COOPERATIVE GAMES: the SHAPLEY VALUE

The description of a cooperative game is still in terms of a **characteristic function** which specifies for every group of players the total payoff that the members of S can obtain by signing an agreement among themselves; this payoff is available for distribution among the members of the group.

DEFINITION. A **coalitional game with transferable payoff** (or characteristic function game) is a pair $\langle N, v \rangle$ where $N = \{1, ..., n\}$ is the set of players and for every subset S of I (called a ***coalition***) $v(S) \in \mathbb{R}$ is the total payoff that is available for division among the members of S (called the ***worth*** of S). We assume that the larger the coalition the higher the payoff (this property is called superadditivity):

$$\text{for all disjoint } S, T \subseteq N, \qquad v(S \cup T) \geq v(S) + v(T)$$

As before, an agreement is a list $(x_1, x_2, \ldots, x_n)$ where x_i is the proposed payoff to individual i. Shapley proposed some conditions (or axioms) that a solutions should satisfy and proved that there is a unique solution that meets those conditions. The solution, known as the **Shapley value**, has a nice interpretation in terms of **expected marginal contribution**. It is calculated by considering all the possible orders of arrival of the players into a room and giving each player his marginal contribution. The following examples illustrate this.

EXAMPLE 1. Suppose that there are two players and v({1}) = 10, v({2}) =12 and v({1,2}) = 23. There are two possible orders of arrival: (1) first 1 then 2, and (2) first 2 then 1.

If 1 comes first and then 2, 1's contribution is v({1}) = 10; when 2 arrives the surplus increases from 10 to v({1,2}) = 23 and therefore 2's marginal contribution is v({1,2}) − v({1}) = 23 − 10 = 13.

If 2 comes first and then 1, 2's contribution is v({2}) = 12; when 1 arrives the surplus increases from 12 to v({1,2}) = 23 and therefore 1's marginal contribution is v({1,2}) − v({2}) = 23 − 12 = 11.

Thus we have the following table:

Probability	Order of arrival	1's marginal contribution	2's marginal contribution
$\frac{1}{2}$	first 1 then 2	10	13
$\frac{1}{2}$	first 2 then 1	11	12

Thus 1's expected marginal contribution is: $\frac{1}{2}10 + \frac{1}{2}\ 11 = 10.5$ and 2's expected marginal contribution is $\frac{1}{2}13 + \frac{1}{2}\ 12 = 12.5$. This is the Shapley value: $x_1 = 10.5$ and $x_2 = 12.5$.

EXAMPLE 2. Suppose that there are three players now and v({1}) = 100, v({2}) =125, v({3}) = 50, v({1,2}) = 270, v({1,3}) = 375, v({2,3}) = 350 and v({1,2,3}) = 500. Then we have the following table:

v({1}) = 100, v({2}) =125, v({3}) = 50, v({1,2}) = 270, v({1,3}) = 375, v({2,3}) = 350 and v({1,2,3}) = 500

Probability	Order of arrival	1's marginal contribution	2's marginal contribution	3's marginal contribution
$\frac{1}{6}$	first 1 then 2 then 3: 123	v({1}) = 100	v({1,2}) − v({1}) = 270 − 100 = 170	v({1,2,3}) − v({1,2}) = 500 − 270 = 230
$\frac{1}{6}$	first 1 then 3 then 2: 132	v({1}) = 100	v({1,2,3}) − v({1,3}) = 500 − 375 = 125	v({1,3}) − v({1}) = 375 − 100 = 275
$\frac{1}{6}$	first 2 then 1 then 3: 213	v({1,2}) − v({2}) = 270 − 125 = 145	v({2}) =125	v({1,2,3}) − v({1,2}) = 500 − 270 = 230
$\frac{1}{6}$	first 2 then 3 then 1: 231	v({1,2,3}) − v({2,3}) = 500 − 350 = 150	v({2}) =125	v({2,3}) − v({2}) = 350 − 125 = 225
$\frac{1}{6}$	first 3 then 1 then 2: 312	v({1,3}) − v({3}) = 375 − 50 = 325	v({1,2,3}) − v({1,3}) = 500 − 375 = 125	v({3}) = 50
$\frac{1}{6}$	first 3 then 2 then 1: 321	v({1,2,3}) − v({2,3}) = 500 − 350 = 150	v({2,3}) − v({3}) = 350 − 50 = 300	v({3}) = 50

Thus 1's expected marginal contribution is: $\frac{1}{6}\left(100+100+145+150+325+150\right)=\frac{970}{6}$

2's expected marginal contribution is $\frac{1}{6}170+\frac{1}{6}125+\frac{1}{6}125+\frac{1}{6}125+\frac{1}{6}125+\frac{1}{6}300=\frac{970}{6}$

3's expected marginal contribution is $\frac{1}{6}230+\frac{1}{6}275+\frac{1}{6}230+\frac{1}{6}225+\frac{1}{6}50+\frac{1}{6}50=\frac{1060}{6}$

The sum, of course, is $\frac{3000}{6}=500=v(\{1,2,3\})$