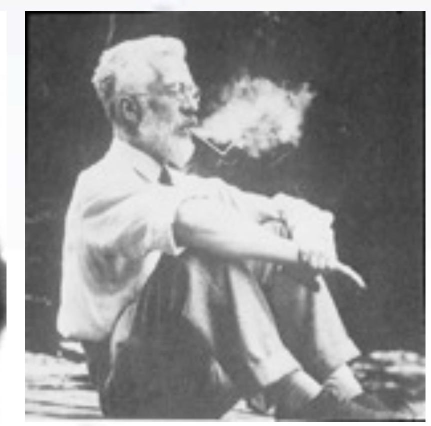
Applied Statistics

Principle of maximum likelihood





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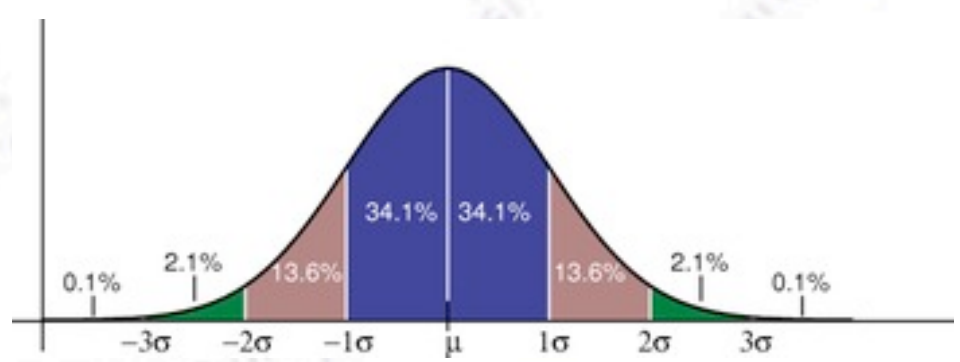


"Statistics is merely a quantization of common sense"

Likelihood function

"I shall stick to the principle of likelihood…"

[Plato, in Timaeus]

Likelihood function

Given a PDF f, what is the chance that with N observations, x_i falls in the interval $[x_i, x_i + dx_i]$?

$$\mathcal{L}(\theta) = \prod_i f(x_i, \theta) dx_i$$

Likelihood function

Given a set of measurements **x**, and parameter(s) **θ**, the likelihood function is defined as:

$$\mathcal{L}(x_1, x_2, \ldots, x_N; \theta) = \prod_i p(x_i, \theta)$$

The **principle of maximum likelihood** for parameter estimation consist of maximizing the likelihood of parameter(s) (here **θ**) given some data (here **x**).

The likelihood function plays a central role in statistics, as it can be shown to be:

- Consistent (converges to the right value!)
- Asymptotically normal (converges with Gaussian errors).
- Efficient (reaches the Cramer-Rao lower bound for large N).

To some extend, this means that the likelihood function is "optimal", that is, if it can be applied in practice.

Likelihood vs. Chi-Square

For computational reasons, it is often much easier to minimize the logarithm of the likelihood function:

$$\left.\frac{\partial \ln \mathcal{L}}{\partial \theta}\right|_{\theta=\bar{\theta}} = 0$$

In problems with Gaussian errors, it turns out that the **likelihood function** boils down to the **Chi-Square** with a constant offset and a factor -2 in difference.

In practice, the likelihood comes in two versions:

- Binned likelihood (using Poisson).
- Unbinned likelihood (using PDF).

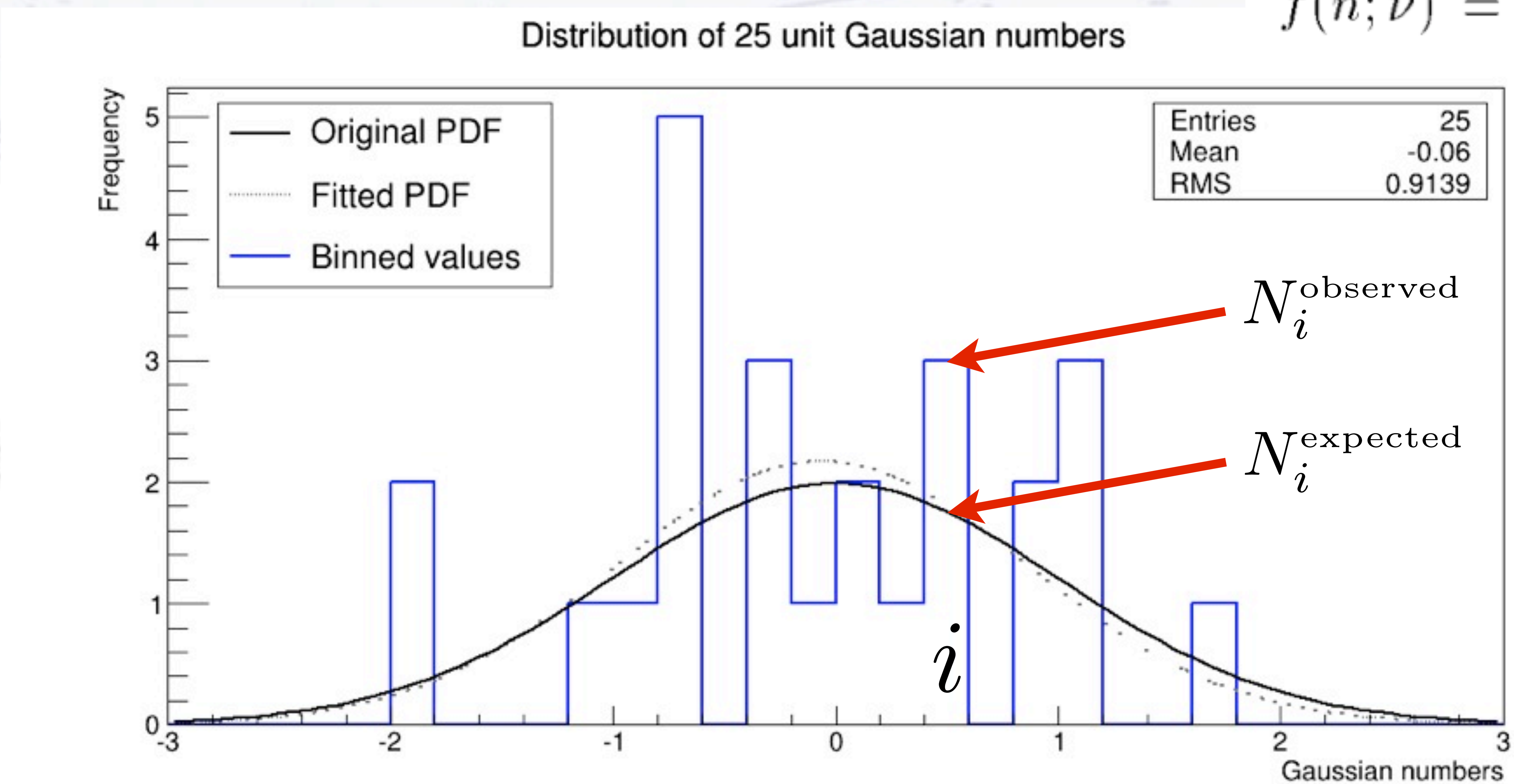
The “trouble” with the likelihood is, that it is unlike the Chi-Square, there is NO simple way to obtain a probability of obtaining certain likelihood value!

Binned Likelihood

The binned likelihood is a sum over bins in a histogram:

$$\mathcal{L}(\theta)_{\text{binned}} = \prod_{i}^{N_{\text{bins}}} \text{Poisson}(N_i^{\text{expected}}, N_i^{\text{observed}})$$

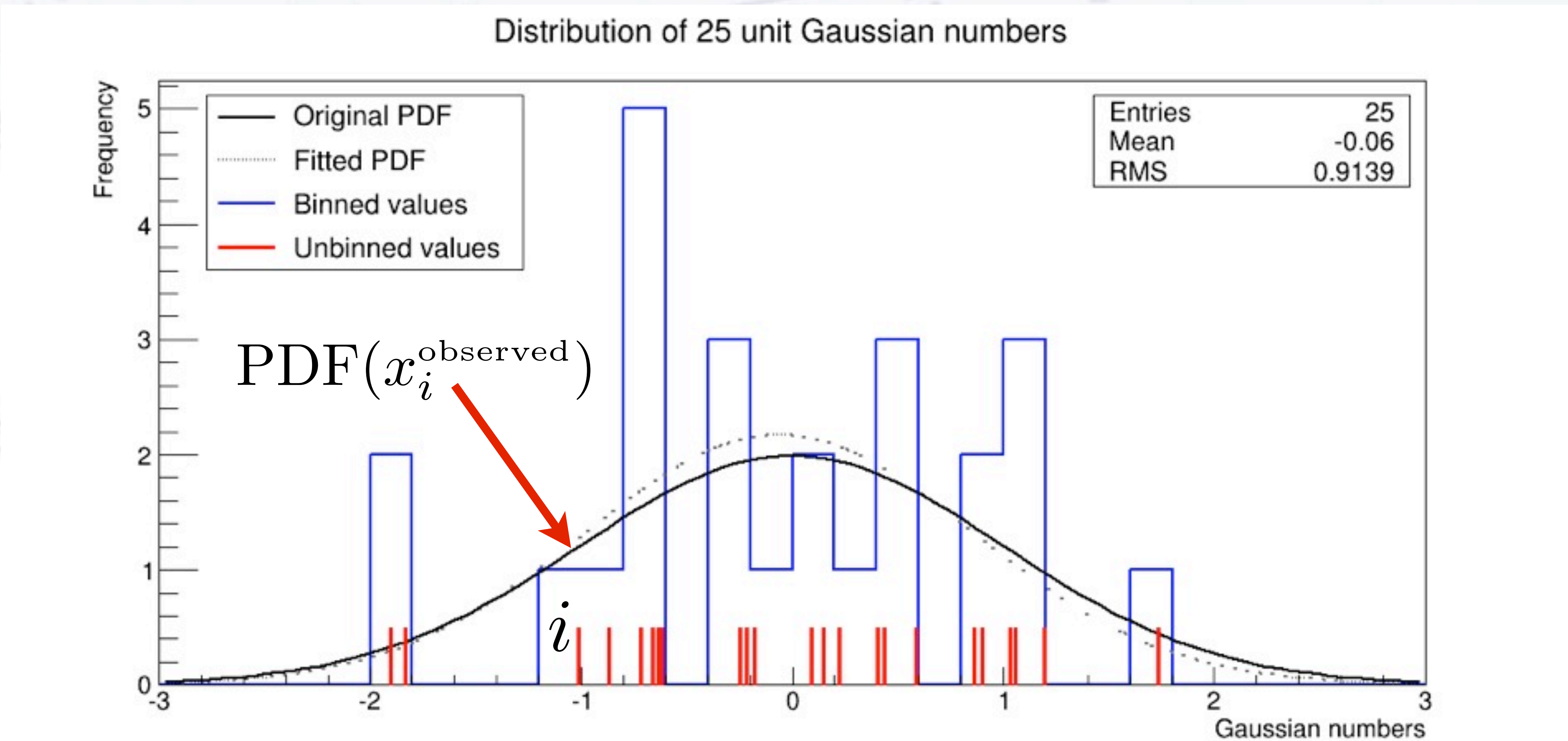
$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu}$$



Unbinned Likelihood

The binned likelihood is a sum over single measurements:

$$\mathcal{L}(\theta)_{\text{unbinned}} = \prod_{i}^{N_{\text{meas.}}} \text{PDF}(x_i^{\text{observed}})$$



Notes on the likelihood

For a large sample, the maximum likelihood (ML) is indeed unbiased and has the minimum variance - that is hard to beat! However...

For the ML, you have to know your PDF. This is also true for the Chi-Square, but unlike for the Chi-Square, you get no goodness-of-fit measure!

Also, the small statistics, the ML is not necessarily unbiased (nor is the ChiSquare)! Careful with this. The way to avoid this problem is using simulation - more to follow.

The likelihood ratio test

Not unlike the Chi-Square, were one can compare χ^2 values, the likelihood between two competing hypothesis can be compared (SAME offset constant/factor!).

While their individual LLH values do not say much, their RATIO says everything!

As with the likelihood, one often takes the logarithm and multiplies by -2 to match the Chi-Square, thus the "test statistic" becomes:

$$D = -2 \ln \left(\frac{\text{likelihood for null model}}{\text{likelihood for alternative model}} \right)$$
$$= -2 \ln(\text{likelihood for null model}) + 2 \ln(\text{likelihood for alternative model})$$

If the two hypothesis are simple (i.e. no free parameters) then the **Neyman-Pearson Lemma** states that this is the best possible test one can make.

If the alternative model is not simple, this difference behaves like a Chi-Square distribution with $N_{dof} = N_{dof}(\text{alternative}) - N_{dof}(\text{null})$