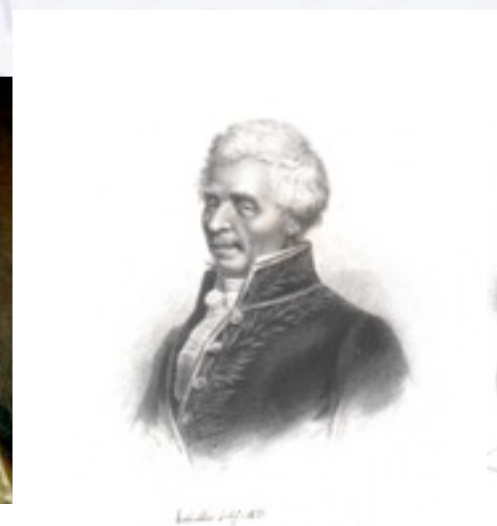
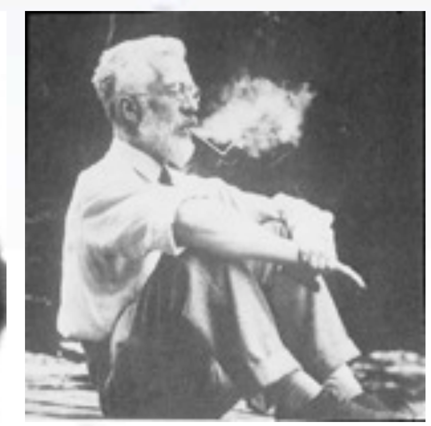
Applied Statistics

Principle of maximum likelihood





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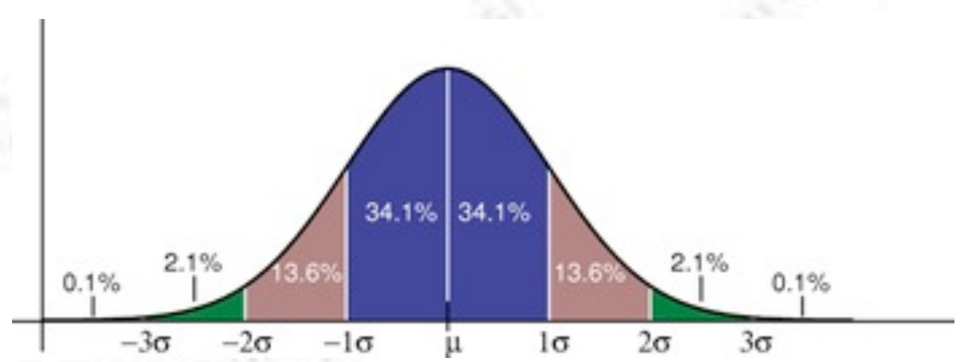


"Statistics is merely a quantization of common sense"

Likelihood function

I shall stick to the principle of likelihood...

[Plato, in Timaeus]

Likelihood function

Given a PDF f, what is the chance that with N observations, x_i falls in the interval $[x_i, x_i + dx_i]$?

$$\mathcal{L}(\theta) = \prod_i f(x_i, \theta) dx_i$$

Likelihood function

Given a set of measurements **x**, and parameter(s) θ, the likelihood function is defined as:

$$\mathcal{L}(x_1, x_2, \ldots, x_N; \theta) = \prod_i p(x_i, \theta)$$

The **principle of maximum likelihood** for parameter estimation consist of maximizing the likelihood of parameter(s) (here θ) given some data (here **x**).

The likelihood function plays a central role in statistics, as it can be shown to be:
- Consistent (converges to the right value!)
- Asymptotically normal (converges with Gaussian errors).
- Efficient (reaches the Cramer-Rao lower bound for large N).

To some extend, this means that the likelihood function is “optimal”, that is, if it can be applied in practice.

Likelihood vs. Chi-Square

For computational reasons, it is often much easier to minimize the logarithm of the likelihood function:

$$\left.\frac{\partial \ln \mathcal{L}}{\partial \theta}\right|_{\theta=\bar{\theta}} = 0$$

In problems with Gaussian errors, it turns out that the **likelihood function** boils down to the **Chi-Square** with a constant offset and a factor -2 in difference.

In practice, the likelihood comes in two versions:

- Binned likelihood (using Poisson).
- Unbinned likelihood (using PDF).

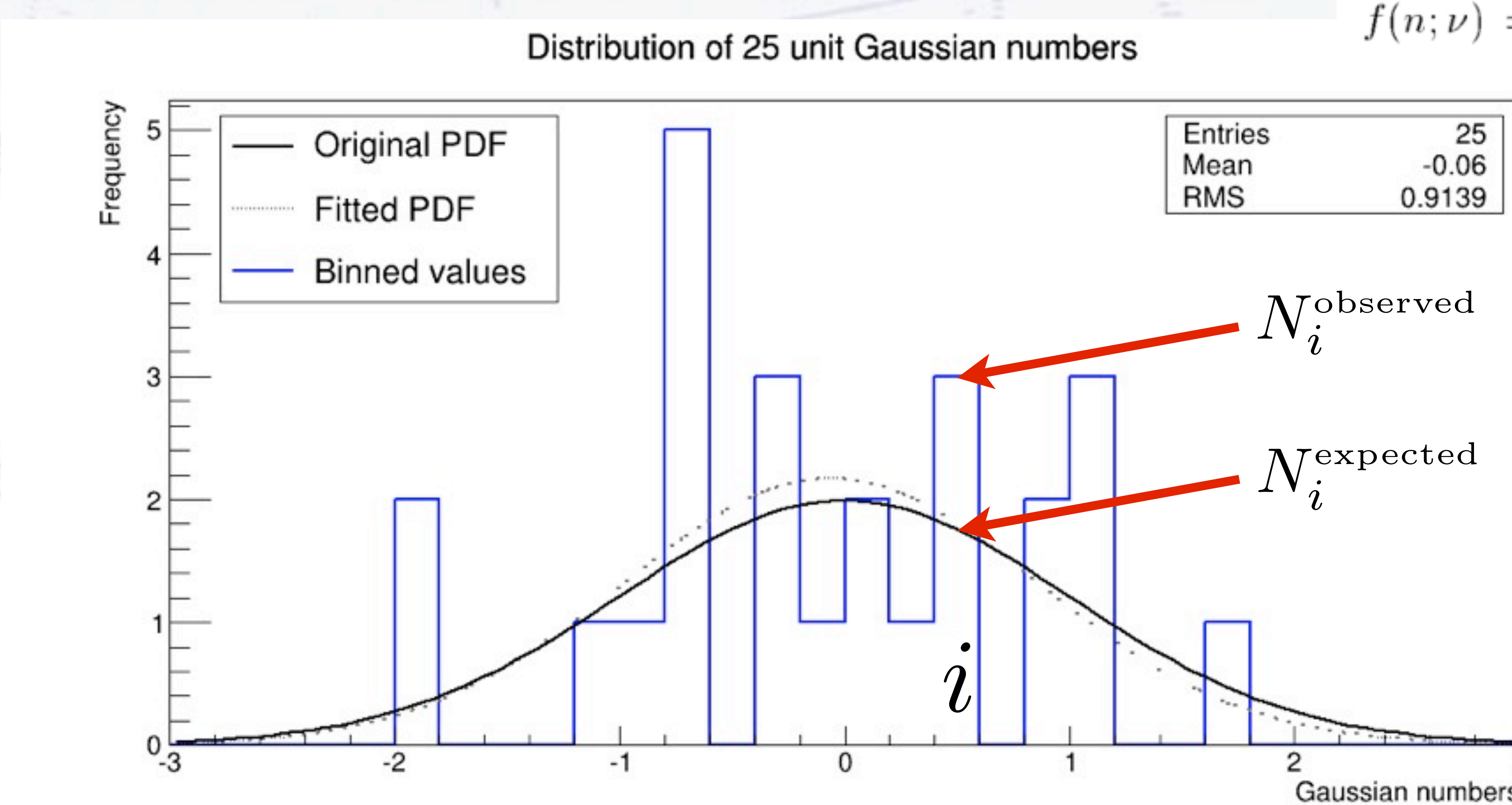
The "trouble" with the likelihood is, that it is unlike the Chi-Square, there is NO simple way to obtain a probability of obtaining certain likelihood value!

Binned Likelihood

The binned likelihood is a sum over bins in a histogram:

$$\mathcal{L}(\theta)_{\text{binned}} = \prod_{i}^{N_{\text{bins}}} \text{Poisson}(N_i^{\text{expected}}, N_i^{\text{observed}})$$

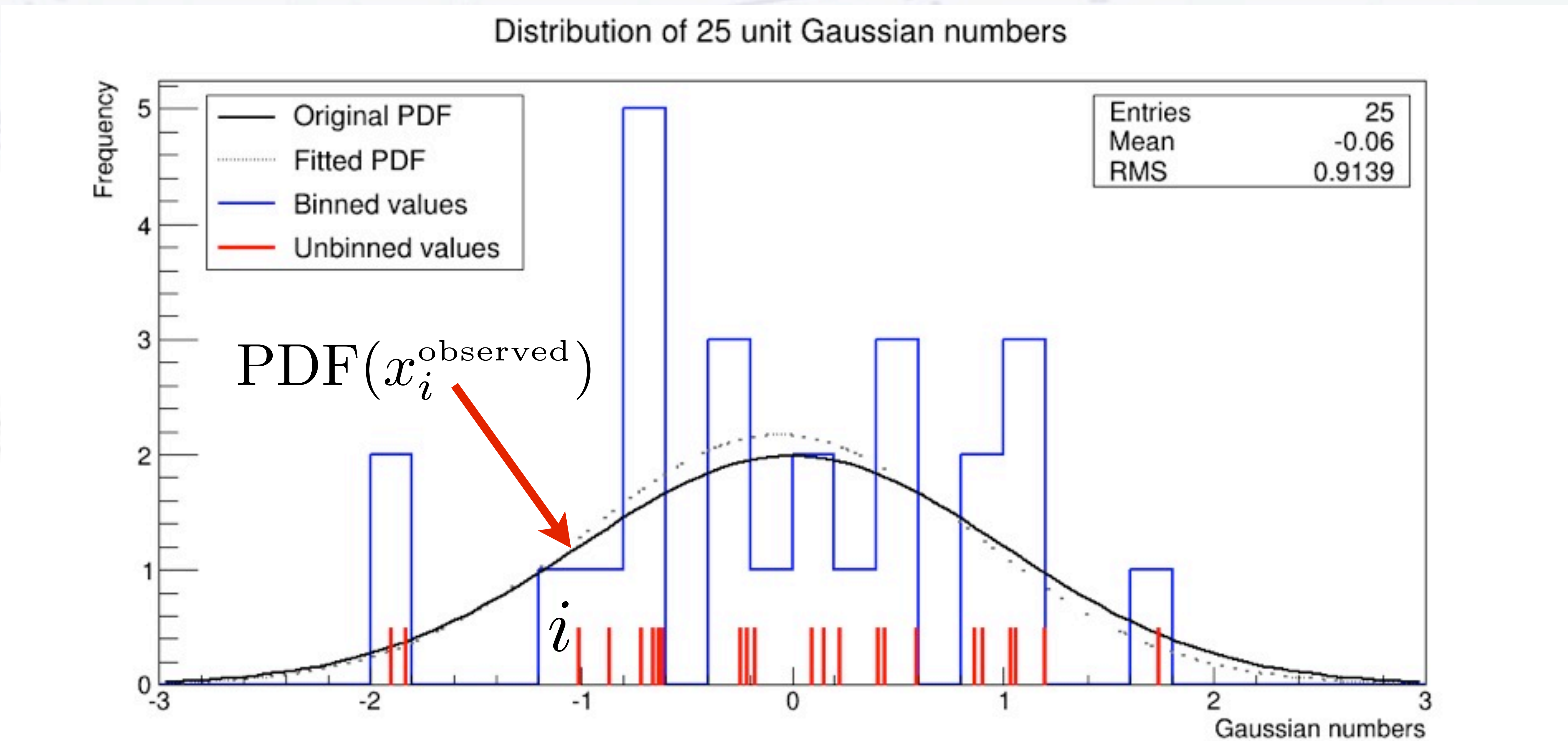
$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu}$$



Unbinned Likelihood

The binned likelihood is a sum over single measurements:

$$\mathcal{L}(\theta)_{\text{unbinned}} = \prod_{i}^{N_{\text{meas.}}} \text{PDF}(x_i^{\text{observed}})$$



Notes on the likelihood

For a large sample, the maximum likelihood (ML) is indeed unbiased and has the minimum variance - that is hard to beat! However...

For the ML, you have to know your PDF. This is also true for the Chi-Square, but unlike for the Chi-Square, you get no goodness-of-fit measure!

Also, the small statistics, the ML is not necessarily unbiased! Careful with this. The way to avoid this problem is using simulation - more to follow.

The likelihood ratio test

No unlike the Chi-Square, were one can compare χ^2 values, the likelihood between two competing hypothesis can be compared.

While their individual LLH values do not say much, their RATIO says everything!

As with the likelihood, one often takes the logarithm and multiplies by -2 to match the Chi-Square, thus the "test statistic" becomes:

$$\begin{aligned} D &= -2\ln\left(\frac{\text{likelihood for null model}}{\text{likelihood for alternative model}}\right) \\ &= -2\ln(\text{likelihood for null model}) + 2\ln(\text{likelihood for alternative model}) \end{aligned}$$

If the two hypothesis are simple (i.e. no free parameters) then the **Neyman-Pearson Lemma** states that this is the best possible test one can make.

If the alternative model is not simple, this difference behaves like a Chi-Square distribution with $N_{dof} = N_{dof}(\text{alternative}) - N_{dof}(\text{null})$