uplift we form a warped product of the original metric, which we write as $\tilde{g}_{\mu\nu}$ with coordinates x^μ, and q flat extra dimensions with coordinates y^m,

$$ds^2 = e^{2\alpha\phi(x)}\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + 2e^{2\beta\phi(x)}\delta_{mn}(y)dy^m dy^n. \tag{2.55}$$

We write a $(q+1)$-from electrical ansatz for the gauge potential in such a way that, upon compactification is a Maxwell potential, and with the q indices lying in the added extended directions,

$$B_{[q+1]} = \tilde{B}_{[1]} \wedge dy^1 \wedge \ldots \wedge dy^p, \tag{2.56}$$

where $\tilde{B}_{[1]}$ is the original gauge field. The dilaton ϕ is taken to be proportional to the original one by $\phi = \gamma\tilde{\phi}$, where γ is a undetermined coefficient.

Now, writing the new Ricci scalar in terms of the one of the original solution and the dilaton and ensuring that the d-dimensional solution also obeys the $D = n + q + 3$-dimensional equations of motion we are able to solve for the dilaton coupling and the other different coefficients. We will not write the explicit determination of the coefficients here. The uplifted $(n + q + 3)$-dimensional solution we end up with has the metric,

$$ds^2 = -h^{-A}(fdt^2 + d\vec{y}^2) + h^B(f^{-1}dr^2 + r^2 d\Omega^2_{(n+1)}), \tag{2.57}$$

with

$$A = \frac{4n}{2n(q+1) + (n+q+1)a^2}, \quad B = \frac{4(q+1)}{2n(q+1) + (n+q+1)a^2}. \tag{2.58}$$

The new $(q + 1)$-form gauge field reads

$$B_{[q+1]} = -\sqrt{A+B}\frac{r_0^n}{r^n h(r)} \sinh\alpha \cosh\alpha dt \wedge \epsilon, \tag{2.59}$$

with $\epsilon \equiv dy^1 \wedge \ldots \wedge dy^p$. Finally, the dilaton is given by

$$\phi = -\frac{1}{4}(A+B)a \ln h(r). \tag{2.60}$$

Note that the coefficients A and B always enter as $A+B$ so instead of writing them separately we simply identify them with $N = A + B$.

The next step to obtaining the black p-brane solutions with diluted general q-brane charge ($q \leq p$) is to uplift the solution obtained above and uplifting it by $p - q$ extra extended directions. We will denote the new extra dimensions collectively by coordinates z^a. A dimensional reduction is performed along the y^m directions and the uplift procedure for the new directions is precisely the same as we just did above. We will simply write the solution for the higher-form charged black p-branes. The $D = n + p + 3$ dimensional metric reads

$$ds^2 = -h^{-A}(fdt^2 + d\vec{y}^2) + h^B(f^{-1}dr^2 + r^2 d\Omega^2_{(n+1)} + d\vec{z}^2), \tag{2.61}$$

with

$$N = A + B = \frac{4(n+p+1)}{2(q+1)(n+p-q) + (n+p+1)a^2}. \tag{2.62}$$

The solution for the gauge field is

$$B_{[q+1]} = -\sqrt{N}\frac{r_0^n}{r^n h(r)} \sinh\alpha \cosh\alpha dt \wedge dy^1 \wedge \ldots \wedge dy^q, \tag{2.63}$$

and the dilaton solution reads

$$\phi = -\frac{1}{4} N a \ln h(r). \tag{2.64}$$

As expected, this solution reduced to the previous one when $p = q$ and setting $q = 0$ one obtains the general p-brane solutions arising from EMD theory.

We conclude our discussion of the blackfold approach for charged branes and move on to the extension of the blackfold method to 1st order in multipole expansion of the stress-energy tensor, incorporating the finite thickness effects of charged black branes.

Chapter 3

Pole-dipole order charged blackfolds

To the first order of derivative expansion in the blackfold approach the dynamics of black branes can be approximated in a way similar to the approximation of planetary bodies following geodesics as point particles. Higher order corrections yield effects incorporating a minute thickness and gravitational effects such as backreaction. This chapter will follow the works done in [13] and [14], where the blackfold approach is extended to first order in derivative expansion to demonstrate dipole effects arising from bending a black brane. As well as accounting for dipole moments the correction takes into account the internal degrees of freedom. The formalism for the bending effects of charged black branes are arrived at assuming a linear response theory (this assumption is later verified), analogous to classical fluid dynamics and material science.

Up until now we have worked with a zero thickness stress-energy tensor with a support on its world-volume. To leading order the ADM stress-energy tensor of black p-branes is approximated as

$$T^{\mu\nu} = \int d^{p+1}x\sqrt{-\gamma}T^{\mu\nu}_{(0)}\frac{\delta^D(x^\alpha - X^\alpha(\sigma^a))}{\sqrt{-g}}, \tag{3.1}$$

where γ is the induced metric on the brane, $T^{\mu\nu}_{(0)}$ is the monopole source of the stress-energy, x^α are the spacetime coordinates, and the location of the brane world-volume, $\mathcal{W}_{p+1}$, in the spacetime is given by $X^\alpha(\sigma^a)$. Our goal is to extend on this monopole approximation to capture higher order effects. We will only focus on the fine structure corrections. In order to do that we require the gravitational self-reaction effects to be subleading to the ones that incorporate the bending effects. The corrections are characterized by power expansions in $(r_0/R)^n$, where n is the codimension. For codimension $n > 2$ it has been shown that the self-gravitational corrections are subleading to the fine structure corrections.

3.1 The effective stress-energy tensor and effective current

The equations of motion for the first order corrected charged pole-dipole branes are arrived at by solving the conservation equations for the appropriate effective stress-energy tensor and effective current, so our starting point of this discussion will be to take a look at the multipole expansion of those quantities. Earlier in our discussion of the blackfold approach, the branes were taken to be infinitely thin. The problem is that the effects of bending don't manifest themselves unless the brane is of finite thickness. To visualize this we can picture a metal rod being bent into the shape of a circular arc. The material of which the rod consists of will stretch at the outer points from the center of curvature and the inner points will condense, effectively resulting in a dipole distribution of the material. Allowing higher orders in the multipole expansion (3.1) of the effective stress-energy tensor will include the finite thickness effects that we wish to analyze,

$$\begin{aligned} T^{\mu\nu}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\Bigg[T^{\mu\nu}_{(0)}(\sigma^a)\frac{\delta^D(x^\alpha - X^\alpha(\sigma^a))}{\sqrt{-g}} - \\ \nabla_\rho\left(T^{\mu\nu\rho}_{(1)}(\sigma^a)\frac{\delta^D(x^\alpha - X^\alpha(\sigma^a))}{\sqrt{-g}}\right) + \ldots\Bigg]. \end{aligned} \tag{3.2}$$

In the expansion, $T^{\mu\nu}_{(0)}$ is the conventional monopole source of the stress-energy and $T^{\mu\nu\rho}_{(1)}$ is the new structure term that encodes the finite thickness effects. This expansion is analogous to a conventional multipole expansion used in electrodynamics for a distribution of charge. Truncating the expression to zeroth order we obtain the zero thickness expression (3.1) which leads us to the blackfold equations discussed in the previous chapter but here we are considering a truncation to the first order which will lead us to the dynamics of pole-dipole branes in a curved background spacetime.

The multipole expansion (3.2) is written in a covariant way, invariant under spacetime diffeomorphisms and reparametrizations of world-volume. However, $T^{\mu\nu}$ enjoys two other additional symmetries. It is invariant under two gauge transformations coined by the authors of [30] as "extra symmetry 1" and "extra symmetry 2". A description of them is in order before we continue as they are an important part of understanding pole-dipole brane physics. These symmetries will be explained further in the next section.

The stress-energy tensor should fulfill the covariant conservation equations

$$\nabla_\nu T^{\nu\mu} = 0, \tag{3.3}$$

and we wish to write out the equations of motion following from them using the expansion (3.2). For later convenience let's decompose $T_{(1)}^{\mu\nu\rho}$ in tangential and orthogonal components,

$$T_{(1)}^{\mu\nu\rho} = u_b^{(\mu} j^{(b)\nu)\rho} + u_a^\mu u_b^\nu d^{ab\rho} + u_a^\rho T_{(1)}^{\mu\nu a}. \tag{3.4}$$

The reason for the parenthesis around "b" means that it is insensitive to the symmetrization of the greek indices. Two quantities have been introduced in this expression, $j^{b\nu\rho}$ and $d^{ab\rho}$. We give $j^{b\nu\rho}$ the interpretation of a current density of transverse angular momenta, where we have defined $j^{a\mu\nu} = 2T_{(1)}^{a\mu\nu}$. The components of the angular momenta have the properties $j^{b\nu\rho} = j^{b[\nu\rho]}$ and $u_\nu^a j^{b\nu\rho} = 0$. Both greek indices should be read to be orthogonal to the brane world-volume. The other, $d^{ab\rho}$, has the interpretation of the bending moment of the brane. The bending moment components have the properties $d^{ab\rho} = d^{(ab)\rho}$ and $u_\rho^c d^{ab\rho} = 0$. The second property of the quantities is just a consequence of the greek indices being orthogonal to the world-volume. In section 3.4 the physical interpretation of the various quantities will be discussed further.

The branes we consider from here on are generally charged under a $(q+1)$-form gauge field, except when we take explicit examples leading up to the $(q \leq p)$-form charge. In the same way as we expanded the stress-energy tensor in multipoles we should expand the total anti-symmetric current tensor $J^{\mu_1 \cdots \mu_{q+1}}$ which lives on the brane world-volume [31]:

$$\begin{aligned} J^{\mu_1 \cdots \mu_{q+1}}(x^\alpha) = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma \sqrt{-\gamma} \Bigg[& J_{(0)}^{\mu_1 \cdots \mu_{q+1}}(\sigma^a) \frac{\delta^D(x^\alpha - X^\alpha(\sigma^a))}{\sqrt{-g}} \\ & - \nabla_\rho \left(J_{(1)}^{\mu_1 \cdots \mu_{q+1}\rho}(\sigma^a) \frac{\delta^D(x^\alpha - X^\alpha(\sigma^a))}{\sqrt{-g}} \right) + \ldots \Bigg]. \end{aligned} \tag{3.5}$$

Just the same as with the stress-energy tensor $J_{(0)}$ encodes the monopole source of the charged q-brane current and $J_{(1)}$ encodes the finite thickness. The extra symmetries mentioned for the expansion of the stress-energy also apply in a very similar manner for this expansion of the current tensor, which we will make clear in the next section. The equations of motion for the current follow from the conservation equation

$$\nabla_{\mu_1} J^{\mu_1 \cdots \mu_{q+1}} = 0. \tag{3.6}$$

3.2 Symmetry properties of the derivative expansion

The two forementioned additional symmetry properties enjoyed by the stress-energy tensor and the charge current will now be given explicit descriptions.

- Extra symmetry 1

The first extra symmetry is a consequence of the $p+1$ integrations over the δ-functions in expression (3.2). The symmetry property not only valid to 1st order but to all orders of the multipole expansion. Derivatives in the world-volume directions are integrated out, a consequence of the multipole expansion being an expansion in transverse derivatives. This means that there are components of $T^{\mu\nu\rho}_{(1)}$ which have a gauge freedom. The action of this symmetry on the monopole and dipole terms in the expansion is expressed as

$$\delta_1 T^{\mu\nu}_{(0)} = -\nabla_a \epsilon^{\mu\nu a}, \qquad \delta_1 T^{\mu\nu\rho}_{(1)} = \epsilon^{\mu\nu a} u^{\rho}_{a}. \tag{3.7}$$

We have introduced, $\epsilon^{\mu\nu a}$, symmetric in μ and ν, as free parameters except on the boundary of the world-volume where they are required to vanish,

$$\hat{n}_a \epsilon^{\mu\nu a}|_{\partial \mathcal{W}_{p+1}} = 0. \tag{3.8}$$

We use $\hat{n}_a$ as a normal vector to the boundary of the brane world-volume. Now we can use this transformation to show that the tangential components of $T^{\mu\nu\rho}_{(1)}$ (that is $T^{\mu\nu\rho}_{(1)} u^a_\rho$), can be gauged away everywhere except for the boundary of the world-volume,

$$\delta_1 (T^{\mu\nu\rho}_{(1)} u^a_\rho) = \epsilon^{\mu\nu a}. \tag{3.9}$$

Hence, we see that the degrees of freedom for the dipole corrections live only on the boundary of the world-volume.

The action of this symmetry on the current expansion has the same form as the one on the structure components of $T^{\mu\nu}$ except for an additional minus sign in the second expression of (3.7). And so, the dipole structure component of the current can also be gauged away everywhere except for the boundary.

- Extra symmetry 2

Extra symmetry 2 is a perturbative symmetry, so in fact it is approximate, not valid to all orders of the multipole expansion. To each of the contributions in the multipole expansion, $T^{\mu\nu}_{(0)}$ and $T^{\mu\nu\rho}_{(1)}$ we associate an order parameter $\tilde{\varepsilon}$ and write

$$T^{\mu\nu}_{(0)} = \mathcal{O}(1), \qquad T^{\mu\nu\rho}_{(1)} = \mathcal{O}(\tilde{\varepsilon}). \tag{3.10}$$

As an example of the order parameter, for a brane of horizon thickness r_0 bent over a submanifold of curvature radius R, we have $\tilde{\varepsilon} = r_0/R$. When we consider elastic perturbations to the monopole approximation of the stress-energy tensor (i.e. dipole corrections) we truncate expression (3.2) to first order and the brane acquires a bending moment. In this case, an ambiguity arises in the position of the world-volume boundary because of the finite thickness. This reparametrization invariance of the world-volume surface is expressed as "extra symmetry 2". The action of the symmetry on each of the structure components of $T^{\mu\nu}$ is a displacement of $\mathcal{O}(\tilde{\varepsilon})$,

$$X^{\alpha}(\sigma^a) \to X^{\alpha}(\sigma^a) + \tilde{\varepsilon}^{\alpha}(\sigma^a). \tag{3.11}$$

This action to first order demands the following transformation rules for the structure components:

$$\delta_2 T^{\mu\nu}_{(0)} = -T^{\mu\nu}_{(0)} u^a_\rho \nabla_a \tilde{\varepsilon}^\rho - 2T^{\lambda(\mu}_{(0)} \Gamma^{\nu)}_{\lambda\rho} \tilde{\varepsilon}^\rho, \qquad \delta_2 T^{\mu\nu\rho}_{(1)} = -T^{\mu\nu}_{(0)} \tilde{\varepsilon}^\rho. \tag{3.12}$$

If we took the single pole approximation we would see that $\delta_2 T^{\mu\nu}_{(0)} = 0$, making it clear that if our object is infinitely thin we would have no freedom in choosing the world-volume surface.

Extra symmetry 2 acts in the same way on the current expansion as on the stress-energy tensor.

3.3 Equations of motion

Only dipole corrections will be of interest to us here so we will take $j^{b\nu\rho} = 0$ to avoid having to deal with the properties that arise when the brane has an angular momentum. Then, from (3.3) and (3.4) we can write the equations of motion as [13] [32] [14]

$$\nabla_a \hat{T}^{ab} + u^b_\mu \nabla_a \nabla_c d^{ac\mu} = d^{ac\mu} R^b_{ac\mu}, \tag{3.13}$$

$$\hat{T}^{ab} K^\rho_{ab} + \perp^\rho_\mu \nabla_a \nabla_b d^{ab\mu} = d^{ab\mu} R^\rho_{ab\mu}. \tag{3.14}$$

Here we have defined the quantity $\hat{T}^{ab} = T^{ab}_{(0)} + 2d^{(ac\mu} K^{b)}_{c\mu}$. Taking the dipole term to zero we recover Carter's extrinsic equations of motion in the absence of external forces (2.9) and if one assumes the monopole structure term to be of perfect fluid form we would reobtain the leading order blackfold equations. These equations of motion are relativistic generalizations of thin elastic branes [32]. We must supplement them with the integrability condition $d^{ab[\mu} K^{\rho]}_{ab}$ and boundary conditions on the world-volume surface:

$$d^{ab\rho} \hat{n}_a \hat{n}_b |_{\mathcal{W}_{p+1}} = 0, \qquad \left(\hat{T}^{ab} u^\mu_b - d^{ac\rho} K^b_{c\rho} u^\mu_b + \perp^\mu_\rho \nabla_b d^{ab\rho} \right) \hat{n}_a |_{\mathcal{W}_{p+1}} = 0. \tag{3.15}$$

In this section we will go into the equations of motion stepwise higher from ($q = 0$) Maxwell charge to ($q = 1$) string charge and finally the $0 < q \leq p$ higher-form charge.

- Pole-dipole branes carrying $q = 0$ charge

In the point charge case, ($q = 0$), of pole-dipole branes carrying Maxwell charge we have a one-form current J^μ following the form of (3.5). In [14] the steps of the method to arrive at the equations of motion is presented in detail. The procedure closely follows the methods of [30] where an arbitrary tensor field $f_\mu(x^\alpha)$ of compact support is introduced such that

$$\int d^D x \sqrt{-g} f_\nu(x^\alpha) \nabla_\mu J^{\mu\nu} = 0 \quad . \tag{3.16}$$

Derivatives of $f(x^\alpha)$ are decomposed in parallel and orthogonal components to the world-volume, then that decomposition and the expansion of J^μ are plugged into the integral. Requiring the resulting expression to vanish for

for each component of f results a set of three equations of motion. We are also left with a boundary term of the integral which manifestly vanishes. Splitting f up into boundary terms and requiring the terms proportional to them to vanish leads to boundary conditions on the world-volume. In order to solve the equations of motion, the structures $J^{\mu}_{(0)}$ and $J^{\mu\nu}_{(1)}$ are decomposed into tangential and orthogonal components,

$$J^{\mu}_{(0)} = J^{a}_{(0)} u^{\mu}_{a} + J^{\mu}_{\perp(1)}, \qquad J^{\mu\nu}_{(1)} = m^{\mu\nu} + u^{\mu}_{a} p^{a\nu} + J^{\mu a}_{(1)} u^{\nu}_{a}, \tag{3.17}$$

such that $m^{\mu\nu}$ and $p^{a\rho}$ are transverse in their spacetime indices. Note that the term $J^{\mu a}_{(1)}$ has the property $J^{[ab]}_{(1)} = 0$. For this specific case of $(q = 0)$ we make a requirement that $J^{ab}_{(1)} = J^{(ab)}_{(1)}$. This is because we want the equations of motion to be invariant under extra symmetries 1 and 2. In later cases of $(q > 0)$ this requirement is not needed since the effective current enters with more than just one index. The last term in the decomposition (3.17) of $J^{\mu\nu}_{(1)}$ is left neither parallel nor orthogonal to the world-volume for the reason that it can be gauged away everywhere except on the boundary due to extra symmetry 1. The expression we can read out from plugging the decomposition into the equations of motion is the constraint $m^{\mu\nu} = m^{[\mu\nu]}$. Without writing details of the derivation, the equations of motion are found to be

$$J^{\mu}_{\perp(1)} = \perp^{\mu}_{\nu} \nabla_a \big(p^{a\nu} + J^{\nu a}_{(1)}\big), \tag{3.18}$$

$$\nabla_a \big(\hat{J}^a + p^{b\mu} K_{b\mu}^{\;\;a}\big) = 0, \tag{3.19}$$

where we introduce $\hat{J}^a = J^{a}_{(0)} - u^{a}_{\mu} \nabla_b J^{\mu b}_{(1)}$. The second equation of the last two is the world-volume current conservation. With the decomposition, the boundary conditions become

$$\big(p^{a\mu} + J^{\mu a}_{\perp(1)}\big) \hat{n}_a |_{\partial \mathcal{W}_{p+1}} = 0, \tag{3.20}$$

$$J^{ab}_{(1)} \hat{n}_a \hat{n}_b |_{\partial \mathcal{W}_{p+1}} = 0, \tag{3.21}$$

$$\big[\nabla_{\hat{a}} J^{\hat{a}}_{(1)} - \hat{n}_a \big(\hat{J}^a + p^{a\mu} K_{b\mu}^{\;\;a}\big)\big] |_{\partial \mathcal{W}_{p+1}} = 0, \tag{3.22}$$

where $J^{\hat{a}}_{(1)} = J^{ab}_{(1)} v^{\hat{a}}_{b} \hat{n}_a$ are defined as the boundary degrees of freedom. One can observe that the $m^{\mu\nu}$ used in the decomposition does not enter further into the equations of motion nor the boundary conditions that we have written.

Now we can write out the explicit actions of extra symmetry 1 on each of the quantities we have introduced here to characterize the charge current to further demonstrate that it leaves the equations of motion invariant:

$$\delta_1 \hat{J}^a = 0, \qquad \delta_1 p^{a\mu} = 0, \qquad \delta_1 J^{\hat{a}}_{(0)} = 0. \tag{3.23}$$

Explicit application of extra symmetry 2 on the characterizing structure leads to

$$\begin{aligned} \delta_2 \hat{J}^a &= -J^{\mu}_{(0)} u^b_\rho \nabla_b \tilde{\varepsilon}^\rho - u^a_\rho J^b_{(0)} \nabla_b \tilde{\varepsilon}^\rho + \nabla_b \big(J^a_{(0)} \tilde{\varepsilon}^\rho \big), \\ \delta_2 p^{a\mu} &= -J^a_{(0)} \tilde{\varepsilon}^\mu, \qquad \delta_2 J^{\hat{a}}_{(1)} = -J^b_{(0)} v^{\hat{a}}_b \tilde{\varepsilon}^a \hat{n}_a, \end{aligned} \tag{3.24}$$

where the hatted indices signify that they point in normal directions. The transformations leave the equations of motion invariant.

- Pole-dipole branes carrying $q = 1$ charge

Pole-dipole order branes carrying ($q = 1$) string charge are characterized by the two-form structure components $J^{\mu\nu}_{(0)}$ and $J^{\mu\nu\rho}_{(1)}$. Following the same procedure as for the ($q = 0$) case we can directly get the equations of motion for this case. Again, we decompose the structures into orthogonal and parallel components,

$$J^{\mu\nu}_{(0)} = u^\mu_a u^\nu_b J^{ab}_{(0)} + 2u^{[\mu}_b J^{\nu]b}_{\perp(1)} + J^{\mu\nu}_{\perp(1)}, \tag{3.25}$$

$$J^{\mu\nu\rho}_{(1)} = 2u^{[\mu}_a m^{a\nu]\rho} + u^\mu_a u^\nu_b p^{ab\rho} + J^{\mu\nu a}_{(1)} u^\rho_a. \tag{3.26}$$

As before $m^{a\mu\nu}$ and $p^{ab\rho}$ are orthogonal to the brane world-volume in their spacetime indices. We will not go explicitly into the extra symmetries in this case and for charge $q > 1$ as they are natural generalizations of the Maxwell charge case. That is, the extra symmetries will leave the equations of motion invariant under the action on the individual components that characterize the current and the last term in the decomposition of $J^{\mu\nu\rho}_{(1)}$ can be gauged away everywhere except on the boundary. The last two components in $J^{\mu\nu}_{(0)}$ are related through the dipole contributions of $J^{\mu\nu\rho}_{(1)}$ via

$$J^{\mu b}_{\perp(1)} = u^b_\rho \perp^\mu_\lambda \nabla_c (J^{\rho\lambda c}_{(1)} - m^{c\rho\lambda}), \quad J^{\mu\nu}_{\perp(1)} = \perp^\mu_\lambda \perp^\nu_\rho \nabla_c (J^{\rho\lambda c}_{(1)} - m^{c\rho\lambda}). \tag{3.27}$$

Using the decomposition we can now obtain the equations of motion and boundary conditions. Without going into details we will write the results. The current conservation equation takes the form

$$\nabla_a \big(\hat{J}^{ab} - 2p^{c[a(\mu)} K^{b]}_{c\mu} \big) = 0, \tag{3.28}$$

where $\hat{J}^{ab} = J^{ab}_{(0)} - u^a_\mu u^b_\nu \nabla_c J^{\mu\nu c}_{(1)}$ is the world-volume effective current that obeys $\hat{J}^{ab} = \hat{J}^{[ab]}$. The boundary conditions for bent branes carrying string charge are

$$\big(p^{ba\mu} + 2\perp^\mu_\lambda J^{\lambda ab}_{(1)}\big)\hat{n}_b|_{\partial\mathcal{W}_{p+1}} = 0, \tag{3.29}$$

$$J^{\mu ab}_{(1)}\hat{n}_a\hat{n}_b|_{\partial\mathcal{W}_{p+1}} = 0, \tag{3.30}$$

$$\big[v^b_{\hat{b}}\nabla_{\hat{a}}J^{\hat{a}\hat{b}}_{(1)} - \hat{n}_a\big(\hat{J}^{ab} - 2p^{c[a\mu}K_{c\mu}{}^{b]}\big)\big]|_{\partial\mathcal{W}_{p+1}} = 0, \tag{3.31}$$

where $J^{\hat{a}\hat{b}}_{(1)} = J^{\mu\nu c}_{(1)}u^\rho_c\hat{n}_{(\rho}v^{\hat{a}}_{\nu)}v^{\hat{b}}_\mu$ is defined as the boundary degrees of freedom with $v^{\hat{a}}_\mu = u^a_\mu v^{\hat{a}}_a$. Branes carrying string charge are characterized by the world-volume effective current, the dipole moment and a boundary current.

- Pole-dipole branes carrying $q > 1$ charge

Branes carrying general higher form fields $q > 1$ are characterized by the structures $J^{\mu_1\ldots\mu_{q+1}}_{(0)}$ and $J^{\mu_1\ldots\mu_{q+1}\rho}_{(1)}$. The steps would be the same as before, we make a general decomposition of both the structure components, and analogously introduce the tensors m and p that characterize the dipole contributions that obey certain antisymmetrization requirements. In [14] the authors conjecture the form of the conservation equations of motion to be of the form

$$\nabla_{a_1}\big(\hat{J}^{a_1\ldots a_{q+1}} + (-1)^q[(q+1)!/q!]p^{c[a_1\ldots a_q(\mu)}K_{c\mu}{}^{a_{q+1}]}\big) = 0. \tag{3.32}$$

The boundary conditions are analogous to the ones presented in the string charge case

$$\big(p^{a_1\ldots a_{q+1}\mu} + (-1)^q q!\perp^\mu_\lambda J^{\lambda a_1\ldots a_{q+1}}\big)\hat{n}_{a_{q+1}}|_{\partial\mathcal{W}_{p+1}} = 0, \tag{3.33}$$

$$J^{\mu_1\ldots\mu_q a_{q+1}b}_{(1)}\hat{n}_{a_{q+1}}\hat{n}_b|_{\partial\mathcal{W}_{p+1}} = 0, \tag{3.34}$$

$$\big[v^b_{\hat{b}}\nabla_{\hat{a}_1}J^{\hat{a}_1\ldots\hat{a}_{q+1}}_{(1)} - \hat{n}_{a_1}\big(\hat{J}^{a_1\ldots a_{q+1}} + (-1)^q[(q+1)!/q!]p^{c[a_1\ldots a_q(\mu)}K_{c\mu}{}^{a_{q+1}]}\big)\big]|_{\partial\mathcal{W}_{p+1}} = 0. \tag{3.35}$$

Where we have introduced the effective world-volume current $\hat{J}^{a_1\ldots a_{q+1}}$ and the boundary degrees of freedom $J^{\hat{a}_1\ldots\hat{a}_{q+1}}_{(1)}$.

3.4 Physical interpretation

In the previous sections we have introduced some quantities in the equations of motion that should be given a physical interpretation. Let us start out by discussing the angular momentum, then move on to the interpretation of the stress-energy dipole moment and the electric dipole moment. The physical interpretations we review here are given in [13] [32] [14].

3.4.1 Physical quantities

We start out by assuming a flat spacetime and look at uniform p-branes extended in $x^0, ..., x^p$ directions. The total angular momentum $\mathcal{J}_\perp^{\mu\nu}$ on the transverse plane, labeled by μ, ν, is defined as a constant time slice integral Σ in the bulk spacetime,

$$\mathcal{J}_\perp^{\mu\nu} = \int_\Sigma d^{D-1}x(T^{0\mu}x^\nu - T^{0\nu}x^\mu) = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}(2T^{0\mu\nu}_{(1)\perp}) + \text{boundary terms.} \tag{3.36}$$

The monopole structure term does not enter the expression, we thus see that in the monopole approximation there are no intrinsic angular momenta. The boundary terms is the expression have been ignored and we will continue to leave them out, without further noting, in the interpretation of the other quantities. The intrinsic angular momenta are exclusively encoded by the dipole structure $T^{a\mu\nu}_{(1)\perp}$. The interpretation of $j^{a\mu\nu}$ as a current density of transverse angular momenta of the pole-dipole branes is justified through the relation

$$j^{a\mu\nu} = 2u^a_\rho \perp^\mu_\sigma \perp^\nu_\lambda T^{\rho[\sigma\lambda]}_{(1)} = 2T^{a\mu\nu}_{(1)\perp}, \tag{3.37}$$

where μ and ν are orthogonal to the world-volume. The conservation of angular momentum follows from the conservation of the current density of the angular momenta.

The dipole term introduced before, $d^{ab\rho}$, accounts for the bending moment of the brane. The total bending moment from the stress-energy tensor can be computed by

$$D^{ab\rho} = \int_\Sigma d^{D-1}x\sqrt{-g}T^{\mu\nu}u^a_\mu u^b_\nu x^\rho = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}T^{ab\rho}_{(1)\perp}, \tag{3.38}$$

where the ρ index is orthogonal to the world-volume. The quantity $d^{ab\rho}$ is given the interpretation of a current density that describes the dipole deformations to the intrinsic stress-energy tensor,

$$d^{ab\rho} = u^a_\mu u^b_\nu \perp^\rho_\lambda T^{\mu\nu\lambda}_{(1)} = T^{\rho ab}_{(1)\perp}. \tag{3.39}$$

Note that the definition of the dipole moment (3.38) is the same as the usual way of writing the classical electrically induced dipole with a charge density $\rho(x)$, $\vec{D} = \int_\Sigma d^{D-1}x\vec{x}\rho(x)$. In the single pole approximation $d^{ab\rho}$ does have a non-vanishing component, $d^{\tau\tau\rho}$, where τ is the proper time of the world-line. However, because of extra symmetry 2 of the stress-energy tensor, the gauge parameter $\tilde{\varepsilon}^\rho$ can be chosen in such a way that $d^{\tau\tau\rho}$ is eliminated. Single-pole branes can thus not possess a bending moment.

In much the same way as we have interpreted the previous quantities we can assign an intuitive interpretation to $p^{a\rho}$ and $m^{\mu\nu}$, by taking the case, $q = 0$ as an example. The electric dipole moment $P^{a\rho}$ of a charged brane is obtained by evaluating

$$P^{a\rho} = \int_\Sigma d^{D-1}x\sqrt{-g}J^\mu u^a_\mu x^\rho = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}p^{a\rho}. \tag{3.40}$$

The structure $p^{a\rho}$ is thus interpreted as a world-volume electric dipole moment density. Just as for $d^{ab\rho}$, there is one non-vanishing component in the ($p = 0$) case that can however be gauged away because of the extra symmetry 2 property of J. For cases $p \geq 1$, the dipole terms cannot be gauged away.

In the case of a point particle ($p = 0$) carrying a Maxwell charge ($q = 0$) the object can possess a magnetic dipole moment $M^{\mu\nu}$ obtained by

$$M^{\mu\nu} = \int_\Sigma d^{D-1}x\sqrt{-g}(J^\mu x^\nu - J^\nu x^\mu) = \int_{\mathcal{B}_p} d^p\sigma\sqrt{-\gamma}m^{\mu\nu}. \tag{3.41}$$

The quantity $m^{\mu\nu}$ is interpreted as a world-volume density of a magnetic dipole moment associated with a dynamic charge. It is most natural to assume $m^{\mu\nu}$ to be proportional to the spin current $j^{\tau\mu\nu}$. We can write this relation generically for a p-brane with smeared 0-brane charge as $m^{\mu\nu} = \lambda(\sigma^b)u_a j^{a\mu\nu}$, for a world-volume function λ. This can be naturally extended to general p-branes carrying smeared q-brane charge. Since we are not considering any angular momentum in the range of this thesis there will be no further analysis done on the magnetic dipole moment.

Assuming a linear response theory, the finite thickness quantities discussed above can be characterized by a set of linear response coefficients that determine the (electro)elasticity of charged pole-dipole oder black branes.

3.4.2 The Young modulus and piezoelectric moduli

In classical mechanics the Young modulus is the ratio of stress to strain of elastic materials. The bending of an elastic rod will result in a dipole of stress in the material distribution. The strain of an object in its transverse direction is captured by the extrinsic curvature tensor $K_{\mu\nu}^{\rho}$ and the stress induced by the deformation is encoded by the dipole moment, $d_{ab\rho}$. In [13] a relativistic generalization of the Young modulus is introduced. Assuming that we have a brane that will behave according to classical elastic theory, that is, by a linear response theory, we can write the stress/strain relation as

$$d^{ab\rho} = \tilde{Y}^{abcd} K_{cd}^{\rho}, \tag{3.42}$$

where $\tilde{Y}^{abcd}$ has been introduced as the Young modulus of the brane. Omitting tensor indices, $\tilde{Y} = YI$, where Y is the normalized Young modulus and I signifies the moment of inertia with respect to the world-volume surface choice. We note that the Young modulus should display the extra symmetry 2 ambiguity of the dipole current density $d^{ab\rho}$, according to the definition (3.39). The Young modulus obeys the same symmetry property in its indices as the classical elastic tensor would, $\tilde{Y}^{abcd} = \tilde{Y}^{(ab)(cd)} = \tilde{Y}^{cdab}$. For isotropic stationary p-branes carrying Maxwell charge the Young modulus takes the form [32]

$$\begin{aligned}\tilde{Y}^{abcd} = & -2\big(\lambda_1(\mathbf{k};T,\Phi_H)\gamma^{ab}\gamma^{cd} + \lambda_2(\mathbf{k};T,\Phi_H)\gamma^{a(c}\gamma^{d)b} \\ & + \lambda_3(\mathbf{k};T,\Phi_H)\mathbf{k}^{(a}\gamma^{b)(c}\mathbf{k}^{d)} + \lambda_4(\mathbf{k};T,\Phi_H)\frac{1}{2}(\mathbf{k}^a\mathbf{k}^b\gamma^{cd} + \gamma^{ab}\mathbf{k}^c\mathbf{k}^d) \\ & + \lambda_5(\mathbf{k};T,\Phi_H)\mathbf{k}^a\mathbf{k}^b\mathbf{k}^c\mathbf{k}^d\big).\end{aligned} \tag{3.43}$$

With $\mathbf{k}^a$ being the Killing vector field of the fluid, moving with $\mathbf{k} = |-\gamma_{ab}\mathbf{k}^a\mathbf{k}^b|^{\frac{1}{2}}$. We have introduced the dependency of the response coefficients on the global temperature T, and the global chemical potential Φ_H, determined by the stationarity of the solution and its effective blackfold fluid form. Due to the extra symmetry 2 property that the Young modulus must have, some of the terms in its expression are gauge dependent, so not all of the λ coefficients are independent with respect to the equations of motion. In chapter 4 we will see this explicitly where the derivation of the response coefficients is presented for charged black brane solutions.

As shown in the previous chapter, for the simplest anisotropic case of p-branes carrying a string charge, we need to supplement our expressions by a normalized velocity vector v^a, orthogonal to the velocity vector of the brane fluid u^a. For this reason the expression for the Young modulus, written above, should be supplemented by extra terms. The classical symmetries that apply to $\tilde{Y}^{abcd}$ do not necessarily apply to the extra terms. In [14] the response coefficients are calculated for particular cases of charged black branes carrying a $q > 0$ charge. It is found that because of the extra symmetry property, the extra terms can be transformed away, and thus well described by $\tilde{Y}^{abcd}$ in the isotropic case. In the case of $q > 0$ we do however have to introduce a new non-normalized space-like vector ζ^a, where $v^a = \zeta^a/\zeta$ and $\zeta = |\zeta^a\zeta^b\gamma_{ab}|^{1/2}$.

In analogy with the introduction to the relativistic Young modulus we now turn to the case of the electric dipole moment. If we assume the branes follow classical electroelastic theory, we can write the form of the dipole moment as

$$p^{a\rho} = \tilde{\kappa}^{abc} K_{bc}^{\rho}. \tag{3.44}$$

This expression is a covariant generalization of the electric dipole moment of classical piezoelectrics [14]. The piezoelectric moduli obey the symmetry property $\tilde{\kappa}^{abc} = \tilde{\kappa}^{a(bc)}$ and enjoys extra symmetry 2. In the $q = 0$ case, that leads to the form

$$\tilde{\kappa}^{abc} = -2\big(\kappa_1(\mathbf{k}; T, \Phi_H)\gamma^{a(b}\mathbf{k}^{c)} + \kappa_2(\mathbf{k}; T, \Phi_H)\mathbf{k}^a\mathbf{k}^b\mathbf{k}^c + \kappa_3(\mathbf{k}; T, \Phi_H)\mathbf{k}^a\gamma^{bc}\big). \tag{3.45}$$

As for the Young modulus, the response coefficients of piezoelectric moduli are not all independent.

The introduction of the piezoelectric moduli can naturally be extended to cases of smeared $q > 0$ charge, where we have $p^{a_1 \ldots a_{q+1}\rho}$. The electric dipole moment will have the same interpretation as for the $q = 0$ case. We must, as before, note that the response coefficients have an additional dependency on ζ^a. The higher-form piezoelectric moduli inherit the symmetry properties of $p^{a_1 \ldots a_{q+1}\rho}$, $\tilde{\kappa}^{a_1 \ldots a_{q+1} bc} = \tilde{\kappa}^{[a_1 \ldots a_{q+1}]bc}$ and $\tilde{\kappa}^{a_1 \ldots a_{q+1} bc} = \tilde{\kappa}^{a_1 \ldots a_{q+1}(bc)}$. The antisymmetry property in the first $q + 1$ indices does not have any parallel in classical piezoelectrics.

3.5 Response coefficients

In [14] a class of bent black brane geometries is constructed and their response coefficients is measured. This section will outline the framework used to obtain the coefficients from the large r-asymptotics of a charged bent black brane solution and in chapter 4 their derivation for stationary configurations will be presented.

We work from a special class of solutions to the action,

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\bigg(R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2(p+2)!}e^{a\phi}H_{q+2}^2\bigg). \tag{3.46}$$

Again, we see the relevance of the setting of this action as we have mentioned it in the introduction of black branes in chapter 1 and also in chapter 2 where the Gibbons-Maeda class of solutions was derived for leading order p-branes carrying higher-form smeared charge. Now, however we wish to examine the dipole structures of the solutions. The equations of motions of motion that lead from the action in the presence of sources are

$$\begin{gathered} G_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi \\ -\frac{1}{2(q+1)!}e^{a\phi}\bigg(H_{\mu\rho_1\ldots\rho_{q+1}}H_\nu^{\rho_1\ldots\rho_{q+1}} - \frac{1}{2(q+2)}H^2 g_{\mu\nu}\bigg) = 8\pi G T_{\mu\nu}, \end{gathered} \tag{3.47}$$

$$\nabla_\nu(e^{a\phi}H^{\nu\mu_1\ldots\mu_{q+1}}) = -16\pi G J^{\mu_1\ldots\mu_{q+1}}, \quad \Box\phi - \frac{a}{2(q+1)!}e^{a\phi}H^2 = 0. \tag{3.48}$$

The stress-energy tensor and the current are given by the multipole expansions (3.2) and (3.5). To measure the response coefficients for the Young modulus and the piezoelectric moduli we look at the large r-asymptotics. Far away from the solution, the geometry, and the gauge field are replaced by effective stress-energy sources and current. The bending moments for the stress-energy and electric current are related to dipole corrections of the fields as they approach spatial infinity. The dipole moments for the bending and the gauge field are related through (3.47) and (3.48) to the dipole corrections we obtain for the fields approaching spatial infinity, where they have a fall-off behaviour $\mathcal{O}(r^{-n-1})$ by definition [13]. For convenience, we decompose the spacetime metric into the Minkowski metric, and monopole and dipole contributions, and perform the same decomposition for the gauge field,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(M)} + h_{\mu\nu}^{(D)} + \mathcal{O}(r^{-n-2}), \tag{3.49}$$

$$A_{\mu_1\ldots\mu_{q+1}} = A^{(M)}_{\mu_1\ldots\mu_{q+1}} + A^{(D)}_{\mu_1\ldots\mu_{q+1}} + \mathcal{O}(r^{-n-2}). \tag{3.50}$$

The labels (M) and (D) indicate the monopole and dipole contributions. The procedure of extracting the response coefficients is done by considering the linearized equations of motion of the action (3.46), assuming that there are no background fields and that the background metric is asymptotically flat. The dilaton field receives no bending corrections and is thus unnecessary for the discussion.

Plugging in the decomposition of the stress-energy tensor into the right-hand side of the equation of motion (3.47) we see that the dipole contribution to the metric should satisfy linearized equations of motion

$$\nabla_{\perp}^2 \bar{h}_{\mu\nu}^{(D)} = 16\pi G d_{\mu\nu}^{r_\perp} \partial_{r_\perp} \delta^{n+2}(r), \quad \nabla_\mu \bar{h}^\mu_\nu = 0, \tag{3.51}$$

where

$$\bar{h}_{\mu\nu}^{(D)} = h_{\mu\nu}^{(D)} - \frac{h^{(D)}}{2}\eta_{\mu\nu}, \quad h^{(D)} = \eta^{\mu\nu} h_{\mu\nu}^{(D)}. \tag{3.52}$$

The Laplacian is in the transverse directions to the brane world-volume, and $r_\perp = r\cos\theta$ is transverse to the bending direction of the brane. Writing out the explicit r- and θ-dependence of the asymptotic dipole contributions will yield a convenient form to write the dipole contributions to the metric. We write

$$\begin{aligned} h_{ab}^{(D)} = f_{ab}^{(D)} \cos\theta \frac{r_0^{n+2}}{r^{n+1}}, \quad h_{rr}^{(D)} = f_{rr}^{(D)} \cos\theta \frac{r_0^{n+2}}{r^{n+1}}, \\ h_{ij}^{(D)} = r^2 g_{ij} f_{\Omega\Omega}^{(D)} \cos\theta \frac{r_0^{n+2}}{r^{n+1}}. \end{aligned} \tag{3.53}$$

With a little bit of rewriting, using the transverse gauge condition, leads to an expression of the dipole contributions to the metric in terms of the newly defined f corrections,

$$\hat{d}_{ab} = \bar{f}_{ab}^{(D)} = f_{ab}^{(D)} - f_{\Omega\Omega}^{(D)} \eta_{ab}. \tag{3.54}$$

Through this expression the Young modulus can then be obtained by $d^{ab\rho} = \tilde{Y}^{abcd} K_{cd}^{\rho}$ after the f's are determined in the asymptotic region. The hatted notation of the dipole moments means that we have written $d_{ab} = \frac{\Omega_{(n+1)} r_0^n}{16\pi G} r_0^2 \hat{d}_{ab}$ and omitted writing the transverse index $r_\perp$ and we will do the same for the electric dipole moment.

Using the analogous approach, we can arrive at the electric dipole moment for a ($q \leq p$) charged bent brane through a linearized version of the multipole expansion of the brane current and the structure decomposition. We introduce asymptotic gauge field coefficients $a^{(D)}_{\mu_1 \ldots \mu_{q+1}}$ that are independent of r and θ. We write electric dipole moment as

$$\hat{p}_{a_1 \ldots a_{q+1}} = a^{(D)}_{a_1 \ldots a_{q+1}}. \tag{3.55}$$

The piezoelectric moduli can then be read from $p^{a_1 \ldots a_{q+1}\rho} = \tilde{\kappa}^{a_1 \ldots a_{q+1}bc} K_{bc}^{\rho}$ after evaluating the asymptotic gauge field and dipole contributions.

In [14], a special class of solutions of the action is considered, namely generalized Gibbons-Maeda solutions. A bent version is then obtained for a subset of those solutions. In the last subsection of chapter 2 we presented the steps that lead to the smeared q-brane charged p-brane solution and the resulting metric was

$$ds^2 = h^{-A}(-f dt^2 + d\vec{y}) + h^B(f^{-1}dr^2 + r^2 d\Omega^2_{(n+1)} + d\vec{z}), \tag{3.56}$$

$$f(r) = 1 - \frac{r_0^n}{r^n} \quad , \quad h(r) = 1 + \frac{r_0^n}{r^n} \sinh^2 \alpha, \tag{3.57}$$

with $\vec{y}$ labeling the q directions of the gauge field, $\vec{z}$ labeling the $p-q$ smeared directions, and r_0 and α labeling the horizon radius and charge parameter respectively. The gauge field is given by

$$A_{[q+1]} = -\sqrt{N} \frac{r_0^n}{r^n h(r)} \cosh \alpha \sinh \alpha dt \wedge dy^1 \wedge \ldots \wedge dy^q, \tag{3.58}$$

and the dilaton reads

$$\phi = \frac{1}{2} N a \log h(r), \tag{3.59}$$

with $N = A + B$. The constants A and B depend on p, q, n and a.

The goal is to arrive at the dipole corrections to the metric and gauge field. The first step towards obtaining them is to generate bent black brane solutions with smeared q-brane charge and Kaluza-Klein dilaton coupling by taking an elastically perturbed neutral black brane [13] as a seed solution for a series of solution generating techniques. The first class considered is obtained by the $m + 1$ dimensional uplift and Kaluza-Klein reduction on the boosted spatial direction, this gives us a bent brane solution carrying a Maxwell charge. The second class of solutions considered are to type-II string theory, obtained by a sequence of T-duality transformations, resulting in higher-form gauge fields. The solutions considered are required to have $N = 1$. In the next chapter, the steps of the construction of the seed solution will be presented along with the solution generating techniques used to obtain the desired bent solutions. From there on the response coefficients will be read off the large-r asymptotics using the relations we have outlined in this section.

3.6 Thermodynamics

The stress-energy tensor $T^{ab}_{(0)}$ and world-volume electric current $J^{a_1...a_{q+1}}_{(0)}$ of the generated solutions are characterized by thermodynamic quantities which we present here briefly. The leading order effective stress-energy tensor has the form

$$T^{ab}_{(0)} = \varepsilon u^a u^b + P_\perp \left(\gamma^{ab} + u^a u^b - \sum_{i=1}^{q} v^a_{(i)} v^b_{(i)} \right) + P_\parallel \sum_{i=1}^{q} v^a_{(i)} v^b_{(i)}, \tag{3.60}$$

and the leading order world-volume effective current takes the form

$$J^{a_1...a_{q+1}}_{(0)} = (q+1)! \mathcal{Q} u^{[a_1} v^{a_2}_{(1)} ... v^{a_{q+1}]}_{(q)}. \tag{3.61}$$

The form of the effective world-volume stress-energy tensor and the effective current have the same form as they would have to leading order as they receive no corrections to 1st order [9] [33] [31]. The interest is in stationary solutions and it follows that the energy density ε, the parallel, and transverse pressure, $P_\parallel$ and $P_\perp$ take the form of leading order results [26] presented for the blackfold effective fluid in chapter 2 with (2.34) and (2.35). The only difference is in the pressure, the parallel pressure obtains the same form as the one in (2.34), but the transverse pressure is

$$P_\perp = -\frac{\Omega_{(n+1)}}{16\pi G} r_0^n. \tag{3.62}$$

The transverse pressure has the same form as the Gibbs free energy except that it has the opposite sign. The local temperature $\mathcal{T}$, the local entropy density s, the local charge density $\mathcal{Q}$ and the local chemical potential Φ all have the same form as before.

Chapter 4

Derivation of the response coefficients

In this chapter the techniques and steps leading to an explicit expression of the response coefficients of the relativistic generalization of the Young modulus and piezoelectric moduli for charged black p-branes are presented. The relativistic Young modulus was first measured for neutral black strings in [13]. The analysis was expanded upon in [33], to more general bent neutral p-brane solutions. In practice it turns out, in general, to be very difficult to perturb charged branes to obtain their bent versions. Another way exists to arrive at these solutions. To obtain the solutions for charged bent black p-branes a series of solution generating techniques, with a stationary bent neutral black brane as a seed solution, are used to produce bent solutions carrying a Maxwell charge, then smeared Dq-brane solutions pertaining to type-II string theory. We will start this chapter with an introduction to how bent neutral p-brane solutions are obtained, then present the explicit steps needed for the seed solution that we will use. The derivation of the Maxwell charged- and higher-form charged solutions, generated from the seed solution is presented, then the response coefficients are read from their large-r asymptotics.

4.1 The seed solution

4.1.1 Neutral bent black brane solutions

As shown in chapter 1, a basic form of neutral flat black p-brane solutions to the Einstein equations can be written as

$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + f^{-1} dr^2 + r^2 d\Omega^2_{(n+1)}, \quad f = 1 - \frac{r_0^n}{r^n}. \quad (4.1)$$

In [30] the world sheet equations and boundary conditions in pole-dipole order are obtained for neutral p-branes of finite thickness. A important technique used in obtaining a bent solutions is matched asymptotic expansion (MAE for short). The method was introduced in [34], and later refined for use in [35] [36] [37] [38]. Let us go into the procedure of how to obtain bent brane solutions by looking at the example of how the perturbed black string is constructed.

The solution we describe the steps for is a bent Schwarzschild string, computed by first order in r_0/R, where r_0 is the horizon thickness and R is the radius of curvature. This solution is constructed in detail in [9]. The Einstein equations are solved by using matched asymptotic expansion, where solution is obtained separately for the near horizon region $r << R$ and the far, weak field approximation $r >> r_0$. These two zones are then matched with each other by their boundary conditions. Note that we require that an overlap zone exists, that is when the horizon thickness is much smaller than the curvature radius, $r_0 << R$. The geometry of a straight boosted Schwarzschild black string reads

$$\begin{aligned} ds^2 = &\frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2\left(d\theta^2 + \sin^2\theta d\Omega^2_{(n)}\right) \\ &- \left(1 - \frac{r_0^n}{r^n}\right)(\cosh\alpha dt + \sinh\alpha dz)^2 + (\cosh\alpha dz + \sinh\alpha dt)^2. \end{aligned} \quad (4.2)$$

The steps we take in the method of MAE are as follows, well summarized in [13]: In the near horizon zone the spacetime geometry is that of the straight boosted Schwarzschild black string. Like we did to obtain conserved quantities in chapter 1, we linearize the Einstein equations with the black string stress-energy tensor and obtain the metric in the far region $r >> r_0$.

As the source of the linearized gravity we take the ADM stress-energy tensor of the string to zeroth order in the multipole expansion, $T_{\mu\nu} = T^{(0)}_{\mu\nu}\delta^{n+2}(r)$:

$$
\begin{aligned}
T^{(0)}_{tt} &= \frac{\Omega_{(n+1)} r_0^n}{16\pi G}(n\cosh^2\alpha + 1), \\
T^{(0)}_{tz} &= \frac{\Omega_{(n+1)} r_0^n}{16\pi G} n\cosh\alpha\sinh\alpha, \\
T^{(0)}_{zz} &= \frac{\Omega_{(n+1)} r_0^n}{16\pi G}(n\sinh^2\alpha - 1).
\end{aligned}
\tag{4.3}
$$

This stress-energy tensor enters into the linearized approximation for the far zone, $\Box\bar{h}_{\mu\nu} = -16\pi G T^{(0)}_{\mu\nu}$. The large-$r$ field is now bent along a manifold of curvature $1/R$ because it is corrected to first order in $1/R$. Note that only stress-energy components satisfying the conservation equation, $\nabla^\mu T^{(0)}_{\mu\nu} = 0$, can enter as they are the only ones that can couple to the gravitational field. Next the near zone is perturbed to order $1/R$ by using the far field approximation as a boundary condition. Leading from the first order corrected near zone we can obtain the corrected stress-energy tensor source for the far zone. The corrected stress-energy tensor can then be used to compute the corrected far field solution. In principal we could go on and obtain higher order corrections but we limit ourselves to the first order corrections.

The 1st order corrected stress-energy tensor for the bent boosted Schwarzschild string was derived in [9], and in [13] the next step was taken for the far field geometry sourced by that stress-energy was computed.

In order to find the corrections to the far field the metric obtained to 1st order correction is decomposed into the flat Minkowski metric, monopole contributions, and dipole contributions (as we did in chapter 3),

$$
g_{\mu\nu} = \eta_{\mu\nu} + h^{(M)}_{\mu\nu} + h^{(D)}_{\mu\nu} + \mathcal{O}(r^{-(n+2)}). \tag{4.4}
$$

The monopole correction to the flat metric is the correction from the linearized approximation sourced by the boosted Schwarzschild black string. The task is then to solve for the dipole corrections to the metric, $h^{(D)}_{\mu\nu}$.

We now see how bent black brane solutions can be constructed with the method of MAE. We move on by reviewing the derivation of the stationary bent black p-brane seed solution that we later use to generate bent charged black branes.

4.1.2 Obtaining the seed solution

The deformations of the brane geometry in its transverse coordinates is characterized by the extrinsic curvature tensor. This subsection will review the construction detailed in [33], which is also reviewed in the appendix of [14]. It is in order to define a coordinate system in which will be convenient to work in for the purpose of the derivation. The coordinate system most adapt to arriving at a bent neutral solution suitable for the solution generating techniques is analogous to a Fermi normal coordinate system. Fermi normal coordinates is a localized coordinate system in the neighborhood of a given point. The first derivatives of the metric generally vanish at the point which the normal coordinate system is localized around (therefore, also the Christoffel connections). There is a constraint on this system, the world-volume that the coordinate system is adapted to cannot be geodesically embedded in the background spacetime. Some of the first derivatives do not vanish as they encode the embedding of the brane in the world-volume.

Our first step is to construct the background spacetime metric in Fermi normal coordinates. Let us choose normal world-volume coordinates σ^a in a way that all the Christoffel symbols are vanishing on the world-volume, $\Gamma^c_{ab} = 0$. It is convenient to write the transverse directions to the world-volume separately as y^i, $i = p+1, \ldots, D$, such that the brane sits at the origin $y^i = 0$. The transverse indices we have introduced are raised, lowered, and contracted using the Cartesian delta symbol δ_{ij}. Ignoring corrections to first order, the flat metric takes the form

$$ds^2 = \eta_{ab} d\sigma^a d\sigma^b + dy_i dy^i + \mathcal{O}(\sigma^2/R^2_{int}). \tag{4.5}$$

Here we have omitted that there are in fact two characteristic lengths, the extrinsic- and extrinsic curvature radius. They are typically of similar order [39] so only the intrinsic curvature radius is left in the higher order corrections $\mathcal{O}(\sigma^2/R^2_{int})$.

Let us now look at corrections to the metric of first order in y/R. The only corrections characterizing the brane bending is the extrinsic curvature tensor K^i_{ab} [33], and so the induced metric on the brane can be written in terms of it. The Fermi normal coordinates for the neutral bent p-brane take the form

$$ds^2 = (\eta_{ab} - 2K^i_{ab} y_i) d\sigma^a d\sigma^b + dy_i dy^i + \mathcal{O}(y^2/R^2). \tag{4.6}$$

This is the first derivative order form of the metric that describes the background surrounding the brane.

Now we can write the slowly fluctuating metric of a first order perturbed p-brane embedded in a background spacetime with a generic extrinsic perturbation,

$$ds^2 = \left(\eta_{ab} - 2K^i_{ab}y_i + \frac{r_0^n}{r^n}u_a u_b\right)d\sigma^a d\sigma^b + f^{-1}dr^2 + r^2 d\Omega^2_{(n+1)} \\ + h_{\mu\nu}(y^i)dx^\mu dx^\nu + \mathcal{O}(r^2/R^2). \tag{4.7}$$

The additional perturbating terms $h_{\mu\nu}(y^i)$ are dependent on r_0 and u^a and are linear in K^i_{ab}, thus of first order in $1/R$. In (4.7) f is defined in the same way as in (4.1). The radial coordinate r is orthogonal to the world-volume, defined as $r = \sqrt{y_i y^i}$. With Fermi normal coordinates the transverse perturbations, induced by the extrinsic curvature, in the directions y^i decouple from each other, so we can view them separately in each normal direction. Thus we continue on, and confine ourselves to the case where K^i_{ab} is non-zero in only one transverse direction, which we label $i = \hat{i}$. The coordinate $y^{\hat{i}}$ is identified with a direction cosine as

$$y^{\hat{i}} = r\cos\theta \quad . \tag{4.8}$$

The line element (4.7) now takes the form

$$ds^2 = \left(\eta_{ab} - 2K^i_{ab}r\cos\theta + \frac{r_0^n}{r^n}u_a u_b\right)d\sigma^a d\sigma^b + f^{-1}dr^2 + r^2 d\theta \\ + r^2 d\Omega^2_{(n+1)} + h_{\mu\nu}(r,\theta)dx^\mu dx^\nu + \mathcal{O}(r^2/R^2). \tag{4.9}$$

Furthermore, since the perturbations are of dipole nature we can write $h_{\mu\nu}(r,\theta) = \cos\theta\hat{h}_{\mu\nu}(r)$ [9]. The problem of finding the solutions to the Einstein equations is now distilled to solving a set of coupled ordinary differential equations of the form

$$\hat{h}_{\mu\nu}(r)dx^\mu dx^\nu = \hat{h}_{ab}(r)d\sigma^a d\sigma^b + \hat{h}_{rr}(r)dr^2 + \hat{h}_{\Omega\Omega}(r)(d\theta^2 + \sin^2\theta d\Omega^2_{(n)}). \tag{4.10}$$

Direct computation of the Einstein tensor $G_{\mu\nu}$, specifically components $G_{r\theta}$ and G_{rr}, of (4.9) leads us to the leading order extrinsic equation of motion

$$T^{ab}K^{\hat{i}}_{ab} = 0. \tag{4.11}$$

The leading order stress-energy tensor for the neutral p-brane is

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n(nu^a u^b - \eta^{ab}). \tag{4.12}$$

We look for solutions of the perturbations $\hat{h}_{\mu\nu}(r)$ so that the horizon remains regular and satisfy the above equation extrinsic equation (4.11) with the stress-energy tensor (4.12) with $d^{ab\rho} = 0$. Taking the stress-energy tensor into account, this condition is equivalent to $nu^a u^b K_{ab} = K$. We have dropped the $\hat{i}$ notation and will continue to do so from here on. Since $\hat{h}_{\mu\nu}$ are of linear order in the extrinsic curvature they must be proportional to structures of order $1/R$, constructed with K_{ab}, η_{ab}, and u^a. With use of the extrinsic equations of motion, independent terms entering the metric corrections can be written in terms of five functions

$$\hat{h}_{ab}(r) = K_{ab}\mathrm{h}_1(r) + u^c u_{(a}K_{b)c}\mathrm{h}_2(r) + K u_a u_b h_\gamma(r), \tag{4.13}$$

$$\hat{h}_{rr}(r) = K f^{-1} h_r(r), \tag{4.14}$$

$$\hat{h}_{\Omega\Omega}(r) = K r^2 h_\Omega(r). \tag{4.15}$$

A coordinate transformation invariance allows us to express the perturbations in terms of four coordinate gauge invariant functions by taking certain combinations of them. The function $h_\gamma(r)$, however, is gauge dependent so a specific gauge choice is needed to specify the complete metric. It is subject to an asymptotic boundary condition and its value at $r = r_0$ is constrained by the requirement of horizon regularity. The specific gauge-choice for h_γ is made only for convenience as it will not affect the value of the response coefficients.

We focus on large r-asymptotics of these four functions that characterize the metric perturbations in order to obtain the bending moment of the stress-energy tensor. In the large-r region the asymptotics of the remaining functions for the neutral bent p-branes with $n > 2$ are [14]

$$\begin{aligned}
\mathrm{h}_1(r) &= \frac{1}{n}\frac{r_0^n}{r^{n-1}} - \frac{\xi_2(n)}{n+2}\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}),\\
\mathrm{h}_2(r) &= -2\frac{r_0^n}{r^{n-1}} - 2\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}),\\
\mathrm{h}_r(r) &= \frac{2}{n}r - \frac{3}{n^2}\frac{r_0^n}{r^{n-1}} + \frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}),\\
\mathrm{h}_\Omega(r) &= \frac{2}{n}r - \frac{n-3}{n^2(n-1)}\frac{r_0^n}{r^{n-1}} - \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}),
\end{aligned} \tag{4.16}$$

where

$$\xi_2(n) = \frac{\Gamma[\frac{n-2}{n}]\Gamma[\frac{n+1}{n}]^2}{\Gamma[\frac{n+2}{n}]\Gamma[\frac{n-1}{n}]^2} = \frac{n\tan(\pi/n)}{4\pi r_0^2}A^2. \tag{4.17}$$

Here, A has been defined as the constant $A = 2r_0\frac{\Gamma[\frac{n+1}{n}]^2}{\Gamma[\frac{n+2}{n}]}$. For the computations at hand it is convenient to parameterize the asymptotics of h_γ by coefficients b_0, b_1, k_1, b_4, and k_2. By choosing $b_0 = 0$ and $b_1 = \frac{1}{2}$ we are able to eliminate some leading order terms in h_r and h_Ω, we do not need them as they do not characterize the bending effects. Their asymptotic behaviour can then be written,

$$\begin{aligned} h_r(r) =& \left[\frac{n^2-6+(4n^2-8n)b_4}{2n^2}\right]\frac{r_0^n}{r^{n-1}} \\ &+ \left[(k_1+2k_2) + \frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-n-2}), \end{aligned} \tag{4.18}$$

$$\begin{aligned} h_\Omega(r) =& \left[\frac{3-2b_4(n^2-2n)}{n^2(n-1)}\right]\frac{r_0^n}{r^{n-1}} \\ &+ \left[\frac{2(k_1-nk_2)}{n(n+1)} + \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\right]\frac{r_0^{n+2}}{r^{n+1}} + \mathcal{O}(r^{-n-2}). \end{aligned} \tag{4.19}$$

For later convenience we should read off the dipole contributions $f^{(D)}$, described in the previous chapter (3.53), that are used to determine the bending moment. Later, this will make it possible for us to write the dipole contributions of the charged solutions in terms of the neutral seed solution dipole contributions. From the large-r asymptotics of the metric perturbations (4.16), (4.18), (4.19), we can write the dipole contributions,

$$\begin{aligned} f_{ab}^{(D)}(r) &= K_{ab}\left(-\frac{\xi_2(n)}{n+2}\right) + u^c u_{(a}K_{b)c}(-2\xi_2(n)) + Ku_a u_b k_1, \\ f_{rr}^{(D)}(r) &= K\left[k_1+2k_2+\frac{4+7n+2n^2}{n^2(n+2)}\xi_2(n)\right], \\ f_{\Omega\Omega}^{(D)}(r) &= K\left[\frac{2(k_1-nk_2)}{n(n+1)} + \frac{4+3n+n^2}{n^2(n+2)(n+1)}\xi_2(n)\right]. \end{aligned} \tag{4.20}$$

Here we could continue, and obtain the dipole moment, from there on read off the response coefficients and obtain the explicit form of the Young modulus for neutral bent p-branes but instead we'll move on to the generating of the charged solutions.

4.2 Generating bent branes carrying Maxwell charge

In chapter 1 it was described how to generate new solutions by an uplift-boost-reduction procedure. The example shows how we can take a neutral seed solution, uplift it by adding $m+1$ extra dimensions, boost the solution in one of the isometry directions, and finally compactify that direction on a circle. The resulting solution is a charged dilatonic p-brane.

Let us write $\tilde{p}$ instead of p and for the associated parameters for the bent neutral brane solution acquired in the previous section. The example presented in chapter 1 can be directly applied by taking the bent neutral $\tilde{p}$-brane solution, shown in the previous section, as the seed solution for

$$ds^2_{d+1} = ds^2_D + \sum_{i=1}^{m}(dy_i)^2 + dx^2. \tag{4.21}$$

The seed solution (4.9) will enter as ds^2_D. This new solution has dimensionality $d = \tilde{p}+m+n+3$. Following the same procedure as in the example we arrive at a extrinsically perturbed solution for black $p = \tilde{p}+m$ brane carrying Maxwell charge. We have to obtain the large asymptotics of the generated solutions to arrive at the dipole corrections, and from the dipole corrections, calculate the response coefficients. The large-r asymptotics of the generated bent Maxwell charged brane solutions are written explicitly in the appendix.

The generated charged bent p-branes constitute a bent version of a subset of Gibbons-Maeda family of solutions which we discussed in chapters 2 and 3. We will denote the critical boost velocity from the neutral seed solution as $\tilde{u}^a$. The generated solution can be identified with the Gibbons-Maeda subset by identifying the dilaton coupling as

$$a^2 = 4(d-1)^2\tilde{a}^2 = \frac{2(\tilde{p}+m+n+2)}{\tilde{p}+m+n+1}. \tag{4.22}$$

The critical boost of the neutral solution can be identified with the boost and charge parameter from the family of solutions in a way that

$$\tilde{u}_t \cosh\kappa = u_t \cosh\alpha, \quad \tilde{u}^i = u^i \cosh\alpha, \quad \sinh^2\alpha = \tilde{u}_t^2 \sinh^2\kappa. \tag{4.23}$$

We can write the exponents from the action of the Gibbons-Maeda solutions correspondingly,

$$A = \frac{d-3}{d-2}, \quad B = \frac{1}{d-2}. \tag{4.24}$$

These identifications all correspond to the case of $N = 1$.

4.3 Generating bent branes carrying higher-form charge

The new solution can be developed further by using T-duality to compactify the remaining m isometry directions. Details of the T-duality symmetries and general examples were presented in chapter 1 and will be employed here to develop extrinsically perturbed p-brane solutions carrying smeared q-brane charge with $q \leq p$. It is necessary to impose the condition that $n+\tilde{p}+m = 7$ and $m \geq 1$. This will make it possible for us to construct configurations in $D = 10$ type-II string theory. The configurations will be consistent with a truncated action where we set the NS-NS B_2 field to zero and only allow one of the R-R fields. The relevant 10-dimensional action is

$$S = \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(q+2)!}e^{\frac{3-q}{2}\phi}H^2_{[q+2]}\right]. \tag{4.25}$$

Before writing up the transformations let us write $\hat{\phi} = -\phi$, where ϕ is the original dilaton, to make an identification of the black brane solution before and the supergravity field. Now, in Einstein frame, the T-duality transformation taken in an isometry direction z takes the form

$$\begin{aligned} g_{\mu\nu} &= e^{\frac{1}{8}\hat{\phi}}(\hat{g}_{zz})^{\frac{1}{4}}\left(\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu z}\hat{g}_{\nu z}}{\hat{g}_{zz}}\right), \\ g_{zz} &= e^{-\frac{7}{8}\hat{\phi}}(\hat{g}_{zz})^{-\frac{3}{4}}, \\ e^{2\phi} &= \frac{e^{\frac{3}{2}\hat{\phi}}}{\hat{g}_{zz}}, \\ A_{[q+2]} &= A_{[q+1]} \wedge dz, \end{aligned} \tag{4.26}$$

where the hatted entries are the fields before transformation. The next step is to apply the transformation first on the $q = 0$ solution generated with the Kaluza-Klein procedure and applying T-duality on the remaining m isometries to generate a bent p-brane solution carrying a $(m+1)$-form gauge field. One ends up with a relation between the m'th transformation and the original fields

$$g_{\mu\nu} = e^{\frac{m}{6}\hat{\phi}}\hat{g}_{\mu\nu}, \qquad g_{y_i y_j} = \delta_{ij}e^{\frac{m-7}{6}\hat{\phi}}, \qquad \phi = \frac{3-m}{3}\hat{\phi}. \tag{4.27}$$

The y_i coordinates come from the compactified isometry directions and the greek indices are the remaining directions. Taking $m = 0$ we reobtain the starting solution. One can see that this results overlaps with the Gibbons-Maeda solutions of (3.56).

We wish to work with the large-r asymptotics of the solution as before. They are obtained from the generated solution in the previous section by plugging (A.2), (A.3), (A.5), into the T-duality transformations (4.27) and noting

$$A = \frac{1}{d-2} \to \frac{q+1}{8}, \quad B = \frac{d-3}{d-2} \to \frac{7-q}{8}. \tag{4.28}$$

4.4 Obtaining the response coefficients

For the solutions obtained for the bent black branes carrying Maxwell- and higher-form charge we will now read off the dipole contributions to the metric and gauge field. From the dipole contributions it is then possible to read off the response coefficients to give the Young modulus and the piezoelectric moduli an explicit form.

4.4.1 The $q = 0$ case

Generically for a solution of the intrinsic equations with $q = 0$, stationarity of the overall configuration is required $u^a = \mathbf{k}^a/\mathbf{k}$, where $\mathbf{k}^a$ is a Killing vector along the brane fluid. Accordingly, the global horizon temperature T and global horizon potential are set to $T = |\mathbf{k}|\mathcal{T}$ and $\Phi_H = |\mathbf{k}|\Phi$ [26]. Stationarity, however, is ensured by the fact that the seed solution was constructed to be stationary. Because of the stationarity of the solution, we can write the horizon thickness r_0 and charge parameter α in terms of global thermodynamic quantities,

$$r_0 = \frac{n}{4\pi T}|\mathbf{k}|\left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{1}{2}}, \quad \tanh\alpha = \frac{\Phi_H}{|\mathbf{k}|}. \tag{4.29}$$

The world-volume stress-energy tensor components to order $\mathcal{O}(\tilde{\varepsilon})$ are

$$T_{(0)}^{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n\cosh^2\alpha u^a u^b - \eta^{ab}\right), \quad T_{(0)}^{y_i y_j} = P_\perp \delta^{y_i y_j}, \tag{4.30}$$

with $a = (t, z_i)$.

Before arriving at the response coefficients we need to write the dipole contributions $f^{(D)}_{\mu\nu}$. The dipole contributions can be read from the asymptotic form of the metric (A.3) and gauge field (A.5) obtained by the Kaluza-Klein procedure. We will denote the dipole contributions of the charged solution with a hat, $\hat{f}^{(D)}_{\mu\nu}$. For the charged solution, the dipole contributions can be written in terms of the dipole contributions of the neutral solution according to (3.53) and (3.54),

$$\begin{aligned}
\hat{f}^{(D)}_{tt} &= \left(1 + \frac{d-3}{d-2}s_\kappa^2\right) f^{(D)}_{tt}, \quad \hat{f}^{(D)}_{tz_i} = c_\kappa f^{(D)}_{tz_i}, \\
\hat{f}^{(D)}_{z_iz_i} &= \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{z_iz_i}, \quad \hat{f}^{(D)}_{y_iy_i} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt}, \\
\hat{f}^{(D)}_{rr} &= \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{rr}, \quad \hat{f}^{(D)}_{\Omega\Omega} = \frac{1}{d-2}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{\Omega\Omega}.
\end{aligned} \tag{4.31}$$

If one takes the rapidity $\kappa \to 0$, the original neutral solution contributions are reobtained. The transverse gauge condition gives us the constraint,

$$(\eta^{ab}\hat{f}_{ab} + m\hat{f}_{yy}) + \hat{f}_{rr} + (n-1)\hat{f}_{\Omega\Omega} = 0. \tag{4.32}$$

This leads to a relation, along with a specific gauge choice for k_2, which we write

$$k_1 = \frac{2}{1-n}(\xi_2(n) + nk_2), \quad -\tilde{k}\xi_2(n) - \frac{(n+1)(n-4)}{n^2(n^2+n-2)}\xi_2(n) = k_2\left(\frac{2}{1-n}\right). \tag{4.33}$$

Through the harmonic gauge condition we can see that $\hat{f}^{(D)} = 2\hat{f}_{\Omega\Omega}$, and the gauge choice shows us that $f^{(D)}_{\Omega\Omega} = -\xi_2(n)\tilde{k}K$ is purely a gauge artifact and hence, also the dipole terms in those directions.

Expressing the non-vanishing dipole moments in terms of the neutral dipole coefficients and choosing a specific gauge for $\tilde{k}$ to make contact with the generalized Gibbons-Maeda solutions one obtains the world-volume bending moment,

$$\begin{aligned}
\hat{d}_{ab} = &- \xi_2(n)\cosh^2\alpha\left[\frac{K_{ab}}{(n+2)\cosh^2\alpha} + 2u^c u_{(a}K_{b)c} + \frac{3n+4}{n^2(n+2)}\eta_{ab}K\right] \\
&- \bar{k}\xi_2(n)[n\cosh^2\alpha u_a u_b - \eta_{ab}]K.
\end{aligned} \tag{4.34}$$

This form of the bending moment is only valid under the assumption that $K_{ta} = 0$ for all a. From the assumption that the bending moment behaves according to classical elastic theory,

$$d^{ab\rho} = \tilde{Y}^{abcd} K_{cd}{}^{\rho}, \tag{4.35}$$

we can now read off the Young modulus of the Maxwell charged bent black brane. The covariant form of the Young modulus is found to be [14]

$$\begin{aligned} \tilde{Y}_{ab}^{cd} =& P_{\perp} r_0^2 \xi_2(n) \cosh^2\alpha \left[\frac{3n+4}{n^2(n+2)} \eta_{ab}\eta^{cd} + \frac{1}{(n+2)\cosh^2\alpha} \delta^c_{(a}\delta^d_{b)} + 2u_{(a}\delta^{(c}_{b)}u^{d)} \right] \\ & - \bar{k}\xi_2(n) r_0^2 \left[T_{ab}^{(0)} \eta^{cd} + \eta_{ab} T^{cd}_{(0)} \right]. \end{aligned} \tag{4.36}$$

From this result and the relation of the solution parameters r_0 and α we can read off the four non-vanishing response coefficients, only three of which are found to be independent. The response coefficients obtained from (4.29) and (3.43) of the bending effects are as follows,

$$\begin{aligned} \lambda_1(\mathbf{k}; T, \Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n+2} \left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2} \right)^{\frac{n}{2}} \left(\frac{3n+4}{2n^2(n+2)} - \bar{k} \left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2} \right) \right), \\ \lambda_2(\mathbf{k}; T, \Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n+2} \left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2} \right)^{\frac{n}{2}+1} \frac{1}{2(n+2)}, \\ \lambda_3(\mathbf{k}; T, \Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G} \xi_2(n) \left(\frac{n}{4\pi T} \right)^{n+2} |\mathbf{k}|^{n} \left(1 - \frac{\Phi_H^2}{|\mathbf{k}|^2} \right)^{\frac{n}{2}}, \\ \lambda_4(\mathbf{k}; T, \Phi_H) &= \lambda_3(\mathbf{k}; T, \Phi) n\bar{k}. \end{aligned} \tag{4.37}$$

We now turn to the dipole contributions to the gauge field. The non-vanishing dipole terms for the asymptotic gauge field are from (A.5)

$$a_t^{(D)} = c_\kappa s_\kappa f_{tt}^{(D)}, \quad a_{z_i}^{(D)} = s_\kappa f_{tz_i}^{(D)}. \tag{4.38}$$

From this and the relations to the Gibbons-Maeda solutions on can construct the electric dipole moment (3.44),

$$\hat{p}_a = -\xi_2(n) \cosh\alpha \sinh\alpha [u^c K_{ca} + \bar{k} u_a K]. \tag{4.39}$$

Using the bending moment and the electric dipole moment the form of the piezoelectric moduli can be written a covariant form

$$\tilde{\kappa}_a^{bc} = -\xi_2(n) r_0^2 \left(\frac{\mathcal{Q}}{n} \delta_a^{(b} u^{c)} + \bar{k} J_a^{(0)} \eta^{bc} \right). \tag{4.40}$$

The two response coefficients can be read from the piezoelectric moduli, of which only one of them is found to be independent. From (3.45) and (4.29) the electric response coefficients are as follows,

$$\begin{aligned}\kappa_1(\mathbf{k};T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\frac{\xi_2(n)}{2}\left(\frac{n}{4\pi T}\right)^{n+2}\Phi_H|\mathbf{k}|^n\left(1-\frac{\Phi_H^2}{|\mathbf{k}|^2}\right)^{\frac{n}{2}},\\ \kappa_3(\mathbf{k};T,\Phi_H) &= \kappa_1(\mathbf{k};T,\Phi_H)n\bar{k}.\end{aligned} \tag{4.41}$$

4.4.2 The $q \leq p$ case

The T-duality transformations allowed us to generate a higher-form charged bent brane solution for type-II string theory $D = 10$. The generated solutions are solutions of the truncated action (4.25) for black p-branes carrying smeared Dq-charge. For these solutions the leading order stress-energy tensor takes the form,

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n\left(n\cosh^2\alpha u^a u^b - \eta^{ab}\right), \quad T^{y_iy_j}_{(0)} = P_{\parallel}\delta^{y_iy_j}, \tag{4.42}$$

with $a = (t, z_i)$.

We turn to the calculation of the bending moment and the electric dipole moment for bent black branes carrying higher-form charge, leading to the explicit form of their Young modulus and piezoelectric moduli and their response coefficients. The procedure is the same as in the Maxwell charge case. Again, noting $\hat{f}$ as the dipole contributions of the charged solution we write them in terms of the neutral dipole contributions,

$$\begin{aligned}\hat{f}^{(D)}_{tt} &= \left(1+\frac{7-q}{8}s_\kappa^2\right)f^{(D)}_{tt}, & \hat{f}^{(D)}_{tz_i} &= c_\kappa f^{(D)}_{tz_i},\\ \hat{f}^{(D)}_{z_iz_i} &= \frac{q+1}{8}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{z_iz_i}, & \hat{f}^{(D)}_{y_iy_i} &= -\frac{7-q}{8}s_\kappa^2 f^{(D)}_{tt},\\ \hat{f}^{(D)}_{rr} &= \frac{q+1}{8}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{rr}, & \hat{f}^{(D)}_{\Omega\Omega} &= \frac{q+1}{8}s_\kappa^2 f^{(D)}_{tt} + f^{(D)}_{\Omega\Omega}.\end{aligned} \tag{4.43}$$

To obtain the non-zero bending moment components we follow the same procedure as for the previous case of Maxwell charge. The relations obtained from the transverse gauge condition all remain the same as before after the T-duality transformations. The higher-form charge obtained by the T-duality transformations is always smeared along the bending direction of the brane.

The directions which the q-brane charge lies is always flat and this never critically boosted. In chapter 2 an additional set of vectors $v_a^{(i)}$, $i = 1, ..., q$, is introduced to describe the q directions in which the smeared charge is located. We now write the bending moment in terms of the Gibbons-Maeda boost and charge parameters using (4.23),

$$\begin{aligned}\hat{d}_{ab} = &- \xi_2(n)\cosh^2\alpha\left[\frac{K_{ab}}{(n+2)\cosh^2\alpha} + 2u^c u_{(a}K_{b)c} + \frac{3n+4}{n^2(n+2)}\eta_{ab}K\right] \\ &- \bar{k}\xi_2(n)\left[n(\cosh^2\alpha u_a u_b - \sinh^2\alpha\sum_{i=1}^{q} v_a^{(i)}v_b^{(i)}) - \eta_{ab}\right]K.\end{aligned} \tag{4.44}$$

The same assumptions for the extrinsic curvature are made here as in the previous case. The electric dipole moment is obtained as

$$\hat{p}_{ba_1...a_q} = -(q+1)!\xi_2(n)\cosh\alpha\sinh\alpha\left[u^c v^{(1)}_{[a_1}...v^{(q)}_{a_q}K_{b]c} + \bar{k}u_{[b}v^{(1)}_{a_1}...v^{(q)}_{a_q]}K\right]. \tag{4.45}$$

From this expression, the piezoelectric moduli can be written and the response coefficients read from them.

The general form of the bending moment and the electric dipole moment is the same as in the case of the Maxwell charge, the reason being that the extrinsic perturbations are always along the smeared directions.

Let's first look at the case of bent branes carrying a string charge then examine general higher-form charge. Black branes carrying a string charge $q = 1$ have, to leading order, the stress energy tensor (omitting the index (1) in v)

$$T^{ab}_{(0)} = \frac{\Omega_{(n+1)}}{16\pi G}r_0^n\big(nu^a u^b - \gamma^{ab} - nN\sinh^2\alpha(-u^a u^b + v^a v^b)\big). \tag{4.46}$$

The equilibrium condition to leading order for solutions with $N = 1$ is obtained by solving the 1st order corrected extrinsic equations of motion (3.14), while setting $d^{ab\rho} = 0$,

$$n(u^a u^b\cosh^2\alpha - v^a v^b\sinh^2\alpha)K^i_{ab} = K^i. \tag{4.47}$$

And the leading order intrinsic equation (3.13) is solved for $q = 1$ by requiring, as in the previous case, stationarity and taking the global temperature and potential, $T = |\mathbf{k}|\mathcal{T}$ and $\Phi_H = 2\pi|\zeta||\mathbf{k}|\Phi$. The horizon thickness and charge parameter can then be written in terms of the global thermodynamic quantities,

$$r_0 = \frac{n}{4\pi T}|\mathbf{k}|\left(1 - \frac{1}{2\pi}\frac{\Phi_H^2}{(|\mathbf{k}||\zeta|)^2}\right)^{\frac{1}{2}}, \quad \tanh\alpha = \frac{1}{2\pi}\frac{\Phi_H}{|\mathbf{k}||\zeta|}. \tag{4.48}$$

The form of the Young modulus, calculated from the bending moments for p-branes carrying $q \leq p$ charge, ends up taking the same form as the Young modulus in the $q = 0$ case except that we use the leading order stress-energy tensor $T_{ab}^{(0)}$ written in the beginning of this section (4.42). The non-vanishing response coefficients can then be extracted from the Young modulus and take a form written in terms of the global thermodynamic quantities. Again, since the Young modulus has the same form as before, of the four non-vanishing response coefficients, only three are found to be independent. In terms of (4.48) the response coefficients are as follows,

$$\begin{aligned}
\lambda_1(\mathbf{k},\zeta;T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n+2}\left(1-\frac{1}{2\pi}\frac{\Phi_H^2}{|\mathbf{k}||\zeta|}\right)^{\frac{n}{2}}\left(\frac{3n+4}{2n^2(n+2)} - \bar{k}\left(1-\frac{1}{2\pi}\frac{\Phi_H^2}{|\mathbf{k}||\zeta|}\right)\right)\\
\lambda_2(\mathbf{k},\zeta;T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n+2}\left(1-\frac{1}{2\pi}\frac{\Phi_H^2}{|\mathbf{k}||\zeta|}\right)^{\frac{n}{2}+1}\frac{1}{2(n+2)},\\
\lambda_3(\mathbf{k},\zeta;T,\Phi_H) &= \frac{\Omega_{(n+1)}}{16\pi G}\xi_2(n)\left(\frac{n}{4\pi T}\right)^{n+2}|\mathbf{k}|^{n}\left(1-\frac{1}{2\pi}\frac{\Phi_H^2}{|\mathbf{k}||\zeta|}\right)^{\frac{n}{2}},\\
\lambda_4(\mathbf{k},\zeta;T,\Phi_H) &= \lambda_3(\mathbf{k},\zeta;T,\Phi)n\bar{k}.
\end{aligned} \tag{4.49}$$

The piezoelectric moduli obtained from the electric dipole moment takes the form

$$\tilde{\kappa}_{ab}^{cd} = -\xi_2(n)r_0^2\left(2\frac{\mathcal{Q}}{n}\delta_{[a}^{(c}v_{b]}u^{d)} + \bar{k}J_{ab}^{(0)}\eta^{cd}\right), \tag{4.50}$$

and $\tilde{\kappa}_{ab}^{cd} = \kappa_{[ab]}^{(cd)}$. This form of the piezoelectric moduli is a natural generalization of the one obtained for the case of $q = 0$. The non-vanishing response coefficients corresponding to the piezoelectric moduli only have one independent component. They read

$$\begin{aligned}
\kappa_1(\mathbf{k},\zeta;T,\Phi_H) &= \frac{\Omega_{(n+)}}{16\pi G}\frac{\xi_2(n)}{2}\left(\frac{n}{4\pi T}\right)^{n+2}\Phi_H|\mathbf{k}|^{n}\left(1-\frac{1}{2\pi}\frac{\Phi_H^2}{|\mathbf{k}||\zeta|}\right)^{\frac{n}{2}},\\
\kappa_3(\mathbf{k},\zeta;T,\Phi_H) &= \kappa_1(\mathbf{k},\zeta;T,\Phi_H)n\bar{k}.
\end{aligned} \tag{4.51}$$

For branes carrying charges $1 < q < p$, the Young modulus is again, simply the same as before except with the general q stress-energy tensor (3.60). Naturally, the number of independent response coefficients will again be three. The piezoelectric moduli is a natural generalization to arbitrary q,

$$\tilde{\kappa}^{cd}_{ba_1...a_q} = -\xi_2(n) r_0^2 \left((q+1)! \frac{\mathcal{Q}}{n} \delta^{(c}_{[b} v^{(1)}_{a_1} ... v^{(q)}_{a_q]} u^{d)} + \bar{k} J^{(0)}_{ba_1...a_q} \eta^{cd} \right), \tag{4.52}$$

with the property $\tilde{\kappa}^{cd}_{ba_1...a_q} = \tilde{\kappa}^{(cd)}_{[ba_1...a_q]}$. There is only one independent response coefficient characterizing the piezoelectric moduli.

Discussion

In chapter 1 we took together the tools needed to go from general relativity and string theory to the discussion of the theory of blackfolds. The general idea of what a p-brane is was introduced and it was explained how string dualities can be used to generate new solutions, which turned out to be of great use for the developments in [14], reviewed in chapters 3 and 4. A simple example of smearing a brane solution was presented for the purpose of clarifying the concept of smeared q-charges carried by a black p-brane.

The main topic of the thesis was the bending effects on charged fluid black branes embedded in a background spacetime, characterized by two widely separated horizon length scales. In other words the bending effects on charged blackfolds. The blackfold approach for charged black p-branes was reviewed in chapter 2, mainly the work of [26] and [27]. Necessary concepts concerning the mathematics of embedding of p-branes in a background spacetime were presented and it was shown how the conservation of the stress-energy tensor leads to the blackfold equations. The effective stress-energy tensor and effective world-volume current was discussed for the cases of branes carrying Maxwell charge and a higher-form charge and the thermodynamic properties arising from them.

Following the introduction of the blackfold theory of charged black branes, the framework of the blackfold approach in the 1st order correction of the multipole expansion of the stress-energy tensor and charge current was reviewed, based on [13] [14]. A generalization of the neutral pole-dipole branes obtained in [13] and [30] was presented for the case of electrically charged branes. When the blackfold approach is expanded to 1st order correction, while ensuring that backreaction effects are subleading and assuming a linear response theory, it is shown that the bending effects of stationary charged (an)isotropic fluid branes are captured by a relativistic generalization of concepts in classical elastic theory. That is, the Young modulus and piezoelectric moduli, each of which is described by a set of response coefficients that characterize the bending of the brane.

With the response coefficients of the charged black branes introduced in chapter 3 we moved on to the explicit derivation of them for a special class of solutions in chapter 4. Perturbing charged black brane solutions to first order, to obtain their bent versions, turns out to be a difficult task, so instead, the route taken to arrive at those solutions was to feed a bent neutral black brane solution as a seed solution for a series of solution generating techniques. In chapter 4 we began by introducing the method of matched asymptotic expansion, used to find bent black brane solutions, then we showed the specific construction of the solution used to obtain the charged bent black branes. A uplift-boost-reduction procedure, introduced in chapter 1, was used to obtain the isotropic bent Maxwell charged brane by compactifying one of the uplift directions. Following from that, an anisotropic solution of type-II string theory, carrying smeared q-charge, was obtained by successive T-duality compactifications of the remaining uplifted directions. From the fact that these solutions were obtained by use of solution generating techniques it follows that the charge obtained is always smeared in the bending directions of the branes.

It is, in fact, natural that fluid branes in the blackfold approach possess attributes described by generalizations of concepts in fluid dynamics and material sciences. The blackfold approach is a long wavelength theory and fluid dynamics and material science are universal long wavelength theories, so in the blackfold limit we expect them to be analogous to each other.

Finally, the dipole contributions to the metric and gauge field in the large-r asymptotics of the solutions was derived for the case of 0-brane charge and $(q \leq p)$-brane charge. From the dipole contributions it was possible to use the framework from chapter 3 to write the Young modulus and piezoelectric moduli for each case. Expressing the horizon thickness and charge parameter, r_0 and α, in terms of global thermodynamic quantities made possible to write every non-vanishing independent response coefficient in terms of numbers characterizing the branes. In each case the bending effects were captured by a total of $3 + 1 = 4$ response coefficients, three for the Young modulus, and one for the piezoelectric moduli. It is of interest to see how more general bent black brane geometries, where the bending is not only in the smeared directions but also in the directions which the brane is charged.

The work reviewed in the thesis sheds important light into further understanding of finite thickness effects of the blackfold approach to charged branes in supergravity. A complete characterization of stationary blackfolds with $q > 1$ smeared charge has not been developed yet. In that relation, expanding the approach to higher order approximation is of obvious interest.

The authors of [14] mention the interesting developments that can be made in context of AdS/CFT, which we briefly mentioned in the thermodynamics section of chapter 1. Finding the metric of a bent D3-brane in type-IIB string theory, we would expect a measurable extra contribution to the dipole electric (magnetic) moment which would make it possible to measure electric (magnetic) susceptibilities. Further, in the context of AdS/CFT, it is worth examining whether the response coefficients provide clues regarding the microscopics of black holes and branes. Generally, it holds potential in shedding further light on flat space holography.

Yet to be computed explicitly are the magnetic response coefficients, that should arise if the angular momentum of the branes considering hadn't been put to zero. Naturally, there would be an induced magnetic dipole caused by the angular motion of the charged brane. Another kind of magnetic response coefficient can also be derived. Working in a magnetic dual frame, by calculating the hodge dual of the produced gauge fields $\tilde{A} = *A$, we could obtain the magnetic dual of the electric dipole response coefficients.

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Appendix A

Large r-asymptotics of the bent Maxwell charged solution

For convenience we define the object

$$H_{\mu\nu} = \cos\theta\Big[\hat{h}_{\mu\nu} - 2rK_{\mu\nu}\Big], \tag{A.1}$$

where $\hat{h}_{\mu\nu}$ is given by (4.13)-(4.16) and (4.18) and (4.19). The large-r asymptotic of the dilaton solution is

$$e^{-2\tilde{a}\phi} = 1 + \frac{s_\kappa^2 \tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n} + \frac{s_\kappa^2}{d-2}F_{tt}\left(1 - \frac{d-3}{d-2}\frac{r_0^n}{r^n}s_\kappa^2\tilde{u}^2\right) + \mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right). \tag{A.2}$$

The large-r asymptotics of the generated metric components are

$$
\begin{aligned}
g_{tt} &= \eta_{tt} + \left(1+\frac{d-3}{d-2}s_\kappa^2\right)\frac{r_0^n}{r^n}\tilde{u}_t^2 + H_{tt}\left(1+\frac{d-3}{d-2}s_\kappa^2+\frac{r_0^n}{r^n}\mathcal{C}_{tt}\right)+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
g_{z_iz_j} &= \eta_{ij} + \left(\tilde{u}_i\tilde{u}_j+\eta_{ij}\frac{s_\kappa^2\tilde{u}_t^2}{d-2}\right)\frac{r_0^n}{r^n} + \left(H_{ij}+\eta_{ij}\frac{s_\kappa^2 F_{tt}}{d-2}\right)+\frac{r_0^n}{r^n}\mathcal{C}_{z_iz_j}+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
g_{tz_i} &= c_\kappa\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_i + H_{ti}c_\kappa - 2c_\kappa s_\kappa^2\tilde{u}_t\tilde{u}_{(t}H_{i)t}\frac{d-3}{d-2}\frac{r_0^n}{r^n}+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
g_{y_iy_i} &= 1+\frac{s_\kappa^2\tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n}+\frac{s_\kappa^2}{d-2}H_{tt}\left(1-\frac{d-3}{d-2}\frac{r_0^n}{r^n}s_\kappa^2\tilde{u}_t^2\right)+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
g_{rr} &= 1+\left(1+\frac{s_\kappa^2\tilde{u}_t^2}{d-2}\right)\frac{r_0^n}{r^n}+H_{rr}+\frac{s_\kappa^2}{d-2}H_{tt}+\frac{r_0^n}{r^n}\mathcal{C}_{rr}+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
g_{\Omega\Omega} &= g_{\xi_i\xi_j}\left(1+\frac{s_\kappa^2\tilde{u}_t^2}{d-2}\frac{r_0^n}{r^n}\right)+H_{\Omega\Omega}+\frac{g_{\xi_i\xi_j}s_\kappa^2}{d-2}H_{tt}+\frac{r_0^n}{r^n}\mathcal{C}_{\Omega\Omega}+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),
\end{aligned}
\tag{A.3}
$$

where

$$
\begin{aligned}
\mathcal{C}_{tt} &= s_\kappa^2\tilde{u}_t^2\frac{d-3}{d-2}\left(\frac{s_\kappa^2}{d-2}-2c_\kappa^2\right),\\
\mathcal{C}_{z_iz_j} &= \frac{s_\kappa^2}{d-2}\left[\tilde{u}_t^2 H_{ij}+\left(\tilde{u}_i\tilde{u}_j-\eta_{ij}\frac{d-3}{d-2}s_\kappa^2\tilde{u}_t^2\right)H_{tt}\right]-2s_\kappa^2\tilde{u}_t\tilde{u}_{(i}H_{j)t},\\
\mathcal{C}_{rr} &= \frac{s_\kappa^2}{d-2}\left[\tilde{u}_t^2 H_{rr}+\left(1-\frac{d-3}{d-2}s_\kappa^2\tilde{u}_t^2\right)H_{tt}\right],\\
\mathcal{C}_{\Omega\Omega} &= \frac{s_\kappa^2\tilde{u}_t^2}{d-2}\left(H_{\Omega\Omega}-g_{\xi_i\xi_j}H_{tt}\frac{d-3}{d-2}s_\kappa^2\right),
\end{aligned}
\tag{A.4}
$$

with $H_{ij} = H_{z_iz_j}$, $\eta_{ij} = \eta_{z_iz_j}$ and $\tilde{u}_i = \tilde{u}_{z_i}$. We have written $g_{\xi_i\xi_j}$ as the metric for a $(n+1)$-sphere of radius r. In terms of the parameters of the neutral solution, the large-r asymptotics of the gauge field generated is

$$
\begin{aligned}
A_t &= s_\kappa c_\kappa\left[\frac{r_0^n}{r^n}\tilde{u}_t^2+H_{tt}\left(1-2s_\kappa^2\frac{r_0^n}{r^n}\tilde{u}_t^2\right)\right]+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right),\\
A_{z_i} &= s_\kappa\left[\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_i+H_{ti}-2s_\kappa^2\frac{r_0^n}{r^n}\tilde{u}_t\tilde{u}_{(t}H_{i)t}\right]+\mathcal{O}\left(\frac{r_0^{n+2}}{r^{n+2}}\right).
\end{aligned}
\tag{A.5}
$$