Applied Statistics

Error propagation






Troels C. Petersen (NBI)

"Statistics is merely a quantisation of common sense"

Error propagation

Imagine that y is a function of x_i , and that we wish to find the error on y from the errors on x_i . Making a Taylor expansion of the function y gives:

$$y(\bar{x}) \simeq y(\bar{\mu}) + \sum_{i}^{n} \frac{\partial y}{\partial x_i}(x_i - \mu_i)$$

In order to get the uncertainty of y as a function of the variables x_i we calculate:

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = E[x^2] - E^2[x]$$

$$E[y(\bar{x})] \simeq y(\bar{\mu})$$

$$E[y^2(\bar{x})] \simeq y^2(\bar{\mu}) + \sum_{i,j}^{n} \left[\frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right] V_{ij}$$

Error propagation formula

Subtracting the two formulae, we obtain:

$$\sigma_y^2 = \sum_{i,j}^{n} \left[\frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right]_{\bar{x}=\bar{y}} V_{ij}$$

If there are no correlations, only the diagonal (individual errors) enter:

$$\sigma_y^2 = \sum_{i}^{n} \left[\frac{\partial y}{\partial x_i} \right]_{\bar{x}=\bar{y}}^2 \sigma_i^2$$

Specific error propagation formula

Addition

Specific formula:

$$x = u + v$$

$$\sigma_x^2 = \sigma_u^2 + \sigma_v^2 + 2V_{uv}$$

General formula:

$$x = au + bv$$

$$\sigma_x^2 = a^2\sigma_u^2 + b^2\sigma_v^2 + 2abV_{uv}$$

"When adding numbers, their errors add in quadrature"

Specific error propagation formula

Multiplication

$$x = uv$$

$$\sigma_x^2 = (v\sigma_u)^2 + (u\sigma_v)^2 + 2uvV_{uv}$$

Dividing by x^2 to get relative terms, we obtain:

$$\frac{\sigma_x^2}{x^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2} + 2\frac{V_{uv}}{uv}$$

"When multiplying numbers, their RELATIVE errors add in quadrature"

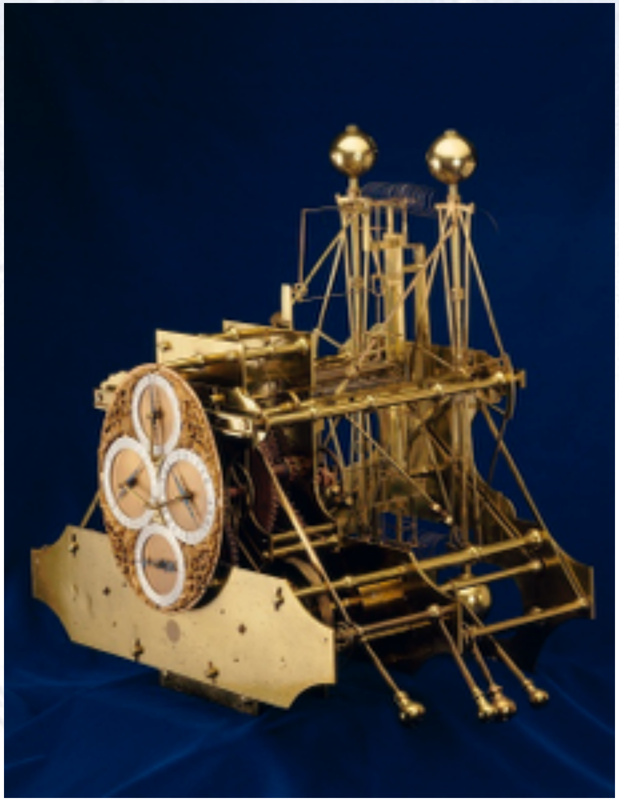
Error propagation at work...

John Harrison (24 March 1693 – 24 March 1776)

British clockmaker extraordinaire

“Won” the Longitude Act prize (3 sec/day).

Harrison's first sea clock (H1)

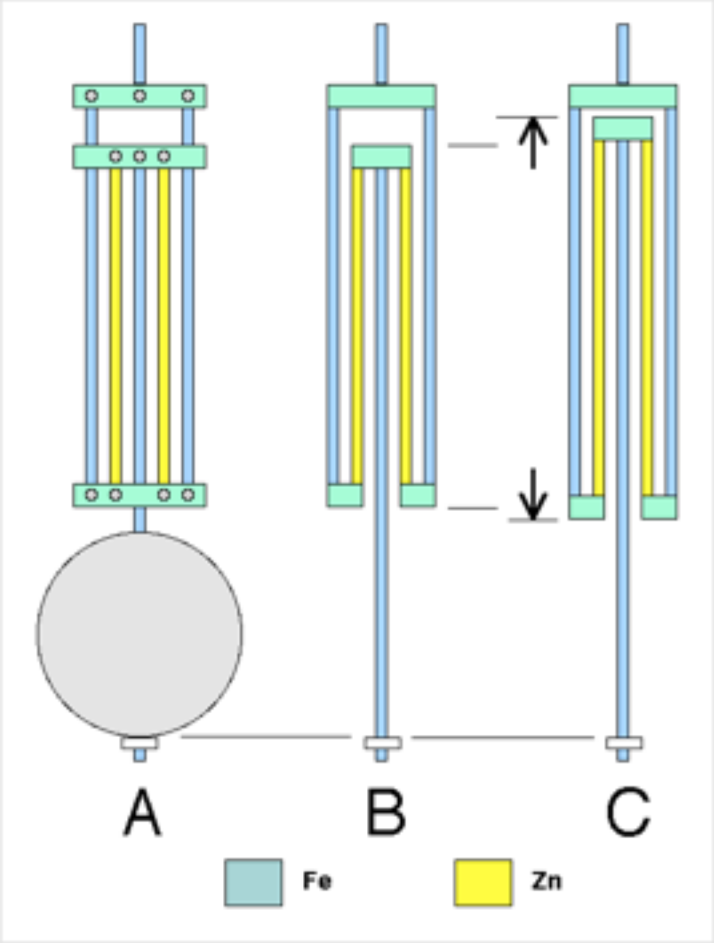


Harrison build H1-H5.

K1 (Copy of H4) was used by James Cook.

Error propagation at work...

Harrison's Gridiron pendulum is designed to cancel the change in length (in fact moment of inertia) with temperature.



Coefficient of thermal expansion:
Iron = 11.8×10^{-6}/C° Zinc = 30.2×10^{-6}/C°

Error propagation at more work...

Analysis of tiny differences in Uranus' orbit from Newtonian prediction led to the prediction and discovery of Neptune!

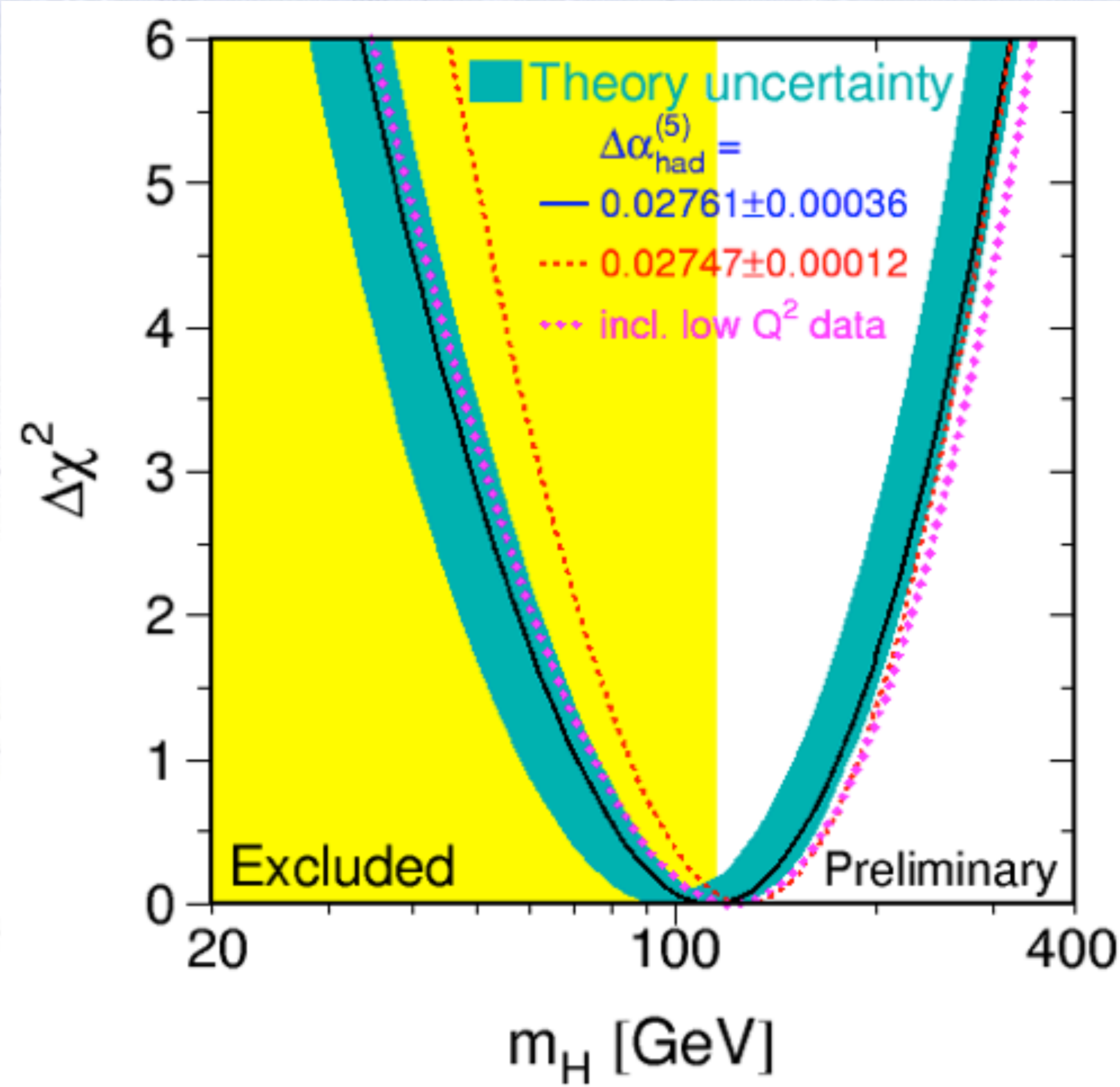
Continuing with Mercury...

TABLE II. Contributions to the motion of the perihelia of Mercury and the earth.

Cause	m^{-1}	Motion of perihelion: Mercury	Motion of perihelion: Earth
Mercury	6 000 000 ±1 000 000	0″.025±0″.00	−13″.75±2″.3
Venus	408 000 ± 1 000	277.856±0.68	345.49±0.8
Earth	329 390 ± 300	90.038±0.08	
Mars	3 088 000 ± 3 000	2.536±0.00	97.69±0.1
Jupiter	1 047.39 ± 0.03	153.584±0.00	696.85±0.0
Saturn	3 499 ± 4	7.302±0.01	18.74±0.0
Uranus	22 800 ± 300	0.141±0.00	0.57±0.0
Neptune	19 500 ± 300	0.042±0.00	0.18±0.0
Solar oblateness		0.010±0.02	0.00±0.0
Moon			7.68±0.0
General precession (Julian century, 1850)		5025.645±0.50	5025.65±0.5
Sum		5557.18 ±0.85	6179.1 ±2.5
Observed motion		5599.74 ±0.41	6183.7 ±1.1
Difference		42.56 ±0.94	4.6 ±2.7
Relativity effect		43.03 ±0.03	3.8 ±0.0

Urbain Le Verrier (1811-1877)

Advanced example of error propagation (Higgs particle mass):



Reporting uncertainties

The systematic uncertainties of a measurement should be reported in a table, and if measurements are combined, the correlation needs consideration.

CDF II preliminary $L = 200\ pb^{-1}$

m_T Uncertainty [MeV]	Electrons	Muons	Common
Lepton Scale	30	17	17
Lepton Resolution	9	3	0
Recoil Scale	9	9	9
Recoil Resolution	7	7	7
$u_{\|}$ Efficiency	3	1	0
Lepton Removal	8	5	5
Backgrounds	8	9	0
$p_T(W)$	3	3	3
PDF	11	11	11
QED	11	12	11
Total Systematic	39	27	26
Statistical	48	54	0
Total	62	60	26