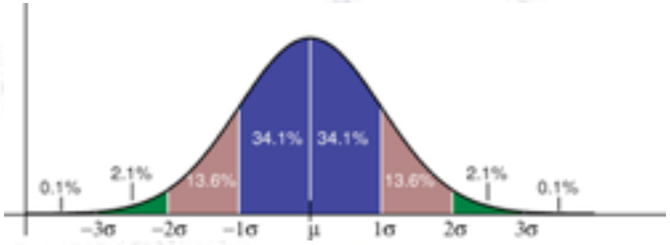
Applied Statistics

Binomial, Poisson, and Gaussian






Troels C. Petersen (NBI)



"Statistics is merely a quantisation of common sense"

Probability Density Functions

A Probability Density Function (PDF) f(x) describes the probability of an outcome x:

probability to observe x in the interval [x, x+dx] = f(x) dx

PDFs are required to be normalized:

$$\int_S f(x)dx = 1$$

The expectation value (aka. mean) and the variance (i.e. standard deviation squared) are then defined as follows:

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

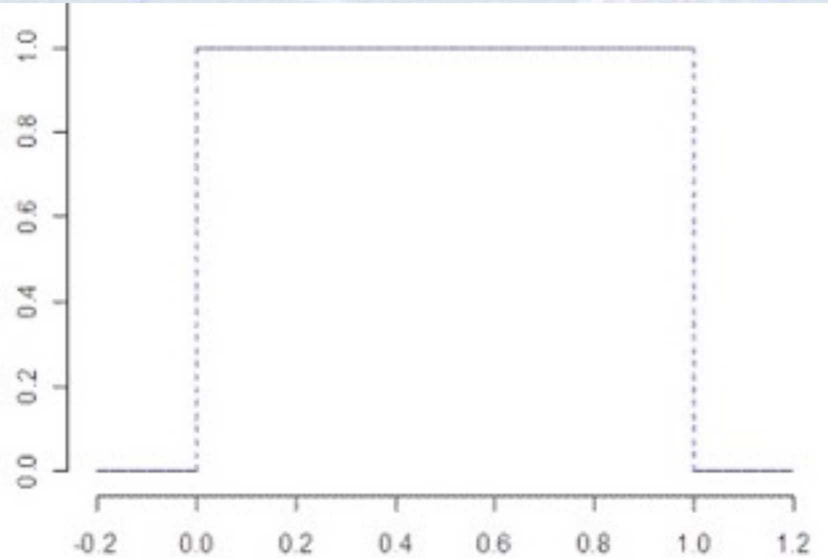
$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Probability Density Functions

<u>Example:</u>

Consider a uniform distribution:

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & else \end{cases}$$



Calculating the mean and width:

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_0^1 (x - \frac{1}{2})^2 dx =$$

$$[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

Probability Density Functions

The number of PDFs is infinite, and nearly so is the list of known ones:

Discrete distributions [edit source | edit beta]

With finite support [edit source | edit beta]

- The Bernoulli distribution, which takes value 1 with
- The Rademacher distribution, which takes value 1
- The binomial distribution, which describes the numb
- The beta-binomial distribution, which describes the
- The degenerate distribution at x_0, where X is certa random variables in the same formalism.
- The discrete uniform distribution, where all element shuffled deck.
- The hypergeometric distribution, which describes th there is no replacement.
- The Poisson binomial distribution, which describes
- Fisher's noncentral hypergeometric distribution
- Wallenius' noncentral hypergeometric distribution
- Benford's law, which describes the frequency of th

With infinite support [edit source | edit beta]

- The beta negative binomial distribution
- The Boltzmann distribution, a discrete distribution i analogue. Special cases include:
  - The Gibbs distribution
  - The Maxwell–Boltzmann distribution
- The Borel distribution
- The extended negative binomial distribution
- The extended hypergeometric distribution
- The generalized log-series distribution
- The generalized normal distribution
- The geometric distribution, a discrete distribution w
- The hypergeometric distribution
- The logarithmic (series) distribution
- The negative binomial distribution or Pascal distribu
- The parabolic fractal distribution
- The Poisson distribution, which describes a very la Poisson, the hyper-Poisson, the general Poisson b
  - The Conway–Maxwell–Poisson distribution, a tw
- The Polya-Eggenberger distribution
- The Skellam distribution, the distribution of the diff
- The skew elliptical distribution
- The skew normal distribution
- The Yule–Simon distribution
- The zeta distribution has uses in applied statistics
- Zipf's law or the Zipf distribution. A discrete power-
- The Zipf–Mandelbrot law is a discrete power law dis

Continuous distributions [edit source | edit beta]

Supported on a bounded interval [edit source | edit

- The Arcsine distribution on [a,b], which is a speci
- The Beta distribution on [0,1], of which the unifor
- The Logitnormal distribution on (0,1).
- The Dirac delta function although not strictly a fur but the notation treats it as if it were a continuou
- The continuous uniform distribution on [a,b], when
  - The rectangular distribution is a uniform distrib
- The Irwin-Hall distribution is the distribution of the
- The Kent distribution on the three-dimensional sph
- The Kumaraswamy distribution is as versatile as t
- The logarithmic distribution (continuous)
- The PERT distribution is a special case of the bet
- The raised cosine distribution on $[\mu - s, \mu + s]$
- The reciprocal distribution
- The triangular distribution on [a, b], a special case
- The truncated normal distribution on [a, b].
- The U-quadratic distribution on [a, b].
- The von Mises distribution on the circle.
- The von Mises-Fisher distribution on the N-dimens
- The Wigner semicircle distribution is important in t

Supported on semi-infinite intervals, usually [0,∞)

- The Beta prime distribution
- The Birnbaum-Saunders distribution, also known
- The chi distribution
  - The noncentral chi distribution
- The chi-squared distribution, which is the sum of t
  - The inverse-chi-squared distribution
  - The noncentral chi-squared distribution
  - The Scaled-inverse-chi-squared distribution
- The Dagum distribution
- The exponential distribution, which describes the t
- The F-distribution, which is the distribution of the ratio of two chi-squared variates which are not no
  - The noncentral F-distribution
- Fisher's z-distribution
- The folded normal distribution
- The Fréchet distribution
- The Gamma distribution, which describes the time
  - The Erlang distribution, which is a special case
  - The inverse-gamma distribution
- The generalized Pareto distribution
- The Gamma/Gompertz distribution
- The Gompertz distribution
- The half-normal distribution
- Hotelling's T-squared distribution
- The inverse Gaussian distribution, also kn
- The Lévy distribution
- The log-Cauchy distribution
- The log-gamma distribution
- The log-Laplace distribution
- The log-logistic distribution
- The log-normal distribution, describing vari
- The Mittag-Leffler distribution
- The Nakagami distribution
- The Pareto distribution, or "power law" dist
- The Pearson Type III distribution
- The phased bi-exponential distribution is c
- The phased bi-Weibull distribution
- The Rayleigh distribution
- The Rayleigh mixture distribution
- The Rice distribution
- The shifted Gompertz distribution
- The type-2 Gumbel distribution
- The Weibull distribution or Rosin Rammler grinding, milling and crushing operations.

Supported on the whole real line [edit sour

- The Behrens–Fisher distribution, which ari
- The Cauchy distribution, an example of a resonance energy distribution, impact and
- Chernoff's distribution
- The Exponentially modified Gaussian distri
- The Fisher–Tippett, extreme value, or log-
  - The Gumbel distribution, a special case
- Fisher's z-distribution
- The generalized logistic distribution
- The generalized normal distribution
- The geometric stable distribution
- The Holtsmark distribution, an example of
- The hyperbolic distribution
- The hyperbolic secant distribution
- The Johnson SU distribution
- The Landau distribution
- The Laplace distribution
- The Lévy skew alpha-stable distribution or distribution, Lévy distribution and normal d
- The Linnik distribution
- The logistic distribution
- The map-Airy distribution
- The normal distribution, also called the Ga independent, identically distributed variabl
- The Normal-exponential-gamma distributior
- The Pearson Type IV distribution (see Pea
- The skew normal distribution
- Student's t-distribution, useful for estimating
  - The noncentral t-distribution
- The type-1 Gumbel distribution
- The Voigt distribution, or Voigt profile, is the c
- The Gaussian minus exponential distribution i

With variable support [edit source | edit beta]

- The generalized extreme value distribution has parameter
- The generalized Pareto distribution has a sup
- The Tukey lambda distribution is either suppo
- The Wakeby distribution

Mixed discrete/continuous distributions [edi

- The rectified Gaussian distribution replaces n

Joint distributions [edit source | edit beta]

For any set of independent random variables the

Two or more random variables on the same sa

- The Dirichlet distribution, a generalization of th
- The Ewens's sampling formula is a probability
- The Balding–Nichols model
- The multinomial distribution, a generalization o
- The multivariate normal distribution, a generali
- The negative multinomial distribution, a genera
- The generalized multivariate log-gamma distrib

Matrix-valued distributions [edit source | edit b

- The Wishart distribution
- The inverse-Wishart distribution
- The matrix normal distribution
- The matrix t-distribution

Non-numeric distributions [edit source | edit

- The categorical distribution

newton distribution

Miscellaneous distributions [edit source | edit

- The Cantor distribution
- The generalized logistic distribution family
- The Pearson distribution family
- The phase-type distribution

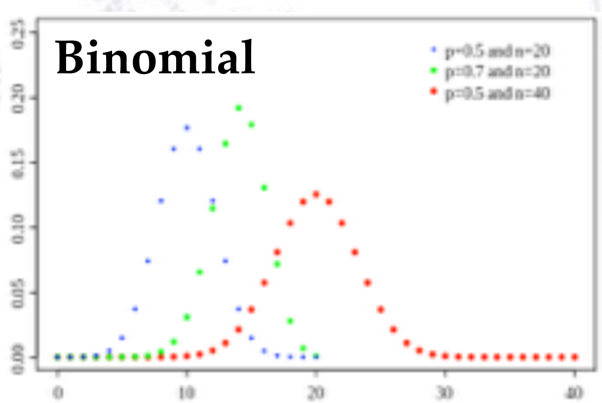
And surely more!

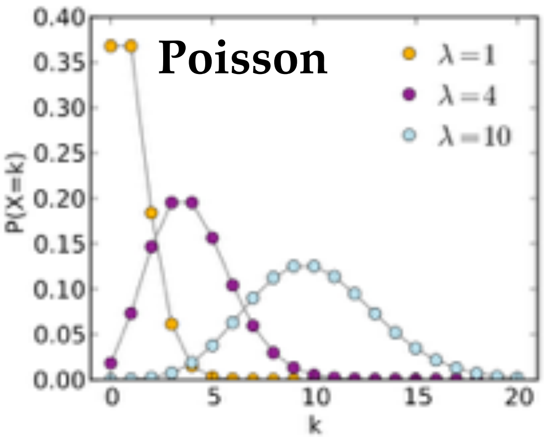
Probability Density Functions

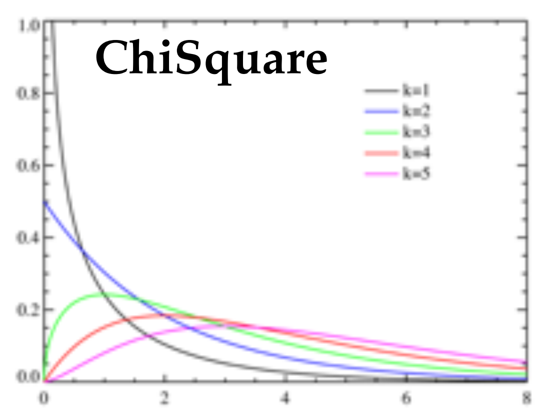
An almost complete list of those we will deal with in this course is:

- **Gaussian** (aka. Normal)
- **Poisson**
- **Binomial** (and also Multinomial)
- Students t-distribution
- Uniform
- ChiSquare
- Exponential
- Error function (integral of Gaussian)

You should already know most of these, and the rest will be explained.



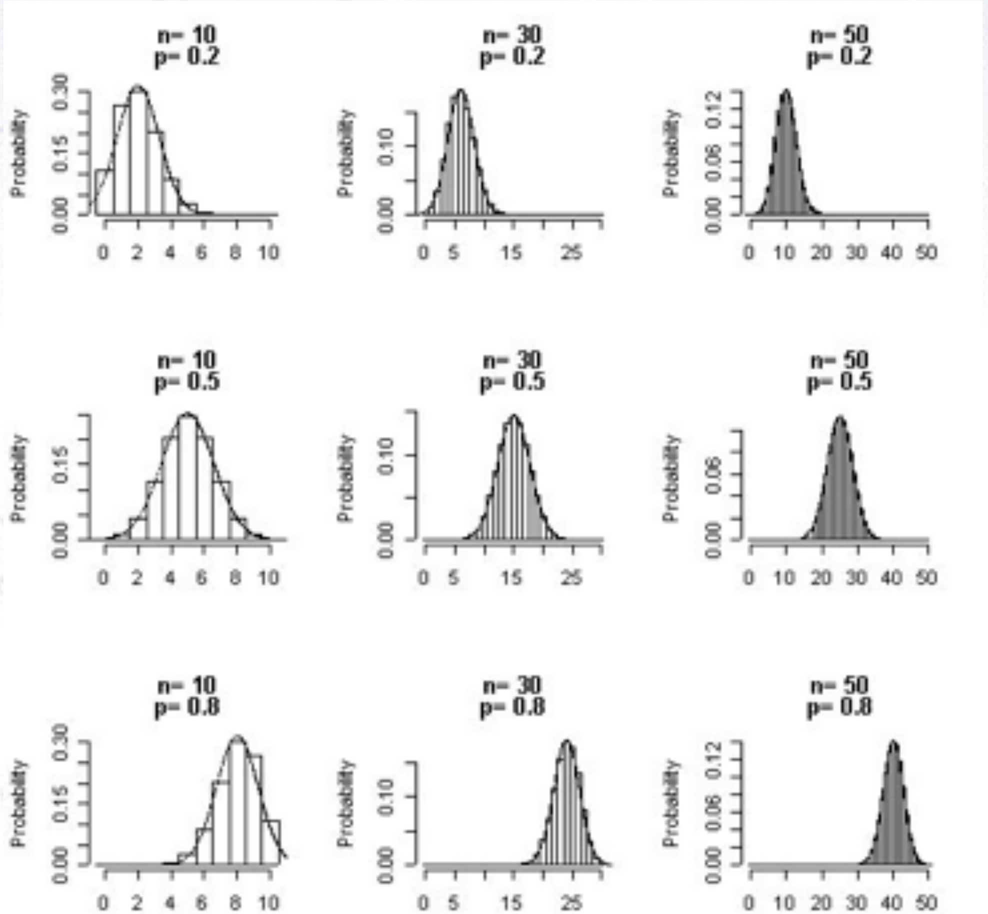




Binomial, Poisson, Gaussian

$$f(n;N,p) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}$$

Given **N trials** each with **p chance of success**, how many **successes n** should you expect in total?



This distribution is... **Binomial:**

$$f(n;N,p) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}$$

Mean = Np
Variance = Np(1-p)

This means, that the error on a fraction f = n/N is:

$$\sigma(f) = \sqrt{\frac{f(1-f)}{N}}$$

Binomial, Poisson, Gaussian

You count 100 cars, and see 15 red cars. What is your estimate of the fraction (i.e. probability) of red cars and its uncertainty?

a) 0.150 ± 0.050
b) 0.150 ± 0.026
c) 0.150 ± 0.036
d) 0.125 ± 0.030
e) 0.150 ± 0.081

From previous page: $$\sigma(f) = \sqrt{\frac{f(1-f)}{N}}$$

A friend tells you, that 8% of the cars on Blegdamsvej are red. What is the chance of that? Could he be right?

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Binomial, Poisson, Gaussian

Requirements to be Binomial:

- Fixed number of trials, N
- Independent trials.
- Only two outcomes (success/failure).
- Constant probability of success/failure.

If number of possible outcomes is more than two ⇒ **Multinomial distribution**.

Examples of Binomial experiments:

- Tossing a coin 20 times counting number of tails.
- Asking 200 people if they watch sports on TV.
- Rolling a die to see if a 6 appear (Multinomial for all outcomes!).
- Asking 100 die-hards from Enhedslisten, if they would vote for Konservative at the next election!

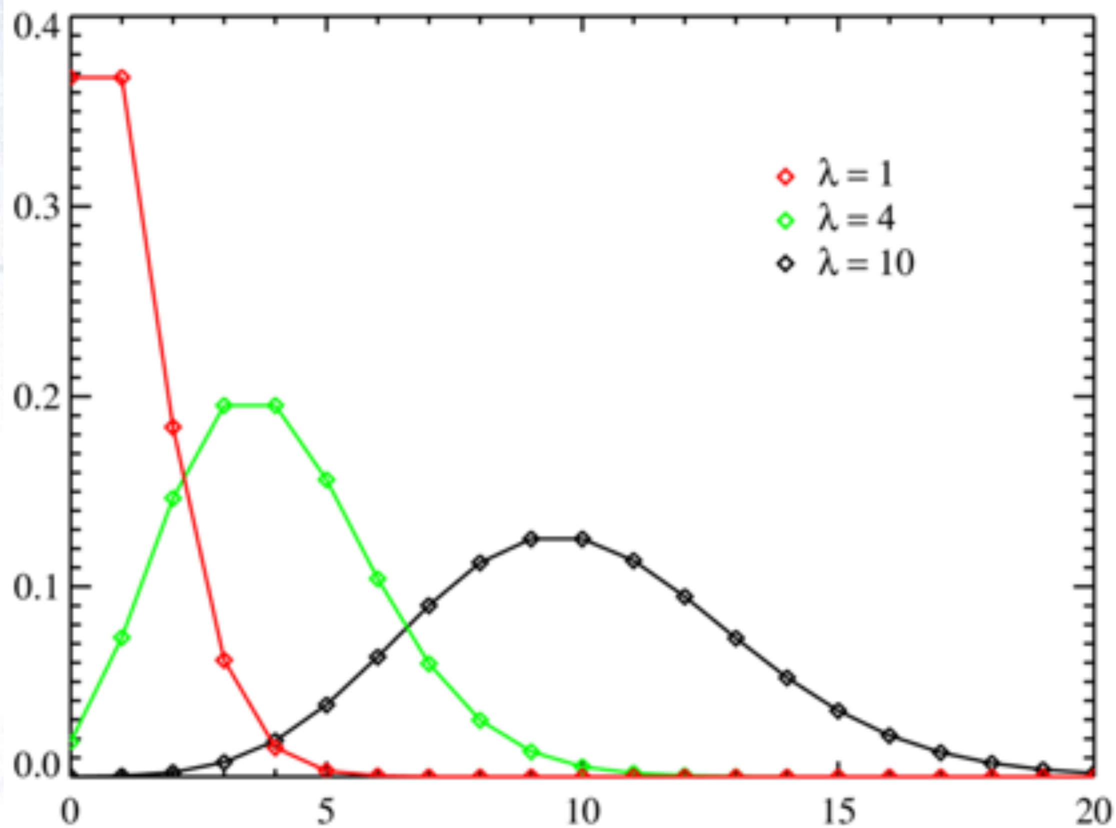
Examples which aren't Binomial experiments:

- Rolling a die until a 6 appears (not fixed number of trials).
- Drawing 5 cards for a poker hand (no replacement ⇒ not independent)

Binomial, **Poisson**, Gaussian

If N → ∞ and p → 0, but Np → λ then a Binomial approaches a Poisson:

$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu}$$



In reality, the approximation is already quite good at e.g. N=50 and p=0.1.

The Poisson distribution only has one parameter, namely λ.
Mean = λ
Variance = λ

So the error on a number is...

...the square root of that number!

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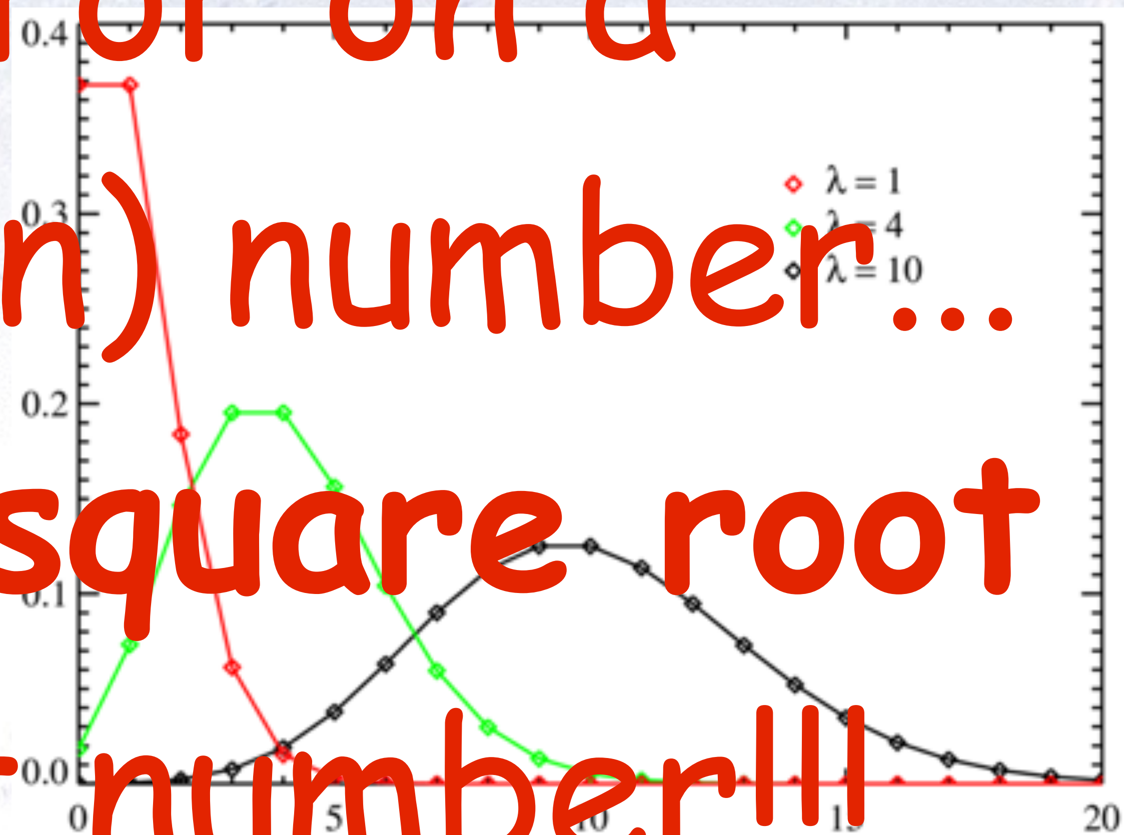

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The error on a (Poisson) number... is the square root of that number!!!

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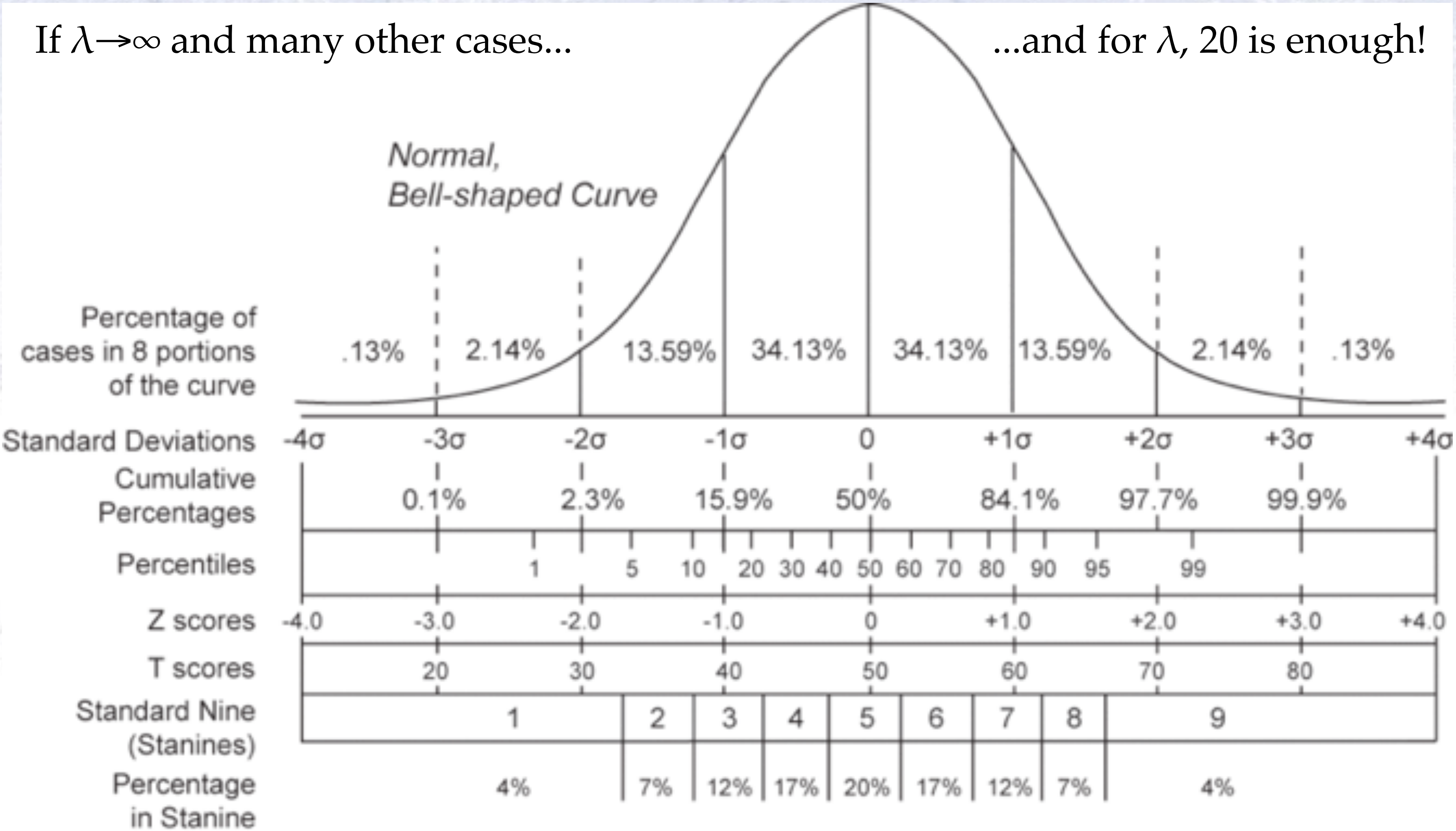
is the square root of that number!!!

Note: The sum of two Poissons with λ_a and λ_b is a new Poisson with $\lambda = \lambda_a + \lambda_b$.
(See Barlow pages 33-34 for proof)

Binomial, Poisson, **Gaussian**

If $\lambda \to \infty$ and many other cases... ...and for λ, 20 is enough!



Binomial, Poisson, **Gaussian**

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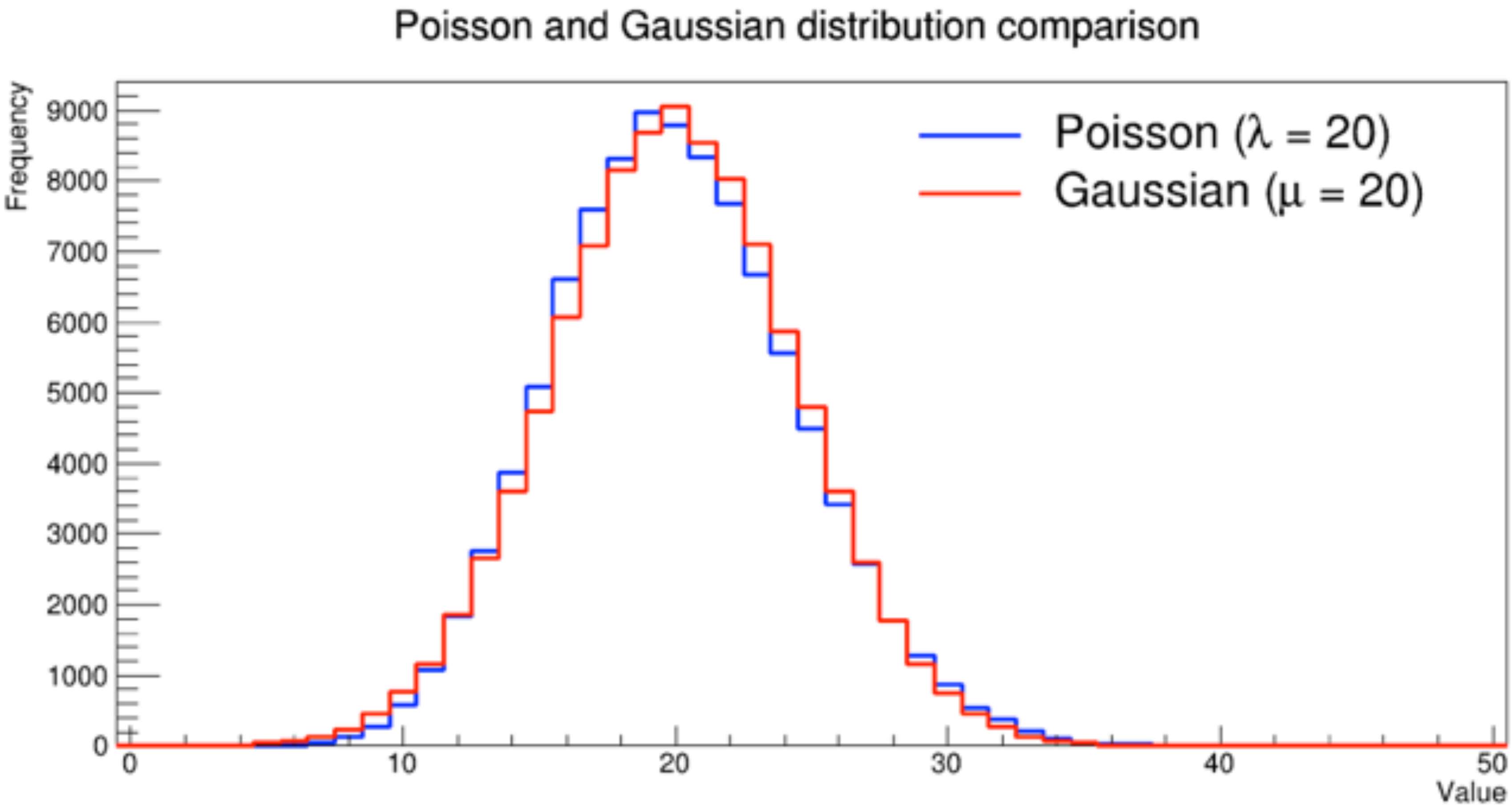

Binomial, Poisson, **Gaussian**

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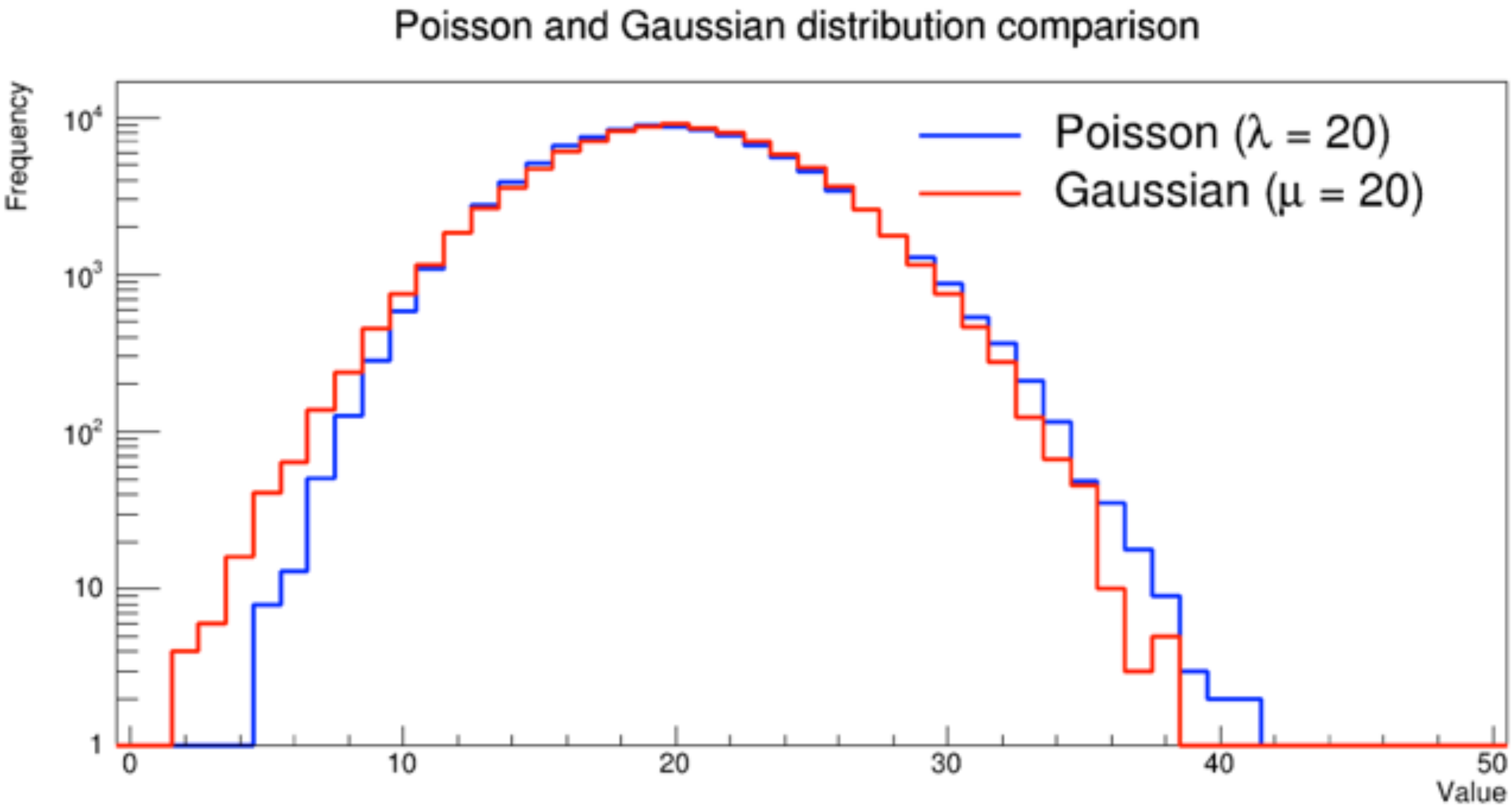
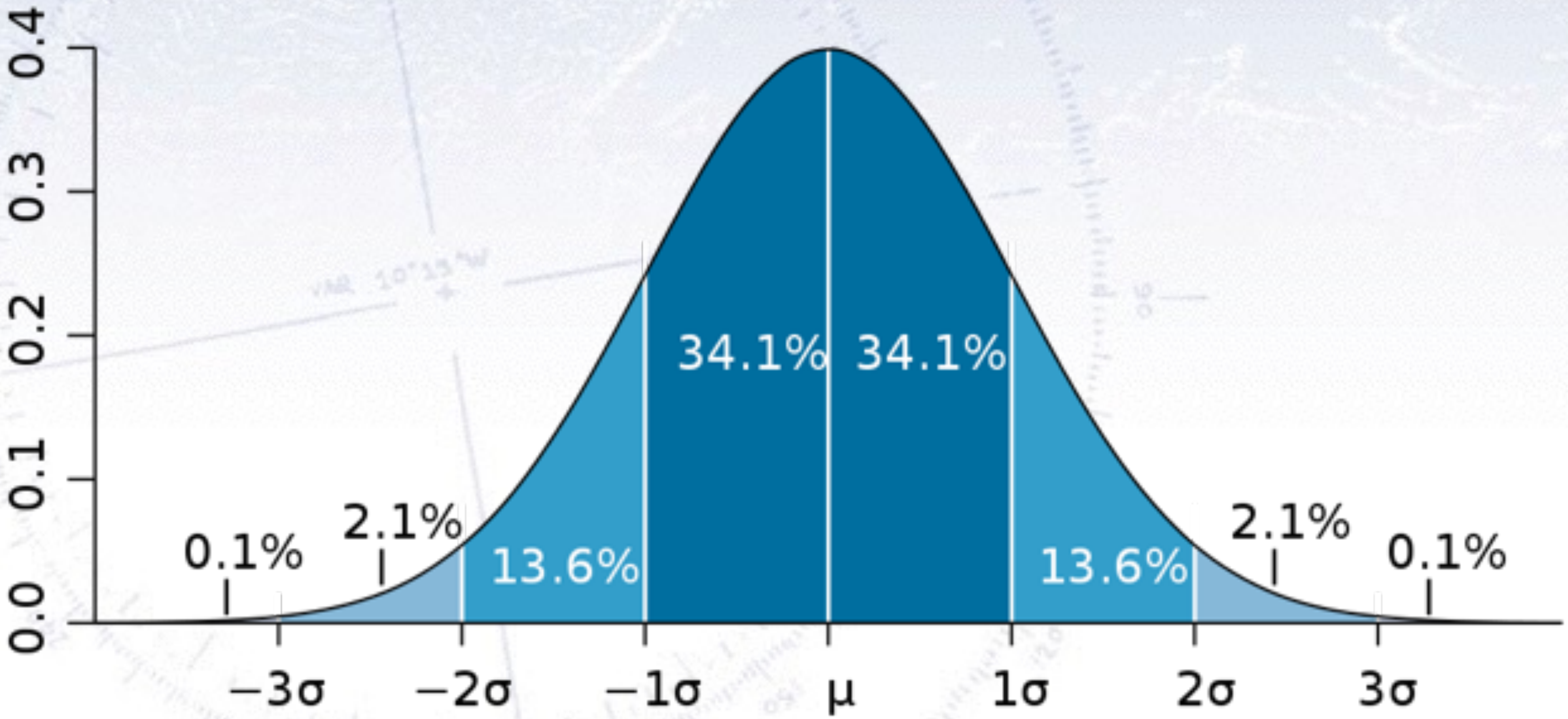


Binomial, Poisson, **Gaussian**

“If the Greeks had known it, they would have deified it.”

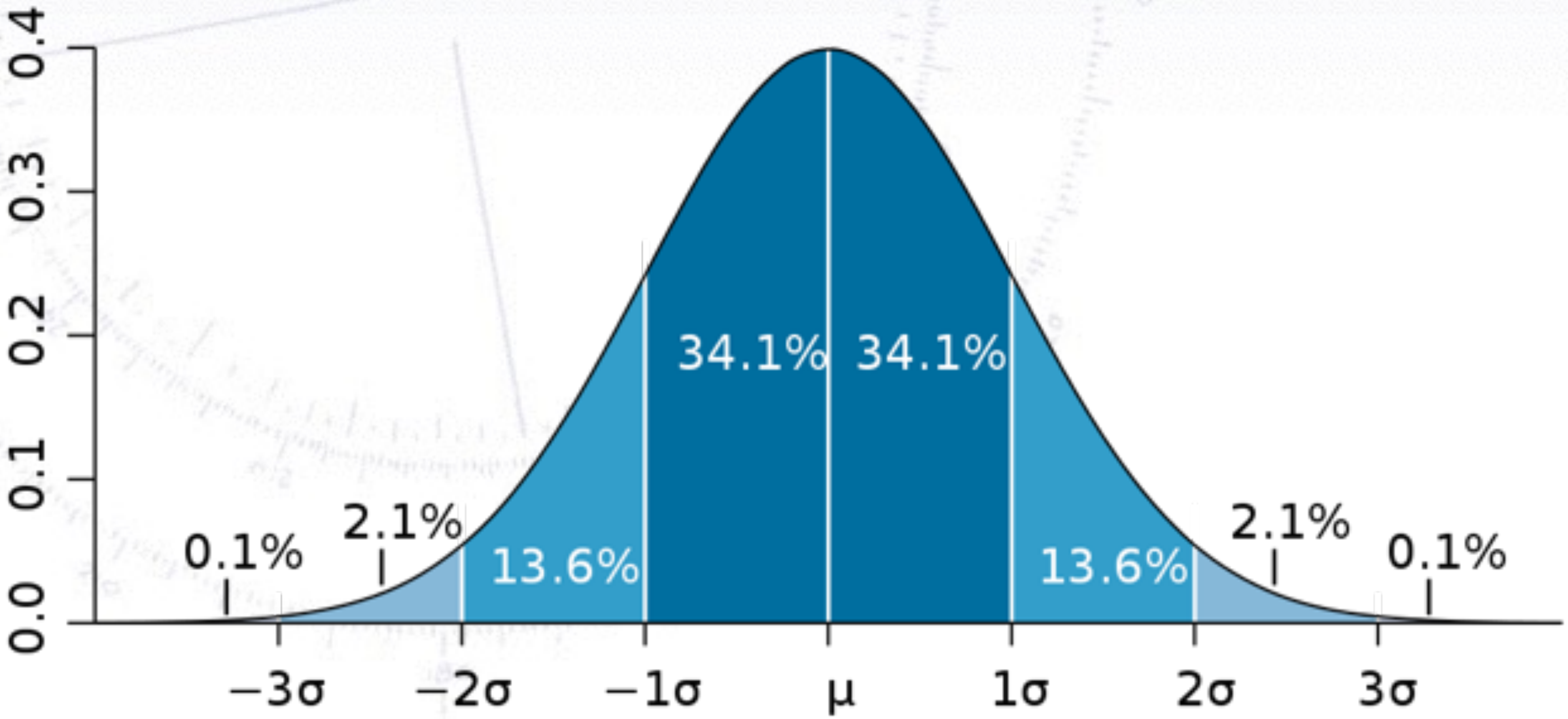


“If the Greeks had known it, they would have deified it. It reigns with serenity and in complete self-effacement amids the wildest confusion. The more huge the mob and the greater the apparent anarchy, the more perfect is its sway. It is the supreme Law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshaled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to be latent all along.” [Karl Pearson]

Binomial, Poisson, **Gaussian**

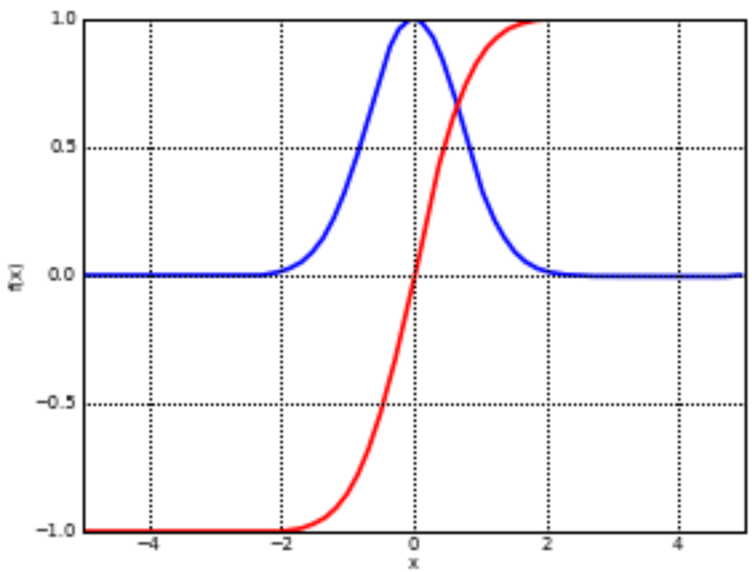
The Gaussian **defines** the way we consider uncertainties.

Range	Inside	Outside
$\pm 1\sigma$	**68** %	32 %
$\pm 2\sigma$	**95** %	5 %
$\pm 3\sigma$	**99.7** %	0.3 %
$\pm 5\sigma$	99.99995 %	0.00005 %



Error function

Imagine taking the integral of a Gaussian:



The error function is "almost" that, only it is defined slightly differently, namely as:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

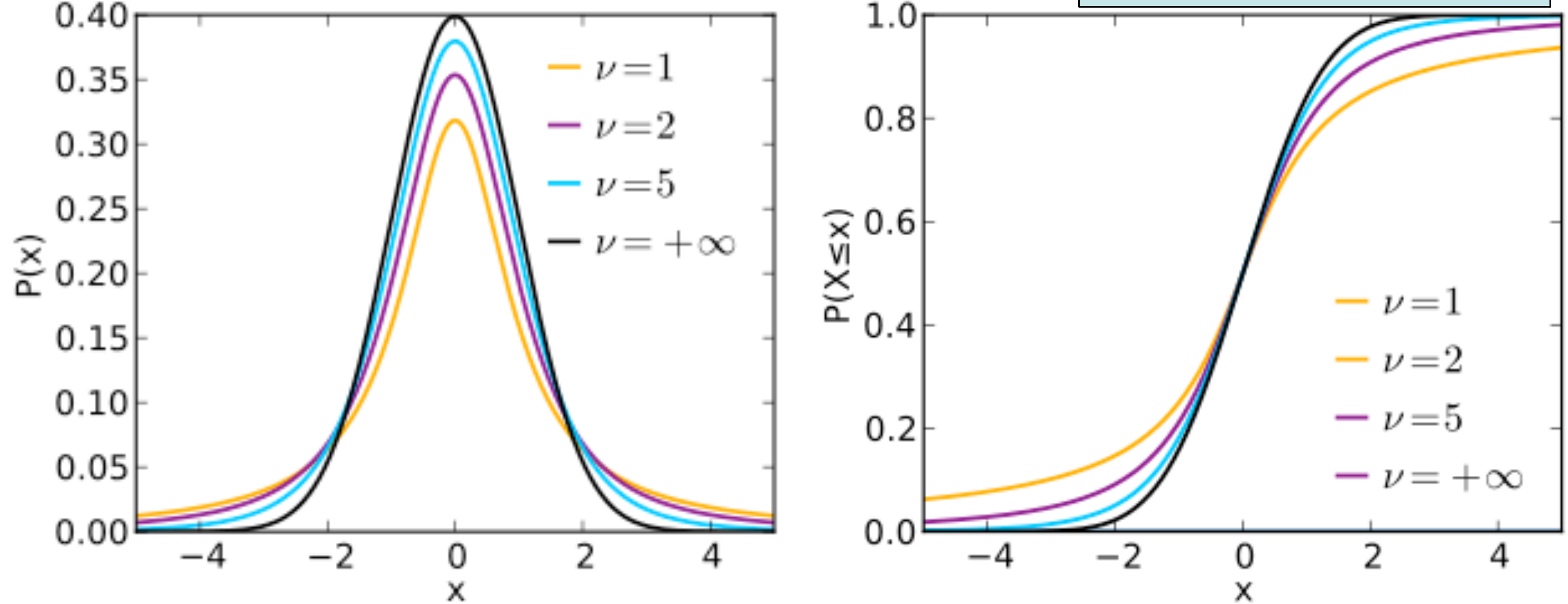
Likewise, there is a complementary error function, which is 1 minus the error function. The functions are used to evaluate Gaussian integrals, i.e. typically "how many sigmas" or "what p-value" does this correspond to.

They also belong to the general class of "sigmoids", i.e. onset functions.

Student's t-distribution

Discovered by William Gosset (who signed "student"), student's t-distribution takes into account lacking knowledge of the variance.

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$



When variance is unknown, estimating it from sample gives additional error:

Gaussian: $$z = \frac{x-\mu}{\sigma}$$

Student's: $$t = \frac{x-\mu}{\hat{\sigma}}$$