Aspects of Black Hole Physics

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Abstract: This project examines some of the exact solutions to Einstein's theory, the theory of linearized gravity, the Komar definition of mass and angular momentum in general relativity and some aspects of (four dimensional) black hole physics. The project assumes familiarity with the basics of general relativity and differential geometry, but is otherwise intended to be self contained.

The project was written as a "self-study project" under the supervision of Niels Obers in the summer of 2008.

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Preface and acknowledgement

First version: As a master's degree student in physics at the Niels Bohr Institute (Copenhagen University), I am required to complete 7.5 ects points of independent work ("selvstændigt element")[†]. This project accounts for 5 of these points. The project was written during the late spring/summer of 2008.
Second version: In the second version I corrected some minor mistakes, redraw one of the figures and changed the general layout.

The project is divided into three parts. The first part examines spacetime symmetries and with a fairly high degree of mathematic rigour the stationary, axisymmetric solutions to Einstein's equation. This part is mainly based on [Wal84], [Car04] and [tH02].

The second part investigates how it is possible to assign mass, charge and angular momentum to the spacetimes introduced in the first part. Einstein's equation in the weak field limit is examined and a definition of mass and angular momentum of axisymmetrical, stationary spacetimes is given (Komar integrals). This part is maily based [Wal84], [Car04] and [MP86].

The third part collects the results from the two first parts and applies it to the to black holes. The concepts of event horizons, Killing horizons and surface gravity for black holes are explained. Finally, the laws of black hole mechanics are examined and interpreted in terms of thermodynamics. This part is mainly based on [Wal84], [Ole07], [Car04] and [Tow97].

The project assumes the reader to have the same prerequisites as I had when I started writing it: The project presupposes that the reader has read and understood the lecture notes on general relativity [Ole07] by Poul Olesen (or the equivalent). Moreover, it is assumed that the reader knows the fundamentals of differential and Riemannian geometry, as treated, for example, in [Sch06], [Lee97,

[†]Note added in the second version: It has come to my attention that the requirement of 7.5 ects points of independent work has suddenly been dropped by the university in the autumn of 2008.

chapters 1-4, 8-10, 12-14] and [Lee00, chapters 1-7]. In particular, the reader should be conversant with manifold theory, tangent vectors, tensors, n-forms, metrics and connections, the theory of geodesics, curvature, Killing vectors and Killing's equation, integration on manifolds and Frobenius' theorem. The latter three subjects are accounted for, in a relatively understandable manner, in [Wal84, appendices B and C] and [Car04, appendices B, D and E].

I would like to thank my academic advisor Niels Obers for helpful guidance, illuminating discussions and for taking the time to answer my questions.

Units, conventions and notation

We will work in units where $c = 1$ and $\varepsilon_0 = 1/4\pi$. In these units Maxwell's equations take the form

$$\partial_\mu F^{\nu\mu} = 4\pi j^\nu$$
$$\partial_{[\mu} F_{\nu\rho]} = 0$$

We will use the mostly positive metric signature, i.e., the signature $(- + + +)$. Moreover, we use the same mathematical notation as [Wal84] and [Car04], especially we denote the μ-component of the covariant derivative of a tensor $T^{\nu\cdots}{}_{\rho\cdots}$ by

$$\nabla_\mu T^{\nu\cdots}{}_{\rho\cdots}$$

We use the $(\cdot,\cdot)$- and $[\cdot,\cdot]$-notation to denote respectively the (normalized) symmetrize and anti-symmetrize operation on tensors. For example for a two-tensor $A_{\mu\nu}$;

$$A_{(\mu\nu)} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \quad \text{and} \quad A_{[\mu\nu]} \equiv \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$$

and so on.

Finally, throughout this text, Killing vectors (and only Killing vectors) are denoted by gothic letters.

This text is typeset in $\LaTeX\,2_\varepsilon$.

1. Stationary solutions to Einstein's equation

1.1. Introduction

The aim of the first part of the project is to explore some of the few exact (and highly symmetric) solutions to Einstein's field equations of general relativity. Before we do this let us *briefly* recall Einstein's wonderful theory

The main assumption in general relativity (GR) is that the universe (the collection of all events) can be modeled as a four dimensional manifold M equipped with a Lorentzian (or pseudo-Riemannian) metric $\mathrm{g}_{\mu\nu}$, i.e., a metric with signature $- \ + \ + \ +$. This manifold is usually referred to as *spacetime*. According to Einstein's theory, the metric $\mathrm{g}_{\mu\nu}$ is in principle completely determined by the spacetime distribution of matter though *Einstein's equation*

$$\boxed{R_{\mu\nu} - \frac{1}{2} R \mathrm{g}_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{1.1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = R^{\mu}{}_{\mu} = \mathrm{g}^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar. The tensor on the left-hand side of Einstein's equation $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \mathrm{g}_{\mu\nu}$ is known as the *Einstein tensor*. The Einstein tensor is a *geometrical quantity*, i.e., it is completely determined by the metric $\mathrm{g}_{\mu\nu}$. The tensor $T_{\mu\nu}$ on the righthand side of Einstein equation is the *energy-momentum tensor*, which describes the distribution and flow of matter in our spacetime (M, g). The reader is assumed to know the physical significance of the energy-momentum tensor in terms of densities and flows of energy and momentum. The energy-momentum tensor for a physical system is usually determined by the laws of special relativity (SR) along with the principle of general covariance. For example, a perfect fluid (that is, a fluid completely characterized by the rest-frame energy density ρ and rest-frame isotropic pressure p and with the four velocity vector field U^{μ}) has the energy-momentum tensor $T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} + p \mathrm{g}_{\mu\nu}$ while the energy-momentum tensor for an electromagnetic field, described by field strength tensor $F_{\mu\nu}$, is given by $T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} \mathrm{g}_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda}$. Both of these formulæ can be obtained using the above described procedure.

Since physics is something that takes place on the spacetime manifold (at least in GR), it is clear that spacetime geometry (i.e., the metric) directly influences the laws of physics. Moreover, the metric determines the causal structure of spacetime. General relativity is therefore a vital tool for understanding *any* physical phenomena in a strong gravitational field.

Notice the enormous complexity of general relativity: In generic coordinates $\{x^{\mu}\}$, the metric has 10 independent components $\mathrm{g}_{\mu\nu}$, i.e., 10 unknown functions of the four coordinates $\{x^{\mu}\}$. These 10 functions are coupled through Einstein's equation, which is a non-linear differential equation! It is straight forward to show that Einstein's equation in the above form is equivalent to the following equation

$$R_{\mu\nu} = 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\Big) \tag{1.1.2}$$

where $T = T^{\mu}_{\ \mu}$. In particular, we have that in vacuum ($T_{\mu\nu} = 0$) Einstein's equation reduces to

$$R_{\mu\nu} = 0 \tag{1.1.3}$$

Using Einstein's theory of general relativity, we seek *stationary* (to be defined below) solutions to Einstein's equation under the following assumptions

I) *The Schwarzschild problem:* Completely spherical symmetric problem and spacetime filled with vacuum, $T_{\mu\nu} = 0$.
II) *The Reissner-Nordström problem:* Completely spherical symmetric problem and spacetime filled with an electromagnetic field, $F_{\mu\nu}$.
III) *The Kerr problem:* Axisymmetric problem and spacetime filled with vacuum, $T_{\mu\nu} = 0$.
IV) *The Kerr-Newman problem:* A general combination of the above three.

The notion of a spacetime being stationary ("time-translation" invariant), spherical symmetric ("rotational" invariant) and axisymmetric ("rotational" invariant around an axis) are easy to understand in SR (using coordinates), however, as we will see they need more careful geometrical definitions in GR. It is exactly these symmetries that will allow us to choose "clever coordinates" $\{x^\mu\}$ in which the metric components

$$\mathrm{g} = \mathrm{g}_{\mu\nu}\ \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \equiv \mathrm{g}_{\mu\nu}\ \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.1.4}$$

will simplify considerable. Having obtained these simplifying coordinates x^μ the Ricci tensor may be calculated in terms of $\mathrm{g}_{\mu\nu}$ and Einstein's equation will finally yield (after a lot of calculations) the metric components $g_{\mu\nu}$. The main point we want to convey is that in the clever coordinates $\{x^\mu\}$, solving Einstein's equation takes a lot of work but is possible, something which is not possible at all in "general" coordinates. So, understanding the mathematics of spacetime symmetries is crucial for finding analytical solutions to Einstein's equation and understanding general relativity.

As we will see, the four above solutions to Einstein's equation introduces three parameters M, Q and a, which all have physical interpretations. The interpretation of the parameters M and Q is pretty straightforward (as should already be well-known from [Ole07]), while understanding the parameter a need some more general considerations. In part 2 we will look into these considerations and justify the physical interpretations of M, Q and a.

1.2. The Schwarzschild solution

We want to solve Einstein's vacuum equation

$$R_{\mu\nu} = 0 \tag{1.2.1}$$

under the assumption that our spacetime M is spherically symmetric and stationary, static. Enough talk, let us now precisely define what it means for a spacetime to be stationary, static and spherically symmetric and see how these symmetries allows us to simplify the metric.

Spacetime symmetries

We start by defining what it means for a spacetime to be stationary.

Definition (Stationary spacetimes) *Suppose that a spacetime* (M, g) *has a one-parameter-group,* φ_t*, whose orbits consist of timelike curves. Such a spacetime is called stationary.*

Since every one-parameter-group of isometries has a corresponding Killing vector field (the (Killing) vector field that generates the one-parameter-group of isometries through its flow) a spacetime is stationary if and only if there exists a timelike Killing vector field $\mathfrak{K}^\mu$ i.e., a timelike vector field obeying Killings equation

$$\nabla_{(\mu}\mathfrak{K}_{\nu)} = 0 \tag{1.2.2}$$

As mentioned above we can think of a stationary spacetime as in some sense being invariant under "time translations". Why is this? Well, time is related to the " $-$ " in the $- \ + \ + +$ signature of the metric. Consider some point $p \in \mathsf{M}$ in the spacetime and consider some particle $\mathcal{P}$ located at p. Assuming that $\mathcal{P}$ is a physical particle, its world line $\gamma \equiv \gamma(\tau)$ may be parameterized by proper time τ so that $\gamma(0) = p$. We now say that an event $q = \gamma(\tau')$ on the world line, with $\tau' > 0$, happened *after* the event p, simply because seen from the particle it did. Recall the crucial property of physical world lines; they are timelike curves meaning that at each point of the curve the tangent vector T^μ

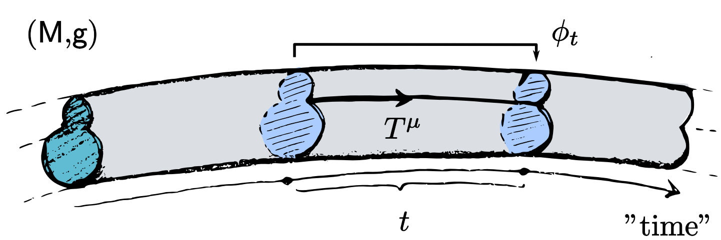

Fig 1. *The spacetime* (M, g) *is invariant under "time translations".*

of the curve is timelike i.e., $\mathrm{g}_{\mu\nu}T^\mu T^\nu < 0$ ($\mathcal{P}$ moves slower than the speed of light), this is precisely where the "$-$" from the metric signature comes to play. In this way see that "time flows" along timelike curves. It should now be clear why a stationary spacetime can be considered as being invariant under "time translations": Since the orbits of the one-parameter-group, φ_t, all are timelike, we can use the parameter t to describe the flow of time. So, the flow of the one-parameter-group of isometries φ_t, described by parameter t, represents the flow of time in M. Since we are dealing with a one-parameter-group of *isometries*, we conclude that the spacetime geometry is invariant under the flow of $t \sim$ flow of time as illustrated in fig. 1.

We will now explain what it means for a spacetime to be static.

Definition (Static spacetimes) *A stationary spacetime* (M, g) *is said to be static if it has a spacelike hypersurface* Σ_0 *which is orthogonal to the timelike orbits of the isometries* φ_t *(from the stationary condition).*

We therefore see that our spacetime M is static if and only if the Killing field $\mathfrak{K}^\mu$ is orthogonal to Σ_0. The characteristic feature of a static spacetime is explored in the following theorem.

Theorem (Static spacetimes) *The existence of the hypersurface* Σ_0 *(from the static condition) induces a family of hypersurfaces,* $\{\Sigma_t\}$*, parameterized by the "time" parameter* t *from the isometries, and all having the same orthogonality property as the original hypersurface* Σ_0*.*

Proof. To show this, assume that the Killing field $\mathfrak{K}^\mu$ is non-vanishing on the hypersurface Σ_0 and consider the integral curves $\{\gamma_p\}_{p\in\Sigma_0}$ of the Killing field $\mathfrak{K}^\mu$ starting at the points in Σ_0. Remember that these integral curves exactly are the orbits (of the isometries φ_t) of the points in Σ_0. This set of integral curves will split up our spacetime (at least in some neighborhood of Σ_0) in the following way: Since $\mathfrak{K}^\mu \neq 0$ on Σ_0, every point q (at least in some neighborhood of Σ_0) will lie on a unique integral curve going through Σ_0. In this way we can define a family of hypersurfaces by

$$\Sigma_t = \{q \in \mathsf{M} \mid q = \gamma_p(t) \text{ for some } p \in \Sigma_0\} \tag{1.2.3}$$

So, Σ_t is simply the collection of points we obtain by considering where the flow of $\mathfrak{K}^\mu$ takes the points of Σ_0 to at the "time" t, in other words; $\Sigma_t = \varphi_t(\Sigma_0)$. As mentioned above, the integral curves $\{\gamma_p\}_{p\in\Sigma_0}$ define a map from Σ_0 to Σ_t which is given by φ_t[1]. This map is one-to-one an onto, under the assumption that $\mathfrak{K}^\mu \neq 0$. To see that the set of hypersurfaces $\{\Sigma_t\}$ are all orthogonal to the orbits

[1]This follows from the correspondence between vector fields, their flows, integral curves and one-parameter-groups of diffeomorphisms.

of the isometries, again notice that this is true if and only if the hypersurfaces $\{\Sigma_t\}$ are all orthogonal the Killing field $\mathfrak{K}^\mu$. This follows from the fact that $\mathfrak{K}^\mu$ is invariant under its own flow, that Σ_t is the image of Σ_0 under φ_t and that φ_t is an isometry i.e., it preserves the inner product and especially orthogonality. □

This theorem enables us to use the hypersurfaces $\{\Sigma_t\}$ to give our spacetime M coordinates: We start by giving Σ_0 arbitrary "spatial" coordinates $\{x^i\}$ (where we consider Σ_0 as a three-dimensional embedded submanifold of M). We write the metric of Σ_0 in these coordinates as

$$\mathrm{d}\ell^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1.2.4}$$

A point $q \in \mathsf{M}$ will now lie in one of the hypersurfaces Σ_t for some unique t (at least in a neighborhood of Σ_0) and according to the above discussion there is a unique point $p \in \Sigma_0$ so that $\varphi_t(p) = q$. Now denote the coordinates of the point $p \in \Sigma_0$ by (x^1, x^2, x^3). We simply then give the point $q \in \mathsf{M}$ the coordinates (t, x^1, x^2, x^3), see figure 2. In these coordinates the metric $\mathrm{g}_{\mu\nu}$ takes the form

$$\mathrm{d}s^2 = -f(x^1, x^2, x^3)\mathrm{d}t^2 + h_{ij}(x^1, x^2, x^3)\mathrm{d}x^i\mathrm{d}x^j \tag{1.2.5}$$

where $-f = \mathrm{g}_{\mu\nu}\mathfrak{K}^\mu\mathfrak{K}^\nu = \mathfrak{K}_\mu\mathfrak{K}^\mu$ and where we have used that, since t is a Killing parameter, the metric will be independent of t together with the fact that Σ_t and $\mathfrak{K}^\mu$ are orthogonal, i.e., there will be no cross-terms of the type $\mathrm{d}t\mathrm{d}x^i$ in the expression for $\mathrm{d}s^2$. As we saw above, in order for these coordinates to work on some subset of M it is completely essential that on this subset $\mathfrak{K}^\mu \neq 0$. We therefore expect a breakdown of the above defined coordinates wherever $\mathfrak{K}^\mu = 0$.
Just as we interpreted a stationary spacetime as being invariant under "time translations", a static spacetime has a similar interpretation. We see from the explicit form of the static metric (1.2.5) that it is invariant under the transformation $t \to -t$. We thus see that our spacetime invariant under "time reversal". A

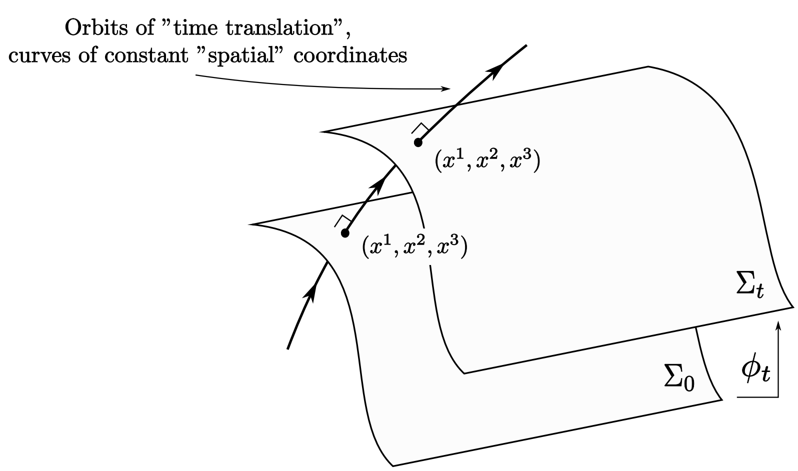

FIG 2. *The orbits of the "time translations" are orthogonal to the hypersurfaces Σ_t.*

stationary spacetime is hence interpreted as describing a physical situation which does not change in time while a static spacetime is interpreted as a describing a truly static physical situation. An example of a stationary but non-static spacetime could be a rotating star: Clearly such a spacetime is invariant under time translations while reversing time would reverse the direction of rotation and thus the spacetime geometry. This also means that a non-rotating star is described by a static spacetime.
Finally we mention that using the theory of hypersurfaces along with Frobenius' theorem, it can be showed that a spacetime is static if and only if the timelike Killing field $\mathfrak{K}^{\mu}$ fulfills the following condition

$$\mathfrak{K}_{[\mu}\nabla_{\nu}\mathfrak{K}_{\rho]} = 0 \tag{1.2.6}$$

The simplification of the metric to its static form (1.2.5) is the best we can do (which is already pretty good) without invoking spatial symmetries such as spherical symmetry. We will therefore now introduce the concept of spherical symmetry. Just as with time-symmetries we must define spherical symmetry in a coordinate independent way - we will do this algebraically.

Definition (Spherical symmetry) *A spacetime is said to be spherically symmetric if its isometry group (the set of all isometries equipped with the obvious composition) contains a subgroup isomorphic to the rotation group* SO(3).

Equivalently a spacetime is spherically symmetric if it possesses three Killing vector fields $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ that satisfy

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c} \quad , \quad [\mathfrak{b}, \mathfrak{c}] = \mathfrak{a} \quad , \quad [\mathfrak{c}, \mathfrak{a}] = \mathfrak{b} \tag{1.2.7}$$

We immediately recognize this algebra as the rotation algebra (recall if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are the infinitesimal rotations around respectively the x, y and z axis, then they will satisfy (1.2.7)). This is simply because the Killing vector fields precisely are the *infinitesimal generators* of their corresponding isometries. This means that we can identify the infinitesimal SO(3) generators, i.e., the rotation group Lie algebra $\mathfrak{so}(3)$ with the Killing vector fields generating the SO(3) isometries. Especially they satisfy the same algebra, i.e., the one given above. Now Frobenius' theorem tells us that the set $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ of vector fields fit together in such a way that they generate integral submanifolds (through their integral curves). This means that if we look at the collection of integral curves for respectively $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ through some arbitrary point, they will make out a (integral) submanifold. Since $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ corresponds to the rotation algebra this suggests that these submanifolds exactly will be (diffeomorphic to) 2-spheres. We will not go further into this discussion, since it relies on some pretty deep theorems from differential geometry, but we will simply assume that the orbits of SO(3) (resulting from the group action on our spacetime manifold M) are diffeomorphic to S^2, keeping in mind that this

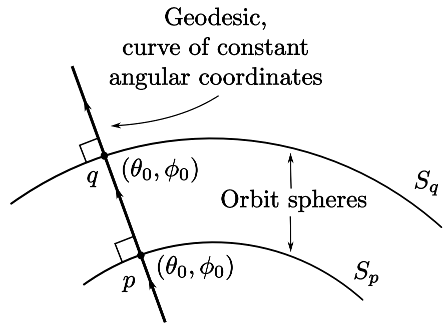

Fig 3. *The geodesics orthogonal to the orbit spheres determine the angular coordinates throughout* Σ_0.

is a reasonable assumption. Since the SO(3) action leaves the orbits $\sim$ 2-spheres invariant, the action of SO(3) can then be physically interpreted as rotations. Having identified the SO(3) action with rotations, we see that a spherically symmetric spacetime exactly is a spacetime which is invariant under rotations.

Now consider our original static, spherically symmetric spacetime M. As we have argued a point $p \in \mathsf{M}$ will lie on a unique orbit 2-sphere S_p and on a unique hypersurface Σ_t. As we will see now the entire 2-sphere S_p is contained in Σ_t. In order to show this, assume that the Killing field $\mathfrak{K}^\mu$ is unique and notice that a rotated timelike Killing field (to be completely precise: The pushforward of a timelike Killing under the isometry corresponding to the given rotation) is still a timelike Killing field since rotations are isometries. Since we have assumed that the Killing field $\mathfrak{K}^\mu$ is unique, we conclude that it must be invariant under rotations[2]. Since $\mathfrak{K}^\mu$ is invariant under rotations its projection onto S_p must also be invariant. However, the only vector field on $S_p \sim S^2$ which is invariant under *all* rotations is the zero vector field. We thus conclude that the $\mathfrak{K}^\mu$ is orthogonal to S in all points. Since S_p is obtained the collection of all the orbits of the point p under all rotations and we just showed that these orbits all are orthogonal to $\mathfrak{K}^\mu$, we conclude that all the orbits and thus S_p is contained in Σ_t.

We are now almost finished with simplifying the metric components. The final step consists of using the orbit spheres contained in (and covering) Σ_0 to define "clever" coordinates $\{x^i\}$ on Σ_0, which is, according to (1.2.4) a 3-dimensional Riemannian manifold.

As we now will see it is possible to construct "spherical coordinates" on a spherically symmetric space. Again we must be very careful in setting up coordinates since we do not know anything about Σ_0 except that it is spherically symmetric. Recall that ordinary spherical coordinates (r, θ, ϕ) on $\mathbf{R}^3$ have the property that

[2]Alternatively we could skip the assumption of uniqueness and just assume that $\mathfrak{K}^\mu$ is invariant under rotations. This is certainly reasonable since we in (1.2.5) have separated the "time" t and "space" and $\mathfrak{K}^\mu$ is related to t while rotations are related to space.

r is the distance to the origin and the straight lines (geodesics) emerging from the origin have $\theta, \phi = const.$ We will define spherical coordinates on Σ_0 in a similar manner.

Let $p \in \Sigma_0$ be some point. We start by assigning a "radial" coordinate r to p. As we have argued, the point p will lie on some orbit sphere $S_p \sim S^2$ which is contained in Σ_0. Now consider S_p as a (Riemannian) submanifold of Σ_0. Since S_p has SO(3) in its isometry group and it is diffeomorphic to S^2 it must be isometric to rS^2 (the 2-sphere of radius r with the canonical metric) for some $r > 0$. Since every point in Σ_0 will lie on some orbit sphere, this allows us to assign the "radial" coordinate r to the point p. Moreover, since $S_p \cong rS^2$ we can choose coordinates (θ, ϕ) on S_p, so that the metric on S_p takes the form

$$r^2 \mathrm{d}\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.2.8}$$

This defines spherical coordinates on S_p for all $p \in \Sigma_0$. This is, however, not good enough, since we have not specified how coordinates for different points on different orbit spheres are related. We must now carefully align the spherical coordinate systems on each of the different orbit spheres, to this end we will use geodesics. On the 2-dimensional orbit sphere S_p let p have the coordinates (θ_0, ϕ_0) and the coordinates (r, θ_0, ϕ_0) on Σ_0. Notice that, if we on each orbit sphere S_q, $q \in \Sigma_0$, specify what point should be assigned angular coordinates (θ_0, ϕ_0), we have defined spherical coordinates throughout Σ_0. This specification is done in the same way as for ordinary spherical coordinates on $\mathbf{R}^3$: We consider a geodesic through p which is orthogonal to S_p. We then *define* all the points along this geodesic to have angular coordinates (θ_0, ϕ_0) and their radial coordinate is determined by the "radius" of the orbit sphere on which they are located (as explained above), see fig. 3. This defines spherical coordinates on Σ_0 (at least in a neighborhood of S_p). Clearly, the assignment of spherical coordinates is only well-defined as long $\nabla_\mu r \neq 0$, since if $\nabla_\mu r = 0$ the radial coordinate will be degenerate. Notice that in these coordinates *all* the curves with $\theta, \phi = const.$ will be geodesics orthogonal to the orbit spheres they pass trough. This follows from rotational symmetry: Since rotations take geodesics to geodesics, preserve the inner product (since they all are isometries) and act transitively on $S_p \sim rS^2$, we see that any curve $\theta, \phi = const.$ through S_p can by obtained by rotating the geodesic through p orthogonal to S_p. Hence all the curves $\theta, \phi = const.$ will be geodesics orthogonal to S_p. Due to rotational symmetry, to show that all these curves will be orthogonal to any orbit sphere they pass through it is enough to show that this is true the geodesic trough p; $(\theta, \phi) = (\theta_0, \phi_0)$. To this end, notice that a rotation R around the vector orthogonal to S_p at p will send each point on the geodesic $(\theta, \phi) = (\theta_0, \phi_0)$ to itself (again because rotations (isometries) rotate geodesics into geodesics and preserves orthogonality). Now consider some point $q \in S_q$ on the geodesic $(\theta, \phi) = (\theta_0, \phi_0)$ where $q \notin S_p$, i.e., $S_q \neq S_p$. We want to show that the tangent vector $v \in T_q\Sigma_0$ to the geodesic $(\theta, \phi) = (\theta_0, \phi_0)$ in q is in the orthogonal complement of T_qS_q. So if we write $v = v^{\parallel} + v^{\perp}$ where $v^{\parallel} \in T_qS_q$ and

$v^\perp \in T_q^\perp S_q$, we must show that $v^\parallel = 0$. Since R sends each point on the geodesic $(\theta, \phi) = (\theta_0, \phi_0)$ to itself, we have that $Rv = v$, so $(v^\parallel - Rv^\parallel) + (v^\perp - Rv^\perp) = 0$. Notice that since R leaves q invariant it will also leave $T_q S_q$ and $T_q^\perp S_q$ invariant, i.e., $v^\parallel = Rv^\parallel$ and $v^\perp = Rv^\perp$. Again since R leaves q invariant $v^\perp = Rv^\perp$ automatically while $v^\parallel = Rv^\parallel$ requires that $v^\parallel = 0$.

In the above constructed spherical coordinates the metric h_{ij} on Σ_0 will take a particular simple form. Since by construction the vectors $\partial_\theta, \partial_\phi \in T_p S_p$ and according to the above argument $\partial_r \in T_p^\perp S_p$ we have that in the coordinates (r, θ, ϕ) the metric components h_{ij} take the form

$$\begin{aligned} \mathrm{d}\ell^2 &= h(r)\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \\ &= h(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \end{aligned} \tag{1.2.9}$$

where $h = h_{rr}\ \big(= h(\partial_r, \partial_r)\big)$ is constant on each of the orbit spheres (due to rotational symmetry), i.e., it cannot depend on the angular coordinates θ and ϕ.

Having obtained the "spatial" coordinates on Σ_0, by equation (1.2.5) we have found that the metric for a static, spherically symmetric spacetime, in the above defined coordinates, is given by (the function f can only depend on the radial coordinate r by rotational symmetry)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + h(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{1.2.10}$$

where we keep in mind that there might be regions of spacetime where these coordinates break down (as explained above).

The solution

Having carefully examined the symmetries of a static, spherically symmetric spacetime, we have reduced the problem of solving Einstein's equation to a problem of determining the two functions f and h, which are only functions of the radial coordinate r.
The metric (1.2.10) is the best we can do using symmetries and geometrical considerations - we must now solve Einstein's vacuum equation. In order to determine the two unknown functions f and h, we must go through the exercise of expressing the Ricci tensor $R_{\mu\nu}$ in terms of h and g. There are several methods for computing the curvature, Ricci tensor *etc*. The simplest method[3] is simply just first to express the Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ in terms of the metric components

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} \mathrm{g}^{\rho\lambda} \big\{ \partial_\mu \mathrm{g}_{\nu\lambda} + \partial_\nu \mathrm{g}_{\mu\lambda} - \partial_\lambda \mathrm{g}_{\mu\nu} \big\} \tag{1.2.11}$$

[3]A more elegant (but sophisticated) method is the so called tetrad (or vierbein) approach for computing curvature, see e.g. [Wal84].

and then use the expression for the Ricci tensor in terms of the Christoffel symbols (the derivation of this relation can be found in [Ole07])

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu \log g - \partial_\lambda \Gamma^\lambda{}_{\mu\nu} + \Gamma^\eta{}_{\mu\lambda}\Gamma^\lambda{}_{\lambda\nu} - \frac{1}{2}\Gamma^\eta{}_{\mu\nu}\partial_\eta \log g \tag{1.2.12}$$

Since the reader is assumed to have seen (and carried out) this final step of the derivation of the Schwarzschild solution, in for example [Ole07], we will skip this step and just write down the result: By solving Einstein's vacuum equation $R_{\mu\nu} = 0$ for the two unknown functions f and h, we obtain the following general solution for the metric of a static, spherically symmetric spacetime:

$$\boxed{\mathrm{d}s^2 = -\Big(1 - \frac{2MG}{r}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2MG}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2} \tag{1.2.13}$$

with

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \tag{1.2.14}$$

as the usual metric on S^2 and where M is a real positive parameter. This is the *Schwarzschild solution*.

Notice that the Schwarzschild solution (1.2.13) is *asymptotically flat* meaning that "infinitely far away" from the spacetime disturbance (i.e., for $r \to \infty$) the Schwarzschild metric is just the ordinary flat Minkowski metric $\eta_{\mu\nu}$ expressed in spherical coordinates[4]. Also notice that if we let $M \to 0$ ds^2 reduces to the flat Minkowski metric expressed in ordinary spherical coordinates.

As mentioned above the number M is a parameter of our solution, it comes about as an integration constant in the derivation of the Schwarzschild metric. However, it is possible to give M a physical interpretation (we look further into this identification in part 2): The parameter M can be identified with the physical mass of the (spherically symmetric) gravitational source responsible for the spacetime disturbance. Inside the mass distribution we of course have that $T_{\mu\nu} \neq 0$, so the Schwarzschild solution (1.2.13) is only valid outside the mass distribution. We will now briefly discuss some of the features of the Schwarzschild solution.

The first thing one notices are the singularities appearing in the Schwarzschild metric. We see that the metric components become singular in the set of points with radial coordinates $r = 2MG$ and $r = 0$. According to the above discussion there are two possibilities for the origin for such singularities

- ⋆ A breakdown of Schwarzschild *coordinates*.
- ⋆ A genuine singularity in the Schwarzschild spacetime geometry.

[4]From a strictly mathematical point of view this definition of asymptotic flatness is not a good definition, since it directly refers to the specific coordinate r. To define asymptotic flatness properly we need to define the concept of *spatial infinity* without referring to coordinates (actually we do not even know that large $r \sim$ spatial infinity, here we have to rely on our physical intuition). This *can* be done using so called conformal transformations (and diagrams). This is a rather technical discussion which we will not pursue any further.

As should be well-known to the reader, the singularity at $r = 2MG$ is associated with a coordinate breakdown while the singularity at $r = 0$ is a true singularity. The radial coordinate for the coordinate singularity

$$R_s = 2MG \tag{1.2.15}$$

is called the *Schwarzschild radius*. We completely understand why the coordinates breakdown at the Schwarzschild radius: Recall that the t-coordinate is associated with the timelike Killing vector field $\mathfrak{K}^\mu$ through the integral curves of $\mathfrak{K}^\mu$. In this way the metric components become independent of the "time" t. It was, however, an essential part of the construction of the Schwarzschild coordinates that the *timelike* Killing vector field $\mathfrak{K}^\mu$ was non-zero, i.e., $\mathfrak{K}^\mu \mathfrak{K}_\mu \neq 0$, since if $\mathfrak{K}^\mu \mathfrak{K}_\mu = 0$, the integral curves for $\mathfrak{K}^\mu$ cannot be used to carry the coordinates from the spacelike surface Σ_0. Now recall that

$$\mathfrak{K}^\mu \mathfrak{K}_\mu = g_{tt}(r) = \frac{2MG}{r} - 1 \tag{1.2.16}$$

so we see at $r = 2MG$ we have $\mathfrak{K}^\mu \mathfrak{K}_\mu = 0$, so the coordinate breakdown at the Schwarzschild radius is perfectly understood. As is well-known to the reader the singularity at $r = R_s$ is indeed a coordinate singularity meaning that it is possible to find coordinates in which there are no singularities at $r = R_s$. An example of such coordinates are the Kruskal coordinates. The Kruskal coordinates are constructed by an analysis of the Schwarzschild geodesics. This allows us to write up the Schwarzschild metric in a new set of coordinates; the Kruskal coordinates. The Kruskal coordinates have the property that they allow an analytical continuation for $r < R_S$ called the Kruskal extension (see e.g. [Ole07], [Car04] or [Wal84]). In other words, the Kruskal extension allows us to understand the global structure of the Schwarzschild spacetime. Similar constructions can be done with the Reissner-Nordström and Kerr spacetimes (where the global spacetime structures are very "strange"). The point is that the part of spacetime $r > R_S$ is just one region of a complicated global structure. However, since this region is where all the physics we know and can measure is located (cf. the discussion on event horizons below), we will not concern ourselves with the global extensions in this project. Moreover, it does not seem clear if the global extensions have any physical significance or if they are just pure math.

We will now discuss the $r = 0$ singularity. This was also to be expected, since the Schwarzschild solution physically corresponds to the gravitational field around a point particle of mass M. This means that at $r = 0$ we have "$T_{00} = \infty$". This singularity is reminiscent of the singularity of the electric field around a point charge in electrodynamics. Notice that the $r = 0$ singularity is a true (or geometrical) singularity, i.e., it is not possible to get rid of by some clever coordinate transformation. This can be seen by computing the curvature *invariant*

$$R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} = \frac{48M^2G^2}{r^6} \tag{1.2.17}$$

which is indeed singular at $r = 0$. Also, notice that $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ displays no singular behavior at $r = R_S$ in accordance with the fact the $r = R_S$ singularity corresponds to a breakdown of Schwarzschild coordinates, not spacetime. In order to understand the physics near the $r = 0$ singularity one has to examine the interior solutions to Einstein's equations, i.e., the static , spherically symmetric solution with e.g. a perfect fluid energy-momentum tensor. This is exactly the analysis in we need to go through in order to understand the spacetime inside a star or a planet. It would be rather unsatisfying if such a physically realistic solution also yielded a singularity at $r = 0$ (we do not expect a singularity at the center of Earth). We will not go into this analysis here (see [Wal84]) but merely state that the static, interior solutions describing objects such as stars, do not have singularities present at $r = 0$.

As noted above, the Schwarzschild solution will only be valid *outside* a static, spherically symmetric matter distribution e.g. (to good approximation) a non-rotating star. This means that, if the coordinate singularity at $r = 2MG$ is located within the matter distribution, it should not be taken too seriously, since it is a feature of the *vacuum* solution. For example, the Schwarzschild radius for the sun is 2.96 km, which is deep inside sun's interior where the vacuum solution is not valid. Indeed, just as with the singularity at $r = 0$, one can show that [Wal84] the Schwarzschild radius singularity is not present if it is located inside a spherically symmetric stable mass distribution. There is, however, the theoretical possibility that all the matter is located within the Schwarzschild radius. In this case, the vacuum solution is valid all the way in to the Schwarzschild radius and the coordinate singularities will be present in the metric. Such an object is the (Schwarzschild) *black hole*. A black hole is characterized by having an *event horizon*; an event horizon is a boundary in spacetime surrounding the black hole that only allows information to flow *into* it. This means that what happens inside the event horizon cannot affect what happens outside the event horizon. However, the opposite is not true, in other words, you can fall into a black hole but never come out. Especially, light emitted from inside the event horizon can never pass the event horizon and will therefore never reach an outside observer. Moreover, radiation emitted close to, but outside, the event horizon will become extremely redshifted. Hence, black holes are pitch black, thus the name. As we will explain in part 3, the presence of event horizons and the presence of coordinate singularities in the metric[5] are closely related. For example, the event horizon for the Schwarzschild black hole is exactly located at the Schwarzschild radius $R_s = 2MG$. In part 3 we look more into the different horizons related to black holes, moreover there is a few words on the formation of black holes in nature. Notice that a black hole is an *extremely* dense objects. By a simple rewriting of

[5]More precisely, singularities for the metric component g_{rr} when expressed in "asymptotic Minkowski" coordinates

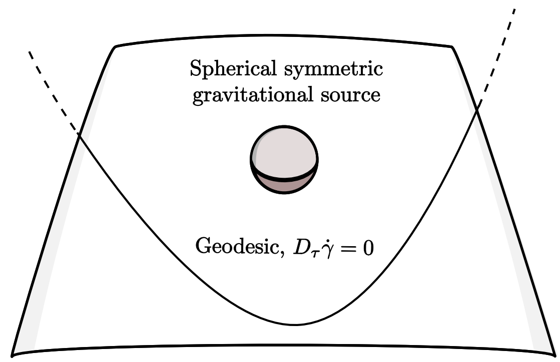

FIG 4. *Geodesics in the equatorial plane $\theta = \pi/2$ stay in the equatorial plane for all times.*

the Schwarzschild radius we have

$$R_s = 2.96 \left(\frac{M}{M_\odot}\right) \text{ km} \tag{1.2.18}$$

This means that in order to obtain a black hole from e.g. the Sun or the Earth, one should compress the Sun to a sphere of radius 2.96 km while one should compress the mass of the Earth to a sphere of radius of approximately 1 cm!

Finally we will briefly discuss the geodesics of the Schwarzschild spacetime. The first thing we notice is that the equatorial plane $\theta = \pi/2$ is totally geodesic, meaning that any geodesic with the following properties

- ⋆ It goes through the plane $\theta = \pi/2$ in a point p.
- ⋆ It is tangent to the plane $\theta = \pi/2$ in the point p.

will stay in the plane $\theta = \pi/2$ for all times. By rotational symmetry, this means that any geodesic in the Schwarzschild spacetime can be obtained by rotating an equatorial geodesic. We can therefore, with no loss of generality, assume that $\theta = \pi/2$ when examining the Schwarzschild geodesics. Let us now take such a equatorial geodesic γ with coordinates

$$\gamma : \ \gamma^\mu(\tau) \equiv \big(t(\tau), r(\tau), \pi/2, \phi(\tau)\big) \tag{1.2.19}$$

where τ is proper time if γ is timelike and some arbitrary non-affine parameter if γ is null. Here we have (as usual $\dot{\star} \equiv d\star/d\tau$)

$$\begin{aligned} -\alpha &= \mathrm{g}_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \\ &= -(1-2MG/r)\dot{t}^2 + (1-2MG/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \end{aligned} \tag{1.2.20}$$

where the number α is equal to 1 for γ timelike (massive particles) and equal to 0 for γ null (massless particles). We will now present a very useful theorem when determining the geodesic equations of motion

Theorem: Suppose that $\mathfrak{X}^\mu$ is a Killing vector field and that $\gamma \equiv \gamma(\tau)$ is a geodesic (parameterized by proper time τ[6]) with tangent vector $\dot\gamma^\mu \equiv \dot\gamma^\mu(\tau)$. Then

$$\boxed{\mathfrak{X}_\mu \dot\gamma^\mu = \text{constant along } \gamma} \tag{1.2.21}$$

Proof. The map $\tau \to \mathfrak{X}_\mu \dot\gamma^\mu$ is only defined along γ so we must be careful to use covariant derivatives *along* γ. In order to show that $\mathfrak{X}_\mu \dot\gamma^\mu$ is constant along γ, all we need to show is that $d/d\tau(\mathfrak{X}_\mu \dot\gamma^\mu) = 0$. We have that[7]

$$\frac{d}{d\tau}(\mathfrak{X}_\mu \dot\gamma^\mu) = D_\tau \mathfrak{X}_\mu \ \dot\gamma^\mu + \mathfrak{X}_\mu \ D_\tau \dot\gamma^\mu \tag{1.2.22}$$

here $D_\tau \dot\gamma^\mu$ denotes the covariant derivative *along* γ acting on $\dot\gamma$ (which is only defined along γ) and similarly for $\mathfrak{X}^\mu$. Now since γ is a geodesic we have by definition, that the velocity of γ, i.e., $\dot\gamma$ is parallel along γ, so $D_\tau \dot\gamma^\mu = 0$. Since the Killing field $\mathfrak{X}^\mu$ is a genuine vector field (i.e., not only defined along γ), the covariant derivative $D_\tau \mathfrak{X}^\mu(\tau)$ along γ is obtained by taking the covariant derivative of $\mathfrak{X}^\mu$ in the direction of $\dot\gamma^\mu(\tau)$. We therefore have

$$\frac{d}{d\tau}(\mathfrak{X}_\mu \dot\gamma^\mu) = D_\tau \mathfrak{X}_\mu \ \dot\gamma^\mu = \mathrm{g}_{\mu\nu} D_\tau \mathfrak{X}^\nu \ \dot\gamma^\mu = \mathrm{g}_{\mu\nu} \dot\gamma^\rho \nabla_\rho \mathfrak{X}^\nu \ \dot\gamma^\mu = \dot\gamma^\mu \dot\gamma^\nu \nabla_\mu \mathfrak{X}_\nu \tag{1.2.23}$$

Now, since $\mathfrak{X}^\mu$ is a Killing field we have from Killings equation that $\nabla_\mu \mathfrak{X}_\nu = -\nabla_\nu \mathfrak{X}_\mu$, so we are left with a sum consisting of a completely symmetric and a completely anti-symmetric part which is therefore equal to zero. Thus
$d/d\tau(\mathfrak{X}_\mu \dot\gamma^\mu) = 0$, which completes the proof. □

We therefore conclude that a one-parameter-group of spacetime symmetries gives rise a conserved quantity for massive particles and light rays (compare to Noether's theorem). Finding the geodesics of the Schwarzschild spacetime beautifully shows how powerful this theorem is. The Schwarzschild spacetime (or more precisely, the equatorial plane $\theta = \pi/2$) possesses two Killing fields, namely $\mathfrak{K}^\mu = \partial_t$ and $\mathfrak{R}^\mu = \partial_\phi$. In coordinates these are

$$\mathfrak{K}^\mu = (1, 0, 0, 0) \quad \text{and} \quad \mathfrak{R}^\mu = (0, 0, 0, 1) \tag{1.2.24}$$

Along the (arbitrary) geodesic γ, we therefore have the following two conserved quantities

$$E = -\mathfrak{K}_\mu \dot\gamma^\mu = -\mathrm{g}_{\mu\nu} \mathfrak{K}^\nu \dot\gamma^\mu = \Big(1 - \frac{2MG}{r}\Big)\dot t \tag{1.2.25}$$

[6]This is of course only possible if γ is not null. It is, however, easy to see that if γ is null we can repeat the proof for the theorem even if τ is not an affine parameter.

[7]Actually this is a property of the Levi-Civita connection (i.e., the unique, torsion free connection which is compatible with g) which is always assumed in (classical) GR.

and

$$L = \mathfrak{R}_\mu \dot{\gamma}^\mu = g_{\mu\nu}\mathfrak{R}^\nu \dot{\gamma}^\mu = \dot{r}^2 \dot{\phi} \tag{1.2.26}$$

the conserved quantity E has the interpretation of the total energy (kinetic + gravitational) per unit rest mass for the particle following the geodesic, relative to a stationary observer at infinity. Observe that we have defined the energy with a "$-$". This is because both $\dot{\gamma}^\mu$ and the time-translational field $\mathfrak{K}^\mu$ are future directed timelike[8], i.e., the product $\mathfrak{K}_\mu \dot{\gamma}^\mu$ will be negative, however, we want energy to be positive. The conserved quantity L has the interpretation of angular momentum per unit rest mass (compare to Kepler's second law). Notice, that, of course, the definitions of E and L coincides with those from SR in the asymptotically flat region.

The equations (1.2.20), (1.2.25) and (1.2.26) completely determine the geodesic equations of motion in the Schwarzschild spacetime (that was easy!). As is well-known, these equations account for the anomalous precession of the planet Mercury[9], describes the bending of light around the sun and so forth. For a discussion of these phenomena see any relativity text - Both [Ole07] and [Wal84] have very good subsections devoted to this.

1.3. The Reissner-Nordström solution

Suppose that instead of vacuum $T_{\mu\nu} = 0$, we consider a spacetime which contains a source-free ($j_\mu = 0$) electromagnetic field. We now want to solve Einstein's equation for such a system, under the assumption the problem is static and spherically symmetric, i.e., that spacetime is static and spherically symmetric and that the electromagnetic field tensor $F_{\mu\nu}$ is invariant under time-translations and rotations (these transformations were defined above).

Symmetries

Since electromagnetism involves long-range forces, we do not in general have $T_{\mu\nu}(p) = 0$ even though $j_\mu = 0$ in the point p. From special relativity and the principle of covariance we do, however, exactly know how to write up the energy-momentum tensor related to an electromagnetic field $F_{\mu\nu} = -F_{\nu\mu}$, it is in our

[8] As we will see in part 3, there is actually a region of the Kerr spacetime where $\mathfrak{K}^\mu$ becomes spacelike.

[9] Which is the best test we have of Einstein's theory to date.

units given by[10]

$$T_{\mu\nu} = \frac{1}{4\pi}\Big\{F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big\} \tag{1.3.1}$$

Notice that $T = T^{\mu}_{\ \mu} = 0$ - this simplifies the field equations considerably. The electromagnetic field tensor $F_{\mu\nu}$ is of course governed by the source-free Maxwell equations

$$g^{\mu\nu}\nabla_{\mu}F_{\nu\rho} = 0 \tag{1.3.2}$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0 \tag{1.3.3}$$

Notice how the Maxwell equations involve the metric $g_{\mu\nu}$; the propagation of electromagnetic radiation depends on the spacetime geometry. This is of course a very reasonable result, since the (free) movement of a particle is taking place along a geodesic (which is null in the case of a photon) and geodesics are purely determined by geometry. So, the presence of an electromagnetic field affects the spacetime curvature while the spacetime curvature affects how the electromagnetic field behaves.

This means that solving Einstein's equation in the presence of an electromagnetic field is extremely complicated (even more complicated than the general vacuum equation!). However, we have assumed that spacetime is static and spherically symmetric. This assumption immediately allows us to write up the most general metric respecting these symmetries. As we saw above, it is given by

$$ds^2 = -f(r)dt^2 + h(r)d^2 + r^2 d\Omega^2 \tag{1.3.4}$$

Under the assumption of spherical symmetry and time translation invariance, let us now examine how the most general electromagnetic field tensor $F_{\mu\nu}$ looks like. To this end introduce four new vector fields given by

$$\begin{aligned} e_0 &= \partial_t \\ e_1 &= \partial_r \\ e_2 &= \partial_\theta \\ e_3 &= \frac{1}{\sin\theta}\partial_\phi \end{aligned}$$

Using the metric (1.3.4) along with the construction of the coordinate vector field $\partial_t, \partial_r, \partial_\theta$ and ∂_ϕ we see that: Rotations leave the two vector fields e_0 and e_1 invariant, while the two vector fields e_3 and e_4 are not left invariant by rotations but rotate amongst each other, meaning that there exists a rotation that rotates

[10]This expression is obtained from special relativity and the principle of covariance: Recall that in SR we have $T^{00} = \frac{1}{2\times 4\pi}(\vec{E}^2 + \vec{B}^2)$ (energy density) and $T^{0i} = 1/4\pi(\vec{E}\times\vec{B})_i$, $i = 1,2,3$ (Poynting vector), which combines to $T_{\mu\nu} = 1/4\pi(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$. By the principle of general covariance (1.3.1) is then obtained by replacing $\eta_{\mu\nu} \to g_{\mu\nu}$. Alternatively this result can also be reached by a variation principle.

e_2 to e_3 and vice versa (this is exactly why we need the scaling factor $1/\sin\theta$ in e_3). We conclude, under the assumption that $F_{\mu\nu}$ is invariant under rotations, that the only possible non-zero components (in terms of the vector field e_i, $i = 0, 1, 2, 3$) of $F_{\mu\nu} = -F_{\nu\mu}$ are

$$F_{01} \quad \text{and} \quad F_{23} \tag{1.3.5}$$

Moreover, spherical symmetry and time translation invariance implies that F_{01} and F_{23} cannot depend on either the time t or the angular coordinates θ, ϕ. The dual vector field corresponding to e_i, $i = 1, 2, 3, 4$, are of course given by respectively $\mathrm{d}r, \mathrm{d}t, \mathrm{d}\theta$ and $\sin\theta\mathrm{d}\phi$. The most general static, spherically symmetric electromagnetic field tensor therefore is

$$F_{\mu\nu} = A(r)\big[\mathrm{d}r \otimes \mathrm{d}t - \mathrm{d}t \otimes \mathrm{d}r\big] + B(r)\sin\theta\big[\mathrm{d}\theta \otimes \mathrm{d}\phi - \mathrm{d}\phi \otimes \mathrm{d}\theta\big] \tag{1.3.6}$$

So, in terms of the coordinates t, r, θ and ϕ, the electromagnetic field tensor only has two independent, non-vanishing components, which are given by

$$\begin{aligned} F_{tr} &= A(r) = -F_{rt} \\ F_{\theta\phi} &= B(r)\sin\theta = -F_{\phi\theta} \end{aligned}$$

The solution

Having obtained the non-vanishing components of the electromagnetic field tensor it is now possible from equation (1.3.1) to express the energy-momentum tensor $T_{\mu\nu}$ in terms of the two functions A and B. Using Einstein's equation, the expression for $T_{\mu\nu}$ in the coordinates (t, r, θ, ϕ), the metric (1.3.4) and Maxwell's equations (1.3.2), one obtain a solvable set of ordinary differential equations for the functions f, h, A and B. Again we will not go explicitly through the computations (which can be found in [Ole07]), but merely state the result. The solution to Einstein's equation in the presence of a source-free electromagnetic field, under the assumption that the problem spherical symmetric and static, is given by

$$\boxed{\mathrm{d}s^2 = -\Pi\ \mathrm{d}t^2 + \Pi^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2} \tag{1.3.7}$$

where

$$\Pi(r; M, Q^2) = 1 - \frac{2MG}{r} + \frac{GQ^2}{r^2} \tag{1.3.8}$$

This is the *Reissner-Nordström solution*. The Reissner-Nordström solution introduces two real parameters M and Q^2. As for the Schwarzschild solution, the parameter M can be interpreted at the total mass of the gravitational source while the parameter Q^2 is related to the total electric and magnetic (monopole) charge of the Reissner-Nordström spacetime: By going through the derivation of the Reissner-Nordström solution, one finds that the two functions A and B are given by $A = -e/r^2$ and $B = m$, where e and m are integration constants. Now recall that the radial component of respectively the electric and magnetic field are given by $E_r = F_{rt}$ and the "dual relation" $B_r = F_{\theta\phi}/(r^2 \sin\theta)$. Therefore

$$E_r = \frac{e}{r^2} \quad \text{and} \quad B_r = \frac{m}{r^2} \tag{1.3.9}$$

This shows that we may identify the parameters e and m with respectively the total electric charge and the total magnetic charge[11] of the Reissner-Nordström spacetime. The parameter Q^2 is related to the two physical charges e and m through the relation

$$Q^2 = e^2 + m^2 \tag{1.3.10}$$

We see that the Reissner-Nordström solution is symmetric in the electric and magnetic charge, e and m. This is of course a very reasonable result, since gravity couples to energy and not to charge and in the units we use the electric and magnetic field contains the same amount of energy per volume per unit charge). In part 2 we will justify this identification, moreover we will see that it is possible to give a general definition of electric and magnetic charge for a spacetime manifold. We will now look into some of the features of the Reissner-Nordström spacetime.

The first thing one notices about the Reissner-Nordström solution is that it is, just as the Schwarzschild solution, asymptotically flat. This is of course a very reasonable result. Furthermore, the Reissner-Nordström solution is of course only valid *outside* the spherically symmetric charged object that it describes.

As mentioned above, the Schwarzschild solution is used to describe spacetime around e.g. stars. We may now ask, does the Reissner-Nordström metric have the same astrophysical significance as the Schwarzschild metric? Since electrostatic forces are much stronger than gravity, it is very hard for a astrophysical body, such as a star to build up any significant charge, simply because it takes a very small accumulated charge for the electric repulsion to exceed the gravitational attraction. A simple argument shows that the expected maximal electric charge e an astrophysical body of mass M can obtain is determined by

$$\frac{e}{\sqrt{G}M} \approx 10^{-18} \tag{1.3.11}$$

where the factor $(10^{18})^{-1}$ comes from the charge-mass ratio of the proton $q_p/m_p = 10^{18}$ (which is three orders of magnitude smaller than the charge mass ratio of the electron). Therefore, it seems that the gravitational field around any spherically symmetric astrophysical body is perfectly described by the Schwarzschild metric. This does, however, not mean that the Reissner-Nordström solution is not interesting - exact solutions to Einstein's theory are always interesting! It is possible[12]

[11] Of course in ordinary electrodynamics *i.e.* Maxwell's electrodynamics we have $m = 0$. It is however nice to keep the possibility of a non-zero m for theoretical reasons.

[12] I am guessing here.

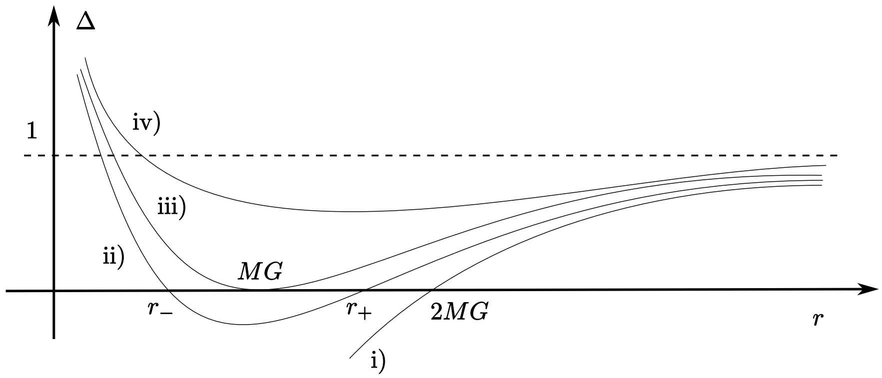

FIG 5. *The four types of Reissner-Nordström solutions.*

that the Reissner-Nordström solution is vital in higher dimensional gravity, quantum gravity and for understanding how gravity couples to more advanced gauge field.

We will now turn to a discussion of the singularities of the Reissner-Nordström metric. Just as with the Schwarzschild solution, the singularity at $r = 0$ is a true physical singularity (which is reasonable). The reader can convince himself of this by calculating $R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda}$ and seeing that this quantity does indeed blow up near $r = 0$. The Reissner-Nordström metric contains more singularities than the one in $r = 0$. However, just as with the Schwarzschild solution, these singularities are all coordinate singularities, i.e., we can get rid of them by choosing another set of coordinates (we will not do this here). The number of singularities for a given Reissner-Nordström metric depends on the ratio Q^2/M^2 - this separates the Reissner-Nordström solution into four distinct cases (see fig. 5)

i) $Q = 0$: This is just the Schwarzschild solution.

ii) $GM^2 > Q^2$: The coefficient Π has two roots r_+ and r_-,i.e., there are two singularities in the metric with radial coordinates $r_+ > r_-$. A straight forward calculation shows that the two radii r_+ and r_- are given by

$$r_\pm = MG \pm \sqrt{M^2G^2 - GQ^2} \tag{1.3.12}$$

As we will see in part 3, the two singularities r_+ and r_- corresponds to two event horizons; the *inner event horizon* located at r_- and the *outer event horizon* located at r_+.

iii) $GM^2 = Q^2$: This solution is know as the *extreme Reissner-Nordström solution*. The factor Π has *one* root corresponding to one event horizon. The extreme Reissner-Nordström solution gives rise to one of the most complex known exact solutions to Einstein's equation; the multi-extremal Reissner-Nordström solution (see [Car04]).

iv) $GM^2 < Q^2$: The coefficient Π has *no* roots, i.e., there are no singularities in the metric. However, the singularity in $r = 0$ is still present. This is an example of a so called naked singularity, since the $r = 0$ singularity is not hidden behind a metric singularity in the "asymptotic Minkowski" (spherical) coordinates (t, r, θ, ϕ), corresponding to an event horizon (more on this in part 3). This solution is in general considered unphysical. Of course most microscopical objects (e.g., an electron) fall under this category. However, such objects also obey the laws of quantum mechanics in which Einstein's classical theory does not make much sense.

So solutions of the type iv) are thought to be unphysical while the solutions of type iii) are unstable (and "extreme"). We will therefore always assume that the type of solutions we see in nature are of the type ii) or i), i.e., that

$$GM^2 > Q^2 \tag{1.3.13}$$

Using the same methods as we did with the Schwarzschild spacetime, it should be possible to examine both the pure geodesic movement (here explicitly presented in coordinates)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{1.3.14}$$

and the equation of motion of a charged particle with charge q and mass m

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau} \tag{1.3.15}$$

since, as opposed to the Schwarzschild spacetime, $F_{\mu\nu} \neq 0$. As mentioned above, the charge-mass ratio of a realistic astrophysical body seems to be extremely small. However, it is possible that the presence of even a small non-zero net charge could have an appreciable effect on the orbits of a charged test particle due to the Lorentzian force - we will not go through such an analysis here.

Before we go on to discussing the Kerr spacetimes, we will take a minute to mention Birkhoff's theorem regarding spherically symmetric spacetimes.

Theorem (Birkhoff) *Any spherically symmetric solution of the Maxwell/Einstein electro-vacuum field equations is static and asymptotically flat.*

This means that the spacetime around a spherically symmetric charged object will always be Reissner-Nordström. Notice that this theorem is a generalization of a well-known Newtonian theorem: The Newtonian gravitational field around a spherically symmetric mass distribution is the same as if all the mass were concentrated in the center. A similar result holds for the electric field around a spherically symmetric charge distribution. This especially means that the gravitational and electrical field around a time dependent spherically symmetric mass/charge distribution will be time independent - Birkhoff's theorem is the relativistic generalization of this. For a "proof" of Birkhoff's theorem see [Car04][13].

[13]The proof relies of the assumption that a spherically symmetric spacetime can be foliated

1.4. The Kerr solution

We will now look at the Kerr solution which describes a stationary, cylindrical symmetrically spacetime around a rotating body. Such a spacetime is clearly not static, since it is not invariant under time reflection, $t \to -t$. This means that Kerr solution must inevitable have cross-terms of the type $\mathrm{d}t\mathrm{d}x^i$. We will not go into a detailed derivation as with the Schwarzschild solution but for the sake of completeness take moment to talk about axisymmetry.

Axisymmetry

Following the discussion on spherical symmetry, it would natural to define a *stationary, axisymmetric* spacetime to be a stationary spacetime whose isometry group contains S^1. This definition is, however, not precise enough to be useful. We are nevertheless interested in having continuous isometries whose orbits look like S^1, i.e. closed curves. We will use this as our defining property.
A spacetime (M, g) is said to axisymmetric if it has a one-parameter-group of isometries ψ_ϕ whose orbits consist of *closed* spacelike curves. Equivalently a spacetime (M, g) is axisymmetric if and only if it has a spacelike Killing field $\mathfrak{R}^\mu$ whose integral curves all are closed curves.

If a spacetime (M, g), in extend of being axisymmetric, is also stationary *and* the two one-parameter-groups from the stationary and axisymmetric conditions commute, i.e. satisfy

$$\varphi_t \circ \psi_\phi = \psi_\phi \circ \varphi_t \tag{1.4.1}$$

for all t, ϕ, (M, g) is said to be *stationary, axisymmetric*. Equivalently a stationary and axisymmetric spacetime is stationary, axisymmetric if

$$[\mathfrak{K}, \mathfrak{R}] = 0 \tag{1.4.2}$$

The condition (1.4.1) should be interpreted as a space/time-decoupling condition. For example, the condition (1.4.1) implies that in an asymptotically flat spacetime there will be a curve Z on which $\mathfrak{R}^\mu$ vanishes [Wal84], i.e., this curve is left invariant under the isometries ψ_ϕ. In this way, the one-parameter-group of isometries ψ_ϕ can be interpreted as "rotations" with the curve Z as the "axis of rotation" see fig. 6.

The solution

We now seek the stationary, axisymmetric solutions to Einstein's vacuum equation $R_{\mu\nu} = 0$. Going through the derivation of these solutions from first principles

into 2-spheres (which we have showed is possible if the spacetime is spherically symmetric and static). This is, however, a rather strong assumption.

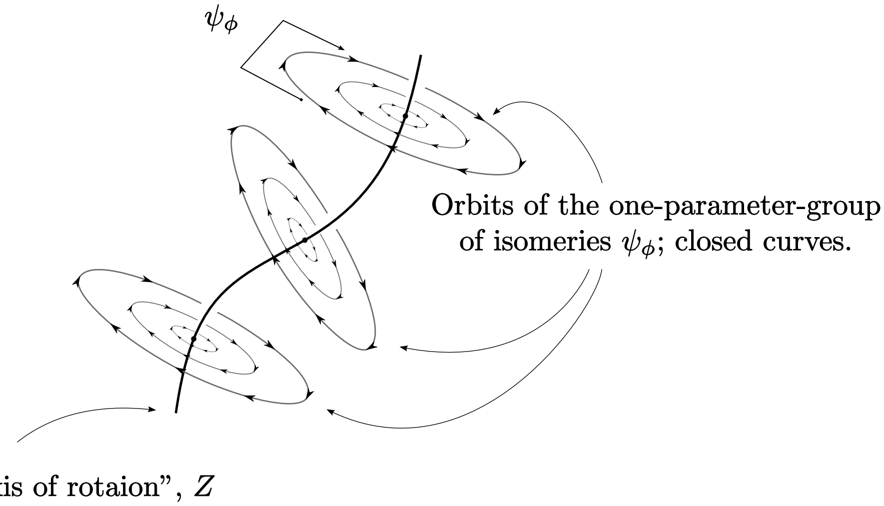

FIG 6. *The Killing field $\mathfrak{R}^\mu$ vanishes on the axis of rotation Z, i.e., the points on Z are left invariant under the isometries ψ_ϕ.*

requires a lot of heavy mathematics which is beyond the scope of this project, so we will just state the result. The stationary, axisymmetric solutions to Einstein's vacuum equation $R_{\mu\nu} = 0$ are given by

$$\boxed{\begin{aligned} \mathrm{d}s^2 = -\Big(1 - \frac{2MGr}{\rho^2}\Big)\mathrm{d}t^2 - \frac{2MGra\sin^2\theta}{\rho^2}\big(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t\big) \\ + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\big[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\big]\mathrm{d}\phi^2 \end{aligned}} \tag{1.4.3}$$

where a and M are two real numbers and the two functions ρ and Δ are defined by

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta(r;M,a) = r^2 - 2MGr + a^2 \tag{1.4.4}$$

The coordinates (t,r,θ,ϕ) in the Kerr solution are known as *Boyer-Lindquist coordinates*. The spacetime (1.4.3) is axisymmetric: The coordinate vector field ∂_t (just as with the Schwarzschild solution) corresponds to the timelike Killing field $\mathfrak{K}$ (the metric is independent of t) and the coordinate vector field ∂_ϕ corresponds to the Killing field $\mathfrak{R}$ (the metric is independent of the ϕ and ϕ coordinates are identified modulo $2\pi \sim$ closed orbits of ∂_ϕ). Clearly we have

$$[\partial_t, \partial_\phi] = 0 \tag{1.4.5}$$

Plugging the expression (1.4.3) for the Kerr metric into e.g. Maple confirms that (1.4.3) indeed is a solution to Einstein's vacuum equation $R_{\mu\nu} = 0$.

The parameter M has the same physical interpretation as with the Schwarzschild solution; it is the total mass (energy) of the gravitational source. The parameter a also has a physical interpretation. It is related to the total angular momentum of the gravitational source (recall that the Kerr solution describes the gravitational field around a rotating object). Define a new parameter J in terms of M and a by

$$J = Ma \tag{1.4.6}$$

This parameter J has the interpretation of the total angular momentum of the gravitational source. Also, notice that the line element (1.4.3) is invariant under each of the transformations $(t \to -t, a \to -a), (\phi \to -\phi, a \to -a)$ and $(t \to -t, \phi \to -\phi)$, which it of course should, since each of the transformations leave the (component of) angular momentum unchanged. Because of the above symmetries, *we will from now on assume that* $a \geq 0$.

With the identifications of the parameters M and $J = Ma$, which are justified in part 2, we see that, from an astrophysical point of view, the Kerr solution seems somewhat more fundamental the Reissner-Nordström solution. This is simply because, as opposed to the charge-mass ratio, there is not an upper limit on the parameter $a = J/M$ an astrophysical objects (and if there is one, it is rather large). For example, one could imagine a binary star system consisting of a main star and a lighter companion star, orbiting around their common center of mass. Suppose that the companion star comes close enough to the main star so that matter from the companion star starts flowing towards the main star[14], see fig. 7. Eventually the companion star will be completely absorbed by the main star. Of course the total angular momentum of the system is conserved, therefore the main star + absorbed companion star can obtain a larger angular momentum-mass ratio than the stars in the original system, since the original orbital angular momentum has now been converted into spin angular momentum.

If one has the patience, or a computer, it is straight forward to calculate the

[14]The volume where this mass transfer is possible (roughly speaking) is called the Roche Lobe and the process known as Roche Lobe overflow.

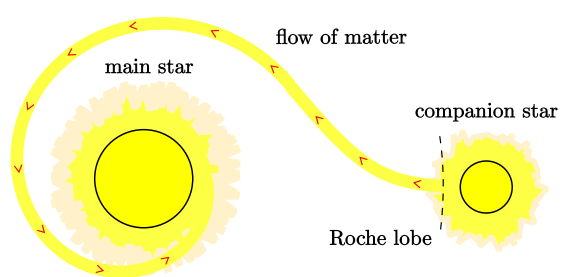

FIG 7. *Illustration of mass and angular momentum transfer in a binary star system trough Roche Lobe overflow.*

curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and just as with the Schwarzschild and Reissner-Nordström solutions, there is a point (actually now more than just one point) where it is singular - this means that the Kerr spacetime has a true (i.e., geometrical) singularity. The location of this singularity is *not* determined by $r = 0$, but rather ($a \neq 0$) by the equation

$$\rho = r^2 + a^2 \cos^2\theta = 0 \tag{1.4.7}$$

which is satisfied if and only if $r = 0$ and $\theta = \pi/2$. This might seem somewhat strange, however, the nature of this singularity is better understood if we keep the parameter a fixed and let $M \to 0$. By doing this, one sees that the Kerr metric (1.4.3) reduces to the flat Minkowski metric expressed in ellipsoidal coordinates (this is certainly reasonable since in this limit there is no mass ($M = 0$) and there only vacuum), defined by

$$\begin{aligned} x^0 &= t \\ x^1 &= (r^2 + a^2)^{1/2} \sin\theta \cos\phi \\ x^2 &= (r^2 + a^2)^{1/2} \sin\theta \sin\phi \\ x^3 &= r\cos\theta \end{aligned} \tag{1.4.8}$$

where $\{x^\mu\}$ denotes ordinary Minkowski coordinates. Here the points $\rho = 0$ corresponds to a ring of radius a in the $z = 0$ plane and the $\rho = 0$ singularity is, of course *here*, nothing more than a coordinate singularity. However, this shows that the Boyer-Lindquist coordinates should be understood as being ellipsoidal coordinates rather than spherical coordinates. This is of course very reasonable since the Kerr spacetime is *not* spherically symmetric. We therefore see that the *true* singularity at $\rho = 0$ ($M, a \neq 0$) should be interpreted as a ring singularity but rather as a point singularity (as with the Schwarzschild and Reissner-Nordström spacetimes). Having understood the true singularities of the Kerr spacetime, we may now look at the coordinate singularities. The coordinate singularities are determined by the real roots of the function Δ. Therefore the analysis of the coordinate singularities we did for the Reissner-Nordström spacetime also pertains to the Kerr spacetime (with the substitution $GQ^2 \to a^2$): The Kerr spacetime becomes Schwarzschild for $a = 0$, has a naked singularity (no coordinate singularity/event horizon) for $G^2M^2 < a^2$, has one singularity/event horizon for $G^2M^2 = a^2$ (the extreme Kerr solution) and has two singularities/event horizons for $G^2M^2 > a^2$. Just as with Reissner-Nordström spacetime, the only physical Kerr spacetimes are thought to be the ones with

$$G^2M^2 > a^2 \tag{1.4.9}$$

since naked singularities are not thought to exist in nature. The locations of the coordinate singularities for the physical Kerr spacetime are given by

$$r_\pm = MG \pm \sqrt{M^2G^2 - a^2} \tag{1.4.10}$$

and as explained in part 3, these two radii will correspond to event horizons.

Determining the geodesics of the Kerr spacetime follows the same recipe as with the Schwarzschild spacetime; we use the spacetime symmetries to find a set of first integrals which allows us to reduce the geodesic equation of motion to a simple low-dimensional problem. There is, however, a significant difference: The Kerr spacetime is not spherically symmetric. This means that the two constants of motion E and L (corresponding to respectively the time-translation and rotational Killing fields) are not enough to determine the equation of motion. It is, nevertheless, still possible to use the above explained method. This is due to the fact that the Kerr spacetime has a Killing tensor $\mathfrak{K}^{\mu\nu}$ (the tensor generalization of a Killing vector). Such a Killing tensor provides us with an extra constant of motion along geodesics $C = \mathfrak{K}_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu$ (in a similar manner as with Killing vectors) and a corresponding first integral. These equations are enough to determine the geodesics of the Kerr spacetime - see e.g. [Wal84].

1.5. The Kerr-Newman solution

Finally it is possible to find the most general stationary, axisymmetric gravitational and electromagnetic fields that solve the coupled Einstein/Maxwell equations - this is the Kerr-Newman solution (or just the charged Kerr solution). Recall that the Reissner-Nordström solution could be obtained from the Schwarzschild solution by the substitution $2MGr \to 2MGr - GQ^2$. The Kerr-Newman solution is obtained from the Kerr metric by the same substitution, i.e., by the following expression

$$\boxed{\begin{aligned} ds^2 = -\Big[\frac{\Delta - a^2\sin^2\theta}{\rho^2}\Big]\mathrm{d}t^2 - \frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) \\ + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\big[(r^2+a^2)^2 - \Delta\ a^2\sin^2\theta)\big]\mathrm{d}\phi^2 \end{aligned}} \tag{1.5.1}$$

where

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta(r;M,Q^2,a) = r^2 - 2MGr + Q^2 + a^2 \tag{1.5.2}$$

moreover the electromagnetic vector potential is given by (assuming that there is no magnetic charge, $Q = e$)

$$A_\mu = -\frac{er}{\rho^2}[(\mathrm{d}t)_\mu - a\sin^2\theta(\mathrm{d}\phi)_\mu] \tag{1.5.3}$$

We thus see that the rotation of the black hole also gives rise to a magnetic dipole potential in addition to the usual electrostatic potential. If we set $Q^2 = 0$ in the Kerr-Newman solution we obtain the Kerr solution while setting $a = 0$

renders the Reissner-Nordström solution. The Kerr-Newman solution represents the spacetime around a rotating, charged object of mass M, "charge" Q and total angular momentum $J = Ma$. The same considerations on singularities pertains to the Kerr-Newman spacetime as with the Reissner-Nordström and Kerr spacetimes; the Kerr-Newman metric has a true ring like singularity located at $\rho = 0$. Moreover the coordinate singularities are determined by the real roots of Δ. This leads to the physical condition that

$$M^2G^2 > Q^2 + a^2 \tag{1.5.4}$$

in order to avoid naked singularities. The coordinate singularities (corresponding to event horizons) are given by

$$r_{\pm} = MG \pm \sqrt{M^2G^2 - Q^2 - a^2} \tag{1.5.5}$$

2. Mass, charge and angular momentum (stationary spacetimes)

2.1. Introduction

As we saw in part 1, the three *stationary, axisymmetric* solutions to Einstein's equation introduce three parameters M, Q and J. As already mentioned, it is possible to identify these three parameters with respectively the mass (energy), charge[15] and angular momentum of the given gravitational source. Before we can understand this, we must first clarify exactly what we mean by the mass, charge and angular momentum in GR.

Generally, the notions of mass, charge and angular momentum are highly non-trivial to define for a generic spacetime manifold. However, recall that the spacetimes we have considered all are stationary and asymptotically flat. Also recall that we can view an asymptotically flat spacetime as a region of curved space surrounded by flat Minkowski space. This means that in principle, we can consider the curved space surrounding a gravitational source as a *point* situated in Minkowski flat space. Mass, charge and angular momentum are all perfectly-well defined and understood in special relativity. Moreover, we have seen that for a stationary, asymptotically flat spacetime, it was possible to define the energy and angular momentum (relative to a stationary observer at spatial infinity) for a point particle by the expression (1.2.21). Since any gravitational source can be considered a collection of point particles, we will therefore assume that is has an associated total mass (energy) $\mathcal{M}$, charge $\mathcal{Q}$ and angular momentum $\mathcal{J}$. By looking at the Schwarzschild, Reissner-Nordström and Kerr solutions we see that the three parameters M, Q and J have dimensions of respectively mass, charge and angular momentum. This means that, apart from maybe some constants, there is really no other possibility than that M, Q and J *are* respectively $\mathcal{M}$,$\mathcal{Q}$ and $\mathcal{J}$. In the following we will justify this identification further.

We will start by looking into the *weak field approximation*. That is, we examine Einstein's equation for small perturbations of the flat metric $\eta_{\mu\nu}$. Since, in the weak field approximation, we know the relation between the energy-momentum tensor $T_{\mu\nu}$ and mass, charge and angular momentum, this will allow us to compare the parameters M, Q and J appearing the exact solutions to mass, charge and angular momentum. In order to do this we need to examine Einstein's equation to lowest (linear) order in the metric perturbation.

2.2. Linearized Gravity

We will now examine the structure of Einstein's equation when spacetime is almost Minkowskian. We start by considering ordinary Minkowski space (M, η). Since Minkowski space is flat it can be covered by *global* inertial coordinates $\{x^\mu\}$

[15] Understood in the sense that $Q = \sqrt{e^2 + m^2}$.

where the metric η has the components $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ - this is, of course, noting but special relativity. Now suppose that we introduce some matter to our Minkowski space (Minkowski space corresponds to the spacetime of a universe without any matter present) which is some sense not too dense. The introduction of a "small" energy-momentum tensor $T_{\mu\nu}$ will result in a small perturbation of flat spacetime (M, η) to some new spacetime $\tilde{\mathsf{M}}$ with metric g. Since we assume that the perturbed spacetime $\tilde{\mathsf{M}}$ is almost flat, the metric g can be written

$$\mathrm{g} = \eta + \mathrm{h} \tag{2.2.1}$$

where the tensor h describing the perturbation of η, is, in some sense, assumed small relative to η. Since $\tilde{\mathsf{M}}$ is a slightly deformed version of M, we assume that we can carry the global Minkowski coordinates $\{x^\mu\}$ defined on M to $\tilde{\mathsf{M}}$ as illustrated in fig. 8. In these coordinates equation (2.2.1) takes the form

$$\mathrm{g}_{\mu\nu} = \eta_{\mu\nu} + \mathrm{h}_{\mu\nu} \tag{2.2.2}$$

where the assumption that h is "in some sense" small relative to η is translated to the condition $|\mathrm{h}_{\mu\nu}| \ll 1$. The *linearized gravity* scheme consists of approximating various equations by calculating to first (linear) order in the perturbation $\mathrm{h}_{\mu\nu}$. It is important to realize that in the linearized gravity scheme (at least as it is presented here) we expand the metric around the canonical Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

So, we assume that there exist coordinates $\{x^\mu\}$ on $\tilde{M}$ in which the metric has components $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ plus some small perturbation $\ll 1$. Now, are these coordinates unique? In other words, are there other coordinates on $\tilde{M}$ where the metric also takes the form $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ plus some small perturbation $\ll 1$? Clearly this is not the case for any set of coordinates. If we for example define spherical coordinates (t, r, θ, ϕ) on $\tilde{M}$ from the coordinates $\{x^\mu\}$, we see that in these coordinates the metric takes the form $\text{diag}(1, 1, r^2, r^2 \sin^2\theta)$ plus

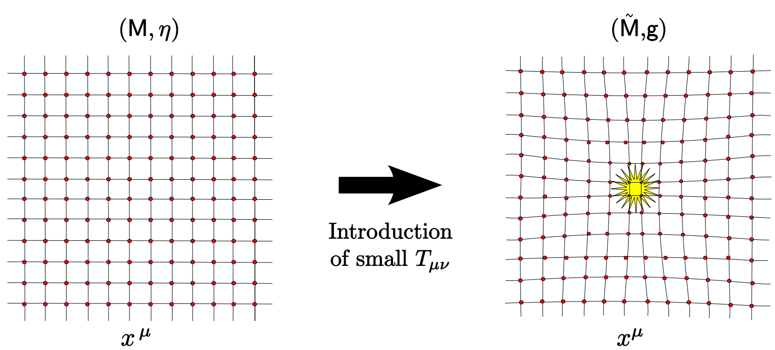

FIG 8. *We carry the coordinates on flat Minkowski spacetime to the slightly deformed spacetime* $(\tilde{\mathsf{M}}, g)$.

some small perturbation, i.e., an expansion around the flat metric in spherical coordinates. However, since, we are only interested in coordinates where the metric takes the form diag$(-1, 1, 1, 1)$ plus some small perturbation, we must restrict ourselves to coordinates $\{x'^\mu\}$ which differ from $\{x^\mu\}$ by a transformation which, to zeroth order in $\mathrm{h}_{\mu\nu}$, is the identity.
We therefore consider a coordinate transformation of the type

$$x'^\mu(x) = x^\mu - \underbrace{\chi^\mu(x)}_{\text{order } \mathrm{h}_{\mu\nu}} \tag{2.2.3}$$

where $\chi^\mu(x)$ is of order $\mathrm{h}_{\mu\nu}$, i.e, *very* small. To zeroth order in $\mathrm{h}_{\mu\nu}$, the transformation (2.2.3) corresponds to the identity, just as we wanted. Let us now see how the components of $\mathrm{g}_{\mu\nu}$ transform under this transformation. Using the tensor transformation law, we have to first order in $\mathrm{h}_{\mu\nu}$ (remember that we assume that $\chi^\mu \sim \mathcal{O}(\mathrm{h}_{\mu\nu})$)

$$\begin{aligned} \mathrm{g}'_{\mu\nu} &= \mathrm{g}_{\rho\sigma}\ \partial x^\rho/\partial x'^\mu \partial x^\sigma/\partial x'^\nu \\ &= \mathrm{g}_{\rho\sigma}(\delta^\rho_\mu + \partial_\mu\chi^\rho)(\delta^\sigma_\nu + \partial_\nu\chi^\sigma) \\ &= (\eta_{\rho\sigma} + \mathrm{h}_{\rho\sigma})(\delta^\rho_\mu + \partial_\mu\chi^\rho)(\delta^\sigma_\nu + \partial_\nu\chi^\sigma) \\ &= \eta_{\rho\sigma}\delta^\rho_\mu\delta^\sigma_\nu + \eta_{\rho\sigma}\partial_\mu\chi^\rho\delta^\sigma_\nu + \eta_{\rho\sigma}\delta^\rho_\mu\partial_\nu\chi^\sigma + \mathrm{h}_{\rho\sigma}\delta^\rho_\mu\delta^\sigma_\nu \\ &= \eta_{\mu\nu} + \mathrm{h}_{\mu\nu} + \partial_\mu\chi_\nu + \partial_\nu\chi_\mu \end{aligned} \tag{2.2.4}$$

where we used that to order $\mathrm{h}_{\mu\nu}$ we have that $\partial x^\rho/\partial x'^\mu = \delta^\rho_\mu + \partial\chi^\rho/\partial x^\mu \equiv \delta^\rho_\mu + \partial_\mu\chi^\rho$. We therefore have

$$\begin{aligned} \mathrm{g}'_{\mu\nu}(x') &= \eta_{\mu\nu}(x) + \mathrm{h}_{\mu\nu}(x) + \partial_\mu\chi_\nu(x) + \partial_\nu\chi_\mu(x) \\ &= \eta_{\mu\nu}(x') + \mathrm{h}_{\mu\nu}(x') + \partial_\mu\chi_\nu(x') + \partial_\nu\chi_\mu(x') \end{aligned} \tag{2.2.5}$$

where we used that the components $\eta_{\mu\nu}$ are all constant functions on $\tilde{M}$ and that to linear order in $\mathrm{h}_{\mu\nu}$ we have $\mathrm{h}_{\mu\nu}(x') = \mathrm{h}_{\mu\nu}(x)$ etc. We thus conclude that the transformation of $\mathrm{h}_{\mu\nu}$ given by

$$\boxed{\mathrm{h}_{\mu\nu} \to \mathrm{h}_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}} \tag{2.2.6}$$

describes the same physical situation. This freedom in choice of $\mathrm{h}_{\mu\nu}$ is clearly reminiscent of the gauge freedom $A_\mu \to A_\mu + \partial_\mu\chi$ in electrodynamics. As we will see now, the *gauge freedom in linearized gravity* will aid us in simplifying Einstein's equation in the linearized gravity scheme. In order for the gauge freedom in linearized gravity to be really useful, notice that tensors of order $\mathrm{h}_{\mu\nu}$ are *gauge invariant*. This is because the gauge transformation (2.2.16) corresponds to the coordinate transformation (2.2.3), so to linear order in $\mathrm{h}_{\mu\nu}$ the components of a tensor of order $\mathrm{h}_{\mu\nu}$ will not change. Specifically, since the energy-momentum tensor $T_{\mu\nu}$ is of order $\mathrm{h}_{\mu\nu}$ (by consistency of Einstein's equation), it is gauge invariant in the linearized gravity scheme.

We will now derive the *linearized Einstein field equations*. To this end notice that, to linear order in the perturbation $\mathrm{h}^{\mu\nu}$, we have

$$\mathrm{g}^{\mu\nu} = \eta^{\mu\nu} - \mathrm{h}^{\mu\nu} \tag{2.2.7}$$

where $\mathrm{h}^{\mu\nu}$ is $\mathrm{h}_{\mu\nu}$ with both indices raised with $\eta^{\mu\nu}$ (*not* $\mathrm{g}^{\mu\nu}$). Just as expected, the inverse metric $\mathrm{g}^{\mu\nu}$ is equal to $\eta^{\mu\nu}$ plus some small perturbation $-\mathrm{h}^{\mu\nu}$ of order $\mathrm{h}_{\mu\nu}$.
Remember that the Christoffel symbols $\Gamma^{\rho}{}_{\mu\nu}$ in general are determined by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\rho\sigma}\big\{\partial_\mu \mathrm{g}_{\nu\sigma} + \partial_\nu \mathrm{g}_{\mu\sigma} - \partial_\sigma \mathrm{g}_{\mu\nu}\big\} \tag{2.2.8}$$

We therefore see that to linear order in $\mathrm{h}_{\mu\nu}$, the Christoffel symbols are given by

$$^{(1)}\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\big\{\partial_\mu \mathrm{h}_{\nu\sigma} + \partial_\nu \mathrm{h}_{\mu\sigma} - \partial_\sigma \mathrm{h}_{\mu\nu}\big\} \tag{2.2.9}$$

We can now use this expression to compute the Ricci tensor $R_{\mu\nu}$ to linear order in $\mathrm{h}_{\mu\nu}$. In terms of the Christoffel symbols the Ricci tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\mu \Gamma^{\rho}{}_{\rho\nu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\rho}{}_{\alpha\rho} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\alpha\mu} \tag{2.2.10}$$

We therefore see that to linear order in $\mathrm{h}_{\nu\sigma}$ we may discard to two last terms and to linear order we therefore have

$$\begin{aligned} ^{(1)}R_{\mu\nu} &= \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\mu \Gamma^{\rho}{}_{\rho\nu} \\ &= \frac{1}{2}\eta^{\rho\sigma}\big\{\partial_\rho\partial_\mu \mathrm{h}_{\nu\sigma} + \partial_\rho\partial_\nu \mathrm{h}_{\mu\sigma} - \partial_\rho\partial_\sigma \mathrm{h}_{\mu\nu}\big\} \\ &\qquad - \frac{1}{2}\eta^{\rho\sigma}\big\{\partial_\mu\partial_\rho \mathrm{h}_{\nu\sigma} + \partial_\mu\partial_\nu \mathrm{h}_{\rho\sigma} - \partial_\mu\partial_\sigma \mathrm{h}_{\rho\nu}\big\} \end{aligned} \tag{2.2.11}$$

The first and fourth term cancel while the second and sixth term combine to $\partial^\sigma\partial_{(\nu}\mathrm{h}_{\mu)\sigma}$. All in all we thus get

$$^{(1)}R_{\mu\nu} = \partial^\rho\partial_{(\nu}\mathrm{h}_{\mu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho \mathrm{h}_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \tag{2.2.12}$$

where we have defined $h \equiv \mathrm{h}^{\mu}{}_{\mu}$. The Ricci scalar R to linear order in $\mathrm{h}_{\mu\nu}$ is now readily obtained

$$\begin{aligned} ^{(1)}R &= {}^{(1)}R^{\nu}{}_{\nu} \\ &= \eta^{\nu\mu}\,{}^{(1)}R_{\mu\nu} \\ &= \eta^{\nu\mu}\big(\partial^\rho\partial_{(\nu}\mathrm{h}_{\mu)\rho} - \frac{1}{2}(\partial^\rho\partial_\rho \mathrm{h}_{\mu\nu} + \partial_\mu\partial_\nu h)\big) \\ &= \partial^\rho\partial^\sigma \mathrm{h}_{\rho\sigma} - \partial^\sigma\partial_\sigma h \end{aligned} \tag{2.2.13}$$

Having computed both the Ricci tensor and scalar we finally obtain the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$ to linear order in $\mathrm{h}_{\mu\nu}$

$$\begin{aligned} {}^{(1)}G_{\mu\nu} &\equiv {}^{(1)}R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\ {}^{(1)}R \\ &= \partial^\rho\partial_{(\nu}\mathrm{h}_{\mu)\rho} - \frac{1}{2}\partial^c\partial_c\mathrm{h}_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma\mathrm{h}_{\rho\sigma} - \partial^\sigma\partial_\sigma h) \end{aligned} \tag{2.2.14}$$

This expression can be simplified by introducing a "shifted" $\mathrm{h}_{\mu\nu}$, defined by

$$\overline{\mathrm{h}}_{\mu\nu} \equiv \mathrm{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.2.15}$$

Clearly we have $\overline{\mathrm{h}} \equiv \overline{\mathrm{h}}^\nu{}_\nu = -h$, for this reason $\overline{\mathrm{h}}_{\mu\nu}$ is referred to as the *trace reversed perturbation*. After a bit of algebra Einstein's equation $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ in linearized scheme can then be written as

$$\boxed{-\frac{1}{2}\partial^\rho\partial_\rho\overline{\mathrm{h}}_{\mu\nu} + \partial^\rho\partial_{(\nu}\overline{\mathrm{h}}_{\mu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma\overline{\mathrm{h}}_{\rho\sigma} = 8\pi GT_{\mu\nu}} \tag{2.2.16}$$

This is the general *linearized Einstein field equation*.

As we discussed above, the linearized gravity scheme contains the gauge freedom (2.2.16). We see that under the gauge transformation (2.2.16) $\overline{\mathrm{h}}_{\mu\nu}$ transforms as

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} - \eta_{\mu\nu}\partial^\rho\chi_\rho \tag{2.2.17}$$

Notice that

$$\partial^\nu\left[\overline{h}_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} - \eta_{\mu\nu}\partial^\rho\chi_\rho\right] = \partial^\nu\overline{\mathrm{h}}_{\mu\nu} + \partial^\nu\partial_\nu\chi_\mu \tag{2.2.18}$$

Now suppose that we perform the gauge transformation (2.2.16), where the gauge parameter χ^μ is chosen such that it satisfies

$$\partial^\nu\partial_\nu\chi_\mu = -\partial^\nu\overline{\mathrm{h}}_{\mu\nu} \tag{2.2.19}$$

In this specific gauge the linearized Einstein equation is particularly simple. We see that in the gauge (2.2.19) we have

$$\partial^\nu\overline{\mathrm{h}}_{\mu\nu} = 0 \tag{2.2.20}$$

by the equations (2.2.17), (2.2.18) and the condition (2.2.19). This is analogous to the Lorentz gauge from electrodynamics: $\partial^\mu A_\mu = 0$. It should, however, be noted that, just as in electrodynamics, this "Lorentz" gauge does not fix $\overline{\mathrm{h}}_{\mu\nu}$ completely. We are still free to perform gauge transformations (2.2.16) as long as (2.2.20) is satisfied. In the gauge (2.2.20), we see that the general linearized Einstein equation simplifies to (introducing the usual notation $\Box \equiv \partial^\mu\partial_\mu$)

$$\boxed{\Box\,\overline{\mathrm{h}}_{\mu\nu} = -16\pi T_{\mu\nu}} \tag{2.2.21}$$

As we have argued, the right-hand of this equation is gauge invariant. This equation is therefore very similar to Maxwell's equations expressed in Lorentz gauge $\Box A^\mu = -4\pi J^\mu$. Notice the simplicity of the linearized Einstein equation in the "Lorentz" gauge (2.2.21): Using the machinery of Greens functions (for $\Box$), we see that finding the solutions for the trace reversed perturbation $\overline{\mathrm{h}}_{\mu\nu}$ is reduced to computing integrals.

The linearized Einstein field equation has many applications. It is for example used to study gravitational radiation: According to Einstein's theory, dense non-static systems such as binary star systems, neutron stars, and black holes will disturb spacetime. These curvature fluctuations will propagate through spacetime as ripples, governed by the two equations

$$\partial^\nu \overline{\mathrm{h}}_{\mu\nu} = 0 \tag{2.2.22}$$

$$\Box\, \overline{\mathrm{h}}_{\mu\nu} = 0 \tag{2.2.23}$$

Here the linear approximation is assumed to hold, but the metric perturbation is assumed to be large relative to the energy-momentum tensor of "ordinary" matter in the universe e.g. the Sun or the Earth, i.e., we set $T_{\mu\nu} = 0$. The solution to the two above equations are simple wave functions and, in principle, it should be possible to the detect such gravitational waves here on earth[16]. However, as of 2008, gravitational waves are yet to be observed (at least directly).
Unfortunately, it is beyond the scope of this project to go into the fascinating topic of gravitational radiation. There is a lot of good material on this to be found in [Wal84], [tH02] and especially in [Car04].

2.3. The weak field approximation

We will now use the linearized field equations to interpret the parameters M, Q and J from the Kerr solution in terms of the mass $\mathcal{M}$, charge $\mathcal{Q}$ and angular momentum $\mathcal{J}$ of the given gravitational source. First a few words about the general solution for $\overline{\mathrm{h}}_{\mu\nu}$ (and thereby $\mathrm{h}_{\mu\nu}$).

In the Lorentz gauge we need to solve

$$\Box\, \overline{\mathrm{h}}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{2.3.1}$$

This is exactly (for *each* component) the equation governing a massless scalar field in the presence of a source[17]. As is well-known from relativistic field theory, the

[16] The theory/sensitivty of gravitational wave detectors is completely amazing! For more on the experimental side of gravitational waves see for example [OS06]

[17] In fact, this tensor equation (i.e., not viewed componentwise) is the equation that governs a massless spin-2 field, [Wal84].

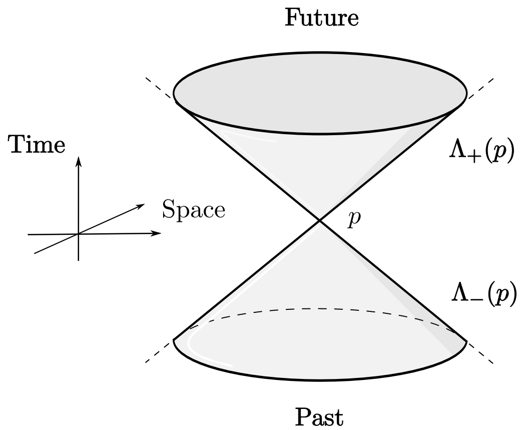

FIG 9. *The future and past light cones for the spacetime point p. The (weak field approximation) gravitational field in p only depends on $\Lambda_-(p)$ and only affects what happens in $\Lambda_+(p)$.*

solution for such a field can be obtained by using the theory of Greens functions. The Greens function for the d'Alembertian operator $\square$ is the scalar function $G \equiv G(x^\mu - x'^\mu)$ satisfying

$$\square_x \; G(x^\mu - x'^\mu) = \delta(x^\mu - x'^\mu) \tag{2.3.2}$$

where the subscript x denotes differentiation wrt. the non-primed coordinates x^μ. These Greens functions exist and can be classified as being either *advanced* or *retarded*. Going through this analysis (which is well-known from relativistic field theory) shows that the metric field $\overline{\mathrm{h}}_{\mu\nu}$ in the spacetime point x^μ only depends on the points on the past light cone $\Lambda_-(x^\mu)$ of x^μ, i.e., the points satisfying

$$\Lambda_-(x^\mu) = \{y^\mu \in \tilde{\mathsf{M}} \bigm| x^0 - y^0 = |\vec{x} - \vec{y}| \;,\; y^0 < x^0\} \tag{2.3.3}$$

and can only affect events on the future light cone (fig. 15)

$$\Lambda_+(x^\mu) = \{y^\mu \in \tilde{\mathsf{M}} \bigm| x^0 - y^0 = |\vec{x} - \vec{y}| \;,\; y^0 > x^0\} \tag{2.3.4}$$

This explicitly shows that the well-known prediction that gravity in the weak field limit propagates with the speed of light.
Using the Greens function G, the general solution for $\overline{\mathrm{h}}_{\mu\nu}$ is now given by

$$\overline{\mathrm{h}}_{\mu\nu} = -16\pi G \int d^4x' \;\; G(x^\mu - x'^\mu) T_{\mu\nu}(x'^\mu) \tag{2.3.5}$$

As a consistency check, notice that this solution fulfills the Lorentz gauge condition

$$\partial^\mu \overline{\mathrm{h}}_{\mu\nu} = 16\pi G \int d^4x' \Big(\partial'^\mu \big(G(x^\mu - x'^\mu) T_{\mu\nu} \big) - G(x^\mu - x'^\mu) \partial'^\mu T_{\mu\nu})(x'^\mu) \Big) = 0 \tag{2.3.6}$$

where we used that $\partial'^{\mu} G(x^\mu - x'^\mu) = -\partial^\mu G(x^\mu - x'^\mu)$, $G(x^\mu - x'^\mu) = 0$ at infinity and energy-momentum conservation $\partial^\mu T_{\mu\nu} = 0$.
In the case of a stationary spacetime, the problem of finding $\overline{\mathrm{h}}_{\mu\nu}$ simplifies somewhat. Since here, the metric components do not depend on time, the equation (2.3.1) simplifies to

$$\triangle\, \overline{\mathrm{h}}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{2.3.7}$$

where $\triangle = \partial^i \partial_i$ denotes the usual Laplace differential operator. Now the problem only depends on the spatial coordinates x^i and the Greens function for $\triangle$ is given by

$$G(\vec{x}, \vec{x}') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \tag{2.3.8}$$

The solution for $\overline{\mathrm{h}}_{\mu\nu}$ is then obtained as above with the only difference that we now only integrate over space. We will now analyze the time independent solutions for $\mathrm{h}_{\mu\nu}$ for different energy-momentum tensors to identify M, Q and J.

2.3.1. The effect of a mass distribution on spacetime

We will now examine how the presence of a *static* mass distribution ρ affects flat spacetime. We therefore take $T_{00} = T^{00} = \rho$ while the rest of the components are set equal to zero.

In terms of the Greens function G, the solution to $\overline{\mathrm{h}}_{00}$ is then given by

$$\begin{aligned} \overline{\mathrm{h}}_{00}(\vec{x}) &= -16\pi G \int d^3x' \ \rho(x') G(r, r') = 4G \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \\ &= -4\Phi(\vec{x}) \end{aligned} \tag{2.3.9}$$

where we have identified $-1/4\ \overline{\mathrm{h}}_{00}$ with the classical Newtonian gravitational potential Φ, i.e., the scalar function satisfying Poisson's equation

$$\triangle\, \Phi = 4\pi G \rho \tag{2.3.10}$$

We see that, in principle, $\overline{\mathrm{h}}_{00}$ and Φ are only determined up to some constant (of order $\overline{\mathrm{h}}_{00}$). We therefore conclude that the gauge freedom in Newtonian gravitational physics comes from the gauge freedom of Einstein's equations for linearized gravity.

The rest of the components $\overline{\mathrm{h}}_{\mu\nu}$, $\mu\nu \neq 00$, fulfills

$$\triangle\, \overline{\mathrm{h}}_{\mu\nu} = 0 \tag{2.3.11}$$

We can therefore choose all $\overline{\mathrm{h}}_{\mu\nu} = 0$, $\mu\nu \neq 00$. This means that

$$h = -\overline{h} = -\eta^{\mu\nu} \overline{\mathrm{h}}_{\mu\nu} = \overline{\mathrm{h}}_{00} \tag{2.3.12}$$

So, the only non-vanishing components of $\mathrm{h}_{\mu\nu}$ are the diagonal entries and we have

$$\mathrm{h}_{00} = \overline{\mathrm{h}}_{00} + \frac{1}{2}\eta_{00}h = \frac{1}{2}\overline{\mathrm{h}}_{00} = -2\Phi \tag{2.3.13}$$

and

$$\mathrm{h}_{ii} = \frac{1}{2}\eta_{ii}h = \frac{1}{2}\overline{\mathrm{h}}_{00} = -2\Phi \tag{2.3.14}$$

The weak field metric is therefore given by the following expression

$$\mathrm{d}s^2 = -\Big(1 + 2\Phi\Big)\mathrm{d}t^2 + \Big(1 - 2\Phi\Big)\big[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big] \tag{2.3.15}$$

It follows quite easy from the geodesic equation of motion, that in the weak field approximation the motion of a test particle is governed by the equation $d^2\vec{x}/dt^2 = -\vec{\nabla}\Phi$, [Wal84]. Since Φ is the Newtonian gravitational potential this is, of course, nothing but the classical equation of motion of a test particle moving in a Newtonian gravitational field. We have therefore recovered the well known result: Einstein's theory of gravitation reproduces Newton's theory in the weak field limit.
Notice that it is also possible to derive the component g_{00} of the perturbed metric by directly comparing the geodesic equation to the classical equation $d^2\vec{x}/dt^2 = -\nabla\Phi$. This is how the weak field limit is presented in for example [Ole07]. This (simpler) method, however, has its drawbacks; it does not provide us with the spatial components of the perturbed metric. These components are essential for understanding the parameter J.

Since we want to compare the weak field metric (2.3.15) to the general Schwarzschild metric, let us now take the mass distribution to be spherically symmetric $\rho(\vec{x}) \equiv \rho(r)$. Here the gravitational potential is given by the well-known formula

$$\Phi = -\frac{\mathcal{M}G}{r} \tag{2.3.16}$$

where $\mathcal{M} = \int d^3x\, \rho(x)$ is the total mass of the gravitational source. Now switching to spherical coordinates

$$\begin{aligned} x^0 &= t \\ x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta \end{aligned} \tag{2.3.17}$$

and redefining the radial coordinate $r^2 \to r^2(1+2MG/r)$, we get to leading order in the perturbation (i.e., in MG), that the metric can be written as

$$\mathrm{d}s^2 = -\Big(1 - \frac{2\mathcal{M}G}{r}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2\mathcal{M}G}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{2.3.18}$$

This is readily compared to the Schwarzschild solution (1.2.13) and we conclude that we can make the identification

$$M \rightsquigarrow \mathcal{M} \tag{2.3.19}$$

In other words, *the parameter M in the Schwarzschild solution can be identified with the physical mass $\mathcal{M}$ of the spherically symmetric gravitational source.*

2.3.2. The effect of a charged mass distribution on spacetime

Let us now see how the presence of a charge distribution (in addition to a mass distribution) affects spacetime. Recall that the 00 component of the energy-momentum tensor (the energy density) of an electromagnetic field is given by

$$T_{00}^{em} = \frac{1}{8\pi}\big(\vec{E}^2 + \vec{B}^2\big) \tag{2.3.20}$$

For a static problem, the magnetic field vanishes due to the non-existence of magnetic charge (monopoles) in Maxwell's theory. There is, however, nothing that stops us from including the possibility of the existence of magnetic charge in our theory. We therefore define the total charges of the matter distribution

$$e = \int d^3x \; \sigma_{electric} \quad , \quad m = \int d^3x \; \sigma_{magnetic} \tag{2.3.21}$$

where respectively $\sigma_{electric}$ and $\sigma_{magnetic}$ denotes the density of electric and magnetic charge. Notice that this of course only makes sense in the weak field approximation: We approximate the energy-momentum tensor with the one we know from special relativity. As we noted above, this approximation is valid to linear order in the perturbation.

Now, in a spherically symmetric problem, we see that the only non-vanishing component of $T_{\mu\nu}^{em}$ is T_{00}^{em} (since $\vec{E} \times \vec{B} = 0$) and it is given by

$$T_{00}^{em}(\vec{x}) = \frac{1}{8\pi}\frac{\mathcal{Q}^2}{r^4} \; , \quad r = |\vec{x}| \tag{2.3.22}$$

where we have defined $\mathcal{Q}^2 = e^2 + m^2$. This means that, in the presence of a spherically symmetric charge distribution, the linearized Einstein equation for the 00 component of the trace inverted perturbation $\overline{\mathrm{h}}_{00}$ is modified to

$$\triangle \, \overline{\mathrm{h}}_{00}(\vec{x}) = -16\pi T_{00} = -16\pi\rho(r) - \frac{2\mathcal{Q}^2}{r^4} \tag{2.3.23}$$

Using that $\triangle(1/r^2) = 2/r^4$, we see that the solution for $\overline{\mathrm{h}}_{00}$ now is modified to

$$\overline{\mathrm{h}}_{00}(\vec{x}) = \frac{2\mathcal{M}G}{r} - \frac{G\mathcal{Q}^2}{r^2} \tag{2.3.24}$$

All the computations needed to derive the weak-field metric are exactly the same as the last section only with the difference that $2\mathcal{M}G/r$ should be substituted with $2\mathcal{M}G/r - G\mathcal{Q}^2/r^2$. To linear order in the perturbation, we therefore obtain the following expression for the perturbed metric

$$\mathrm{d}s^2 = \Big(1 - \frac{2\mathcal{M}G}{r} + \frac{G\mathcal{Q}^2}{r^2}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2\mathcal{M}G}{r} + \frac{G\mathcal{Q}^2}{r^2}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \quad (2.3.25)$$

Comparing to the Reissner-Nordström solution, we then see that we can make the identification

$$Q^2 \leftrightsquigarrow \mathcal{Q}^2 \quad (2.3.26)$$

The conclusion therefore is that *the parameter Q in the Reissner-Nordström solution can be identified with the physical (electrical + magnetic) charge $\mathcal{Q}$ of the static, spherically symmetric gravitational source.*

Alternatively, as we saw, this result could also be reached by explicitly looking at the electromagnetic field tensor $F_{\mu\nu}$ (i.e., the two functions A and B) and comparing it to the electromagnetic field tensor for a static, spherically symmetric charge distribution in flat space, i.e., examining $F_{\mu\nu}$ for $r \to \infty$.

2.3.3. The effect of a rotating mass distribution on spacetime

We have seen that it is possible to give the parameters M and Q the physical interpretation of respectively mass and charge. We mentioned that we do not really need the linearized gravity formalism to derive these results, simply because both mass and charge give rise to classical potentials. This is, however, not the case for the parameter J. The rotation of a body does not give rise to an additional classical potential! So, we have introduced the linearized gravity scheme for two reasons. First of all it is an important theoretical tool in general relativity. Moreover, it is vital for understanding the parameter J in the Kerr solution. To linear order, the effects of a rotation is seen in the $\mathrm{d}x^i\mathrm{d}t$ cross-terms (as opposed to the $\mathrm{d}t^2$ and $(\mathrm{d}x^i)^2$ terms). In order to analyze this, we need the linearized gravity scheme - we start out with some rather general considerations.

Consider a matter distribution described by an energy-momentum tensor $T_{\mu\nu}$. Since we want to describe a rotating system, i.e., a non-static system, the components T^{0i} representing flow of momentum are non-zero. However, since the system is assumed non-relativistic, time derivatives will be much smaller than spatial derivatives (remember that $c = 1$). Using energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$, we therefore see that we can order the components of the energy-momentum tensor as

$$|T_{00}| \gg |T_{0i}| \gg |T_{ij}| \quad (2.3.27)$$

This shows that the effect on the spacetime perturbation from rotation, is expected to be much smaller that the one coming from the pure presence of a mass but much bigger than the contribution from the sheer components. We are only interested in looking at the effect of rotation, so we consider the system in its rest frame, i.e., the system in which

$$\int d^3x \; T^{0i} = 0 \qquad (i = 1, 2, 3) \tag{2.3.28}$$

There can be some confusion whether to use up- or down-stairs indices for the energy-momentum tensor: The energy-momentum tensor representing physical quantities (such as energy density, momentum density etc.) is the one with both indices up, i.e., $T^{\mu\nu}$. This means that, the quantity $\int d^3x \; T^{0i}$ represents the integral over the momentum density flow in the i'th direction, i.e., the total momentum in the i'th direction, which is exactly equal to zero in the rest frame.

For this general, non-relativistic system, we now define an angular four-momentum tensor by

$$J^{\mu\nu} = \int d^3x \; (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \tag{2.3.29}$$

from which we obtain the usual angular three-momentum in the given system [18] by

$$\mathcal{J}^i = \frac{1}{2}\epsilon_{ijk} J^{jk} = \int d^3x \; \epsilon_{ijk} x^j T^{0k} \qquad (i = 1, 2, 3) \tag{2.3.30}$$

Using energy-momentum conservation, along with a partial integration, we see that[19]

$$\int d^3x \; x^i T^{j0} = -\int d^3x \; x^j T^{i0} \tag{2.3.32}$$

This concludes our general considerations.

Let us now look at a very specific system, namely an axisymmetric stationary system that is rotating around the z-axis. We therefore take the only non-vanishing components of $T^{\mu\nu}$ to be T^{00}, T^{01} and T^{02} and assume them to be time independent. Such a system has angular momentum

$$\mathcal{J}^3 = 2\int d^3x \; xT^{20} \tag{2.3.33}$$

while

$$\mathcal{J}^1 = 2\int d^3x \; yT^{30} = 0 \quad \text{and} \quad \mathcal{J}^2 = 2\int d^3x \; xT^{30} = 0 \tag{2.3.34}$$

[18] This is clearly not a Lorentz-invariant definition.

[19] Just use that

$$\int d^3x \; x^i T^{j0} = \int d^3x \; \partial_j(x^i x^j) T^{j0} \tag{2.3.31}$$

do the partial integration and require energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$.

So this system does indeed describe a pure rotation around the z-axis with total angular momentum $\mathcal{J} = \mathcal{J}^3 = 2\int d^3x\; xT^{20} = -2\int d^3x\; yT^{10}$. Let us now determine the metric perturbation $\mathrm{h}_{\mu\nu}$. Since the only difference from the calculation we did above concerns the off-diagonal elements $\overline{\mathrm{h}}_{0i}$, we see that the diagonal elements of $\mathrm{h}_{\mu\nu}$ will be the same as in the non-rotating case (i.e., determined by the total mass $\mathcal{M}$). For the off-diagonal elements we have

$$\mathrm{h}^{01} = \overline{\mathrm{h}}^{\;01} = 4G\int d^3x'\; \frac{T^{01}(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad \text{and} \quad \mathrm{h}^{02} = \overline{\mathrm{h}}^{\;02} = 4G\int d^3x'\; \frac{T^{02}(\vec{x}')}{|\vec{x}-\vec{x}'|} \tag{2.3.35}$$

we may now perform a multipole expansion of the function $1/|\vec{x}-\vec{x}'|$. We have

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{xx'+yy'+zz'}{r^3} + \dots \tag{2.3.36}$$

For $r \equiv |\vec{x}| \gg |\vec{x}'|$ we may ignore the higher order terms represented by the dots and we thus get

$$\mathrm{h}_{01} = \frac{2Gy}{r^3}\,\mathcal{J} \quad \text{and} \quad \mathrm{h}_{02} = -\frac{2Gx}{r^3}\,\mathcal{J} \tag{2.3.37}$$

Here we used the rest frame condition (2.3.28), the relation (2.3.30) and finally lowered the indices (which just gives a factor of -1 on the components). Now in order to compare to the Kerr solution, we must again transform to spherical coordinates (2.3.17)[20]. Now the metric perturbation transforms according to the tensor transformation law

$$\mathrm{h}_{\nu_1'\nu_2'} = \frac{\partial x^{\mu_1}}{\partial x^{\nu_1'}}\frac{\partial x^{\mu_2}}{\partial x^{\nu_2'}}\mathrm{h}_{\mu_1\mu_2} \tag{2.3.38}$$

where the primed coordinates now denote spherical coordinates, i.e., $x'^{\mu} \equiv (t, r, \theta, \phi)$ and the non-primed coordinates are our original Minkowski coordinates, i.e., $x^\mu \equiv (x^0, \vec{x})$. The result of this transformation gives the same expression for the diagonal terms as before + some new cross-terms. As we see, the only (possible) non-vanishing cross-terms are h_{tr}, $\mathrm{h}_{t\theta}$ and $\mathrm{h}_{t\phi}$. They are given by

$$\mathrm{h}_{t\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}\mathrm{h}_{0\nu} \tag{2.3.39}$$

If we believe the Kerr solution to be true, the two components h_{tr} and $\mathrm{h}_{t\theta}$ *must* vanish. This is indeed the case, since r and θ enters the same way in x and y, we have

$$\begin{aligned}
\mathrm{h}_{tr} &= \frac{1}{r}\Big(x\mathrm{h}_{01} + y\mathrm{h}_{02}\Big) = 0 \\
\mathrm{h}_{t\theta} &= \cot\theta\Big(x\mathrm{h}_{01} + y\mathrm{h}_{02}\Big) = 0 \\
\mathrm{h}_{t\phi} &= -y\mathrm{h}_{01} + x\mathrm{h}_{02} = -\frac{2G(x^2+y^2)}{r^3}\,\mathcal{J} = -\frac{2G\mathcal{J}\sin^2\theta}{r}
\end{aligned}$$

[20] Note that to linear order, the Boyer-Lindquist coordinates reduce to ordinary spherical coordinates.

Having found the expression for the metric correction $\mathrm{h}_{\mu\nu}$ we are thus left with the following expression for the full metric $\mathrm{g}_{\mu\nu} = \eta_{\mu\nu} + \mathrm{h}_{\mu\nu}$

$$\begin{aligned} \mathrm{d}s^2 = -\Big(1 - \frac{2\mathcal{M}G}{r}\Big)\mathrm{d}t^2 - \frac{2G\mathcal{J}\sin^2\theta}{r}\big[\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t\big] \\ + \Big(1 - \frac{2\mathcal{M}G}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \end{aligned} \quad (2.3.40)$$

We therefore conclude that to linear order, the effects of rotation show up in the $\mathrm{d}\phi\mathrm{d}t$ cross-term. This was pretty much expected: We expected the breaking of time reversal symmetry to show up in $\mathrm{d}t\mathrm{d}x^i$ cross-terms, but cross-terms of the type $\mathrm{d}t\mathrm{d}\theta$, $\mathrm{d}t\mathrm{d}r$ would break axisymmetry.

Having found the weak field approximation for a rotating massive body, we may now compare it to the exact Kerr solution in order to identify the parameter $J = aM$. By comparing the coefficients of the $\mathrm{d}t\mathrm{d}\phi$ cross-term and the diagonal terms, we conclude that if we identify

$$\begin{aligned} M &\leftrightsquigarrow \mathcal{M} \\ Ma = J &\leftrightsquigarrow \mathcal{J} \end{aligned}$$

we will have $a = \mathcal{J}/\mathcal{M} \ll 1$, since the system is assumed non-relativistic. This means that we can discard all the terms with a^2. In the non-relativistic limit we therefore have $\rho^2(r) = r^2$ and $\Delta(r) = r^2 - 2\mathcal{M}Gr$. Plugging this into the exact Kerr solution reproduces the result (2.3.40). We therefore conclude, *the parameter J can be identified with the physical angular momentum $\mathcal{J}$ of the Kerr spacetime.*

2.4. Conserved currents in general relativity

In this section we will discuss, on a general spacetime manifold, how a conserved current gives rise to a corresponding conserved charge integral. We will see that a Killing field has an associated conserved current and this will allow us to give a new definition of spacetime mass and angular momentum.

We start by showing how a conserved current J^μ, i.e., a vector field satisfying

$$\nabla_\mu J^\mu = 0 \ , \quad (2.4.1)$$

gives rise to a conserved charge. Let us recall how this is done in special relativity. In SR, spacetime is flat, i.e., spacetime has the structure $\mathsf{M} = \mathbf{R} \times \mathbf{R}^3$ and the condition $\nabla_\mu J^\mu = 0$ reduces to

$$\partial_\mu J^\mu = -\partial_t J^0 + \partial_i J^i = 0 \quad (2.4.2)$$

Now define a charge by

$$Q = \int d^3x J^0 \quad (2.4.3)$$

This charge is conserved since

$$\partial_t Q = \int d^3x \partial_t J^0 = \int d^3x \partial_i J^i = \int d^3x \partial_i J^i = \int dA_i J^i = 0 \tag{2.4.4}$$

where we used Stokes' theorem:

$$\int_M \nabla \cdot \mathbf{F} \, dV = \oint_{\partial M} \mathbf{F} \cdot d\mathbf{A} \tag{2.4.5}$$

along with the usual assumption that the current J^μ vanishes at spatial infinity. It should be underlined that the charge integral $Q = \int d^3x J^0$ is not only conserved but also Lorentz invariant. This is because the measure d^3x and the zeroth component of J^μ transform in "opposite" ways, so the product $J^0 d^3x$ is Lorentz invariant.

It is possible to mimic this derivation on a general spacetime manifold. Of course, a general spacetime does not separate time and space as $\mathbf{R} \times \mathbf{R}^3$, so already here one must be a bit careful. The key ingredient in the derivation is the fact that Stokes' theorem does in fact hold on a general manifold. As is well-known to the reader, on a general n-dimensional manifold, the theory of integration is *not* build up around functions but around n-forms. The most general form of Stokes' theorem reads that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \tag{2.4.6}$$

Here M is some domain of M (with some suitable niceness properties), ∂M is the boundary of M, ω is an $n-1$ form and $\mathrm{d}\omega$ is the exterior derivative of ω and therefore an n-form. Notice that, since M is an n-dimensional manifold and ∂M is an $(n-1)$-dimensional manifold, this makes sense. As mentioned integration on a general manifold is only defined for n-forms. On a (oriented) Riemannian (or pseudo-Riemannian) manifold it is, however, possible to define integration of functions, using the metric tensor g. The Riemannian volume element dV is an n-form with the property that $dV(E_1, \ldots, E_n) = 1$, whenever $\{E_1, \ldots, E_n\}$ is an (oriented) orthonormal basis for the tangent space $T_p\mathsf{M}$. It is possible to show that, the Riemannian volume element is unique and that if $\{x^\mu\}$ is a right-handed coordinate chart, locally dV can be written as

$$dV = \sqrt{|g|} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{2.4.7}$$

where $g \equiv \det(g_{\mu\nu})$. The integral of a continuous function f over some domain $M \subset \mathsf{M}$ is then defined as

$$\int_M f \equiv \int_M f dV \tag{2.4.8}$$

where the right-hand side is now a (well-understood) integral of an n-form. Especially this implies that

$$\int_M f \equiv \int_M f dV = \int \sqrt{|g|} \prod_{i=1}^n dx^i \; f(x^1, \ldots, x^n) \tag{2.4.9}$$

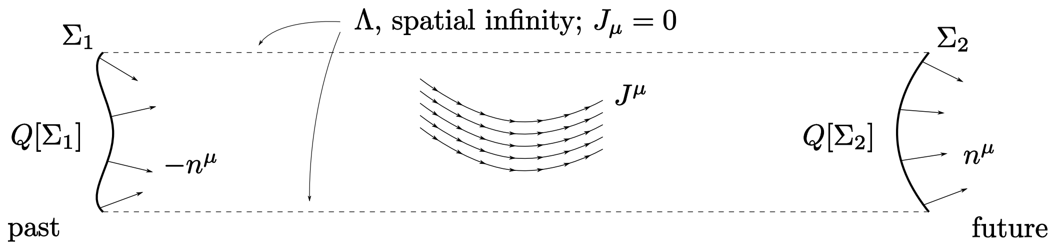

FIG 10. *Definition of the future directed unit normal n^μ; notice on Σ_1 it is the vector $-n^\mu$ that is the unit normal pointing out of M.*

when M is contained in *one* coordinate chart $\{x^\mu\}$. This is nothing but an Euclidian integral and this is how the integral of a function is defined in e.g. [Ole07]. If M is contained in more than one chart, we can use a similar expression for each of the charts and then sum all the contributions from each chart (a point in an overlap is counted with weight 1, technically this is a partition of unity). Integration theory is diffeomorphism invariant, this means that the various integrals are invariant under coordinate transformations. The point we wish to convey to the reader is that the "naive" (intuitive) integration (2.4.9) is well-defined, coordinate transformation invariant and when M is contained in more than one chart, the integral (2.4.9) is indeed modified in the intuitive (obvious) way. Usually we will write the integral $\int_M f$ as (2.4.9) even if M is not necessarily contained in one chart.

Back to Stokes' theorem. Using the various definitions, it is possible to show that Stokes' theorem implies that

$$\int_M d^n x\sqrt{|g|}\nabla_\mu V^\mu = \int_{\partial M} d^{n-1}y\sqrt{|\gamma|}n_\mu V^\mu \tag{2.4.10}$$

where y denotes coordinates on ∂M, γ is the metric on ∂M induced from the metric g and n^μ is a unit normal to the boundary ∂M consistent with the orientation of M: Since ∂M is an $(n-1)$-dimensional hypersurface the unit normal is uniquely determined up to a sign. Since the orientation on ∂M is the one it inherits from M, the unit normal n^μ must be chosen inward-pointing (outward-pointing) (i.e., pointing into (out of) M) if ∂M is timelike (spacelike).

We will now use Stokes' theorem in the form (2.4.10) to see that a conserved current gives rise to a conserved charge. Suppose that we are given some spacelike hypersurface Σ and a current J^μ. We can define a charge associated with this spacelike hypersurface by

$$Q[\Sigma] = -\int_\Sigma d^{n-1}y\sqrt{|\gamma|}n_\mu J^\mu \tag{2.4.11}$$

where n^μ is a *future directed* timelike unit normal ($n_\mu n^\mu = -1$) to Σ. This integral physically represents the flow of current through the surface Σ. The minus sign in the definition (2.4.11) is needed to assure that the charge (2.4.11) coincides with the usual definition of charge in SR (see below). Now suppose that the current J^μ is conserved, i.e., $\nabla_\mu J^\mu = 0$. We will now show that Q_Σ is independent of the spacelike surface Σ and can thus be interpreted as being independent of "time", i.e., conserved. To this end, consider two spacelike surfaces Σ_1 and Σ_2 and assume that Σ_2 is in the future region of Σ_1. Moreover, consider a surface Λ that connects the boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$ as in fig. 10. The surface $\{\Sigma_1 + \Sigma_2 + \Lambda\}$ is a closed surface with Λ located at spatial infinity where we assume that the current J^μ vanishes. Using Stokes' theorem we therefore have

$$\begin{aligned}\int_{\mathrm{int}(\Sigma_1+\Sigma_2+\Lambda)} d^n x \sqrt{|g|}\nabla_\mu J^\mu &= \int_{\Sigma_1+\Sigma_2+\Lambda} d^{n-1}y\sqrt{|\gamma|}n_\mu J^\mu \\ &= \int_{\Sigma_2} d^{n-1}y\sqrt{|\gamma|}n_\mu J^\mu - \int_{\Sigma_1} d^{n-1}y\sqrt{|\gamma|}(-n_\mu)J^\mu \\ &= Q[\Sigma_2] - Q[\Sigma_1]\end{aligned} \tag{2.4.12}$$

Here we used that, since Σ_1 and Σ_2 are spacelike, the unit normal vector n^μ used in Stokes' theorem must point *out of* the surface $\{\Sigma_1 + \Sigma_2 + \Lambda\}$. Therefore, on the surface Σ_2, the unit normal n^μ is future directed timelike while on Σ_1 it is the unit normal $-n^\mu$ that is future directed timelike (see fig. 10). Since $\nabla_\mu J^\mu = 0$, we therefore conclude

$$Q[\Sigma_2] = Q[\Sigma_1] \tag{2.4.13}$$

So the charge $Q[\Sigma]$ is indeed independent of the spacelike hypersurface Σ. Before we invoke this result to give an expression for the mass and angular momentum of a spacetime, we will take a few seconds to introduce and prove the so-called Killing vector lemma.

Theorem (Killing vector lemma) *Suppose that $\mathfrak{X}^\mu$ is a Killing vector field, then*

$$\nabla_\mu \nabla_\nu \mathfrak{X}_\rho = R_{\rho\nu\mu}{}^\lambda \mathfrak{X}_\lambda = -R_{\nu\rho\mu}{}^\lambda \mathfrak{X}_\lambda \tag{2.4.14}$$

where $R_{\mu\nu\rho\lambda}$ is the Riemann curvature tensor.

Proof. The (components of the) Riemann tensor fulfills that[21]

$$\nabla_\mu \nabla_\nu \mathfrak{X}_\rho - \nabla_\nu \nabla_\mu \mathfrak{X}_\rho = R_{\mu\nu\rho}{}^\lambda \mathfrak{X}_\lambda \tag{2.4.15}$$

[21] This is how the Riemann tensor is defined in e.g. [Wal84], i.e., as a measure of how second covariant derivatives fail to commute on one-forms. Usually the Riemann tensor is defined as a measure of how second covariant derivatives fail to commute on vectors. The two definitions are, however, equivalent if one assumes the Levi-Civata connection (as we always do in classical GR), since here the covariant derivative of the metric vanishes. Therefore, we can raise (lower) indices *inside* covariant derivatives by multiplication of $g^{\mu\nu}$ ($g_{\mu\nu}$) *outside* the covariant derivative.

Now by Killings equation $\nabla_{(\mu}\mathfrak{X}_{\nu)} = 0$ we can rewrite this equation to

$$\nabla_\mu \nabla_\nu \mathfrak{X}_\rho + \nabla_\nu \nabla_\rho \mathfrak{X}_\mu = R_{\mu\nu\rho}{}^\lambda \mathfrak{X}_\lambda \tag{2.4.16}$$

The Riemann tensor fulfills some very specific symmetries which, of course, the left-hand side of the above equation must also satisfy. We will use this observation to show the desired result, more specifically we will use that the Riemann tensor fulfills

$$R_{\mu\nu\rho}{}^\lambda + R_{\nu\rho\mu}{}^\lambda + R_{\rho\mu\nu}{}^\lambda = 0 \tag{2.4.17}$$

By equation (2.4.16) we then have

$$\begin{aligned} 2\nabla_\nu \nabla_\rho \mathfrak{X}_\mu &= \big(\nabla_\mu \nabla_\nu \mathfrak{X}_\rho + \nabla_\nu \nabla_\rho \mathfrak{X}_\mu\big) + \big(\nabla_\nu \nabla_\rho \mathfrak{X}_\mu + \nabla_\rho \nabla_\mu \mathfrak{X}_\nu\big) - \big(\nabla_\rho \nabla_\mu \mathfrak{X}_\nu + \nabla_\mu \nabla_\nu \mathfrak{X}_\rho\big) \\ &= \big(R_{\mu\nu\rho}{}^\lambda + R_{\nu\rho\mu}{}^\lambda - R_{\rho\mu\nu}{}^\lambda\big)\mathfrak{X}_\lambda \\ &= -2R_{\rho\mu\nu}{}^\lambda \mathfrak{X}_\lambda \end{aligned} \tag{2.4.18}$$

where we used the symmetry (2.4.17). □

The electric charge integral

As a nice example of the use of the above theory, let us give an expression for the total electric charge of a spacetime. Recall, the electric current associated to the electromagnetic field tensor $F_{\mu\nu}$ is given by

$$j_e^\mu = \frac{1}{4\pi}\nabla_\nu F^{\mu\nu} \tag{2.4.19}$$

This current is conserved,

$$\nabla_\mu j_e^\mu = 0 \tag{2.4.20}$$

To show this, it is sufficient to show that the commutator $[\nabla_\mu, \nabla_\nu]$ applied to $F^{\mu\nu}$ vanishes, since $F^{\mu\nu}$ is antisymmetric. Of course, in general $[\nabla_\mu, \nabla_\nu] \neq 0$, however for an arbitrary covariant two-tensor $X^{\alpha\beta}$ we have

$$[\nabla_\mu, \nabla_\nu]X^{\alpha\beta} = R^\alpha{}_{\lambda\mu\nu}X^{\lambda\beta} + R^\beta{}_{\lambda\mu\nu}X^{\alpha\lambda} \tag{2.4.21}$$

Using the antisymmetry of $F^{\mu\nu}$ along with the usual symmetries of the Riemann curvature tensor, we have

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]F^{\nu\mu} &= R^\nu{}_{\lambda\mu\nu}F^{\lambda\mu} + R^\mu{}_{\lambda\mu\nu}F^{\nu\lambda} \\ &= R^\nu{}_{\lambda\mu\nu}F^{\lambda\mu} - R^\mu{}_{\lambda\mu\nu}F^{\lambda\nu} \\ &= R^\nu{}_{\lambda\mu\nu}F^{\lambda\mu} - R^\mu{}_{\lambda\nu\mu}F^{\lambda\nu} \\ &= 0 \end{aligned} \tag{2.4.22}$$

Since $\nabla_\mu j^\mu = 0$, we may now associate an electric charge to spacelike surface Σ by

$$\begin{aligned} Q_e &= -\int_\Sigma d^3x\sqrt{|\gamma|}n_\mu j_e^\mu \\ &= -\frac{1}{4\pi}\int_\Sigma d^3x\sqrt{|\gamma|}n_\mu \nabla_\nu F^{\mu\nu} \end{aligned} \tag{2.4.23}$$

which is conserved. Notice, this definition of electric charge coincides with the one from special relativity. In SR we usually take the spacelike hypersurface Σ to

$$\Sigma_t = \{x^\mu \mid t\ constant\} \cong \mathbf{R}^3 \tag{2.4.24}$$

with $\gamma = \mathrm{diag}(1,1,1)$ and $n^\mu = (1,0,0,0)$, i.e., $n_\mu = (-1,0,0,0)$. Furthermore in SR we identify $j^\mu = (\rho, \vec{J})$, where ρ is the electric charge density. This means that

$$Q_e^{SR} = -\int_{\mathbf{R^3}} d^3x\sqrt{|\gamma|}n_\mu j^\mu = \int_{\mathbf{R^3}} d^3x\ \rho(\vec{x}) = \text{total electric charge in SR} \tag{2.4.25}$$

This explains the origin of the minus in the definition (2.4.11).

Since the 3-dimensional volume integral (2.4.23), representing the charge Q_e, is an integral over a divergence of an antisymmetric tensor, it is possible (again using Stokes' theorem (2.4.10)) to rewrite it to a surface integral as

$$\begin{aligned} Q_e &= -\frac{1}{4\pi}\int_\Sigma d^3x\sqrt{|\gamma|}n_\mu \nabla_\nu F^{\mu\nu} \\ &= -\frac{1}{4\pi}\int_{\partial\Sigma} d^2x\sqrt{|\alpha|}n_\mu \sigma_\nu F^{\mu\nu} \end{aligned} \tag{2.4.26}$$

where the two-dimensional surface $\partial\Sigma$ is the boundary of Σ at spatial infinity (typically a 2-sphere of "infinite" radius), α_{ij} is the induced metric on $\partial\Sigma$ and σ^μ is the (outward-pointing) unit normal to $\partial\Sigma$. Using this formula, it is straight forward to show that the total electric charge of the Reissner-Nordström spacetime is indeed given by the parameter $e = Q$ ($m = 0$): If we choose the timelike surface as in (2.4.24) and $\partial\Sigma$ to be a 2-sphere at spatial infinity, we see that in coordinates, the two unit normals are given by

$$n^\mu = (\Pi^{-1/2}, 0, 0, 0) \quad \text{and} \quad \sigma^\mu = (0, \Pi^{1/2}, 0, 0) \tag{2.4.27}$$

where we used that the Reissner-Nordström metric is diagonal. This also means that

$$n_\mu = (-\Pi^{1/2}, 0, 0, 0) \quad \text{and} \quad \sigma_\mu = (0, \Pi^{-1/2}, 0, 0) \tag{2.4.28}$$

moreover on $\partial\Sigma$

$$\sqrt{|\alpha|} = r^2 \sin\theta \tag{2.4.29}$$

Therefore

$$Q_e = -\frac{1}{4\pi}\int_{\partial\Sigma} d^2x\sqrt{|\alpha|}n_\mu\sigma_\nu F^{\mu\nu} = \frac{1}{4\pi}\int_{S^2_r} d\theta d\phi r^2 \sin\theta F^{tr}\Bigg|_{r=\infty} \tag{2.4.30}$$

As mentioned above, we have that

$$F^{tr} = \frac{e}{r^2} \tag{2.4.31}$$

We therefore have

$$Q_e = e \tag{2.4.32}$$

So the electric charge integral constructed from the Maxwell equation (2.4.19) does indeed coincide with the parameter e. Using the "dual" of the Maxwell equation (2.4.19), a similar construction can be done for the magnetic charge.

2.4.1. Komar integrals

Just as with the electric charge integral, it would be nice if we could define respectively the mass and angular momentum of a spacetime in terms of currents. To this end we now introduce the Komar integral associated with a Killing vector $\mathfrak{X}^\mu$. For a Killing vector field $\mathfrak{X}^\mu$ consider a current defined by

$$J^\mu[\mathfrak{X}] = \mathfrak{X}_\nu R^{\mu\nu} \tag{2.4.33}$$

Furthermore, assume that spacetime is not "too exotic", i.e., assume that our spacetime does not contain any singularities etc.[22] - (for now) think of the spacetime we are considering in the following, as the spacetime around a star or a planet. By virtue of Einstein's equation (1.1.1), the current $J^\mu[\mathfrak{X}]$ can be written as

$$J^\mu[\mathfrak{X}] = 8\pi G\mathfrak{X}_\nu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big) \tag{2.4.34}$$

We will now show that $J^\mu[\mathfrak{X}]$ is conserved. To realize this, recall that $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\rho \mathrm{g}^{\mu\nu} = 0$, therefore

$$\begin{aligned}\nabla_\mu J^\mu[\mathfrak{X}] &= 8\pi G\Big\{\nabla_\mu\mathfrak{X}_\nu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big) + \mathfrak{X}_\nu\nabla_\mu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big)\Big\} \\ &= 4\pi G\mathfrak{X}^\mu\nabla_\mu T \\ &= -\frac{1}{2}\mathfrak{X}^\mu\nabla_\mu R \\ &= 0\end{aligned} \tag{2.4.35}$$

[22]For example, in the Schwarzschild spacetime the current (2.4.33) would be everywhere equal to 0 and thus not very interesting.

Here we used that $\mathfrak{X}^\mu$ is Killing and that the covariant derivative of the Ricci scalar R in the direction of $\mathfrak{X}^\mu$ vanishes. This is seen by using the Killing vector lemma along with Killings equation. It is, of course, a reflection of the fact that geometry does not change along Killing fields. Since $\nabla_\mu J^\mu[\mathfrak{X}] = 0$, the charge

$$\begin{aligned} Q[\mathfrak{X}] &= -\int_\Sigma d^{n-1}y\sqrt{|\gamma|}n_\mu J^\mu[\mathfrak{X}] \\ &= -\int_\Sigma d^{n-1}y\sqrt{|\gamma|}n_\mu \mathfrak{X}^\nu R^{\mu\nu} \\ &= -8\pi G\int_\Sigma d^{n-1}y\sqrt{|\gamma|}n_\mu \mathfrak{X}_\nu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big) \end{aligned} \tag{2.4.36}$$

is conserved. Just as we did with the electric charge integral, it is possible to rewrite the expression for $Q[\mathfrak{X}]$ to a surface integral. This follows from the contracted Killing vector lemma:

$$\nabla_\nu\nabla^\mu\mathfrak{X}^\nu = R^{\mu\nu}\mathfrak{X}_\nu \tag{2.4.37}$$

This means that the current $J^\mu[\mathfrak{X}]$ can be written as

$$J^\mu[\mathfrak{X}] = \nabla_\nu\big[\nabla^\mu\mathfrak{X}^\nu\big] \tag{2.4.38}$$

Now since the two tensor $\nabla^\mu\mathfrak{X}^\nu$ is antisymmetric ($\mathfrak{X}^\mu$ is Killing), the 3-dimensional integral (2.4.36) can be written as 2-dimensional integral

$$Q[\mathfrak{X}] = -\int_{\partial\Sigma} d^2x\sqrt{|\alpha|}n_\mu\sigma_\nu\nabla^\mu\mathfrak{X}^\nu \tag{2.4.39}$$

This is the *Komar integral* associated with the Killing vector field $\mathfrak{X}^\mu$. So, *each Killing vector field has an associated conserved charge, which can be written as a 2-dimensional surface integral at spatial infinity.*

Now recall Noether's theorem from classical physics. It states that

Continuous symmetry of the action of a physical system →
Conserved charge (the Noether charge).

This leads to the well-known result that, if a physical system in invariant under time translations, the conserved Noether charge is the total energy, while if it is invariant under rotations around an axis, the conserved Noether charge is the angular momentum wrt. the given axis. Let us assume that such relations also hold in general relativity. We thus conjecture that Noether's theorem generalizes to general relativity and that the Noether charge (energy, angular momentum *etc.*) associated with the symmetry, represented by a Killing field $\mathfrak{X}^\mu$, is given by the Komar integral for $\mathfrak{X}^\mu$ (up to some normalization).

Now, a stationary spacetime is characterized by having a timelike killing field $\mathfrak{K}^\mu$. Of course this Killing field is only determined up to a multiplicative constant. This means that if we want to associate a well-defined charge with the timelike killing field $\mathfrak{K}^\mu$, we must specify a normalization of $\mathfrak{K}^\mu$. We normalize the time translational Killing field $\mathfrak{K}^\mu$ in the obvious way: In an asymptotically flat stationary spacetime (with associated time translational Killing field $\mathfrak{K}^\mu$) we choose the normalization of $\mathfrak{K}^\mu$ by requiring that $\mathfrak{K}^\mu$ is future directed and that

$$-\mathfrak{K}_\mu \mathfrak{K}^\mu \to 1 \quad \text{at spatial infinity } (r \to \infty) \tag{2.4.40}$$

With these normalization conditions, we see that the Killing field $\mathfrak{K}^\mu$ exactly corresponds to the Killing field that generates the forward time translations at spatial infinity. Therefore, in the spacetimes we have considered $\mathfrak{K}^\mu$ is normalized to $\mathfrak{K} = \partial_t$ (this was also the normalization we used to define the "point mass" energy (1.2.25)). As we have argued, a stationary spacetime can be interpreted as being invariant under time translations and the Killing field representing this symmetry is exactly $\mathfrak{K}^\mu$. Following the above discussion, this means that, the total mass (energy) of a stationary spacetime is given by the following Komar integral

$$\boxed{\mathcal{M}_{\text{Komar}} = \frac{1}{4\pi G} \int_{\partial\Sigma} \mathrm{d}A \; n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu} \tag{2.4.41}$$

where $\mathrm{d}A$ denotes the area element $\sqrt{|\alpha|}d^2x$ on $\partial\Sigma$. The normalization factor of $-4\pi G$ is justified below.

An axisymmetric spacetime is characterized by possessing a spacelike Killing field $\mathfrak{R}^\mu$ whose orbits consist of closed curves. Just as with the Killing field $\mathfrak{K}^\mu$, we must specify a normalization of $\mathfrak{R}^\mu$. We do this by requiring that $\mathfrak{R}^\mu$ generates rotations at spatial infinity, i.e, in an axisymmetrical asymptotically flat spacetime $\mathfrak{R}^\mu$ is normalized so that the flow ψ_ϕ corresponding to $\mathfrak{R}^\mu$ fulfills that $\psi_\phi = \psi_{\phi+2\pi}$. Therefore if the spacetime is Kerr, then $\mathfrak{R} = \partial_\phi$ (this was also the normalization we used to define the "point mass" angular momentum (1.2.26)). As we have seen, the Killing field $\mathfrak{R}^\mu$ has the interpretation of being the generator of rotations around the axis of rotation (where $\mathfrak{R}^\mu = 0$). This means that, the total angular momentum around the axis of rotation is given by the following Komar integral

$$\boxed{\mathcal{J}_{\text{Komar}} = -\frac{1}{8\pi G} \int_{\partial\Sigma} \mathrm{d}A \; n_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu} \tag{2.4.42}$$

Notice that the normalization factor of $8\pi G$ differs from the one we used in for the Komar mass by a factor of -2. The formulas for the Komar mass and angular momentum were derived under the assumption that spacetime is not "too exotic". However, the two expressions (2.4.41) and (2.4.42) only depend on *asymptotic* data. This means that we can change the gravitational source in any way we want without changing M and J, provided that the *asymptotic* metric is unchanged (which is certainly a reasonable assumption). We therefore take the two surface integrals (2.4.41) and (2.4.42) to be valid *in general*, especially this means that we can apply them to black holes (which we will do in part 3).

We will now justify the normalization factors in the Komar expressions. We do this by requiring that in the classical limit, the Komar mass and angular momentum coincides with the usual classical expressions. This is most easily done using the Komar charge expressed as a 3-dimensional volume integral, i.e., using the expression (2.4.36). In the Newtonian limit we choose Σ as (2.4.24) and here we have

$$n^\mu = (1,0,0,0) \quad , \quad \mathfrak{K}^\mu = (1,0,0,0) \quad , \quad \mathfrak{R}^\mu = (0,0,0,1) \tag{2.4.43}$$

in ordinary spherical coordinates (t, r, θ, ϕ). We therefore get (let $\mathrm{d}V$ denote the volume element on Σ)

$$\mathcal{M}_{\text{Komar}} = -\frac{8\pi G}{N_{\mathcal{M}}}\int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{K}_\nu \Big(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T\Big) \approx -\frac{4\pi G}{N_{\mathcal{M}}}\int_\Sigma \mathrm{d}V\ T^{tt} \tag{2.4.44}$$

and

$$\mathcal{J}_{\text{Komar}} = -\frac{8\pi G}{N_{\mathcal{J}}}\int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{R}_\nu \Big(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T\Big) \approx \frac{8\pi G}{N_{\mathcal{M}}}\int_\Sigma \mathrm{d}V\ T^{t\phi} \tag{2.4.45}$$

where we used the ordering relation of the energy-momentum tensor (2.3.27). Now, recall the two Newtonian expressions for respectively mass and angular momentum in terms of the energy-momentum tensor;

$$\mathcal{M}_{\text{Newton}} = \int \mathrm{d}V\ T^{tt} \quad \text{and} \quad \mathcal{J}_{\text{Newton}} = \int \mathrm{d}V\ T^{t\phi} \tag{2.4.46}$$

where the latter relation is realized by using the usual relation (2.3.30) and transforming to spherical coordinates. We therefore see that we must indeed choose $N_{\mathcal{M}} = -4\pi G$ and $N_{\mathcal{J}} = 8\pi G$.

Having obtained the expression for the Komar mass, let us now try to calculate the Komar mass of the Schwarzschild spacetime (to get a feel of the math). Just as we did with the Reissner-Nordström charge integral, we take the two-dimensional surface at spatial infinity to be a 2-sphere of infinite radius. The unit normals n^μ and σ^μ are therefore the same as before (with $Q = 0$). We must now work out the factor $n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu$. This is, however, not too hard, in fact, all we need is the Christoffel symbol Γ^r_{tt}. It is given by

$$\Gamma^r_{tt} = \frac{MG}{r^2}\Big\{1 - \frac{2MG}{r}\Big\} \tag{2.4.47}$$

We now have

$$n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu = -\nabla^t \mathfrak{K}^r = -\mathrm{g}^{tt}\nabla_t \mathfrak{K}^r \tag{2.4.48}$$

We now use the usual formula for the coordinate expression of the covariant derivative of a vector field;

$$\mathrm{g}^{tt}\nabla_t \mathfrak{K}^r = \mathrm{g}^{tt}\left(\partial_t \mathfrak{K}^r + \Gamma^r_{t\mu}\mathfrak{K}^\mu\right) = \mathrm{g}^{tt}\Gamma^r_{tt} \tag{2.4.49}$$

So

$$n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu = \frac{MG}{r^2} \tag{2.4.50}$$

Therefore

$$\mathcal{M}_{\text{Komar}} = \frac{1}{4\pi G}\int_{\partial\Sigma} \mathrm{d}A\ n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu = \frac{1}{4\pi G}\int_{S^2_r} d\theta d\phi r^2 \sin\theta \frac{MG}{r^2}\bigg|_{r=\infty} = M \tag{2.4.51}$$

So, the Komar mass of the Schwarzschild spacetime is exactly equal to the parameter M (it would have been rather unsatisfying if this were not the case!). A similar calculation (much longer) shows that the Komar mass and angular momentum of the charged Kerr spacetime are given by respectively M and J.

2.5. Energy conditions

Consider Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}R\mathrm{g}_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.5.1}$$

In Einstein's original theory, the only real condition on the energy-momentum tensor is that it is conserved, i.e.,

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.5.2}$$

Apart from this, we are in principle free to use any energy-momentum tensor we want. However, it turns out that it is possible to find (conserved) energy-momentum tensors that produce pathological results [Car04], when plugged into Einstein's equation. For example, consider the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\mathrm{g}_{\mu\nu} \tag{2.5.3}$$

This energy-momentum tensor is conserved for *any* pressure p, energy density ρ and unit time-like vector field U^μ. It is, nevertheless, not clear if any p and ρ produce physically sound results (for example, can ρ be negative?) and if these values are acquirable for "realistically" matter. To resolve these problems, one introduces the so called *energy conditions*. The energy conditions describe properties common to all (or almost all) states of matter and nongravitational fields. The energy conditions can therefore be thought of as a set of conditions we impose on the energy-momentum tensor in order for it to describe realistic matter.

Moreover, imposing the energy conditions will eliminate most of the unphysical solutions to Einstein's equation [Car04]. The energy conditions are useful when one works with a non-specified energy-momentum tensor and are vital for proving certain singularity theorems and the laws of lack hole dynamics (see part 3). There are several (currently around 6) more or less mathematically independent energy conditions in use, each suitable for its own type of matter field. In this section we will focus on the so called *dominant energy condition* (which in general is thought to be the most fundamental energy condition).

Definition (the dominant energy condition) *An energy-momentum tensor field $T_{\mu\nu}$ is said to satisfy the dominant energy condition if for all future-directed timelike vector fields t, the vector field*

$$\mathcal{T}(t) = -t^{\mu} T_{\mu}{}^{\nu}\, \partial_{\nu} \tag{2.5.4}$$

is future-directed timelike or null (possible zero).

This condition might seem somewhat arbitrary at first sight, but it has a very nice physical interpretation. The dominant energy condition (DEC) can be interpreted as the condition that energy and momentum should not be able to flow faster than the speed of light nor should it be able to appear out of nothing [Wal84]. The DEC (which is a mathematic statement) is in other words equivalent to the physical statement that all reasonable matter should respect causality. It is possible to make the DEC a bit more transparent in terms of the energy-momentum tensor. This follows from the fact that the energy-momentum tensor $T_{\mu\nu}$ is symmetric in its indices and is therefore diagonalizable[23]. This means that there exists an orthogonal basis $\{v^{\mu}, x^{\mu}, y^{\mu}, z^{\mu}\}$ consisting of eigenvectors of T^{μ}_{ν}, so that

$$T_{\mu\nu} = \rho v_{\mu} v_{\nu} + p_1 x_{\mu} x_{\nu} + p_2 y_{\mu} y_{\nu} + p_3 z_{\mu} z_{\nu} \tag{2.5.5}$$

where v^{μ} is future directed timelike and ρ, p_1, p_2, p_3 are the real eigenvalues. The quantity ρ has the interpretation of rest energy density while the three "spatial" eigenvalues p_1, p_2 and p_3 are the so-called principal pressures. For example, consider the ideal fluid energy-momentum tensor (2.5.3). It is straight forward to find an orthonormal basis that diagonalizes $T_{\mu\nu}$ - simply go to inertial orthonormal coordinates where the fluid is at rest, i.e., $U^{\mu} = (1, \vec{0})$. In these coordinates, the energy-momentum tensor takes the form (remember that we work in inertial

[23] Actually, it is T^{μ}_{ν} that needs to be diagonalizable since it represents a linear map from vectors to vectors. There could be a problem if one of the eigenvectors where null, however energy-momentum tensors of physical realistically tends always to be diagonalizable [Wal84].

coordinates where the components of the metric tensor are $\eta_{\mu\nu}$)

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.5.6}$$

We therefore see that the ρ-eigenvalue corresponds to the rest energy of the ideal fluid while the three principal pressures are all equal to p = isotropy. Let us now return to a non-specified energy-momentum tensor $T_{\mu\nu}$. Using that $\{v^\mu, x^\mu, y^\mu, z^\mu\}$ is an orthonormal basis, we see that any future directed timelike vector t^μ can be written as

$$t^\mu = c_0 v^\mu + c_1 x^\mu + c_2 y^\mu + c_3 z^\mu \tag{2.5.7}$$

where $c_0 > 0$ and $c_0^2 > \sum_{i=1}^3 c_i^2$. Now using the orthonormal decomposition of $T_{\mu\nu}$, we see that for any future directed timelike vector t^μ, we have

$$-t^\mu T_\mu{}^\nu = c_0 \rho v^\mu - c_1 p_1 x^\mu - c_2 p_2 y^\mu - c_3 p_3 z^\mu \tag{2.5.8}$$

Now this vector is future directed timelike or null (possible zero) if and only if

$$t^\mu t^\lambda T_\mu{}^\nu T_{\lambda\nu} \leq 0 \tag{2.5.9}$$

and

$$v_\nu[-t^\mu T_\mu{}^\nu] \leq 0 \tag{2.5.10}$$

This is clearly equivalent to the condition

$$\rho \geq 0 \quad \text{and} \quad \rho \geq |p_i| \qquad (i = 1, 2, 3) \tag{2.5.11}$$

The DEC therefore says that the energy density for realistic matter cannot be negative, however, negative pressure is allowed as long as its magnitude does not exceed the energy density. The latter condition has a clear physical interpretation; if the pressure is too large compared to the energy density, the flow of energy and momentum will be too fast, i.e., non-causal. In a similar manner, it is possible to show that the energy-momentum tensor of electromagnetism and scalar fields respects the DEC.

For completeness we now mention a few of the other energy conditions in use: The weak energy condition (WEC) says that

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \tag{2.5.12}$$

for all timelike t^μ. The weak energy condition has the physical interpretation that the energy density of matter (described by $T_{\mu\nu}$) cannot be negative. Notice that the DEC implies the WEC. Another energy condition is the strong energy condition (SEC). The SEC states that

$$\Big(T_{\mu\nu} + \frac{1}{2}T\mathrm{g}_{\mu\nu}\Big)t^{\mu}t^{\nu} \geq 0 \tag{2.5.13}$$

for all timelike t^{μ}. We will not try to justify this energy condition, but only note this is a rather strong condition (an inflationary universe will not respect the SEC [Tow97]), also note that the SEC does *not* imply the WEC.

3. Black holes

3.1. Introduction

The discussion of the Schwarzschild black hole on p. 15 can be generalized to the Kerr spacetime. The Kerr spacetime describes the spacetime *outside* an axisymmetric, stationary body. Again, we can imagine an object which has all of its matter located within surface $r = r_+$. For such an object the Kerr spacetime is valid all the way into the coordinate singularity

$$r = r_+ \ , \tag{3.1.1}$$

the so-called event horizon. Such an object, i.e., the spacetime behind the surface is the *Kerr black hole*. In this part we will look into the physics of black holes.

As we saw in part 2, the parameters M, Q and J of the Kerr solution had the interpretation of respectively mass, charge and angular momentum of the Kerr spacetime. Moreover, we saw that the Komar mass and angular momentum for the Kerr spacetime exactly were equal to M and J. Therefore, from now on, we will simply denote the black hole mass, charge and angular momentum by respectively M, Q and J.

3.2. Event horizons

We will now go into a bit more detailed discussion regarding event horizons. Just as with the concept of spatial infinity, asymptotically flat spacetimes etc., the concept of event horizons also has a very sophisticated coordinate-independent definition [Wal84]. In this project we will be satisfied with defining event horizons in a coordinate dependent manner. We will do this by first examining what the process of falling into a Schwarzschild black hole looks like for respectively a freely falling observer and a distant observer. This will point out the physical significance of the event horizons and will aid us in giving a more mathematical definition of event horizons.

Falling into a Schwarzschild black hole

Suppose that a long time ago in a galaxy far, far away a spaceship[24] discovers a black hole. The spaceship chooses to send a probe carrying a clock into the black hole in order to examine why the black hole is so black. We will now describe how the descend towards the black hole looks from respectively the spaceship and the probe. We assume that the spaceship is at rest wrt. the black hole and that it is located very far away from the black hole where spacetime

[24] Whose crew consists of a race that has mastered interstellar travel but does not know anything about general relativity

can be considered Minkowskian. Furthermore, we assume that the probe is falling freely, i.e., moving on a geodesic. For simplicity, let us assume that the geodesic is completely radial and is in the equatorial plane (this is possible since the subspace $\theta = \pi/2, \phi = const.$ is totally geodesic, which follows from (1.2.26) - if a geodesic has $\dot{\phi} = 0$ at some time, it will always have $\dot{\phi} = 0$). According to our discussion of the Schwarzschild geodesics the (timelike) geodesic radial motion of the probe is governed by the equations (1.2.20), (1.2.25), i.e., the two equations

$$1 = (1 - 2MG/r)\dot{t}^2 - (1 - 2MG/r)^{-1}\dot{r}^2 \tag{3.2.1}$$

and

$$E = \Big(1 - \frac{2MG}{r}\Big)\dot{t} \tag{3.2.2}$$

where the quantity E is conserved along the geodesic. Notice that there are three times in play here

- ⋆ The proper time along the geodesic describing the motion of the probe, $\tau = \int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. The proper time τ is the time measured by a freely falling clock, i.e., the clock carried by the probe.
- ⋆ The proper time $\tilde{\tau}$ along the motion of the spaceship (which is at rest in (local) Minkowski spacetime). This is the time measured by the clock carried by the spaceship.
- ⋆ The coordinate time t. A priori, this time has no physical meaning, it must be identified in terms of the physical proper time. Since the spacetime is at rest in (almost) flat space we can make the identification $\tilde{\tau} = t$.

If we assume that the probe starts out at rest when it leaves the spaceship we have that $E = 1$ (in general E will be determined by the initial velocity $|\vec{v}|$ of the probe by $E = \gamma(\vec{v}) = 1/\sqrt{1 - \vec{v}^2}$). We therefore have the following two simple equations

$$\dot{t} = \Big(1 - \frac{2MG}{r}\Big)^{-1} \quad \text{and} \quad \dot{r}^2 = \frac{2MG}{r} \tag{3.2.3}$$

Solving these equations for the radial coordinate r as respectively a function of proper time τ and the coordinate time t is a straight forward. A straight forward integration of the second equation yields (notice, this equation has the same structure of that governing the motion of a freely falling object in Newtonian mechanics)

$$\tau(r) = 2\frac{2MG}{3}\left[\Big(\frac{r_0}{2MG}\Big)^{3/2} - \Big(\frac{r}{2MG}\Big)^{3/2}\right] \tag{3.2.4}$$

where r_0 is defined so that $\tau(r_0) = 0 = t(r_0)$. We therefore see that the probe reaches the Schwarzschild radius $R_s = 2MG$ in a finite amount of proper time (and it will continue toward the singularity). Therefore, reaching the Schwarzschild radius is nothing special seen from the prope[25]. This, however, is drastically different from how the spaceship views the journey. Near $r = 2MG$ we have

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\frac{1}{2MG}(r - 2MG), \quad \text{for } 2MG \lessapprox r \tag{3.2.5}$$

with the solution

$$r(t) = 2MG + \Big\{(r_0 - 2MG)\exp(-t/2MG)\Big\} \tag{3.2.6}$$

We therefore see that from the point of view of the distant spaceship, the Schwarzschild radius is never reached! Seen from the spaceship, the probe will simply keep slowing down in such a manner that the Schwarzschild radius is never passed. Moreover, we see that any radiation emitted from the probe will become increasingly redshifted. This means that at some point the probe will become invisible since all its radiation will become extremely redshifted. Actually, no radiation emitted from behind the Schwarzschild radius can escape to the spaceship. This follows from the equation (1.2.20). For light we have $\alpha = 0$, so we see

$$\left|\frac{dr}{dt}\right| = 1 - \frac{2MG}{r} \tag{3.2.7}$$

Since radiation emitted from behind the Schwarzschild radius would have to pass the Schwarzschild radius in order to reach the spaceship, we see that at the Schwarzschild radius it would simply stop, i.e., never reach the spaceship and the surrounding universe. We thus see that the distant observers in the spaceship will never find out what happens behind the Schwarzschild radius. The only way to find out what happens behind the Schwarzschild radius is to go behind it and once the surface $r = 2MG$ is passed there is no way one can escape from it again. For this reason the surface $r = R_s = 2MG$ is called the *event horizon* of the Schwarzschild black hole.

Coordinate singularities and event horizons

Here we wish to treat event horizons a bit more in general. To this end we will take a few seconds discussing the notion of null hypersurfaces.

Definition (null hypersurface) *Let Σ be hypersurface in a Lorentzian manifold* (M, g) *with an associated normal vector field σ. If the normal vector field σ is everywhere spacelike, timelike or null, the hypersurface is said to be respectively timelike, spacelike or null*[26].

[25] Of course, close to the Schwarzschild radius the gravitational tidal forces are very strong and would be properly tear the probe apart.

[26] This definition coincides with the usual definition of spacelike (timelike) hypersurfaces, since if σ is timelike (spacelike) the tangents vectors to Σ will be spacelike (timelike).

Now, any hypersurface can be written in the form (at least locally[27])

$$f(x) = const. \tag{3.2.8}$$

for some function f. If a hypersurface Σ is specified as in (3.2.8), we can write down a normal field of Σ in terms of f, it is given by

$$\sigma = \mathrm{g}^{\mu\nu}\nabla_\nu f \partial_\mu \tag{3.2.9}$$

so σ has the components

$$\sigma^\mu = \mathrm{g}^{\mu\nu}\nabla_\nu f \tag{3.2.10}$$

Proving this claim is easy; since the function f is constant on Σ, we know that if we differentiate f in the direction of Σ we will get zero, i.e., if v^μ is a tangent vector to Σ, then $v^\mu \nabla_\nu f = 0$. We therefore have for any tangent v^μ to Σ

$$\sigma_\mu v^\mu = v^\nu \mathrm{g}_{\mu\nu}\mathrm{g}^{\mu\lambda}\nabla_\lambda f = v^\nu \nabla_\nu f = 0 \tag{3.2.11}$$

Notice that if Σ is a null hypersurface with normal vector σ^μ, then since σ^μ is null

$$\sigma_\mu \sigma^\mu = 0 \tag{3.2.12}$$

we therefore conclude that σ^μ also is tangent to Σ. We therefore see that the null hypersurfaces exactly are the hypersurfaces where the normal field is also tangent.

We will now explain the significance of null hypersurfaces in the context of event horizons. As mentioned before, an event horizon is defined as a "region of no escape", i.e., a region of spacetime where it is impossible for a causal curve to escape to spatial infinity. This definition implies that event horizons are always guarantied to be null hypersurfaces but the opposite is certainly not true (for example, there are a lot of null hypersurfaces in flat Minkowski space). However, the spacetimes we have considered in part 1 all have the property that their *topological* structure *outside* the coordinate singularities can be written as

$$\underbrace{\mathbf{R}}_{t-\text{coor.}} \times \underbrace{\mathbf{R}_+}_{r-\text{coor.}} \times \underbrace{S^2}_{\theta,\phi-\text{coor.}} \tag{3.2.13}$$

At spatial infinity ($r \to \infty$) the 2-spheres are clearly spacelike but as r decrease they become "less and less" spacelike[28]. Now suppose that when the radial coordinate r reaches some value R_E the corresponding hypercylinder $\mathbf{R} \times S^2$ becomes null. This hypercylinder must be an event horizon; this is realized by looking at fig. 11. As the radial coordinate approaches R_E, the light cones in (t, r)-coordinates close up and on the actual null cylinder $r = R_E$, the light cones are totally closed

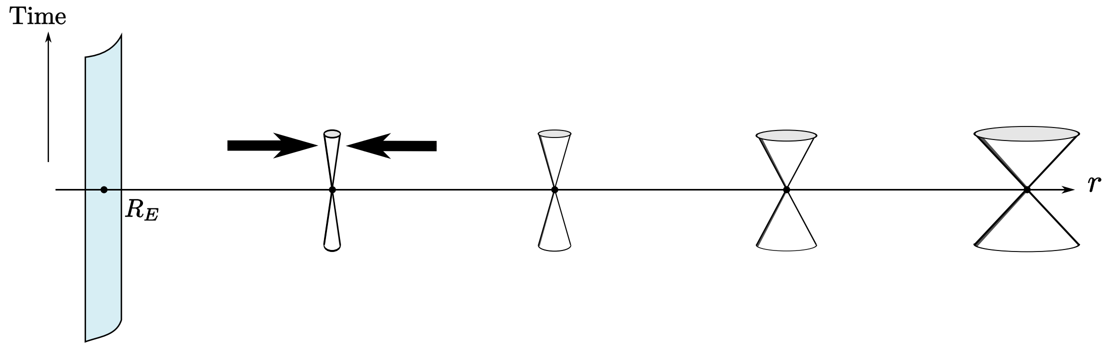

FIG 11. *The light cones close up near the event horizon.*

up and pointing tangent to $r = R_E$ (simply because the null-direction is also tangent to the hypercylinder $r = R_E$ since it is null). Since all (tangents to) causal curves are confined to the light cones, it is now clear that in (t, r)-coordinates nothing will be able to enter or exit the null hypersurface $r = R_E$, i.e., it is an event horizon. Of course there exist curves that reach the event horizon in a finite amount of proper time. Therefore, it is possible to enter the event horizon by falling into it, however, as seen from spatial infinity, this process takes an infinite amount of time.

This means that in order to find the event horizons of the spacetimes possessing the structure (3.2.13), we must find out at which radii the cylinder $r = const.$ becomes null. According to the above discussion, the normal field to $r = const.$ is given by $\mathrm{g}^{\mu\nu}\nabla_\nu r = \mathrm{g}^{\mu r}$. Therefore, to find the radii for which $r = const.$ becomes null we must find out when the vector field $\mathrm{g}^{\mu r}$ becomes null, i.e., we must solve the equation

$$0 = \mathrm{g}_{\mu\nu}\mathrm{g}^{\mu r}\mathrm{g}^{\nu r} = \delta^r_\nu \mathrm{g}^{\nu r} = \mathrm{g}^{rr} \tag{3.2.14}$$

Since we are considering axisymmetric spacetimes there are no cross-terms of the type $\mathrm{d}r\mathrm{d}t$... in the metric. We therefore have that $\mathrm{g}^{rr} = 1/\mathrm{g}_{rr}$, we thus conclude that *the location of the event horizon(s) of a axisymmetric spacetime is located where the metric component g_{rr} becomes singular.* To find the event horizons, we must therefore identify the coordinate singularities of the component g_{rr}, i.e., in general solve

$$\Delta(r; M, Q^2, a) = 0 \tag{3.2.15}$$

which, as we saw, had the solution(s)

$$\boxed{R_E = r_\pm = MG \pm \sqrt{M^2G^2 - Q^2 - a^2}} \tag{3.2.16}$$

[27] The hypersurfaces used in GR are often defined in this way

[28] Of course such a statement does not make sense, since a hypersurface is either spacelike, timelike or null, but hopefully the meaning is clear when one looks at fig. 11

3.2.1. The no-hair theorem and Hawking's area theorem

We have seen that the stationary, axisymmetric spacetimes coupled to electromagnetism have the possibility of containing an event horizon, i.e., containing a black hole. We saw that these black hole solutions were characterized by the mass M, charge Q and angular momentum J of the black hole. The different solutions were described according to

Black hole solutions	**Not rotating** $J = 0$	**Rotating** $J \neq 0$
Not charged $Q = 0$	*Schwarzschild*	*Kerr*
Charged $Q \neq 0$	*Reissner-Nordström*	*Kerr-Newman*

We may now ask how general these solution are? We have argued that it is possible to assign a definite mass, charge and angular momentum to any asymptotically flat, axisymmetric, stationary spacetime. Now suppose that such a spacetime contains an event horizon with no singularities present outside the event horizon (i.e., the spacetime contains only one black hole). Since the spacetime is stationary, all the matter responsible for the black hole has fallen inside the event horizon. According to our discussion of event horizons, no information is able to escape from inside of the event horizon to the outside (while the opposite is certainly possible). This *suggests* that the spacetime structure (at least outside the event horizon) must be independent of what happens inside the event horizon. This therefore indicates that all black hole solutions fall in the four above categories. We are thus lead to suggest the *no-hair theorem*

Theorem (No-hair theorem) *The stationary spacetimes describing a black holes are either axisymmetric or static. A black hole is therefore always Kerr (possible charged) and is therefore completely characterized by its mass M, charge Q and angular momentum J.*

It is possible to give a mathematical proof for this theorem. It requires *very* careful analysis of event horizons, black holes and general singularities[29]. Furthermore, it relies on the *cosmic censorship conjecture* which conjectures that it is impossible for naked singularities to form in nature; in nature singularities will always be behind an event horizon and thus hidden from a distant observer.

The no-hair theorem tells us that, if two spacetimes undergo complete gravitational collapse, so that they end up with the same external observables (M, Q and J), they will end up in the same state, no matter their initial data - "the black hole has no hairs". For this reason, from now on when we talk about a black hole we will always assume that it is Kerr (possible charged) and thus completely

[29] An analysis I cannot claim to have understood, yet.

determined by its mass, charge and angular momentum.

We will now state another important theorem in general relativity. It is the so-called area theorem which roughly states that the area of a black hole event horizon cannot decrease with time. In order to understand this statement we must first understand what we mean with "the area of an event horizon". First of all, the event horizon of a black hole is not a two dimensional manifold but a three dimensional submanifold of spacetime - this means that an event horizon does not have an area but rather a volume. However, we usually think of the term "event horizon" as the spatial part of the actual event horizon (event horizon $\sim$ time $\times$ spatial part of event horizon). Therefore, the area of an event horizon is the area of the two dimensional embedded manifold we obtain by looking at the event horizon "at a fixed time". On a general spacetime manifold the slices of spacetime "at a fixed time" are known as Cauchy surfaces. Again, this is a rather technical discussion, but let us be content by thinking of a Cauchy surface as a timelike three-dimensional hypersurface on which specifying initial conditions completely determines the future (and the past) $\sim$ spacetime "at a fixed time". We are now able to state and understand the area theorem

Theorem (Hawking's area Theorem) *Suppose that $\mathcal{H}$ is the event horizon of a black hole in a spacetime that is asymptotically flat (+ some additional technical requirements) and suppose that Σ_f and Σ_p are two Cauchy surfaces where Σ_f is in the future region of Σ_p. Now consider the two spatial parts (spacelike hypersurfaces), $H_f = \mathcal{H} \cap \Sigma_f$ and $H_p = \mathcal{H} \cap \Sigma_p$, of the event horizon $\mathcal{H}$ corresponding to respectively Σ_f and Σ_p. If the matter in the spacetime manifold respects the WEC, then the area of H_f is greater or equal to the area of H_p.*

The proof of the area theorem (which can be found in [Wal84] and to some extend in [Tow97]) again relies on a careful analysis of event horizons and of null hypersurfaces under the assumption of the cosmic censorship conjecture. The area theorem, as it is stated above, is not of much direct use, however, the black hole coordinates (Boyer-Lindquist for a given set M, Q^2 ad J) provides us with a surface of the type $H = \mathcal{H} \cap \Sigma$, determined by $r = r_+$ and $t = const.$. Let us see how this works for the Schwarzschild black hole. The induced metric on the "spatial part" of the event horizon $r = R_s$ is given by (simply set $\mathrm{d}r = \mathrm{d}t = 0$ in the Schwarzschild metric)

$$d\ell^2 = R_s^2 d\Omega^2 \tag{3.2.17}$$

The area of the Schwarzschild event horizon is thus given by

$$A_s = R_s^2 \int d\phi d\theta \sin\theta = 4\pi R_s^2 \tag{3.2.18}$$

or in terms of the physical mass

$$A_s = 16\pi M^2 G^2 \tag{3.2.19}$$

The area of a general black hole will also get contributions from the black hole charge and angular momentum, but in order to get an idea of how powerful the area theorem is, let us see what it implies for the Schwarzschild black holes: Since $A_s = 16\pi M^2 G^2$ the area theorem tells us that if a black hole of mass M_1 is somehow disturbed (spacetime becomes non-stationary), then when it has settled down again (spacetime becomes stationary again), it will be a new black hole of mass M_2, where $M_2 \geq M_1$, since the black hole area has increased. The physical content of this is clear; the presence of an event horizon allows energy to pass through the event horizon but never escape (however, as we will see, it is possible to extract energy from a rotating black hole). The area theorem also gives us a better understanding of what happens in a black hole "collision"; suppose that two Schwarzschild black holes of mass M_1 and M_2 collide and form a new Schwarzschild black hole as depicted in fig. 12.[30] What is the mass M_3 of the new black hole? Of course the answer to this question depends on the specific configuration of the problem but energy is carried away by gravitational radiation so $M_3 < M_1 + M_2$; The area theorem allows us to put a lower limit on M_3, since $A_3 \geq A_1 + A_2$

$$M_3 \geq \sqrt{M_1^2 + M_2^2} \tag{3.2.20}$$

the maximal efficiency $\eta = 1 - M_3/(M_1 + M_2)$ for the mass $\rightarrow$ gravitational radiation process is therefore

$$\eta \leq 1 - 1/\sqrt{2} \approx 29\% \tag{3.2.21}$$

so the area theorem somehow puts an upper limit of the amount of energy we can extract from a black hole.

Finally notice the resemblance to a thermodynamical system. Once we impose the thermodynamical equilibrium condition ($\sim$ stationary condition on the spacetime) on a thermodynamical system, the system is completely determined by its thermodynamic parameters (e.g. temperature, volume, pressure). For example, the state of a monatomic gas is determined by the internal energy U and volume V - it does, so to speak, not care about how it ended up in the state (U, V). Just as both internal energy and volume are extensive variables, so are respectively the black hole mass M, charge Q and angular momentum J. In principle, we could convert the energy contained in gravitational radiation to work. As have seen it is impossible to extract all the energy from a black hole because of the area theorem, i.e., it is impossible to convert all the energy in a black hole to work. This again bears resemblance to a thermodynamical system where the entropy is introduced as a measure of the systems unavailability to convert internal energy into work. At the end of this part, we will look a more into the mathematical analogy between black hole physics and thermodynamics.

[30] Clearly, this requires that the two black holes have no angular momentum wrt. each other, i.e., no orbital angular momentum.

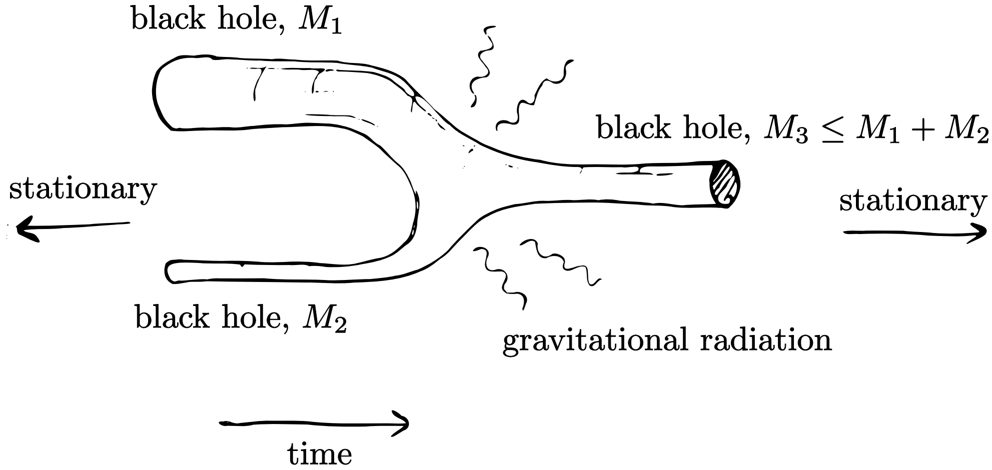

FIG 12. *The merging of two Schwarzschild black holes.*

3.3. More on horizons

In this section we introduce the rather abstract concept of Killing horizons on a spacetime manifold. A priori Killing horizons have nothing to do with event horizons, however, as we will see it is possible to interpret the event horizon of a Kerr black hole as a Killing horizon. We end this section off by looking at the so-called ergosphere which is a region of the Kerr spacetime where it is impossible for physical particles to stay stationary.

3.3.1. Killing Horizons

We will now introduce the important concept of *Killing horizons*. As we shall see these geometrical constructions will turn out to provide us with another link between black hole physics and thermodynamics.

Definition (Killing horizon) *Suppose that $\mathfrak{X}^\mu$ is a Killing vector field and that Σ is a null hypersurface. If $\mathfrak{X}^\mu$ is normal to Σ, we say that Σ is a Killing horizon for the Killing vector field $\mathfrak{X}^\mu$.*

More generally, a null hypersurface Σ is said to be a Killing horizon if it is Killing horizon for some Killing vector field (i.e., we do not always specify the Killing field). For a general Killing horizon, it is possible to attach a scalar function in the following way. Since the Killing field $\mathfrak{X}^\mu$ is normal to Σ we have that

$$\mathfrak{X}^\mu \mathfrak{X}_\mu \big|_\Sigma = 0 \tag{3.3.1}$$

This means that Σ can be specified by setting the function $\mathfrak{X}^\mu \mathfrak{X}_\mu$ equal to zero (=constant). We therefore see that the vector field

$$\mathrm{g}^{\mu\nu} \nabla_\nu \big(\mathfrak{X}^\rho \mathfrak{X}_\rho\big) = \nabla^\mu \big(\mathfrak{X}^\nu \mathfrak{X}_\nu\big) \tag{3.3.2}$$

is normal to Σ, but this implies that on the horizon Σ

$$\nabla^\mu\big(\mathfrak{X}^\nu\mathfrak{X}_\nu\big) = -2\kappa\ \mathfrak{X}^\mu \tag{3.3.3}$$

for some scalar function κ (where the factor of -2 is, of course, purely conventional). The scalar function κ is called the *surface gravity* of Σ. Using Killings equation notice that (3.3.3) implies

$$\mathfrak{X}^\nu\nabla_\mu\mathfrak{X}_\nu = -\mathfrak{X}^\nu\nabla_\nu\mathfrak{X}_\mu = -\kappa\ \mathfrak{X}_\mu \tag{3.3.4}$$

As the surface gravity is defined now, it is a rather abstract quantity. Our first task is therefore to find a formula for the surface gravity κ in terms of the Killing field $\mathfrak{X}^\mu$. To this end notice that, since the vector field $\mathfrak{X}^\mu$ is normal to Σ we know by Frobenius' theorem that on the horizon

$$\mathfrak{X}_{[\mu}\nabla_\nu\mathfrak{X}_{\rho]} = 0 \tag{3.3.5}$$

Moreover, since $\mathfrak{X}^\mu$ is Killing we have

$$\nabla_{[\mu}\mathfrak{X}_{\nu]} = \nabla_\mu\mathfrak{X}_\nu \tag{3.3.6}$$

Thus

$$\begin{aligned}\mathfrak{X}_{[\mu}\nabla_\nu\mathfrak{X}_{\rho]} &= \frac{1}{3}\Big\{\mathfrak{X}_\mu\nabla_{[\nu}\mathfrak{X}_{\rho]} - \mathfrak{X}_\nu\nabla_{[\mu}\mathfrak{X}_{\rho]} + \mathfrak{X}_\rho\nabla_{[\mu}\mathfrak{X}_{\nu]}\Big\}\\ &= \frac{1}{3}\Big\{\mathfrak{X}_\mu\nabla_\nu\mathfrak{X}_\rho - \mathfrak{X}_\nu\nabla_\mu\mathfrak{X}_\rho + \mathfrak{X}_\rho\nabla_{[\mu}\mathfrak{X}_{\nu]}\Big\}\\ &= 0\end{aligned} \tag{3.3.7}$$

So

$$\mathfrak{X}_\rho\nabla_\mu\mathfrak{X}_\nu = \mathfrak{X}_\rho\nabla_{[\mu}\mathfrak{X}_{\nu]} = -2\mathfrak{X}_{[\mu}\nabla_{\nu]}\mathfrak{X}_\rho \tag{3.3.8}$$

Now multiply this equation by $\nabla^\nu\mathfrak{X}^\nu$ and contract the indices. This gives

$$\begin{aligned}(\nabla^\mu\mathfrak{X}^\nu)\mathfrak{X}_\rho(\nabla_\mu\mathfrak{X}_\nu) &= -2(\nabla^\mu\mathfrak{X}^\nu)\mathfrak{X}_{[\mu}\nabla_{\nu]}\mathfrak{X}_\rho\\ &= -2\mathfrak{X}_\mu(\nabla^\mu\mathfrak{X}^\nu)(\nabla_\nu\mathfrak{X}_\rho)\\ &= -2\kappa\mathfrak{X}^\nu(\nabla_\nu\mathfrak{X}_\rho)\\ &= -2\kappa^2\mathfrak{X}_\rho\end{aligned} \tag{3.3.9}$$

where we used that if $B^{\mu\nu}$ is antisymmetric then for any $A_{\mu\nu}$ we have $A_{[\mu\nu]}B^{\mu\nu} = A_{\mu\nu}B^{\mu\nu}$ along with the equation (3.3.4). This means that we have obtained the following simple expression for the surface gravity

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu\mathfrak{X}^\nu)(\nabla_\mu\mathfrak{X}_\nu)\Big|_\Sigma \tag{3.3.10}$$

This is indeed a simple formula, since obtaining the surface gravity κ is now reduced to taking covariant derivatives (not even in the direction of Σ).

Several comments are in order concerning general Killing horizons. First of all, as mentioned above Killing horizons does not a priori have anything to do with event horizons. It is for example not hard to realize that ordinary Minkowski spacetime is filled with Killing Horizons. Of course, ordinary Minkowski spacetime contains no event horizons. Furthermore, notice that if Σ is a Killing horizon with associated Killing vector $\mathfrak{X}^\mu$ and with surface gravity κ, it will also be a Killing horizon for $k\mathfrak{X}^\mu$ (k constant) but with surface gravity $k^2\kappa$. We therefore conclude that the surface gravity of a Killing horizon Σ is not an intrinsic (i.e., geometrical) quantity for Σ. However, if we consistently choose a normalization of the Killing field $\mathfrak{X}^\mu$, this will uniquely fix the surface gravity of the Killing horizon Σ. As explained below such a normalization can be achieved by choosing a normalization of respectively the time translational Killing field $\mathfrak{K}^\mu$ and the rotational Killing field $\mathfrak{R}^\mu$. We will use the same normalization for $\mathfrak{K}^\mu$ and $\mathfrak{R}^\mu$ as we did when we defined the Komar mass and angular momentum (explained in section 2.4.1). As we will also see below, this normalization allows us to give a physical interpretation of the surface gravity. Finally, we mention that it seems that Killing horizons play an important role in understanding the causal structure of spacetime[31]. We will not try to justify this claim but keep it in mind for further studies. Before we move on to looking at how event horizons can be considered Killing horizons, we will state an important theorem regarding surface gravity

Theorem *Suppose that Σ is a Killing horizon and that the energy-momentum tensor obeys the dominant energy condition. It then holds that the surface gravity κ of the Killing horizon Σ is constant on Σ.*

Proof. We include this (sketchy) proof to show the use of the dominant energy-condition. The derivation relies on the fact that on the Killing horizon it holds that

$$R_{\mu\nu}\mathfrak{X}^\mu\mathfrak{X}^\nu = 0 \tag{3.3.11}$$

It is beyond the scope of this project to justify this, but it follows from a detailed analysis of the so-called generators of null horizons[32] and Killing's equation. Now using Einstein's equation along with $\mathfrak{X}_\mu\mathfrak{X}^\mu = 0$, we see that on Σ

$$T_{\mu\nu}\mathfrak{X}^\mu\mathfrak{X}^\nu = \mathcal{T}_\mu(\mathfrak{X})\mathfrak{X}^\mu = 0 \tag{3.3.12}$$

Since the energy-momentum tensor $T_{\mu\nu}$ satisfies the DEC and $\mathfrak{X}^\mu$ is null, the vector $\mathcal{T}^\mu(\mathfrak{X})$ is timelike or null. The equation (3.3.12) can therefore only be satisfied if the vector $\mathcal{T}^\mu(\mathfrak{X})$ is proportional to $\mathfrak{X}^\mu$. This implies that $\mathfrak{X}_{[\mu}\mathcal{T}_{\nu]}(\mathfrak{X}) =$

[31] For example, cosmological De Sitter space contains a Killing horizon.

[32] This is the theory of geodesic congruences. Both [Wal84] and [Tow97] have a chapter devoted to this.

0 (go to coordinates where $\mathfrak{X}^\mu = (1,0,0,0)$ and use that if a tensor has *all* components equal to zero, it must be the zero tensor). Therefore

$$\mathfrak{X}_{[\rho} T_{\mu]\nu} \mathfrak{X}^\nu = 0 \tag{3.3.13}$$

By use of Einstein's equation and the equations (3.3.4), (3.3.8) along with the Killing vector lemma, it is straight forward (for details, see [Wal84]) to show that

$$\mathfrak{X}_{[\rho} \nabla_{\mu]} \kappa = -\mathfrak{X}_{[\rho} T_{\mu]\nu} \mathfrak{X}^\nu \tag{3.3.14}$$

We therefore conclude that $\mathfrak{X}_{[\mu} \nabla_{\nu]} \kappa = 0$, which in turn means that $\nabla_\nu \kappa$ is proportional to $\mathfrak{X}_\mu$ and therefore normal to Σ. Thus for any tangent t^μ to Σ we have

$$t^\mu \nabla_\mu \kappa = 0 \tag{3.3.15}$$

i.e., κ is constant on Σ. □

3.3.2. Event horizons as Killing horizons

Consider the (outer) event horizon $\mathcal{H}$ of a charged rotating black hole $r = r_+$. As we have argued the event horizon is a null hypersurface. This means that if we can find a Killing field which is everywhere normal to $\mathcal{H}$, the event horizon $\mathcal{H}$ is a Killing horizon for which we may associate a unique (after normalization) surface gravity. Let us start by looking at the Schwarzschild black hole. As we have seen

$$\mathrm{g}_{tt} \to 0 \quad \text{as} \quad r \to R_S \tag{3.3.16}$$

Therefore $\mathfrak{K}^\mu$ is normal to the Schwarzschild event horizon $\mathcal{H}$. We therefore conclude that the Schwarzschild event horizon $\mathcal{H}$ is a Killing horizon for $\mathfrak{K}^\mu$. Let us now return to the general Kerr black hole. Recall that the Kerr spacetime has two Killing vector fields: The time translational Killing field $\mathfrak{K}^\mu$ and the rotational Killing field $\mathfrak{R}^\mu$. The first thing we notice is that $\mathfrak{K}^\mu$ is *not* normal to $\mathcal{H}$ for a rotating black hole, since $\mathrm{g}_{t\phi} \neq 0$ on $\mathcal{H}$. However, since the Kerr black hole is rotating we could imagine that we could choose a local rotating frame in which the event horizon is static[33], i.e., in the rotating frame, the event horizon looks Schwarzschild. Now let $\mathfrak{X}^\mu$ be the Killing field that generates time translations in the rotating frame. Since the event horizon looks Schwarzschild in the rotating frame, we expect $\mathfrak{X}^\mu$ to be normal to the event horizon. The vector field $\mathfrak{X}^\mu$ represents "time translations" in the rotated frame so it must be given by a "time translation" + "rotation" in non-rotating frame. We therefore expect the Kerr

[33] In other words, we imagine that the event horizon of the Kerr black hole rotates in some sense. We then choose a frame which rotates along the rotating event horizon so that this region of spacetime in this particular frame looks static. Such a transformation can only make a region of spacetime look static otherwise we would conclude that the Kerr spacetime is static which it is most certainly not. Notice how this is in accordance with Mach's principle.

event horizon to be a Killing horizon of a vector field $\mathfrak{X}^\mu$ of the type (where $\Omega_\mathcal{H}$ is a constant)

$$\mathfrak{X}^\mu = \mathfrak{K}^\mu + \Omega_\mathcal{H}\mathfrak{R}^\mu \tag{3.3.17}$$

Of course any (constant!) linear combination of Killing fields is again a Killing field, so the vector field $\mathfrak{X}^\mu$, given by the above expression, is indeed Killing. Notice that our normalization of $\mathfrak{K}^\mu$ and $\mathfrak{R}^\mu$ ($\mathfrak{K} = \partial_t$, $\mathfrak{R} = \partial_\phi$) fixes the normalization of $\mathfrak{X}^\mu$. The number $\Omega_\mathcal{H}$ is determined by requiring that $\mathfrak{X} = \partial_t + \Omega_\mathcal{H}\partial_\phi$ is normal to $\mathcal{H}$. Since the tangent space to the hypersurface $\mathcal{H}$ is spanned by ∂_t, ∂_θ and ∂_ϕ we see that $\mathfrak{X}^\mu$ is normal to $\mathcal{H}$ if the following two equations

$$\mathrm{g}_{tt} + \Omega_\mathcal{H}\ \mathrm{g}_{t\phi} = 0 \tag{3.3.18}$$

and

$$\mathrm{g}_{t\phi} + \Omega_\mathcal{H}\ \mathrm{g}_{\phi\phi} = 0 \tag{3.3.19}$$

are satisfied on all of $\mathcal{H}$. Apart from the above plausibility argument it is not evident that both these equations can be solved and that they can be solved with the same $\Omega_\mathcal{H}$. This is, however, the case, as the reader can easily verify. This means that the event horizon $\mathcal{H}$ is a Killing horizon for the Killing field

$$\boxed{\mathfrak{X}^\mu = \mathfrak{K}^\mu + \Omega_\mathcal{H}\mathfrak{K}^\mu} \tag{3.3.20}$$

where the constant

$$\boxed{\Omega_\mathcal{H} = -\frac{\mathrm{g}_{t\phi}}{\mathrm{g}_{\phi\phi}}\bigg|_{r=r_+} = \frac{a}{r_+^2 + a^2}} \tag{3.3.21}$$

is called the *angular velocity* for the rotating black hole. Having found the (unique determined) Killing vector field that makes the event horizon $\mathcal{H}$ into a Killing horizon and satisfies the normalization conditions, we can calculate the surface gravity associated with $\mathcal{H}$. To see how this is done, we will now calculate the surface gravity for the Schwarzschild black hole, i.e., $a = Q = 0$. To this end we will use the formula (3.3.10) which can be rewritten as

$$\kappa^2 = -\frac{1}{2}\mathrm{g}^{\mu\rho}\mathrm{g}_{\nu\lambda}(\nabla_\rho\mathfrak{X}^\nu)(\nabla_\mu\mathfrak{X}^\lambda)\Big|_\mathcal{H} \tag{3.3.22}$$

For the Schwarzschild black hole $\Omega_\mathcal{H} = 0$ so the Killing vector field $\mathfrak{X}^\mu$ is given by $\mathfrak{X}^\mu = \mathfrak{K}^\mu$. Therefore in coordinates we have

$$\mathfrak{X}^\mu = (1, 0, 0, 0) \tag{3.3.23}$$

Recall the usual formula for the covariant derivative of a vector field in coordinates

$$\nabla_\mu\mathfrak{X}^\nu = \partial_\mu\mathfrak{X}^\nu + \Gamma^\nu_{\mu\rho}\mathfrak{X}^\rho \tag{3.3.24}$$

Thus

$$\nabla_\mu \mathfrak{X}^\nu = \Gamma^\nu_{\mu\rho} \mathfrak{X}^\rho = \Gamma^\nu_{\mu t} \tag{3.3.25}$$

The only non-vanishing Christoffel symbols of the type $\Gamma^\nu_{\mu t}$ are Γ^t_{rt} and Γ^r_{tt} and they are given by

$$\Gamma^t_{rt} = \frac{MG}{r^2(1-2MG/r)} \quad \text{and} \quad \Gamma^r_{tt} = \frac{MG(1-2MG/r)}{r^2} \tag{3.3.26}$$

Using this along with the fact that the Schwarzschild metric is diagonal we obtain

$$\kappa^2 = -\frac{1}{2}\Big\{ \mathrm{g}^{tt}\mathrm{g}_{rr}(\Gamma^r_{tt})^2 + \mathrm{g}^{rr}\mathrm{g}_{tt}(\Gamma^t_{rt})^2 \Big\}\Big|_{r=2MG} \tag{3.3.27}$$

We have

$$\mathrm{g}^{rr}\mathrm{g}_{tt} = \left(\mathrm{g}^{tt}\mathrm{g}_{rr}\right)^{-1} = -\mathrm{g}_{rr}^{-2} = -(1-2mG/r)^2 \tag{3.3.28}$$

Therefore the surface gravity for the Schwarzschild black hole is given by

$$\kappa = \frac{1}{4MG} \tag{3.3.29}$$

Notice that κ decreases with M. A straight forward but long and tedious computation shows that in general, the surface gravity for the Kerr black hole is given by

$$\boxed{\kappa = \frac{(M^2-a^2-Q^2)^{1/2}}{2M[M+(M^2-a^2-Q^2)^{1/2}]-Q^2}} \tag{3.3.30}$$

3.3.3. The ergosphere

We will now explain what it means for a path (in a stationary spacetime) to be stationary. A stationary path is a one-dimensional curve which is invariant under the "time translations" φ_t defining the stationary spacetime. Equivalently a stationary path is an orbit of the one-parameter-group φ_t. Physically a stationary path can be interpreted as a curve where only time flows, i.e., the spatial part does not change. Clearly for such a path we have for the tangent U^μ (in some parameterization)

$$U^\mu = V^{-1}(x)\mathfrak{K}^\mu \tag{3.3.31}$$

for some normalization factor V^{-1}. A physical particle is said to be stationary if its world line γ is a stationary path. The tangent $\dot\gamma^\mu$ for a physical (massive) particle in the affine parameterization $\gamma \equiv \gamma(\tau)$ fulfills that $\dot\gamma^\mu\dot\gamma_\mu = -1$. Therefore, we conclude that for a stationary particle we have

$$\dot\gamma^\mu = V^{-1}\mathfrak{K}^\mu, \quad V^2 = -\mathfrak{K}_\mu\mathfrak{K}^\mu \tag{3.3.32}$$

We therefore see that if a stationary asymptotically flat spacetime contains a region where the time translational Killing field $\mathfrak{K}^\mu$ becomes spacelike, $\mathfrak{K}_\mu\mathfrak{K}^\mu \geq 0$, it will be impossible for a physical particle located in this region to stay stationary. This is also seen if we calculate the acceleration a^μ of a particle of mass m that is held stationary (clearly stationary paths will not, in general, be geodesics since freely falling particles certainly moves in space as time goes.). We have

$$a^\mu = D_\tau \dot{\gamma}^\mu = \dot{\gamma}^\nu \nabla_\nu (V\mathfrak{K})^\mu = -\frac{1}{V^2}\mathfrak{K}_\nu \nabla^\mu \mathfrak{K}^\nu \tag{3.3.33}$$

so

$$a^\mu = \nabla^\mu \log V \tag{3.3.34}$$

here we used that $V^2 = -\mathfrak{K}_\mu \mathfrak{K}^\mu$ implies that $\mathfrak{K}_\mu \nabla_\nu \mathfrak{K}^\mu = -V\nabla_\nu V$ along with Killing's equation. The stationary particle is thus under the influence of a force F^μ of the magnitude

$$F = \frac{m}{V}\sqrt{(\nabla_\mu V)(\nabla^\mu V)} \tag{3.3.35}$$

which goes to infinity when $V \to 0$. The level surface $V = 0$ for is called the stationary limit surface (for obvious reasons). If we use the usual normalization of $\mathfrak{K}^\mu$ we therefore see that the function V (called the redshift factor) goes from 1 at spatial infinity to 0 at the stationary limit surface. The stationary limit surface of the Schwarzschild spacetime is just the event horizon $\mathcal{H}$. This means that as long as we stay outside the event horizon, it is always possible to stay stationary. However, notice that for the general rotating ($a \neq 0$) charged Kerr black hole we have

$$-V^2 = \mathfrak{K}_\mu \mathfrak{K}^\mu = \mathrm{g}_{tt} = \frac{a^2 \sin^2\theta - \Delta}{\rho^2} \tag{3.3.36}$$

Now since $\Delta = r^2 + a^2 + GQ^2 - 2MGr$, the right-hand side becomes positive when

$$r^2 + a^2\cos^2\theta + GQ^2 - 2MGr < 0 \tag{3.3.37}$$

We therefore see that in the region defined by (referred to as the *ergosphere*)

$$r_+ < r < MG + (M^2G^2 - GQ^2 - a^2\cos^2\theta)^{1/2} \tag{3.3.38}$$

it is impossible for a physical particle to stay stationary. In this way the ergosphere is no different than the black hole itself (i.e., the spacetime inside the event horizon) there is, however, one crucial difference: It is possible for a physical path to enter the ergosphere and exit it again. As is easy to show, such a path *must* have ($a > 0$)

$$\frac{d\phi}{d\tau} > 0 \tag{3.3.39}$$

when it is inside the ergosphere. The different physical and non-physical paths are depicted in fig. 13.

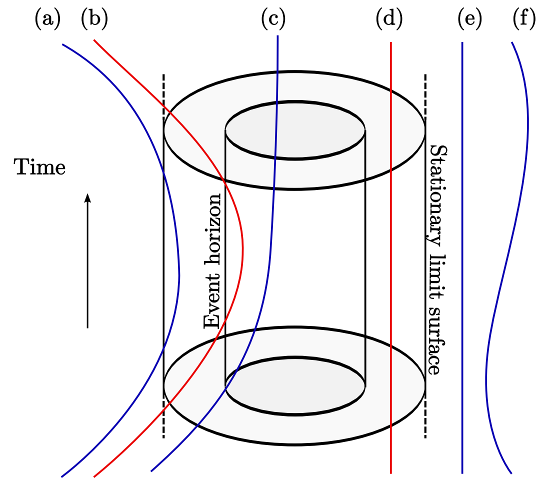

FIG 13. *The paths (a), (c), (e) and (f) are all physical while the paths (b) and (d) cannot be obtained by a physical particle.*

Physical interpretation of surface gravity and angular velocity

We will now give a physical interpretation of respectively the surface gravity and angular velocity of a black hole. We start by looking at surface gravity.
Usually the term "surface gravity" of an object is used for the gravitational force (per unit mass) experienced by a test particle on the surface of the given object. However, a meaningfull/intuitive physical interpretation of something related to a very strong gravitational field, such as a black hole, should be in terms observables that can be measured at spatial infinity (relativity is relative and we are (hopefully) always located far away from a black hole). Consider the stationary particle of mass m from before. The particle is under the influence of the force

$$F = ma^{\mu} \tag{3.3.40}$$

However, an observer located at spatial infinity will not measure that the particle is influenced by the force F^{μ} but instead a force redshifted by a factor V. This statement can be understood if one imagines that the distant observer is holding the particle stationary by a (massless) string. The force the distant observer needs to apply to the sting in order to keep the particle stationary has the magnitude[34]

$$F_{\infty} = ma_{\infty} = VF, \quad a_{\infty} = \sqrt{(\nabla_{\mu}V)(\nabla^{\mu}V)} \tag{3.3.41}$$

which is, as we will see now, finite and in fact equal to the surface gravity κ (times m). To prove this claim, first notice that for a general Killing field $\mathfrak{X}^{\mu}$ with an

[34]According to (1.2.21) (with $\dot{\gamma}^{\mu} = 1/V\mathfrak{K}^{\mu}$, since the point mass m is stationary), the energy as measured from infinity is redshifted by a factor of V. By using conservation of energy, we see that the force at the end of the string (at spatial infinity) must also be redshifted by the factor V.

associated Killing horizon Σ we have the following identity

$$3(\mathfrak{X}^{[\mu}\nabla^{\nu}\mathfrak{X}^{\rho]})(\mathfrak{X}_{[\mu}\nabla_{\nu}\mathfrak{X}_{\rho]}) = \mathfrak{X}^{\mu}(\nabla^{\nu}\mathfrak{X}^{\rho})\mathfrak{X}_{\mu}(\nabla_{\nu}\mathfrak{X}_{\rho}) - 2\mathfrak{X}^{\mu}(\nabla^{\nu}\mathfrak{X}^{\rho})\mathfrak{X}_{\nu}(\nabla_{\mu}\mathfrak{X}_{\rho}) \quad (3.3.42)$$

which is straight forward to show if one writes out the left-hand side and uses Killing's equation. Now this identity holds on all of the spacetime manifold but by equation (3.3.7) we know the left-hand side goes to zero as we approach the horizon Σ. Also, since Σ is a Killing horizon for $\mathfrak{X}^{\mu}$ we know that $\mathfrak{X}^{\mu}\mathfrak{X}_{\mu}$ goes to zero as we approach the horizon Σ. This means that we cannot just divide the left-hand side by $\mathfrak{X}^{\mu}\mathfrak{X}_{\mu}$ and evaluate on the horizon. However, the derivative of the left-hand side vanishes on the horizon Σ (since it is a product of two factors that each vanish on Σ) while the derivative of $\mathfrak{X}^{\mu}\mathfrak{X}_{\mu}$ is given by $-2\kappa\mathfrak{X}_{\mu}$ which is non-zero provided $\kappa \neq 0$. This means that, the limit of the ratio between the left-hand side of equation (3.3.42) and $\mathfrak{X}^{\mu}\mathfrak{X}_{\mu}$ goes to zero as we approach Σ (l'Hospital's rule). So

$$0 = \lim_{p\to\mathcal{H}^+}\left\{(\nabla^{\nu}\mathfrak{X}^{\rho})(\nabla_{\nu}\mathfrak{X}_{\rho}) + 2\frac{(\mathfrak{X}_{\nu}\nabla^{\nu}\mathfrak{X}^{\rho})(\mathfrak{X}^{\mu}\nabla_{\mu}\mathfrak{X}_{\rho})}{-\mathfrak{X}^{\mu}\mathfrak{X}_{\mu}}\right\} \quad (3.3.43)$$

Here $\lim_{p\to\mathcal{H}^+}$ denotes the limit as we approach the horizon from the outside where the Killing field $\mathfrak{X}^{\mu}$ is assumed to be timelike, i.e., $-\mathfrak{X}^{\rho}\mathfrak{X}_{\rho} > 0$. Now using the relation (3.3.10) along with the expression for the "physical acceleration" a^{μ} for an orbit of $\mathfrak{X}^{\mu}$

$$a^{\mu} = \frac{\mathfrak{X}^{\nu}\nabla_{\nu}\mathfrak{X}^{\mu}}{-\mathfrak{X}^{\rho}\mathfrak{X}_{\rho}} \quad (3.3.44)$$

we get that

$$\kappa = \lim_{p\to\mathcal{H}^+}(Va) \quad (3.3.45)$$

where $a \equiv (a^{\mu}a_{\mu})^{1/2}$ and $V \equiv (-\mathfrak{X}^{\mu}\mathfrak{X}_{\mu})^{1/2}$. Let us now return to black holes where we have chosen $\mathfrak{X}^{\mu}$ so that the event horizon $\mathcal{H}$ is a Killing horizon for $\mathfrak{X}^{\mu}$. According to the above analysis, which lead to the equation (3.3.45), the surface gravity is related to the acceleration of the physical orbits of $\mathfrak{X}^{\mu}$ near $\mathcal{H}$. The force F_{∞} was however related to the acceleration of the physical orbit of $\mathfrak{K}^{\mu}$ near $\mathcal{H}$, which is only defined outside the ergosphere. On a general rotating black hole, the only point p where $\mathcal{H}$ and the stationary limit surface intersect is on the rotation axis (i.e., the point $(r = r_+, \theta = 0)$). Now observe that the orbits of respectively $\mathfrak{K}^{\mu}$ and $\mathfrak{X}^{\mu}$ through p are the same (since $\mathfrak{R}^{\mu}$ vanishes on the axis of rotation). Therefore, the surface gravity can be interpreted as the force (per unit mass), that is needed to keep a particle located on the rotation axis just outside $\mathcal{H}$ stationary (see fig. 14).

We will now give an interpretation of the angular velocity $\Omega_{\mathcal{H}}$. Notice that for the Killing field $\mathfrak{X}^{\mu} = \mathfrak{K}^{\mu} + \Omega_{\mathcal{H}}\mathfrak{R}^{\mu}$ we have

$$\mathfrak{X}^{\mu}\nabla_{\mu}(\phi - \Omega_{\mathcal{H}}t) = 0 \quad (3.3.46)$$

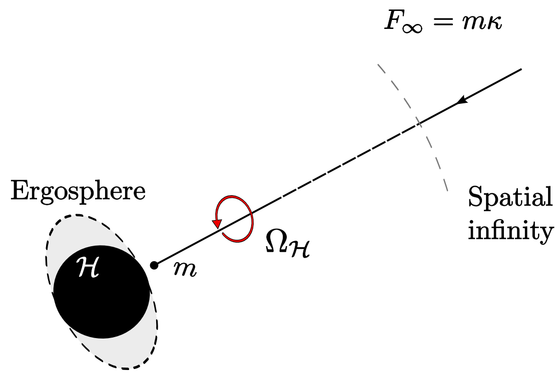

FIG 14. *Physical interpretation of the surface gravity κ.*

this means that the function $\phi - \Omega_{\mathcal{H}} t$ is constant on the orbits of $\mathfrak{X}^\mu$. As we have argued the Killing field $\mathfrak{X}^\mu$ can be interpreted as being the generator of time translations in a local rotating frame which is rotating along the event horizon. This suggests that we can interpret the quantity $\Omega_{\mathcal{H}}$ as the angular velocity of the "spatial part" of the event horizon as measured from spatial infinity. This interpretation is also supported by the fact that the minimum angular velocity (in the direction of ∂_ϕ) of a particle located on the horizon is given by $\Omega_{\mathcal{H}}$ (for the simple argument see [Car04, p. 266-267]). Therefore spacetime is dragged along the rotation of the rotating black hole, this phenomenon is know as *dragging of inertial frames*[35].

3.4. Black hole dynamics

3.4.1. The Penrose Mechanism

In this section we will see that it is possible to extract energy from a rotating black hole. To this end, we will discuss the so-called Penrose mechanism (or Penrose process). We start out with some general considerations.

A particle of mass m following a path $\gamma(\tau)$ has the four-momentum

$$p^\mu = m\dot{\gamma}^\mu \tag{3.4.1}$$

On a stationary, axisymmetric spacetime, we were able to define respectively the energy and angular momentum of the particle by the expressions

$$E = -\mathfrak{K}_\mu p^\mu \quad \text{and} \quad p^\mu = \mathfrak{R}_\mu p^\mu \tag{3.4.2}$$

On the Kerr spacetime (1.4.3), the two quantities take the form

$$E = m\Big(1 - \frac{2MGr}{\rho^2}\Big)\,\dot{t} + \frac{2mGMar}{\rho^2}\sin^2\theta\;\dot{\phi} \tag{3.4.3}$$

[35]This is an effect of all rotating bodies, not only the "extreme" case of a Kerr black hole.

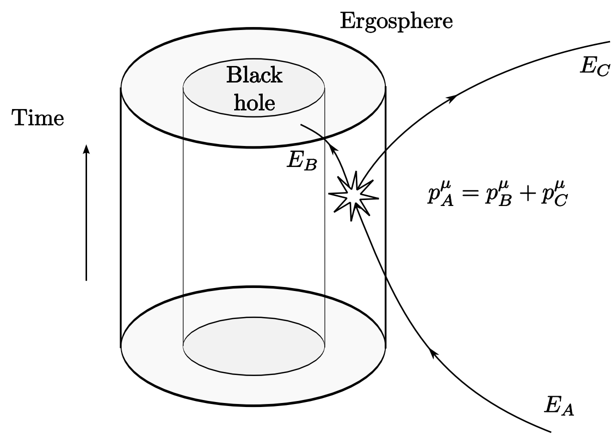

FIG 15. *The Penrose mechanism.*

and

$$L = -\frac{2mMGar}{\rho^2}\sin^2\theta\ \dot{t} + \frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta\ \dot{\phi} \tag{3.4.4}$$

Far away from the rotating black hole, the Killing field $\mathfrak{K}^\mu$ is timelike and the energy E is positive as it should be. However, recall that the Kerr spacetime has a region (the ergosphere) where the Killing field $\mathfrak{K}^\mu$ becomes spacelike. This means that, inside the ergosphere it is possible to have

$$E < 0 \tag{3.4.5}$$

As we will now see, the fact that inside the ergosphere energy can be negative, but outside the ergosphere must be positive, will allow us to extract energy from a rotating black hole. The idea is very simple: Suppose that a particle A, located at spatial infinity, is send toward a rotating black hole. Denote the four-momentum of the particle A by p^μ_A - the particle A therefore has the energy

$$E_A = -\mathfrak{K}_\mu p^\mu_A > 0 \tag{3.4.6}$$

Recall the crucial property of the quantities E and L; they are conserved along geodesics. This means that, as the particle A freely falls toward the rotating black hole, its energy will be conserved. Now suppose that, once the particle A has entered the ergosphere (but is still outside the event horizon), it splits up in two particles B and C, in such a way that the particle B has negative energy and enters the black hole while the particle C escapes from the ergosphere and to spatial infinity[36]. Denote the four-momentum of the two particles B and C

[36]Of course, the real trick consists in showing that two such geodesic trajectories exist. However, Penrose was able to show that this is indeed the case, thus the name of the mechanism.

by respectively p_B^μ and p_C^μ. By conservation of four-momentum, we have at the moment of the split up $A \to B + C$ that

$$p_A^\mu = p_B^\mu + p_C^\mu \tag{3.4.7}$$

and by contraction with $\mathfrak{K}^\mu$, we see

$$E_A = E_B + E_C \tag{3.4.8}$$

Now since the particle A started outside the ergosphere and particle C escapes from the ergosphere, we conclude that $E_A, E_C > 0$. However, we arranged the split up so that $E_B < 0$, therefore

$$E_C = E_A + |E_B| \tag{3.4.9}$$

We therefore see that the particle C has more energy than the particle A started out with! By total conservation of energy, the particle C must have carried away an amount $|E_B|$ of energy away from the black hole. In other words, when the black hole has settled down, after having absorbed the negative energy particle, we have

$$\delta M = -|E_B| \tag{3.4.10}$$

This shows that it is possible to extract energy from a rotating black hole. However, if we are to believe the cosmic censorship conjecture, there must be some limit on the amount of energy we can extract from a rotating black hole. If this were not the case, we would be able to extract energy from a rotating black hole until the event horizon disappears and the $r = 0$ singularity becomes naked. Such a limit does indeed exist, it follows from the fact that a negative energy particle *must* carry negative angular momentum. In order to realize this, notice that the four-momentum vector p_B^μ (as always) is future directed timelike while the Killing field ξ^μ, by construction, is future directed null on the event horizon. This means that, the particle B entering the rotating black hole obeys

$$p_B^\mu \xi_\mu = p_B^\mu (\mathfrak{K}_\mu + \Omega_{\mathcal{H}} \mathfrak{R}_\mu) = -E_B + \Omega_{\mathcal{H}} L_B < 0 \tag{3.4.11}$$

which shows that if E_B is negative, then L_B must also be negative. Since $\delta M = E_B$ and $\delta J = L_B$, we therefore conclude that the changes in the black hole mass and angular momentum satisfy

$$\delta J < \frac{\delta M}{\Omega_{\mathcal{H}}} \tag{3.4.12}$$

Hence, if we carry away from a black hole, we must also carry away angular momentum, which in turn will slow down the rotation of the black hole. At some point, all the black hole angular momentum has been carried away and the black hole becomes Schwarzschild, i.e., the ergosphere disappears and no more energy extraction can take place. The condition (3.4.12) can be rewritten in terms of the so called *irreducible mass*, defined by

$$M_{\mathrm{irr}}^2 = \frac{1}{2}\Big(M^2 + \sqrt{M^4 - (J/G)^2}\Big) \tag{3.4.13}$$

As is straight forward to check, it holds that

$$\delta M_{\mathrm{irr}} = \frac{a}{4GM_{\mathrm{irr}}\sqrt{G^2M^2 - a^2}}(\delta M/\Omega_{\mathcal{H}} - \delta J) \tag{3.4.14}$$

The condition (3.4.12) can therefore be written

$$\delta M_{\mathrm{irr}} > 0 \tag{3.4.15}$$

This shows that it is impossible to decrease the irreducible mass via the Penrose mechanism. It fact this is true for any physical process. This is seen using the area theorem: The induced metric on the event horizon is given by

$$\alpha_{ij}\mathrm{d}x^i\mathrm{d}x^j = \rho^2(r_+)\mathrm{d}\theta^2 + \frac{(r_+^2 + a^2)^2\sin^2\theta}{\rho^2(r_+)}\mathrm{d}\phi^2 \tag{3.4.16}$$

The area for the Kerr black hole is therefore

$$A = \int \sqrt{|\alpha|}\; d\theta d\phi = 4\pi(r_+^2 + a^2) \tag{3.4.17}$$

Now, the black hole area A is related to the irreducible mass through

$$A = 16\pi G^2 M_{\mathrm{irr}}^2 \tag{3.4.18}$$

Therefore, if we could decrease the irreducible mass in a physical process, we would be able to decrease the black hole area, which is not possible by the Hawking area theorem.

Consider an "ideal process" with $\delta M_{\mathrm{irr}} = 0$ in which we extract energy and angular momentum from a black hole. We see from (3.4.13) that the black hole mass cannot be reduced below M_{irr}. The quantity

$$M - M_{\mathrm{irr}} \tag{3.4.19}$$

therefore has the interpretation of "rotational energy" of the rotating black hole.

3.4.2. The laws of black hole dynamics

The Penrose process showed us that it is possible to extract energy and angular momentum from a black hole. The Penrose process therefore teaches us that a black hole is not just an absorber - in other words it is a much more dynamical object than first thought. Of course there is nothing special about the Penrose process other than its simplicity[37]: A black hole is an object that can exchange energy, angular momentum and charge with the surrounding universe through all sorts of physical processes.

As we will now see, it is possible to write down a formula that encapsulates the dynamics of black hole mechanics. To do this, we will start out by re-expressing the angular momentum J and total mass M (energy) of an axisymmetric, stationary spacetime containing a black hole. Recall that we were able to express the angular momentum of an axisymmetric spacetime as a surface integral over a surface S at spatial infinity

$$J = -\frac{1}{8\pi G}\int_S \mathrm{d}A\ n_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu \tag{3.4.20}$$

where n^μ and σ^μ are the two unit normals to S (with n^μ being the future directed timelike normal) and $\mathrm{d}A$ is the area element on S. Furthermore, recall that this formula was derived using the (normalized) Komar current

$$J^\mu[\mathfrak{R}] = \mathfrak{R}_\nu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big) = \frac{1}{8\pi G}\nabla_\nu(\nabla^\mu \mathfrak{R}^\nu) \tag{3.4.21}$$

along with Stokes' theorem (see section 2.4.1). Now let Σ be a asymptotically flat spacelike hypersurface in the exterior black hole spacetime with outer boundary S and with inner boundary H consisting of the intersection between the event horizon $\mathcal{H}$ and Σ (see fig. 16). Furthermore suppose that we choose Σ so that it intersects the horizon $\mathcal{H}$ on a 2-sphere, i.e, $H \sim S^2$. We can nu use the two expressions for the Komar current (3.4.21) along with Stokes' theorem to equate a surface integral at the boundary of Σ, i.e., the two surfaces S and H, with a volume integral over Σ. By Stokes' theorem we have

$$\int_S \mathrm{d}A\ n_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu + \int_H \mathrm{d}A\ n_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu = 8\pi G\int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{R}_\nu\Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\Big) \tag{3.4.22}$$

where $\mathrm{d}V$ denotes the volume element on Σ. Several comments are in order. First, since the surface H is null, the two normals n^μ and σ^μ cannot be chosen to be unit vectors. However, Stokes' theorem still applies if we choose the following normalization of n^μ and σ^μ (and use that $H \sim S^2$)

$$n_\mu \sigma^\mu = -1 \tag{3.4.23}$$

Second, the Killing vector $\mathfrak{X}^\mu$ is timelike (null) future directed on H, this means that we can identify $n^\mu = \mathfrak{X}^\mu$ *on the boundary of the event horizon* H. Using the

[37]For example, it is possible to carry energy away from a rotating black hole using a scalar field wave incident upon the black hole [Wal84, page 327-328].

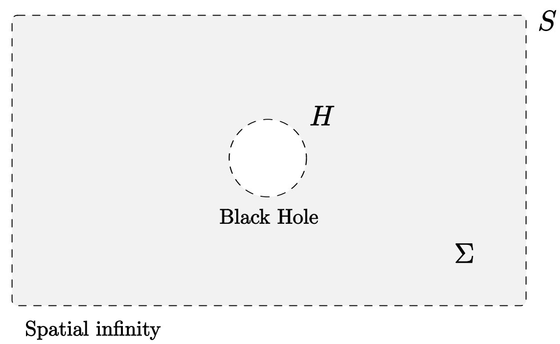

FIG 16. *The spacelike hypersurface Σ has outer boundary S and inner boundary H.*

expression (3.4.20), we therefore see that the angular momentum can be written as

$$J = \frac{1}{8\pi G}\int_H \mathrm{d}A\ \mathfrak{X}_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu - \int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{R}_\nu \Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu} T\Big) \tag{3.4.24}$$

We can write up a similar expression for the spacetime mass. As we have argued, the mass of an axisymmetric, stationary spacetime is given by the following surface (Komar) integral

$$M = \frac{1}{4\pi G}\int_S \mathrm{d}A\ n_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu \tag{3.4.25}$$

Using the same reasoning as above, we then see that we have

$$M = -\frac{1}{4\pi G}\int_H \mathrm{d}A\ \mathfrak{X}_\mu \sigma_\nu \nabla^\mu \mathfrak{K}^\nu + \int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{K}_\nu \Big(2T^{\mu\nu} - \mathrm{g}^{\mu\nu} T\Big) \tag{3.4.26}$$

Recall the relationship between the three Killing fields $\mathfrak{K}^\mu, \mathfrak{R}^\mu$ and $\mathfrak{X}^\mu$

$$\mathfrak{K}^\mu = \mathfrak{X}^\mu - \Omega_{\mathcal{H}} \mathfrak{R}^\mu \tag{3.4.27}$$

where the *constant* $\Omega_{\mathcal{H}}$ is the angular velocity of the black hole. Using this we get

$$\begin{aligned} M = &-\frac{1}{4\pi G}\int_H \mathrm{d}A\ \mathfrak{X}_\mu \sigma_\nu \nabla^\mu \mathfrak{X}^\nu + \int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{X}_\nu \Big(2T^{\mu\nu} - \mathrm{g}^{\mu\nu} T\Big) \\ &+ 2\Omega_{\mathcal{H}} \left\{ \frac{1}{8\pi G}\int_H \mathrm{d}A\ \mathfrak{X}_\mu \sigma_\nu \nabla^\mu \mathfrak{R}^\nu - \int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{R}_\nu \Big(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu} T\Big) \right\} \end{aligned} \tag{3.4.28}$$

We now use the expression (3.4.24) for the angular momentum to obtain

$$M = -\frac{1}{4\pi G}\int_H \mathrm{d}A\ \mathfrak{X}_\mu \sigma_\nu \nabla^\mu \mathfrak{X}^\nu + \int_\Sigma \mathrm{d}V\ n_\mu \mathfrak{X}_\nu \Big(2T^{\mu\nu} - \mathrm{g}^{\mu\nu} T\Big) + 2\Omega_{\mathcal{H}} J \tag{3.4.29}$$

The first term can now be expressed in terms of the black hole horizon area A and the surface gravity κ. Using the relation (3.3.4) and the normalization condition (3.4.23), we see that

$$\frac{1}{4\pi G}\int_H \mathrm{d}A\,\mathfrak{X}_\mu\sigma_\nu\nabla^\mu\mathfrak{X}^\nu = \frac{1}{4\pi G}\int_H \mathrm{d}A\,\kappa\sigma_\nu\mathfrak{X}^\nu = -\frac{\kappa}{4\pi G}\int_H \mathrm{d}A = -\frac{\kappa A}{4\pi G} \quad (3.4.30)$$

where we used that the surface gravity is constant over the horizon H. We have now found an expression for the spacetime mass in terms of the horizon area, surface gravity, angular velocity and spacetime angular momentum:

$$\boxed{M = \frac{\kappa A}{4\pi G} + 2\Omega_{\mathcal{H}}J + \int_\Sigma \mathrm{d}V\ n_\mu\mathfrak{X}_\nu\Big(2T^{\mu\nu} - \mathrm{g}^{\mu\nu}T\Big)} \quad (3.4.31)$$

As interesting as this formula is, it is not quite what we are looking for. In the following we wish to find an expression for the differential dM (exactly what we mean with this is explained below). First, note that if the matter field $T_{\mu\nu}$ is of electromagnetic nature, the trace $T = T^\mu_{\ \mu}$ vanishes. Therefore, in this case

$$M = \frac{\kappa A}{4\pi G} + 2\Omega_{\mathcal{H}}J + 2\int_\Sigma \mathrm{d}V\ n_\mu\mathfrak{X}_\nu T^{\mu\nu} \quad (3.4.32)$$

If the black hole is surrounded by vacuum $T_{\mu\nu} = 0$, which we will assume below, the above formula will reduces to

$$\boxed{M = \frac{\kappa A}{4\pi G} + 2\Omega_{\mathcal{H}}J} \quad (3.4.33)$$

Before we move on, notice that the formula (3.4.31) is *very* general. The equation (3.4.31) tells us the relationship between the mass and angular momentum of an stationary, axisymmetric spacetime containing a black hole. The matter field $T_{\mu\nu}$ is completely unspecified (of course it must respect the DEC, since the DEC $\Rightarrow \kappa$ constant on H). Especially, the equation (3.4.31) does not assume that the black hole has "no hairs"[38].

A formula for the differential mass dM

Consider a stationary vacuum black hole of mass M and angular momentum J (for simplicity, suppose that $Q = 0$). Now suppose that we make a *small* stationary, axisymmetric change in the metric. This will change the black parameters M and J slightly to $M + dM$ and $J + dJ$. We want to find an expression for the differential mass dM. A straight forward variation of equation (3.4.33) shows that

$$dM = \frac{1}{4\pi}(Ad\kappa + \kappa dA) + 2(Jd\Omega_{\mathcal{H}} + \Omega_{\mathcal{H}}dJ) \quad (3.4.34)$$

[38] The no-hair theorem only applies if $T_{\mu\nu}$ describes an electromagnetic field or vacuum

As we will now see, the two terms $Ad\kappa$ and $Jd\Omega_{\mathcal{H}}$ combine in a very special way. An analysis of the behavior of the terms $Ad\kappa$ and $Jd\Omega_{\mathcal{H}}$ can be found in [Wal84]. This analysis relies, not only on the definition of the Komar mass (2.4.41), but also on the so-called ADM-energy[39]. Defining and understanding the ADM-energy is beyond the scope of this project but we note that the ADM energy is an alternative definition of the energy of a (asymptotically flat) spacetime and (under certain assumptions) is equivalent to the Komar energy [Car04].

Fortunately, there exists a very nice little derivation of dM that does not rely on the ADM-energy (among other things!) but rather the no-hair theorem: By the no-hair theorem, i.e., uniqueness, we know that the mass M of a black hole is completely determined in terms of the area A and the angular momentum J;

$$M = M(A, J) \tag{3.4.35}$$

Now choose geometric units ($c = G = 1$). Here both A and J have dimensions of M^2. This means that the function $M = M(A, J)$ *must* exhibit the following scaling behavior

$$\alpha M = M(\alpha^2 A, \alpha^2 J) \tag{3.4.36}$$

The function M is therefore a homogenous function of degree $1/2$[40]. Thus by Euler's theorem for homogenous functions

$$A\frac{\partial M}{\partial A} + J\frac{\partial J}{\partial A} = \frac{1}{2}M \tag{3.4.38}$$

Therefore by equation (3.4.33) we have

$$A\Big\{\frac{\partial M}{\partial A} - \frac{\kappa}{8\pi}\Big\} + J\Big\{\frac{\partial M}{\partial J} - \Omega_{\mathcal{H}}\Big\} = 0 \tag{3.4.39}$$

Since the parameters A and J are free, this identity can only be true if $\partial M/\partial A = \dfrac{\kappa}{8\pi G}$ (reintroducing G) and $\partial M/\partial J = \Omega_{\mathcal{H}}$. Therefore

$$\boxed{dM = \frac{\kappa}{8\pi G}dA + \Omega_{\mathcal{H}}dJ} \tag{3.4.40}$$

In section 3.2.1 we noted an analogy between the black hole theorems (the no-hair and area theorem) and thermodynamics. The equation for dM makes this analogy even more apparent.

[39] Short for Richard Arnowitt, Stanley Deser and Charles Misner. For a relatively understandable introduction the ADM-energy, see [Tow97].

[40] Recall, a vector function f is said to be homogeneous of degree k if

$$f(\alpha\mathbf{v}) = \alpha^k f(\mathbf{v}) \tag{3.4.37}$$

Recall the first law of thermodynamics

$$dE = TdS + \text{work terms} \tag{3.4.41}$$

Here E is the energy, T is the temperature, S is the entropy and the "work terms" denote the work done on the system when we change the configuration parameters defining the system (such as volume, magnetization etc.). Recall that the terms on the right-hand side of the first law of thermodynamics all have the form

$$\underbrace{I}_{\substack{\text{intensive}\\ \text{variable}}} \times \underbrace{dE}_{\substack{\text{extensive}\\ \text{variable}}} \tag{3.4.42}$$

Let us now compare the first law of thermodynamics to the formula for the differential mass (3.4.40): Both the formulæ tell us how the energy of their respective systems (a black hole and a thermodynamical system) changes if we make small changes in their parameters. The first thing we notice is that the term $\Omega_{\mathcal{H}} dJ$ in (3.4.40) is precisely the work term we would expect a rotating body to have (the work we have to do on a rotating Newtonian body in order to change its angular momentum by an infinitesimal amount dJ is exactly given by ΩdJ). Moreover, the terms in the formula for the differential mass dM all possesses the structure (3.4.42). This suggests that we can identify the term $\kappa/(8\pi G dA)$ as a "heat term" and in fact consider a black hole as a thermodynamical system (!) with the following identifications (up to a multiplicative constant for A and κ)

$$\begin{aligned} E &\leftrightsquigarrow M \\ S &\leftrightsquigarrow A \\ T &\leftrightsquigarrow \frac{\kappa}{8\pi G} \end{aligned}$$

Of course, if a black hole really is a thermodynamical system, it must also respect the zeroth and second law of thermodynamics (for all we know, the "first law" 3.4.40 could be a pure coincidence). The zeroth law of thermodynamics states that if a thermodynamical body is in equilibrium, the temperature T is constant throughout the body. According to the above identifications, we identify the temperature of a black hole with its surface gravity (up to a multiplicative constant), however, we found that the surface gravity κ is constant over the event horizon of the black hole (see p. 67). Therefore, the zeroth law is satisfied! What about the second law of thermodynamics? The second law of thermodynamics states that the entropy of an isolated body cannot decrease with time. Now recall the area theorem due to Hawking: "The area A of a black hole horizon cannot decrease with time". Since we identify the entropy of a black hole with its horizon area (up to a multiplicative constant), we see that the second law of thermodynamics is also satisfied! Finally, recall that a thermodynamical object emits radiation corresponding to its temperature T. This means that if the surface gravity really is to be interpreted as a black hole temperature, a black hole must emit thermal radiation corresponding to $\kappa/8\pi$ (times some constant). Since a black hole is not able to emit radiation from its horizon, it seems unlikely that it should radiate at all. However, we have not taken quantum effects into account and since the gravitational field is extremely strong near the horizon, it is possible that quantum fluctuations cannot be ignored here. This is exactly the case, as Hawking was able to show with his famous semiclassical calculation (1974): A black hole emits thermal radiation (*Hawking radiation*) corresponding to the temperature (in units $k_B = c = \hbar = 1$)

$$T = \frac{\kappa}{2\pi} \tag{3.4.43}$$

All of these considerations strongly suggests that the identification of black holes as thermodynamical systems should be taken serious: Since a black hole has a temperature it also has an entropy. From the relation $TdS = (\kappa/8\pi G)dA$ we conclude from the above expression for T that

$$S = \frac{k_{\mathrm{B}} A}{4\ell_{\mathrm{P}}^2} \tag{3.4.44}$$

where we have reintroduced $k_B,\ c$ and $\hbar$ and where

$$\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3} \tag{3.4.45}$$

is the Planck length. Notice how the no-hair theorem poses two problems now. First of all, the entropy of a system is related to the (logarithm of the) number of accessible microstates of the given system. However, the no-hair theorem indicates that a black hole has only *one* accessible microstate. Therefore, the no-hair theorem suggests that the entropy of a black hole should be very low (well, in fact zero)! However, if we calculate the entropy of a solar mass black hole, we see that the entropy (3.4.44) will contain a factor $(3\mathrm{km}/\ell_{\mathrm{P}})^2$ (very big!).

Moreover, taking Hawking radiation into account, we see that a black hole will eventually evaporate[41]. From the no-hair theorem we expect the Hawking radiation to be completely independent of the material entering the black hole. Therefore it seems that the information contained in a black hole simply disappears as the black hole evaporates! This is of course in deep conflict with the most fundamental principles of modern physics. It is therefore the hope that a quantum theory for gravity will be able resolve these problems (among others).

[41]Is this is violation with the area theorem? No, since the quantum field responsible for Hawking radiation does not respect the WEC

References

[Car04] Sean M. Carroll. *Spacetime and Geometry, An Introduction to General Relativity*. Addison-Wesley publising company, 2004.

[Lee97] John M. Lee. *Riemannian Manifolds, An Introduction to Curvature*. Spring-Verlag, 1997.

[Lee00] John M. Lee. *Introduction to Smooth Manifolds*. Spring-Verlag, 2000.

[MP86] R. C. Myers and M. J. Perry. Black holes in higher dimensional spacetimes. *Annals of Physics*, 172, 1986.

[Ole07] Poul Olesen. *General Relativity and Cosmology*, lecture notes, http://www.nbi.dk/~polesen/GRC.pdf, 2007.

[OS06] Scully; M. O. and Zubairy; M. S. *Quantum Optics*. Cambridge University Press, 2006.

[Sch06] Henrik Schlichtkrull. *Differential Manifolds*, lecture notes for geometry 2, 2006.

[tH02] Gerardus 't Hooft. *Introduction to General Relativity*, http://www.phys.uu.nl/~thooft/lectures/genrel.pdf, 2002.

[Tow97] P.K. Townsend. *Black Holes*, arxiv:gr-qc/9707012v1 http://www.citebase.org/abstract?id=oai:arXiv.org:gr-qc/9707012, 1997.

[Wal84] Robert M. Wald. *General Relativity*. The University of Chicago Press, 1984.