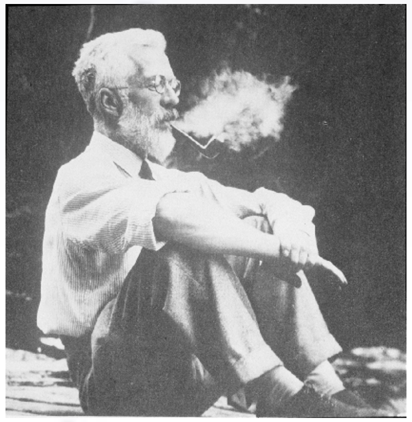
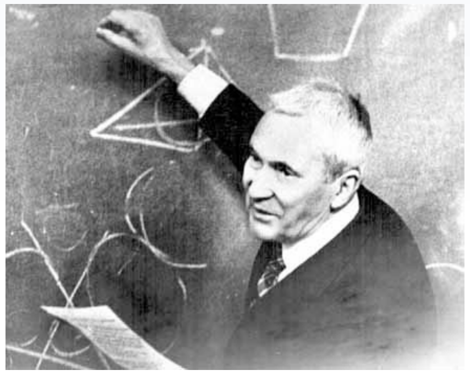
Applied Statistics

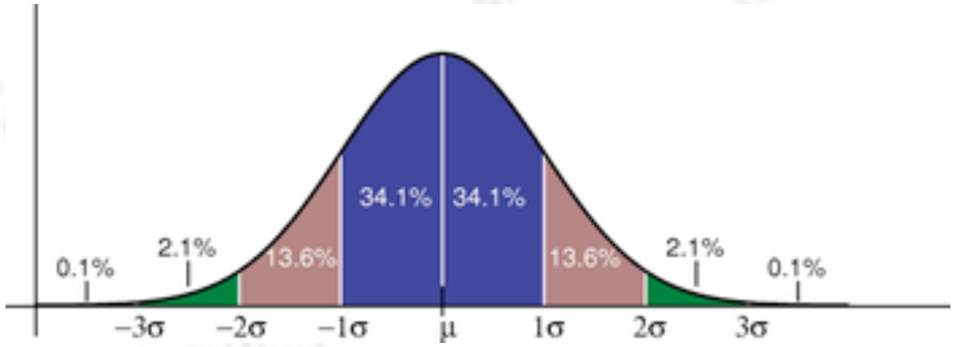
Bayes' Theorem





Troels C. Petersen (NBI)



"Statistics is merely a quantisation of common sense"

Problem

Suppose a drug test can be characterised as follows:
- 99% positive results for users (99% sensitive, i.e. 1% Type I errors).
- 99% negative results for non-users (99% specific, i.e. 1% Type II errors).

If 0.5% of a population is using the drug, and a random person tests positive, what is the chance that he/she is using the drug?

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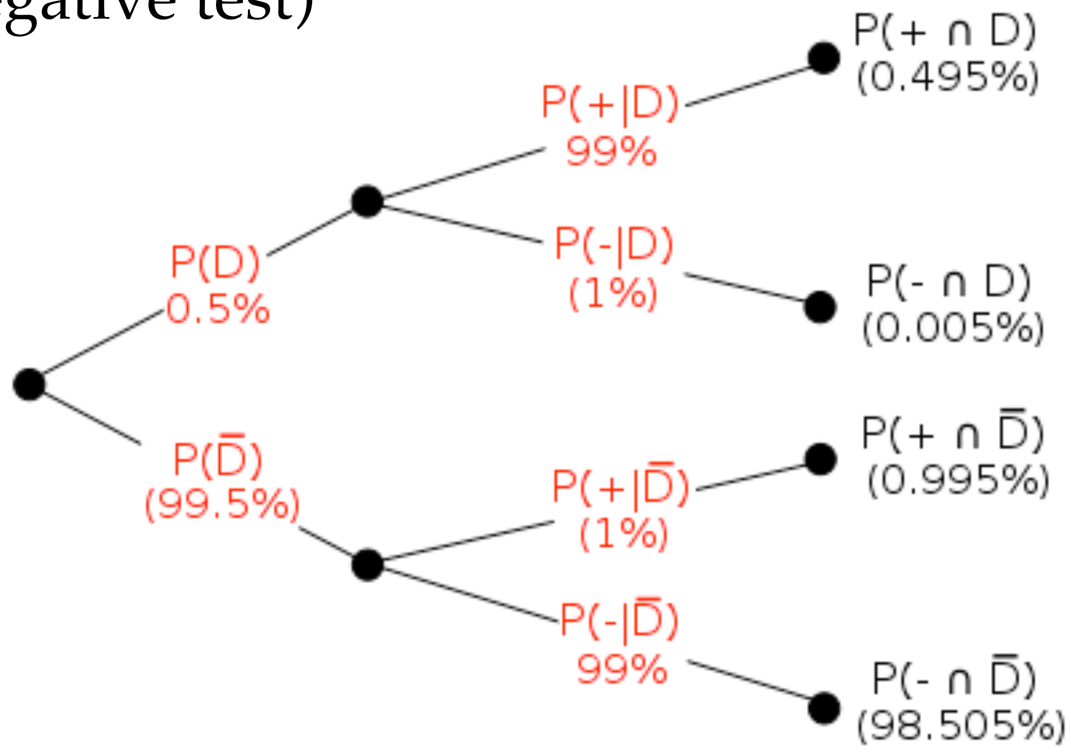
The answer is 33.2%, i.e. NOT very high! The reason is the **prior probability**. False positives (0.995%) are large compared to true positives (0.495%).
(D = user, $\overline{\mathrm{D}}$ = non=user, + = positive test, - = negative test)

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|\bar{D})P(\bar{D})}$$
$$= \frac{0.99 \times 0.005}{0.99 \times 0.005 + 0.01 \times 0.995}$$
$$= 33.2\%.$$



Different versions...

The "original" version of Bayes' Theorem was stated as follows:

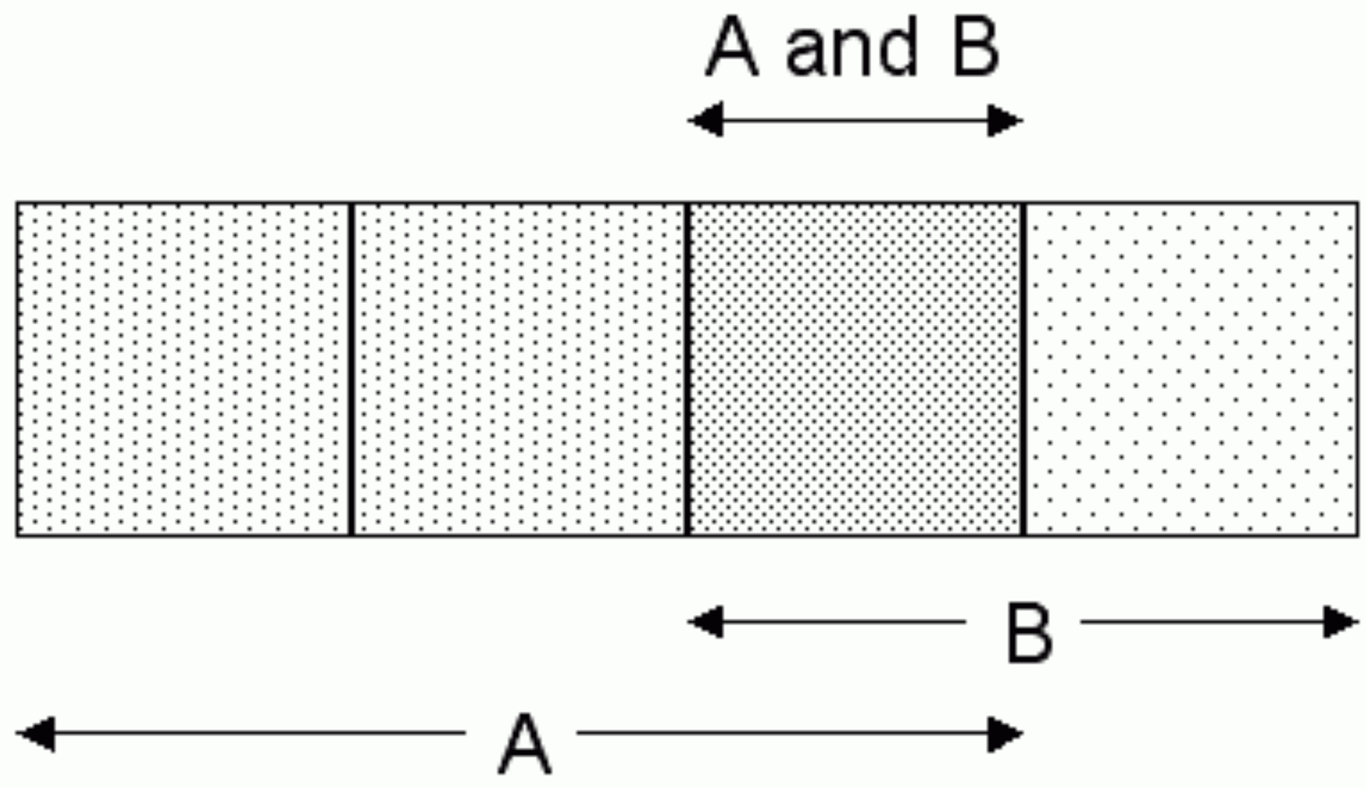
$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}.$$

However, it can be expanded (using the total law of probability) to:

$$P(A|B) = \frac{P(B|A)\,P(A)}{\sum_i P(B|A_i)P(A_i)}.$$

It is in this form, that Bayes' Theorem is most often used.

Bayes' Theorem illustrated



P(A) = 3/4

P(B) = 2/4

P(A and B) = P(AB) = 1/4

P(A|B) = P(AB) / P(B) = (1/4) / (2/4) = 1/2

P(B|A) = P(AB) / P(A) = (1/4) / (3/4) = 1/3

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B|A)}{P(B)}$$

Overview

★ Apply Bayes' theory to our the measurement of a parameter x

- We determine $P(\text{data};x)$, i.e. the likelihood function
- We want $P(x;\text{data})$, i.e. the PDF for x in the light of the data
- Bayes' theory gives:

$$P(x;\text{data}) = \frac{P(\text{data};x)P(x)}{P(\text{data})}$$

$P(\text{data};x)$ the likelihood function, i.e. what we measure

$P(x;\text{data})$ the posterior PDF for x, i.e. in the light of the data

$P(\text{data})$ { prior probability of the data. Since this doesn't depend on x it is essentially a normalisation constant

$P(x)$ { prior probability of x, i.e. encompassing our knowledge of x before the measurement

★ Bayes' theory tells us how to modify our knowledge of x in the light of new data

Bayes' theory is the formal basis of Statistical Inference

From: M. A. Thomson

Example of priors influence

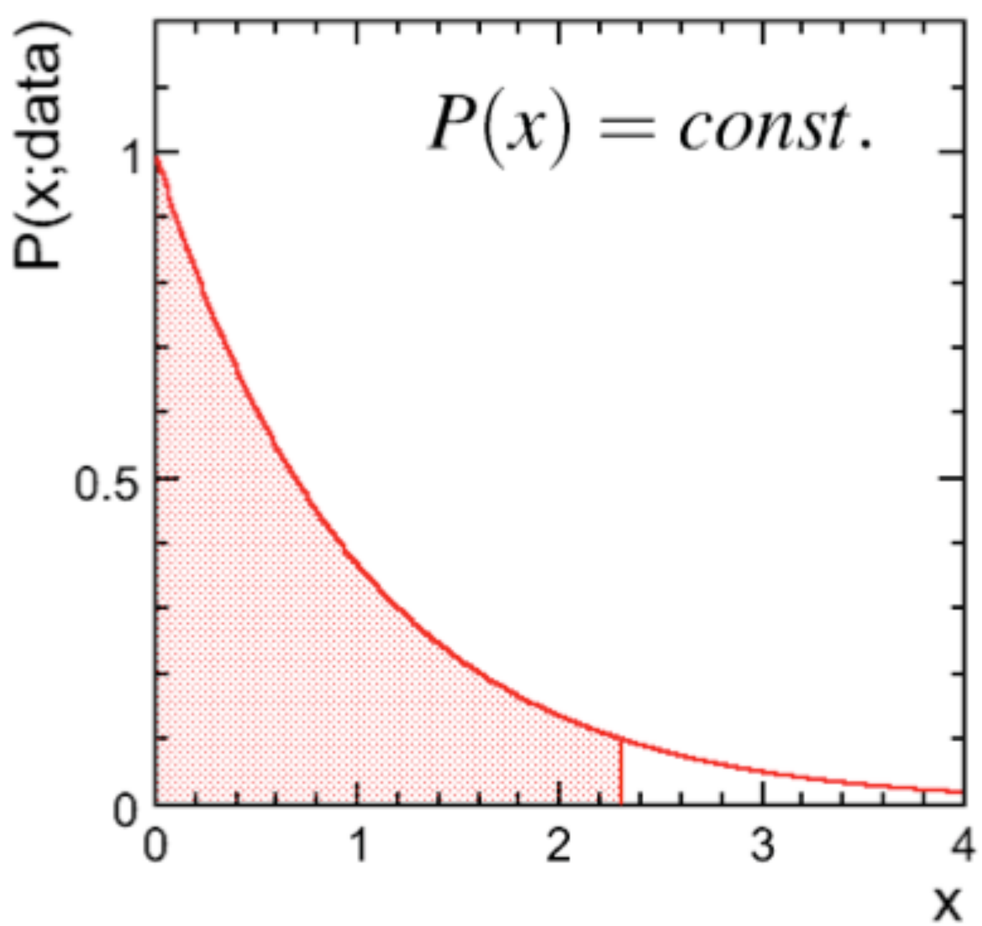
★ See no events…

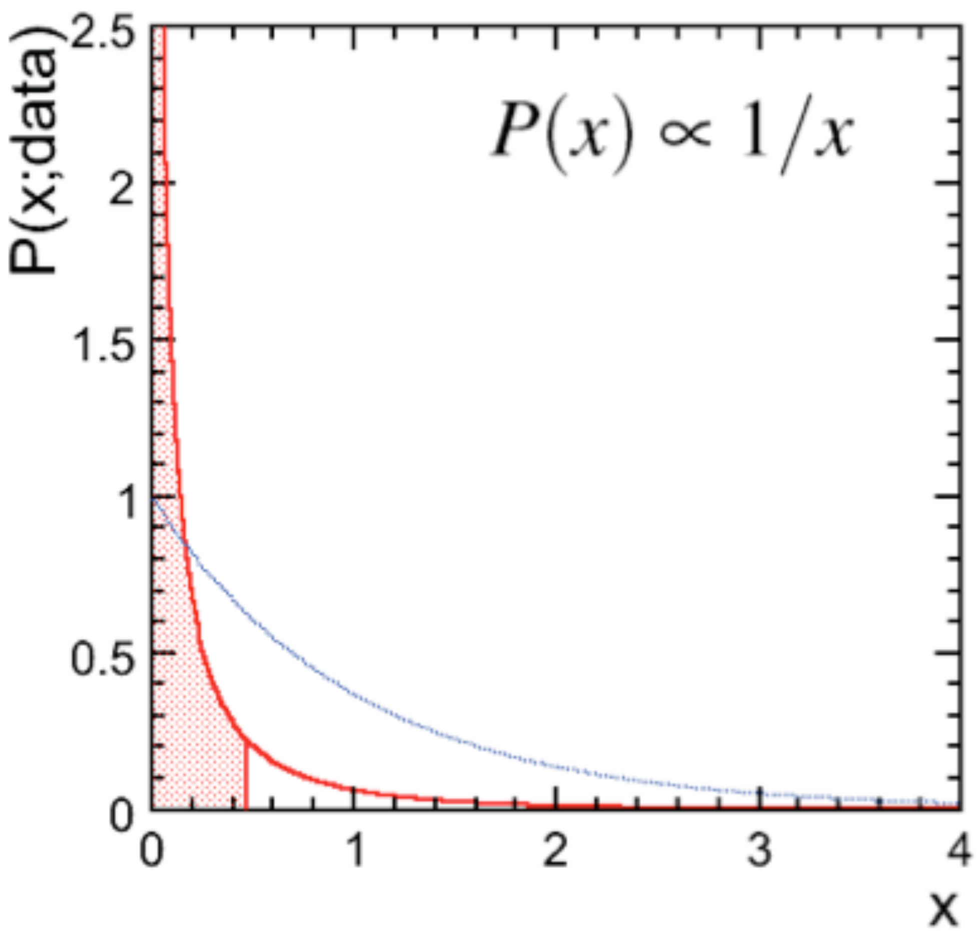
$$P(\text{data};x) = P(0;x) = e^{-x}$$

Poisson prob. for observing 0

Prior flat prior in x : $P(x) = const.$

Prior flat prior in lnx : $P(\ln x) = const.$





★ The Conclusions are very different. Compare regions containing **90 %** of probability

$x < 2.3$

$x < 0.46$

- In this case, the choice of prior is important

From: M. A. Thomson

Frequentist problems...

One of the reasons for Bayesian statistics, is the following problem for frequentist!

Imagine that you observe **5 events in data**, when expecting **0.9 background events**.
Then you can say with 68% confidence, that the signal is in the range **[2.8, 8.4]**.

But what if the expectation was **10.9 background events**?
Then you would say with 68% confidence, that the signal is in the range **[-8.1, -2.5]**.

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While this is technically correct...

it is completely stupid!

We of course knew ahead of time, that the signal is either zero or positive.

The possible solution is to include the prior information, that the number is positive.
But exactly what prior to use? That is the problem.

Interpretations

One way Bayes' Theorem is often used in normal thinking is:

$$P(\text{theory}|\text{data}) \propto P(\text{data}|\text{theory}) \cdot P(\text{theory}).$$

Here, P(data) has been omitted (doesn't depend on parameters, so normalisation).

The trouble is, that it is hard to define P(theory) = a "degree of belief" in a theory.

Perhaps Glen Cowan sums it up best (chapter 1):

> Bayesian statistics provides no fundamental rule for assigning the prior probability to a theory, but once this has been done, it says how one's degree of belief should change in the light of experimental data.

"When the facts change, I change my opinion. What do you do, sir?"

[John Maynard Keynes]

Bonus Slides

The Monty Hall problem

A famous problem involving Bayes' Theorem is the "Monty Hall problem".

There are 3 doors. Behing one door there is a car (prize); behind the other two there are two goats. You are asked to decide the door you want to open. You make your choice, let that be the first one for example. A this point the game dealer (who knows where the car and the goats are) opens one of the other two doors and shows a goat (he has to show you a goat of course, not the car). Then he askes you if you want to change your mind and open the other door instead of your original choice.

The Monty Hall solution I

Solution with Bayes Theorem

Let A_1 the event that the car is behind the first door, and similarly A_2 and A_3.
Let B the event that the game dealer opens the last door.
$P(A_1) = P(A_2) = P(A_3) = 1/3$
The following stand:

- $P(B|A_1) = 1/2$, the game dealer has the choice of opening the door 2 or 3, since you picked the car.
- $P(B|A_2) = 1$, the game dealer is forced to show you the goat behind the door 3, since you picked the first door and the car is behind the second one.
- $P(B|A_3) = 0$, since the game dealer cannot show you the car.

The overall probability of the game dealer opening the third door is:

$$P(B) = \sum_{i=0}^{3} P(B|A_i)P(A_i)$$

$= (1/2)(1/3)+1(1/3)+0 = 1/2$

The Monty Hall solution II

Now, from Bayes theorem:

- $P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P_B} = \frac{(1/2)(1/3)}{(1/2)} = 1/3$
- $P(A_2|B) = \frac{P(B|A_2)P(A_2)}{P_B} = \frac{(1)(1/3)}{(1/2)} = 2/3$
- $P(A_3|B) = \frac{P(B|A_3)P(A_3)}{P_B} = \frac{(0)(1/3)}{(1/2)} = 0$

By compairing the two possibilities, in the case the game dealer opens the third door (event B), by changing our initial guess we double the probability of winning. The problem is symmetrical with the initial guess, of course.
The non-intuitive asymmetry (1/3 against 2/3) lies in the fact the game dealer knows where the car is ($P(B|A_i)$ are different).
The problem can be soled in other way, for example by looking at the 'tree' development.