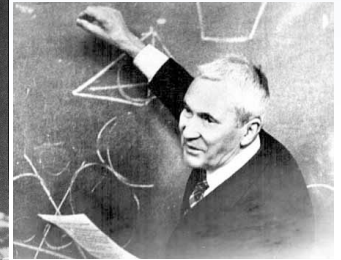
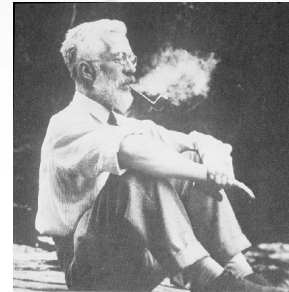
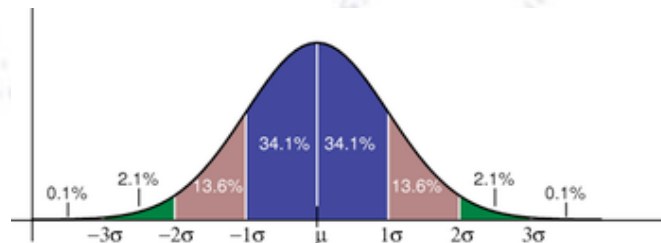


# Applied Statistics

## Error propagation



Troels C. Petersen (NBI)



*"Statistics is merely a quantisation of common sense"*

# Error propagation

Imagine that  $y$  is a function of  $x_i$

$$y(x_i)$$

and that we wish to find the error on  $y$  from the errors on  $x_i$ .

$$\sigma(x_i) = 0.8$$

$$\sigma(y(x_i)) = ?$$

# Error propagation

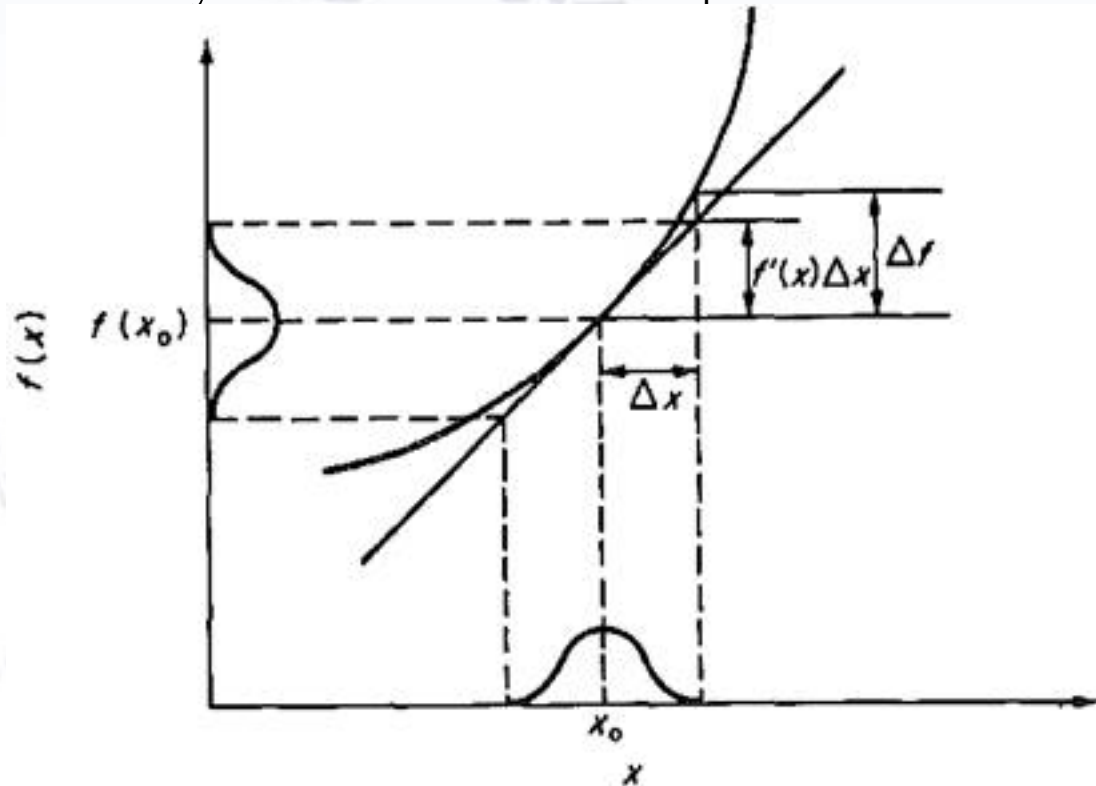
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# Error propagation

Imagine that  $y$  is a function of  $x_i$

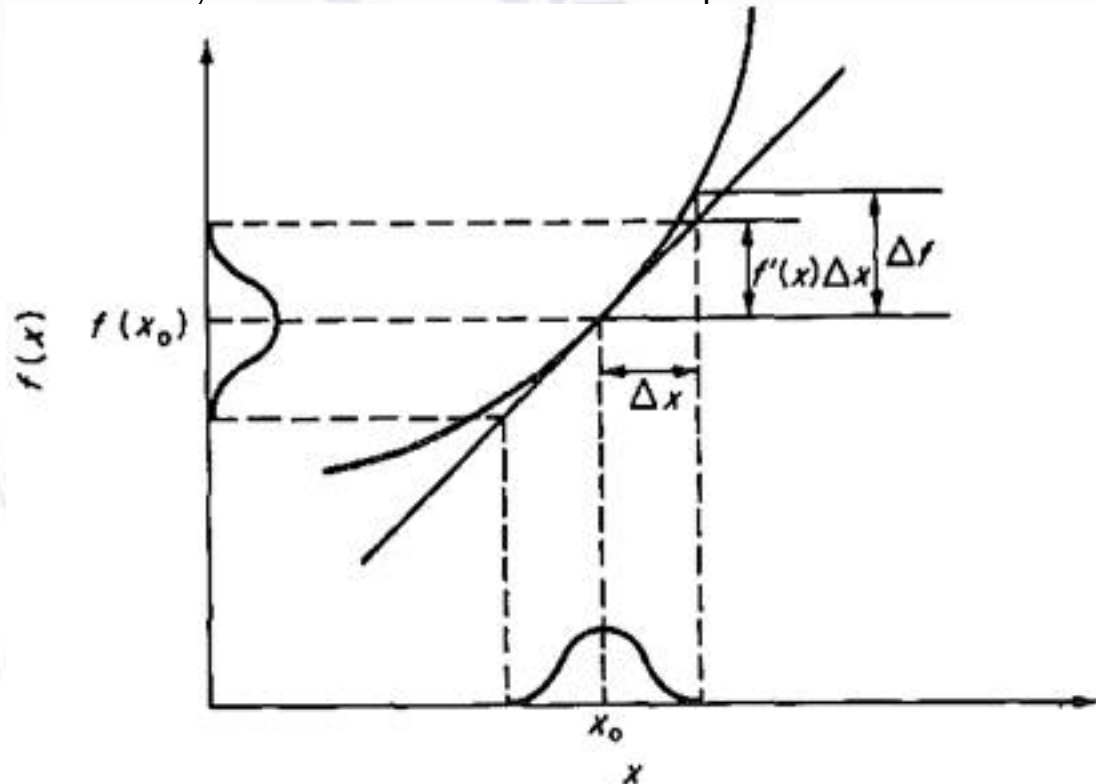
$$y(x_i)$$

and that we wish to find the error on  $y$  from the errors on  $x_i$ .

$$\sigma(x_i) = 0.8$$

$$\sigma(y(x_i)) =$$

$$\frac{\partial y}{\partial x_i} \times 0.8$$





# Error propagation

Imagine that  $y$  is a function of  $x_i$ , and that we wish to find the error on  $y$  from the errors on  $x_i$ . Making a Taylor expansion of the function  $y$  gives:

$$y(\bar{x}) \simeq y(\bar{\mu}) + \sum_i^n \frac{\partial y}{\partial x_i} (x_i - \mu_i)$$

In order to get the uncertainty of  $y$  as a function of the variables  $x_i$  we calculate:

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = E[x^2] - E^2[x]$$

$$E[y(\bar{x})] \simeq y(\bar{\mu})$$

$$E[y^2(\bar{x})] \simeq y^2(\bar{\mu}) + \sum_{i,j}^n \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right] V_{ij}$$

# Error propagation formula

Subtracting the two formulae, we obtain:

$$\sigma_y^2 = \sum_{i,j}^n \left[ \frac{\partial y}{\partial x_i} \quad \frac{\partial y}{\partial x_j} \right]_{\bar{x}=\bar{y}} V_{ij}$$

If there are no correlations, only the diagonal (individual errors) enter:

$$\sigma_y^2 = \sum_i^n \left[ \frac{\partial y}{\partial x_i} \right]_{\bar{x}=\bar{y}}^2 \sigma_i^2$$

# Specific error propagation formula

## Addition

Specific formula:

$$x = u + v$$

$$\sigma_x^2 = \sigma_u^2 + \sigma_v^2 + 2V_{uv}$$

General formula:

$$x = au + bv$$

$$\sigma_x^2 = a^2\sigma_u^2 + b^2\sigma_v^2 + 2abV_{uv}$$

*“When adding numbers, their errors add in quadrature”*

# Specific error propagation formula Multiplication

$$x = uv$$

$$\sigma_x^2 = (v\sigma_u)^2 + (u\sigma_v)^2 + 2uvV_{uv}$$

Dividing by  $x^2$  to get relative terms, we obtain:

$$\frac{\sigma_x^2}{x^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2} + 2\frac{V_{uv}}{uv}$$

*“When multiplying numbers, their RELATIVE errors add in quadrature”*



# Error propagation at work...

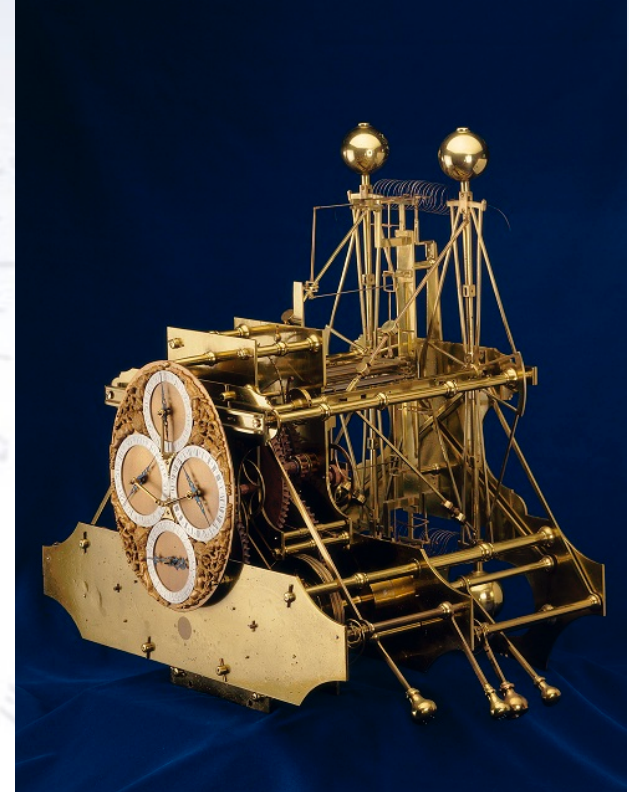


**John Harrison** (24 March 1693 – 24 March 1776)

British clockmaker extraordinaire

“Won” the Longitude Act prize (3 sec/day).

Harrison's first sea clock (H1)

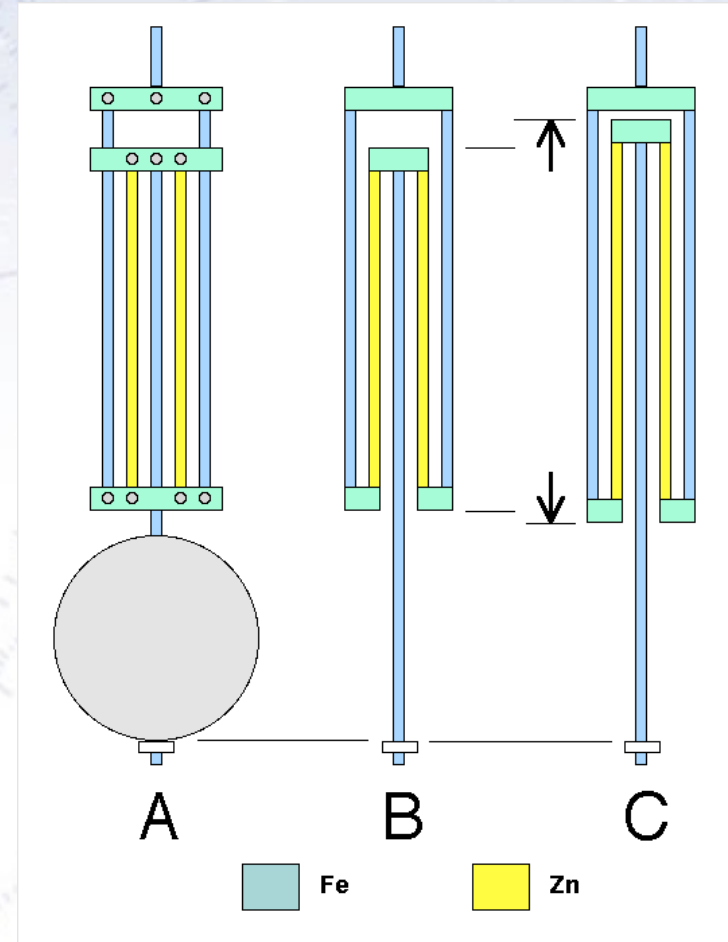
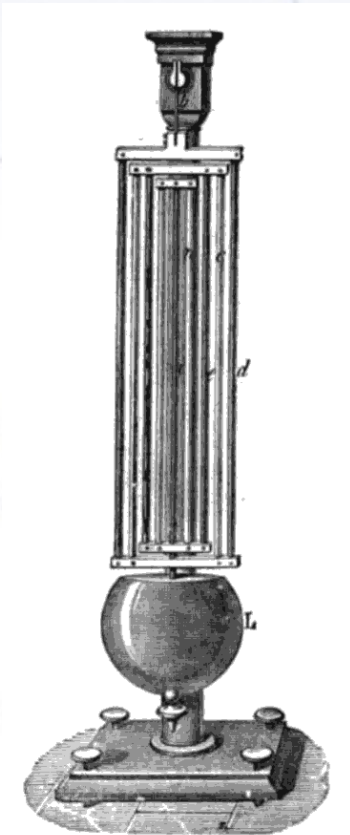


**Harrison build H1-H5.**

K1 (Copy of H4) was used by James Cook.

# Error propagation at work...

Harrison's Gridiron pendulum is designed to cancel the change in length (in fact moment of inertia) with temperature.



Coefficient of thermal expansion:  
Iron =  $11.8 \times 10^{-6} / C^{\circ}$  Zinc =  $30.2 \times 10^{-6} / C^{\circ}$



# Error propagation at more work...

Analysis of tiny differences in Uranus' orbit from Newtonian prediction led to the prediction and discovery of Neptune!

Continuing with Mercury...

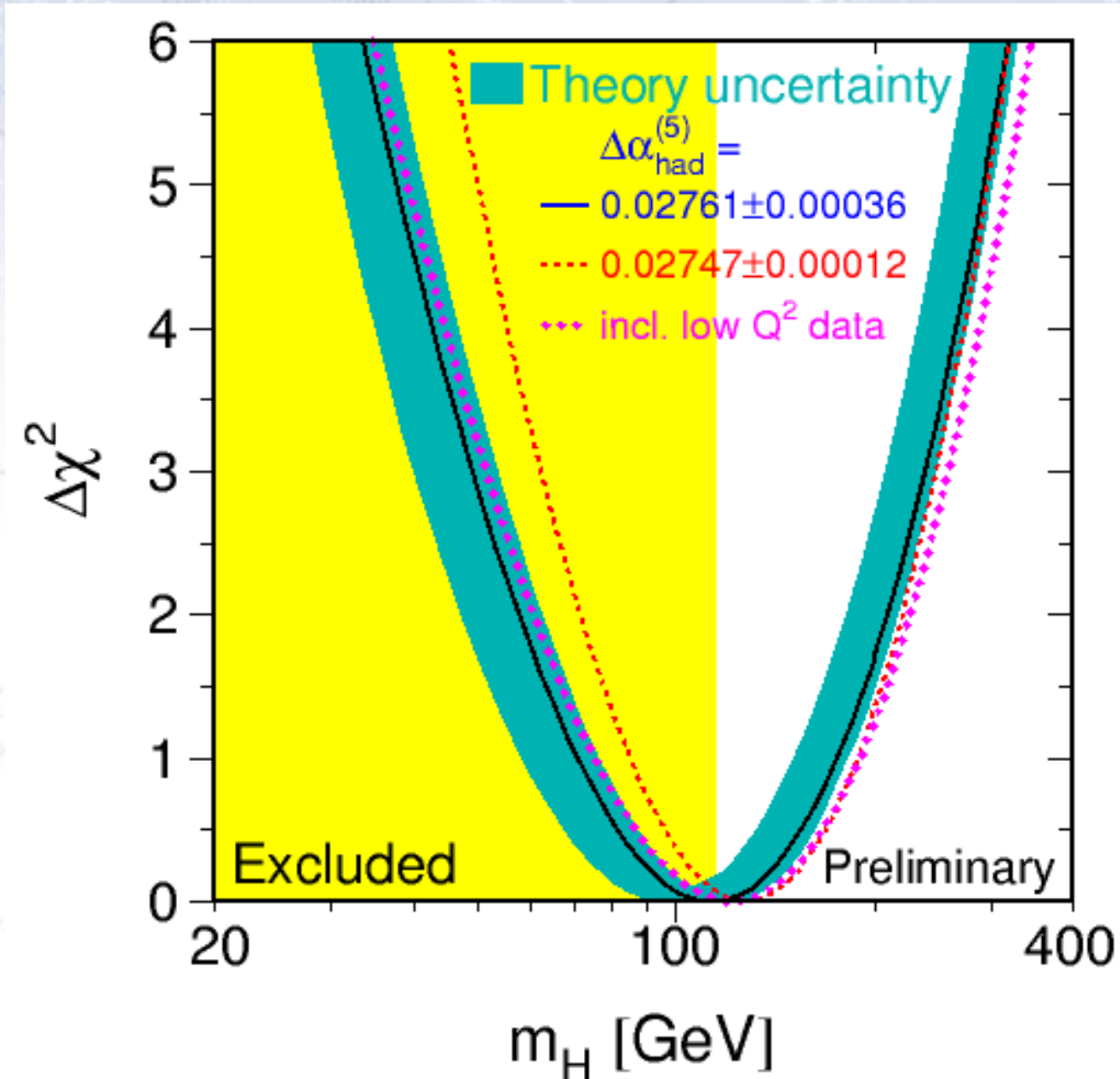
TABLE II. Contributions to the motion of the perihelia of Mercury and the earth.

Cause	$m^{-1}$		Motion of perihelion	
			Mercury	Earth
Mercury	6 000 000	$\pm 1\ 000\ 000$	$0''.025 \pm 0''.00$	$-13''.75 \pm 2''.3$
Venus	408 000	$\pm 1\ 000$	$277.856 \pm 0.68$	$345.49 \pm 0.8$
Earth	329 390	$\pm 300$	$90.038 \pm 0.08$	
Mars	3 088 000	$\pm 3\ 000$	$2.536 \pm 0.00$	$97.69 \pm 0.1$
Jupiter	$1\ 047.39 \pm 0.03$		$153.584 \pm 0.00$	$696.85 \pm 0.0$
Saturn	3 499	$\pm 4$	$7.302 \pm 0.01$	$18.74 \pm 0.0$
Uranus	22 800	$\pm 300$	$0.141 \pm 0.00$	$0.57 \pm 0.0$
Neptune	19 500	$\pm 300$	$0.042 \pm 0.00$	$0.18 \pm 0.0$
Solar oblateness			$0.010 \pm 0.02$	$0.00 \pm 0.0$
Moon				$7.68 \pm 0.0$
General precession (Julian century, 1850)			$5025.645 \pm 0.50$	$5025.65 \pm 0.5$
Sum			$5557.18 \pm 0.85$	$6179.1 \pm 2.5$
Observed motion			$5599.74 \pm 0.41$	$6183.7 \pm 1.1$
Difference			$42.56 \pm 0.94$	$4.6 \pm 2.7$
Relativity effect			$43.03 \pm 0.03$	$3.8 \pm 0.0$



Urbain Le Verrier (1811-1877)

Advanced example of error propagation (Higgs particle mass):





# Reporting uncertainties

The systematic uncertainties of a measurement should be reported in a table, and if measurements are combined, the correlation needs consideration.

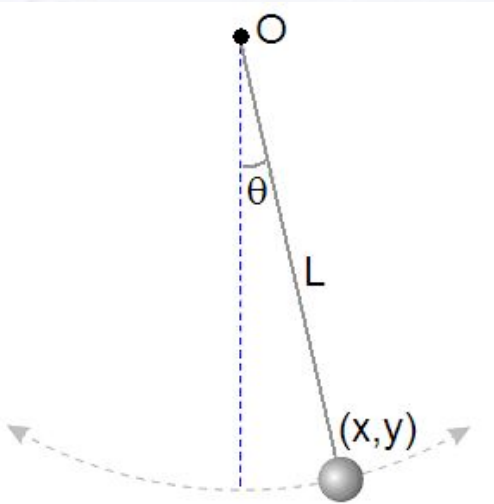
CDF II preliminary L = 200 pb<sup>-1</sup>

m <sub>T</sub> Uncertainty [MeV]	Electrons	Muons	Common
Lepton Scale	30	17	17
Lepton Resolution	9	3	0
Recoil Scale	9	9	9
Recoil Resolution	7	7	7
u <sub>  </sub> Efficiency	3	1	0
Lepton Removal	8	5	5
Backgrounds	8	9	0
p <sub>T</sub> (W)	3	3	3
PDF	11	11	11
QED	11	12	11
<b>Total Systematic</b>	<b>39</b>	<b>27</b>	<b>26</b>
<b>Statistical</b>	<b>48</b>	<b>54</b>	<b>0</b>
<b>Total</b>	<b>62</b>	<b>60</b>	<b>26</b>

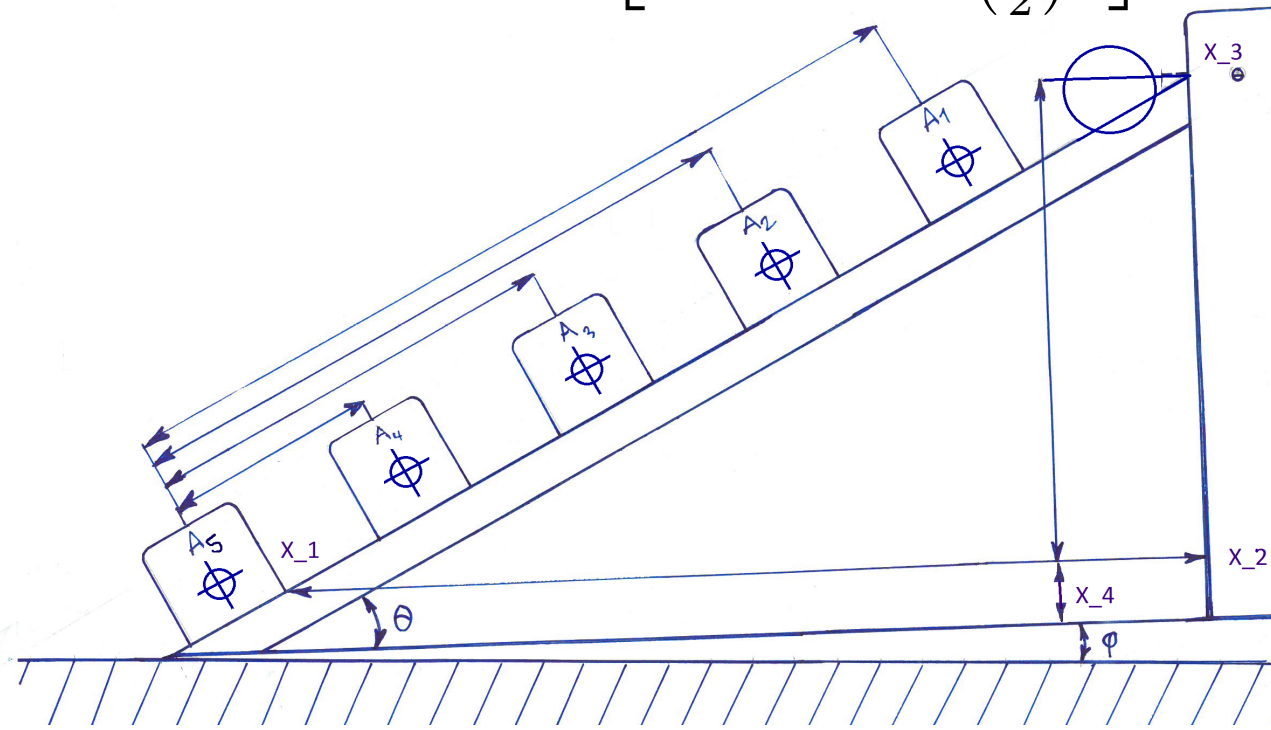
# Applying error propagation

For the project, we'll be working with two measurements, which result from two formulae. Work out the error propagation formula for these two cases, and use these, when we discuss typical size of uncertainties.

$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2} \right]$$



$$g = L \left( \frac{2\pi}{T} \right)^2$$



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$$g = \frac{a}{\sin(\theta \pm \Delta\theta)} \left[ 1 + \frac{2}{5} \frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2} \right]$$

Yes, do this today or soon after, possibly in groups.

$$g = L \left( \frac{2\pi}{T} \right)^2$$

